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Elliptic Genera of Pure Gauge Theories in Two Dimensions with Semisimple Non-Simply-Connected Gauge Groups

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Abstract: In this paper we describe a systematic method to compute elliptic genera of $(2,2)$ supersymmetric gauge theories in two dimensions with gauge group G/Γ (for G semisimple and simply-connected, Γ a subgroup of the center of G) with various discrete theta angles. We apply the technique to examples of pure gauge theories with low-rank gauge groups. Our results are consistent with expectations from decomposition of two-dimensional theories with finite global one-form symmetries and with computations of supersymmetry breaking for some discrete theta angles in pure gauge theories. Finally, we make predictions for the elliptic genera of all the other remaining pure gauge theories by applying decomposition and matching to known supersymmetry breaking patterns.

Contents

1. Introduction

The low energy infrared (IR) limits of gauge theories have been of interest for many years. Pure gauge theories in two dimensions with $\mathcal{N} = (2, 2)$ supersymmetry have long been believed to be gapless, as a result of the chiral R-symmetry and anomalous two-point functions [1, section 3]. The paper [2] made a more refined conjecture: that the IR limit of a (2,2) supersymmetric pure G gauge theory, G semisimple and simply-connected, should be a theory of free twisted chiral multiplets, as many as the rank of G, with R-charges proportional to Casimir degrees. Using nonabelian mirrors [3] it was checked in [3–5] that the IR theory contains as many twisted chirals as the rank, and in pure G/Γ gauge theories for Γ a subgroup of the center of G, that one gets an identical free theory for one value of the discrete theta angle, and supersymmetry breaking in the IR for other values of the discrete theta angle.

All that said, the work [3–5] did not compute elliptic genera, which would provide a very explicit concrete check of R-charges of free IR twisted chirals. For a pure (2,2) supersymmetric G gauge theory for G simply-connected, methods to compute elliptic genera exist (see e.g. [6–9]), and it is being checked [10,11], that those elliptic genera match the expectations of [2].

The purpose of this paper is to develop the technology to compute elliptic genera of pure (2,2) supersymmetric G/Γ gauge theories for various discrete theta angles. The elliptic genus is given by a sum of Jeffrey-Kirwan residues of a meromorphic form over the moduli space of flat G/Γ-connections the torus using supersymmetric localization [6,7]. The meromorphic form is obtained by evaluating the one-loop determinants corresponding to G/Γ-bundles with non-trivial characteristic classes. We combine the results from different components of the moduli space, weighted by phases from the discrete theta angle, to determine the elliptic genus.

In Sect. 2 we review known results for elliptic genera of pure supersymmetric gauge theories in two dimensions. In Sect. 3 we describe the procedure we will use to compute elliptic genera of pure supersymmetric gauge theories with semisimple but non-simply-connected gauge groups. The remainder of this paper is spent working out low-rank examples. We begin in Sect. 4 by discussing pure $SO(3)$ gauge theories. For these, the elliptic genera in question were previously derived in [12, appendix A], but this case acts as a test and demonstration of our strategy. In Sect. 5, we compute elliptic genera of pure $SU(3)/\mathbb{Z}_3$ gauge theories; in Sect. 6, pure $SO(4)$ gauge theories; in Sect. 7, pure $\text{Spin}(4)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ gauge theories; in Sect. 8, pure $SO(5)$ gauge theories; and, in Sect. 9, pure $Sp(6)/\mathbb{Z}_2$ gauge theories. In each case, the elliptic genus vanishes (and supersymmetry is broken) unless the discrete theta angle takes the value described in [3–5]. We conclude by making predictions for elliptic genera of all other pure gauge theories with semisimple non-simply-connected gauge groups, in Sect. 10.

