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Abundance of Observable Lyapunov Irregular Sets

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Abstract: Lyapunov exponent is widely used in natural science to find chaotic signal, but its existence is seldom discussed. In the present paper, we consider the problem of whether the set of points at which Lyapunov exponent fails to exist, called the Lyapunov irregular set, has positive Lebesgue measure. The only known example with the Lyapunov irregular set of positive Lebesgue measure is a figure-8 attractor by the work of Ott and Yorke (Phys Rev 78, 056203, 2008), whose key mechanism (homoclinic loop) is easy to be broken by small perturbations. In this paper, we show that surface diffeomorphisms with a robust homoclinic tangency given by Colli and Vargas (Ergod Theory Dyn Syst 21, 1657–1681, 2001), as well as other several known nonhyperbolic dynamics, have the Lyapunov irregular set of positive Lebesgue measure. We can construct such positive Lebesgue measure sets both as the time averages exist and do not exist on it.

1. Introduction

Lyapunov exponent is a quantity to measure sensitivity of an orbit to initial conditions and natural scientists often compute it to find chaotic signal. However, the existence of Lyapunov exponent is seldom discussed. The aim of this paper is to investigate the abundance of dynamical systems whose Lyapunov exponents fail to exist on a physically observable set, that is, a *positive Lebesgue measure* set.

Let M be a compact Riemannian manifold and $f : M \to M$ a differential map. A point $x \in M$ is said to be *Lyapunov irregular* if there is a non-zero vector $v \in T_x M$ such that the Lyapunov exponent of x for v,

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\|, \tag{1.1}$$

does not exist. When we would like to emphasize the dependence on v, we call it Lyapunov irregular for v. Similarly a point x is said to be *Birkhoff irregular* if there is a continuous function $\varphi : M \to \mathbb{R}$ such that the time average $\lim_{n\to\infty}(\sum_{j=0}^{n-1} \varphi(f^j(x)))/n$

does not exist. Otherwise, we say that x is *Birkhoff regular*. Moreover, we call the set of Lyapunov (resp. Birkhoff) irregular points the *Lyapunov* (resp. *Birkhoff*) *irregular set* of f. We borrowed these terminologies from Abdenur–Bonatti–Crovisier [1], while they studied the *residuality* of Lyapunov/Birkhoff irregular sets, which is not the scope of the present paper. Indeed, the residuality of irregular sets is a generic property ([1, Theorem 3.15]) while the positivity of Lebesgue measure of irregular sets does not hold for Axiom A diffeomorphisms, see e.g. [23]. The terminology *historic behavior* by Ruelle [19] is also commonly used for the forward orbit of a point to mean that the point is Birkhoff irregular, in particular in the study of the positivity of Lebesgue measure of Birkhoff irregular sets after Takens [21], see e.g. [10,11] and references therein.

Due to the Oseledets multiplicative ergodic theorem, the Lyapunov irregular set of f is a zero measure set for any *invariant* measure. However, this tells nothing about whether the Lyapunov irregular set is of positive Lebesgue measure in general. In fact, the Birkhoff ergodic theorem ensures that the Birkhoff irregular set has zero measure with respect to any invariant measure, but for a wide variety of dynamical systems the Birkhoff irregular set is known to have positive Lebesgue measure, see e.g. [10,11,19,21] and references therein. Furthermore, the positivity of Lebesgue measure of the Birkhoff irregular set for these examples are strongly related with *non-hyperbolicity* of the systems, and the two complementary conjectures given by Palis [18] and Takens [21] for the abundance of dynamics with the Birkhoff irregular set of positive Lebesgue measure opened a deep research field in smooth dynamical systems theory. So, it is naturally expected that finding a large class of dynamical systems with the Lyapunov irregular set of positive Lebesgue measure would be a significant subject.

Yet, the known example whose Lyapunov irregular set has positive Lebesgue measure is only a surface flow with an attracting homoclinic loop, called a figure-8 attractor ([17]), see Sect. 1.1.1 for details. However, the homoclinic loop is easy to be broken by small perturbations. Therefore, in this paper we give surface diffeomorphisms with a $\mathcal{C}^r$-*robust homoclinic tangency* ($r \geq 2$) and the Lyapunov irregular set of positive Lebesgue measure. Recall that Newhouse [16] showed that, when M is a closed surface, any homoclinic tangency yields a $\mathcal{C}^r$-diffeomorphism f with a robust homoclinic tangency associated with a thick basic set Λ, that is, there is a neighborhood $\mathcal{O}$ of f in the set $\mathrm{Diff}^r(M)$ of $\mathcal{C}^r$-diffeomorphisms such that for every $g \in \mathcal{O}$ the continuation Λ_g of Λ has a homoclinic tangency. Such an open set $\mathcal{O}$ is called a *Newhouse open set*.

We finally remark that if f is a $\mathcal{C}^1$-diffeomorphism whose Lyapunov irregular set has positive Lebesgue measure and $\tilde{f}$ is conjugate to f by a $\mathcal{C}^1$-diffeomorphism h, that is, $\tilde{f} = h^{-1} \circ f \circ h$, then the Lyapunov irregular set of $\tilde{f}$ also has positive Lebesgue measure. Our main theorem is the following.

Theorem A. *There exists a diffeomorphism g in a Newhouse open set of* $\mathrm{Diff}^r(M)$ *of a closed surface M and $2 \leq r < \infty$ such that for any small $\mathcal{C}^r$-neighborhood $\mathcal{O}$ of g one can find an uncountable set $\mathcal{L} \subset \mathcal{O}$ satisfying the following:*

(1) *Every f and $\tilde{f}$ in $\mathcal{L}$ are not topologically conjugate if $f \neq \tilde{f}$;*
(2) *For any $f \in \mathcal{L}$, there exist open sets $U_f \subset M$ and $V_f \subset \mathbb{R}^2$, under the identification of TU_f with $U_f \times \mathbb{R}^2$, such that any point $x \in U_f$ is Lyapunov irregular for any non-zero vector $v \in V_f$.*

Furthermore, $\mathcal{L}$ can be decomposed into two uncountable sets $\mathcal{R}$ and $\mathcal{I}$ such that any point in U_f is Birkhoff regular for each $f \in \mathcal{R}$ and any point in U_f is Birkhoff irregular for each $f \in \mathcal{I}$.

Remark. (Generalization of Theorem A) It is a famous folklore result known to Bowen that a surface flow with heteroclinically connected two dissipative saddle points has the Birkhoff irregular set of positive Lebesgue measure (see Sect. 1.1.1), and its precise proof was given by Gaunersdorfer [8], see also Takens [20]. However, again, the heteroclinic connections are easily broken by small perturbations, and thus Takens asked in [21] whether the Birkhoff irregular set can have positive Lebesgue measure in a persistent manner. In [11], the first and fourth authors affirmatively answered it by showing that there is a *dense* subset of any $\mathcal{C}^r$-Newhouse open set of surface diffeomorphisms with $2 \leq r < \infty$ such that any element of the dense set has an open subset in the Birkhoff irregular set, by extending the technology developed for a special surface diffeomorphism with a robust homoclinic tangency given by Colli and Vargas [5]. Furthermore, we adopt the Colli–Vargas diffeomorphism to prove Theorem A. Therefore, it is likely that Theorem A can be extended to surface diffeomorphisms in a dense subset of any Newhouse open set. The main technical difficulty might be the control of higher order terms of the return map of diffeomorphisms in the dense set, which do not appear for the return map of the Colli–Vargas diffeomorphism, see the expression (1.10).

Furthermore, the above result [11] was recently extended in [4] to the $\mathcal{C}^\infty$ and $\mathcal{C}^\omega$ categories by introducing a geometric model, and Colli–Vargas' result was extended in [12] to a 3-dimensional diffeomorphism with a $\mathcal{C}^1$-robust homoclinic tangency derived from a blender-horseshoe. Hence, we expect that Theorem A holds for $r = \infty, \omega$ and for $r = 1$ when the dimension of M is three. We also remark that [3,13] extended the result of [11] to 3-dimensional flows and higher dimensional diffeomorphisms.

Remark. (Irregular vectors) Ott and Yorke [17] asserted that they constructed an open set U any point of which is Lyapunov irregular for *any* non-zero vectors, but we believe that their proof has a gap. What one can immediately conclude from their argument is that any point in U is Lyapunov irregular for non-zero vectors in the *flow* direction (and thus, the set of irregular vectors are not observable); see Sect. 1.1.1 for details. In Sect. 1.1.3, we further show that a surface diffeomorphism with a figure-8 attractor introduced by Guarino–Guihéneuf–Santiago [9] has an open set every element of which is Lyapunov irregular for *any* non-zero vectors.

Remark. (Relation with Birkhoff irregular sets) One can find differences between Birkhoff irregular sets and Lyapunov irregular sets, other than Theorem A, in the literature. Indeed, it was already pointed out in Ott-Yorke [17] that the figure-8 attractor has a positive Lebesgue measure set on which the time averages exist but the Lyapunov exponents do not exist (see also [7]). Conversely, diffeomorphisms whose Birkhoff irregular set has positive Lebesgue measure but Lyapunov irregular set has zero Lebesgue measure were exhibited in [6]. We also remark that, in contrast to the deterministic case, under physical noise both Birkhoff and Lyapunov irregular sets of any diffeomorphism have zero Lebesgue measure by [2] and [14].

In the rest of Sect. 1, we explain that several nonhyperbolic systems in the literature also have Lyapunov irregular sets of positive Lebesgue measure (see, in particular, Sect. 1.1). The purpose of the attention to these examples are not to increase the collection of dynamics with observable Lyapunov irregular sets, but rather to understand the mechanism making observable Lyapunov irregular sets, which is especially discussed in Sect. 1.2.

1.1. Other examples.

1.1.1. Figure-8 attractor Ott and Yorke showed in [17] that a figure-8 attractor has the Lyapunov irregular set of positive Lebesgue measure as follows. Let $(f^t)_{t\in\mathbb{R}}$ be a smooth flow on $\mathbb{R}^2$ generated by a vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$ with an equilibrium point p of saddle type with homoclinic orbits, that is, the unstable manifold of p coincides with the stable manifold of p and consists of $\{p\}$ and two orbits γ_1, γ_2. We also assume that the loops $\gamma_1 \cup \{p\}$ and $\gamma_2 \cup \{p\}$ are attracting in the sense that $\alpha_- > \alpha_+$, where α_+ and $-\alpha_-$ are eigenvalues of the linearized vector field of V at p with $\alpha_\pm > 0$. Due to the assumption, one can find open sets U_1 and U_2 inside and near the loops $\gamma_1 \cup \{p\}$ and $\gamma_2 \cup \{p\}$, respectively, such that the ω-limit set of $(f^t(x))_{t\in\mathbb{R}}$ is $\gamma_i \cup \{p\}$ for all $x \in U_i$ with $i = 1, 2$. In this setting, $\gamma_1 \cup \gamma_2 \cup \{p\}$ is called a *figure-8 attractor*.

It is easy to see that the Birkhoff irregular set of the figure-8 attractor is empty inside $U_1 \cup U_2$: in fact, if $x \in U_1 \cup U_2$, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \varphi \circ f^s(x)ds = \varphi(p) \quad \text{for any continuous function } \varphi : \mathbb{R}^2 \to \mathbb{R}$$

(cf. [9]). On the other hand, Ott and Yorke showed in [17] that any point x in $U_1 \cup U_2$ is Lyapunov irregular for the vector $V(x)$, that is, the Lyapunov irregular set has positive Lebesgue measure (in fact, they implicitly put an additional assumption for simple calculations, see Sect. 2).

As previously mentioned, they also asserted that $x \in U_1 \cup U_2$ is Lyapunov irregular for any non-zero vector v, because $(\frac{1}{t} \log \det(Df^t(x)V(x)\ Df^t(x)v))_{t\in\mathbb{R}}$ converges to $\alpha_+ - \alpha_-$ as $t \to \infty$. However, the oscillation of $(\frac{1}{t} \log \|Df^t(x)v\|)_{t\in\mathbb{R}}$ is not a direct consequence of this fact and the oscillation of $(\frac{1}{t} \log \|Df^t(x)V(x)\|)_{t\in\mathbb{R}}$ when v is not parallel to $V(x)$ because the angle between $Df^t(x)V(x)$ and $Df^t(x)v$ can also oscillate.

1.1.2. Bowen flow In [17], Ott and Yorke also indicated the oscillation of Lyapunov exponents for a vector along the flow direction for a special Bowen flow by a numerical experiment. By following the argument of [17] for a figure-8 attractor, we can rigorously prove that the Lyapunov irregular set has positive Lebesgue measure for any Bowen flow.

Let $(f^t)_{t\in\mathbb{R}}$ be a smooth flow on $\mathbb{R}^2$ generated by a vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$ of class $\mathcal{C}^{1+\alpha}$ $(\alpha > 0)$ with two equilibrium points p and $\hat{p}$ and two heteroclinic orbits γ_1 and γ_2 connecting the points, which are included in the unstable and stable manifolds of p respectively, such that the closed curve $\gamma := \gamma_1 \cup \gamma_2 \cup \{p\} \cup \{\hat{p}\}$ is attracting in the following sense: if we denote the expanding and contracting eigenvalues of the linearized vector field around p by α_+ and $-\alpha_-$, and the ones around p_2 by β_+ and $-\beta_-$, then

$$\alpha_-\beta_- > \alpha_+\beta_+.$$

In this setting, one can find an open set U inside and near the closed curve γ such that the ω-limit set of $(f^t(x))_{t\in\mathbb{R}}$ is γ for all $x \in U$. As explained, it was proven in [8,20] that any point in U is Birkhoff irregular. In fact, if $x \in U$, then one can find time sequences $(\tau_n)_{n\in\mathbb{N}}$, $(\hat{\tau}_n)_{n\in\mathbb{N}}$ (given in Sect. 2) such that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{\tau_n} \int_0^{\tau_n} \varphi \circ f^s(x)ds &= \frac{r\varphi(p) + \varphi(\hat{p})}{1+r}, \\ \lim_{n\to\infty} \frac{1}{\hat{\tau}_n} \int_0^{\hat{\tau}_n} \varphi \circ f^s(x)ds &= \frac{\varphi(p) + \hat{r}\varphi(\hat{p})}{1+\hat{r}} \end{aligned} \tag{1.2}$$

for any continuous function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, where $r = \frac{\alpha_-}{\beta_+}$ and $\hat{r} = \frac{\beta_-}{\alpha_+}$. According to Takens [20] we call such a flow a *Bowen flow*. We can show the following proposition for the Lyapunov irregular set, whose proof will be given in Sect. 2.

