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On the Harish-Chandra Homomorphism for Quantum Superalgebras

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Abstract: In this paper, we introduce the Harish-Chandra homomorphism for the quantum superalgebra $\mathrm{U}_q(\mathfrak{g})$ associated with a simple basic Lie superalgebra $\mathfrak{g}$ and give an explicit description of its image. We use it to prove that the center of $\mathrm{U}_q(\mathfrak{g})$ is isomorphic to a subring of the ring $J(\mathfrak{g})$ of exponential super-invariants in the sense of Sergeev and Veselov, establishing a Harish-Chandra type theorem for $\mathrm{U}_q(\mathfrak{g})$. As a byproduct, we obtain a basis of the center of $\mathrm{U}_q(\mathfrak{g})$ with the aid of quasi-R-matrix.

Contents

1. Introduction

Harish-Chandra introduced a homomorphism, known as the *Harish-Chandra homomorphism*, for semisimple Lie algebras in the study of unitary representations of semisimple Lie groups in 1951 [19]. Later on, the Harish-Chandra homomorphism was developed for infinite dimensional Lie algebras [28,36], Lie superalgebras [28,40,41] and quantum groups [3,9,25,38,43].

Knowledge about the invariants and the center of quantum superalgebras is not merely of mathematical interest but is also physically important. On one hand, the study of the centralizer of a (quantized) universal enveloping (super)algebra is an indispensable part of its representation theory. On the other hand, the study of physical theories to a large extent involves the exploration of the invariants of the symmetry algebras, which usually correspond to certain physical observables. The *Harish-Chandra homomorphism* reveals many connections between the center of the enveloping (super)algebras or their quantization and the (super)symmetric polynomials as well as the highest weight representations of the corresponding algebras, and it has been one of the most inspiring themes in Lie theory.

Let $\mathfrak{g}$ be a semisimple Lie algebra (resp., a basic Lie superalgebra) over $\mathbb{C}$ with triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}^+$ (resp., $\mathfrak{n}^-$) is the positive (resp., negative) part of $\mathfrak{g}$ corresponding to a positive root system Φ^+. Using the PBW Theorem, we have the decomposition $\mathrm{U}(\mathfrak{g}) = \mathrm{U}(\mathfrak{h}) \oplus \big(\mathfrak{n}^-\mathrm{U}(\mathfrak{g}) + \mathrm{U}(\mathfrak{g})\mathfrak{n}^+\big)$. Let $\pi : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{h}) = \mathrm{S}(\mathfrak{h})$ be the associated projection. The restriction of π to the center $\mathcal{Z}(\mathrm{U}(\mathfrak{g}))$ of $\mathrm{U}(\mathfrak{g})$ is an algebra homomorphism, and the composite $\gamma_{-\rho} \circ \pi : \mathcal{Z}(\mathrm{U}(\mathfrak{g})) \to \mathrm{S}(\mathfrak{h})$ of π with a "shift" by the Weyl vector ρ is called the *Harish-Chandra homomorphism* of $\mathrm{U}(\mathfrak{g})$. The famous Harish-Chandra isomorphism theorem says that $\gamma_{-\rho} \circ \pi$ induces an isomorphism from $\mathcal{Z}(\mathrm{U}(\mathfrak{g}))$ to the algebra of W-invariant polynomials if $\mathfrak{g}$ is a semisimple Lie algebra or the algebra of W-invariant supersymmetric polynomials if $\mathfrak{g}$ is a classical Lie superalgebra. More details can be found in [7, Chap. 11] for classical Lie algebras, and [8, Sect. 2.2], [35, Chapt. 13] for classical Lie superalgebras.

Quantum groups, first appearing in the theory of quantum integrable system, were formalized independently by Drinfeld and Jimbo as certain special Hopf algebras around 1984 [11,24], including deformations of universal enveloping algebras of semisimple Lie algebras and coordinate algebras of the corresponding algebraic groups. In 1990, by the aid of the Universal R-matrix, Rosso [38] defined a significant ad-invariant bilinear form on $\mathrm{U}_q(\mathfrak{g})$ at a generic value q of the parameter. The form, often referred to as the Rosso form or quantum Killing form, could also be obtained by using Drinfeld double construction. Tanisaki [43,44] described this form by skew-Hopf pairing between the positive part and the negative part of the quantum algebra and obtained the quantum analogue of the Harish-Chandra isomorphism between $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ and the subalgebra of W-invariant Laurent polynomials. As an application, the generators and the defining relations for $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ have been obtained in [5,10,33].

Associated with the generalization of Lie algebras to Lie superalgebras, many researchers have investigated the quantization of universal enveloping superalgebras in recent years. Drinfeld-Jimbo quantum superalgebras [45,51] are a class of quasi-triangular Hopf superalgebras, depending on the choice of Borel subalgebras, which were introduced in the early 1990s. As a supersymmetric version of quantum groups, quantum superalgebras have a natural connection with supersymmetric integrable lattice models and conformal field theories. They have been found applications in various areas, including in the study of the solution of quantum Yang-Baxter Eq. [18], construction of topological invariants of knots and 3-mainfolds [49,50,53] and so on. Quantum superalgebras have been investigated extensively by many authors in aspects such as Serre relations, PBW basis, universal R-matrix [45,46], crystal bases [30,31], invariant theory [32], highest weight representations [15,54,55] and so on.

The following questions for quantum superalgebras are natural and fundamental comparing to Lie (super)algebras and quantum groups: What is the Harish-Chandra isomorphism for quantum superalgebras? How to determine the center of quantum superalgebras? The purpose of the present work is to answer these questions.

Let $\mathfrak{g}$ be a simple basic Lie superalgebra, except for $A(1,1)$, with root system $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$, and let $\mathrm{U} = \mathrm{U}_q(\mathfrak{g})$ be the associated quantum superalgebra over $k = K(q^{\frac{1}{2}})$, where K is a field of characteristic 0 and q is an indeterminate. The Weyl group and Weyl vector are denoted by W and ρ, respectively. Let $\Lambda = \left\{\lambda \in \mathfrak{h}^* \,\middle|\, \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z},\ \forall \alpha \in \Phi_{\bar{0}}\right\}$ be the integral weight lattice, where $\mathfrak{h}^*$ is the dual space of the cartan subalgebra $\mathfrak{h}$.

The Cartan subalgebra U^0 is the group ring of $\mathbb{Z}\Phi$ with basis $\left\{\mathbb{K}_\mu \,\middle|\, \mu \in \mathbb{Z}\Phi\right\}$ and multiplication $\mathbb{K}_\mu \mathbb{K}_\nu = \mathbb{K}_{\mu+\nu}$ for all $\mu, \nu \in \mathbb{Z}\Phi$. For each $\lambda \in \Lambda$, we define an automorphism $\gamma_\lambda : \mathrm{U}^0 \to \mathrm{U}^0$ by $\gamma_\lambda(\mathbb{K}_\mu) = q^{(\lambda,\mu)}\mathbb{K}_\mu$ for all $\mu \in \mathbb{Z}\Phi$.

Let Π be the simple roots of distinguished borel subalgebra if $\mathfrak{g} = A(n,n)$ with $n \neq 1$, and let $\mathbb{Z}\tilde{\Phi}$ be the free abelian group with $\mathbb{Z}$-basis Π. We set

$$Q = \begin{cases} \mathbb{Z}\tilde{\Phi}, & \text{for } \mathfrak{g} = A(n,n), \\ \mathbb{Z}\Phi, & \text{otherwise.} \end{cases}$$

Thus, the root system of $A(n,n)$ is $\mathbb{Z}\Phi = \mathbb{Z}\tilde{\Phi}/\mathbb{Z}\gamma$ for some γ. Define the standard partial order relation on Q by $\lambda \leqslant \mu \Leftrightarrow \mu - \lambda \in \sum_{i\in\mathbb{I}} \mathbb{Z}_+\alpha_i$.

There is a triangular decomposition $\mathrm{U} = \mathrm{U}^-\mathrm{U}^0\mathrm{U}^+$, where U^- and U^+ are the negative and positive parts of U, respectively. Clearly U, U^- and U^+ are all Q-graded algebras. The triangular decomposition implies a direct sum decomposition

$$\mathrm{U}_0 = \mathrm{U}^0 \oplus \bigoplus_{\nu>0} \mathrm{U}^-_{-\nu}\mathrm{U}^0\mathrm{U}^+_\nu,$$

where U_0 is the component of degree 0 of U, and U^+_ν (resp., $\mathrm{U}^-_{-\nu}$) is the component of degree ν (resp., $-\nu$) of U^+ (resp., U^-) for $\nu > 0$. Note that the projection map $\pi : \mathrm{U}_0 \to \mathrm{U}^0$ is an algebra homomorphism. From now on, we do not make a distinction between the element in $\mathbb{Z}\Phi$ and Q if no confusion emerges.

We observe that the center $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ of $\mathrm{U}_q(\mathfrak{g})$ is contained in U_0 by Corollary 3.7. Inspired by the quantum group case, we define the *Harish-Chandra homomorphism* $\mathcal{HC}$ of $\mathrm{U}_q(\mathfrak{g})$ to be the composite

$$\mathcal{HC} : \mathcal{Z}(\mathrm{U}_q(\mathfrak{g})) \hookrightarrow \mathrm{U}_0 \xrightarrow{\pi} \mathrm{U}^0 \xrightarrow{\gamma_{-\rho}} \mathrm{U}^0.$$

To establish the Harish-Chandra type theorem for quantum superalgebras, we need to describe the image of $\mathcal{HC}$. Recall that a root $\alpha \in \Phi$ is isotropic if $(\alpha, \alpha) = 0$, and the set of isotropic roots is denoted by Φ_{iso}. Set

$$(\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}} = \left\{ \sum_{\mu \in 2\Lambda \cap \mathbb{Z}\Phi} a_\mu \mathbb{K}_\mu \in \mathrm{U}^0 \middle| a_{w\mu} = a_\mu,\ \forall w \in W; \right.$$
$$\left. \sum_{\mu \in A^\alpha_\nu} a_\mu = 0,\ \forall \alpha \in \Phi_{\mathrm{iso}} \text{ with } (\nu, \alpha) \neq 0 \right\},$$

where $A^\alpha_\nu = \{\nu + n\alpha \mid n \in \mathbb{Z}\}$ for each $\nu \in \Lambda$ and $\alpha \in \Phi_{\mathrm{iso}}$. The notation is consistent with the one in quantum groups [23, Sect. 6.6] and the one in classical Lie superalgebras [8, Sect. 2.2.4]. Then the image of $\mathcal{HC}$ is contained in $(\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$, which is essentially derived from character formulas of Verma modules and simple modules of $\mathrm{U}_q(\mathfrak{g})$, certain automorphisms of $\mathrm{U}_q(\mathfrak{g})$ and nontrivial homomorphisms between Verma modules; see Lemmas 5.2, 5.3, 5.4.

Now we can state our main theorem.

Theorem A. *The Harish-Chandra homomorphism $\mathcal{HC}$ for the quantum superalgebra* $\mathrm{U}_q(\mathfrak{g})$ *associated to a simple basic Lie superalgebra* $\mathfrak{g}$ *induces an isomorphism from* $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ *to* $(\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$.

The Lie superalgebra $\mathfrak{g} = A(1, 1)$ is very special. The image of $\mathcal{HC}$ is contained in $(\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$, while whether the $\mathcal{HC}$ is surjective is not known to us yet; see Remark 5.8.

We noticed that Batra and Yamane have introduced the generalized quantum group $U(\chi, \pi)$ associated with a bi-character χ and established a Harish-Chandra type theorem for describing its (skew) center in [3]. Furthermore, they conjectured a basis of the skew center of generalized quantum groups indexed by irreducible highest weight modules [4]. While the quantum superalgebra $\mathrm{U}_q(\mathfrak{s})$ of a basic classical Lie superalgebra $\mathfrak{s}$ has been identified with a subalgebra of $\hat{U}^\sigma$ involving a new generator σ, so does the image of Harish-Chandra homomorphism (see [3]). It is not known whether one can derive the Harish-Chandra type theorem for quantum superalgebra $\mathrm{U}_q(\mathfrak{s})$ from [3].

As an application of Theorem A, we obtain a basis of $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ by using quasi-R-matrix.

Theorem B. *The center* $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ *has a basis, which is constructed by using quasi-R-matrix and parametrized by* $\left\{\lambda \in \Lambda \cap \frac{1}{2}\mathbb{Z}\Phi \,\middle|\, \dim L(\lambda) < \infty\right\}$*, where* $L(\lambda)$ *is an irreducible module of Lie superalgebra* $\mathfrak{g}$ *with the highest weight* λ.

To prove Theorem A, it suffices to prove $\mathcal{HC}$ is injective and the image $\mathcal{HC}$ is equal to $(\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$. For the injectivity, we establish a key Proposition 3.4 by using the character formula of typical finite-dimensional modules of $\mathrm{U}_q(\mathfrak{g})$, which is a super version of Tanisaki's result for quantum algebras [43, Sect. 3.2].

The difficulty is proving the image of $\mathcal{HC}$ is equal to $(\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$. With the help of the well-known classical Lie theory of semisimple Lie algebras, one can prove the surjectivity for quantum groups by using induction on the weights. However, the technique does not apply to quantum superalgebras, where one encounters two main obstacles:

1): There are infinitely many $\Phi^+_{\bar{0}}$-dominant weights less than a given $\Phi^+_{\bar{0}}$-dominant weight with respect to the standard partial order if $\mathfrak{g}$ is of type I.

2): Besides the condition of the $\Phi_{\bar{0}}^{+}$-dominant integral, an extra condition for the finiteness of the dimension of an irreducible $\mathfrak{g}$-module $L(\lambda)$ is that λ satisfies the hook partition if $\mathfrak{g}$ is of type II.

We notice that the close connection between $K(\mathfrak{g})$, $J(\mathfrak{g})$ and $K(\mathrm{U}_q(\mathfrak{g}))$ will help us to overcome the obstacles, where $K(\mathfrak{g})$ and $K(\mathrm{U}_q(\mathfrak{g}))$ are the Grothendieck rings of $\mathfrak{g}$ and $\mathrm{U}_q(\mathfrak{g})$, respectively, and $J(\mathfrak{g})$ is the ring of Laurent supersymmetric polynomials (also called *ring of exponential super-invariants* in [42]). Recall Sergeev and Veselov's isomorphism [42] $\mathrm{Sch}: K(\mathfrak{g}) \xrightarrow{\sim} J(\mathfrak{g})$, where Sch is the supercharater map, and the injective algebra homomorphism $\jmath: K(\mathfrak{g}) \hookrightarrow K(\mathrm{U}_q(\mathfrak{g}))$ is induced by taking deformation, which is implicitly given by Geer in [15]. The main ingredient of our proof can be illustrated in the following commutative diagram:

$$\begin{tikzcd}
 & k\otimes_{\mathbb{Z}} K(\mathrm{U}_q(\mathfrak{g})) & & k\otimes_{\mathbb{Z}} K(\mathfrak{g}) \arrow[ll, hook', "k\otimes_{\mathbb{Z}}\jmath"'] \arrow[dr, "k\otimes_{\mathbb{Z}}\mathrm{Sch}", "\cong"'] & \\
k\otimes_{\mathbb{Z}} K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g})) \arrow[ur, hook] \arrow[dr, dashed, "\Psi_{\mathcal{R}}"'] & & k\otimes_{\mathbb{Z}} K_{\mathrm{ev}}(\mathfrak{g}) \arrow[ll, hook'] \arrow[ur, hook] \arrow[dr, dashed, "\cong"'] & & k\otimes_{\mathbb{Z}} J(\mathfrak{g}) \\
 & \mathcal{Z}(\mathrm{U}_q(\mathfrak{g})) \arrow[rr, dashed, "\mathcal{HC}"] & & (\mathrm{U}_{\mathrm{ev}}^0)^W_{\mathrm{sup}} = k\otimes_{\mathbb{Z}} J_{\mathrm{ev}}(\mathfrak{g}) \arrow[ur, hook, dashed, "\iota"'] &
\end{tikzcd}$$

First, we identify $(\mathrm{U}_{\mathrm{ev}}^0)^W_{\mathrm{sup}}$ with a subring of $k\otimes_{\mathbb{Z}} J(\mathfrak{g})$ by some ι, and the key idea is to reformulate $(\mathrm{U}_{\mathrm{ev}}^0)^W_{\mathrm{sup}}$ as $k\otimes_{\mathbb{Z}} J_{\mathrm{ev}}(\mathfrak{g})$, which embeds into $k\otimes_{\mathbb{Z}} J(\mathfrak{g})$ in a natural way; see Eq. 3.2 and Proposition 5.6. One can prove that under the isomorphism $k\otimes_{\mathbb{Z}}\mathrm{Sch}$, the ring $(\mathrm{U}_{\mathrm{ev}}^0)^W_{\mathrm{sup}}$ is isomorphic to $k\otimes_{\mathbb{Z}} K_{\mathrm{ev}}(\mathfrak{g})$, where $K_{\mathrm{ev}}(\mathfrak{g})$ is a subring of $K(\mathfrak{g})$ consisting of modules with all weights contained in $\Lambda \cap \frac{1}{2}\mathbb{Z}\Phi$.

Second, $\jmath$ induces an injection $k\otimes_{\mathbb{Z}} K_{\mathrm{ev}}(\mathfrak{g}) \hookrightarrow k\otimes_{\mathbb{Z}} K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g}))$, where $K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g}))$ is the subring of $K(\mathrm{U}_q(\mathfrak{g}))$ consisting of modules with all weights contained in $\Lambda\cap\frac{1}{2}\mathbb{Z}\Phi$.

Third, analogous to quantum groups [23, Chap. 6], [38,44], by using the Rosso form and the quantum supertrace for quantum superalgebras, we define a linear map $\Psi_{\mathcal{R}}: k\otimes_{\mathbb{Z}} K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g})) \to \mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$; see Proposition 5.7. This involves lengthy computations and some subtle constructions. We remark that $\Psi_{\mathcal{R}}$ is an algebra isomorphism, but not in an obvious way.

Now the surjectivity of $\mathcal{HC}$ follows from the commutative diagram easily. Moreover, we show that $\mathcal{HC}\circ\Psi_{\mathcal{R}}$ is injective, and combined with the injectivity of $\mathcal{HC}$, we can prove that homomorphisms occurring in the bottom left parallelogram are all isomorphisms of algebras. Consequently, the restriction $\jmath: K_{\mathrm{ev}}(\mathfrak{g}) \to K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g}))$ is an isomorphism.

By definition, $k\otimes_{\mathbb{Z}} K_{\mathrm{ev}}(\mathfrak{g})$ has a basis $\left\{[L(\lambda)] \,\middle|\, \lambda \in \Lambda \cap \frac{1}{2}\mathbb{Z}\Phi,\ \dim L(\lambda) < \infty\right\}$ and $k\otimes_{\mathbb{Z}} K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g}))$ has a basis $\left\{[L_q(\lambda)] \,\middle|\, \lambda \in \Lambda \cap \frac{1}{2}\mathbb{Z}\Phi,\ \dim L_q(\lambda) < \infty\right\}$, where $L(\lambda)$ and $L_q(\lambda)$ are the irreducible $\mathfrak{g}$-module and the irreducible $\mathrm{U}_q(\mathfrak{g})$-module with the highest weight λ, respectively. We remark that if $\lambda \in \Lambda \cap \frac{1}{2}\mathbb{Z}\Phi$, then $\dim L(\lambda) < \infty$ if and only if $\dim L_q(\lambda) < \infty$. Then the desired basis of $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ in Theorem B is obtained by applying the isomorphism $\Psi_{\mathcal{R}}$, and here we rely heavily on an alternating construction of $\Psi_{\mathcal{R}}$ by using quasi-R-matrix as in [17].

The paper is organized as follows: In Sect. 2, we review some basic facts related to contragredient Lie superalgebras and quantum superalgebras. In Sect. 3, we show several useful results on representations of quantum superalgebras, which seem to be well-known among experts. In particular, we give a super version of a Tanisaki's result for quantum superalgebras (see Proposition 3.4), which has been used to prove the injectivity of $\mathcal{HC}$. In Sect. 4, we recall that the quantum superalgebra can be realized as a Drinfeld double. As a consequence, a non-degenerate ad-invariant bilinear form on $\mathrm{U}_q(\mathfrak{g})$ (Theorem 4.6) is obtained, which serves for proving the surjectivity of $\mathcal{HC}$. In Sect. 5, first we define the Harish-Chandra homomorphism for quantum superalgebras and prove its injectivitity. Then we prove that the image of $\mathcal{HC}$ is contained in $(\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$ and then explicitly describe its image $J_{\mathrm{ev}}(\mathfrak{g})$, which will be used to prove our main theorem for quantum superalgebras; see Theorem A. In Sect. 6, we construct an explicit central element C_M associated with each finite-dimensional $\mathrm{U}_q(\mathfrak{g})$-module M by using the quasi-R-matrix of quantum superalgebras. As an application of the Harish-Chandra theorem, we obtain a basis for the center of quantum superalgebras.

Notations and terminologies:

Throughout this paper, we will use the standard notations $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ that represent the sets of integers, non-negative integers and positive integers, respectively. The Kronecker delta δ_{ij} is equal to 1 if $i = j$ and 0 otherwise.

We write $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. For a homogeneous element x of an associative or Lie superalgebra, we use $|x|$ to denote the degree of x with respect to the $\mathbb{Z}_2$-grading. Throughout the paper, when we write $|x|$ for an element x, we will always assume that x is a homogeneous element and automatically extend the relevant formulas by linearity (whenever applicable). All modules of Lie superalgebras and quantum superalgebras are assumed to be $\mathbb{Z}_2$-graded. The tensor product of two superalgebras A and B carries a superalgebra structure by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2.$$

2. Lie Superalgebras and Quantum Superalgebras

2.1. Lie superalgebras. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional complex simple Lie superalgebra of type A-G such that $\mathfrak{g}_{\bar{1}} \neq 0$, and let $\Pi = \{\alpha_1, \alpha_2, \ldots \alpha_r\}$, with r the rank of $\mathfrak{g}$, be the simple roots of $\mathfrak{g}$. Also let (A, τ) be the corresponding Cartan matrix, where $A = (a_{ij})$ is a $r \times r$ matrix and τ is a subset of $\mathbb{I} = \{1, 2, \ldots, r\}$ determining the parity of the generators. Kac showed that the Lie superalgebra $\mathfrak{g}(A, \tau)$ is characterized by its associated Dynkin diagrams (equivalent Cartan matrix A, and a subset τ); see [26]. These Lie superalgebras are called basic. For convenience (see remark 2.3), we will restrict our attention to the simplest case and only consider root systems related to a special Borel sub-superalgebra with at most one odd root, called distinguished root system, denoted by $\mathfrak{g}(A, \{s\})$ or simply $\mathfrak{g}$ in no confusion. The explicit description of root systems can be found in Appendix A. The Cartan matrix A is symmetrizable, that is, there exist non-zero rational numbers $d_1, d_2, \ldots d_r$ such that $d_i a_{ij} = d_j a_{ji}$. Without loss of generality, we assume $d_1 = 1$, since there exists a unique (up to constant factor) non-degenerate supersymmetric invariant bilinear form (-, -) on $\mathfrak{g}$ and the restriction of this form to Cartan subalgebra $\mathfrak{h}$ is also non-degenerate. Let Φ be the root system of $\mathfrak{g}$, and denote the sets of even and odd roots, respectively, as $\Phi_{\bar{0}}$ and $\Phi_{\bar{1}}$. In order to define quantum superalgebra associated with a Lie superalgebra $\mathfrak{g}(A, \{s\})$, we first review the generators-relations presentation of Lie superalgebra $\mathfrak{g}(A, \{s\})$ given by Yamane [46] and Zhang [57].

Definition 2.1 [57, Theorem 3.4]. Let $(A, \{s\})$ be the Cartan matrix of a contragredient Lie superalgebra in the distinguished root system. Then $\mathrm{U}(\mathfrak{g}(A, \{s\}))$ (simplify for $\mathrm{U}(\mathfrak{g})$) is generated by $e_i, f_i, h_i (i = 1, 2, \ldots r)$ over $\mathbb{C}$, where e_s and f_s are odd and the rest are even, subject to the quadratic relations:

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_j, \tag{2.1}$$

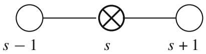
and the additional linear relation $\sum_{i=1}^{r} J_i h_i = 0$ if $\mathfrak{g} = A\left(\frac{r-1}{2}, \frac{r-1}{2}\right)$ for odd r, where $J = (J_1, J_2, \cdots, J_r)$ such that $JA = 0$ (more explicitly, $J = \Big(1, 2, \cdots, \frac{r+1}{2}, -\frac{r-1}{2}, -\frac{r-3}{2}, \cdots, -1\Big)$), and the *standard Serre relations*

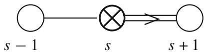
$$\begin{aligned} &e_s^2 = f_s^2 = 0, \quad \text{if } (\alpha_s, \alpha_s) = 0, \\ &(\mathrm{ad} e_i)^{1-a_{ij}} e_j = (\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0, \quad \text{if } i \neq j, \text{ with } a_{ii} \neq 0, \text{ or } a_{ij} = 0, \end{aligned}$$

and *higher order Serre relations*

$$[e_s, [e_{s-1}, [e_s, e_{s+1}]]] = 0, \quad [f_s, [f_{s-1}, [f_s, f_{s+1}]]] = 0, \tag{2.2}$$

if the Dynkin diagram of A contains a full sub-diagram of the form



, or


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We refer the reader to [57] for undefined terminology and the presentation for each simple basic Lie superalgebra in an arbitrary root system.