We will also note in each case that the results are consistent with decomposition [13–15]. (See also e.g. [16,17] for four-dimensional versions and related analyses.) Briefly, decomposition is the statement that a two-dimensional theory with a finite global 1-form symmetry (such as a two-dimensional gauge theory in which a finite center acts trivially) decomposes[1] into a disjoint union of theories which individually do not have a 1-form symmetry. In the case of a pure G gauge theory for G simply-connected, with Γ a finite subgroup of the center, the G gauge theory has a global one-form Γ symmetry (sometimes denoted $B\Gamma$), and so decomposes into a disjoint union of G/Γ

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[1] This is a stronger statement than just superselection. For example, only in infinite volume does one get a selection rule from superselection sectors, whereas decomposition holds at finite volume. This distinction is discussed in greater detail in [16].

gauge theories with various discrete theta angles, which we write schematically as

$$G = \oplus_{\theta \in \hat{\Gamma}} (G/\Gamma)_\theta. \tag{1.1}$$

In particular, the elliptic genus of a pure G gauge theory should be the sum of elliptic genera of pure G/Γ gauge theories with various discrete theta angles. The result that the elliptic genera of G/Γ gauge theories vanish except for a single discrete theta angle, for which the elliptic genus matches that of the G gauge theory, is consistent with the decomposition above.

Also, although we will not emphasize this perspective in this paper, in principle these computations have a mathematical understanding. Elliptic genera of pure G gauge theories should, in principle, match [18–20] elliptic genera of classifying stacks BG, the G-equivariant elliptic genera of points [21,22], and so we are also making predictions for those elliptic genera.

2. Review and Overview

Pure $\mathcal{N} = (2, 2)$ supersymmetric G gauge theory can be described in terms of vector multiplet consisting of a gauge field A_μ, gauginos λ and $\bar{\lambda}$, scalars $\sigma, \bar{\sigma}$, and a real auxiliary scalar D. The gauge field strength is a twisted chiral superfield Σ with lowest component σ. The Euclidean Yang–Mills Lagrangian is

$$\mathcal{L}_{YM} = \text{Tr}\left(F_{12}^2 + D^2 + D_\mu \bar{\sigma} D^\mu \sigma + iD[\sigma, \bar{\sigma}] - i\bar{\lambda}\gamma^\mu D_\mu \lambda - i\bar{\lambda}P_+[\sigma, \lambda] - i\bar{\lambda}P_-[\bar{\sigma}, \lambda]\right), \tag{2.1}$$

where

$$P_\pm = \frac{1 \pm \gamma_3}{2}. \tag{2.2}$$

The classical potential is proportional to $\text{Tr}\left[\sigma, \sigma^\dagger\right]^2$. The classical vacua occur at the minimum of the potential and satisfy $\left[\sigma, \sigma^\dagger\right] = 0$. Equivalently, the classical Coulomb branch of vacua can be described by the vacuum expectation values of the gauge invariant polynomials in σ. It is a classical result that this ring of functions is freely generated by rank(G) generators. However, the potential receives quantum corrections, so the IR behavior is potentially more complex.

2.1. Prediction for simply-connected semisimple G. The paper [2] proposed that for G semisimple and simply-connected, the IR theory should be a free theory of twisted chiral multiplets, $Y_i(\Sigma)$, $i = 1, \ldots, \text{rank}(G)$, built out of the generators of the invariant functions on Σ, with axial R-charges r_i given by twice the Casimir degrees[2] d_i of G computed from and in one-to-one correspondence with the possible Casimirs (of which there are as many as the rank). The contribution of a single twisted chiral multiplet $Y(\Sigma)$ with axial R-charge r to the elliptic genus is [7, equ'n (2.11)]

$$\text{Tr}_{RR} (-1)^F q^{H_L} \bar{q}^{H_R} y^J = \frac{\theta_1(\tau|(1 - r/2)z)}{\theta_1(\tau| - (r/2)z)}, \tag{2.3}$$

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[2] This follows from the Harish-Chandra isomorphism that relates Casimirs to symmetric invariants.

where $q = \exp(2\pi i \tau)$, $y = \exp(2\pi i z)$, J is the left-moving $U(1)_R$ charge, and the genus is computed for periodic left-moving fermions. Since the low energy theory is a theory of free twisted chiral multiplets, the elliptic genus is expected to be

$$\prod_i \frac{\theta_1(\tau|(1 - r_i/2)z)}{\theta_1(\tau| - (r_i/2)z)}. \tag{2.4}$$

For simply-connected G, this will be demonstrated by explicit computation in [11].