Proposition 1.1. *For the Bowen flow $(f^t)_{t\in\mathbb{R}}$ with the open set U given above, any point x in U is Lyapunov irregular for the vector $V(x)$.*

Remark. For the time sequences in (1.2) for which the time averages oscillate, we will see that

$$\lim_{n\to\infty} \frac{1}{\tau_n} \log \|Df^{\tau_n}(x)\| = \lim_{n\to\infty} \frac{1}{\hat{\tau}_n} \log \|Df^{\hat{\tau}_n}(x)\| = 0 \tag{1.3}$$

for any $x \in U$. That is, the mechanism causing the oscillation of Lyapunov exponents is different from the one leading to oscillation of time averages; see Sect. 1.2 for details.

1.1.3. Guarino–Guihéneuf–Santiago's simple figure-8 attractor A disadvantage of the arguments in Sects. 1.1.1 and 1.1.2 is that, although it follows the arguments that a point x in the open set $U_1 \cup U_2$ or U is Lyapunov irregular for the vector $V(x)$ generating the flow, it is unclear whether x is also Lyapunov irregular for a vector which is not parallel to $V(x)$, because the derivative $Df^t(x)$ at the return time t to neighborhoods of p or $p \cup \hat{p}$ is not explicitly calculated in the arguments (instead, the fact that $Df^t(x)V(x) = V(f^t(x))$ is used). On the other hand, Guarino, Guihéneuf and Santiago in [9] constructed a surface diffeomorphism with a pair of saddle connections forming a figure of eight and whose return map is affine (see Proposition (1.4)). By virtue of this simple form of the return map, it is quite easy to prove that the diffeomorphism has an open set each element of which is Lyapunov irregular for *any* non-zero vectors. Furthermore, we will see in Sect. 1.2 that the calculation is a prototype of the proof of Theorem A.

Fix a constant $\sigma > 1$ and numbers a, b such that $1 < a < b < \sigma$. Let $I = [a, b]$ and denote the map $\mathbb{R}^2 \ni (x, y) \mapsto (\sigma^{-2}x, \sigma y)$ by H. For every $n \in \mathbb{N}$, let $S_n = I \times \sigma^{-n} I$ and $U_n = \sigma^{-n} I \times I$, so that

$$H^n(S_n) = U_{2n} \quad \text{and} \quad H^n : S_n \to U_{2n} \text{ is a diffeomorphism.}$$

See Fig. 1. Furthermore, let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map which is a rotation of $-\frac{\pi}{2}$ around the point $(\frac{a+b}{2}, \frac{a+b}{2})$, i.e.

$$R(x, y) = (a + b - y, x).$$

We say that a diffeomorphism of the plane is said to be *compactly supported* if it equals the identity outside a ball centered at the origin O, and moreover the diffeomorphism has a *saddle (homoclinic) connection* if it has a separatrix of the stable manifold coinciding with a separatrix of the unstable manifold associated with a saddle periodic point O, so that it bounds an open 2-disk. Specially, we call the union of O and a pair of saddle connections associated with O a *figure-8 attractor* at O, and it satisfies $W^u(O) = W^s(O)$.

Proposition 1.2 ([9, Proposition 3.4]). *There exists a compactly supported $\mathcal{C}^\infty$-diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ which has a saddle connection of a saddle fixed point $O = (0, 0)$, and moreover there are positive integers n_0, k_0 such that the following holds:*

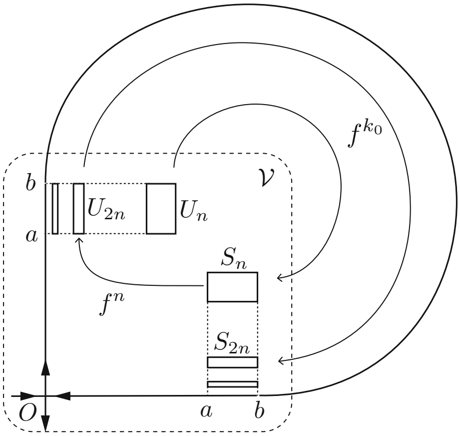

Fig. 1. Guarino–Guihéneuf–Santiago's diffeomorphism

(a) *There is a neighborhood* $\mathcal{V}$ *of* O *such that*

$$\bigcup_{n\geq n_0}\bigcup_{0\leq \ell\leq n} f^{\ell}(S_n)\subset \mathcal{V} \quad \textit{and} \quad f|_{\mathcal{V}}=H.$$

(b) $f^{k_0}(U_n)=S_n$ *for all* $n\geq n_0$ *and*

$$f^{k_0}(x,y)=R(x,y) \quad \textit{for all } (x,y)\in[0,\sigma^{-2n_0}]\times I.$$

In particular, for every $n\geq n_0$*,*

$$f^{n+k_0}(x,y)=(a+b-\sigma^n y,\sigma^{-2n}x)\in S_{2n} \quad \textit{for all } (x,y)\in S_n. \tag{1.4}$$

Remark. If we suppose that $f|_{V_3}=s_h\circ f|_{V_1}\circ s_v$, where V_i is the i-th quadrant of $\mathbb{R}^2$, and $s_v, s_h:\mathbb{R}^2\to\mathbb{R}^2$ are symmetry maps with respect to the vertical and horizontal axes, respectively, f has a figure-8 attractor at O, see [9].

Although the dynamics in Proposition 1.2 is defined on $\mathbb{R}^2$, one can easily embed the restriction of f on the support of f into any compact surface. It follows from [9, Corollary 3.5] that if $z\in S_{n_0}$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(z))=\varphi(O) \quad \text{for any continuous function } \varphi:\mathbb{R}^2\to\mathbb{R}.$$

In particular, any point in S_{n_0} is Birkhoff regular. Our result for the Lyapunov irregular set is the following, whose proof will be given in Sect. 3.

Theorem 1.3. *For the diffeomorphism* f *and the rectangle* S_{n_0} *given in Proposition 1.2, any point* z *in* S_{n_0} *is Lyapunov irregular for any non-zero vector.*

Remark. We note that the piecewise expanding map on a surface constructed by Tsujii [22] has a return map around the origin whose form is quite similar to one of the diffeomorphism of Theorem 1.3. So, it is natural to expect that (a slightly modified version of) the map in [22] has an open set consisting of Lyapunov irregular points for any non-zero vectors.

1.2. Idea of proofs of Theorems A and 1.3: anti-diagonal matrix form of the return map.

1.2.1. The figure-8 attractor We start from the Guarino–Guihéneuf–Santiago's figure-8 attractor. Let f be the diffeomorphism given in Proposition 1.2. Then, it follows from (1.4) that for any $n \geq n_0$ and $z \in S_n$, $Df^{n+k_0}(z)$ is an anti-diagonal matrix,

$$Df^{n+k_0}(z) = \begin{pmatrix} 0 & -\sigma^n \\ \sigma^{-2n} & 0 \end{pmatrix}, \tag{1.5}$$

so $Df^{(2n+k_0)+(n+k_0)}(z) = Df^{2n+k_0}(f^{n+k_0}(z))Df^{n+k_0}(z)$ is a diagonal matrix. Hence, if we define the d-th return time $N(d)$ from S_{n_0} to $\bigcup_{n\geq n_0} S_n$ with $d \geq 1$ by

$$N(d) = \sum_{d'=1}^{d} n(d'), \quad n(d') = 2^{d'-1}n_0 + k_0 \tag{1.6}$$

(notice that $f^{N(d)}(S_{n_0}) \subset S_{2^d n_0}$), then it follows from a chain of calculations that for any $z \in S_{n_0}$

$$\begin{aligned} Df^{N(2d-1)}(z) &= (-1)^{d-1} \begin{pmatrix} 0 & -\sigma^{n_0} \\ \sigma^{-2^{2d-1}n_0} & 0 \end{pmatrix}, \\ Df^{N(2d)}(z) &= (-1)^d \begin{pmatrix} 1 & 0 \\ 0 & -\sigma^{(-2^{2d}+1)n_0} \end{pmatrix}, \end{aligned} \tag{1.7}$$

and thus, for any $v \notin \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cup \mathbb{R}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$\lim_{d\to\infty} \frac{1}{N(d)} \log \left\| Df^{N(d)}(z)v \right\| = 0. \tag{1.8}$$

Furthermore, one can see by a direct calculation that with the function $\vartheta : [0, 1] \to \mathbb{R}$ given by $\vartheta(\zeta) = -(1-\zeta)/(1+\zeta)$ if $\zeta \geq 1/3$ and $\vartheta(\zeta) = -2\zeta/(1+\zeta)$ if $\zeta < 1/3$, it holds that for any $\zeta \in [0, 1]$,

$$\lim_{d\to\infty} \frac{1}{N(4d) + \lfloor \zeta 2^{4d} n_0 \rfloor} \log \left\| Df^{N(4d)+\lfloor \zeta 2^{4d} n_0 \rfloor}(z)v \right\| = \vartheta(\zeta)\log\sigma, \tag{1.9}$$

where $\lfloor a \rfloor$ for $a \in \mathbb{R}$ is the greatest integer less than or equal to a. Note that $N(4d) = 2^{4d}n_0 + (4dk_0 - n_0)$, so $N(4d) + \lfloor \zeta 2^{4d} n_0 \rfloor$ over $\zeta \in [0, 1]$ essentially realizes all times from $N(4d)$ to $N(4d+1)$. A detailed calculation will be given in Sect. 3.

1.2.2. The Newhouse open set Next we consider the diffeomorphisms in the Newhouse open set given in Theorem A. Colli and Vargas constructed in [5] a diffeomorphism g in a Newhouse open set with constants $0 < \lambda < 1 < \sigma$ such that for any $\mathcal{C}^r$-neighborhood $\mathcal{O}$ of g and any increasing sequence $(n_k^0)_{k\geq 0}$ of integers with $\limsup_{k\to\infty} n_{k+1}/n_k < \infty$, one can find a diffeomorphism f in $\mathcal{O}$ together with a sequence of rectangles $(R_k)_{k=1}^{\infty}$ and a sequence of increasing sequence $(\tilde{n}_k)_{k\geq 1}$ of integers with $\tilde{n}_k = O(k)$ such that $f^{n_k+2}(R_k) \subset R_{k+1}$ and for each $(\tilde{x}_k + x, y) \in R_k$,

$$f^{n_k+2}(\tilde{x}_k + x, y) = (\tilde{x}_{k,1} - \sigma^{2n_k}x^2 - \lambda^{n_k}y, \sigma^{n_k}x), \tag{1.10}$$

where $n_k = n_k^0 + \tilde{n}_k$ and $(\tilde{x}_k, 0)$ is the center of R_k, see Theorem 4.1 for details. Thus, the derivative of the return map has the form

$$Df^{n_k+2}(\tilde{x}_k + x, y) = \begin{pmatrix} -2\sigma^{2n_k} x & -\lambda^{n_k} \\ \sigma^{n_k} & 0 \end{pmatrix}. \tag{1.11}$$

Compare this formula with (1.5) for $n = n(d) - k_0$ and note that $\lim_{d\to\infty}(n(d+1) - k_0)/(n(d) - k_0) = 2$.

The biggest obstacle in (1.11) to repeat the above calculation for Guarino–Guihéneuf–Santiago's figure-8 attractor is the term $-2\sigma^{2n_k}x$: the absolute value of the term should be as small as the absolute value of $-\lambda^{n_k}$ of (1.11), while σ^{2n_k} may be much larger than λ^{n_k} because $0 < \lambda < 1 < \sigma$. Therefore, the key point in the proof is to find a subset U_k of R_k such that any $x \in U_k$ satisfies the required condition $|-2\sigma^{2n_k}x| < \xi|-\lambda^{n_k}|$ with a positive constant ξ independently of k (Lemma 4.5), and to show $f^{n_k+2}(U_k) \subset U_{k+1}$ (Lemma 4.4).

1.2.3. Some technical observations Finally, we give a couple of (more technical) remarks on the similarity of mechanics leading to observable Lyapunov irregular sets for the dynamics of this paper.

Remark. To understand the time scale $N(4d)+\lfloor\zeta 2^{4d} n_0\rfloor$ of (1.9), calculations of (partial) Lyapunov exponents for the Bowen flow might be helpful. Let $(f^t)_{t\in\mathbb{R}}$, V, p, $\hat{p}$, U be as in Sect. 1.1.2. Let N and $\hat{N}$ be small neighborhoods of p and $\hat{p}$, respectively, such that $N \cap \hat{N} = \emptyset$. Fix $z \in U$ and let τ_n and $\hat{\tau}_n$ be the n-th return time of z to N and $\hat{N}$, respectively (see Sect. 2 for their precise definition). Then, since $Df^t(z)V(z) = V(f^t(z))$ for each $t \geq 0$, both $\|Df^{\tau_n}(z)V(z)\|$ and $\|Df^{\hat{\tau}_n}(z)V(z)\|$ are bounded from above and below uniformly with respect to n, which implies (1.3) (while (1.2) is a consequence of [20]).