2.2. Quantum superalgebras. Let $k = K(q^{\frac{1}{2}})$, where K is a field of characteristic 0 and q is an indeterminate, and we set $q_i = q^{d_i}$, then $q_i^{a_{ij}} = q_j^{a_{ji}}$ for all $i, j = 1, 2, \ldots, r$. Set

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \begin{cases} \prod_{i=1}^{n} \frac{(q^{m-i+1} - q^{i-m-1})}{(q^i - q^{-i})}, & \text{if } m > n > 0, \\ 1, & \text{if } n = m, 0. \end{cases}$$

Definition 2.2 [14,32,45]. Let $(A, \{s\})$ be the Cartan matrix of a simple basic Lie superalgebra $\mathfrak{g}$ in the distinguished root system. The quantum superalgebra $\mathrm{U}_q(\mathfrak{g})$ is defined over k in q generated by $\mathbb{K}_i^{\pm 1}, \mathbb{E}_i, \mathbb{F}_i, i \in \mathbb{I}$ (all generators are even except for $\mathbb{E}_s$ and $\mathbb{F}_s$, which are odd), subject to the following relations:

$$\mathbb{K}_i \mathbb{K}_j = \mathbb{K}_j \mathbb{K}_i, \qquad \mathbb{K}_i \mathbb{K}_i^{-1} = \mathbb{K}_i^{-1} \mathbb{K}_i = 1, \tag{2.3}$$

$$\mathbb{K}_i \mathbb{E}_j \mathbb{K}_i^{-1} = q^{(\alpha_i, \alpha_j)} \mathbb{E}_j, \quad \mathbb{K}_i \mathbb{F}_j \mathbb{K}_i^{-1} = q^{-(\alpha_i, \alpha_j)} \mathbb{F}_j, \tag{2.4}$$

$$\mathbb{E}_i \mathbb{F}_j - (-1)^{|\mathbb{E}_i||\mathbb{F}_j|} \mathbb{F}_j \mathbb{E}_i = \delta_{ij} \frac{\mathbb{K}_i - \mathbb{K}_i^{-1}}{q_i - q_i^{-1}}, \tag{2.5}$$

$$\mathrm{Ad}_{\mathbb{E}_i}^{1-a_{ij}}(\mathbb{E}_j) = 0 \text{ for } i \neq j \text{ with } a_{ii} \neq 0 \text{ or } a_{ij} = 0, \tag{2.6}$$

$$\mathrm{Ad}_{\mathbb{F}_i}^{1-a_{ij}}(\mathbb{F}_j) = 0 \text{ for } i \neq j \text{ with } a_{ii} \neq 0 \text{ or } a_{ij} = 0, \tag{2.7}$$

$$(\mathbb{E}_s)^2 = (\mathbb{F}_s)^2 = 0, \text{ if } a_{ss} = 0, \tag{2.8}$$

and higher order quantum Serre relations, and

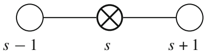
$$\prod_{i=1}^{r} \mathbb{K}_i^{d_i J_i} = 1 \text{ if } \mathfrak{g} = A\left(\frac{r-1}{2}, \frac{r-1}{2}\right) \text{ for odd } r.$$

where

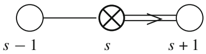
$$\mathrm{Ad}_{\mathbb{E}_i}(x) = \mathbb{E}_i x - (-1)^{|\mathbb{E}_i||x|}\mathbb{K}_i x \mathbb{K}_i^{-1}\mathbb{E}_i; \tag{2.9}$$

$$\mathrm{Ad}_{\mathbb{F}_i}(x) = \mathbb{F}_i x - (-1)^{|\mathbb{F}_i||x|}\mathbb{K}_i^{-1} x \mathbb{K}_i \mathbb{F}_i. \tag{2.10}$$

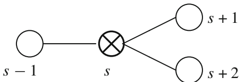
For the distinguished root data [57, Appendix A.2.1], higher order Serre relations appear if the Dynkin diagram contains a sub-diagram of the following types:

(i) , the higher order quantum Serre relations are

$$\mathbb{E}_s\mathbb{E}_{s-1,s,s+1} + \mathbb{E}_{s-1,s,s+1}\mathbb{E}_s = 0, \quad \mathbb{F}_s\mathbb{F}_{s-1,s,s+1} + \mathbb{F}_{s-1,s,s+1}\mathbb{F}_s = 0; \tag{2.11}$$

(ii) , the higher order quantum Serre relations are

$$\mathbb{E}_s\mathbb{E}_{s-1;s;s+1} + \mathbb{E}_{s-1;s;s+1}\mathbb{E}_s = 0, \quad \mathbb{F}_s\mathbb{F}_{s-1;s;s+1} + \mathbb{F}_{s-1;s;s+1}\mathbb{F}_s = 0; \tag{2.12}$$

(iii) , the higher order quantum Serre relations are

$$\begin{aligned}
&\mathbb{E}_s\mathbb{E}_{s-1;s;s+1} + \mathbb{E}_{s-1;s;s+1}\mathbb{E}_s = 0, \quad \mathbb{F}_s\mathbb{F}_{s-1;s;s+1} + \mathbb{F}_{s-1;s;s+1}\mathbb{F}_s = 0,\\
&\mathbb{E}_s\mathbb{E}_{s-1;s;s+2} + \mathbb{E}_{s-1;s;s+2}\mathbb{E}_s = 0, \quad \mathbb{F}_s\mathbb{F}_{s-1;s;s+2} + \mathbb{F}_{s-1;s;s+2}\mathbb{F}_s = 0;
\end{aligned} \tag{2.13}$$

where

$$\mathbb{E}_{s-1;s;j} = \mathbb{E}_{s-1}\left(\mathbb{E}_s\mathbb{E}_j - q_j^{a_{js}}\mathbb{E}_j\mathbb{E}_s\right) - q_{s-1}^{a_{s-1,s}}\left(\mathbb{E}_s\mathbb{E}_j - q_j^{a_{js}}\mathbb{E}_j\mathbb{E}_s\right)\mathbb{E}_{s-1},$$

$$\mathbb{F}_{s-1;s;j} = \mathbb{F}_{s-1}\left(\mathbb{F}_s\mathbb{F}_j - q_j^{a_{js}}\mathbb{F}_j\mathbb{F}_s\right) - q_{s-1}^{a_{s-1,s}}\left(\mathbb{F}_s\mathbb{F}_j - q_j^{a_{js}}\mathbb{F}_j\mathbb{F}_s\right)\mathbb{F}_{s-1}.$$

For the other root data of $\mathfrak{g}$, the higher order quantum Serre relations vary considerably with the choice of the root datum; thus, we will not spell them out explicitly here.

Remark 2.3. The definition of the quantum superalgebra above is dependent on the choice of the Borel subalgebras. Although the quantum superalgebras defined by non-conjugacy Borel subalgebras of a Lie superalgebra are not isomorphic as Hopf superalgebras, they are isomorphic as superalgebras; see [29] or [47, Proposition 7.4.1].

There is a unique automorphism ω of $\mathrm{U}_q(\mathfrak{g})$ such that $\omega(\mathbb{E}_i) = (-1)^{|\mathbb{E}_i|}\mathbb{F}_i$, $\omega(\mathbb{F}_i) = \mathbb{E}_i$ and $\omega(\mathbb{K}_i) = \mathbb{K}_i^{-1}$ for $i \in \mathbb{I}$. The quantum superalgebra $\mathrm{U}_q(\mathfrak{g})$ has the structure of a $\mathbb{Z}_2$-graded Hopf algebra. The co-multiplication

$$\Delta\colon \mathrm{U}_q(\mathfrak{g}) \to \mathrm{U}_q(\mathfrak{g}) \otimes \mathrm{U}_q(\mathfrak{g})$$

is given by

$$\Delta(\mathbb{E}_i) = \mathbb{K}_i \otimes \mathbb{E}_i + \mathbb{E}_i \otimes 1, \quad \Delta(\mathbb{F}_i) = 1 \otimes \mathbb{F}_i + \mathbb{F}_i \otimes \mathbb{K}_i^{-1}, \quad \Delta(\mathbb{K}_i^{\pm 1}) = \mathbb{K}_i^{\pm 1} \otimes \mathbb{K}_i^{\pm 1}, \tag{2.14}$$

for $i \in \mathbb{I}$ and the co-unit $\varepsilon\colon \mathrm{U}_q(\mathfrak{g}) \to k$ is defined by

$$\varepsilon(\mathbb{E}_i) = \varepsilon(\mathbb{F}_i) = 0, \quad \varepsilon(\mathbb{K}_i^{\pm 1}) = 1, \text{ for } i \in \mathbb{I},$$

then the corresponding antipode $S\colon \mathrm{U}_q(\mathfrak{g}) \to \mathrm{U}_q(\mathfrak{g})$ is given by

$$S(\mathbb{E}_i) = -\mathbb{K}_i^{-1}\mathbb{E}_i, \quad S(\mathbb{F}_i) = -\mathbb{F}_i\mathbb{K}_i, \quad S(\mathbb{K}_i^{\pm 1}) = \mathbb{K}_i^{\mp 1}, \text{ for } i \in \mathbb{I}, \tag{2.15}$$

which is a $\mathbb{Z}_2$-graded algebra anti-automorphism, i.e., $S(xy) = (-1)^{|x||y|}S(y)S(x)$.

Denote by $\mathrm{U}^{\geqslant 0}$ (resp., $\mathrm{U}^{\leqslant 0}$) the sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$ generated by all $\mathbb{E}_i$, $\mathbb{K}_i^{\pm 1}$ (resp., $\mathbb{F}_i$, $\mathbb{K}_i^{\pm 1}$), set U^0 equal to the sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$ generated by all $\mathbb{K}_i^{\pm 1}$, and denote by U^+ (resp., U^-) the sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$ generated by all $\mathbb{E}_i$ (resp., $\mathbb{F}_i$), it is well-known that $\mathrm{U}^+ \otimes \mathrm{U}^0 \cong \mathrm{U}^{\geqslant 0}$ (resp., $\mathrm{U}^- \otimes \mathrm{U}^0 \cong \mathrm{U}^{\leqslant 0}$) by the multiplication map. And the multiplication map $\mathrm{U}^- \otimes \mathrm{U}^0 \otimes \mathrm{U}^+ \to \mathrm{U}$ is an isomorphism as super vector spaces.

Remark 2.4. Analogous to the quantum group, the quantum Serre relations and the higher order quantum Serre relations can be explained from the view of skew primitive elements in the quantum superalgebras. For example,

$$\begin{aligned}
&\Delta(u_{ij}^+) = u_{ij}^+ \otimes 1 + \mathbb{K}_i^{1-a_{ij}}\mathbb{K}_j \otimes u_{ij}^+, \quad \Delta(u_{ij}^-) = u_{ij}^- \otimes \mathbb{K}_i^{a_{ij}-1}\mathbb{K}_j^{-1} + 1 \otimes u_{ij}^-,\\
&\Delta(u_B^+) = u_B^+ \otimes 1 + \mathbb{K}_{s-1}\mathbb{K}_s^3 \otimes u_B^+, \quad \Delta(u_B^-) = 1 \otimes u_B^- + u_B^- \otimes \mathbb{K}_{s-1}^{-1}\mathbb{K}_s^{-3},\\
&\Delta(u^+) = u^+ \otimes 1 + \mathbb{K}_{s-1}\mathbb{K}_s^2\mathbb{K}_j \otimes u^+, \quad \Delta(u^-) = 1 \otimes u^- + u^- \otimes \mathbb{K}_{s-1}^{-1}\mathbb{K}_s^{-2}\mathbb{K}_j^{-1}.
\end{aligned}$$

where $u_{ij}^{\pm}$ (resp. $u_B^{\pm}$) is on the left side of Eqs. (2.6) and (2.7) for $i \neq j$ and even α_i (resp., for non-isotropic odd root α_i with $a_{ij} \neq 0$ for $i \neq j$), and $u^{\pm}$ is on the left side of Eqs. (2.11)-(2.13).

For any $\mu = \sum_{i=1}^{r} m_i\alpha_i \in \mathbb{Z}\Phi$, set $\mathbb{K}_\mu = \prod_{i=1}^{r} \mathbb{K}_i^{m_i}$. Thus, $\mathbb{K}_\mu\mathbb{K}_{\mu'} = \mathbb{K}_{\mu+\mu'}$ for all $\mu, \mu' \in \mathbb{Z}\Phi$. Therefore, $\{\mathbb{K}_\mu\}_{\mu\in\mathbb{Z}\Phi}$ spans U^0 as a vector space, and

$$\mathbb{K}_\mu\mathbb{E}_i\mathbb{K}_\mu^{-1} = q^{(\mu,\alpha_i)}\mathbb{E}_i, \ \mathbb{K}_\mu\mathbb{F}_i\mathbb{K}_\mu^{-1} = q^{-(\mu,\alpha_i)}\mathbb{F}_i.$$

The quantum superalgebra $\mathrm{U}_q(\mathfrak{g})$ is $\mathbb{Z}\Phi$-graded. And the gradation is given by

$$\deg \mathbb{K}_\mu = 0, \ \deg \mathbb{E}_i = \alpha_i, \ \deg \mathbb{F}_i = -\alpha_i,$$

for all $\mu \in \mathbb{Z}\Phi$ and $i \in \mathbb{I}$. We denote that U_ν is the $\nu \in \mathbb{Z}\Phi$-component if $\mathfrak{g} \neq A(n, n)$.

Note that if $\mathfrak{g} = A(n, n)$, the simple roots for distinguished Borel subalgebra are not linearly independent (that is, $\gamma = \sum_{i=1}^{2n+1} d_i J_i\alpha_i = 0$). This causes some technical difficulties. However, the quantum superalgebra $\mathrm{U}_q(\mathfrak{g})$ is also $\mathbb{Z}\tilde{\Phi}$-graded, where $\mathbb{Z}\tilde{\Phi}$ is a free abelian group generated by all simple roots $\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}$. Obviously, $\mathbb{Z}\Phi = \mathbb{Z}\tilde{\Phi}/\mathbb{Z}\gamma$.

Denote $\mathrm{U}|_\mu$ (resp. U_ν) as the μ-component (resp. ν-component) with respect to $\mathbb{Z}\Phi$-gradation (resp. $\mathbb{Z}\tilde{\Phi}$-gradation). From now on, we do not make a distinction between the elements in $\mathbb{Z}\Phi$ and $\mathbb{Z}\tilde{\Phi}$ if no confusion emerges. Hence, $\mathrm{U}|_\mu = \bigoplus_{k\in\mathbb{Z}} \mathrm{U}_{\mu+k\gamma}$. Set

$$Q = \begin{cases} \mathbb{Z}\tilde{\Phi}, & \text{for } \mathfrak{g} = A(n,n), \\ \mathbb{Z}\Phi, & \text{otherwise.} \end{cases}$$

Note that $\mathfrak{h}^* = \mathbb{C}\Phi$. If $\mathfrak{g} \neq A(n,n)$, define the standard partial order relation on $\mathfrak{h}^*$ by $\lambda \leqslant \mu \Leftrightarrow \mu - \lambda \in \sum_{i\in\mathbb{I}} \mathbb{Z}_+\alpha_i$. This breaks down if $\mathfrak{g} = A(n,n)$ because $\gamma = 0$ and $d_i J_i \in \mathbb{Z}_+$ for all $i \in \mathbb{I}$. However, we can define a similar partial order on $\mathbb{C}\tilde{\Phi}$. From now on, we will use the partial order on $\mathbb{C}\tilde{\Phi}$ if necessary for $\mathfrak{g} = A(n,n)$.

Remark 2.5. The Lie superalgebra $A(n,n)$ is rather special, and the restriction of the Harish-Chandra projection determined by the distinguish triangular decomposition to the zero weight space (with respect to $\mathbb{Z}\Phi$-gradation) is not an algebra homomorphism; for more details, see [16, Sect. 6.1.4]. For this reason, we do not expect that the projection from $\mathrm{U}|_0$ to U^0 is an algebra homomorphism. However, the projection $\pi : \mathrm{U}_0 \to \mathrm{U}^0$ is an algebra homomorphism. Fortunately, we can prove that $\mathcal{Z}$ is contained in U_0; see Corollary 3.7. Therefore, we can establish the Harish-Chandra homomorphism for $\mathfrak{g} = A(n,n)$.

3. Representation of Quantum Superalgebras

3.1. Representations. We will recall some basic facts about the representation theory of the quantum superalgebra $\mathrm{U}_q(\mathfrak{g})$. The bilinear form (-, -) on $\mathbb{Z}\Phi$ can be linearly extended to $\mathfrak{h}^*$. For any $\lambda, \mu \in \mathfrak{h}^*$ with $(\mu,\mu) \neq 0$, denote $\langle\lambda,\mu\rangle = \frac{2(\lambda,\mu)}{(\mu,\mu)}$. Let $\Lambda = \left\{\lambda \in \mathfrak{h}^* \mid \langle\lambda,\alpha\rangle \in \mathbb{Z},\ \forall\alpha \in \Phi_{\bar{0}}\right\}$ be the integral weight lattice, and denote by $\Lambda^+ = \left\{\lambda \in \mathfrak{h}^* \mid \langle\lambda,\alpha\rangle \in \mathbb{Z}_+,\ \forall\alpha \in \Phi_{\bar{0}}^+\right\}$ the set of $\Phi_{\bar{0}}^+$-dominant integral weights.

A $\mathrm{U}_q(\mathfrak{g})$-module M is called *a weight module* if it admits a weight space decomposition

$$M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda, \quad \text{where } M_\lambda = \left\{u \in M \mid \mathbb{K}_i u = q^{(\lambda,\alpha_i)}u,\ \forall i \in \mathbb{I}\right\}. \tag{3.1}$$

In this paper, all module are weight module and type **1**. Denote by $\mathrm{wt}(M)$ the set of weights of the finite-dimensional $\mathrm{U}_q(\mathfrak{g})$-module M. A weight module M is called a *highest weight module* with the highest weight λ if there exists a unique non-zero vector $v_\lambda \in M$, which is called a *highest weight vector* such that $\mathbb{K}_i v_\lambda = q^{(\lambda,\alpha_i)}$, $\mathbb{E}_i v_\lambda = 0$ for all $i \in \mathbb{I}$ and $M = \mathrm{U}_q(\mathfrak{g})v_\lambda$.

Let $J_\lambda = \sum_{i=1}^r \mathrm{U}_q(\mathfrak{g})\mathbb{E}_i + \sum_{i=1}^r \mathrm{U}_q(\mathfrak{g})(\mathbb{K}_i - q^{(\lambda,\alpha_i)})$ for $\lambda \in \Lambda$, and set $\Delta_q(\lambda) = \mathrm{U}_q(\mathfrak{g})/J_\lambda$. This is a $\mathrm{U}_q(\mathfrak{g})$-module generated by the coset of 1; also denote this coset by v_λ. Obviously, $\mathbb{E}_i v_\lambda = 0$ and $\mathbb{K}_i v_\lambda = q^{(\lambda,\alpha_i)}v_\lambda$ for $i \in \mathbb{I}$. We call $\Delta_q(\lambda)$ the *Verma module* of the highest weight λ. It has the following universal property: If M is a $\mathrm{U}_q(\mathfrak{g})$-module and $v \in M_\lambda$ with $\mathbb{E}_i v = 0$ for all $i \in \mathbb{I}$, then there is a unique homomorphism of $\mathrm{U}_q(\mathfrak{g})$-modules $\varphi : \Delta_q(\lambda) \to M$ with $\varphi(v_\lambda) = v$. The Verma module $\Delta_q(\lambda)$ has a unique maximal submodule, thus, $\Delta_q(\lambda)$ admits a unique simple quotient $\mathrm{U}_q(\mathfrak{g})$-module $L_q(\lambda)$.

Lemma 3.1. *Let $\lambda \in \Lambda$ with $(\lambda, \alpha_s) = 0$. Then there is a homomorphism of* $\mathrm{U}_q(\mathfrak{g})$-*modules* $\varphi\colon \Delta_q(\lambda - \alpha_s) \to \Delta_q(\lambda)$ *with* $\varphi(v_{\lambda-\alpha_s}) = \mathbb{F}_s v_\lambda$.

Proof. We have $\mathbb{F}_s v_\lambda \in \Delta_q(\lambda)_{\lambda-\alpha_s}$. Therefore, the universal property of $\Delta_q(\lambda - \alpha_s)$ implies that it is enough to show that $\mathbb{E}_j\mathbb{F}_s v_\lambda = 0$ for all $j \in \mathbb{I}$. This is obvious for $j \neq s$ because $\mathbb{E}_j$ and $\mathbb{F}_s$ commute. For $j = s$, we have $\mathbb{E}_s\mathbb{F}_s v_\lambda = [\mathbb{E}_s, \mathbb{F}_s]v_\lambda - \mathbb{F}_s\mathbb{E}_s v_\lambda = \frac{\mathbb{K}_s - \mathbb{K}_s^{-1}}{q_s - q_s^{-1}} v_\lambda - 0 = 0.$ □

The finite-dimensional irreducible representations of $\mathrm{U}_q(\mathfrak{g})$ can be classified into two types: typical and atypical. The representation theory of $\mathrm{U}_q(\mathfrak{g})$ at generic q is rather similar to the Lie superalgebra $\mathfrak{g}$, as well. Geer proved the theorem that each irreducible highest weight module of a Lie superalgebra of Type A-G can be deformed to an irreducible highest weight module over the corresponding Drinfeld-Jimbo algebra; see [15, Theorem 1.2]. We also refer to [54, Proposition 3], [55, Proposition 1] and [30, Theorem 4.2] for quantum superalgebras of type $\mathrm{U}_q(\mathfrak{gl}_{m|n})$, $\mathrm{U}_q(\mathfrak{osp}_{2|2n})$ and $\mathrm{U}_q(\mathfrak{osp}_{m|2n})$, respectively.

Theorem 3.2. *For $\lambda \in \mathfrak{h}^*$, let $L(\lambda)$ be the irreducible highest weight module over $\mathfrak{g}$ of highest weight λ. Then there exists an irreducible highest weight module $L_q(\lambda)$ of highest weight λ which is a deformation of $L(\lambda)$. Moreover, the classical limit of $L_q(\lambda)$ is $L(\lambda)$, and their (super)characters are equal* [1].

3.2. Grothendieck ring. Let A-**mod** be the category of finite-dimensional modules of a Hopf superalgebra A over k. There is a parity reversing functor on this category. For an A-module $M = M_{\bar{0}} \oplus M_{\bar{1}}$, define

$$\Pi(M) = \Pi(M)_{\bar{0}} \oplus \Pi(M)_{\bar{1}}, \quad \Pi(M)_i = \Pi(M)_{i+\bar{1}}, \ \forall i \in \mathbb{Z}_2.$$

Then $\Pi(M)$ is also an A-module with the action $am = (-1)^{|a|}m$. Let $k\pi$ be a 1-dimensional odd vector space with basis $\{\pi\}$, then $k\pi$ can be views as a trivial A-module and $\Pi(M) \cong k\pi \otimes M$ as A-modules. Define the Grothendieck group $K(A)$ of A-**mod** to be the abelian group generated by all objects in A-**mod** subject to the following two relations: (i) $[M] = [L] + [N]$; (ii) $[\Pi(M)] = -[M]$, for all A-modules L, M, N which satisfying a short exact sequence $0 \to L \to M \to N \to 0$ with even morphisms.

It is easy to see that the Grothendieck group $K(A)$ is a free $\mathbb{Z}$-module with the basis corresponding to the classes of the irreducible modules. Furthermore, if A is a Hopf superalgebra, then for any A-modules M and N, one can define the A-module structure on $M \otimes N$. Using this, we define the product on $K(A)$ by the formula

$$[M][N] = [M \otimes N].$$

Since all modules are finite-dimensional, this multiplication is well-defined on the Grothendieck group $K(A)$ and introduces the ring structure on it. The corresponding ring is called the *Grothendieck ring* of A. The Grothendieck ring of $\mathrm{U}(\mathfrak{g})$ is denoted by $K(\mathfrak{g})$. Let $K_{\mathrm{ev}}(\mathfrak{g})$ (resp. $K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g}))$) be the subring of $K(\mathfrak{g})$ (resp. $K(\mathrm{U}_q(\mathfrak{g}))$) generated by all objects in $\mathrm{U}(\mathfrak{g})$-**mod** (resp. $\mathrm{U}_q(\mathfrak{g})$-**mod**), whose weights are contained in $\Lambda \cap \frac{1}{2}\mathbb{Z}\Phi$.

[1] However, the inverse of the theorem is not true in general [2]. For example, there are many finite-dimensional irreducible modules (spinorial modules) of quantum superalgebras of type $\mathrm{U}_q(\mathfrak{osp}_{1|2})$ without classical limit; see [52] for more details.

Let M be a finite-dimensional representation of $\mathfrak{g}$ or $\mathrm{U}_q(\mathfrak{g})$. We define the *character map* and the *supercharacter map* as:

$$\mathrm{ch}(M) = \sum_{\lambda} \dim M_\lambda e^\lambda, \qquad \mathrm{Sch}(M) = \sum_{\lambda} \mathrm{sdim} M_\lambda e^\lambda,$$

where sdim is the superdimension defined for any $\mathbb{Z}_2$-graded vector space $W = W_0 \oplus W_1$ as the difference of usual dimensions of graded components: $\mathrm{sdim}\, W = \dim W_0 - \dim W_1$.

Proposition 3.3. *There is an injective ring homomorphism $\jmath\colon K(\mathfrak{g}) \to K(\mathrm{U}_q(\mathfrak{g}))$, which preserves (super)characters.*

Proof. By Theorem 3.2, we can define $\jmath([L(\lambda)]) = [L_q(\lambda)]$ for all finite-dimensional irreducible $\mathfrak{g}$-modules $L(\lambda)$. This then induces an abelian group homomorphism from $K(\mathfrak{g})$ to $K(\mathrm{U}_q(\mathfrak{g}))$. The map preserves (super)characters, so $\jmath$ is a ring homomorphism. Suppose there exist nonzero $a_i \in \mathbb{Z}$ and distinct $\lambda_i \in \mathfrak{h}^*$ for $i = 1, 2 \cdots, n$ such that $\jmath(\sum_{i=1}^n a_i[L(\lambda_i)]) = 0$. Then $\mathrm{Sch}(\sum_{i=1}^n a_i[L(\lambda_i)]) = 0$. Choose λ_j maximal in $\{\lambda_i \in \mathfrak{h}^* | i = 1, 2, \cdots, n\}$ for some j, then $a_j = 0$ since $\dim(L(\lambda_i))_{\lambda_j} = \delta_{ij}$. This contradicts $a_j \neq 0$. Thus, $\sum_{i=1}^n a_i[L(\lambda_i)] = 0$. □

Sergeev and Veselov proved that the Grothendieck ring $K(\mathfrak{g})$ is isomorphic to the *ring of exponential super-invariants* $J(\mathfrak{g}) = \Big\{ f \in \mathbb{Z}[P_0]^{W_0} \Big| D_\alpha f \in (e^\alpha - 1)$ for any isotropic root $\alpha \Big\}$ for $\mathfrak{g} \neq A(1, 1)$, where $D_\alpha(e^\lambda) = (\lambda, \alpha)e^\lambda$, $\{e^\lambda \,|\, \lambda \in P_0\}$ is a $\mathbb{Z}$-free basis of $\mathbb{Z}[P_0]$, and here $P_0 = \Lambda$ and $W_0 = W$, more details could be found in [42].

Set

$$J_{\mathrm{ev}}(\mathfrak{g}) = \Big\{ \sum_{\mu \in 2\Lambda \cap \mathbb{Z}\Phi} a_\mu \mathbb{K}_\mu \in \mathrm{U}^0 \Big| a_{w\mu} = a_\mu,\ \forall w \in W a_\mu \in \mathbb{Z},\ \forall \mu;\ D_\alpha(u) \in (\mathbb{K}_\alpha^2 - 1),\ \forall \alpha \in \Phi_{\mathrm{iso}} \Big\}, \tag{3.2}$$

where $D_\alpha(\mathbb{K}_\mu) = (\mu, \alpha)\mathbb{K}_\mu$.