For later use, we collect in Table 1 the degrees of Casimirs for simple Lie algebras, each of which is half the R-charge of a corresponding twisted chiral in Eq. (2.4). For example, the elliptic genus of a pure G_2 gauge theory is predicted to be

$$\frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \frac{\theta_1(\tau| - 5z)}{\theta_1(\tau| - 6z)}. \tag{2.5}$$

As a consistency check, the Casimir degrees d_i and the dimension of the group G are related by

$$\dim G = \sum_i (2d_i - 1). \tag{2.6}$$

In passing, identifying R-charges $r_i = 2d_i$, we can apply the central charge formula[3] [23][equ'n (15)] to see that

$$\frac{c_{\text{eff}}}{3} = \sum_i (1 - r_i) = -\dim G, \tag{2.7}$$

where c_{eff} is an effective central charge, differing from the ordinary central charge as [24, equ'n (13)]

$$c_{\text{eff}} = c - 24h_{\min}, \tag{2.8}$$

for $h_{\min}$ the smallest conformal dimension appearing in the theory, as relevant to theories with continuous spectra [25,26]. We can get the same result from the modular transformation properties. Applying [7, equ'n (2.7)]

$$Z\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \exp\left[\frac{c_{\text{eff}}}{3} \frac{\pi i}{\tau} z^2\right] Z(\tau, z) \tag{2.9}$$

and the modular transformation property [7, equ'n (A.8)]

$$\theta_1\left(-\frac{1}{\tau}\bigg|\frac{z}{\tau}\right) = -i\sqrt{-i\tau} \exp(\pi i z^2/\tau)\theta_1(\tau|z), \tag{2.10}$$

we see that under $\tau \mapsto -1/\tau$, $z \mapsto z/\tau$, the elliptic genus of a twisted chiral with R-charge r (equ'n (2.3)) picks up a phase

$$\exp\left(\pi i(1 - r)z^2/\tau\right), \tag{2.11}$$

and the elliptic genus of a pure G gauge theory (2.4) picks up a phase

$$\exp\left(\pi i \sum_i (1 - r_i)z^2/\tau\right) = \exp\left(-\pi i (\dim G)z^2/\tau\right). \tag{2.12}$$

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[3] In conventions in which the superpotential obeys $W(\lambda^{r_i} x_i) = \lambda^2 W(x_i)$.

Table 1. List of Casimir degrees for various gauge groups, each corresponding to half an R-charge

Gauge group	Dimension	Casimir degrees
$SU(n+1)(A_n)$	$(n+1)^2 - 1$	$2, 3, 4, \cdots, n+1$
$\text{Spin}(2n+1)(B_n)$	$n(2n+1)$	$2, 4, 6, \cdots, 2n$
$Sp(2n)(C_n)$	$n(2n+1)$	$2, 4, 6, \cdots, 2n$
$\text{Spin}(2n)(D_n)$	$n(2n-1)$	$n; 2, 4, 6, \cdots, 2n-2$
G_2	14	2, 6
F_4	52	2, 6, 8, 12
E_6	78	2, 5, 6, 8, 9, 12
E_7	133	2, 6, 8, 10, 12, 14, 18
E_8	248	2, 8, 12, 14, 18, 20, 24, 30

See e.g. [33, table 5a]

This phase is determined by the smallest conformal weight $h_{\min}$ appearing in the theory, following [24]. In any event, we see again that the (left-moving) effective central charge is given by

$$\frac{c_{\text{eff}}}{3} = -\dim G. \tag{2.13}$$

Intuitively, for theories formulated on S^2, the sign of the central charge above is surely related to the fact that for R charge greater than two, the action has a curvature-dependent term of the wrong sign [27, section 3.4].