We further define ρ_n as the time t in $[0, \tau_{n+1} - \tau_n]$ at which $f^{\tau_n+t}(z)$ makes the closest approach to p (that is, ρ_n is the the minimizer of $\|f^{\tau_n+t}(z) - p\|$ over $0 \leq t \leq \tau_{n+1} - \tau_n$). Then, since the vector field V is zero at p, it can be expected that $\|Df^{\tau_n+\rho_n}(z)V(z)\| = \|V(f^{\tau_n+\rho_n}(z))\|$ decays rapidly as n increases. In fact, we can show that

$$\lim_{n\to\infty} \frac{1}{\tau_n + \rho_n} \log \|Df^{\tau_n+\rho_n}(z)V(z)\| = \frac{\alpha_+\beta_+ - \alpha_-\beta_-}{\alpha_+ + \beta_+ + \alpha_- + \beta_-} < 0,$$

which is $\frac{\alpha_+ - \alpha_-}{2}$ when $\alpha_+ = \beta_+$ and $\alpha_- = \beta_-$. On the other hand, $\vartheta(\zeta)$ in (1.9) takes the minimum $-\frac{1}{2}$ at $\zeta = \frac{1}{3}$, so the minimum of (1.9) is

$$\lim_{d\to\infty} \frac{1}{N(4d) + \lfloor\frac{2^{4d}n_0}{3}\rfloor} \log \left\| Df^{N(4d)+\lfloor\frac{2^{4d}n_0}{3}\rfloor}(z)v \right\| = -\frac{1}{2}\log\sigma = \frac{\log\sigma + \log\sigma^{-2}}{2}.$$

Remark. We emphasize that the choice of $(n_k^0)_{k\in\mathbb{N}}$ in (1.10) is totally free except the condition $\limsup_{k\to\infty} n_{k+1}^0/n_k^0 < \infty$, while $(n(d))_{d\in\mathbb{N}}$ in (1.6) must satisfy $\lim_{d\to\infty} n(d+1)/n(d) = 2$. This freedom makes the construction of the oscillation of (partial) Lyapunov exponents of f a bit simpler. Indeed, in the proof of Theorem A we take $(n_k)_{k\in\mathbb{N}}$ as

$$\lim_{p\to\infty} \frac{n_{2p+1}}{n_{2p}} < \lim_{p\to\infty} \frac{n_{2p}}{n_{2p-1}} < \infty,$$

which enables us to conclude that for any z in an open subset of R_κ with some large integer κ and any vector v in an open set,

$$\lim_{p\to\infty} \frac{1}{N_{2p-1}} \log \left\| Df^{N_{2p-1}}(z)v \right\| = \frac{\log\lambda + \alpha\log\sigma}{1+\alpha} < \lim_{p\to\infty} \frac{1}{N_{2p}} \log \left\| Df^{N_{2p}}(z)v \right\| = \frac{\log\lambda + \beta\log\sigma}{1+\beta},$$

where $\alpha = \lim_{p\to\infty} n_{2p+1}/n_{2p}$, $\beta = \lim_{p\to\infty} n_{2p}/n_{2p-1}$ and $N_j = (n_\kappa+2)+(n_{\kappa+1}+2)+\cdots+(n_{\kappa+j}+2)$ (so the "time at closest approach" $N(4d)+\lfloor\zeta 2^{4d} n_0\rfloor$ with $\zeta\in(0,1)$ for Guarino–Guihéneuf–Santiago's figure-8 attractor is not necessary).

Remark. We outline why the open set V_f in Theorem A is not easy to be replaced by $\mathbb{R}^2\setminus\{0\}$ by our argument. Again, Guarino–Guihéneuf–Santiago's figure-8 attractor might be useful to understand the situation. Let v be the unit vertical vector. Then, it follows from (1.7) that $\left\|Df^{N(2d)}(z)v\right\| = \sigma^{(-2^{2d}+1)n_0}$, which is much smaller than the lower bound $1-\sigma^{(-2^{2d}+1)n_0}$ of $\left\|Df^{N(2d)}(z)v'\right\|$ for any non-zero vector v' being not parallel to v, and thus (1.8) does not hold for this v (see (3.1) for details). For the diffeomorphism of Theorem A, this special situation on the vertical line may be spread to a vertical cone $\mathcal{K}_v := \{(v_1,v_2)\in\mathbb{R}^2 \mid |v_1|\le K^{-1}|v_2|\}$ with a constant $K>1$ (see (4.9)) and it is hard to repeat the above calculation on the cone due to the higher order term $-2\sigma^{2n_k}x$ of (1.11). A similar difficulty occurs on a horizontal cone $\mathcal{K}_h := \{(v_1,v_2)\in\mathbb{R}^2 \mid |v_2|\le K^{-1}|v_1|\}$, and the open set V_f of Theorem A is given as $\mathbb{R}\setminus(K_v\cup K_h)$.

2. Proof of Proposition 1.1

We follow the argument [17] for the figure-8 attractor,[1] so the reader familiar with this subject can skip this section. Let $(f^t)_{t\in\mathbb{R}}$ be the Bowen flow given in Sect. 1.1.2. Let N and $\hat{N}$ be neighborhoods of p and $\hat{p}$, respectively, such that there are linearizing coordinates $\phi : N\to\mathbb{R}^2$ and $\hat{\phi} : \hat{N}\to\mathbb{R}^2$ satisfying that both $\phi(N)$ and $\hat{\phi}(\hat{N})$ include $(0,1]^2$ and

$$\phi\circ f^t\circ\phi^{-1}(r,s) = (e^{-\alpha_- t}r, e^{\alpha_+ t}s), \quad \hat{\phi}\circ f^t\circ\hat{\phi}^{-1}(r,s) = (e^{-\beta_- t}r, e^{\beta_+ t}s) \tag{2.1}$$

on $(0,1]^2$. Fix $(x,y)\in U$. Let $\hat{T}_0$ be the hitting time of (x,y) to $\{\phi^{-1}(1,s) \mid s\in(0,1]\}$, i.e. the smallest positive number t such that $f^t(x,y) = \phi^{-1}(1,s)$ with some $s\in(0,1]$. Let s_1 be the second component of $\phi\circ f^{\hat{T}_0}(x,y)$. We inductively define sequences $(t_n, T_n, \hat{t}_n, \hat{T}_n)_{n\in\mathbb{N}}$, $(s_n, r_n, \hat{s}_n, \hat{r}_n)_{n\in\mathbb{N}}$ of positive numbers as

- t_n is the hitting time of $\phi^{-1}(1,s_n)$ to $\{\phi^{-1}(r,1) \mid r\in(0,1]\}$, and r_n is the first component of $\phi\circ f^{t_n}\circ\phi^{-1}(1,s_n)$,
- T_n is the hitting time of $\phi^{-1}(r_n,1)$ to $\{\hat{\phi}^{-1}(1,s) \mid s\in(0,1]\}$, and $\hat{s}_n$ is the second component of $\hat{\phi}\circ f^{T_n}\circ\phi^{-1}(r_n,1)$,
- $\hat{t}_n$ is the hitting time of $\hat{\phi}^{-1}(1,\hat{s}_n)$ to $\{\hat{\phi}^{-1}(r,1) \mid r\in(0,1]\}$, and $\hat{r}_n$ is the first component of $\hat{\phi}\circ f^{t_n}\circ\hat{\phi}^{-1}(1,\hat{s}_n)$,

[1] They implicitly ignored the higher order terms of the transient map of the flow, i.e. assumed that $\hat{s}_n = c r_n$ and $s_{n+1} = \hat{c}\hat{r}_n$ instead of (2.3) below.

- $\hat{T}_n$ is the hitting time of $\hat{\phi}^{-1}(\hat{r}_n, 1)$ to $\{\phi^{-1}(1, s) \mid s \in (0, 1]\}$, and s_{n+1} is the second component of $\phi \circ f^{T_n} \circ \hat{\phi}^{-1}(\hat{r}_n, 1)$.

Then, from

$$(e^{-\alpha_- t_n}, e^{\alpha_+ t_n} s_n) = (r_n, 1), \quad (e^{-\beta_- \hat{t}_n}, e^{\beta_+ \hat{t}_n} \hat{s}_n) = (\hat{r}_n, 1)$$

it follows that

$$t_n = -\frac{\log s_n}{\alpha_+}, \quad r_n = s_n^a, \quad \hat{t}_n = -\frac{\log \hat{s}_n}{\beta_+}, \quad \hat{r}_n = \hat{s}_n^b. \tag{2.2}$$

with $a := \frac{\alpha_-}{\alpha_+}$ and $b := \frac{\beta_-}{\beta_+}$. On the other hand, it is straightforward to see that both T_n and $\hat{T}_n$ are bounded from above and below uniformly with respect to n, and thus, since the vector field V is of class $\mathcal{C}^{1+\alpha}$, one can find positive numbers c and $\hat{c}$ (which are independent of n) such that

$$\hat{s}_n = c r_n + o(r_n^{1+\alpha}), \quad s_{n+1} = \hat{c}\hat{r}_n + o(\hat{r}_n^{1+\alpha}). \tag{2.3}$$

Moreover, we set

$$\tau_n := \hat{T}_0 + \sum_{k=1}^{n-1}(t_k + T_k + \hat{t}_k + \hat{T}_k), \quad \hat{\tau}_n := \hat{T}_0 + \sum_{k=1}^{n-1}(t_k + T_k + \hat{t}_k + \hat{T}_k) + t_n + T_n,$$

that is, the n-th return time to N and $\hat{N}$, respectively. Notice that $Df^t(x, y)V(x, y) = V(f^t(x, y))$ for each $t \geq 0$. Hence, we have

$$\lim_{n\to\infty} \frac{1}{\tau_n} \log \|Df^{\tau_n}(x, y)V(x, y)\| = \lim_{n\to\infty} \frac{1}{\tau_n} \log \|V(1, s_n)\| = 0$$

because $\|V(1, s)\|$ is bounded from above and below uniformly with respect to $s \in (0, 1]$.

From now on, we identify $\phi(x, y)$ and $\hat{\phi}(x, y)$ with (x, y) if it makes no confusion. We further define a sequence $(\rho_n)_{n\in\mathbb{N}}$ of positive numbers as ρ_n is the minimizer of

$$\|f^t(1, s_n) - p\|^2 = e^{-2\alpha_- t} + e^{2\alpha_+ t} s_n^2$$

(under the linearizing coordinate ϕ) over $0 \leq t \leq t_n$, that is, the time at which $f^t(1, s_n)$ makes the closest approach to p over $0 \leq t \leq t_n$. Then, it follows from a straightforward calculation that

$$\rho_n = -\frac{\log s_n}{\alpha_+ + \alpha_-} + C_1, \quad \|L(f^{\rho_n}(1, s_n))\| = C_1' s_n^{\alpha_-/(\alpha_+ + \alpha_-)}, \tag{2.4}$$

where $C_1 := \frac{\log \alpha_- - \log \alpha_+}{2(\alpha_+ + \alpha_-)}$, $C_1' := \sqrt{\alpha_-^2 e^{-2\alpha_- C_1} + \alpha_+^2 e^{2\alpha_+ C_1}}$ and L is the linearized vector sub-field of V around p corresponding to (2.1), i.e $L(x, y) = (-\alpha_- x, \alpha_+ y)$. We show that

$$\limsup_{n\to\infty} \frac{1}{\tau_n + \rho_n} \log \|Df^{\tau_n + \rho_n}(x, y)V(x, y)\| \leq \frac{\alpha_+\beta_+ - \alpha_-\beta_-}{\alpha_+ + \beta_+ + \alpha_- + \beta_-}. \tag{2.5}$$

Fix $\epsilon > 0$. Then, it follows from (2.3) that one can find n_0 such that

$$c_- r_n \leq \hat{s}_n \leq c_+ r_n, \quad \hat{c}_- \hat{r}_n \leq s_{n+1} \leq \hat{c}_+ \hat{r}_n$$

for any $n \geq n_0$, where $c_\pm = (1 \pm \epsilon)c$ and $\hat{c}_\pm = (1 \pm \epsilon)\hat{c}$. Therefore, by induction, together with (2.2), it is straightforward to see that

$$c_-^{b\Lambda_n} \hat{c}_-^{\Lambda_n} s_{n_0}^{(ab)^n} \leq s_{n_0+n} \leq (c_+^b)^{\Lambda_n} \hat{c}_+^{\Lambda_n} s_{n_0}^{(ab)^n},$$
$$c_- \left(c_-^{b\Lambda_n} \hat{c}_-^{\Lambda_n} s_{n_0}^{(ab)^n} \right)^a \leq \hat{s}_{n_0+n} \leq c_+ \left(c_+^{b\Lambda_n} \hat{c}_+^{\Lambda_n} s_{n_0}^{(ab)^n} \right)^a$$

for any $n \geq 0$, where $\Lambda_n = 1 + ab + \cdots + (ab)^{n-1} = \frac{(ab)^n - 1}{ab-1}$. Fix $n \geq n_0$ and write $N := n_0 + n$ to avoid heavy notations. Then, it holds that

$$(ab)^n \log \left(s_{n_0} C_- \right) - C_2 \leq \log s_N \leq (ab)^n \log \left(s_{n_0} C_+ \right) + C_2,$$
$$a(ab)^n \log \left(s_{n_0} C_- \right) - C_2 \leq \log \hat{s}_N \leq a(ab)^n \log \left(s_{n_0} C_+ \right) + C_2$$

with some constant $C_2 > 0$, where $C_\pm := c_\pm^{b/(ab-1)} \hat{c}_\pm^{1/(ab-1)}$. Thus, by (2.2) we have

$$\tau_N \geq \sum_{k=1}^{n-1} \left(-\frac{1}{\alpha_+} - \frac{a}{\beta_+} \right) (ab)^k \log \left(s_{n_0} C_+ \right) + C_{n_0} + nC_3$$
$$= -\frac{\alpha_- + \beta_+}{\alpha_- \beta_- - \alpha_+ \beta_+} (ab)^n \log \left(s_{n_0} C_+ \right) + C'_{n_0} + nC_3$$

with some constants C_{n_0}, C'_{n_0} and C_3. Furthermore, it follows from (2.4) that

$$\rho_N \geq -\frac{(ab)^n \log \left(s_{n_0} C_+ \right)}{\alpha_+ + \alpha_-} + C'_3,$$

so that

$$\tau_N + \rho_N \geq C''_{n_0} + nC_3 + \frac{\alpha_-(\alpha_+ + \beta_+ + \alpha_- + \beta_-)}{(\alpha_+ \beta_+ - \alpha_- \beta_-)(\alpha_+ + \alpha_-)} (ab)^n \log \left(s_{n_0} C_+ \right)$$

with some constants C'_3, C''_{n_0}. On the other hand, by (2.4) it holds that

$$\log \| V(f^{\tau_N + \rho_N}(x, y)) \| = \log \| L(f^{\rho_N(1, s_N)}) \| \leq \frac{\alpha_-}{(\alpha_+ + \alpha_-)} (ab)^n \log \left(s_{n_0} C_- \right) + C'_3$$

with some constants C_3, C'_3. Therefore,

$$\limsup_{n \to \infty} \frac{1}{\tau_n + \rho_n} \log \| Df^{\tau_n + \rho_n}(x, y) V(x, y) \| \leq \frac{\alpha_+ \beta_+ - \alpha_- \beta_-}{\alpha_+ + \beta_+ + \alpha_- + \beta_-} \cdot \frac{\log(s_{n_0} C_-)}{\log(s_{n_0} C_+)}.$$