Obviously, there is an injective homomorphism $\iota\colon J_{\mathrm{ev}}(\mathfrak{g}) \to J(\mathfrak{g})$ with $\iota(\mathbb{K}_\mu) = e^{-\mu/2}$. This induces an isomorphism from $K_{\mathrm{ev}}(\mathfrak{g})$ to $J_{\mathrm{ev}}(\mathfrak{g})$, hence we have the following commutative diagram:

$$\begin{tikzcd}
K(\mathrm{U}_q(\mathfrak{g})) & K(\mathfrak{g}) \arrow[l, hook', "\jmath"'] \arrow[r, "\cong", "\mathrm{Sch}"'] & J(\mathfrak{g}) \\
K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g})) \arrow[u, hook] & K_{\mathrm{ev}}(\mathfrak{g}) \arrow[l, hook'] \arrow[u, hook] \arrow[r, dashed, "\cong"] & J_{\mathrm{ev}}(\mathfrak{g}) \arrow[u, hook, "\iota"']
\end{tikzcd}$$

We remark that the above diagram is not true for $\mathfrak{g} = A(1, 1)$. In Appendix B, we describe $J_{\mathrm{ev}}(\mathfrak{g})$ in sense of Sergeev and Veselov [42] and illustrate why $K_{\mathrm{ev}}(\mathfrak{g}) \ncong J_{\mathrm{ev}}(\mathfrak{g})$ if $\mathfrak{g} = A(1, 1)$.

3.3. Some important propositions. In this subsection, we investigate some important propositions, which show that the center of $\mathrm{U}_q(A(n,n))$ is contained in U^0 and will be used to prove the injectivity of $\mathcal{HC}$.

If $\mathfrak{g}$ is of type II, there exists a unique $\delta \in \Phi_{\bar{0}}^+$ such that $(\Pi \setminus \{\alpha_s\}) \cup \{\delta\}$ is a simple root system of $\Phi_{\bar{0}}^+$. By writing $\delta = \sum_{i=1}^{r} c_i \alpha_i$, we can get $c_s = 2$. The following proposition is a super version of [43, Sect. 3.2] for quantum superalgebra $\mathrm{U}_q(\mathfrak{g})$ associated with a simple basic Lie superalgebra.

Proposition 3.4. *Set* $\beta = \sum_{i=1}^{r} m_i \alpha_i \in \mathbb{Z}_+\Pi$*, and let* $L_q(\lambda)$ *be a typical finite-dimensional irreducible module. Suppose* λ *satisfies*

(i) $\langle \lambda, \alpha_i \rangle \geqslant m_i$ *for all* $i \neq s$*;*
(ii) *an extra condition* $2\langle \lambda + \rho, \delta \rangle \geqslant m_s + 1$ *when* $\mathfrak{g}$ *is of type II, then* $\mathrm{U}_{-\beta}^- \to L_q(\lambda)_{\lambda-\beta}$ *with* $u \mapsto uv_\lambda$ *is bijective.*

Proof. In the proof of this proposition, we choose $\lambda \in \mathbb{C} \otimes_{\mathbb{Z}} Q$ since the Verma module and simple module can be viewed as Q-graded modules. Notice that the partial order is well-defined on Q.

The canonical map from $\Delta_q(\lambda)$ to $L_q(\lambda)$ is surjective, which follows that every finite-dimensional irreducible module is a quotient of a Verma module. So we only need to prove $\dim \Delta_q(\lambda)_{\lambda-\beta} = \dim L_q(\lambda)_{\lambda-\beta}$, since $\dim \mathrm{U}_{-\beta}^- = \dim \Delta_q(\lambda)_{\lambda-\beta}$. The $\dim \Delta_q(\lambda)_{\lambda-\beta}$ is the coefficient of $e^{\lambda-\beta}$ in $\mathrm{ch}\Delta_q(\lambda)$, and $\dim L_q(\lambda)_{\lambda-\beta}$ is the coefficient of $e^{\lambda-\beta}$ in $\mathrm{ch} L_q(\lambda)$.

The following character formulas of a Verma module and a typical finite-dimensional irreducible $\mathrm{U}_q(\mathfrak{g})$-module with the highest weight λ are given by [27, Theorem 1] and Theorem 3.2:

$$\mathrm{ch}\Delta_q(\lambda) = \frac{\prod_{\alpha \in \Phi_{\bar{1}}^+}(1+e^{-\alpha})}{\prod_{\beta \in \Phi_{\bar{0}}^+}(1-e^{-\beta})} e^{\lambda},$$
$$\mathrm{ch} L_q(\lambda) = \frac{\prod_{\alpha \in \Phi_{\bar{1}}^+}(1+e^{-\alpha})}{\prod_{\beta \in \Phi_{\bar{0}}^+}(1-e^{-\beta})} \sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}.$$

Hence, it is sufficient to show $w(\lambda+\rho) - \rho - (\lambda - \beta) \notin \mathbb{Z}_+\Pi$ for all $w \neq 1$. Let us prove it by induction on $l(w)$.

If $\mathfrak{g}$ is of type I and $l(w) = 1$, then we have $w = s_i$ for some $i \neq s$, and hence

$$w(\lambda+\rho) - \rho - (\lambda - \beta) = -(\langle \lambda, \alpha_i \rangle + 1)\alpha_i + \beta \notin \mathbb{Z}_+\Pi.$$

Assume that $l(w) \geqslant 2$. There exists some $j \neq s$ and $w' \in W$ such that $w = s_j w'$ with $l(w') = l(w) - 1$, and then it is known that $w'^{-1}(\alpha_j) \in \Phi_{\bar{0}}^+$. We have

$$w(\lambda+\rho) - \rho - (\lambda - \beta) = w'(\lambda+\rho) - \rho - (\lambda - \beta) - \langle \lambda + \rho, w'^{-1}(\alpha_j) \rangle \alpha_j,$$

$w'(\lambda+\rho) - \rho - (\lambda - \beta) \notin \mathbb{Z}_+\Pi$ by induction and $\langle \lambda + \rho, w'^{-1}(\alpha_j) \rangle \geqslant 0$ since $\lambda + \rho$ is $\Phi_{\bar{0}}^+$-dominant, so $w(\lambda+\rho) - \rho - (\lambda - \beta) \notin \mathbb{Z}_+\Pi$ for all $w \neq 1$.

If $\mathfrak{g}$ is of type II and $l(w)=1$, then we have $w=s_i$ for some $i\neq s$ or $w=s_\delta$. By the same argument as above, we only need to consider $w=s_\delta$. Indeed,

$$\begin{aligned} w(\lambda+\rho)-\rho-(\lambda-\beta) &= -\langle\lambda+\rho,\delta\rangle\delta+\beta \\ &= \sum_{i=1}^{r}(-\langle\lambda+\rho,\delta\rangle c_i+m_i)\alpha_i \notin \mathbb{Z}_+\Pi \end{aligned}$$

since $c_s=2$ and $2\langle\lambda+\rho,\delta\rangle\geqslant m_s+1$. Assume $l(w)\geqslant 2$. There exists some $j\neq s$ and $w'\in W$ such that $w=s_jw'$ or $w=s_\delta w'$ with $l(w')=l(w)-1$. Then it is known that $w'^{-1}(\alpha_j)$ or $w'^{-1}(\delta)$ belongs to $\Phi_{\bar 0}^+$. The proof is similar to type I when $w=s_jw'$, so we omit it here. If $w=s_\delta w'$, then

$$w(\lambda+\rho)-\rho-(\lambda-\beta)=w'(\lambda+\rho)-\rho-(\lambda-\beta)-\langle\lambda+\rho,w'^{-1}(\delta)\rangle\delta.$$

Once again, $w'(\lambda+\rho)-\rho-(\lambda-\beta)\notin\mathbb{Z}_+\Pi$ by induction and $\langle\lambda+\rho,w'^{-1}(\alpha_j)\rangle\geqslant 0$ since $\lambda+\rho$ is $\Phi_{\bar 0}^+$-dominant, so $w(\lambda+\rho)-\rho-(\lambda-\beta)\notin\mathbb{Z}_+\Pi$ for all $w\neq 1$. □

Let $\lambda\in\Lambda$ be a typical weight such that $L_q(\lambda)$ is finite-dimensional, then we can define a twisted action on $L_q(\lambda)$ via the automorphism ω of $\mathrm{U}_q(\mathfrak{g})$, denoted by $L_q^\omega(\lambda)$. Set v_λ by v'_λ when considered as an element of $L_q^\omega(\lambda)$. We then have $\mathbb{K}_\mu v'_\lambda=q^{-(\mu,\lambda)}v'_\lambda$ for all $\mu\in\mathbb{Z}\Phi$. Furthermore, we have $\mathbb{F}_iv'_\lambda=0$ for all $i\in\mathbb{I}$, and $x\mapsto xv'_\lambda$ maps each U_ν^+ onto $L_q^\omega(\lambda)_{-\lambda+\nu}$.

Similarly, if $\langle\lambda,\alpha_i\rangle\geqslant m_i$, $\forall i\neq s$ and λ satisfies an extra condition $2\langle\lambda+\rho,\delta\rangle\geqslant m_s+1$ for $\mathfrak{g}$ is of type II, then the map $\mathrm{U}_\nu^+\to L_q^\omega(\lambda)_{-\lambda+\nu}$ with $x\mapsto xv'_\lambda$ is bijective.

Theorem 3.5. *Let $u\in\mathrm{U}$. If u annihilates all finite-dimensional U-modules, then $u=0$.*

Proof. For any typical weights $\lambda,\lambda'\in\Lambda$ such that $L_q(\lambda)$ and $L_q^\omega(\lambda')$ are finite-dimensional, the tensor product $L_q(\lambda)\otimes L_q^\omega(\lambda')$ is also a finite-dimensional $\mathrm{U}_q(\mathfrak{g})$-module. Suppose that $u\in\mathrm{U}_q(\mathfrak{g})$ annihilates all these tensor products, in particular $u(v_\lambda\otimes v'_{\lambda'})=0$ for all λ and λ'. We show that this implies $u=0$.

Choose bases $(x_i)_i$ of U^+ and $(y_j)_j$ of U^- consisting of homogeneous weight vectors, say $x_i\in\mathrm{U}^+_{\nu(i)}$ and $y_j\in\mathrm{U}^-_{-\nu'(j)}$ with $\nu(i)$ and $\nu'(j)$ in $\mathbb{Z}_+\Pi$. Write

$$u=\sum_j\sum_\mu\sum_i a_{j,\mu,i}y_j\mathbb{K}_\mu x_i$$

with $a_{j,\mu,i}\in k$, which is a finite sum. Suppose that $u\neq 0$. Let $\nu_0\in\mathbb{Z}_+\Pi$ be maximal among the weights ν such that there exist i,μ,j with $a_{j,\mu,i}\neq 0$ and $\nu=\nu(i)$.

So we have

$$\mathbb{K}_\mu x_i(v_\lambda\otimes v'_{\lambda'})=q^{(\nu(i),\lambda)+(\mu,\lambda-\lambda'+\nu(i))}v_\lambda\otimes x_iv'_{\lambda'}.$$

Each $\Delta(y_j)$ is equal to $y_j\otimes\mathbb{K}^{-1}_{\nu'(j)}$ plus a sum of terms in $\mathrm{U}^-\otimes\mathrm{U}^0\mathrm{U}^-_{<0}$. This implies that

$$y_j\mathbb{K}_\mu x_i(v_\lambda\otimes v'_{\lambda'})=q^{(\nu(i),\lambda)+(\mu,\lambda-\lambda'+\nu(i))-(\nu'(j),-\lambda'+\nu(i))}y_jv_\lambda\otimes x_iv'_{\lambda'}+(*),$$

where $(*)$ is a sum of terms from a certain $L_q(\lambda)\otimes L_q^\omega(\lambda')_{-\lambda'+\nu}$ with $\nu\neq\nu(i)$.

The maximality of ν_0 implies that $y_j \mathbb{K}_\mu x_i(v_\lambda \otimes v'_{\lambda'})$ has a component in $L_q(\lambda) \otimes L_q^\omega(\lambda')_{-\lambda'+\nu_0}$ only for $\nu(i) = \nu_0$. Therefore, the projection of $u(v_\lambda \otimes v'_{\lambda'})$ onto $L_q(\lambda) \otimes L_q^\omega(\lambda')_{-\lambda'+\nu_0}$ is equal to

$$\sum_{j,\mu,i;\nu(i)=\nu_0} a_{j,\mu,i} q^{(\nu_0,\lambda)(\mu,\lambda-\lambda'+\nu_0)-(\nu'(j),-\lambda'+\nu_0)} y_j v_\lambda \otimes x_i v'_{\lambda'}, \tag{3.3}$$

since we assume that $u(v_\lambda \otimes v'_{\lambda'}) = 0$, this projection is also equal to 0.

We can find an integer $N > 0$ such that

$$\nu_0 < \sum_{\alpha \in \Pi} N\alpha \quad \text{and} \quad \nu'(j) < \sum_{\alpha \in \Pi} N\alpha$$

for all j. Set

$$\Lambda_N^+ = \left\{ \lambda \in \Lambda \;\middle|\; \begin{array}{l} \lambda \text{ is typical, } L_q(\lambda) \text{ is finite-dimensional, } \langle \lambda, \alpha_i \rangle > N \text{ for all } i \neq s \\ \text{and plus an extra condition } 2\langle \lambda + \rho, \delta \rangle > N + 1 \text{ if } \mathfrak{g} \text{ is of type II} \end{array} \right\}.$$

By the same argument before the proposition, we know that the map $\mathrm{U}_{\nu_0}^+ \to L_q^\omega(\lambda')_{\lambda'-\nu_0}$, $x \mapsto x v'_{\lambda'}$ is bijective for all $\lambda' \in \Lambda_N^+$. Thus, the elements $x_i v'_{\lambda'}$ with $\nu(i) = \nu_0$ are linearly independent. Therefore, the vanishing of the sum in (3.3) implies (for all $\lambda' \in \Lambda_N^+$)

$$\sum_{j,\mu} a_{j,\mu,i} q^{(\nu_0,\lambda)+(\mu,\lambda-\lambda'+\nu_0)-(\nu'(j),-\lambda'+\nu_0)} y_j v_\lambda = 0, \tag{3.4}$$

for all i with $\nu(i) = \nu_0$.

The statement before this theorem implies that all $y_j v_\lambda$ with nonzero coefficients $a_{j,\mu,i}$ occurring in (3.4) are linearly independent for all $\lambda \in \Lambda_N^+$. So we get from (3.4)

$$\sum_{\mu} a_{j,\mu,i} q^{(\nu_0,\lambda)+(\mu,\lambda-\lambda'+\nu_0)-(\nu'(j),-\lambda'+\nu_0)} = 0, \tag{3.5}$$

for all i, j with $\nu(i) = \nu_0$. We can cancel the (nonzero) factor $q^{(\nu_0,\lambda)-(\nu'(j),-\lambda'+\nu_0)}$ in (3.5), which does not depend on μ, and get

$$\sum_{\mu} a_{j,\mu,i} q^{(\mu,\nu_0-\lambda')} q^{(\mu,\lambda)} = 0, \tag{3.6}$$

for all i, j with $\nu(i) = \nu_0$ and all $\lambda, \lambda' \in \Lambda_N^+$. Now, fix λ' and notice that (-, -) on $\mathbb{Z}\Phi \times \Lambda_N^+$ is non-degenerate in the first component for all N, thus the coefficients $a_{j,\mu,i} q^{(\mu,\nu_0-\lambda')}$ in (3.6) are all equal to 0. This implies that $a_{j,\mu,i} = 0$ for all i, j, μ with $\nu(i) = \nu_0$, contradicting the choice of ν_0. Therefore, $u = 0$. □

One can check Proposition 3.4 and Theorem 3.5 hold if $\mathfrak{g} = A(n, n)$ since $\mathbb{Z}\tilde{\Phi}$ has a partial order. Next, we strengthen Theorem 3.5 for $\mathfrak{g} = A(n, n)$.

Theorem 3.6. *Let $u \in \mathrm{U}_q(A(n, n))$. If u annihilates all typical finite-dimensional irreducible $\mathrm{U}_q(A(n, n))$-modules, then $u = 0$.*

Proof. It is known that if a typical irreducible module $L_q(\lambda)$ is a composition factor of a finite-dimensional module M, then $L_q(\lambda)$ is a direct summand of M [16, Sect. 3.2]. By the proof of Theorem 3.5, we only need to prove the following claim.

For all $N > n$, there exists $\lambda \in \Lambda_N^+$ such that the set

$$\left\{\lambda' \in \Lambda_N^+ \middle| \, L_q(\lambda) \otimes L_q^\omega(\lambda') \text{ is completely reducible}\right\}$$

could linearly span $\mathfrak{h}^*$.

If it is true, then $L_q(\lambda) \otimes L_q^\omega(\lambda')$ is completely reducible if all weights in $\lambda + \mathrm{wt}\left(L_q^\omega(\lambda')\right)$ are typical. Because the composition factors of $L_q(\lambda) \otimes L_q^\omega(\lambda')$ are the form of $L_q(\bar{\lambda})$ with $\bar{\lambda} \in \lambda + \mathrm{wt}\left(L_q^\omega(\lambda')\right)$ [39, Corollary 5.2].

Proof of the claim. Let $\tilde{\lambda} = \sum_{i=1}^{n+1} \big((n+1-i)(N+2)+2\big)\varepsilon_i - \sum_{j=1}^{n}(j-1)(N+2)\delta_j - (nN+4n+2)\delta_{n+1} \in \Lambda_{N+1}^+$. Then $\tilde{\lambda}+\alpha_i \in \Lambda_N^+$ for all $i \in \mathbb{I}$. There exists a positive integer κ such that it is bigger than $\pm(\mu, \varepsilon_j)$ and $\pm(\mu, \delta_k)$ for any $\mu \in \mathrm{wt}\big(L_q^\omega(\tilde{\lambda}+\alpha_i)\big)$ with $i \in \mathbb{I}$, $j, k = 1, 2, \cdots, n+1$. Let $a = 8\kappa$ and $\lambda = \sum_{i=1}^{n+1}(n+\frac{5}{2}-i)a\varepsilon_i - \sum_{j=1}^{n} ja\delta_j - \frac{3(n+1)}{2}a\delta_{n+1} \in \Lambda$. Then $\lambda \in \Lambda_N^+$ and $\lambda + \mu$ are typical weights for all $\mu \in \mathrm{wt}\big(L_q^\omega(\tilde{\lambda}+\alpha_i)\big)$ with $i \in \mathbb{I}$. So $L_q(\lambda) \otimes L_q^\omega(\tilde{\lambda}+\alpha_i)$ are completely reducible for all $i \in \mathbb{I}$. Since $\{\tilde{\lambda}+\alpha_i | i \in \mathbb{I}\}$ could linearly span $\mathfrak{h}^*$, the claim holds. □

Corollary 3.7. *The Center $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ is contained in* U_0.

Proof. If $\mathfrak{g} \neq A(n,n)$, note that $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ is $\mathbb{Z}\Phi$-graded since $\mathrm{U}_q(\mathfrak{g})$ is $\mathbb{Z}\Phi$-graded. Assuming that $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g})) \cap \mathrm{U}_q(\mathfrak{g})_\nu \neq 0$ for some $\nu \in \mathbb{Z}\Phi$, we will show that $\nu = 0$. Pick $0 \neq z \in \mathcal{Z}(\mathrm{U}_q(\mathfrak{g})) \cap \mathrm{U}_q(\mathfrak{g})_\nu$. Then $z = \mathbb{K}_i z \mathbb{K}_i^{-1} = q^{(\nu,\alpha_i)}z$ for all $i \in \mathbb{I}$; hence $(\nu, \alpha_i) = 0$ for all $i \in \mathbb{I}$, and $\nu = 0$ since (-, -) is non-degenerate.

For $\mathfrak{g} = A(n,n)$, the quantum superalgebra $\mathrm{U}_q(\mathfrak{g})$ is $\mathbb{Z}\tilde{\Phi}$-graded. Similar to the argument above, if $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g})) \cap \mathrm{U}_q(\mathfrak{g})_\nu \neq 0$ with $\nu \in \mathbb{Z}\tilde{\Phi}$, then ν is contained in the radical of (-, -). Thus, $\nu = k\gamma$ for some $k \in \mathbb{Z}$. We need to prove $k = 0$. Otherwise assume $k \neq 0$. Let M be an arbitrary finite-dimensional irreducible module with the highest weight λ and highest weight vector v_λ and lowest weight λ' and lowest weight vector $v_{\lambda'}$. Then $zv_\lambda \in M_{\lambda+k\gamma} = 0$ if $k > 0$ since $k\gamma > 0$. Furthermore, $zv_{\lambda'} \in M_{\lambda'+k\gamma} = 0$ if $k < 0$ since $k\gamma < 0$. Thus $zM = 0$ and hence $z = 0$ by Theorem 3.6, which contradicts the choice of z. □

Remark 3.8. It is not known to us whether the projection from $\mathrm{U}|_0$ to U^0 is an algebra homomorphism or not, see Remark 2.5. However, the projection π from U_0 to U^0 is an algebra homomorphism, then $\mathcal{HC}$ is an algebra homomorphism automatically. Moreover, Corollary 3.7 is also crucial in our proof of the injectivity of $\mathcal{HC}$ which relies on the decomposition $\mathrm{U}_0 = \bigoplus_{\nu \geqslant 0} \mathrm{U}_{-\nu}^-\mathrm{U}^0\mathrm{U}_\nu^+$, see Lemma 5.1.

4. Drinfeld Double and Ad-Invariant Bilinear Form

4.1. The Drinfeld double. In order to establish the Harish-Chandra homomorphism for quantum superalgebras, we need to construct the quantum Killing form or Rosso form for quantum superalgebras. Our approach to obtaining this takes advantage of the Drinfeld double for $\mathbb{Z}_2$-graded Hopf algebras [18].

Definition 4.1. A bilinear mapping $(\ ,\ )\colon \mathcal{B} \times \mathcal{A} \mapsto k$ is called a *skew-pairing* of the $\mathbb{Z}_2$-graded Hopf algebras $\mathcal{A}$ and $\mathcal{B}$ over k if for all $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$ we have

$$\begin{aligned}&(b, 1) = \varepsilon(b), \quad (1, a) = \varepsilon(a),\\ &(bb', a) = (-1)^{|b'||a_{(1)}|} \sum (b, a_{(1)})(b', a_{(2)}), \quad (b, aa') = \sum (b_{(1)}, a')(b_{(2)}, a).\end{aligned} \tag{4.1}$$

Proposition 4.2. ([18, Proposition 4]) *Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbb{Z}_2$-graded Hopf algebras equipped with a skew-pairing $(\ ,\ )\colon \mathcal{B} \times \mathcal{A} \mapsto k$. Then the vector space $\mathcal{A} \otimes \mathcal{B}$ becomes a superalgebra with multiplication defined by*

$$(a \otimes b)(a' \otimes b') = \sum (-1)^{(|a'_{(1)}|+|a'_{(2)}|)(|b_{(2)}|+|b_{(3)}|)} (S(b_{(1)}), a'_{(1)})(b_{(3)}, a'_{(3)}) a a'_{(2)} \otimes b_{(2)} b', \tag{4.2}$$

for $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. With the tensor product co-algebra and antipode $S(a \otimes b) = (-1)^{|a||b|}(1 \otimes S(b))(S(a) \otimes 1)$ structure of $\mathcal{A} \otimes \mathcal{B}$, this superalgebra is also a $\mathbb{Z}_2$-graded Hopf algebra, called the Drinfeld double of $\mathcal{A}$ and $\mathcal{B}$ and denoted it by $\mathcal{D}(\mathcal{A}, \mathcal{B})$.

The existence of a dual pairing of $\mathrm{U}^{\geqslant 0}$ and $(\mathrm{U}^{\leqslant 0})^{\mathrm{op}}$ was observed by Drinfeld [11]. In our exposition, we followed Tanisaki [44, Proposition 2.1.1] for quantum groups and Lehrer, Zhang, Zhang [32, Sect. 3] for quantum superalgebra $\mathrm{U}_q(\mathfrak{gl}_{m|n})$. We have the following proposition.

Proposition 4.3. *There is a unique non-degenerate skew-pairing between the $\mathbb{Z}_2$-graded Hopf algebras $\mathrm{U}^{\geqslant 0}$ and $\mathrm{U}^{\leqslant 0}$ with*

$$(\mathbb{K}_i, \mathbb{K}_j) = q^{-(\alpha_i, \alpha_j)}, \quad (\mathbb{F}_i, \mathbb{E}_j) = -\delta_{ij} \frac{1}{q_i - q_i^{-1}} \text{ and } (\mathbb{K}_i, \mathbb{E}_j) = 0, \quad (\mathbb{F}_i, \mathbb{K}_j) = 0. \tag{4.3}$$

Proof. The skew-pairing is well-defined follows from [14] or Remark 2.4, and the non-degeneracy of skew-pairing can be obtained from the following: for $\mu \in \mathbb{Z}\Phi$ with $\mu > 0$ and $u \in \mathrm{U}^-_{-\mu}$ with $[\mathbb{E}_i, u] = 0$ for all $i \in \mathbb{I}$, then $u = 0$. Similarly, if $u \in \mathrm{U}^+_{\mu}$ with $[\mathbb{F}_i, u] = 0$ for all $i \in \mathbb{I}$, then $u = 0$. The fact can be proven in a similar way to Lemma 5.1, which we omit here. □

Remark 4.4. Geer [14] extended Lusztig's [34] results to the Etingof-Kazhdan quantization of Lie superalgebras $\mathrm{U}_h^{DJ}(\mathfrak{g})$ and checked directly that the extra quantum Serre-type relations are in the radical of the bilinear form. Indeed, the radical of the bilinear form is generated by the extra quantum Serre-type relations and higher order Serre relations.