Mathematically, this has a simple understanding. A pure G-gauge theory is a sigma model on [18–20] the stack $BG = [\text{point}/G]$, and this stack has dimension (see e.g. [28, section 7], [29, example 2.44])

$$\dim [\text{point}/G] = -\dim G, \tag{2.14}$$

matching $c_{\text{eff}}/3$ above.

2.2. Non-simply-connected G. In this paper, we will compute elliptic genera of pure supersymmetric gauge theories with gauge groups G/Γ, where G is simply-connected and Γ is a subgroup of the center of G. Now, a principal G/Γ bundle on worldsheet T^2 admits a characteristic class we shall denote $w \in H^2(T^2, \Gamma) \cong \Gamma$. (For example, for $SO(k)$ bundles, w is the Stiefel–Whitney class w_2.) Such theories admit analogues of theta angles, known as discrete theta angles, in which the path integral is weighted by phases of the form $\exp(i\theta \cdot w)$ for θ a (log of a) character of Γ, the set of which we shall denote $\hat{\Gamma}$.

The papers [3–5] have looked at IR behavior of two-dimensional pure (2,2) supersymmetric gauge theories with non-simply-connected gauge groups G/Γ. (See also [12,30,31] for computations of elliptic genera in some examples related to Hori's dualities [32].) Briefly, these papers found

- If the gauge group is not simply-connected, then for precisely one value of the discrete theta angle, the IR limit is a theory of free twisted chirals, as many as the rank (and as many as IR limit of a pure gauge theory with corresponding simply-connected gauge group). For other values of the discrete theta angle, there are no supersymmetric vacua, hence supersymmetry is broken in the IR.
- For the one nontrivial case, the IR theory is a theory of as many twisted chiral multiplets as the rank, matching the IR behavior of a pure G gauge theory.

Table 2. List of distinguished discrete theta angles for various non-simply-connected gauge groups, for which a pure gauge theory admits supersymmetric vacua, summarizing results from [3–5]

Gauge group	Discrete theta angle for which susy unbroken
$SU(k)/\mathbb{Z}_k$	$-(1/2)k(k-1) \mod k$
$\text{Spin}(2k+1)/\mathbb{Z}_2$	$1 \mod 2$
$\text{Spin}(4k)/\mathbb{Z}_2 \times \mathbb{Z}_2$	$k(2k-1) \mod 2, 0 \mod 2$
$\text{Spin}(4k+2)/\mathbb{Z}_4$	$2k(2k-1) \mod 4$
$Sp(2k)/\mathbb{Z}_2$	$(1/2)k(k+1) \mod 2$
$E_6/\mathbb{Z}_3$	$0 \mod 3$
$E_7/\mathbb{Z}_2$	$1 \mod 2$

This structure is consistent with the predictions of decomposition [13–15] for two-dimensional theories with one-form symmetries, as discussed in [3–5].

In this paper, we will compute elliptic genera to check these claims for more general theories.

To understand some of the quantum subtleties that will arise when studying pure G/Γ gauge theories, let us briefly review such theories more concretely. The Lagrangian for such a theory can be written in (2,2) superspace in the form[4]

$$-\frac{1}{4g^2} \int d^4\theta\, \text{Tr}\, \overline{\Sigma}\Sigma \,+\, \left(-r+i\frac{\theta}{2\pi}\right) \int d\theta^+ d\overline{\theta}^- \,\text{Tr}\Sigma|_{\theta^-=\overline{\theta}^+=0} \,+\, c.c., \qquad (2.15)$$

where Σ is a twisted chiral superfield encoding the gauge field strength, r is a Fayet–Iliopoulos parameter, and θ the theta angle. In analyzing the low-energy behavior of such theories one often works on the Coulomb branch, along which there is a twisted one-loop effective superpotential which for a pure G/Γ gauge theory with G simply-connected and Γ a subgroup of the center, takes the form