Since ϵ is arbitrary, we get (2.5) (notice that $\frac{\log(s_{n_0} C_-)}{\log(s_{n_0} C_+)}$ converges to 1 from below as ϵ goes to zero). In a similar manner, one can show that

$$\liminf_{n \to \infty} \frac{1}{\tau_n + \rho_n} \log \| Df^{\tau_n + \rho_n}(x, y) V(x, y) \| \geq \frac{\alpha_+ \beta_+ - \alpha_- \beta_-}{\alpha_+ + \beta_+ + \alpha_- + \beta_-},$$

and we complete the proof of Proposition 1.1. □

3. Proof of Theorem 1.3

Let f be the Guarino–Guihéneuf–Santiago diffeomorphism of Proposition 1.2 and $N(d)$ the d-th return time given in (1.6). Fix $z \in S_{n_0}$. By induction with respect to d we first show (1.7). It immediately follows from (1.4) that the first equality of (1.7) is true for $d = 1$. Then let us assume that the first equality of (1.7) is true for a given positive integer d. Since $N(2(d+1)-1) = N(2d+1) = N(2d-1) + n(2d) + n(2d+1)$, by the chain rule and the inductive hypothesis,

$$
\begin{aligned}
Df^{N(2(d+1)-1)}(z) &= Df^{n(2d+1)}(f^{N(2d)}(z))Df^{n(2d)}(f^{N(2d-1)}(z))Df^{N(2d-1)}(z)\\
&= \begin{pmatrix} 0 & -\sigma^{2^{2d}n_0} \\ \sigma^{-2^{2d+1}n_0} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sigma^{2^{2d-1}n_0} \\ \sigma^{-2^{2d}n_0} & 0 \end{pmatrix}\\
&\quad \times (-1)^{d-1}\begin{pmatrix} 0 & -\sigma^{n_0} \\ \sigma^{-2^{2d-1}n_0} & 0 \end{pmatrix}\\
&= (-1)^{d-1}\begin{pmatrix} 0 & \sigma^{n_0} \\ -\sigma^{-2^{2d+1}n_0} & 0 \end{pmatrix} = (-1)^{d}\begin{pmatrix} 0 & -\sigma^{n_0} \\ \sigma^{-2^{2d+1}n_0} & 0 \end{pmatrix}.
\end{aligned}
$$

That is, the first equality of (1.7) holds for $d+1$. In a similar manner, by induction with respect to d, we can prove the second equality of (1.7).

We next prove that z is Lyapunov irregular for any nonzero horizontal vector $v = \begin{pmatrix} s \\ 0 \end{pmatrix}$. By the first equality of (1.7), we obtain

$$
\frac{\log \left\| Df^{N(2d-1)}(z)v \right\|}{N(2d-1)} = \frac{-2^{2d-1}n_0 \log \sigma + \log |s|}{(2^{2d-1}-1)n_0 + (2d-1)k_0} \xrightarrow[d\to\infty]{} -\log \sigma. \tag{3.1}
$$

On the other hand, it follows from the second equality of (1.7) that

$$
\frac{\log \left\| Df^{N(2d)}(z)v \right\|}{N(2d)} = \frac{\log |s|}{(2^{2d}-1)n_0 + (2d)k_0} \xrightarrow[d\to\infty]{} 0.
$$

In a similar manner, we can show that z is Lyapunov irregular for any nonzero vertical vector.

Finally, we will prove (1.8) and (1.9), which immediately implies that z is Lyapunov irregular for any nonzero vector $v \notin \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cup \mathbb{R}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. For simplicity, we assume that $\zeta 2^{4d} n_0$ is an integer. Essentially, the proof of (1.8) is included in the discussion until now. Thus, we show (1.9). By (1.7) and the item (a) of Proposition 1.2,

$$
\begin{aligned}
Df^{N(4d)+\zeta 2^{4d}n_0}(z) &= \begin{pmatrix} \sigma^{-2\zeta\cdot 2^{4d}n_0} & 0 \\ 0 & -\sigma^{-(1-\zeta)2^{4d}n_0+n_0} \end{pmatrix}\\
&= \sigma^{-(1-\zeta)2^{4d}n_0}\begin{pmatrix} \sigma^{(1-3\zeta)2^{4d}n_0} & 0 \\ 0 & -\sigma^{n_0} \end{pmatrix}.
\end{aligned}
$$

Fix a vector $v = \begin{pmatrix} s \\ u \end{pmatrix}$ with $su \neq 0$. If $1 - 3\zeta \leq 0$, then

$$
\lim_{d\to\infty} \left\| \begin{pmatrix} \sigma^{(1-3\zeta)2^{4d}n_0} & 0 \\ 0 & -\sigma^{n_0} \end{pmatrix} v \right\| = 1.
$$

Hence, since $N(4d) + \zeta 2^{4d} n_0 = (1+\zeta)2^{4d} n_0 + (4dk_0 - n_0)$, we get

$$\lim_{d\to\infty} \frac{1}{N(4d)+\zeta 2^{4d} n_0} \log \left\| Df^{N(4d)+\zeta 2^{4d} n_0}(z)v \right\| = -\frac{1-\zeta}{1+\zeta} \log \sigma.$$

On the other hand, if $1 - 3\zeta > 0$, then

$$\lim_{d\to\infty} \left\| \begin{pmatrix} \sigma^{(1-3\zeta)2^{4d} n_0} & 0 \\ 0 & -\sigma^{n_0} \end{pmatrix} v \right\| \cdot \sigma^{-(1-3\zeta)2^{4d} n_0} = 1.$$

Thus we get

$$\begin{aligned} \lim_{d\to\infty} \frac{1}{N(4d)+\zeta 2^{4d} n_0} \log \left\| Df^{N(4d)+\zeta 2^{4d} n_0}(z)v \right\| &= \frac{-(1-\zeta)+(1-3\zeta)}{1+\zeta} \log \sigma \\ &= -\frac{2\zeta}{1+\zeta} \log \sigma. \end{aligned}$$

This completes the proof of Theorem 1.3. □

4. Proof of Theorem A

In this section, we give the proof of Theorem A. In Sect. 4.1 we briefly recall a small perturbation of a diffeomorphism with a robust homoclinic tangency introduced by Colli and Vargas [5]. In Sect. 4.2 we establish key lemmas to control the higher order term in (1.5), and prove the positivity of Lebesgue measure of Lyapunov irregular sets in Sect. 4.3. Finally, in Sect. 4.4, we discuss the Birkhoff (ir)regularity of the set.

4.1. Dynamics. Let us start the proof of Theorem A by remembering the Colli–Vargas model with a robust homoclinic tangency introduced in [5]. The reader familiar with this subject can skip this section. Let M be a closed surface including $[-2, 2]^2$, and a diffeomorphism $g \equiv g_\mu : M \to M$ with a real number μ satisfying the following.

(1) (Affine horseshoe) There exist constants $0 < \lambda < \frac{1}{2}$ and $\sigma > 2$ such that

$$g(x, y) = \left(\pm\sigma\left(x \pm \frac{1}{2}\right), \pm\lambda y \mp \frac{1}{2}\right) \quad \text{if } \left|x \pm \frac{1}{2}\right| \le \frac{1}{\sigma}, |y| \le 1$$

and $\lambda\sigma^2 < 1$;

(2) (Quadratic tangency) For any (x, y) near a small neighborhood of $(0, -1)$,

$$g^2(x, y) = (\mu - x^2 - y, x).$$

Then, it was proven by Newhouse [15] that there is a μ such that g has a $\mathcal{C}^2$-robust homoclinic tangency on $\{y = 0\}$. See Fig. 2.

Colli and Vargas showed the following.

Theorem 4.1. ([5]) *Let g be the surface diffeomorphism with a robust homoclinic tangency given above. Then, for any $\mathcal{C}^r$-neighborhood $\mathcal{O}$ of g $(2 \le r < \infty)$ and any increasing sequence $(n_k^0)_{k\in\mathbb{N}}$ of integers satisfying $n_k^0 = O((1+\eta)^k)$ with some $\eta > 0$, one can find a diffeomorphism f in $\mathcal{O}$ together with a sequence of rectangles $(R_k)_{k\in\mathbb{N}}$ and an increasing sequence $(\tilde{n}_k)_{k\in\mathbb{N}}$ of integers, satisfying that $\tilde{n}_k = O(k)$ and depends only on $\mathcal{O}$, such that the following holds for each $k \in \mathbb{N}$ with $n_k := n_k^0 + \tilde{n}_k$:*

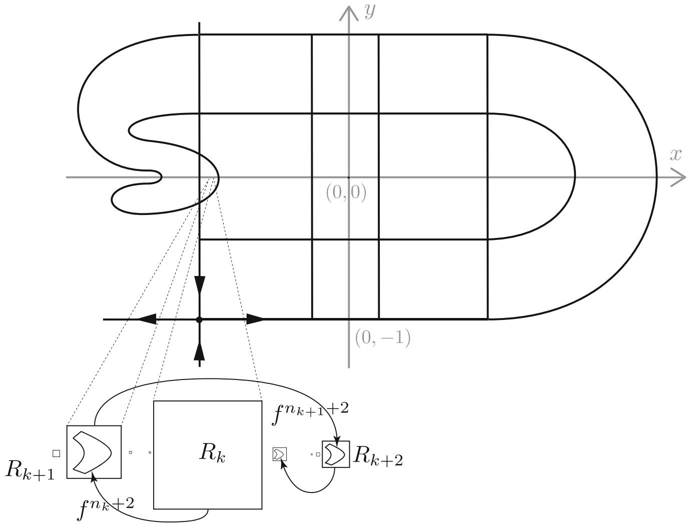

Fig. 2. Colli–Vargas' diffeomorphism

(a) $f^{n_k+2}(R_k) \subset R_{k+1}$;
(b) *For each* $(\tilde{x}_k + x, y) \in R_k$,

$$f^{n_k+2}(\tilde{x}_k + x, y) = (\tilde{x}_{k+1} - \sigma^{2n_k} x^2 \mp \lambda^{n_k} y, \pm\sigma^{n_k} x),$$

where $(\tilde{x}_k, 0)$ *is the center of* R_k.

Refer to the "Conclusion" given in p. 1674 and the "Rectangle lemma" and its proof given in pp. 1975–1976 of the paper [5], where the notation R_k was used to denote a slightly different object that we will not use, and our R_k was written as R_k^*. See Remark 4.2 and Theorem 4.8 for more information.

By the coordinate translation $T_k : (x, y) \mapsto (x - \tilde{x}_k, y)$, which sends $(\tilde{x}_k, 0)$ to $(0, 0)$, the action of $f^{n_k+2}|_{R_k}$ can be rewritten as

$$F_k : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\sigma^{2n_k} x^2 \mp \lambda^{n_k} y \\ \pm\sigma^{n_k} x \end{pmatrix}, \tag{4.1}$$

which sends $(0, 0)$ to $(0, 0)$, that is,

$$f^{n_k+2}(x, y) = T_{k+1}^{-1} \circ F_k \circ T_k(x, y) \quad \text{for every } (x, y) \in R_k.$$

Note that for each $l \geq k$,

$$f^{n_l+2} \circ f^{n_{l-1}+2} \circ \cdots \circ f^{n_k+2} = T_{l+1}^{-1} \circ (F_l \circ F_{l-1} \circ \cdots \circ F_k) \circ T_k,$$

so the oscillation of $(\frac{1}{n} \log \|Df^n(\boldsymbol{x})\boldsymbol{v}\|)_{n\in\mathbb{N}}$ for each $\boldsymbol{x} \in R_k$ with some k and each nonzero vectors $\boldsymbol{v}$ in an open set follows from the oscillation of

$$\left(\frac{1}{(n_k + 2) + \cdots + (n_{l-1} + 2) + (n_l + 2)} \log \|D (F_l \circ F_{l-1} \circ \cdots \circ F_k) (\boldsymbol{x})\boldsymbol{v}\| \right)_{l\in\mathbb{N}}$$

for each $\boldsymbol{x} \in T_k(R_k)$ and each nonzero vectors $\boldsymbol{v}$ in the open set, which we will show in the following.

4.2. Key lemmas. First, let us fix some constants in advance. Fix a small neighborhood $\mathcal{O}$ of g, and let $(\tilde{n}_k)_{k\in\mathbb{N}}$ be the sequece given in Theorem 4.1. Notice that $\lambda\sigma < \lambda\sigma^2 < 1$. Take a sufficiently small $\eta > 0$ and a sufficiently large integer $n_0 \geq 2$ so that

$$\lambda\sigma^{\frac{1+3\eta+8n_0^{-1}}{1-\eta}} < 1,$$

and fix $1 < \alpha < \beta < 1+\eta$ such that

$$\lambda\sigma^{\frac{6\beta-4+8n_0^{-1}}{2-\beta}} < 1, \quad \alpha^2\beta^2 < 2 \quad \text{and} \quad \lambda\sigma^{\alpha} < 1. \tag{4.2}$$

Let $(n_k^0)_{k\in\mathbb{N}}$ be an increasing sequence of integers given by

$$n_{2p}^0 = \lfloor n_0\alpha^p\beta^p\rfloor - \tilde{n}_{2p}, \quad n_{2p+1}^0 = \lfloor n_0\alpha^{p+1}\beta^p\rfloor - \tilde{n}_{2p+1}, \tag{4.3}$$

which are natural numbers for each p by increasing n_0 if necessary. Since $x - 1 < \lfloor x\rfloor \leq x$ and $\tilde{n}_k = O(k)$, by increasing n_0 if necessary, we have

$$\frac{n_{2p+1}^0}{n_{2p}^0} < \frac{n_0\alpha^{p+1}\beta^p - \tilde{n}_{2p+1}}{n_0\alpha^p\beta^p - 1 - \tilde{n}_{2p}} < \alpha + \frac{\alpha(1+\tilde{n}_{2p})}{n_0\alpha^p\beta^p - 1 - \tilde{n}_{2p}} < 1+\eta,$$

$$\frac{n_{2p+2}}{n_{2p+1}} < \frac{n_0\alpha^{p+1}\beta^{p+1} - \tilde{n}_{2p+2}}{n_0\alpha^{p+1}\beta^p - 1 - \tilde{n}_{2p+1}} = \beta + \frac{\beta(1+\tilde{n}_{2p+1})}{n_0\alpha^{p+1}\beta^p - 1 - \tilde{n}_{2p+1}} < 1+\eta,$$

so it holds that $n_k^0 = O((1+\eta)^k)$, which is the only requirement to apply Theorem 4.1. Set $n_k = n_k^0 + \tilde{n}_k$, then we obviously have

$$n_{2p} = \lfloor n_0\alpha^p\beta^p\rfloor, \quad n_{2p+1} = \lfloor n_0\alpha^{p+1}\beta^p\rfloor.$$

Define sequences $(b_k)_{k\in\mathbb{N}}$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ of positive numbers by

$$b_k = \sigma^{-\sum_{i=-1}^{+\infty}\frac{n_{k+1+i}}{2^i}}$$

and

$$\varepsilon_k = \left(\lambda\sigma^{\frac{6\beta-4+8n_k^{-1}}{2-\beta}}\right)^{n_k}.$$

Remark 4.2. Define $\tilde{b}_k$ by

$$\tilde{b}_k = \sigma^{-\sum_{i=0}^{+\infty}\frac{n_{k+1+i}}{2^i}},$$

then R_k of Theorem 4.1 is of the form

$$R_k = \left[\tilde{x}_k - c_k\tilde{b}_k, \tilde{x}_k + c_k\tilde{b}_k\right] \times \left[-20\tilde{b}_k^{\frac{1}{2}}, 20\tilde{b}_k^{\frac{1}{2}}\right]$$

with some constant c_k satisfying that

$$\frac{1}{2} \leq c_k \leq 10,$$

see the “Rectangle lemma” and its proof given in pp. 1975-1976 of [5] (as previously mentioned, in the paper our R_k is written as R_k^* and the notations R_k is used for another object). Note that $b_k < \tilde{b}_k$. Thus, F_k in (4.1) is well-defined on any rectangle of the form

$$[-cb_k, cb_k] \times \left[-c\sqrt{b_k}, c\sqrt{b_k}\right] \quad \text{with } 0 < c \le \frac{1}{2}.$$

In the paper [5] the notation b_k was used to denote $\tilde{b}_k$, but this positive number is not explicitly used in the following argument, so we defined b_k as above for notational simplicity.