Corollary 4.5. *As a superalgebra, $\mathcal{D}(\mathrm{U}^{\geqslant 0}, \mathrm{U}^{\leqslant 0})$ is generated by elements $\mathbb{E}_i, \mathbb{K}_i, \mathbb{K}_i^{-1}$, $\mathbb{F}_i, \mathbb{K}'_i, \mathbb{K}_i'^{-1}$. The defining relations are the relations for the generators $\mathbb{E}_i, \mathbb{K}_i, \mathbb{K}_i^{-1}$, (resp. , $\mathbb{F}_i, \mathbb{K}'_i, \mathbb{K}_i'^{-1}$) of the superalgebra $\mathrm{U}^{\geqslant 0}$ (resp. $\mathrm{U}^{\leqslant 0}$), and the following cross relations:*

$$\mathbb{K}'_i \mathbb{E}_j \mathbb{K}_i'^{-1} = q^{(\alpha_i, \alpha_j)} \mathbb{E}_j, \quad \mathbb{K}_i \mathbb{F}_j \mathbb{K}_i^{-1} = q^{-(\alpha_i, \alpha_j)} \mathbb{F}_j, \tag{4.4}$$

$$\mathbb{K}_i \mathbb{K}'_j = \mathbb{K}'_j \mathbb{K}_i, \quad \mathbb{E}_i \mathbb{F}_j - (-1)^{|\mathbb{E}_i||\mathbb{F}_j|} \mathbb{F}_j \mathbb{E}_i = \delta_{ij} \frac{\mathbb{K}_i - \mathbb{K}_i'^{-1}}{q_i - q_i^{-1}}. \tag{4.5}$$

It is known [14,18] that the sub-superalgebras $\mathrm{U}^{\geqslant 0}$ and $\mathrm{U}^{\leqslant 0}$ of the quantum superalgebras $\mathrm{U}_q(\mathfrak{g})$ form a skew-pairing, and $\mathrm{U}_q(\mathfrak{g})$ is a quotient of quantum double of $\mathcal{D}(\mathrm{U}^{\geqslant 0}, \mathrm{U}^{\leqslant 0})$. More precisely, we set $\mathcal{I}$ to be the two-sided ideal generated by the elements $\mathbb{K}_i - \mathbb{K}_i'^{-1}$, which is also a $\mathbb{Z}_2$-graded Hopf ideal, and we have canonical isomorphism $\mathcal{D}(\mathrm{U}^{\geqslant 0}, \mathrm{U}^{\leqslant 0})/\mathcal{I} \cong \mathrm{U}_q(\mathfrak{g})$ as $\mathbb{Z}_2$-graded Hopf algebras. Recently, Drinfeld doubles have been studied by various authors as a useful tool to recover the quantum groups (see, e.g., [6,12,13,20–22]).

4.2. Rosso form. Now we can define an ad-invariant and non-degenerate bilinear form on quantum superalgebras by using skew-pairing between $\mathrm{U}^{\geqslant 0}$ and $\mathrm{U}^{\leqslant 0}$.

Theorem 4.6. *Define a bilinear form* $\langle\ ,\ \rangle\colon \mathrm{U}_q(\mathfrak{g}) \times \mathrm{U}_q(\mathfrak{g}) \to k$ *by*

$$\langle (y\mathbb{K}_\nu)\mathbb{K}_\lambda x, (y'\mathbb{K}_{\nu'})\mathbb{K}_{\lambda'} x'\rangle = (-1)^{|y|}(y', x)(y, x')q^{(2\rho,\nu)}q^{-(\lambda,\lambda')/2}, \tag{4.6}$$

for $x \in \mathrm{U}^+_\mu$, $x' \in \mathrm{U}^+_{\mu'}$, $y \in \mathrm{U}^-_{-\nu}$, $y' \in \mathrm{U}^-_{-\nu'}$, $\lambda, \lambda' \in \mathbb{Z}\Phi$ *and* $\mu, \mu', \nu, \nu' \in Q$. *The bilinear form is ad-invariant, i.e.,* $\langle \mathrm{ad}(u)v, v'\rangle = (-1)^{|u||v|}\langle v, \mathrm{ad}(S(u))v'\rangle$.

By the use of the duality pairing, Tanisaki [44] described the Killing form of the quantum algebra, which is first constructed by Rosso [38], then used it to investigate the center of quantum algebra. Similar techniques could be applied in the case when $\mathfrak{g}$ is a Lie superalgebra of type A-G. Perhaps the proof of this theorem is known by several specialists, but it seems difficult to find in the existing literature. It is fundamental to prove the surjectivity of Harish-Chandra homomorphism throughout this paper, so we write down the details to make the paper more accessible. Here we need some tedious computations, which are also essential for Sect. 6.

For $x \in \mathrm{U}^+_\mu$ and $y \in \mathrm{U}^-_{-\mu}$, we know $\Delta(x) \in \bigoplus_{0\leqslant\nu\leqslant\mu} \mathrm{U}^+_{\mu-\nu}\mathbb{K}_\nu \otimes \mathrm{U}^+_\nu$ and $\Delta(y) \in \bigoplus_{0\leqslant\nu\leqslant\mu} \mathrm{U}^-_{-\nu} \otimes \mathrm{U}^-_{-(\mu-\nu)}\mathbb{K}_\nu^{-1}$, thus for each $\alpha_i \in \Pi$, we can define elements $r_i(x), r_i'(x)$ in $\mathrm{U}^+_{\mu-\alpha}$ and $r_i(y), r_i'(y)$ in $\mathrm{U}^-_{-(\mu-\alpha)}$ to satisfy the following equations:

$$\Delta(x) = x \otimes 1 + \sum_{i=1}^{r} r_i(x)\mathbb{K}_i \otimes \mathbb{E}_i + \cdots = \mathbb{K}_\mu \otimes x + \sum_{i=1}^{r} \mathbb{E}_i\mathbb{K}_{\mu-\alpha_i} \otimes r_i'(x) + \cdots, \text{ and}$$

$$\Delta(y) = y \otimes \mathbb{K}_\mu^{-1} + \sum_{i=1}^{r} r_i(y) \otimes \mathbb{F}_i\mathbb{K}^{-1}_{\mu-\alpha_i} + \cdots = 1 \otimes y + \sum_{i=1}^{r} \mathbb{F}_i \otimes r_i'(y)\mathbb{K}^{-1}_{\alpha_i} + \cdots.$$

Then for all $x \in \mathrm{U}^+_\mu$, $x' \in \mathrm{U}^+_{\mu'}$ and $y \in \mathrm{U}^-$, we have

$$\begin{aligned}
r_i(xx') &= xr_i(x') + (-1)^{|\mathbb{E}_i||x'|}q^{(\mu',\alpha_i)}r_i(x)x',\\
r_i'(xx') &= (-1)^{|x||\mathbb{E}_i|}q^{(\mu,\alpha_i)}xr_i'(x') + r_i'(x)x',\\
(\mathbb{F}_i y, x) &= (-1)^{|r_i'(x)||\mathbb{E}_i|}(\mathbb{F}_i, \mathbb{E}_i)(y, r_i'(x)),\\
(y\mathbb{F}_i, x) &= (-1)^{|\mathbb{F}_i||r_i(x)|}(\mathbb{F}_i, \mathbb{E}_i)(y, r_i(x)).
\end{aligned}$$

Similarly, for all $y \in \mathrm{U}^-_{-\mu}$, $y' \in \mathrm{U}^-_{-\mu'}$ and $x \in \mathrm{U}^+$, we have

$$\begin{aligned} r_i(yy') &= q^{(\mu,\alpha_i)} y r_i(y') + (-1)^{|\mathbb{F}_i||y'|} r_i(y) y', \\ r'_i(yy') &= (-1)^{|y||\mathbb{F}_i|} y r'_i(y') + q^{(\mu',\alpha_i)} r'_i(y) y', \\ (y, \mathbb{E}_i x) &= (\mathbb{F}_i, \mathbb{E}_i)(r_i(y), x), \\ (y, x\mathbb{E}_i) &= (\mathbb{F}_i, \mathbb{E}_i)(r'_i(y), x). \end{aligned}$$

Thus, we have the following lemma.

Lemma 4.7. *For all $x \in \mathrm{U}^+_\mu$ and $y \in \mathrm{U}^-_{-\mu}$, we have*

$$[x, \mathbb{F}_i] = x\mathbb{F}_i - (-1)^{|x||\mathbb{F}_i|}\mathbb{F}_i x = (q_i - q_i^{-1})^{-1}\big(r_i(x)\mathbb{K}_i - (-1)^{|r'_i(x)||\mathbb{F}_i|}\mathbb{K}_i^{-1} r'_i(x)\big), \tag{4.7}$$

$$[\mathbb{E}_i, y] = \mathbb{E}_i y - (-1)^{|y||\mathbb{E}_i|} y\mathbb{E}_i = (q_i - q_i^{-1})^{-1}\big((-1)^{|\mathbb{E}_i||r_i(y)|}\mathbb{K}_i r_i(y) - r'_i(y)\mathbb{K}_i^{-1}\big). \tag{4.8}$$

Proof. We only prove Eqs. (4.8), and (4.7) is similar. For $y = 1$ and $y = \mathbb{F}_i$ the formula follows from definition, so it is enough to show that if Eq. (4.8) holds for $y \in \mathrm{U}^-_{-\mu}$ and $y' \in \mathrm{U}^-_{-\mu'}$, then Eq. (4.8) holds for yy'. This can be derived as follows.

$$\begin{aligned} (q_i - q_i^{-1})[\mathbb{E}_i, yy'] &= (q_i - q_i^{-1})\big([\mathbb{E}_i, y]y' + (-1)^{|\mathbb{E}_i||y|} y[\mathbb{E}_i, y']\big) \\ &= (-1)^{|\mathbb{E}_i||r_i(y)|}\big(\mathbb{K}_i r_i(y) - r'_i(y)\mathbb{K}_i^{-1}\big)y' \\ &\quad + (-1)^{|\mathbb{E}_i||y|} y\big((-1)^{|\mathbb{E}_i||r_i(y')|}\mathbb{K}_i r_i(y') - r'_i(y')\big)\mathbb{K}_i^{-1} \\ &= (-1)^{|\mathbb{E}_i||r_i(yy')|}\mathbb{K}_i\big((-1)^{|\mathbb{E}_i||y'|} r_i(y)y' + q^{(\mu,\alpha_i)} y r_i(y')\big) \\ &\quad - \big(q^{(\mu',\alpha_i)} r'_i(y)y' + (-1)^{|\mathbb{E}_i||y|} y r'_i(y')\big)\mathbb{K}_i^{-1} \\ &= (-1)^{|\mathbb{E}_i||r_i(yy')|}\mathbb{K}_i r_i(yy') - r'_i(yy')\mathbb{K}_i^{-1}. \end{aligned}$$

□

Combining the above lemma, we get the following equations, which are very useful when proofing Theorem 4.6.

$$\begin{aligned} \mathrm{ad}(\mathbb{E}_i)(y\mathbb{K}_\lambda x) &= \mathbb{E}_i y\mathbb{K}_\lambda x - (-1)^{|\mathbb{E}_i|(|x|+|y|)}\mathbb{K}_i y\mathbb{K}_\lambda x\mathbb{K}_i^{-1}\mathbb{E}_i \\ &= [\mathbb{E}_i, y]\mathbb{K}_\lambda x + (-1)^{|y||\mathbb{E}_i|} y\mathbb{E}_i\mathbb{K}_\lambda x - (-1)^{|\mathbb{E}_i|(|x|+|y|)}\mathbb{K}_i y\mathbb{K}_\lambda x\mathbb{K}_i^{-1}\mathbb{E}_i \\ &= (q_i - q_i^{-1})^{-1}\big((-1)^{|\mathbb{E}_i||r_i(y)|}\mathbb{K}_i r_i(y)\mathbb{K}_\lambda x - r'_i(y)\mathbb{K}_i^{-1}\mathbb{K}_\lambda x\big) \\ &\quad + (-1)^{|y||\mathbb{E}_i|} q^{(\lambda,-\alpha_i)} y\mathbb{K}_\lambda\mathbb{E}_i x - (-1)^{|\mathbb{E}_i|(|x|+|y|)} q^{(\mu-\nu,\alpha_i)} y\mathbb{K}_\lambda x\mathbb{E}_i \\ &= (q_i - q_i^{-1})^{-1}\big((-1)^{|\mathbb{E}_i||r_i(y)|} q^{(\nu-\alpha_i,-\alpha_i)} r_i(y)\mathbb{K}_{\lambda+\alpha_i} x - r'_i(y)\mathbb{K}_{\lambda-\alpha_i} x\big) \\ &\quad + (-1)^{|y||\mathbb{E}_i|} q^{(\lambda,-\alpha_i)} y\mathbb{K}_\lambda\mathbb{E}_i x - (-1)^{|\mathbb{E}_i|(|x|+|y|)} q^{(\mu-\nu,\alpha_i)} y\mathbb{K}_\lambda x\mathbb{E}_i. \end{aligned}$$

Now, we are ready to prove Theorem 4.6.

Proof of Theorem 4.6. It is enough to take u to be generators, i.e., $\mathbb{E}_i$, $\mathbb{F}_i$ and $\mathbb{K}_i$. Furthermore, we may assume that

$$v = (y\mathbb{K}_\nu)\mathbb{K}_\lambda x \quad \text{and} \quad v' = (y'\mathbb{K}_{\nu'})\mathbb{K}_{\lambda'} x',$$

with $\lambda, \lambda' \in \mathbb{Z}\Phi$ and $x \in \mathrm{U}^+_{\mu}$, $x' \in \mathrm{U}^+_{\mu'}$, $y \in \mathrm{U}^-_{-\nu}$, $y' \in \mathrm{U}^-_{-\nu'}$ with weights $\mu, \mu', \nu, \nu' \in Q$.

It is obvious for $u = \mathbb{K}_i$. For $u = \mathbb{E}_i$, then

$$\begin{aligned}
\mathrm{ad}(\mathbb{E}_i)(v) &= (q_i - q_i^{-1})^{-1}\big((-1)^{|\mathbb{E}_i||r_i(y)|}q^{(\nu-\alpha_i,-\alpha_i)}r_i(y)\mathbb{K}_{\lambda+\nu+\alpha_i}x - r_i'(y)\mathbb{K}_{\lambda+\nu-\alpha_i}x\big)\\
&\quad + (-1)^{|y||\mathbb{E}_i|}q^{(\lambda+\nu,-\alpha_i)}y\mathbb{K}_{\lambda+\nu}\mathbb{E}_i x\\
&\quad - (-1)^{|\mathbb{E}_i|(|x|+|y|)}q^{(\mu-\nu,\alpha_i)}y\mathbb{K}_{\lambda+\nu}x\mathbb{E}_i, \text{ and}\\
\mathrm{ad}(S(\mathbb{E}_i))(v') &= -\mathrm{ad}(\mathbb{K}_i^{-1})\mathrm{ad}(\mathbb{E}_i)(v') = -q^{(\mu'+\alpha_i-\nu',-\alpha_i)}\mathrm{ad}(\mathbb{E}_i)(v')\\
&= (q_i - q_i^{-1})^{-1}\big(-(-1)^{|\mathbb{E}_i||r_i(y')|}q^{(\mu',-\alpha_i)}r_i(y')\mathbb{K}_{\lambda'+\nu'+\alpha_i}x'\\
&\quad + q^{(\mu'+\alpha_i-\nu',-\alpha_i)}r_i'(y')\mathbb{K}_{\lambda'+\nu'-\alpha_i}x'\big)\\
&\quad - (-1)^{|y'||\mathbb{E}_i|}q^{(\lambda'+\mu'+\alpha_i,-\alpha_i)}y'\mathbb{K}_{\lambda'+\nu'}\mathbb{E}_i x'\\
&\quad + (-1)^{|\mathbb{E}_i|(|x'|+|y'|)}q^{(\alpha_i,-\alpha_i)}y'\mathbb{K}_{\lambda'+\nu'}x'\mathbb{E}_i.
\end{aligned}$$

Now the problem can be split into two cases. First, if $\mu = \nu'$ and $\mu' + \alpha_i = \nu$, then

$$\begin{aligned}
\langle \mathrm{ad}(\mathbb{E}_i)v, v'\rangle &= (-1)^{|r_i(y)|}(q_i - q_i^{-1})^{-1}(y', x)q^{(2\rho,\nu-\alpha_i)},\\
&\quad \cdot \big((-1)^{|\mathbb{E}_i||r_i(y)|}q^{(\nu-\alpha_i,-\alpha_i)-1/2(\lambda+2\alpha_i,\lambda')}(r_i(y), x') - q^{-1/2(\lambda,\lambda')}(r_i'(y), x')\big),
\end{aligned}$$

and

$$\begin{aligned}
\langle v, \mathrm{ad}(S(\mathbb{E}_i))v'\rangle &= (-1)^{|y|}(y', x)q^{(2\rho,\nu)}\big(-(-1)^{|y'||\mathbb{E}_i|}q^{(\lambda'+\mu'+\alpha_i,-\alpha_i)-1/2(\lambda,\lambda')}(y, \mathbb{E}_i x')\\
&\quad + (-1)^{|\mathbb{E}_i|(|x'|+|y'|)}q^{(\alpha_i,-\alpha_i)-1/2(\lambda,\lambda')}(y, x'\mathbb{E}_i)\big).
\end{aligned}$$

Therefore, $\langle \mathrm{ad}(\mathbb{E}_i)v, v'\rangle = (-1)^{|\mathbb{E}_i||v|}\langle v, \mathrm{ad}(S(\mathbb{E}_i))v'\rangle$.

Second, if $\mu + \alpha_i = \nu'$ and $\mu' = \nu$, then

$$\begin{aligned}
\langle \mathrm{ad}(\mathbb{E}_i)v, v'\rangle &= (-1)^{|y|}q^{(2\rho,\nu)}(y, x') \cdot \big((-1)^{|y||\mathbb{E}_i|}q^{(\lambda+\nu,-\alpha_i)-1/2(\lambda,\lambda')}\\
&\quad \cdot (y', \mathbb{E}_i x) - (-1)^{|\mathbb{E}_i|(|x|+|y|)}q^{(\mu-\nu,\alpha_i)-1/2(\lambda,\lambda')}(y', x\mathbb{E}_i)\big),
\end{aligned}$$

and

$$\begin{aligned}
\langle v, \mathrm{ad}(S(\mathbb{E}_i))v'\rangle &= (-1)^{|y|}(q_i - q_i^{-1})^{-1}q^{(2\rho,\nu)}(y, x')\\
&\quad \cdot \big(-(-1)^{|\mathbb{E}_i||r_i(y')|}q^{(\mu',-\alpha_i)-1/2(\lambda,\lambda'+2\alpha_i)}\\
&\quad \cdot (r_i(y'), x) + q^{(\mu'+\alpha_i-\nu',-\alpha_i)-1/2(\lambda,\lambda')}(r_i'(y'), x)\big).
\end{aligned}$$

Therefore, $\langle \mathrm{ad}(\mathbb{E}_i)v, v'\rangle = (-1)^{|\mathbb{E}_i||v|}\langle v, \mathrm{ad}(S(\mathbb{E}_i))v'\rangle$. Using a similar procedure, we can check for $u = \mathbb{F}_i$. Thus, we proved the ad-invariance of the bilinear form. □

Proposition 4.8. *Let $u \in \mathrm{U}_q(\mathfrak{g})$. If $\langle v, u\rangle = 0$ for all $v \in \mathrm{U}_q(\mathfrak{g})$, then $u = 0$.*

Proof. Notice that $\mathrm{U}_q(\mathfrak{g})$ is the direct sum of all $\mathrm{U}^-_{-\nu}\mathrm{U}^0\mathrm{U}^+_{\mu} = \mathrm{U}^-_{-\nu}\mathbb{K}_{\nu}\mathrm{U}^0\mathrm{U}^+_{\mu}$ as vector space. Therefore, it is sufficient to show that if $u \in \mathrm{U}^-_{-\nu}\mathrm{U}^0\mathrm{U}^+_{\mu}$ with $\langle v, u\rangle = 0$ for all $v \in \mathrm{U}^-_{-\mu}\mathrm{U}^0\mathrm{U}^+_{\nu}$, then $u = 0$.

Since the skew-pairing between U^- and U^+ is non-degenerate, we can choose an arbitrary basis $u_1^{\mu}, u_2^{\mu}, \cdots, u_{r(\mu)}^{\mu}$ of U^+_{μ} and dual basis $v_1^{\mu}, v_2^{\mu}, \cdots, c_{r(\mu)}^{\mu}$ of $\mathrm{U}^-_{-\mu}$ for any $\mu \in Q$

with respect to skew-pairing, i.e., $(v_i^\mu, u_j^\mu) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant r(\mu)$, where $r(\mu) = \dim \mathrm{U}_\mu^+$.

For any $\mu, \nu \in Q$, we know that $\left\{ (v_i^\nu \mathbb{K}_\nu)\mathbb{K}_\lambda u_j^\mu \,\middle|\, \text{for all } \lambda \in \mathbb{Z}\Phi \text{ and } 1 \leqslant i \leqslant r(\nu), 1 \leqslant j \leqslant r(\mu) \right\}$ is a basis of $\mathrm{U}_{-\nu}^-\mathrm{U}^0\mathrm{U}_\mu^+$. From Eq. (4.6), we have

$$\langle (v_h^\mu \mathbb{K}_\mu)\mathbb{K}_{\lambda'} u_l^\nu, (v_i^\nu \mathbb{K}_\nu)\mathbb{K}_\lambda u_j^\mu \rangle = \delta_{hj}\delta_{li}(-1)^{|\mu|}(q^{1/2})^{-(\lambda,\lambda')}q^{(2\rho,\mu)}. \tag{4.9}$$

Write $u = \sum_{i,j,\lambda} a_{ij\lambda}(v_i^\nu \mathbb{K}_\nu)\mathbb{K}_\lambda u_j^\mu$. The assumption $\langle v, u\rangle = 0$ for all v yields

$$\sum_{\lambda \in \mathbb{Z}\Phi} (-1)^{|\nu|} a_{ij\lambda}(q^{1/2})^{-(\lambda,\lambda')} = 0, \quad \text{for all } i, j, \lambda'. \tag{4.10}$$

Thus, each $a_{ij\lambda} = 0$; hence, $u = 0$ as well. □

4.3. Quantum supertrace. In this subsection, in order to construct explicit central element, we recall the definition of the quantum supertrace.

Let $(A, \Delta, \varepsilon, S)$ be a $\mathbb{Z}_2$-graded Hopf algebra over field k and M, N be two A-modules. Then M^* is an A-module with the action $(af)(m) = (-1)^{|a||f|} f(S(a)m)$ for all $m \in M, a \in A, f \in M^*$. $M \otimes N$ is an A-module with the action $a(m \otimes n) = \sum (-1)^{|a_{(2)}||m|} a_{(1)}m \otimes a_{(2)}n$ for all $a \in A, m \in M, n \in N$ where $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$. $\mathrm{Hom}_k(M, N)$ is an A-module with the action $(af)(m) = \sum (-1)^{|a_{(2)}||f|} a_{(1)} f(S(a_{(2)})m)$ for all $a \in A, m \in M, f \in \mathrm{Hom}_k(M, N)$. Supposing that M is finite-dimensional, we take $\{m_i\}$ to be a homogeneous basis of M and $\{f_i\}$ to be the dual basis with respect to $\{m_i\}$. Then we have $|m_i| = |f_i|$ for all i and the following isomorphism of A-modules:

$$\Phi_{M,N} \colon N \otimes M^* \to \mathrm{Hom}(M, N), \quad n \otimes f \mapsto \varphi_{f,n}, \tag{4.11}$$

with inverse homomorphism $\Psi_{M,N} \colon g \mapsto \sum g(m_i) \otimes f_i$, where $\varphi_{f,n}(m) = f(m)n$ for all $f \in M^*, g \in \mathrm{Hom}(M, N), m \in M, n \in N$. We also have a homomorphism of A-modules $\varepsilon_M \colon M^* \otimes M \to k$ with $\varepsilon_M(f \otimes m) = f(m)$ for all $f \in M^*, m \in M$.

In particular, A is the quantum superalgebra $\mathrm{U}_q(\mathfrak{g})$. Then we have $S^2(u) = \mathbb{K}_{2\rho}^{-1} u \mathbb{K}_{2\rho}$ since $(\rho, \alpha_i) = 2(\alpha_i, \alpha_i)$ for all $i \in \mathbb{I}$. We obtain a homomorphism of A-modules $\psi_M \colon M \to (M^*)^*$ with

$$\big(\psi_M(m)\big)(f) = (-1)^{|f||m|} f(\mathbb{K}_{2\rho}^{-1} m). \tag{4.12}$$

Combined with the previous statements, we have the following homomorphisms of A-modules

$$\mathrm{Str}_q^M \colon \mathrm{End}(M) \xrightarrow{\Psi_{M,M}} M \otimes M^* \xrightarrow{\psi_M \otimes 1_{M^*}} (M^*)^* \otimes M^* \xrightarrow{\varepsilon_{M^*}} k. \tag{4.13}$$

This composition is the so-called *quantum supertrace*, which was used to construct knot and 3-manifold invariants in [56] (we simply replace Str_q^M with Str_q if no confusion appears). More precisely, if $g \in \mathrm{End}(M)$, then

$$\mathrm{Str}_q(g) = \varepsilon_{M^*} \circ (\psi_M \otimes 1_{M^*}) \circ \Psi_{M,M}(g) = (-1)^{|g(m_i)||f_i|} \sum_i f_i\big(\mathbb{K}_{2\rho}^{-1} g(m_i)\big)$$

$$= (-1)^{m_i} \sum_i f_i\big(g(\mathbb{K}_{2\rho}^{-1} m_i)\big).$$

Let A be a $\mathbb{Z}_2$-graded Hopf algebra and define the adjoint representation of A as follows: $\mathrm{ad}(a)(b) = \sum (-1)^{|b||a_{(2)}|} a_{(1)} b S(a_{(2)})$. The map $\mathrm{ad}_M : A \to \mathrm{End}(M)$, which takes $a \in A$ to the action of a on M, is a homomorphism of A-modules and we have

$$\mathrm{Str}_q \circ \mathrm{ad}_M(u) = (-1)^{|m_i|} \sum_i f_i\big(u(\mathbb{K}_{2\rho}^{-1} m_i)\big). \tag{4.14}$$

Indeed, this is the supertrace of $u\mathbb{K}_{2\rho}^{-1}$ acting on M. In particular, we have

$$\mathrm{ad}(\mathbb{E}_i)u = \mathbb{E}_i u - (-1)^{|u||e_i|} \mathbb{K}_i u \mathbb{K}_i^{-1} \mathbb{E}_i, \tag{4.15}$$

$$\mathrm{ad}(\mathbb{F}_i)u = (\mathbb{F}_i u - (-1)^{|u||f_i|} u \mathbb{F}_i)\mathbb{K}_i, \tag{4.16}$$

$$\mathrm{ad}(\mathbb{K}_i)u = \mathbb{K}_i u \mathbb{K}_i^{-1}. \tag{4.17}$$

Noticed that $\mathrm{ad}(\mathbb{E}_i) = \mathrm{Ad}_{\mathbb{E}_i}$, but there is a slightly different between $\mathrm{ad}(\mathbb{F}_i)$ and $\mathrm{Ad}_{\mathbb{F}_i}$; see (2.9) and (2.10).

4.4. Construct central elements. In this subsection, we construct central elements for certain finite-dimensional $\mathrm{U}_q(\mathfrak{g})$-modules following Jantzen's book [23].