$$W_{\text{eff}} \,=\, -\sum_a \Sigma_a \left[-r_a +i\frac{\theta_a}{2\pi} \,+\, \frac{1}{|\Gamma|}\sum_{\tilde{\mu}} \alpha^a_{\tilde{\mu}} \left(\ln\left(\sum_b \Sigma_b \alpha^b_{\tilde{\mu}}\right) - 1\right)\right], \qquad (2.16)$$

where now r_a and θ_a are the FI parameters and theta angles for each of the unbroken $U(1)$'s on the Coulomb branch. (No further corrections exist beyond one-loop order.) The first two terms are the $(-r+i\theta/2\pi)\text{Tr}\Sigma$ of the classical action along the Coulomb branch, and the last is a loop correction, of the same form commonly seen in theories with matter, here ultimately due to W bosons. The $\alpha^a_{\tilde{\mu}}$ are the root vectors of the nonzero roots (indexed by $\tilde{\mu}$) of the Lie algebra of the gauge group. The second term can be simplified, and written as (see e.g. [5, section 2.1])

$$\frac{1}{|\Gamma|}\sum_{\tilde{\mu}} \alpha^a_{\tilde{\mu}} \left(\ln\left(\sum_b \Sigma_b \alpha^b_{\tilde{\mu}}\right) - 1\right) \,=\, \sum_{\tilde{\mu}\,\text{pos}'} \frac{i\pi}{|\Gamma|} \alpha^a_{\tilde{\mu}}, \qquad (2.17)$$

giving what amounts to a gauge-group-dependent shift of the theta angle. (This was first observed in [27, equ'n (10.9)].) These additional phases will play an important role in our computations of elliptic genera of pure G/Γ gauge theories.

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[4] See e.g. [34, section 4.1].

3. Strategy to Compute Elliptic Genera

The elliptic genus of a pure G/Γ-gauge theory reduces to a residue integral over the moduli space $\mathcal{M}$ of flat G/Γ-connections on T^2 [6,7]. Principal G/Γ bundles have a degree-two characteristic class, valued in Γ, which we shall denote $w \in H^2(T^2, \Gamma) \cong \Gamma$, so the moduli space of flat G/Γ connections is a disjoint union of moduli spaces

$$\mathcal{M} = \bigsqcup_{w \in H^2(T^2, \Gamma)} \mathcal{M}_{G/\Gamma, w}. \tag{3.1}$$

In the sector of bundles with $w = 0$, any G/Γ bundle lifts to a G bundle. Essentially as a result, the elliptic genus of a pure G gauge theory matches that of a pure G/Γ gauge theory in the sector $w = 0$, up to a volume factor $1/|\Gamma \times \Gamma|$ and a Jacobian factor $|\Gamma|$[5]:

$$Z(G/\Gamma, w = 0) = \frac{|\Gamma|}{|\Gamma \times \Gamma|} Z(G) = \frac{1}{|\Gamma|} Z(G). \tag{3.2}$$

Now, we turn to a G/Γ gauge theory in a sector in which $w \neq 0$. Computations in these sectors will occupy most of the effort in this paper. To describe such bundles, we pick two holonomies p, q around cycles of the torus, which commute up to an element $w \in \Gamma$:

$$pq = wqp. \tag{3.3}$$

The matrices p and q are the holonomies of any bundle about two cycles of the torus, lifted from G/Γ to G. Put another way, these almost-commuting holonomies are the result of lifting commuties holonomies in G/Γ to pairs in G. Next, we simultaneously diagonalize the adjoint action of p and q on the generators of the Lie algebra in the adjoint representation, writing

$$pT^\alpha p^{-1} = \omega_p^\alpha T^\alpha, \tag{3.4}$$

$$qT^\alpha q^{-1} = \omega_q^\alpha T^\alpha, \tag{3.5}$$

where $\omega_{p,q}^\alpha$ are phases, which enter into the elliptic genus computation. These phases also appeared in the calculation of the four-dimensional Witten index [35,36] . Note that such a diagonalization is not possible for every possible representation in which the T^α may appear; in particular, for the diagonalization above to be possible, one needs for the representation to be acted upon nontrivially[6] by the center detected by p and q. Additionally the phases for the adjoint representation are sufficient to determine the phases for all representations when the center of G/Γ is trivial since the adjoint is a tensor generator of the representation category [37].