By the construction of $(n_k)_{k\in\mathbb{N}}$, we have that $n_l/(n_k+1) < \beta^{l-k}$ for each $k \le l$. Hence, since n_k is increasing,

$$\begin{aligned} 4n_k &< 2n_k + n_{k+1} + \frac{n_{k+2}}{2} + \cdots \\ &< 2(n_k+1)\left(1 + \frac{\beta}{2} + \frac{\beta^2}{2^2} + \cdots\right) = \frac{4(n_k+1)}{2-\beta}. \end{aligned}$$

Therefore we have

$$\sigma^{-\frac{4(n_k+1)}{2-\beta}} < b_k < \sigma^{-4n_k} \quad \text{for each } k \in \mathbb{N} \tag{4.4}$$

and

$$b_{k+1} > \sigma^{-\frac{4(n_{k+1}+1)}{2-\beta}} > \begin{cases} \sigma^{-\frac{4(\alpha n_k+2)}{2-\beta}} & \text{if } k \text{ is even,} \\ \sigma^{-\frac{4(\beta n_k+2)}{2-\beta}} & \text{if } k \text{ is odd.} \end{cases} \tag{4.5}$$

Furthermore, it follows from (4.2) that ε_k can be arbitrarily small by taking k sufficiently large, so there exists a positive integer k_0 such that for any $k \ge k_0$ and $p \ge 0$, we get

$$2\alpha^p\beta^p - n_k^{-1} + \frac{\log 2}{\log \varepsilon_k} > \alpha^{p+2}\beta^{p+2}.$$

Fix such a k_0. Then it immediately holds that for any $k \ge k_0$ and $p \ge 0$,

$$\varepsilon_k^{2\alpha^p\beta^p} < \varepsilon_k^{2\alpha^p\beta^p - n_k^{-1}} < \frac{1}{2}\varepsilon_k^{\alpha^{p+2}\beta^{p+2}} < \frac{1}{2}\varepsilon_k^{\alpha^{p+1}\beta^{p+1}}. \tag{4.6}$$

In the following lemmas, we only consider the case when k is an even number because it is enough to prove Theorem A and makes the statements a bit simpler, but similar estimates hold even when k is an odd number. We first show the following.

Lemma 4.3. *For every even number $k \ge k_0$, $p \in \mathbb{N} \cup \{0\}$ and $j \in \{0, 1\}$,*

$$\lambda^{n_{k+2p+j}}\sqrt{b_{k+2p+j}} \le \varepsilon_k^{\alpha^{p+j}\beta^p - n_k^{-1}} b_{k+2p+1+j}.$$

Proof. Fix an even number $k \geq k_0$. We will prove this lemma by induction with respect to p. For the case $p = 0$, it follows from (4.2), (4.4) and (4.5) that

$$\lambda^{n_k}\sqrt{b_k} \leq \lambda^{n_k}\sigma^{-2n_k} \leq \left(\lambda\sigma^{\frac{6\beta-4+8n_k^{-1}}{2-\beta}}\right)^{n_k}\sigma^{-\frac{4(\alpha n_k+2)}{2-\beta}} < \varepsilon_k^{1-n_k^{-1}} \cdot b_{k+1},$$

and since $\frac{n_{k+1}}{n_k} \geq \frac{n_0\alpha^{k/2+1}\beta^{k/2-1}}{n_0\alpha^{k/2}\beta^{k/2}} \geq \alpha - \frac{1}{n_k}$ by the construction of n_k,

$$\begin{aligned}\lambda^{n_{k+1}}\sqrt{b_{k+1}} &\leq \lambda^{n_{k+1}}\sigma^{-2n_{k+1}} \\ &\leq \left(\lambda\sigma^{\frac{6\beta-4+8n_{k+1}^{-1}}{2-\beta}}\right)^{n_k\cdot\frac{n_{k+1}}{n_k}}\sigma^{-\frac{4(\beta n_{k+1}+2)}{2-\beta}} \leq \varepsilon_k^{\alpha-n_k^{-1}} \cdot b_{k+2}.\end{aligned}$$

Next we assume that the assertion of Lemma 4.3 is true for a given $p \in \mathbb{N} \cup \{0\}$. Then we have

$$\begin{aligned}\lambda^{n_{k+2p+2}}\sqrt{b_{n+2p+2}} &\leq \lambda^{n_{k+2p+2}}\sigma^{-2n_{k+2p+2}} \\ &\leq \left(\lambda\sigma^{\frac{6\beta-4+8n_{k+2p+2}^{-1}}{2-\beta}}\right)^{n_k\cdot\frac{n_{k+2p+2}}{n_k}}\sigma^{-\frac{4(\alpha n_{k+2p+2}+2)}{2-\beta}} \\ &< \varepsilon_k^{\alpha^{p+1}\beta^{p+1}-n_k^{-1}} \cdot b_{k+2p+3}\end{aligned}$$

and

$$\begin{aligned}\lambda^{n_{k+2p+3}}\sqrt{b_{n+2p+3}} &\leq \lambda^{n_{k+2p+3}}\sigma^{-2n_{k+2p+3}} \\ &\leq \left(\lambda\sigma^{\frac{6\beta-4+8n_{k+2p+3}^{-1}}{2-\beta}}\right)^{n_k\cdot\frac{n_{k+2p+3}}{n_k}}\sigma^{-\frac{4(\beta n_{k+2p+3}+2)}{2-\beta}} \\ &< \varepsilon_k^{\alpha^{p+2}\beta^{p+1}-n_k^{-1}} \cdot b_{k+2p+4}.\end{aligned}$$

That is, the assertion of Lemma 4.3 with $p+1$ instead of p is also true. This completes the induction and the proof of Lemma 4.3. □

Define a sequence $(U_{k,m})_{m\geq 0}$ of rectangles with $k \geq k_0$ by

$$U_{k,m} = \left\{(x, y) : \ |x| \leq \varepsilon_k^{(\alpha\beta)^{\lfloor\frac{m}{2}\rfloor}} b_{k+m}, \ |y| \leq \varepsilon_k^{(\alpha\beta)^{\lfloor\frac{m}{2}\rfloor}} \sqrt{b_{k+m}}\right\} \tag{4.7}$$

for each integer $m \geq 0$. Then, by Remark 4.2, $U_{k,m}$ is included in R_{k+m} for any large m (under the translation of $(\tilde{x}_{k+m}, 0)$ to $(0, 0)$), on which F_{k+m} in (4.1) is well-defined. Then we have the following.

Lemma 4.4. *For any even number $k \geq k_0$, $m \in \mathbb{N} \cup \{0\}$ and $\boldsymbol{x} \in U_{k,0}$,*

$$F_{k+m-1} \circ F_{k+m-2} \circ \cdots \circ F_k(\boldsymbol{x}) \in U_{k,m}.$$

Proof. Fix an even number $k \geq k_0$, an integer $m \geq 0$ and $\boldsymbol{x} \in U_{k,0}$, and set $\boldsymbol{x}_{k,m} := F_{k+m-1} \circ F_{k+m-2} \circ \cdots \circ F_k(\boldsymbol{x})$, where $\boldsymbol{x}_{k,0}$ is interpreted as $\boldsymbol{x}$ so that $\boldsymbol{x}_{k,0} \in U_{k,0}$. Denote the first and second coordinate of $\boldsymbol{x}_{k,m}$ by $x_{k,m}$ and $y_{k,m}$, respectively. We will show that $(x_{k,m}, y_{k,m}) \in U_{k,m}$ by induction with respect to $m \in \mathbb{N} \cup \{0\}$.

We first show that $(x_{k,m}, y_{k,m}) \in U_{k,m}$ for $m = 1$. It holds that

$$|x_{k,1}| = |-\sigma^{2n_k}x_k^2 \mp \lambda^{n_k}y_k| \leq \sigma^{4n_k}\varepsilon_k^2 b_k^2 + \lambda^{n_k}\varepsilon_k\sqrt{b_k} \leq \varepsilon_k^2 b_{k+1} + \varepsilon_k^{2-n_k^{-1}} b_{k+1}.$$

In the last inequality, the first term is due to the equality $\sigma^{4n_k} b_k^2 = b_{k+1}$ implied by the definition of b_k, and the second term comes from Lemma 4.3. Hence, it follows from (4.6) that

$$|x_{k,1}| \leq \frac{1}{2}\varepsilon_k^{\alpha\beta} b_{k+1} + \frac{1}{2}\varepsilon_k^{\alpha\beta} b_{k+1} = \varepsilon_k^{\alpha\beta} b_{k+1} \leq \varepsilon_k b_{k+1}$$

and

$$|y_{k,1}| = |\sigma^{n_k} x_k| \leq \sigma^{2n_k}\varepsilon_k b_k = \varepsilon_k \sqrt{b_{k+1}},$$

which concludes that $(x_{k,1}, y_{k,1}) \in U_{k+1}$.

Next we assume that $(x_{k,m}, y_{k,m}) \in U_{k,m}$ for $m = 2p$ and $2p+1$ with a given integer $p \geq 0$. In addition, we assume (as an inductive hypothesis) that

$$|x_{k,2p+1}| \leq \varepsilon_k^{(\alpha\beta)^{p+1}} b_{k+2p+1},$$

which indeed holds in the case when $p = 0$ as seen above. Then it holds that

$$\begin{aligned}
|x_{k,2p+2}| &= |-\sigma^{2n_{k+2p+1}} x_{k,2p+1}^2 \mp \lambda^{n_{k+2p+1}} y_{k,2p+1}| \\
&\leq \sigma^{4n_{k+2p+1}} \varepsilon_k^{2(\alpha\beta)^p} b_{k+2p+1}^2 + \lambda^{n_{k+2p+1}} \varepsilon_k^{(\alpha\beta)^p} \sqrt{b_{k+2p+1}} \\
&\leq \varepsilon_k^{2(\alpha\beta)^p} b_{k+2p+2} + \varepsilon_k^{(\alpha\beta)^p} \cdot \varepsilon_k^{\alpha^{p+1}\beta^p - n_k^{-1}} b_{k+2p+2} \\
&\leq \frac{1}{2}\varepsilon_k^{(\alpha\beta)^{p+1}} b_{k+2p+2} + \frac{1}{2}\varepsilon_k^{(\alpha\beta)^{p+1}} b_{k+2p+2} = \varepsilon_k^{(\alpha\beta)^{p+1}} b_{k+2p+2}, \\
|y_{k,2p+2}| &= |\sigma^{n_{k+2p+1}} x_{k,2p+1}| \\
&\leq \sigma^{2n_{k+2p+1}} \varepsilon_k^{(\alpha\beta)^{p+1}} b_{k+2p+1} = \varepsilon_k^{(\alpha\beta)^{p+1}} \sqrt{b_{k+2p+2}}, \\
|x_{k,2p+3}| &= |-\sigma^{2n_{k+2p+2}} x_{k,2p+2}^2 \mp \lambda^{n_{k+2p+2}} y_{k,2p+2}| \\
&\leq \sigma^{4n_{k+2p+2}} \varepsilon_k^{2(\alpha\beta)^{p+1}} b_{k+2p+2}^2 + \lambda^{n_{k+2p+2}} \varepsilon_k^{(\alpha\beta)^{p+1}} \sqrt{b_{k+2p+2}} \\
&\leq \varepsilon_k^{2(\alpha\beta)^{p+1}} b_{k+2p+3} + \varepsilon_k^{(\alpha\beta)^{p+1}} \cdot \varepsilon_k^{(\alpha\beta)^{p+1} - n_k^{-1}} b_{k+2p+3} \\
&\leq \frac{1}{2}\varepsilon_k^{(\alpha\beta)^{p+3}} b_{k+2p+3} + \frac{1}{2}\varepsilon_k^{(\alpha\beta)^{p+3}} b_{k+2p+3} \\
&\leq \varepsilon_k^{(\alpha\beta)^{p+2}} b_{k+2p+3} \leq \varepsilon_k^{(\alpha\beta)^{p+1}} b_{k+2p+3}, \\
|y_{k,2p+3}| &= |\sigma^{n_{k+2p+2}} x_{k,2p+2}| \\
&\leq \sigma^{2n_{k+2p+2}} \varepsilon_k^{(\alpha\beta)^{p+1}} b_{k+2p+2} = \varepsilon_k^{(\alpha\beta)^{p+1}} \sqrt{b_{k+2p+3}}.
\end{aligned}$$

This shows that $(x_{k,m}, y_{k,m}) \in U_{k,m}$ for $m = 2p+2$ and $2p+3$, which complete the proof of Lemma 4.4. □

Since $0 < \lambda\sigma^{\frac{6\beta-4+8n_k^{-1}}{2-\beta}} < 1$ for any $k \geq 0$ by (4.2), there exists a positive integer m' such that

$$\log \lambda\sigma^{\alpha} > (\alpha\beta)^{\frac{m'}{2}} \log\left(\lambda\sigma^{\frac{6\beta-4+8n_k^{-1}}{2-\beta}}\right)$$

for any $k \geq 0$. Fix such an m'. Fix also a real number $\xi \in (0, 1)$.