Let $\varphi : \mathrm{U}_{-\mu}^- \times \mathrm{U}_\nu^+ \to k$ be a bilinear map and $\lambda \in \mathbb{Z}\Phi$. There is a unique element $u \in (\mathrm{U}_{-\nu}^- \mathbb{K}_\nu)\mathbb{K}_\lambda \mathrm{U}_\mu^+ = \mathrm{U}_{-\nu}^- \mathbb{K}_{\nu+\lambda} \mathrm{U}_\mu^+$ such that for all $x \in \mathrm{U}_\nu^+$, $y \in \mathrm{U}_{-\nu}^-$, $\lambda' \in \mathbb{Z}\Phi$

$$\langle (y\mathbb{K}_\nu)\mathbb{K}_{\lambda'} x, u\rangle = \varphi(y, x)(q^{1/2})^{-(\lambda,\lambda')}. \tag{4.18}$$

Indeed, $u = \sum (-1)^{|y|} \varphi(v_j^\mu, u_i^\nu) q^{-(2\rho,\mu)} (v_i^\nu \mathbb{K}_\nu \mathbb{K}_\lambda u_j^\mu)$ will work and be unique according to Proposition 4.8.

Lemma 4.9. *Let M be a finite-dimensional $\mathrm{U}_q(\mathfrak{g})$-module such that all weights λ of M satisfy $2\lambda \in \mathbb{Z}\Phi$. Then there is for each $m \in M$ and $f \in M^*$ a unique element $u \in \mathrm{U}_q(\mathfrak{g})$ such that $f(vm) = \langle v, u\rangle$ for all $v \in \mathrm{U}_q(\mathfrak{g})$.*

Proof. The uniqueness follows from Proposition 4.8. To prove the existence of u, we may assume that f and m are weight vectors, since $f(\cdot m)$ depends linearly on f and m. Suppose that there are two weights λ and λ' of M with $m \in M_\lambda$ and $f \in (M^*)_{\lambda'}$; i.e., with $f(M_{\lambda''}) = 0$ for all $\lambda'' \neq \lambda$. We have $\mathrm{U}_\nu^+ m \in M_{\lambda+\nu}$ for all ν. As M has only finitely many weights, there are only finitely many ν with $\mathrm{U}_\nu^+ m \neq 0$. Since $\mathrm{U}_{-\mu}^- \mathrm{U}^0 \mathrm{U}_\nu^+ m \subseteq M_{\lambda+\nu-\mu}$ for all μ and ν, we get $f(\mathrm{U}_{-\mu}^- \mathrm{U}^0 \mathrm{U}_\nu^+ m) = 0$ unless $\lambda' = \lambda + \nu - \mu$. This shows that there are only finitely many pairs (μ, ν) with $f(\mathrm{U}_{-\mu}^- \mathrm{U}^0 \mathrm{U}_\nu^+ m) \neq 0$. For all $x \in \mathrm{U}_\nu^+$, $y \in \mathrm{U}_{-\mu}^-$ and $\eta \in \mathbb{Z}\Phi$,

$$f(y\mathbb{K}_\mu \mathbb{K}_\eta xm) = q^{(\eta,\lambda+\nu)} f(y\mathbb{K}_\mu xm) = (q^{1/2})^{(\eta,2\lambda+2\nu)} f(y\mathbb{K}_\mu xm). \tag{4.19}$$

For all μ and ν, the function $(y, x) \mapsto f(y\mathbb{K}_\mu xm)$ is bilinear. We now use that $2(\lambda+\nu) \in \mathbb{Z}\Phi$. We get an element $u_{\nu\mu} \in \mathrm{U}_{-\mu}^- \mathrm{U}^0 \mathrm{U}_\nu^+$ with $\langle v, u_{\nu\mu}\rangle = f(vm)$ for all $v \in \mathrm{U}_{-\mu}^- \mathrm{U}^0 \mathrm{U}_\nu^+$. Then $u = \sum u_{\nu\mu}$ will satisfy our claim. □

Remark 4.10. The condition all weights of M are contained in $\frac{1}{2}\mathbb{Z}\Phi$ is indispensable, since the construction of $u_{\nu\mu}$ depends on the condition $2(\lambda+\nu) \in \mathbb{Z}\Phi$ according to the expression of u in Eq. (4.18). Lemma 4.9 still work without this condition if one enlarge the Cartan subalgebra of quantum superalgebra, also see Remark 6.5.

Lemma 4.11. *Let M be a finite-dimensional $\mathrm{U}_q(\mathfrak{g})$-module such that all weights λ of M satisfy $2\lambda \in \mathbb{Z}\Phi$. Then there is a unique element $z_M \in \mathrm{U}_q(\mathfrak{g})$ such that $\langle u, z_M\rangle$ is equal to the supertrace of $u\mathbb{K}_{2\rho}^{-1}$ acting on M for all $u \in \mathrm{U}_q(\mathfrak{g})$. The element z_M is contained in the center $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ of $\mathrm{U}_q(\mathfrak{g})$.*

Proof. Let $\{m_1, m_2, \cdots, m_r\}$ be a homogeneous basis of M and $\{f_1, f_2, \cdots, f_r\}$ be the dual basis of M^*, then the supertrace of $u\mathbb{K}_{2\rho}^{-1}$ acting on M is equal to $\sum_{i=1}^{r}(-1)^{|m_i|} f_i (u\mathbb{K}_{2\rho}^{-1}m_i) = \langle u, z_M\rangle$. In this way, the existence and uniqueness of z_M follow from Lemma 4.9. Recall that the map $\mathrm{ad}_M : \mathrm{U}_q(\mathfrak{g}) \to \mathrm{End}(M)$ is a homomorphism of $\mathrm{U}_q(\mathfrak{g})$-modules. We notice that $\mathrm{Str}_q^M \circ \mathrm{ad}_M(u)$ is the supertrace of $u\mathbb{K}_{2\rho}^{-1}$ acting on M for all $u \in \mathrm{U}_q(\mathfrak{g})$; i.e., $\mathrm{Str}_q^M \circ \mathrm{ad}_M(u) = \langle u, z_M\rangle$ for all $u \in \mathrm{U}_q(\mathfrak{g})$ by (4.14). This means that for all $u, v \in \mathrm{U}_q(\mathfrak{g})$,

$$\varepsilon(v)\langle u, z_M\rangle = v \cdot (\mathrm{Str}_q^M \circ \mathrm{ad}_M(u)) = \langle \mathrm{ad}(v)u, z_M\rangle = (-1)^{|v||u|}\langle u, \mathrm{ad}(S(v))z_M\rangle. \tag{4.20}$$

Hence, $\varepsilon(v)z_M = (-1)^{|v|(|v|+|z_M|)}\mathrm{ad}(S(v))z_M = (-1)^{|v|}\mathrm{ad}(S(v))z_M$ for all $v \in \mathrm{U}_q(\mathfrak{g})$ by Proposition 4.8. We also have $(-1)^{|v|}\mathrm{ad}(v)z_M = \varepsilon(v)z_M$ by $\varepsilon \circ S = \varepsilon$. Therefore, z_M is central in $\mathrm{U}_q(\mathfrak{g})$. □

5. Harish-Chandra Homomorphism of Quantum Superalgebras

5.1. The Harish-Chandra homomorphism. In the previous section, we used the Drinfeld double to construct an ad-invariant bilinear form in Theorem 4.6, which was also non-degenerate (see Proposition 4.8). By using this form and quantum supertrace, we can construct the central elements of $\mathrm{U}_q(\mathfrak{g})$, which contributes to establish the Harish-Chandra isomorphism for quantum superalgebras $\mathrm{U}_q(\mathfrak{g})$. Now we are ready to define the Harish-Chandra homomorphism.

For each $\lambda \in \Lambda$, there is an algebra homomorphism, also denoted by $\lambda : \mathrm{U}^0 \to \mathbb{C}$, $\lambda(\mathbb{K}_\mu) = q^{(\lambda,\mu)}$ for all $\mu \in \mathbb{Z}\Phi$. Obviously, $(\lambda+\lambda')(h) = \lambda(h)\lambda'(h)$ for $h \in \mathrm{U}^0$ and $\lambda, \lambda' \in \Lambda$.

The triangular decomposition of quantum superalgebra $\mathrm{U}_q(\mathfrak{g})$ implies a direct sum decomposition as follows:

$$\mathrm{U}_0 = \mathrm{U}^0 \oplus \bigoplus_{\nu>0} \mathrm{U}^-_{-\nu}\mathrm{U}^0\mathrm{U}^+_\nu.$$

Let $\pi : \mathrm{U}_0 \to \mathrm{U}^0$ be the projection with respect to this decomposition. One can check that $\bigoplus_{\nu>0} \mathrm{U}^-_{-\nu}\mathrm{U}^0\mathrm{U}^+_\nu$ is a two-sided ideal of U_0. Thus, π is an algebra homomorphism. Denoting the center of $\mathrm{U}_q(\mathfrak{g})$ by $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$[2], we have $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g})) \subseteq \mathrm{U}_0$ by Proposition 3.7.

[2] In general, the center of the Lie superalgebra and quantum superalgebra is $\mathbb{Z}_2$-graded [8, Sect. 2.2]. Similar to the basic Lie superalgebra case, the center of $\mathrm{U}_q(\mathfrak{g})$ consists of only even elements. However, the center contains odd part is also interesting in some aspects; e.g., the skew center of generalized quantum groups [3].

Let $z \in \mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ and write $z = \sum_{\nu \geqslant 0} z_\nu$ where each $z_\nu \in \mathrm{U}^-_{-\nu}\mathrm{U}^0\mathrm{U}^+_\nu$, thus $\pi(z) = z_0$. If we take $v_\lambda \in \Delta_q(\lambda)_\lambda$, then $zv_\lambda = z_0 v_\lambda = \lambda(z_0)v_\lambda$. Since z is the center element of $\mathrm{U}_q(\mathfrak{g})$, this implies $zv = \lambda(z_0)v$, $\forall\, v \in \Delta_q(\lambda)$, so it acts as scalar $\lambda(z_0) = \lambda(\pi(z))$ on $\Delta_q(\lambda)$. We set $\chi_\lambda \colon \mathcal{Z}(\mathrm{U}_q(\mathfrak{g})) \to k$ by $\chi_\lambda(z) = \lambda(\pi(z))$.

For $\lambda \in \Lambda$, we define an algebra automorphism

$$\gamma_\lambda \colon \mathrm{U}^0 \to \mathrm{U}^0 \quad \text{by} \quad \gamma_\lambda(h) = \lambda(h)h, \qquad \text{for all } h \in \mathrm{U}^0.$$

Then

$$\gamma_\lambda(\mathbb{K}_\mu) = q^{(\lambda,\mu)}\mathbb{K}_\mu, \qquad \text{for all } \lambda \in \Lambda,\ \mu \in \mathbb{Z}\Phi.$$

Obviously, γ_0 is the identity map, and

$$\gamma_\lambda \circ \gamma_{\lambda'} = \gamma_{\lambda+\lambda'} \quad \text{and} \quad \lambda'(\gamma_\lambda(h)) = (\lambda+\lambda')(h), \ \text{for all } \lambda, \lambda' \in \Lambda,\ h \in \mathrm{U}^0.$$

Inspired by the quantum group case, we define the *Harish-Chandra homomorphism* $\mathcal{HC}$ of $\mathrm{U}_q(\mathfrak{g})$ to be the composite

$$\mathcal{HC} \colon \mathcal{Z}(\mathrm{U}_q(\mathfrak{g})) \hookrightarrow \mathrm{U}_0 \xrightarrow{\pi} \mathrm{U}^0 \xrightarrow{\gamma_{-\rho}} \mathrm{U}^0.$$

Assume that $h = \mathcal{HC}(z) = \gamma_{-\rho} \circ \pi(z)$, we have $\chi_\lambda(z) = \lambda(\pi(z)) = \lambda(\gamma_\rho(h)) = (\lambda+\rho)(h)$ for all $\lambda \in \Lambda$.

Lemma 5.1. *The Harish-Chandra homomorphism $\mathcal{HC}$ is injective.*

Proof. Suppose $z = \sum_{\mu \geqslant 0} z_\mu \in \mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ with $\mathcal{HC}(z) = \gamma_{-\rho} \circ \pi(z) = 0$ where $z_\mu \in \mathrm{U}^-_{-\mu}\mathrm{U}^0\mathrm{U}^+_\mu$, then $z_0 = \pi(z) = 0$ since $\gamma_{-\rho}$ is an algebra automorphism. If we assume $z \neq 0$, then there exists $z_\mu \neq 0$ for some $\mu \in Q$. Let $\beta \in Q$ be a minimal element satisfying $\beta > 0$ and $z_\beta \neq 0$. Let $\{y_i\}$ and $\{x_k\}$ be bases of $\mathrm{U}^-_{-\beta}$ and U^+_β, respectively, and write

$$z_\beta = \sum_{j,k} y_j h_{jk} x_k, \quad h_{jk} \in \mathrm{U}^0.$$

For all $x \in \mathrm{U}^+_\gamma$, $h \in \mathrm{U}^0$, $y \in \mathrm{U}^-_{-\gamma}$ we have $[\mathbb{E}_i, yhx] = [\mathbb{E}_i, y]hx + (-1)^{|y||\mathbb{E}_i|}y[\mathbb{E}_i, hx]$ with $[\mathbb{E}_i, y]hx \in \mathrm{U}^-_{-(\gamma-\alpha_i)}\mathrm{U}^0\mathrm{U}^+_\gamma$ and $y[\mathbb{E}_i, hx] \in \mathrm{U}^-_{-\gamma}\mathrm{U}^0\mathrm{U}^+_{\gamma+\alpha_i}$ by Eq. (4.8). Since $[\mathbb{E}_i, z] = 0$, we have $\sum_{j,k}[\mathbb{E}_i, y_j]h_{jk}x_k = 0$ by the minimality of β. Hence $\sum_j [\mathbb{E}_i, y_j]h_{jk} = 0$ for any k. Write $\beta = \sum_{i=1}^r m_i\alpha_i$, and let $L_q(\lambda)$ be a finite-dimensional module with the highest weight vector v_λ. Then we have

$$\mathbb{E}_i\Big(\sum_j \lambda(h_{jk})y_j v_\lambda\Big) = \sum_j [\mathbb{E}_i, y_j]h_{jk}v_\lambda = 0,$$

for all $i \in \mathbb{I}$. So $\sum_j \lambda(h_{jk})y_j v_\lambda$ generates a proper submodule of $L_q(\lambda)$, and we get $\sum_j \lambda(h_{jk})y_j v_\lambda = 0$. The linear map $\mathrm{U}^-_{-\beta} \to L_q(\lambda)$ given by $y \mapsto yv_\lambda$ is bijective if λ satisfies the condition of Proposition 3.4. Hence, $\sum_j \lambda(h_{jk})y_j = 0$. Therefore, $h_{jk} = 0$ for any j, k, and $z_\beta = 0$. This contradicts the choice of β with $z_\beta \neq 0$. Thus, $z = 0$ and $\mathcal{HC}$ is injective. □

5.2. Description of the image of the $\mathcal{HC}$. The image of the $\mathcal{HC}$ is much more complicated. We split it into the following three lemmas. Recall that the Weyl group W acts naturally on U^0 as $w(\mathbb{K}_\mu) = \mathbb{K}_{w\mu}$ for all $w \in W$ and $\mu \in \mathbb{Z}\Phi$. We have $(w\lambda)(wh) = \lambda(h)$ for all $w \in W, \lambda \in \Lambda$, and $h \in \mathrm{U}^0$.

Lemma 5.2. *The restriction of the image of Harish-Chandra homomorphism on the center of quantum superalgebra* $\mathrm{U}_q(\mathfrak{g})$ *is contained in the* W*-invariant of* U^0*; i.e.,* $\mathcal{HC}\big(\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))\big) \subset (\mathrm{U}^0)^W$.

Proof. The character of the Verma module $\Delta_q(\lambda)$ with the highest weight $\lambda \in \Lambda$ is given by $\mathrm{ch}\Delta_q(\lambda) = \frac{1}{D}e^{\mu+\rho}$ where $D = \prod_{\beta\in\Phi_{\bar{1}}^+}(e^{\beta/2} - e^{-\beta/2}) / \prod_{\alpha\in\Phi_{\bar{0}}^+}(e^{\alpha/2} - e^{-\alpha/2})$ owing to [27, Theorem 1] and Theorem 3.2.

Since the character of a module is equal to the sum of the characters of its composition factors, we have

$$\mathrm{ch}\Delta_q(\lambda) = \sum_\mu b_{\lambda\mu}\mathrm{ch}L_q(\mu)$$

where $b_{\lambda\mu} \in \mathbb{Z}_+$ and $b_{\lambda\lambda} = 1$. Since $\Delta_q(\lambda)$ is a highest weight module, $b_{\lambda\mu} \neq 0 \Rightarrow \lambda - \mu \in \sum_i \mathbb{Z}_+\alpha_i$ and also $\chi_\lambda = \chi_\mu$. Hence, we have

$$\mathrm{ch}L_q(\lambda) = \sum_\mu a_{\lambda\mu}\mathrm{ch}\Delta_q(\mu) \quad \text{and} \quad D\mathrm{ch}L_q(\lambda) = \sum_\mu a_{\lambda\mu}e^{\mu+\rho}$$

where $a_{\lambda\mu} \in \mathbb{Z}$ with $a_{\lambda\lambda} = 1$, and $a_{\lambda\mu} = 0$ unless $\lambda - \mu \in \sum_i \mathbb{Z}_+\alpha$ and $\chi_\lambda = \chi_\mu$.

Assume for now that $L(\lambda)$ is finite-dimensional. Then $L_q(\lambda)$ is a semisimple $\mathfrak{g}_{\bar{0}}$-module, and $\mathrm{ch}L_q(\lambda)$ is W-invariant as a result. On the other hand, $w(D) = (-1)^{l(w)}D$ for all $w \in W$, and hence $D\mathrm{ch}L_q(\lambda)$ can be written as

$$\sum_{\mu\in X} a_{\lambda\mu} \sum_{w\in W} (-1)^{l(w)} e^{w(\mu+\rho)},$$

where X consists of $\Phi_{\bar{0}}^+$-dominant integral weights such that $a_{\lambda\mu} \neq 0$. Moreover, $a_{\lambda,w(\lambda+\rho)-\rho} = (-1)^{l(w)}a_{\lambda\lambda} = (-1)^{l(w)}$. Hence, we have $\chi_\lambda = \chi_{w(\lambda+\rho)-\rho}$ for all $w \in W, \lambda \in \Lambda_{f.d.}$, where $\Lambda_{f.d.} = \{\lambda \in \Lambda | \dim L_q(\lambda) < \infty\}$.

For $z \in \mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$, we set $h = \mathcal{HC}(z)$. Assuming that $\lambda \in \Lambda$ and $L_q(\lambda)$ is finite-dimensional, we get $(\lambda + \rho)(h) = \chi_\lambda(z) = \chi_{w(\lambda+\rho)-\rho}(z) = (w(\lambda + \rho))(h) = (\lambda + \rho)(wh)$. Hence $\lambda(wh - h) = 0$ for all $w \in W$. Fix w and write $wh - h = \sum_\mu a_\mu\mathbb{K}_\mu$. Then $\lambda\big(\sum_\mu a_\mu\mathbb{K}_\mu\big) = \sum_\mu a_\mu q^{(\lambda,\mu)} = 0$ for all $\lambda \in \Lambda_{f.d.}$. Thus, $wh - h = 0$ and $h \in (\mathrm{U}^0)^W$ because the bilinear form on $\Lambda_{f.d.} \times \mathbb{Z}\Phi$ is non-degenerate in the second component. $\square$

Set

$$(\mathrm{U}^0_{\mathrm{ev}})^W = \left\{ \sum_\mu a_\mu\mathbb{K}_\mu \middle| \mu \in 2\Lambda \cap \mathbb{Z}\Phi \text{ and } a_\mu = a_{w\mu},\ \forall w \in W \right\}. \tag{5.1}$$

Lemma 5.3. *The Harish-Chandra homomorphism $\mathcal{HC}$ maps $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ to $(\mathrm{U}^0_{\mathrm{ev}})^W$.*

Proof. Take an arbitrary $z \in \mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$, we can write $\mathcal{HC}(z) = \sum_\mu a_\mu \mathbb{K}_\mu$ with $a_{w\mu} = a_\mu$ for any $w \in W$. We only need to prove $\langle \mu, \alpha\rangle \in 2\mathbb{Z}$ for all $\mu \in \mathbb{Z}\Phi$ with $a_\mu \neq 0, \alpha \in \Phi_{\bar{0}}$.

For each group homomorphism $\sigma : \mathbb{Z}\Phi \to \{\pm 1\}$, we can define an automorphism $\tilde{\sigma}$ of $\mathrm{U}_q(\mathfrak{g})$ by

$$\tilde{\sigma}(\mathbb{K}_\mu) = \sigma(\mu)\mathbb{K}_\mu, \quad \tilde{\sigma}(\mathbb{E}_i) = \mathbb{E}_i, \quad \tilde{\sigma}(\mathbb{F}_i) = \sigma(\alpha_i)\mathbb{F}_i.$$

Obviously, $\tilde{\sigma}$ maps the center $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ to itself. One can check that $\mathcal{HC} = \gamma_{-\rho} \circ \pi$ commutes with $\tilde{\sigma}$. We already have $\mathcal{HC}(\tilde{\sigma}(z)) = \tilde{\sigma}\big(\sum_\mu a_\mu \mathbb{K}_\mu\big) = \sum_\mu a_\mu \sigma(\mu)\mathbb{K}_\mu$. Since $\tilde{\sigma}(z)$ is central, the sum is in $(\mathrm{U}^0)^W$; so we have $a_\mu\sigma(\mu) = a_{w\mu}\sigma(w\mu) = a_\mu\sigma(w\mu)$ for all $w \in W$. This means: if $a_\mu \neq 0$, then $\sigma(\mu) = \sigma(w\mu)$ for all $w \in W$. Thus, $\sigma(\mu - s_\alpha\mu) = 1$ for all $\alpha \in \Phi^+_{\bar{0}}$, $\mu \in \mathbb{Z}\Phi$. For each α, we can choose σ such that $\sigma(\alpha) = -1$. Therefore, $(-1)^{\langle \mu,\alpha\rangle} = 1$ and $\langle \mu, \alpha\rangle \in 2\mathbb{Z}$. □

For $\nu \in \Lambda$ and $\alpha \in \Phi_{\mathrm{iso}}$, we set $A^\alpha_\nu = \{\nu + n\alpha | n \in \mathbb{Z}\}$. Clearly, $\Lambda = \bigcup_{\nu\in\Lambda} A^\alpha_\nu$. Let

$$(\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}} = \left\{ \sum_\mu a_\mu \mathbb{K}_\mu \in (\mathrm{U}^0_{\mathrm{ev}})^W \;\middle|\; \sum_{\mu\in A^\alpha_\nu} a_\mu = 0,\ \forall \alpha \in \Phi_{\mathrm{iso}} \text{ with } (\nu, \alpha) \neq 0 \right\}. \tag{5.2}$$

Lemma 5.4. *The Harish-Chandra homomorphism $\mathcal{HC}$ maps $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ to $(\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$.*

Proof. We claim that if $\alpha \in \Phi_{\mathrm{iso}}$ and $(\lambda + \rho, \alpha) = 0$, then $\chi_\lambda = \chi_{\lambda-k\alpha}$ for any $k \in \mathbb{Z}$. Indeed, if $\alpha = \alpha_s$ and $(\lambda, \alpha_s) = 0$, then we get a non-trivial homomorphism $\varphi\colon \Delta_q(\lambda-\alpha_s) \to \Delta_q(\lambda)$ according to Lemma 3.1. In this way, $z \in \mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ acts by the same constant on both modules; i.e., $\chi_\lambda(z) = (\lambda+\rho)(h) = (\lambda - \alpha_s + \rho)(h) = \chi_{\lambda-\alpha_s}(z)$ where $h = \mathcal{HC}(z) = \gamma_{-\rho} \circ \pi(z)$. Thus, $\chi_\lambda = \chi_{\lambda-\alpha_s}$.

For any $\alpha \in \Phi_{\mathrm{iso}}$, if $(\lambda + \rho, \alpha) = 0$, then there exists $w \in W$ such that $w(\alpha) = \alpha_s$. Based on the W-invariance of $(\cdot, \cdot)$, we have $\big(w(\lambda + \rho), w(\alpha)\big) = (\lambda + \rho, \alpha) = 0$, so

$$\chi_\lambda = \chi_{w(\lambda+\rho)-\rho} = \chi_{w(\lambda+\rho)-w(\alpha)-\rho} = \chi_{\lambda-\alpha}.$$

This implies $\chi_\lambda = \chi_{\lambda-\alpha}$, so we conclude that $\chi_\lambda = \chi_{\lambda-k\alpha}$ for all $k \in \mathbb{Z}$.