If the phases $\omega_{p,q}$ are different from one, then, those 'directions' in the group are fixed. If they are equal to one, on the other hand, then the group is unconstrained in those directions, and so one must integrate over corresponding Wilson lines, over the corresponding moduli space of flat connections, to get the elliptic genus.

To the latter end, it can be shown that [38,39]

$$\mathcal{M}_{G/\Gamma, w} = \mathcal{M}_{\tilde{G}(w), 1} \tag{3.6}$$

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[5] This arises from the different normalization of the root systems.

[6] A potentially useful reference is [40], describing representations for which such a diagonalization is possible. For a representation in which such a diagonalization is not possible, consider the case $G = SU(2)$, $\Gamma = \mathbb{Z}_2$, with p and q in the **3** of $SU(2)$. It is easy to check that the resulting 3×3 matrices expressing the Lie algebra simply cannot be diagonalized with respect to nontrivial p and q.

Table 3. List of groups $\tilde{G}(w)$ whose moduli space of flat connections matches that of a moduli space of flat G/Γ connections with nontrivial characteristic class $w \in H^2(T^2, \Gamma)$

G/Γ	w	$\tilde{G}(w)$
$A_{n-1} \sim SU(n)/\mathbb{Z}_n$	d	$SU(m), m = gcd(n, d)$
$B_n \sim \text{Spin}(2n+1)/\mathbb{Z}_2$	1	$Sp(2n-2), \text{Spin}(2n-1)$
$C_{2n} \sim Sp(4n)/\mathbb{Z}_2$	1	$Sp(2n), \text{Spin}(2n+1)$
$C_{2n+1} \sim Sp(4n+2)/\mathbb{Z}_2$	1	$Sp(2n), \text{Spin}(2n+1)$
$D_{2n+1} \sim \text{Spin}(4n+2)/\mathbb{Z}_4$	1	$Sp(2n-2), \text{Spin}(2n-1)$
	2	$Sp(4n-2), \text{Spin}(4n-1)$
	3	$Sp(2n-2), \text{Spin}(2n-1)$
$D_{2n} \sim \text{Spin}(4n)/\mathbb{Z}_2 \times \mathbb{Z}_2$	(1, 0)	$Sp(2n), \text{Spin}(2n+1)$
	(0, 1)	$Sp(4n-4), \text{Spin}(4n-3)$
	(1, 1)	$Sp(2n), \text{Spin}(2n+1)$
$E_6/\mathbb{Z}_3$	1	G_2
	2	G_2
$E_7/\mathbb{Z}_2$	1	F_4

In each case, we assume Γ is all of the center of simply-connected G, and not a subgroup. In D_{2n}, the (0, 1) indicates the $\mathbb{Z}_2$ whose quotient of $\text{Spin}(4n)$ is $SO(4n)$. Note that because the ranks and Weyl groups match, the moduli space of flat $\text{Spin}(2k+1)$ connections matches that of flat $Sp(2k)$ connections. This table summarizes results in [41, section 5.4], [42, table 6], and [43, appendix A]

for some other group $\tilde{G}(w)$ that depends upon G/Γ and w, where $\mathcal{M}$ denotes the moduli space of flat connections. Such groups $\tilde{G}(w)$ are listed in[7] [41, section 5.4], [42, table 6], and [43, appendix A], and we summarize their results in Table 3. Roughly speaking, we can think of the groups $\tilde{G}(w)$ as being obtained by folding the affine Dynkin diagram according to the action of $w \in \Gamma$.