Lemma 4.5. *There exist positive integers $k_1 \geq k_0$ and m_0 such that for any even number $k \geq k_1$, any integer $m \geq m_0$ and any $\boldsymbol{x} \in U_{k,0}$,*

$$2|x_{k,m}|\sigma^{2n_{k+m}} \leq \xi\lambda^{n_{k+m}},$$

where $x_{k,m}$ is the first coordinate of $F_{k+m-1} \circ F_{k+m-2} \circ \cdots \circ F_k(\boldsymbol{x})$.

Proof. Since ε_k goes to zero as $k \to \infty$, there exists an even number $k_1 \geq k_0$ such that

$$\varepsilon_k \leq \varepsilon_{k_0}^{(\alpha\beta)^{\frac{m'+2}{2}}}$$

for any $k \geq k_1$. Recall that k_0 is an even number. Note that

$$n_k^{-1}(\alpha\beta)^{-\frac{m+1}{2}}(\log(2\lambda^{-1}\sigma) - \log\xi) \to 0 \quad \text{as } m \to \infty,$$

so by the choice of m', there exists an $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$,

$$\log\lambda\sigma^\alpha \geq (\alpha\beta)^{\frac{m'}{2}} \log\left(\lambda\sigma^{\frac{6\beta-4+8n_{k_0}^{-1}}{2-\beta}}\right) + n_{k_0}^{-1}(\alpha\beta)^{-\frac{m+1}{2}}(\log(2\lambda^{-1}\sigma) - \log\xi).$$

Multiply the inequality by $(\alpha\beta)^{\frac{m+1}{2}}$, then we get

$$(\alpha\beta)^{\frac{m+1}{2}}\log\lambda\sigma^\alpha + n_{k_0}^{-1}\log\xi \geq (\alpha\beta)^{\frac{m'+m+1}{2}}\log\left(\lambda\sigma^{\frac{6\beta-4+8n_{k_0}^{-1}}{2-\beta}}\right) + n_{k_0}^{-1}\log(2\lambda^{-1}\sigma).$$

Hence, it follows that

$$\xi^{n_{k_0}^{-1}}(\lambda\sigma^\alpha)^{(\alpha\beta)^{\lceil\frac{m}{2}\rceil}} \geq \xi^{n_{k_0}^{-1}}(\lambda\sigma^\alpha)^{(\alpha\beta)^{\frac{m+1}{2}}} \geq (2\lambda^{-1}\sigma)^{n_{k_0}^{-1}}\left(\lambda\sigma^{\frac{6\beta-4+8n_{k_0}^{-1}}{2-\beta}}\right)^{(\alpha\beta)^{\frac{m'+m+1}{2}}}$$

because $\frac{m+1}{2} \geq \lceil\frac{m}{2}\rceil$, where $\lceil x\rceil$ denotes the smallest integer which is larger than or equal to x. Raise the above inequality to the n_{k_0}-th power, together with (4.2), then we have

$$\begin{aligned}\lambda\sigma^{-1}\xi(\lambda\sigma^\alpha)^{n_{k_0}(\alpha\beta)^{\lceil\frac{m}{2}\rceil}} &\geq 2\left(\lambda\sigma^{\frac{6\beta-4+8n_{k_0}^{-1}}{2-\beta}}\right)^{n_{k_0}(\alpha\beta)^{\frac{m'+m+1}{2}}} = 2\left(\varepsilon_{k_0}^{(\alpha\beta)^{\frac{m'+2}{2}}}\right)^{(\alpha\beta)^{\frac{m-1}{2}}} \\ &\geq 2\left(\varepsilon_{k_0}^{(\alpha\beta)^{\frac{m'+2}{2}}}\right)^{(\alpha\beta)^{\lfloor\frac{m}{2}\rfloor}} \geq 2\varepsilon_k^{(\alpha\beta)^{\lfloor\frac{m}{2}\rfloor}}\end{aligned}$$

for any $k \geq k_1$.

Fix an even number $k \geq k_1$ and an integer $m \geq m_0$. Then, due to Lemma 4.4, the definition of b_{k+m}, (4.4) and the above inequality, we have

$$\begin{aligned}2|x_{k,m}|\sigma^{2n_{k+m}} &\leq 2\varepsilon_k^{(\alpha\beta)^{\lfloor\frac{m}{2}\rfloor}} b_{k+m}\sigma^{2n_{k+m}} \\ &= 2\varepsilon_k^{(\alpha\beta)^{\lfloor\frac{m}{2}\rfloor}}\sqrt{b_{k+m+1}} \\ &\leq 2\varepsilon_k^{(\alpha\beta)^{\lfloor\frac{m}{2}\rfloor}}\sigma^{-2n_{k+m+1}} \leq 2\varepsilon_k^{(\alpha\beta)^{\lfloor\frac{m}{2}\rfloor}}\sigma^{-n_{k+m+1}} \\ &\leq \lambda\sigma^{-1}\xi(\lambda\sigma^\alpha)^{n_k(\alpha\beta)^{\lceil\frac{m}{2}\rceil}}\sigma^{-n_{k+m+1}}.\end{aligned} \tag{4.8}$$

On the other hand, when $m = 2p$,

$$\lambda^{n_{k+m}} > \lambda^{n_k \alpha^p \beta^p + 1} \quad \text{and} \quad \sigma^{n_{k+m+1}} > \sigma^{n_k \alpha^{p+1} \beta^p - 1},$$

thus

$$\lambda^{n_{k+m}} \sigma^{n_{k+m+1}} > \lambda \sigma^{-1} (\lambda \sigma^\alpha)^{n_k \alpha^p \beta^p}.$$

Similarly, when $m = 2p + 1$,

$$\lambda^{n_{k+m}} > \lambda^{n_k \alpha^{p+1} \beta^p + 1} \text{ and } \sigma^{n_{k+m+1}} > \sigma^{n_k \alpha^{p+1} \beta^{p+1} - 1} > \sigma^{n_k \alpha^{p+2} \beta^p - 1},$$

thus

$$\lambda^{n_{k+m}} \sigma^{n_{k+m+1}} > \lambda \sigma^{-1} (\lambda \sigma^\alpha)^{n_k \alpha^{p+1} \beta^p} > \lambda \sigma^{-1} (\lambda \sigma^\alpha)^{n_k \alpha^{p+1} \beta^{p+1}}.$$

Therefore, we have

$$\lambda^{n_{k+m}} = (\lambda^{n_{k+m}} \sigma^{n_{k+m+1}}) \sigma^{-n_{k+m+1}} \geq \lambda \sigma^{-1} (\lambda \sigma^\alpha)^{n_k (\alpha\beta)^{\lceil m/2 \rceil}} \sigma^{-n_{k+m+1}}.$$

Combining this estimate with (4.8), we get

$$2|x_{k,m}| \sigma^{2n_{k+m}} \leq \xi \lambda^{n_{k+m}},$$

which completes the proof of Lemma 4.5. □

4.3. Lyapunov irregularity. Let k_1 and m_0 be integers given in the previous subsection, and we fix even numbers $k \geq k_1$ and $m \geq m_0$ throughout this subsection.

Fix $\boldsymbol{x} \in U_{k,0}$ and define $\boldsymbol{x}_{k,m+j} = (x_{k,m+j}, y_{k,m+j})$ for each $j \geq 0$ by

$$\boldsymbol{x}_{k,m+j} := F_{k+m+j-1} \circ F_{k+m+j-2} \circ \cdots \circ F_k(\boldsymbol{x}).$$

Recall Lemma 4.5 for $\xi \in (0, 1)$, and set

$$K := \frac{1}{3\xi}.$$

Fix also a vector $\boldsymbol{v}_0 = (v_0, w_0) \in T_{\boldsymbol{x}_{k,m}} M$ with

$$K^{-1} \leq \frac{|v_0|}{|w_0|} \leq K, \tag{4.9}$$

and inductively define $\boldsymbol{v}_j = (v_j, w_j)$ for each $j \geq 0$ by

$$\boldsymbol{v}_{j+1} := DF_{k+m+j}(\boldsymbol{x}_{k,m+j}) \boldsymbol{v}_j.$$

For notational simplicity, we below use

$$\kappa := k + m$$

and

$$(n_p; n_{p+2q}) := n_p + n_{p+2} + n_{p+4} + \cdots + n_{p+2q}$$

for each $p, q \in \mathbb{N}$. For simplicity, we let $(n_p; n_{p-2}) = 0$ for $p \in \mathbb{N}$.

Lemma 4.6. *There exist constants C_j ($j = -2, -1, \ldots$) such that*

$$\boldsymbol{v}_{2p} = \begin{pmatrix} v_{2p} \\ w_{2p} \end{pmatrix} = \begin{pmatrix} C_{2p-1}\lambda^{(n_{\kappa+1};n_{\kappa+2p-1})}\sigma^{(n_\kappa;n_{\kappa+2p-2})}v_0 \\ \pm C_{2p-2}\lambda^{(n_\kappa;n_{\kappa+2p-2})}\sigma^{(n_{\kappa+1};n_{\kappa+2p-1})}w_0 \end{pmatrix}$$
$$\boldsymbol{v}_{2p+1} = \begin{pmatrix} v_{2p+1} \\ w_{2p+1} \end{pmatrix} = \begin{pmatrix} C_{2p}\lambda^{(n_\kappa;n_{\kappa+2p})}\sigma^{(n_{\kappa+1};n_{\kappa+2p-1})}w_0 \\ \pm C_{2p-1}\lambda^{(n_{\kappa+1};n_{\kappa+2p-1})}\sigma^{(n_\kappa;n_{\kappa+2p})}v_0 \end{pmatrix}$$

for every $p \geq 0$, and that $\frac{1}{2} \leq |C_j| \leq \frac{3}{2}$ for every $j \geq -2$.

Proof. We prove Lemma 4.6 by induction. We first show the claim for $p = 0$. The formula for $\boldsymbol{v}_0$ obviously holds with $C_{-2} = C_{-1} = 1$. Due to (4.1), we have

$$DF_\kappa(\boldsymbol{x}_{k,m}) = \begin{pmatrix} -2\sigma^{2n_\kappa}x_{k,m} & \mp\lambda^{n_\kappa} \\ \pm\sigma^{n_\kappa} & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} = DF_\kappa(\boldsymbol{x}_{k,m})\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} -2x_{k,m}\sigma^{2n_\kappa}v_0 \mp \lambda^{n_\kappa}w_0 \\ \pm\sigma^{n_\kappa}v_0 \end{pmatrix}.$$

By Lemma 4.5, (4.9) and the definition of K,

$$|2x_{k,m}\sigma^{2n_\kappa}v_0| \leq \xi\lambda^{n_\kappa}|v_0| \leq \xi K\lambda^{n_\kappa}|w_0| = \frac{1}{3}\lambda^{n_\kappa}|w_0|.$$

In other words,

$$\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} C_0\lambda^{n_\kappa}w_0 \\ \pm C_{-1}\sigma^{n_\kappa}v_0 \end{pmatrix},$$

with a constant C_0 satisfying

$$1 - \frac{1}{3} \leq |C_0| \leq 1 + \frac{1}{3}. \tag{4.10}$$

Next we assume that the claim is true for a given $p \geq 0$, and will show the claim with $p + 1$ instead of p. Note that

$$\boldsymbol{v}_{2p+2} = \begin{pmatrix} v_{2p+2} \\ w_{2p+2} \end{pmatrix} = DF_{\kappa+2p+1}(\boldsymbol{x}_{k,m+2p+1})\begin{pmatrix} v_{2p+1} \\ w_{2p+1} \end{pmatrix}$$
$$= \begin{pmatrix} -2x_{k,m+2p+1}\sigma^{2n_{\kappa+2p+1}}v_{2p+1} \mp \lambda^{n_{\kappa+2p+1}}w_{2p+1} \\ \pm\sigma^{n_{\kappa+2p+1}}v_{2p+1} \end{pmatrix},$$

whose first coordinate is

$$-2x_{k,m+2p+1}\sigma^{2n_{\kappa+2p+1}}C_{2p}\lambda^{(n_\kappa;n_{\kappa+2p})}\sigma^{(n_{\kappa+1};n_{\kappa+2p-1})}w_0$$
$$\mp C_{2p-1}\lambda^{(n_{\kappa+1};n_{\kappa+2p+1})}\sigma^{(n_\kappa;n_{\kappa+2p})}v_0,$$

and second coordinate is

$$\pm C_{2p}\lambda^{(n_\kappa;n_{\kappa+2p})}\sigma^{(n_{\kappa+1};n_{\kappa+2p+1})}w_0,$$

by the inductive hypothesis. On the other hand, it follows from Lemma 4.5, (4.9) and the monotonicity of $(n_l)_{l\in\mathbb{N}}$ that the absolute value of the first term of the first coordinate is bounded by

$$\begin{aligned}
&\xi\lambda^{n_{\kappa+2p+1}}|C_{2p}|\lambda^{(n_\kappa;n_{\kappa+2p})}\sigma^{(n_{\kappa+1};n_{\kappa+2p-1})}|w_0|\\
&\quad\le \xi K\lambda^{n_{\kappa+2p+1}}|C_{2p}|\lambda^{(n_{\kappa-1};n_{\kappa+2p-1})}\sigma^{(n_{\kappa+2};n_{\kappa+2p})}|v_0|\\
&\quad= \frac{1}{3}\frac{\lambda^{n_{\kappa-1}}}{\sigma^{n_\kappa}}|C_{2p}|\lambda^{(n_{\kappa+1};n_{\kappa+2p+1})}\sigma^{(n_\kappa;n_{\kappa+2p})}|v_0|\\
&\quad\le \frac{1}{3}|C_{2p}|\lambda^{(n_{\kappa+1};n_{\kappa+2p+1})}\sigma^{(n_\kappa;n_{\kappa+2p})}|v_0|.
\end{aligned}$$