Now suppose $h = \gamma_{-\rho} \circ \pi(z) = \sum_\mu a_\mu \mathbb{K}_\mu$ for some $z \in \mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ and $\alpha \in \Phi_{\mathrm{iso}}$, by $\chi_\lambda(z) = (\lambda + \rho)\Big(\sum_\mu a_\mu \mathbb{K}_\mu\Big)$ and $\chi_\lambda = \chi_{\lambda-\alpha}$ for all $(\lambda + \rho, \alpha) = 0$. We know

$$(\lambda + \rho + \alpha)\Big(\sum_\mu a_\mu \mathbb{K}_\mu\Big) = (\lambda + \rho)\Big(\sum_\mu a_\mu \mathbb{K}_\mu\Big), \tag{5.3}$$

for all λ such that $(\lambda + \rho, \alpha) = 0$, hence

$$\sum_\mu a_\mu q^{(\lambda+\rho,\mu)} \Big(q^{(\mu,\alpha)} - 1\Big) = 0. \tag{5.4}$$

Notice that $(\lambda+\rho,\nu)=(\lambda+\rho,\nu')$ and $(\nu,\alpha)=(\nu',\alpha)$ if $A^\alpha_\nu=A^\alpha_{\nu'}$. For any $h=\sum_\mu a_\mu \mathbb{K}_\mu \in (\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$, we set $\mathrm{Supp}(h)=\{\mu\in 2\Lambda\cap\mathbb{Z}\Phi \mid a_\mu\neq 0\}$. Suppose the elements of $\mathrm{Supp}(h)$ are listed as

$$\begin{array}{llll} \mu_1, & \mu_1+n_{1,1}\alpha, & \cdots, & \mu_1+n_{1,q_1}\alpha, \\ \mu_2, & \mu_2+n_{2,1}\alpha, & \cdots, & \mu_2+n_{2,q_2}\alpha, \\ \cdots & & & \\ \mu_p, & \mu_p+n_{p,1}\alpha, & \cdots, & \mu_p+n_{p,q_p}\alpha, \end{array}$$

where $A^\alpha_{\mu_i}\neq A^\alpha_{\mu_j}$ if $i\neq j$, and $q_i\geqslant 0$ and $0<n_{i,1}<n_{i,2}<\cdots<n_{i,q_i}$ for each i. Hence, $A^\alpha_{\mu_i}\cap\mathrm{Supp}(h)=\{\mu_i,\mu_i+n_{i,1}\alpha,\cdots,\mu_i+n_{i,q_i}\alpha\}$. Let $X=\{\mu_1,\mu_2,\cdots,\mu_p\}$, we can rewrite Eq. (5.4) as

$$\sum_{\nu\in X}\left(\sum_{\mu\in A^\alpha_\nu} a_\mu\right)\left(q^{(\nu,\alpha)}-1\right)q^{(\lambda+\rho,\nu)}=0 \quad \text{for all } \lambda \text{ such that } (\lambda+\rho,\alpha)=0.$$

Let $\Lambda_\nu=\{\mu\in\Lambda|(\mu,\nu)=0\}$ for all $\nu\in\Lambda$. The bilinear form on Λ induces a bilinear map on $\Lambda/\mathbb{Z}\alpha\times\Lambda_\alpha$ which is non-degenerate in both arguments. Set $Y=\{\mu_i-\mu_j|1\leqslant i<j\leqslant p\}$, hence $\Lambda_\alpha-\Lambda_\nu\neq\varnothing$ for all $\nu\in Y$ and $\Lambda_\alpha-\bigcup_{\nu\in Y}\Lambda_\nu\neq\varnothing$ by induction. Take $\lambda+\rho\in\Lambda_\alpha-\bigcup_{\nu\in Y}\Lambda_\nu$, this means $(\lambda+\rho,\alpha)=0$ and $(\lambda+\rho,\nu)\neq(\lambda+\rho,\nu')$ for all $\nu\neq\nu'$ with $\nu,\nu'\in X$. We get

$$\sum_{i=1}^p\left(\sum_{\mu\in A^\alpha_{\mu_i}} a_\mu\right)\left(q^{(\mu_i,\alpha)}-1\right)q^{(j(\lambda+\rho),\mu_i)}=0,$$

for all $j=1,2,\cdots,p$. Moreover, the Vandermonde matrix $\left(q^{(j(\lambda+\rho),\mu_i)}\right)_{p\times p}$ is invertible since $(\lambda+\rho,\mu_i)\neq(\lambda+\rho,\mu_j)$ for all $1\leqslant i\neq j\leqslant p$. Therefore,

$$\left(\sum_{\mu\in A^\alpha_{\mu_i}} a_\mu\right)\left(q^{(\mu_i,\alpha)}-1\right)=0, \tag{5.5}$$

for all i, and $\sum_{\mu\in A^\alpha_{\mu_i}} a_\mu=0$ if $(\mu_i,\alpha)\neq 0$. The proof is completed. □

Example 5.5. We give some explicit elements in $(\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$ when $\mathfrak{g}$ is of small rank.

(i) Let $\mathfrak{g}=A(1,0)$. In such a case, $\Phi^+_{\bar 1}=\{\alpha_2,\alpha_1+\alpha_2\}$ and $2\Lambda\cap\mathbb{Z}\Phi=\mathbb{Z}\alpha_1+2\mathbb{Z}\alpha_2$. If $\lambda=k_1\alpha_1+2k_2\alpha_2$ is a $\Phi^+_{\bar 0}$-dominant weight, then we have $k_1\geqslant k_2$ and $k_1,k_2\in\mathbb{Z}$. Furthermore, $W\lambda=\{\lambda,\lambda-2(k_1-k_2)\alpha_1\}$. Thus $k_\lambda=\mathbb{K}_\lambda-\mathbb{K}_{\lambda-2\alpha_2}-\mathbb{K}_{\lambda-2\alpha_1-2\alpha_2}+\mathbb{K}_{\lambda-2\alpha_1-4\alpha_2}+\mathbb{K}_{\lambda-2(k_1-k_2)\alpha_1}-\mathbb{K}_{\lambda-2(k_1-k_2)\alpha_1-2\alpha_2}-\mathbb{K}_{\lambda-2(k_1-k_2)\alpha_1-2\alpha_1-2\alpha_2}+\mathbb{K}_{\lambda-2(k_1-k_2)\alpha_1-2\alpha_1-4\alpha_2}\in(\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$.

(ii) Let $\mathfrak{g}=C(2)$. As a result, $\Phi^+_{\bar 1}=\{\alpha_1,\alpha_1+\alpha_2\}$ and $2\Lambda\cap\mathbb{Z}\Phi=2\mathbb{Z}\alpha_1+\mathbb{Z}\alpha_2$. If $\lambda=2k_1\alpha_1+k_2\alpha_2$ is a $\Phi^+_{\bar 0}$-dominant weight, then we have $k_2\geqslant k_1$ and $k_1,k_2\in\mathbb{Z}$. Furthermore, $W\lambda=\{\lambda,\lambda-2(k_2-k_1)\alpha_2\}$. Thus $k_\lambda=\mathbb{K}_\lambda-\mathbb{K}_{\lambda-2\alpha_1}-\mathbb{K}_{\lambda-2\alpha_1-2\alpha_2}+\mathbb{K}_{\lambda-4\alpha_1-2\alpha_2}+\mathbb{K}_{\lambda-2(k_2-k_1)\alpha_2}-\mathbb{K}_{\lambda-2(k_2-k_1)\alpha_2-2\alpha_1}-\mathbb{K}_{\lambda-2(k_2-k_1)\alpha_2-2\alpha_1-2\alpha_2}+\mathbb{K}_{\lambda-2(k_2-k_1)\alpha_2-4\alpha_1-2\alpha_2}\in(\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$.

(iii) Let $\mathfrak{g} = B(1, 1)$. In this case, the positive isotropic roots of $\mathfrak{g}$ are $\{\alpha_1, \alpha_1 + 2\alpha_2\}$ and $2\Lambda \cap \mathbb{Z}\Phi = 2\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$. If $\lambda = \lambda_1\delta_1 + \mu_1\varepsilon_1 \in 2\Lambda \cap Q$ is a $\Phi_{\bar{0}}^{+}$-dominant weight, then we have $\lambda_1 \neq 0$, $\lambda_1 - 2, 2\mu_1 \in 2\mathbb{Z}_+$. Furthermore, $W\lambda = \{\pm\lambda_1\delta_1 \pm \mu_1\varepsilon_1\}$. Thus $k_\lambda = \sum_{w\in W} w(\mathbb{K}_\lambda - \mathbb{K}_{\lambda-2\alpha_1} - \mathbb{K}_{\lambda-2\alpha_1-4\alpha_2} + \mathbb{K}_{\lambda-4\alpha_1-4\alpha_2}) \in (\mathrm{U}_{\mathrm{ev}}^0)_{\mathrm{sup}}^W$.

5.3. Proof of theorem A. In order to prove the surjectivity of $\mathcal{HC}$, we need to investigate the Grothendieck rings $K(\mathfrak{g})$ of finite-dimensional representations of the basic classical Lie superalgebras $\mathfrak{g}$. In the following proposition, we identify the algebra $(\mathrm{U}_{\mathrm{ev}}^0)_{\mathrm{sup}}^W$ with $k \otimes_{\mathbb{Z}} J_{\mathrm{ev}}(\mathfrak{g})$, which plays a crucial role on the surjectivity of $\mathcal{HC}$.

Proposition 5.6. $(\mathrm{U}_{\mathrm{ev}}^0)_{\mathrm{sup}}^W = k \otimes_{\mathbb{Z}} J_{\mathrm{ev}}(\mathfrak{g})$.

Proof. For any $\alpha \in \Phi_{\mathrm{iso}}$, let elements of $\mathrm{Supp}(h)$ and X be same as proof of Lemma 5.4. Furthermore, $n_{i,j}$ are even numbers for all possible i, j since there is an even root β such that $\frac{2(\alpha,\beta)}{(\beta,\beta)} = 1$. Then

$$D_\alpha(h) = \sum_{\mu} a_\mu(\mu, \alpha)\mathbb{K}_\mu = \sum_{\nu\in X}\sum_{k\in\mathbb{Z}_+} a_{\nu+2k\alpha}(\nu, \alpha)\mathbb{K}_{\nu+2k\alpha} \tag{5.6}$$

and

$$\sum_{k\in\mathbb{Z}_+} a_{\nu+2k\alpha}(\nu, \alpha)\mathbb{K}_{\nu+2k\alpha} \in (\mathbb{K}_\alpha^2 - 1), \quad \text{for all } \nu \in X$$

because $\sum_{k\in\mathbb{Z}_+} a_{\nu+k\alpha} = 0$ for all $\nu \in X$ with $(\nu, \alpha) \neq 0$ and $\sum_{k\in\mathbb{Z}_+} a_{\nu+2k\alpha}(\nu, \alpha)\mathbb{K}_{\nu+2k\alpha} = 0$ for all $\nu \in X$ with $(\nu, \alpha) = 0$.

On the other hand, take an element $h = \sum_{\mu} a_\mu\mathbb{K}_\mu \in k \otimes_{\mathbb{Z}} J_{\mathrm{ev}}(\mathfrak{g})$, then

$$D_\alpha(h) = \sum_{\mu} a_\mu(\mu, \alpha)\mathbb{K}_\mu = \sum_{\nu\in X}\sum_{k\in\mathbb{Z}_+} a_{\nu+k\alpha}(\nu, \alpha)\mathbb{K}_{\nu+k\alpha} \in (\mathbb{K}_\alpha^2 - 1),$$

for any $\alpha \in \Phi_{\mathrm{iso}}$. Therefore, $\sum_{k\in\mathbb{Z}_+} a_{\nu+k\alpha}\mathbb{K}_{\nu+k\alpha} \in (\mathbb{K}_\alpha^2 - 1)$ for any $\nu \in X$ if $(\nu, \alpha) \neq 0$. This implies that $\sum_{\mu\in A_\nu^\alpha} a_\mu = \sum_{k\in\mathbb{Z}_+} a_{\nu+k\alpha} = 0$ if $(\nu, \alpha) \neq 0$. □

Proposition 5.7. *There is a linear map $\Psi_{\mathcal{R}}: k \otimes_{\mathbb{Z}} K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g})) \to \mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ such that the diagram in the introduction commutes.*

Proof. Define a map $\Psi_{\mathcal{R}}: k \otimes_{\mathbb{Z}} K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g})) \to \mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ by $\Psi_{\mathcal{R}}([M]) = z_M$ where z_M is defined in Lemma 4.11. We need to prove the map is well-defined and $\iota\circ\mathcal{HC}(z_M) = \mathrm{Sch}([M])$ for all M in U-**mod** with all weights contained in $\Lambda \cap \frac{1}{2}\mathbb{Z}\Phi$.

For every short exact sequences $0 \to L \to M \to N \to 0$ in U-**mod**, choose a homogeneous basis $\{m_1, \cdots, m_k, \cdots, m_l\}$ of M such that $\{m_1, \cdots, m_k\}$ is a basis of L and $\{\bar{m}_{k+1}, \cdots, \bar{m}_l\}$ is a basis of N. Let $\{f_1, \cdots, f_l\}$ be the dual basis of M, then $\{f_1, \cdots, f_k\}$ and $\{\bar{f}_{k+1}, \cdots, \bar{f}_l\}$ can be viewed as dual bases of L and N, respectively. Recall $\Pi(M)$ and π defined in Sect. 3.2, so $\{\pi\otimes m_1, \cdots, \pi\otimes m_l\}$ (resp. $\{\pi\otimes f_1, \cdots, \pi\otimes$

$f_l\}$) is the basis (resp. dual bases) of $\Pi(M)$, and $|\pi\otimes f_i| = |\pi\otimes m_i| = -|m_i| = -|f_i|$ for all i . Hence,

$$\begin{aligned}
\langle u, z_M\rangle &= \sum_{i=1}^{l}(-1)^{|m_i|} f_i(u\mathbb{K}_{2\rho}^{-1}m_i)\\
&= \sum_{i=1}^{k}(-1)^{|m_i|} f_i(u\mathbb{K}_{2\rho}^{-1}m_i) + \sum_{i=k+1}^{l}(-1)^{|m_i|} f_i(u\mathbb{K}_{2\rho}^{-1}m_i)\\
&= \sum_{i=1}^{k}(-1)^{|m_i|} f_i(u\mathbb{K}_{2\rho}^{-1}m_i) + \sum_{i=k+1}^{l}(-1)^{|\bar{m}_i|} \bar{f}_i(u\mathbb{K}_{2\rho}^{-1}\bar{m}_i)\\
&= \langle u, z_L\rangle + \langle u, z_N\rangle = \langle u, z_L + z_N\rangle;\\
\langle u, z_M\rangle &= \sum_{i=1}^{l}(-1)^{|m_i|} f_i(u\mathbb{K}_{2\rho}^{-1}m_i) = -\sum_{i=1}^{l}(-1)^{|\pi\otimes m_i|}(\pi\otimes f_i)\big(u\mathbb{K}_{2\rho}^{-1}(\pi\otimes m_i)\big)\\
&= -\langle u, z_{\Pi(M)}\rangle.
\end{aligned}$$

Therefore, $z_L - z_M + z_N = 0$ and $z_M + z_{\Pi(M)} = 0$ according to Proposition 4.8.

Since z_M is central, we have $z_M = \sum_{\mu\geqslant 0} z_{M,\mu}$ where $z_{M,\mu} \in \mathrm{U}^-_{-\mu}\mathrm{U}^0\mathrm{U}^+_{\mu}$. Write $z_{M,0} = \sum_{\nu} a_\nu \mathbb{K}_\nu$. Then we have

$$\langle \mathbb{K}_{\mu'}, z_M\rangle = \langle \mathbb{K}_{\mu'}, z_{M,0}\rangle = \sum_{\nu} a_\nu \left(q^{1/2}\right)^{-(\nu,\mu')},$$

for all $\mu' \in \mathbb{Z}\Phi$. On the other hand, this is the supertrace of $\mathbb{K}_{\mu'-2\rho}$ acting on M. This means it is equal to

$$\sum_{\lambda'} \mathrm{sdim} M_{\lambda'} q^{(\lambda',\mu'-2\rho)} = \sum_{\lambda'} \mathrm{sdim} M_{\lambda'} q^{-2(\lambda',\rho)} \left(q^{1/2}\right)^{(2\lambda',\mu')}.$$

A comparison of these two formulas shows that

$$z_{M,0} = \sum_{\lambda'} \mathrm{sdim} M_{\lambda'} q^{(-2\lambda',\rho)}\mathbb{K}_{-2\lambda'}.$$

We have $z_{M,0} = \pi(z_M)$, hence

$$\gamma_{-\rho}\circ\pi(z_M) = \sum_{\lambda'} \mathrm{sdim} M_{\lambda'}\mathbb{K}_{-2\lambda'}, \tag{5.7}$$

and $\iota\circ\mathcal{HC}(z_M) = \sum_{\lambda'} \mathrm{sdim} M_{\lambda'} e^{\lambda'} = \mathrm{Sch}([M])$. □

Proof of Theorem A.

$$\begin{tikzcd}
k\otimes_{\mathbb{Z}} K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g})) \arrow[d, dashed, "\Psi_{\mathcal{R}}"'] & & k\otimes_{\mathbb{Z}} K_{\mathrm{ev}}(\mathfrak{g}) \arrow[ll, hook'] \arrow[d, dashed, "\cong"'] \\
\mathcal{Z}(\mathrm{U}_q(\mathfrak{g})) \arrow[r, dashed, "\mathcal{HC}"] & (\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}} \arrow[r, equal] & k\otimes_{\mathbb{Z}} J_{\mathrm{ev}}(\mathfrak{g})
\end{tikzcd}$$

The injectivity of $\mathcal{HC}$ follows from 5.1, so we only need to prove $\mathrm{Im}\mathcal{HC} = (\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$. Based on Proposition 5.7, the above diagram is commutative, so $\mathrm{Im}\mathcal{HC} = (\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$. $\square$

By using $\iota \circ \mathcal{HC} \circ \Psi_{\mathcal{R}}([M]) = \mathrm{Sch}([M])$ for all $[M] \in K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g}))$, we get $\Psi_{\mathcal{R}}$ is injective. All morphisms in the diagram above are algebra isomorphisms as a result. Furthermore, for any $[M] \in K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g}))$, there exists $\sum_i a_i[L(\lambda_i)]$ with $a_i \in k$ such that $\jmath(\sum_i a_i[L(\lambda_i)]) = [M]$, and these λ_i are distinct. Let $X = \{\lambda_i | a_i \notin \mathbb{Z}\}$. Supposing that X is nonempty and taking a maximal element λ_t in X for some t, we get $\dim M_{\lambda_t} = \sum_i a_i \dim L(\lambda_i)_{\lambda_t} \in \mathbb{Z}$ and $\dim L(\lambda_i)_{\lambda_t} = \delta_{it}$. Thus $a_t = \dim M_{\lambda_t}$ is an integer, contradicting $\lambda_t \in X$. Therefore, X is empty and $a_i \in \mathbb{Z}$ for all i. Thus, $K_{\mathrm{ev}}(\mathfrak{g}) \hookrightarrow K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g}))$ is an isomorphism induced by $\jmath$.

Remark 5.8. In Appendix B, we describe the $J_{\mathrm{ev}}(\mathfrak{g})$ in the sense of Sergeev and Veselov [42] and illustrate why $K_{\mathrm{ev}}(\mathfrak{g}) \ncong J_{\mathrm{ev}}(\mathfrak{g})$ if $\mathfrak{g} = A(1,1)$ since $u - v = \mathbb{K}_1 + \mathbb{K}_1^{-1} - \mathbb{K}_3 - \mathbb{K}_3^{-1} \in J_{\mathrm{ev}}(\mathfrak{g})$ and $u - v \notin J(A(1,1))$. Therefore, $k \otimes_{\mathbb{Z}} J(A(1,1)) \subseteq \mathrm{Im}(\mathcal{HC}) \subseteq k \otimes_{\mathbb{Z}} J_{\mathrm{ev}}(\mathfrak{g})$. However, the image of $\mathcal{HC}$ for $\mathfrak{g} = A(1,1)$ has not yet determined.

6. Center of Quantum Superalgebras

6.1. Quasi-R-matrix. In Sect. 5, we established the $\mathcal{HC}$ for quantum superalgebras and proved that the center $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ is isomorphic to $(\mathrm{U}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$, the subalgebra of the ring of exponential super-invariants $J_{\mathrm{ev}}(\mathfrak{g})$. This section studies the structural theorem for the center. Our approach to obtaining a structural theorem for quantum superalgebras takes advantage of the quasi-R-matrix, which is inspired by [49,50]. Recently, based on main results [33], Dai and Zhang [10] used a similar method to investigate explicit generators and relations for the center of the quantum group. They proved that the center $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ of quantum group $\mathrm{U}_q(\mathfrak{g})$ is isomorphic to the subring of Grothendieck algebra $K(\mathrm{U}_q(\mathfrak{g}))$.

For each $\mu \in Q$, we take $u_1^\mu, u_2^\mu, \cdots, u_{r(\mu)}^\mu$ to be a basis of U^+_μ. Since the skew-pairing between the U^+ and U^- is non-degenerate, we can take the dual basis $v_1^\mu, v_2^\mu, \cdots, v_{r(\mu)}^\mu$ of $\mathrm{U}^-_{-\mu}$, with respect to $(v_i^\mu, u_j^\mu) = \delta_{ij}$, for all possible i, j. We have the following proposition.

Proposition 6.1. *Set* $\Theta_\mu = \sum_{i=1}^{r(\mu)} v_i^\mu \otimes u_i^\mu \in \mathrm{U} \otimes \mathrm{U}$. *Then* Θ_μ *does not depend on the choice of the basis* $(u_i^\mu)_i$ *and*

$$(\mathbb{E}_i \otimes 1)\Theta_\mu + (\mathbb{K}_i \otimes \mathbb{E}_i)\Theta_{\mu-\alpha_i} = \Theta_\mu(\mathbb{E}_i \otimes 1) + \Theta_{\mu-\alpha_i}(\mathbb{K}_i^{-1} \otimes \mathbb{E}_i), \tag{6.1}$$

$$(1 \otimes \mathbb{F}_i)\Theta_\mu + (\mathbb{F}_i \otimes \mathbb{K}_i^{-1})\Theta_{\mu-\alpha_i} = \Theta_\mu(1 \otimes \mathbb{F}_i) + \Theta_{\mu-\alpha_i}(\mathbb{F}_i \otimes \mathbb{K}_i), \tag{6.2}$$

$$(\mathbb{K}_i \otimes \mathbb{K}_i)\Theta_\mu = \Theta_\mu(\mathbb{K}_i \otimes \mathbb{K}_i). \tag{6.3}$$

Proof. It is easy to check Θ_μ does not depend on the choice of the basis $(u_i^\mu)_i$ and (6.3). For (6.1), we have

$$(\mathbb{E}_i \otimes 1)\Theta_\mu - \Theta_\mu(\mathbb{E}_i \otimes 1)$$

$$
\begin{aligned}
&= \sum_{j=1}^{r(\mu)} [\mathbb{E}_i, v_j^{\mu}] \otimes u_j^{\mu} \\
&= \sum_{j=1}^{r(\mu)} (q_i - q_i^{-1})^{-1} \big((-1)^{|\mathbb{E}_i||r_i(v_j^{\mu})|} \mathbb{K}_i r_i(v_j^{\mu}) - r_i'(v_j^{\mu}) \mathbb{K}_i^{-1}\big) \otimes u_j^{\mu} \\
&= \sum_{j=1}^{r(\mu)} \sum_{k=1}^{r(\mu-\alpha_i)} (q_i - q_i^{-1})^{-1} \Big((-1)^{|\mathbb{E}_i||r_i(v_j^{\mu})|} \mathbb{K}_i \big(r_i(v_j^{\mu}), u_k^{\mu-\alpha_i}\big) v_k^{\mu-\alpha_i} \\
&\quad - \big(r_i'(v_j^{\mu}), u_k^{\mu-\alpha_i}\big) v_k^{\mu-\alpha_i} \mathbb{K}_i^{-1}\Big) \otimes u_j^{\mu} \\
&= \sum_{j=1}^{r(\mu)} \sum_{k=1}^{r(\mu-\alpha_i)} \Big(- (-1)^{|\mathbb{E}_i||r_i(v_j^{\mu})|} \mathbb{K}_i (\mathbb{F}_i, \mathbb{E}_i) \big(r_i(v_j^{\mu}), u_k^{\mu-\alpha_i}\big) v_k^{\mu-\alpha_i} \\
&\quad + (\mathbb{F}_i, \mathbb{E}_i) \big(r_i'(v_j^{\mu}), u_k^{\mu-\alpha_i}\big) v_k^{\mu-\alpha_i} \mathbb{K}_i^{-1}\Big) \otimes u_j^{\mu} \\
&= \sum_{j=1}^{r(\mu)} \sum_{k=1}^{r(\mu-\alpha_i)} \big(- (-1)^{|\mathbb{E}_i||r_i(v_j^{\mu})|} \mathbb{K}_i (v_j^{\mu}, \mathbb{E}_i u_k^{\mu-\alpha_i}) v_k^{\mu-\alpha_i} \\
&\quad + (v_j^{\mu}, u_k^{\mu-\alpha_i} \mathbb{E}_i) v_k^{\mu-\alpha_i} \mathbb{K}_i^{-1}\big) \otimes u_j^{\mu} \\
&= \sum_{k=1}^{r(\mu-\alpha_i)} -(-1)^{|\mathbb{E}_i||r_i(v_j^{\mu})|} \mathbb{K}_i v_k^{\mu-\alpha_i} \otimes \mathbb{E}_i u_k^{\mu-\alpha_i} + v_k^{\mu-\alpha_i} \mathbb{K}_i^{-1} \otimes u_k^{\mu-\alpha_i} \mathbb{E}_i \\
&= -(\mathbb{K}_i \otimes \mathbb{E}_i) \Theta_{\mu-\alpha_i} + \Theta_{\mu-\alpha_i} (\mathbb{K}_i^{-1} \otimes \mathbb{E}_i).
\end{aligned}
$$

Thus, (6.1) holds. Because the proof for Eq. (6.2) is similar to that for Eq. (6.1), we omit it here. □

There is an algebra automorphism ϕ of $\mathrm{U}_q(\mathfrak{g}) \otimes \mathrm{U}_q(\mathfrak{g})$ defined by

$$
\begin{aligned}
&\phi(\mathbb{K}_i \otimes 1) = \mathbb{K}_i \otimes 1, \quad \phi(\mathbb{E}_i \otimes 1) = \mathbb{E}_i \otimes \mathbb{K}_i^{-1}, \quad \phi(\mathbb{F}_i \otimes 1) = \mathbb{F}_i \otimes \mathbb{K}_i, \\
&\phi(1 \otimes \mathbb{K}_i) = 1 \otimes \mathbb{K}_i, \quad \phi(1 \otimes \mathbb{E}_i) = \mathbb{K}_i^{-1} \otimes \mathbb{E}_i \quad \phi(1 \otimes \mathbb{F}_i) = \mathbb{K}_i \otimes \mathbb{F}_i,
\end{aligned}
$$

and ϕ can be extended to $\mathrm{U}_q(\mathfrak{g}) \widehat{\otimes} \mathrm{U}_q(\mathfrak{g})$, which is a completion of the tensor product $\mathrm{U}_q(\mathfrak{g}) \otimes \mathrm{U}_q(\mathfrak{g})$. Then the quasi-R-matrix is $\sum_{\mu \geqslant 0} \Theta_\mu \in \mathrm{U}_q(\mathfrak{g}) \widehat{\otimes} \mathrm{U}_q(\mathfrak{g})$[3] and it is invertible. Its inverse is denoted by $\mathfrak{R}$. Then, by Proposition 6.1, we have

$$
\mathfrak{R} \Delta(u) = \phi\big(\Delta^{op}(u)\big) \mathfrak{R}, \text{ and } \ \mathfrak{R}^{op} \Delta^{op}(u) = \phi\big(\Delta(u)\big) \mathfrak{R}^{op}.
$$

The universal R-matrix can be derived from the quasi-R-matrix, which is significant because it can induce solutions of the quantum Yang-Baxter equation on any of its modules. This approach is prominent in the study of integrable systems, knot invariants and so on. The following proposition is essential for us to construct the explicit central elements, named Casimir invariants, which have been used to construct a family of Casimir invariants for quantum groups [10], quantum superalgebras $\mathrm{U}_q(\mathfrak{gl}_{m|n})$ and $\mathrm{U}_q(\mathfrak{osp}_{m|2n})$.

[3] More properties about quasi-R-matrix in a super setting can be deduction follows [34, Chap. 4]. For example, $\bar{\mathfrak{R}} = \mathfrak{R}^{-1}$, where the automorphism $^-$ of $\mathrm{U} \widehat{\otimes} \mathrm{U}$ is defined in [34, Chap. 4].

6.2. Constructing central elements using quasi-R-matrix.

Proposition 6.2 [48, Proposition 2]. *Given an operator $\Gamma_M \in \mathrm{End}(M) \otimes \mathrm{U}_q(\mathfrak{g})$ satisfying*

$$[\Gamma_M, \Delta(u)] = 0 \quad \textit{for all } u \in \mathrm{U}_q(\mathfrak{g}), \tag{6.4}$$

the elements

$$C_M^{(k)} := \mathrm{Str}_1\big((\zeta \otimes 1)(\mathbb{K}_{2\rho} \otimes 1)(\Gamma_M)^k\big) \tag{6.5}$$

are central in $\mathrm{U}_q(\mathfrak{g})$*, where* $\mathrm{Str}_1(f \otimes u) = \mathrm{Str}(f)u$ *for* $f \in \mathrm{End}(M)$ *and* $u \in \mathrm{U}_q(\mathfrak{g})$.

Proof. We only need to prove $[C_M^{(k)}, \mathbb{K}_i] = [C_M^{(k)}, \mathbb{E}_i] = [C_M^{(k)}, \mathbb{F}_i] = 0$ for all $i \in \mathbb{I}$. Assume $(\Gamma_M)^k = \sum_j A_j \otimes B_j$, then

$$\begin{aligned}
0 &= \mathrm{Str}_1\big((\mathbb{K}_{2\rho}\mathbb{K}_i^{-1} \otimes 1)[(\Gamma_M)^k, \Delta(\mathbb{K}_i)]\big)\\
&= \mathrm{Str}_1\Big((\mathbb{K}_{2\rho}\mathbb{K}_i^{-1} \otimes 1)\Big[\sum_j A_j \otimes B_j, \mathbb{K}_i \otimes \mathbb{K}_i\Big]\Big)\\
&= \sum_j \mathrm{Str}(\mathbb{K}_{2\rho}\mathbb{K}_i^{-1} A_j \mathbb{K}_i) B_j \mathbb{K}_i - \sum_j \mathrm{Str}(\mathbb{K}_{2\rho} A_j) \mathbb{K}_i B_j\\
&= [C_M^{(k)}, \mathbb{K}_i],
\end{aligned}$$

where the last equation holds by $\mathrm{Str}([x, y]) = 0$ for all $x, y \in \mathrm{End}(M)$. And,

$$\begin{aligned}
0 &= \mathrm{Str}_1\big((\mathbb{K}_{2\rho} \otimes 1)[(\Gamma_M)^k, \Delta(\mathbb{F}_i)]\big)\\
&= \mathrm{Str}_1\Big((\mathbb{K}_{2\rho} \otimes 1)\Big[\sum_j A_j \otimes B_j, \mathbb{F}_i \otimes \mathbb{K}_i^{-1} + 1 \otimes \mathbb{F}_i\Big]\Big)\\
&= \mathrm{Str}_1\Big((\mathbb{K}_{2\rho} \otimes 1)\sum_j \big((-1)^{|B_j||\mathbb{F}_i|} A_j\mathbb{F}_i \otimes B_j\mathbb{K}_i^{-1} + A_j \otimes B_j\mathbb{F}_i\\
&\quad - (-1)^{|\mathbb{F}_i|(|A_j|+|B_j|)}\mathbb{F}_i A_j \otimes \mathbb{K}_i^{-1} B_j - (-1)^{|\mathbb{F}_i||B_j|} A_j \otimes \mathbb{F}_i B_j\big)\Big)\\
&= [C_M^{(k)}, \mathbb{F}_i],
\end{aligned}$$

where the last equation follows from $\Big[\sum_j A_j \otimes B_j, \mathbb{K}_i \otimes \mathbb{K}_i\Big] = 0$ and $\mathrm{Str}([x, y]) = 0$ for all $x, y \in \mathrm{End}(M)$. □

Let M denote a finite-dimensional weight module of $\mathrm{U}_q(\mathfrak{g})$ and let ζ denote the representation afforded by M. Let $P_\eta^M : M \to M_\eta$ be the projection from M to M_η and define the following element in $\mathrm{End}(M) \otimes \mathrm{U}_q(\mathfrak{g})$ as

$$\mathcal{K}_M = \sum_{\eta \in \mathrm{wt}(M)} P_\eta^M \otimes \mathbb{K}_{2\eta}. \tag{6.6}$$

Using the definition of ϕ, we obtain

$$\mathcal{K}_M(\zeta \otimes 1)\big(\phi^2(\Delta(u))\big) = (\zeta \otimes 1)(\Delta(u))\mathcal{K}_M, \quad \forall u \in \mathrm{U}_q(\mathfrak{g}). \tag{6.7}$$

Define $R_M = (\zeta \otimes 1)(\mathfrak{R})$ and $R_M^{op} = (\zeta \otimes 1)(\mathfrak{R}^{op})$, we have

$$\begin{aligned}\mathcal{K}_M \phi(R_M^{op}) R_M (\zeta \otimes 1)(\Delta(u)) &= \mathcal{K}_M (\zeta \otimes 1)\big(\phi(\mathfrak{R}^{op})\mathfrak{R}\Delta(u)\big)\\ &= \mathcal{K}_M (\zeta \otimes 1)\big(\phi^2(\Delta(u))\phi(\mathfrak{R}^{op})\mathfrak{R}\big)\\ &= \mathcal{K}_M (\zeta \otimes 1)\big(\phi^2(\Delta(u))\big)\phi(R_M^{op})R_M\\ &= (\zeta \otimes 1)(\Delta(u))\mathcal{K}_M \phi(R_M^{op})R_M, \qquad \forall u \in \mathrm{U}_q(\mathfrak{g}).\end{aligned}$$

If we take

$$\Gamma_M = \mathcal{K}_M \phi(R_M^{op}) R_M, \tag{6.8}$$

then $[\Gamma_M, (\zeta \otimes 1)(\Delta(u))] = 0$, for all $u \in \mathrm{U}_q(\mathfrak{g})$.

Example 6.3. This example was known in [48,53]. Let $\mathrm{U} = \mathrm{U}_q(A(1,0))$ and $\zeta : \mathrm{U} \to \mathrm{End}(M) = \mathrm{End}(L_q(\varepsilon_1))$ be the vector representation. Let v_1 be its highest weight vector with weight λ_1, and let $v_2 = \mathbb{F}_1 v_1$, $v_3 = \mathbb{F}_2\mathbb{F}_1 v_1$ and λ_2, λ_3 be the corresponding weights associated with v_2, v_3, respectively. $\{v_1, v_2, v_3\}$ is a basis of M. By using of (4.1) and (4.3), $\{-(q_i - q_i)^{-1}\mathbb{F}_i\}$ and $\{\mathbb{E}_i\}$ are two basis-dual basis pairs of $\mathrm{U}^-_{-\alpha_i}$ and $\mathrm{U}^+_{\alpha_i}$ for $i = 1, 2$ and

$$\{(q - q^{-1})\mathbb{F}_1\mathbb{F}_2, (q^{-1} - q)\mathbb{F}_2\mathbb{F}_1\} \text{ and } \{q\mathbb{E}_1\mathbb{E}_2 - \mathbb{E}_2\mathbb{E}_1, \mathbb{E}_1\mathbb{E}_2 - q\mathbb{E}_2\mathbb{E}_1\}$$

is a basis-dual basis pair of $\mathrm{U}^-_{-\alpha_1-\alpha_2}$ and $\mathrm{U}^+_{\alpha_1+\alpha_2}$ with respect to the Drinfeld double. We have $\mathfrak{R} = \overline{\sum_{\mu \geqslant 0} \Theta_\mu}$, which is a generalization of [34, Corollary 4.1.3]. Then

$$\begin{aligned}R_M = (\zeta \otimes 1)(1 \otimes 1 + \sum_{i=1}^{2}(q_i - q_i^{-1})\mathbb{F}_i \otimes \mathbb{E}_i - (q^{-1} - q)\mathbb{F}_2\mathbb{F}_1\\ \otimes(\mathbb{E}_1\mathbb{E}_2 - q^{-1}\mathbb{E}_2\mathbb{E}_1) - (q - q^{-1})\mathbb{F}_1\mathbb{F}_2 \otimes (q^{-1}\mathbb{E}_1\mathbb{E}_2 - \mathbb{E}_2\mathbb{E}_1))\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}\phi(R_M^{\mathrm{op}}) = (\zeta \otimes 1)(1 \otimes 1 + (q^{-1} - q)(\mathbb{E}_1\mathbb{E}_2 - q^{-1}\mathbb{E}_2\mathbb{E}_1)\mathbb{K}_2\mathbb{K}_1 \otimes \mathbb{K}_2^{-1}\mathbb{K}_1^{-1}\mathbb{F}_2\mathbb{F}_1\\ + \sum_{i=1}^{2}(-1)^{\delta_{i2}}(q_i - q_i^{-1})\mathbb{E}_i\mathbb{K}_i \otimes \mathbb{K}_i^{-1}\mathbb{F}_i + (q - q^{-1})\\ (q^{-1}\mathbb{E}_1\mathbb{E}_2 - \mathbb{E}_2\mathbb{E}_1)\mathbb{K}_1\mathbb{K}_2 \otimes \mathbb{K}_2^{-1}\mathbb{K}_1^{-1}\mathbb{F}_1\mathbb{F}_2).\end{aligned} \tag{6.10}$$

because $\zeta(\mathrm{U}^-_{-\nu}) = 0$ if $\nu \neq \alpha_1, \alpha_2, \alpha_1 + \alpha_2$. Substitute (6.6), (6.9) and (6.10) into (6.8) and (6.5). As a result,

$$\begin{aligned}C_M^{(1)} &= \mathrm{Str}_1\big((\zeta \otimes 1)(\mathbb{K}_{2\rho} \otimes 1)\mathcal{K}_M \phi(R_M^{\mathrm{op}})R_M\big)\\ &= \sum_{i=1}^{3}(-1)^{|v_i|}q^{(2\rho,\lambda_i)}\mathbb{K}_{2\lambda_i} + \sum_{i=1}^{2}(q_i - q_i^{-1})^2(-1)^{|v_i|}q^{(\alpha_i,\lambda_{i+1})+(2\rho,\lambda_i)}\mathbb{K}_{2\lambda_i}\mathbb{K}_i^{-1}\mathbb{F}_i\mathbb{E}_i\\ &\quad + (q - q^{-1})^2 q^{(2\rho,\lambda_1)+(\alpha_1+\alpha_2,\lambda_3)}\mathbb{K}_{2\lambda_1}\mathbb{K}_2^{-1}\mathbb{K}_1^{-1}(\mathbb{F}_2\mathbb{F}_1 - q^{-1}\mathbb{F}_1\mathbb{F}_2)(\mathbb{E}_1\mathbb{E}_2 - q^{-1}\mathbb{E}_2\mathbb{E}_1)\\ &= \mathbb{K}_2^{-2} + q^{-2}\mathbb{K}_1^{-2}\mathbb{K}_2^{-2} - q^{-2}\mathbb{K}_1^{-2}\mathbb{K}_2^{-4} + (q - q^{-1})^2\\ &\quad (q^{-1}\mathbb{K}_1^{-1}\mathbb{K}_2^{-2}\mathbb{F}_1\mathbb{E}_1 + q^{-1}\mathbb{K}_1^{-2}\mathbb{K}_2^{-3}\mathbb{F}_2\mathbb{E}_2)\end{aligned}$$

$$+(q-q^{-1})^2 q\mathbb{K}_1^{-1}\mathbb{K}_2^{-3}(\mathbb{F}_2\mathbb{F}_1-q^{-1}\mathbb{F}_1\mathbb{F}_2)(\mathbb{E}_1\mathbb{E}_2-q^{-1}\mathbb{E}_2\mathbb{E}_1),$$

by using

$$\begin{aligned}
2\rho &= \alpha_1-\alpha_2-(\alpha_1+\alpha_2) = -2\alpha_2;\\
\lambda_1 &= \varepsilon_1 = -\varepsilon_2+\delta_1 = -\alpha_2;\\
\lambda_2 &= \varepsilon_2 = -\varepsilon_1+\delta_1 = -\alpha_1-\alpha_2;\\
\lambda_3 &= \delta_1 = -\varepsilon_1-\varepsilon_2+2\delta_1 = -\alpha_1-2\alpha_2.
\end{aligned}$$

There is a k-algebra anti-automorphism τ of U defined by $\tau(\mathbb{E}_i)=\mathbb{F}_i$, $\tau(\mathbb{F}_i)=\mathbb{E}_i$, $\tau(\mathbb{K}_i^{\pm1})=\mathbb{K}_i^{\pm1}$ for $i=1,2$. It is obvious that $C_M^{(1)}$ commutes with $\mathbb{K}_1$ and $\mathbb{K}_2$. One can check directly that $C_M^{(1)}$ commutes with $\mathbb{E}_1$ and $\mathbb{E}_2$. Because $C_M^{(1)}$ is τ-invariant, $C_M^{(1)}$ commutes with $\mathbb{F}_1$ and $\mathbb{F}_2$. Therefore, $C_M^{(1)}\in\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$.

6.3. Proof of theorem B. In the previous subsection, we used the quasi-R-matrix to construct an explicit Γ_M associated with a finite-dimensional $\mathrm{U}_q(\mathfrak{g})$-module M satisfying Proposition 6.8. Thus, we obtained a family of central elements of $\mathrm{U}_q(\mathfrak{g})$. Now, we are ready to prove Theorem B. For convenience, we simplify $C_{L_q(\lambda)}$ for $C_{L_q(\lambda)}^{(1)}$.

Theorem 6.4. $\{C_{L_q(\lambda)}\mid\lambda\in\Lambda\cap\frac{1}{2}\mathbb{Z}\Phi$ *and* $L(\lambda)$ *finite-dimensional* $\}$ *is a basis of* $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$ *if* $\mathfrak{g}\neq A(1,1)$.

Proof. Applying the $\mathcal{HC}$ to $C_{L_q(\lambda)^*}$ results in

$$\begin{aligned}
\mathcal{HC}\left(C_{L_q(\lambda)^*}\right) &= \mathcal{HC}\Big(\mathrm{Str}_1\Big(\big(\zeta(\mathbb{K}_{2\rho})\otimes 1\big)\Gamma_{L_q(\lambda)^*}\Big)\Big)\\
&= \gamma_{-\rho}\circ\pi\Big(\mathrm{Str}_1\Big(\big(\zeta(\mathbb{K}_{2\rho})\otimes 1\big)\mathcal{K}_{L_q(\lambda)^*}\Big)\Big)\\
&= \sum_{\eta\in\mathrm{wt}(L_q(\lambda)^*)}\gamma_{-\rho}\left(\mathrm{Str}(q^{(2\rho,\eta)}P_\eta^{L_q(\lambda)^*})\mathbb{K}_{2\eta}\right)\\
&= \sum_{\mu}\mathrm{sdim}L_q(\lambda)_\mu\mathbb{K}_{-2\mu} = \mathcal{HC}\left(z_{L_q(\lambda)}\right).
\end{aligned}$$

According to Theorem A (i.e., the $\mathcal{HC}=\gamma_{-\rho}\circ\pi$ is an algebra isomorphism), $z_{L_q(\lambda)}=C_{L_q(\lambda)^*}$. Furthermore, $\left\{[L_q(\lambda)]\,\middle|\,\lambda\in\Lambda\cap\frac{1}{2}\mathbb{Z}\Phi \text{ and } L_q(\lambda) \text{ is finite-dimensional}\right\}$ is a basis of $K_{\mathrm{ev}}(\mathrm{U}_q(\mathfrak{g}))$. Hence, $\left\{C_{L_q(\lambda)^*}\,\middle|\,\lambda\in\Lambda\cap\frac{1}{2}\mathbb{Z}\Phi \text{ and } L_q(\lambda) \text{ is finite-dimensional}\right\}$ is a basis of $\mathcal{Z}(\mathrm{U}_q(\mathfrak{g}))$. So is $\left\{C_{L_q(\lambda)}\,\middle|\,\lambda\in\Lambda\cap\frac{1}{2}\mathbb{Z}\Phi \text{ and } L(\lambda) \text{ is finite-dimensional}\right\}$. □

Remark 6.5. One can define a new quantum superalgebra $\tilde{\mathrm{U}}=\tilde{\mathrm{U}}_q(\mathfrak{g})$ associated with a simple Lie superalgebra $\mathfrak{g}$, except for $A(1,1)$, by replacing the cartan subalgebra of quantum superalgebra $\mathrm{U}_q(\mathfrak{g})$ with the group ring $k\Gamma$ if $\mathbb{Z}\Phi\subseteq\Gamma\subseteq\Lambda$, $W\Gamma=\Gamma$ and $q^{(\gamma,\lambda)}\in k$ for all $\gamma\in\Gamma,\lambda\in\Lambda$. Using the same procedure, we can establish the Harish-Chandra isomorphism between $\mathcal{Z}(\tilde{\mathrm{U}})$ and $(\tilde{\mathrm{U}}^0_{\mathrm{ev}})^W_{\mathrm{sup}}$, where

$$(\tilde{\mathrm{U}}^0_{\mathrm{ev}})^W_{\mathrm{sup}} = \left\{\sum_{\mu\in 2\Lambda\cap\Gamma} a_\mu\mathbb{K}_\mu\in\mathrm{U}^0\,\middle|\, a_{w\mu}=a_\mu,\ \forall w\in W;\right.$$

$$\sum_{\mu\in A_\nu^\alpha} a_\mu = 0,\ \forall\alpha\in\Phi_{\mathrm{iso}} \text{ with } (\nu,\alpha)\neq 0\Big\}.$$

In particular, $K(\mathfrak{g}) \cong K_\Lambda(\tilde{\mathrm{U}})$, where $K_\Lambda(\tilde{\mathrm{U}})$ is the subring of $K(\tilde{\mathrm{U}})$ generated by all objects in $\tilde{\mathrm{U}}$-**mod** whose weights are contained in Λ if $\Gamma = \Lambda$.

Remark 6.6. Our approach to obtaining the Harish-Chandra type theorem for quantum superalgebras of type A-G takes advantage of the Rosso form, which cannot be applied to quantum queer superalgebra $\mathrm{U}_q(\mathfrak{q}_n)$ [37] or quantum perplectic superalgebra $\mathrm{U}_q(\mathfrak{p}_n)$ [1]. One immediate problem is to establish the Harish-Chandra type theorems for these quantum superalgebras. We hope to return to these questions in future.

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Appendix A. Dynkin Diagrams in Distinguished Root Systems

The Dynkin diagrams in the distinguished root systems of a simple basic Lie superalgebra of type A-G are listed below, where r is the number of nodes and s is the element of τ. Note that the form of Dynkin diagrams in the distinguished root systems is quite uniform in the literature.

$\boxed{A(m,n)\ \textbf{case}:}$ Let $\mathfrak{h}^*$ be a vector space spanned by $\{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_{m+1} - \delta_1, \delta_j - \delta_{j+1} | 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ satisfies

$$(\varepsilon_1 + \ldots + \varepsilon_{m+1}) - (\delta_1 + \ldots + \delta_{n+1}) = 0.$$

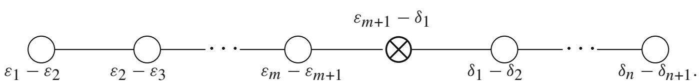
We equip the dual $\mathfrak{h}^*$ with a bilinear form $(\cdot,\cdot)$ such that

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}, \quad (\varepsilon_i, \delta_j) = (\delta_j, \varepsilon_i) = 0, \quad (\delta_i, \delta_j) = -\delta_{ij} \quad \text{for all possible } i, j.$$

The distinguished fundamental system $\Pi = \{\alpha_1, \ \ldots, \ \alpha_{m+n+1}\}$ is given by

$$\{\varepsilon_1 - \varepsilon_2, \ \ldots, \ \varepsilon_m, -\varepsilon_{m+1}, \ \varepsilon_{m+1} - \delta_1, \ \delta_1 - \delta_2, \ \ldots, \ \delta_n - \delta_{n+1}\}.$$

The Dynkin diagram associated with Π is depicted as follows:



In this case $r = m+n+1$, $s = m+1$. The distinguished positive system $\Phi^+ = \Phi^+_{\bar 0} \cup \Phi^+_{\bar 1}$ corresponding to the distinguished Borel subalgebra for $A(m,n)$ is

$$\{\varepsilon_i - \varepsilon_j, \delta_k - \delta_l | 1 \leqslant i < j \leqslant m+1, 1 \leqslant k < l \leqslant n+1\}$$
$$\cup \{\varepsilon_i - \delta_j | 1 \leqslant i \leqslant m+1, 1 \leqslant j \leqslant n+1\}.$$

The Weyl group $W \cong \mathfrak{S}_{m+1} \times \mathfrak{S}_{n+1}$.

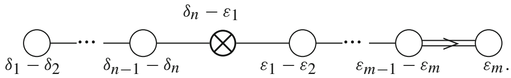
$\boxed{B(m,n)\ \textbf{case:}}$ Let $\mathfrak{h}^*$ be a vector space with basis $\{\varepsilon_i, \delta_j | 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$. We equip the dual $\mathfrak{h}^*$ with a bilinear form $(\cdot,\cdot)$ such that

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}, \quad (\varepsilon_i, \delta_j) = (\delta_j, \varepsilon_i) = 0, (\delta_i, \delta_j) = -\delta_{ij} \quad \text{for all possible } i, j.$$

The distinguished fundamental system $\Pi = \{\alpha_1, \ \ldots, \ \alpha_{m+n}\}$ is given by

$$\{\delta_1 - \delta_2, \ \ldots, \ \delta_{n-1} - \delta_n, \ \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \ \ldots, \ \varepsilon_{m-1} - \varepsilon_m, \ \varepsilon_m\}.$$

The Dynkin diagram associated with Π is depicted as follows:



In this case $r = m+n$, $s = n+1$. The distinguished positive system $\Phi^+ = \Phi^+_{\bar 0} \cup \Phi^+_{\bar 1}$ corresponding to the distinguished Borel subalgebra is

$$\{\delta_i \pm \delta_j, \ 2\delta_p, \ \varepsilon_k \pm \varepsilon_l, \ \varepsilon_q\} \cup \{\delta_p \pm \varepsilon_q, \ \delta_p\},$$

where $1 \leqslant i < j \leqslant n, 1 \leqslant k < l \leqslant m, 1 \leqslant p \leqslant n, 1 \leqslant q \leqslant m$. The Weyl group $W \cong (\mathfrak{S}_n \ltimes \mathbb{Z}_2^n) \times (\mathfrak{S}_m \ltimes \mathbb{Z}_2^m)$.

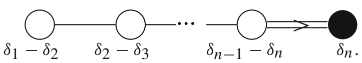
$\boxed{B(0,n)\ \textbf{case:}}$ Let $\mathfrak{h}^*$ be a vector space with basis $\{\delta_i | 1 \leqslant i \leqslant n\}$. We equip the dual $\mathfrak{h}^*$ with a bilinear form $(\cdot,\cdot)$ such that

$$(\delta_i, \delta_j) = -\delta_{ij} \quad \text{for all possible } i, j.$$

The distinguished fundamental system $\Pi = \{\alpha_1, \ \ldots, \ \alpha_n\}$ is given by

$$\{\delta_1 - \delta_2, \ \ldots, \ \delta_{n-1} - \delta_n, \ \delta_n\}.$$

The Dynkin diagram associated with Π is depicted as follows:



In this case, $r = s = n$. The distinguished positive system $\Phi^+ = \Phi^+_{\bar 0} \cup \Phi^+_{\bar 1}$ corresponding to the distinguished Borel subalgebra is

$$\{\delta_i \pm \delta_j, \ 2\delta_p | 1 \leqslant i < j \leqslant n, 1 \leqslant p \leqslant n\} \cup \{\delta_p | 1 \leqslant p \leqslant n\}.$$

The Weyl group $W \cong (\mathfrak{S}_n \ltimes \mathbb{Z}_2^n)$.

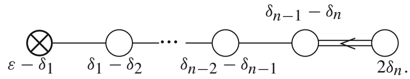
$\boxed{C(n+1)\ \textbf{case:}}$ Let $\mathfrak{h}^*$ be a vector space with basis $\{\varepsilon, \delta_i | 1 \leqslant i \leqslant n\}$. We equip the dual $\mathfrak{h}^*$ with a bilinear form $(\cdot, \cdot)$ such that

$$(\varepsilon, \varepsilon) = 1, \quad (\varepsilon, \delta_i) = (\delta_i, \varepsilon) = 0, \quad (\delta_i, \delta_j) = -\delta_{ij} \quad \text{for all possible } i, j.$$

The distinguished fundamental system $\Pi = \{\alpha_1, \ldots, \alpha_{n+1}\}$ is given by

$$\{\varepsilon - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, 2\delta_n\}.$$

The Dynkin diagram associated with Π is depicted as follows:



In this case $r = n + 1, s = 1$. The distinguished positive system $\Phi^+ = \Phi^+_{\bar{0}} \cup \Phi^+_{\bar{1}}$ corresponding to the distinguished Borel subalgebra is

$$\{\delta_i \pm \delta_j, 2\delta_p | 1 \leqslant i < j \leqslant n, 1 \leqslant p \leqslant n\} \cup \{\varepsilon \pm \delta_p | 1 \leqslant p \leqslant n\}.$$

The Weyl group $W \cong (\mathfrak{S}_n \ltimes \mathbb{Z}_2^n)$.

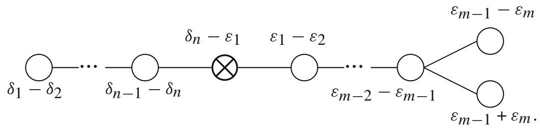
$\boxed{D(m,n)\ \textbf{case:}}$ Let $\mathfrak{h}^*$ be a vector space with basis $\{\varepsilon_i, \delta_j | 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$. We equip the dual $\mathfrak{h}^*$ with a bilinear form $(\cdot, \cdot)$ such that

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}, \quad (\varepsilon_i, \delta_j) = (\delta_j, \varepsilon_i) = 0, \quad (\delta_i, \delta_j) = -\delta_{ij} \quad \text{for all possible } i, j.$$

The distinguished fundamental system $\Pi = \{\alpha_1, \ldots, \alpha_{m+n}\}$ is given by

$$\{\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_{m-1} + \varepsilon_m\}.$$

The Dynkin diagram associated with Π is depicted as follows:



In this case $r = m + n, s = n + 1$. The distinguished positive system $\Phi^+ = \Phi^+_{\bar{0}} \cap \Phi^+_{\bar{1}}$ corresponding to the distinguished Borel subalgebra is

$$\{\delta_i \pm \delta_j, 2\delta_p, \varepsilon_k \pm \varepsilon_l, \} \cup \{\delta_p \pm \varepsilon_q\},$$

where $1 \leqslant i < j \leqslant n, 1 \leqslant k < l \leqslant m, 1 \leqslant p \leqslant n, 1 \leqslant q \leqslant m$. The Weyl group $W \cong (\mathfrak{S}_n \ltimes \mathbb{Z}_2^n) \times (\mathfrak{S}_m \ltimes \mathbb{Z}_2^{m-1})$.