Hence, we can write $\boldsymbol{v}_{2p+2}$ as

$$\begin{pmatrix} v_{2p+2}\\ w_{2p+2}\end{pmatrix} = \begin{pmatrix} C_{2p+1}\lambda^{(n_{\kappa+1};n_{\kappa+2p+1})}\sigma^{(n_\kappa;n_{\kappa+2p})}v_0\\ \pm C_{2p}\lambda^{(n_\kappa;n_{\kappa+2p})}\sigma^{(n_{\kappa+1};n_{\kappa+2p+1})}w_0\end{pmatrix},$$

with a constant C_{2p+1} satisfying that

$$|C_{2p-1}| - \frac{1}{3}|C_{2p}| \le |C_{2p+1}| \le |C_{2p-1}| + \frac{1}{3}|C_{2p}|. \tag{4.11}$$

Similarly,

$$\begin{aligned}
\boldsymbol{v}_{2p+3} = \begin{pmatrix} v_{2p+3}\\ w_{2p+3}\end{pmatrix} &= DF_{\kappa+2p+2}(\boldsymbol{x}_{k,m+2p+2})\begin{pmatrix} v_{2p+2}\\ w_{2p+2}\end{pmatrix}\\
&= \begin{pmatrix} -2x_{k,m+2p+2}\sigma^{2n_{\kappa+2p+2}}v_{2p+2} \mp \lambda^{n_{\kappa+2p+2}}w_{2p+2}\\ \pm\sigma^{n_{\kappa+2p+2}}v_{2p+2}\end{pmatrix},
\end{aligned}$$

whose first coordinate is

$$\begin{aligned}
-2x_{k,m+2p+2}\sigma^{2n_{\kappa+2p+2}}C_{2p+1}\lambda^{(n_{\kappa+1};n_{\kappa+2p+1})}\sigma^{(n_\kappa;n_{\kappa+2p})}v_0&\\
\mp C_{2p}\lambda^{(n_\kappa;n_{\kappa+2p+2})}\sigma^{(n_{\kappa+1};n_{\kappa+2p+1})}w_0&,
\end{aligned}$$

and second coordinate is

$$\pm C_{2p+1}\lambda^{(n_{\kappa+1};n_{\kappa+2p+1})}\sigma^{(n_\kappa;n_{\kappa+2p+2})}v_0$$

by the previous formula. On the other hand, it follows from Lemma 4.5, (4.9) and the monotonicity of $(n_l)_{l\in\mathbb{N}}$ that the absolute value of the first term of the first coordinate is bounded by

$$\begin{aligned}
&\xi\lambda^{n_{\kappa+2p+2}}|C_{2p+1}|\lambda^{(n_{\kappa+1};n_{\kappa+2p+1})}\sigma^{(n_\kappa;n_{\kappa+2p})}|v_0|\\
&\quad\le \xi K\lambda^{n_{\kappa+2p+2}}|C_{2p+1}|\lambda^{(n_\kappa;n_{\kappa+2p})}\sigma^{(n_{\kappa+1};n_{\kappa+2p+1})}|w_0|\\
&\quad= \frac{1}{3}|C_{2p+1}|\lambda^{(n_\kappa;n_{\kappa+2p+2})}\sigma^{(n_{\kappa+1};n_{\kappa+2p+1})}|w_0|.
\end{aligned}$$

Hence, we can write $\boldsymbol{v}_{2p+3}$ as

$$\begin{pmatrix} v_{2p+3}\\ w_{2p+3}\end{pmatrix} = \begin{pmatrix} C_{2p+2}\lambda^{(n_\kappa;n_{\kappa+2p+2})}\sigma^{(n_{\kappa+1};n_{\kappa+2p+1})}w_0\\ \pm C_{2p+1}\lambda^{(n_{\kappa+1};n_{\kappa+2p+1})}\sigma^{(n_\kappa;n_{\kappa+2p+2})}v_0\end{pmatrix},$$

with a constant C_{2p+2} satisfying that

$$|C_{2p}| - \frac{1}{3}|C_{2p+1}| \le |C_{2p+2}| \le |C_{2p}| + \frac{1}{3}|C_{2p+1}|. \tag{4.12}$$

Finally, combining (4.10), (4.11) and (4.12), we get

$$\frac{1}{2} = 1 - \left(\frac{1}{3} + \frac{1}{3^2} + \cdots\right) \le |C_j| \le 1 + \left(\frac{1}{3} + \frac{1}{3^2} + \cdots\right) = \frac{3}{2}$$

for any $j \ge 0$. This completes the proof of Lemma 4.7. □

Given two sequences $(a_p)_{p\ge 0}$ and $(b_p)_{p\ge 0}$ of positive numbers, if there exist constants $c_0,\ c_1 > 0$, independently of p, such that

$$c_0 < \frac{a_p}{b_p} < c_1,$$

then, we say that a_p and b_p are equivalent, denoted by $a_p \sim b_p$.

Lemma 4.7. *For every $p \ge 0$, we have*

$$|v_{2p}| \sim \lambda^{(n_{\kappa+1};n_{\kappa+2p-1})}\sigma^{(n_\kappa;n_{\kappa+2p-2})} < \lambda^{(n_\kappa;n_{\kappa+2p-2})}\sigma^{(n_{\kappa+1};n_{\kappa+2p-1})} \sim |w_{2p}|,$$
$$|v_{2p+1}| \sim \lambda^{(n_\kappa;n_{\kappa+2p})}\sigma^{(n_{\kappa+1};n_{\kappa+2p-1})} < \lambda^{(n_{\kappa+1};n_{\kappa+2p-1})}\sigma^{(n_\kappa;n_{\kappa+2p})} \sim |w_{2p+1}|.$$

Proof. The equivalence relations follow from Lemma 4.6 directly. Since $0 < \lambda < 1$, $\sigma > 1$,

$$\lambda^{(n_{\kappa+1};n_{\kappa+2p-1})} < \lambda^{(n_\kappa;n_{\kappa+2p-2})}, \quad \sigma^{(n_\kappa;n_{\kappa+2p-2})} < \sigma^{(n_{\kappa+1};n_{\kappa+2p-1})},$$

which gives the former formula immediately. In order to prove the later formula, it suffice to notice that

$$\lambda^{(n_\kappa;n_{\kappa+2p})} < \lambda^{(n_{\kappa+2};n_{\kappa+2p})} < \lambda^{(n_{\kappa+1};n_{\kappa+2p-1})},$$
$$\sigma^{(n_{\kappa+1};n_{\kappa+2p-1})} < \sigma^{(n_{\kappa+2};n_{\kappa+2p})} < \sigma^{(n_\kappa;n_{\kappa+2p})}.$$

This completes the proof of Lemma 4.7. □

An immediate consequence of Lemma 4.7 is that

$$\|\boldsymbol{v}_{2p}\| \sim \lambda^{(n_\kappa;n_{\kappa+2p-2})}\sigma^{(n_{\kappa+1};n_{\kappa+2p-1})},$$
$$\|\boldsymbol{v}_{2p+1}\| \sim \lambda^{(n_{\kappa+1};n_{\kappa+2p-1})}\sigma^{(n_\kappa;n_{\kappa+2p})}.$$

Since both k and m are even numbers, for every integer $p \ge 0$, we have

$$|n_{\kappa+2p} - n_0\alpha^{\frac{\kappa}{2}+p}\beta^{\frac{\kappa}{2}+p}| \le 1, \quad |n_{\kappa+2p+1} - n_0\alpha^{\frac{\kappa}{2}+p+1}\beta^{\frac{\kappa}{2}+p}| \le 1.$$

According to Lemma 4.7,

$$\lim_{p\to\infty} \frac{\log \|D(F_{n_{\kappa+2p}} \circ \cdots \circ F_{n_{\kappa+1}} \circ F_{n_\kappa})(\boldsymbol{x}_{k,m})\boldsymbol{v}_0\|}{(n_\kappa + 2) + (n_{\kappa+1} + 2) + \cdots + (n_{\kappa+2p} + 2)}$$
$$= \lim_{p\to\infty} \frac{\log \|\boldsymbol{v}_{2p+1}\|}{n_\kappa + n_{\kappa+1} + \cdots + n_{\kappa+2p} + O(p)}$$

$$
\begin{aligned}
&= \lim_{p\to\infty} \frac{\log \lambda^{(n_{\kappa+1};n_{\kappa+2p-1})}\sigma^{(n_\kappa;n_{\kappa+2p})} + O(p)}{n_\kappa + n_{\kappa+1} + \cdots + n_{\kappa+2p} + O(p)} \\
&= \lim_{p\to\infty} \frac{(n_{\kappa+1} + n_{\kappa+3} + \cdots + n_{\kappa+2p-1}) \log \lambda + (n_\kappa + n_{\kappa+2} + \cdots + n_{\kappa+2p}) \log \sigma + O(p)}{n_\kappa + n_{\kappa+1} + \cdots + n_{\kappa+2p} + O(p)} \\
&= \lim_{p\to\infty} \frac{n_0\alpha^{\frac{\kappa}{2}}\beta^{\frac{\kappa}{2}}[\alpha(1+\alpha\beta+\cdots+(\alpha\beta)^{p-1})\log\lambda + (1+\alpha\beta+\cdots+(\alpha\beta)^p)\log\sigma] + O(p)}{n_0\alpha^{\frac{\kappa}{2}}\beta^{\frac{\kappa}{2}}[1+\alpha+\alpha\beta+\alpha^2\beta+\cdots+(\alpha\beta)^p] + O(p)} \\
&= \frac{\log\lambda + \beta\log\sigma}{1+\beta},
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{p\to\infty} \frac{\log \| D(F_{n_{\kappa+2p-1}} \circ \cdots \circ F_{n_{\kappa+1}} \circ F_{n_\kappa})(\boldsymbol{x}_{k,m})\boldsymbol{v}_0 \|}{(n_\kappa+2) + (n_{\kappa+1}+2) + \cdots + (n_{\kappa+2p-1}+2)} \\
&= \lim_{p\to\infty} \frac{\log \|\boldsymbol{v}_{2p}\|}{n_\kappa + n_{\kappa+1} + \cdots + n_{\kappa+2p-1} + O(p)} \\
&= \lim_{p\to\infty} \frac{\log \lambda^{(n_\kappa;n_{\kappa+2p-2})}\sigma^{(n_{\kappa+1};n_{\kappa+2p-1})} + O(p)}{n_\kappa + n_{\kappa+1} + \cdots + n_{\kappa+2p-1} + O(p)} \\
&= \lim_{p\to\infty} \frac{(n_\kappa + n_{\kappa+2} + \cdots + n_{\kappa+2p-2}) \log \lambda + (n_{\kappa+1} + n_{\kappa+3} + \cdots + n_{\kappa+2p-1}) \log \sigma + O(p)}{n_\kappa + n_{\kappa+1} + \cdots + n_{\kappa+2p-1} + O(p)} \\
&= \lim_{p\to\infty} \frac{n_k\alpha^{\frac{m}{2}}\beta^{\frac{m}{2}}[\alpha(1+\alpha\beta+\cdots+(\alpha\beta)^{p-1})\log\lambda + (1+\alpha\beta+\cdots+(\alpha\beta)^{p-1})\log\sigma] + O(p)}{n_k\alpha^{\frac{m}{2}}\beta^{\frac{m}{2}}[1+\alpha+\alpha\beta+\alpha^2\beta+\cdots+(\alpha\beta)^{p-1}+\alpha^p\beta^{p-1}] + O(p)} \\
&= \frac{\log\lambda + \alpha\log\sigma}{1+\alpha}.
\end{aligned}
$$

Since

$$
\frac{\log\lambda + \beta\log\sigma}{1+\beta} \neq \frac{\log\lambda + \alpha\log\sigma}{1+\alpha},
$$

together with the remark in the end of Sect. 4.1, this completes the proof of the assertion for the Lyapunov irregularity in Theorem A, where U_f and V_f are the interiors of $F_{\kappa-1} \circ F_{\kappa-2} \circ \cdots \circ F_k(U_{k,0})$ (under the coordinate translation) and $\{(v_0, w_0) \mid K^{-1} \leq \frac{|v_0|}{|w_0|} \leq K\}$, respectively.

4.4. Birkhoff (ir)regularity. To show Birkhoff (ir)regularity, as well as the uncountability of f up to conjugacy in Theorem A, we need a more detailed description of Colli–Vargas' theorem as follows. Let g be the surface diffeomorphism of Theorem 4.1 and

$$
\mathbb{B}^u_+ := g([-1,1]^2) \cap ([0,1]\times[-1,1]), \quad \mathbb{B}^u_- := g([-1,1]^2) \cap ([-1,0]\times[-1,1]).
$$

Fo each $l \in \mathbb{N}$ and $\underline{w} = (w_1, w_2, \ldots, w_l) \in \{+,-\}^l$, we let

$$
\mathbb{B}^u_{\underline{w}} := \bigcap_{j=1}^{l} g^{-j+1}(\mathbb{B}^u_{w_j}), \quad \mathbb{G}^u_{\underline{w}} := \mathbb{B}^u_{\underline{w}} \setminus \left(\mathbb{B}^u_{\underline{w}+} \cup \mathbb{B}^u_{\underline{w}-}\right),
$$

where $\underline{w}\pm = (w_1, \ldots, w_l, \pm) \in \{+,-\}^{l+1}$.

Theorem 4.8 ([5]). *Let g be the surface diffeomorphism with a robust homoclinic tangency given in Theorem 4.1. Take*

- *a C^r-neighborhood $\mathcal{O}$ of g with $2 \le r < \infty$,*
- *an increasing sequence $(n_k^0)_{k\in\mathbb{N}}$ of integers satisfying $n_k^0 = O((1+\eta)^k)$ with some $\eta > 0$,*
- *a sequence $(\underline{z}_k^0)_{k\in\mathbb{N}}$ of codes with $\underline{z}_k^0 \in \{+,-\}^{n_k^0}$.*

Then, one can find

- *a diffeomorphism f in $\mathcal{O}$ which coincides with g on $\mathbb{B}_+^u \cup \mathbb{B}_-^u$,*
- *a sequence of rectangles $(R_k)_{k\in\mathbb{N}}$,*
- *increasing sequences $(\hat{n}_k)_{k\in\mathbb{N}}$, $(\hat{m}_k)_{k\in\mathbb{N}}$ of integers satisfying that $\tilde{n}_k := \hat{n}_k + \hat{m}_{k+1} = O(k)$ and depends only on $\mathcal{O}$,*
- *sequences $(\hat{\underline{z}}_k)_{k\in\mathbb{N}}$, $(\hat{\underline{w}}_k)_{k\in\mathbb{N}}$ of codes with $\hat{\underline{z}}_k \in \{+,-\}^{\hat{n}_k}$, $\check{\underline{w}}_k \in \{+,-\}^{\hat{m}_k}$*

such that for each $k \in \mathbb{N}$, (a), (b) in Theorem 4.1 hold and

(c) *$R_k \subset \mathbb{G}^u_{\underline{z}_k}$ for $\underline{z}_k = \hat{\underline{z}}_k \underline{z}_k^0 [\hat{\underline{w}}_{k+1}]^{-1}$, where $[\underline{w}]^{-1} = (w_l, \ldots, w_2, w_1)$ for each $\underline{w} = (w_1, w_2, \ldots, w_l) \in \{+,-\}^l$, $l \in \mathbb{N}$.*

Fix a neighborhood $\mathcal{O}$ of g and a sequence $(n_k^0)_{k\in\mathbb{N}}$ as given in (4.3). To indicate the dependence of $\boldsymbol{z} = (\underline{z}_k^0)_{k\in\mathbb{N}}$ on f and $(R_k)_{k\in\mathbb{N}}$ in Theorem 4.8, we write them as $f_{\boldsymbol{z}}$ and $(R_{k,\boldsymbol{z}})_{k\in\mathbb{N}}$.

We first apply Theorem 4.8 to the sequence $\boldsymbol{z} = (\underline{z}_k^0)_{k\in\mathbb{N}}$ given by

$$\underline{z}_k^0 = (+, +, \ldots, +, z_k'), \quad z_k' \in \{+,-\}$$

for each $k \ge 1$. Then, it is straightforward to see from the item (c) of Theorem 4.8 that for any $k \in \mathbb{N}$, continuous function $\varphi : M \to \mathbb{R}$ and $\epsilon > 0$, there exist integers k_2 and L_0 such that

$$\sup_{\boldsymbol{x}\in R_k} \left|\varphi(f_{\boldsymbol{z}}^n(\boldsymbol{x})) - \varphi(\boldsymbol{p}_+)\right| < \epsilon$$

whenever

$$N(k, k') + L_0 \le n \le N(k, k'+1) - L_0$$

with some $k' \ge k_2$, where $\boldsymbol{p}_+$ is the continuation for $f_{\boldsymbol{z}}$ of the saddle fixed point of g corresponding to the point set $\mathbb{B}^u_{(+,+,\ldots)}$ and

$$N(p, q) := \sum_{k=p}^{q} (n_k + 2)$$

for each $p, q \in \mathbb{N}$ with $p \le q$. Hence, it holds that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_{\boldsymbol{z}}^j(\boldsymbol{x})) = \varphi(\boldsymbol{p}_+)$$

for any $k \in \mathbb{N}$, $\boldsymbol{x} \in R_k$ and continuous function $\varphi : M \to \mathbb{R}$. Since the open set $U_{f_{\boldsymbol{z}}}$ consisting of Lyapunov irregular points constructed in the previous subsection is of the form $f_{\boldsymbol{z}}^n(U_{0,k'})$ with some positive integers n and k', it follows from the remark following (4.7) and the item (a) of Theorem 4.8 that $U_{f_{\boldsymbol{z}}} \subset R_k$ with some k. This implies that any point in $U_{f_{\boldsymbol{z}}}$ is Birkhoff regular.

Notice that the choice of $(z'_1, z'_2, \ldots)$ in $\boldsymbol{z}$ is uncountable. On the other hand, if $\boldsymbol{z} = (\underline{z}_k^0)_{k\in\mathbb{N}}$ and $\boldsymbol{w} = (\underline{w}_k^0)_{k\in\mathbb{N}}$ are of the above form (in particular, $\underline{w}_k^0 = (+, +, \ldots, +, w'_k) \in \{+, -\}^{n_k^0}$ with $w'_k \in \{+, -\}$) and $z'_k \neq w'_k$ for some k, then $f_{\boldsymbol{z}}$ and $f_{\boldsymbol{w}}$ are not topologically conjugate, or $f_{\boldsymbol{z}}$ and $f_{\boldsymbol{w}}$ are topologically conjugate by a homeomorphism h on M and $h(R_{k,\boldsymbol{z}}) \cap R_{k,\boldsymbol{w}} = \emptyset$ for every k, because of the item (c) of Theorem 4.8 and the fact that both $f_{\boldsymbol{z}}$ and $f_{\boldsymbol{w}}$ coincide with g on $\mathbb{B}_+^u \cup \mathbb{B}_-^u$. Therefore, since there can exist at most countably many, mutually disjoint open sets (of positive Lebesgue measure) on M due to the compactness of M, we complete the proof of the claim for the uncountable set $\mathcal{R}$ in Theorem A.

We next apply Theorem 4.8 to the sequence $\boldsymbol{z} = (\underline{z}_k^0)_{k\in\mathbb{N}}$ given by

$$\underline{z}_k^0 = \begin{cases} (+, +, \ldots, +, z'_k) & \text{if } (2p-1)^2 \leq k < (2p)^2 \text{ with some } p \\ (-, -, \ldots, -, z'_k) & \text{if } (2p)^2 \leq k < (2p+1)^2 \text{ with some } p \end{cases}$$

with $z'_k \in \{+, -\}$ for each $k \geq 1$. Then, it follows from the item (c) of Theorem 4.8 that for any $k \in \mathbb{N}$, continuous function $\varphi : M \to \mathbb{R}$ and $\epsilon > 0$, there exist integers k_2 and L_0 such that

$$\sup_{\boldsymbol{x}\in R_k} \left|\varphi(f_{\boldsymbol{z}}^n(\boldsymbol{x})) - \varphi(\boldsymbol{p}_+)\right| < \epsilon$$

whenever

$$N(k, k') + L_0 \leq n \leq N(k, k'+1) - L_0, \quad \max\{k_2, (2p-1)^2\} \leq k' < (2p)^2$$

with some p, and

$$\sup_{\boldsymbol{x}\in R_k} \left|\varphi(f_{\boldsymbol{z}}^n(\boldsymbol{x})) - \varphi(\boldsymbol{p}_-)\right| < \epsilon$$

whenever

$$N(k, k') + L_0 \leq n \leq N(k, k'+1) - L_0, \quad \max\{k_2, (2p)^2\} \leq k' < (2p+1)^2$$

with some p, where $\boldsymbol{p}_-$ is the continuation for $f_{\boldsymbol{z}}$ of the saddle fixed point of g corresponding to the point set $\mathbb{B}^u_{(-,-,\ldots)}$. Hence, if we let

$$\mathbf{N}(\ell) := N(k, (\ell+1)^2) - N(k, \ell^2) = \sum_{k=\ell^2+1}^{(\ell+1)^2} (n_k + 2),$$

then for any $k \in \mathbb{N}$, $\boldsymbol{x} \in R_k$ and continuous function $\varphi : M \to \mathbb{R}$, we have

$$\frac{1}{\mathbf{N}(2p-1)} \sum_{j=N(k,(2p-1)^2)}^{N(k,(2p)^2)-1} \varphi(f_{\boldsymbol{z}}^j(\boldsymbol{x})) = \varphi(\boldsymbol{p}_+) + o(1)$$

and

$$\frac{1}{\mathbf{N}(2p)} \sum_{j=N(k,(2p)^2)}^{N(k,(2p+1)^2)-1} \varphi(f_{\boldsymbol{z}}^j(\boldsymbol{x})) = \varphi(\boldsymbol{p}_-) + o(1).$$

Since $\mathbf{N}(1) + \mathbf{N}(2) + \cdots + \mathbf{N}(\ell - 1) = o(\mathbf{N}(\ell))$, this implies that, with $\ell := \lceil \sqrt{k} \rceil$ which we assume to be an odd number for simplicity,

$$\begin{aligned}
&\frac{1}{N(k, (2p+1)^2)} \sum_{j=0}^{N(k,(2p+1)^2)-1} \varphi(f_z^j(\boldsymbol{x})) \\
&= \frac{1}{N(k, (2p+1)^2) - N(k, \ell^2)} \sum_{j=N(k,\ell^2)}^{N(k,(2p+1)^2)-1} \varphi(f_z^j(\boldsymbol{x})) + o(1) \\
&= \frac{\mathbf{N}(\ell) + \mathbf{N}(\ell+2) + \cdots + \mathbf{N}(2p-1)}{\mathbf{N}(\ell) + \mathbf{N}(\ell+1) + \cdots + \mathbf{N}(2p)} \varphi(\boldsymbol{p}_+) \\
&\quad + \frac{\mathbf{N}(\ell+1) + \mathbf{N}(\ell+3) + \cdots + \mathbf{N}(2p)}{\mathbf{N}(\ell) + \mathbf{N}(\ell+1) + \cdots + \mathbf{N}(2p)} \varphi(\boldsymbol{p}_-) + o(1) \\
&\to \varphi(\boldsymbol{p}_+) \quad (p \to \infty).
\end{aligned}$$

Similarly we have

$$\lim_{p\to\infty} \frac{1}{N(k, (2p)^2)} \sum_{j=0}^{N(k,(2p)^2)-1} \varphi(f_z^j(\boldsymbol{x})) = \varphi(\boldsymbol{p}_-).$$

That is, any point in R_k is Birkhoff irregular. Therefore, repeating the argument for $\mathcal{R}$, we obtain the claim for the uncountable set $\mathcal{I}$ in Theorem A. This completes the proof of Theorem A.

Remark. The proof of Birkhoff (ir)regularity in this subsection essentially appeared in Colli–Vargas [5]. The difference is that our $(n_k^0)_{k\in\mathbb{N}}$ increases exponentially fast because of the requirement (4.3), while their $(n_k^0)_{k\in\mathbb{N}}$ is of order $O(k^2)$.

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Appendix A. Lebesgue Measurability of Irregular Sets

Although it might be a folklore theorem, we have never seen the proof that Birkhoff and Lyapunov irregular sets are Lebesgue measurable. In this appendix we show that Birkhoff and Lyapunov irregular sets are Lebesgue measurable as a corollary of the following proposition.

Proposition A.1. *Let T be a Polish space and $(\theta_n)_{n\in\mathbb{N}}$ a sequence of functions from $M \times T$ to $\mathbb{R}$. Then the irregular set I of $(\theta_n)_{n\in\mathbb{N}}$ over T, given by*

$$I = \left\{ x \in M \mid \text{there exists } t \in T \text{ for which } \lim_{n\to\infty} \theta_n(x, t) \text{ does not exist} \right\},$$

is a Lebesgue measurable set of M.

For simplicity, we assume that M is an open subset of $\mathbb{R}^d$ and identify TM with $M \times \mathbb{R}^d$. The Birkhoff irregular set of a continuous map $f : M \to M$ is the irregular set of $(\theta_n)_{n\in\mathbb{N}}$,

$$\theta_n(x, \varphi) := \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j(x) \quad ((x, \varphi) \in M \times \mathcal{C}^0(M)),$$

over $T = \mathcal{C}^0(M)$, i.e. the space of all continuous functions on M, and the Lyapunov irregular set of a differentiable map $f : M \to M$ is the irregular set of $(\theta_n)_{n\in\mathbb{N}}$,

$$\theta_n(x, v) := \frac{1}{n}\log \|Df^n(x)v\| \quad ((x, v) \in M \times (\mathbb{R}^d \setminus \{0\})),$$

over $T = \mathbb{R}^d \setminus \{0\}$.

Proof. We first note that I is the projection of

$$\widehat{I} := \left\{(x, t) \in M \times T \mid \lim_{n\to\infty} \theta_n(x, t) \text{ does not exist}\right\}$$

along the Polish space T, that is,

$$I = \left\{x \in M \mid \text{there exists } t \in T \text{ such that } (x, t) \in \widehat{I}\right\}.$$

We will show that $\widehat{I}$ is a Borel set. For each $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$, we define open sets $A_n(\alpha)$ and $B_n(\beta)$ of $M \times T$ by

$$A_n(\alpha) = \{(x, t) \in M \times T \mid \theta_n(x, t) > \alpha\},$$
$$B_n(\beta) = \{(x, t) \in M \times T \mid \theta_n(x, t) < \beta\}.$$

Notice that

$$\left\{(x, t) \in M \times T \mid \limsup_{n\to\infty} \theta_n(x, t) \geq \alpha\right\} = \bigcap_{n_0\in\mathbb{N}} \bigcup_{n\geq n_0} A_n(\alpha)$$

and

$$\left\{(x, t) \in M \times T \mid \liminf_{n\to\infty} \theta_n(x, t) \leq \beta\right\} = \bigcap_{n_0\in\mathbb{N}} \bigcup_{n\geq n_0} B_n(\beta).$$

Hence, we get that

$$\widehat{I} = \bigcup_{\substack{(\alpha,\beta)\in\mathbb{Q}^2 \\ \beta<\alpha}} \left(\left(\bigcap_{n_0\in\mathbb{N}} \bigcup_{n\geq n_0} A_n(\alpha)\right) \cap \left(\bigcap_{n_0\in\mathbb{N}} \bigcup_{n\geq n_0} B_n(\beta)\right)\right),$$

which implies that $\widehat{I}$ is a Borel set, as claimed.

Due to the well-known facts that every projection of a Borel set along a Polish space is an analytic set (i.e. the image of a continuous map from a Polish space X to T), and that any analytic set is Lebesgue measurable, the irregular set I is a Lebesgue measurable set. □

Remark. From the above proof, the Birkhoff irregular set for each $\varphi \in \mathcal{C}^0(M)$ and the Lyapunov irregular set for each $v \in \mathbb{R}^d \setminus \{0\}$ are Borel measurable, while it is unclear whether the Birkhoff and Lyapunov irregular sets of f are Borel measurable because we might need to consider a non-denumerable union of Borel measurable sets to find these irregular sets.

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