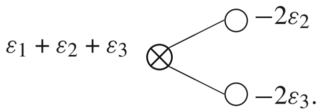
$\boxed{D(2,1;\alpha)\ \textbf{case}:}$ Let $\mathfrak{h}^*$ be a vector space with basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. We equip the dual $\mathfrak{h}^*$ with a bilinear form $(\cdot, \cdot)$ with

$$(\varepsilon_1, \varepsilon_1) = -(1+\alpha), \quad (\varepsilon_2, \varepsilon_2) = 1, \quad (\varepsilon_3, \varepsilon_3) = \alpha \quad \text{and}$$
$$(\varepsilon_i, \varepsilon_j) = 0 \quad \text{for all} \quad 1 \leqslant i \neq j \leqslant 3.$$

The distinguished fundamental system

$$\Pi = \{\alpha_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \alpha_2 = -2\varepsilon_2, \alpha_3 = -2\varepsilon_3\}.$$

The Dynkin diagram associated with Π is depicted as follows:



In this case $r = 3$, $s = 1$. The distinguished positive system $\Phi^+ = \Phi^+_{\bar 0} \cap \Phi^+_{\bar 1}$ corresponding to the distinguished Borel subalgebra is

$$\Phi^+_{\bar 0} = \{2\varepsilon_1, -2\varepsilon_2, -2\varepsilon_3\}, \quad \Phi^+_{\bar 1} = \{\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}.$$

The Weyl group $W \cong \mathbb{Z}_2^3$.

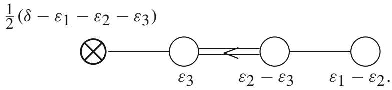
$\boxed{F(4)\ \textbf{case}:}$ Let $\mathfrak{h}^*$ be a vector space with basis $\{\delta, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$.We equip the dual $\mathfrak{h}^*$ with a bilinear form $(\cdot,\cdot)$ such that

$$(\delta,\delta) = -3, \quad (\varepsilon_i,\delta) = (\delta,\varepsilon_i) = 0, \quad (\varepsilon_i,\varepsilon_j) = \delta_{ij} \quad \text{for all } i.$$

The distinguished fundamental system

$$\Pi = \left\{\alpha_1 = \frac{1}{2}(\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \quad \alpha_2 = \varepsilon_3, \quad \alpha_3 = \varepsilon_2 - \varepsilon_3, \quad \alpha_4 = \varepsilon_1 - \varepsilon_2\right\}.$$

The Dynkin diagram associated with Π is depicted as follows:



In this case $r = 4$, $s = 1$. The distinguished positive system $\Phi^+ = \Phi^+_{\bar 0} \cap \Phi^+_{\bar 1}$ corresponding to the distinguished Borel subalgebra is

$$\{\delta,\ \varepsilon_p,\ \varepsilon_i \pm \varepsilon_j | 1 \leqslant i < j \leqslant 3, 1 \leqslant p \leqslant 3\} \cup \left\{\frac{1}{2}(\delta \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3)\right\},$$

The Weyl group $W = \mathbb{Z}_2 \times (\mathfrak{S}_3 \ltimes \mathbb{Z}_2^3)$.

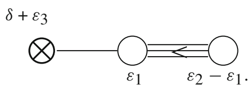
$\boxed{G(3)\ \textbf{case}:}$ Let $\mathfrak{h}^*$ be a vector space with basis $\{\delta, \varepsilon_1, \varepsilon_2\}$ and $\varepsilon_3 = -\varepsilon_1 - \varepsilon_2$. We equip the dual $\mathfrak{h}^*$ with a bilinear form $(\cdot,\cdot)$ such that

$$(\delta,\delta) = -(\varepsilon_i,\varepsilon_i) = -2, \quad (\varepsilon_i,\delta) = (\delta,\varepsilon_i) = 0, \quad (\varepsilon_i,\varepsilon_j) = -1, \quad \text{for all } 1 \leqslant i \neq j \leqslant 3.$$

The distinguished fundamental system

$$\Pi = \{\alpha_1 = \delta + \varepsilon_3, \alpha_2 = \varepsilon_1, \alpha_3 = \varepsilon_2 - \varepsilon_1\}.$$

The Dynkin diagram associated with Π is depicted as follows:



In this case $r = 3$, $s = 1$. The distinguished positive system $\Phi^+ = \Phi^+_{\bar 0} \cap \Phi^+_{\bar 1}$ corresponding to the distinguished Borel subalgebra is

$$\{2\delta,\ \varepsilon_1,\ \varepsilon_2,\ \varepsilon_2 \pm \varepsilon_1,\ \varepsilon_1 - \varepsilon_3,\ \varepsilon_2 - \varepsilon_3\} \cup \{\delta,\ \delta \pm \varepsilon_i | i = 1, 2, 3\},$$

The Weyl group $W = \mathbb{Z}_2 \times D_6$, where D_6 is the dihedral group of order 12.

Appendix B. Explicit Description of the Rings $J_{\mathrm{ev}}(\mathfrak{g})$

Now we give the explicit description of the rings $J_{\mathrm{ev}}(\mathfrak{g})$ for quantum superalgebras, which is inspired by Sergeev and Veselov's description for Lie superalgebras [42, Sects. 7, 8]. Let $x_i = \mathbb{K}_{-\varepsilon_i/2}$ and $y_j = \mathbb{K}_{-\delta_j/2}$ formally. First we need to review the rings $J(\mathfrak{g})$ for $\mathfrak{g}$ is of type A. Let

$$P_0 = \left\{ \sum_{i=1}^{m+1} a_i \varepsilon_i + \sum_{j=1}^{n+1} b_j \delta_j \,\middle|\, a_i, b_j \in \mathbb{C} \text{ and } a_i - a_{i+1}, b_j - b_{j+1} \in \mathbb{Z},\ \forall i, j \right\} \Big/ \mathbb{C}\gamma$$

be the weights of $\mathfrak{sl}_{m+1|n+1}$, where $\gamma = \varepsilon_1 + \cdots + \varepsilon_{m+1} - \delta_1 - \cdots - \delta_{n+1}$ and $x_i = e^{\varepsilon_i}$, $y_j = e^{\delta_j}$ for all possible i, j be the elements of the group ring of $\mathbb{C}[P_0]$. For convenience, we set $\mathbb{C}[x^{\pm}, y^{\pm}] = \mathbb{C}[x_1^{\pm 1}, \cdots, x_{m+1}^{\pm 1}, y_1^{\pm 1}, \cdots, y_{n+1}^{\pm 1}]$, $\mathbb{Z}[x^{\pm}, y^{\pm}] = \mathbb{Z}[x_1^{\pm 1}, \cdots, x_{m+1}^{\pm 1}, y_1^{\pm 1}, \cdots, y_{n+1}^{\pm 1}]$ and then for $(m, n) \neq (1, 1)$

$$\begin{aligned} J(\mathfrak{sl}_{m+1|n+1}) &= \left\{ f \in \mathbb{Z}[P_0]^W \,\middle|\, y_j \frac{\partial f}{\partial y_j} + x_i \frac{\partial f}{\partial x_i} \in (x_i - y_j) \right\} \\ &= \bigoplus_{a \in \mathbb{C}/\mathbb{Z}} J(\mathfrak{sl}_{m+1|n+1})_a, \end{aligned}$$

where

$$J(\mathfrak{sl}_{m+1|n+1})_a = (x_1 \cdots x_{m+1})^a \prod_{i,p} \left(1 - \frac{x_i}{y_p}\right) \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]_0^{\mathfrak{S}_{m+1} \times \mathfrak{S}_{n+1}}$$

if $a \notin \mathbb{Z}$;

$$J(\mathfrak{sl}_{m+1|n+1})_0 = \left\{ f \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]_0^{\mathfrak{S}_{m+1} \times \mathfrak{S}_{n+1}} \,\middle|\, x_i \frac{\partial f}{\partial x_i} + y_j \frac{\partial f}{\partial y_j} \in (x_i - y_j) \right\}$$

and $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]_0^{\mathfrak{S}_{m+1} \times \mathfrak{S}_{n+1}}$ is the quotient of the ring $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]^{\mathfrak{S}_{m+1} \times \mathfrak{S}_{n+1}}$ by the ideal generated by $x_1 \cdots x_{m+1} - y_1 \cdots y_{n+1}$.

$J(A(n, n)) = \bigoplus_{i=0}^{n} J(A(n, n))_i$ for $n \neq 1$, where for $i \neq 0$

$$J(A(n, n))_i = \left\{ f = (x_1 \cdots x_{n+1})^{\frac{i}{n+1}} \prod_{j,p} \left(1 - \frac{x_j}{y_p}\right) g \,\middle|\, g \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]_0^{\mathfrak{S}_{n+1} \times \mathfrak{S}_{n+1}},\ \deg g = -i \right\}$$

and $J(A(n, n))_0$ is the subring of $J(\mathfrak{sl}_{n+1|n+1})_0$ consisting of elements of degree 0.

$J(A(1, 1)) = \{c + (u - v)^2 g | c \in \mathbb{Z}, g \in \mathbb{Z}[u, v]\}$ where $u = \left(\frac{x_1}{x_2}\right)^{\frac{1}{2}} + \left(\frac{x_2}{x_1}\right)^{\frac{1}{2}}$, $v = \left(\frac{y_1}{y_2}\right)^{\frac{1}{2}} + \left(\frac{y_2}{y_1}\right)^{\frac{1}{2}}$.

$\boxed{A(m, n), m \neq n \textbf{ case:}}$ Define

$$J^{m|n} = \left\{ f \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]^{\mathfrak{S}_{m+1} \times \mathfrak{S}_{n+1}} \,\middle|\, x_i \frac{\partial f}{\partial x_i} + y_j \frac{\partial f}{\partial y_j} \in (x_i - y_j) \right\}$$

and

$$J_k^{m|n} = \left\{ f \in J^{m|n} \middle| \deg f = k \right\}.$$

Thus, $J^{m|n} = \bigoplus_{k\in\mathbb{Z}} J_k^{m|n}$.

For any element $\lambda \in \mathfrak{h}^*$, we write $\lambda = \sum_{i=1}^{m+1} a_i\varepsilon_i + \sum_{j=1}^{n+1} b_j\delta_j$, then we have

$$\mathbb{Z}\Phi = \left\{ \lambda \in \mathfrak{h}^* \middle| a_i, b_j \in \mathbb{Z},\ \forall i, j \text{ and } \sum_{i=1}^{m+1} a_i + \sum_{j=1}^{n+1} b_j = 0 \right\},$$

and

$$\Lambda = \left\{ \lambda \in \mathfrak{h}^* \middle| a_i, b_j \in \mathbb{Q},\ a_i - a_{i+1}, b_j - b_{j+1} \in \mathbb{Z},\ \forall i \leqslant m, j \leqslant n \right. \\ \left. \text{and } \sum_{i=1}^{m+1} a_i + \sum_{j=1}^{n+1} b_j = 0 \right\}.$$

By direct computation, we know that

$$2\Lambda \cap \mathbb{Z}\Phi = \begin{cases} 2\mathbb{Z}\Phi + \mathbb{Z}\left(\sum_{i=1}^{m+1} (-1)^{i+1}\varepsilon_i + \sum_{j=1}^{n+1} (-1)^j \delta_j \right), & \text{if } m = 2k, n = 2l, \\ 2\mathbb{Z}\Phi + \mathbb{Z} \sum_{j=1}^{n+1} (-1)^j \delta_j, & \text{if } m = 2k, n = 2l+1, \\ 2\mathbb{Z}\Phi + \mathbb{Z} \sum_{i=1}^{m+1} (-1)^{i+1} \varepsilon_i, & \text{if } m = 2k+1, n = 2l, \\ 2\mathbb{Z}\Phi + \mathbb{Z} \sum_{i=1}^{m+1} (-1)^{i+1} \varepsilon_i + \mathbb{Z} \sum_{j=1}^{n+1} (-1)^j \delta_j, & \text{if } m = 2k+1, n = 2l+1, \end{cases}$$

for some non-negative integers k, l. Then the algebra

$$J_{\mathrm{ev}}(\mathfrak{g}) = \begin{cases} J_0^{m|n} \oplus \prod_i x_i^{\frac{1}{2}} \prod_j y_j^{\frac{1}{2}} J_{-(k+l+1)}^{m|n}, & \text{if } m = 2k, n = 2l, \\ J_0^{m|n} \oplus \prod_j y_j^{\frac{1}{2}} J_{-(l+1)}^{m|n}, & \text{if } m = 2k, n = 2l+1, \\ J_0^{m|n} \oplus \prod_i x_i^{\frac{1}{2}} J_{-(k+1)}^{m|n}, & \text{if } m = 2k+1, n = 2l, \\ J_0^{m|n} \oplus \prod_i x_i^{\frac{1}{2}} J_{-(k+1)}^{m|n} \oplus \prod_j y_j^{\frac{1}{2}} J_{-(l+1)}^{m|n} \oplus \prod_i x_i^{\frac{1}{2}} \prod_j y_j^{\frac{1}{2}} J_{-(k+l+2)}^{m|n}, & \text{if } m = 2k+1, n = 2l+1. \end{cases}$$

for some non-negative integers k, l. So it can be viewed as a subalgebra of $J(\mathfrak{g})$ by $\iota\colon J_{\mathrm{ev}}(\mathfrak{g}) \to J(\mathfrak{g})$ with $\mathbb{K}_i \mapsto e^{-\alpha_i/2}$ and its image is coincide with $\mathrm{Sch}(K_{\mathrm{ev}}(\mathfrak{g}))$.

$\boxed{A(n,n)\ \ (n \neq 1)\ \textbf{case:}}$ In this case, we set

$$J(n)_0 = \left\{ f \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]_{0,0}^{\mathfrak{S}_{n+1}\times\mathfrak{S}_{n+1}} \middle| x_i \frac{\partial f}{\partial x_i} + y_j \frac{\partial f}{\partial y_j} \in (x_i - y_j) \right\}$$

where $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]_{0,0}$ is the quotient of the ring $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ with degree 0 by the ideal $I = \left\langle \frac{x_1 \cdots x_{n+1}}{y_1 \cdots y_{n+1}} - 1 \right\rangle$. Then we have

$$J_{\mathrm{ev}}(\mathfrak{g}) = \begin{cases} J(n)_0 & \text{if } n \text{ is even,} \\ J(n)_0 \oplus \left\{ \overrightarrow{x}^{\frac{1}{2}} \prod\limits_{j,p} \left(1 - \frac{x_j}{y_p}\right) g + I \,\middle|\, g \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]^W,\ \deg g = -\frac{n+1}{2} \right\} & \text{if } n \text{ is odd,} \end{cases}$$

where $\overrightarrow{x} = x_1 x_2 \cdots x_{n+1}$ and $W = \mathfrak{S}_{n+1} \times \mathfrak{S}_{n+1}$. It can be viewed as a subalgebra by $\iota\colon J_{\mathrm{ev}}(\mathfrak{g}) \to J(\mathfrak{g})$ with $\mathbb{K}_i \mapsto e^{-\alpha_i/2}$ and its image is coincide with $\mathrm{Sch}(K_{\mathrm{ev}}(\mathfrak{g}))$.

$\boxed{A(1,1) \textbf{ case:}}$ We have $J_{\mathrm{ev}}(A(1,1)) = \{c + (u - v)g \,|\, g \in \mathbb{Z}[u, v]\}$ where $u = \left(\frac{x_1}{x_2}\right)^{\frac{1}{2}} + \left(\frac{x_2}{x_1}\right)^{\frac{1}{2}}$, $v = \left(\frac{y_1}{y_2}\right)^{\frac{1}{2}} + \left(\frac{y_2}{y_1}\right)^{\frac{1}{2}}$. And $u - v = \mathbb{K}_1 + \mathbb{K}_1^{-1} - \mathbb{K}_3 - \mathbb{K}_3^{-1} \in J_{\mathrm{ev}}(A(1,1))$, but $u - v \notin J(A(1,1))$.

$\boxed{B(m,n), m, n > 0 \textbf{ case:}}$ We set $\lambda = \sum_{i=1}^{m} \lambda_i \varepsilon_i + \sum_{j=1}^{n} \mu_j \delta_j \in \mathfrak{h}^*$, then in this case

$$\mathbb{Z}\Phi = \left\{ \lambda \in \mathfrak{h}^* \,\middle|\, \lambda_i, \mu_j \in \mathbb{Z},\ \forall i, j \right\} \quad \text{and}$$
$$\Lambda = \left\{ \lambda \in \mathfrak{h}^* \,\middle|\, \mu_j \in \mathbb{Z},\ \forall j \text{ and all } \lambda_i \in \mathbb{Z} \text{ or all } \lambda_i \in \mathbb{Z} + \frac{1}{2} \right\}.$$

So $2\Lambda \cap \mathbb{Z}\Phi = 2\Lambda$. Let $u_i = x_i + x_i^{-1}$ and $v_j = y_j + y_j^{-1}$ for all possible i, j, then we have $J_{\mathrm{ev}}(\mathfrak{g}) = J(\mathfrak{g})_0 \oplus J(\mathfrak{g})_{1/2}$, where

$$J(\mathfrak{g})_0 = \left\{ f \in \mathbb{Z}[u_1, \cdots, u_m, v_1, \cdots, v_n]^{\mathfrak{S}_m \times \mathfrak{S}_n} \,\middle|\, u_i \frac{\partial f}{\partial u_i} + v_j \frac{\partial f}{\partial v_j} \in (u_i - v_j) \right\},$$

and

$$J(\mathfrak{g})_{1/2} = \left\{ \prod_{i=1}^{m} (x_i^{1/2} + x_i^{-1/2}) \prod_{i=1}^{m} \prod_{j=1}^{n} (u_i - v_j) g \,\middle|\, g \in \mathbb{Z}[u_1, \cdots, u_m, v_1, \cdots, v_n]^{\mathfrak{S}_m \times \mathfrak{S}_n} \right\}.$$

$\boxed{B(0,n) \textbf{ case:}}$ In this case $\Lambda = \mathbb{Z}\Phi = \left\{ \sum_{j=1}^{n} \mu_j \delta_j \,\middle|\, \mu_j \in \mathbb{Z},\ \forall j \right\}$, so $2\Lambda \cap \mathbb{Z}\Phi = 2\Lambda$ and this algebra $J_{\mathrm{ev}}(\mathfrak{g}) = \mathbb{Z}[v_1, v_2, \cdots, v_n]^{\mathfrak{S}_n}$, where the notation v_i are the same as above.

$\boxed{C(n+1) \textbf{ case:}}$ In this case

$$\Lambda = \left\{ \lambda\varepsilon + \sum_{j=1}^{n} \mu_j \delta_j \,\middle|\, \lambda \in \mathbb{C}, \mu_j \in \mathbb{Z},\ \forall j \right\}$$

and

$$\mathbb{Z}\Phi = \left\{ \lambda\varepsilon + \sum_{j=1}^{n} \mu_j \delta_j \,\middle|\, \lambda, \mu_j \in \mathbb{Z},\ \forall j \text{ and } \lambda + \sum_{j=1}^{n} \mu_j \text{ is even} \right\}.$$

So $2\Lambda \cap \mathbb{Z}\Phi = \left\{\lambda\varepsilon + \sum_{j=1}^{n} \mu_j\delta_j \middle| \lambda, \mu_j \in 2\mathbb{Z},\ \forall j\right\}$ and the algebra

$$J_{\text{ev}}(\mathfrak{g}) = \left\{ f \in \mathbb{Z}[x^{\pm 1}, y_1^{\pm 1}, \cdots, y_{n+1}^{\pm 1}]^W \,\middle|\, y_j \frac{\partial f}{\partial y_j} + x \frac{\partial f}{\partial x} \in (x - y_j) \right\}.$$

$\boxed{D(m,n), m > 1, n > 0 \textbf{ case:}}$ Let $\lambda = \sum_{i=1}^{m} \lambda_i\varepsilon_i + \sum_{j=1}^{n} \mu_j\delta_j \in \mathfrak{h}^*$ and u_i, v_j are as above, then

$$\Lambda = \left\{ \lambda \in \mathfrak{h}^* \,\middle|\, \mu_j \in \mathbb{Z},\ \forall j \text{ and all } \lambda_i \in \mathbb{Z} \text{ or all } \lambda_i \in \mathbb{Z} + \frac{1}{2} \right\}$$

and

$$\mathbb{Z}\Phi = \left\{ \lambda \in \mathfrak{h}^* \,\middle|\, \lambda_i, \mu_j \in \mathbb{Z},\ \forall i, j \text{ and } \sum_{i=1}^{m} \lambda_i + \sum_{j=1}^{n} \mu_j \text{ is even} \right\}.$$

So

$$2\Lambda \cap \mathbb{Z}\Phi = \begin{cases} 2\mathbb{Z}\Phi + \mathbb{Z}\left(\sum_{i=1}^{n} \varepsilon_i\right) + 2\mathbb{Z}\varepsilon_n, & \text{if } m = 2k, \\ 2\mathbb{Z}\Phi + 2\mathbb{Z}\varepsilon_n, & \text{if } m = 2k+1, \end{cases}$$

for some positive integer k. Thus the algebra $J_{\text{ev}}(\mathfrak{g})$ is, respectively, equal to $J(\mathfrak{g})_0 \oplus J(\mathfrak{g})_{1/2}$ for $m = 2k$ and $J(\mathfrak{g})_0$ for $m = 2k+1$, where

$$J(\mathfrak{g})_0 = \left\{ f \in \mathbb{Z}[x_1^{\pm 1}, \cdots, x_m^{\pm 1}, y_1^{\pm 1}, \cdots, y_n^{\pm 1}]^W \,\middle|\, x_i \frac{\partial f}{\partial x_i} + y_j \frac{\partial f}{\partial y_j} \in (x_i - y_j) \right\},$$

and

$$J(\mathfrak{g})_{1/2} = \left\{ \prod_{i,j}(u_i - v_j)\Big((x_1x_2\cdots x_m)^{1/2}\mathbb{Z}[x_1^{\pm 1}, \cdots, x_m^{\pm 1}, y_1^{\pm 1}, \cdots, y_n^{\pm 1}]\Big)^W \right\}.$$

$\boxed{D(2,1,\alpha) \textbf{ case:}}$ In this case,

$$\Lambda = \left\{ \sum_{i=1}^{3} \lambda_i\varepsilon_i \,\middle|\, \lambda_i \in \mathbb{Z},\ \forall i \right\}, \text{ and } \mathbb{Z}\Phi = \left\{ \sum_{i=1}^{3} \lambda_i\varepsilon_i \,\middle|\, \lambda_i \in \mathbb{Z} \text{ and } \lambda_i - \lambda_j \in 2\mathbb{Z},\ \forall i, j \right\}.$$

So $2\Lambda \cap \mathbb{Z}\Phi = 2\Lambda$. Thus the algebra

$$J_{\text{ev}}(\mathfrak{g}) = \begin{cases} \left\{c + \Delta h | c \in \mathbb{Z}, h \in \mathbb{Z}[u_1, u_2, u_3]\right\}, & \text{if } \alpha \text{ is not rational}, \\ \left\{g(w_\alpha) + \Delta h | g \in \mathbb{Z}[\omega], h \in \mathbb{Z}[u_1, u_2, u_3]\right\}, & \text{if } \alpha = p/q \text{ with } p \in \mathbb{Z}, q \in \mathbb{N}, \end{cases}$$

where

$$\Delta = u_1^2 + u_2^2 + u_3^2 - u_1u_2u_3 - 4,\ u_i = x_i + x_i^{-1},\ \text{for } i = 1, 2, 3,$$

and

$$\omega_\alpha = (x_1 + x_1^{-1} - x_2x_3 - x_2^{-1}x_3^{-1})\frac{(x_2^p - x_2^{-p})(x_3^q - x_3^{-q})}{(x_2 - x_2^{-1})(x_3 - x_3^{-1})} + x_2^p x_3^{-q} + x_2^{-p}x_3^q.$$

$\boxed{F(4)\ \textbf{case:}}$ In this case,

$$\Lambda = \left\{\mu\delta + \sum_{i=1}^{3}\lambda_i\varepsilon_i \middle| \text{all } \lambda_i \in \mathbb{Z} \text{ or all } \lambda_i \in \mathbb{Z} + \frac{1}{2},\ 2\mu \in \mathbb{Z}\right\},$$

and

$$\mathbb{Z}\Phi = \left\{\mu\delta + \sum_{i=1}^{3}\lambda_i\varepsilon_i \middle| \text{all } \lambda_i,\ \mu \in \mathbb{Z} \text{ or all } \lambda_i,\ \mu \in \mathbb{Z} + \frac{1}{2}\right\}.$$

So $2\Lambda \cap \mathbb{Z}\Phi = 2\Lambda$, and the algebra

$$J_{\mathrm{ev}}(\mathfrak{g}) = \left\{g(\omega_1, \omega_2) + \Delta h \middle| h \in \mathbb{Z}[x_1^{\pm 2}, x_2^{\pm 2}, x_3^{\pm 2}, (x_1x_2x_3)^{\pm 1}, y^{\pm 1}]^W,\ g \in \mathbb{Z}[\omega_1, \omega_2]\right\},$$

where

$$\Delta = \left(y + y^{-1} - x_1x_2x_3 - x_1^{-1}x_2^{-1}x_3^{-1}\right)\prod_{i=1}^{3}\left(y + y^{-1} - \frac{x_1x_2x_3}{x_i^2} - \frac{x_i^2}{x_1x_2x_3}\right),$$

and

$$\omega_k = \sum_{1\leqslant i<j\leqslant 3}\left(x_i^{2k} + x_i^{-2k} + \frac{1}{2}\right)\left(x_j^{2k} + x_j^{-2k} + \frac{1}{2}\right) - \frac{3}{4} + y^{2k} + y^{-2k} - (y^k + y^{-k})\prod_{i=1}^{3}\left(x_i^k + x_i^{-k}\right)$$

with $k = 1, 2$, and $W = \mathbb{Z}_2 \times (\mathfrak{S}_3 \ltimes \mathbb{Z}_2^3)$.

$\boxed{G(3)\ \textbf{case:}}$ In this case, $\Lambda = \mathbb{Z}\Phi = \left\{\lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \mu\delta | \lambda_1, \lambda_2, \mu \in \mathbb{Z}\right\}$. So $2\Lambda \cap \mathbb{Z}\Phi = 2\Lambda$, and the algebra

$$J_{\mathrm{ev}}(\mathfrak{g}) = \left\{g(\omega) + \prod_{i=1}^{3}(v - u_i)h \middle| h \in \mathbb{Z}[v, u_1, u_2, u_3]^{\mathfrak{S}_3},\ g \in \mathbb{Z}[\omega]\right\},$$

where

$$\omega = v^2 - v(u_1 + u_2 + u_3 + 1) + u_1u_2 + u_1u_3 + u_2u_3.$$

and the notations u_i, v are the same as above.

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