To describe the moduli spaces $\mathcal{M}_{G/\Gamma, w=0}$ more concretely we recall some notions from the theory of compact Lie groups. Let T a maximal torus of G/Γ [8] with corresponding Cartan subalgebra $\mathfrak{h}$. Let Q be the root lattice, P be the weight lattice, and Λ_{char} be the character lattice of G/Γ. Similarly, let $Q^\vee$ be the coroot lattice, $P^\vee$ be the coweight lattice, and $\Lambda_{char}^\vee$ be the co-character lattice. Then the Cartan torus of G/Γ can be identified with $\mathfrak{h}/2\pi \Lambda_{char}^\vee$. The center of and fundamental groups of G/Γ are

$$Z(G/\Gamma) \cong P^\vee/\Lambda_{char}^\vee \cong \Lambda_{char}/Q, \tag{3.7}$$

$$\pi_1(G/\Gamma) \cong \Lambda_{char}^\vee/Q^\vee \cong P/\Lambda_{char}. \tag{3.8}$$

Let

$$\mathfrak{M} = \mathfrak{h}_\mathbb{C}/(\Lambda_{char}^\vee + \tau \Lambda_{char}^\vee), \tag{3.9}$$

then the moduli space of flat G/Γ-connections on T^2 with $w = 0$ is

$$\mathcal{M}_{G/\Gamma, w=0} = \mathfrak{M}/W, \tag{3.10}$$

where W is the Weyl group of G/Γ.

For G simply-connected the cocharacter lattice is equal to the coroot lattice. In the opposite extreme of G/Γ with trivial center, the cocharacter lattice is equal to the coweight lattice. The relations between the cocharacter lattices mean that the moduli space $\mathcal{M}_{G,1}$ is an order $|\Gamma \times \Gamma|$ cover of $\mathcal{M}_{G/\Gamma, w=0}$.

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[7] In addition, the paper [44] relates the moduli spaces $\tilde{G}(w)$ to $\tilde{G}(w = 0)$ by Galois coverings.

[8] Not to be confused with the elliptic curve T^2.

The elliptic genus of a pure G/Γ theory (with bundles of vanishing characteristic class) is given by [7]

$$Z_{T^2}(\tau, z, w = 0) = \frac{1}{|W|} \sum_{u_* \in \mathfrak{M}^*_{\text{sing}}} \text{JK} - \text{Res}_{u=u_*}\big(\mathbf{Q}(u_*), \eta\big) \ Z_{1\text{-loop}}(\tau, z, u) \qquad (3.11)$$

where $|W|$ is the order of the Weyl-group of G.[9] The Jeffrey-Kirwan residue operation [45] $\text{JK} - \text{Res}_{u=u_*}(\mathbf{Q}(u_*), \eta)$ assigns a residue to each pole of $Z_{1\text{-loop}}$ in $\mathfrak{M}^*_{\text{sing}}$ depending on the charge vectors $\mathbf{Q}(u_*)$ responsible for the pole and a covector η. The parameter $q = e^{2\pi i \tau}$ in $Z_{1\text{-loop}}$ specifies the complex structure of the torus T^2 and $y = e^{2\pi i z}$ is the fugacity for the left-moving $U(1)$ R-symmetry. The coordinates u_a on the moduli space $\mathfrak{M}$ can equivalently be described by the coordinates $x_a = e^{2\pi i u_a}$. The contribution of a vector multiplet V with gauge group G/Γ to $Z_{1\text{-loop}}$ for the $w = 0$ characteristic class is

$$Z_{V, G/\Gamma}(\tau, z, u) = \left(\frac{2\pi \eta(q)^3}{\theta_1(q, y^{-1})}\right)^{\text{rank } G} \prod_{\alpha \in G} \frac{\theta_1(q, x^\alpha)}{\theta_1(q, y^{-1} x^\alpha)} \prod_{a=1}^{\text{rank } G} du_a \ . \qquad (3.12)$$

The product is over the roots α of the gauge group and $\eta(q)$ is the Dedekind eta function.

For bundles with non-trivial characteristic classes w, the contribution to $Z_{1\text{-loop}}$ is modified. Using the eigenvalues $\omega^\alpha_{p,q}$, one can then construct an elliptic genus for bundles of fixed characteristic class w as a product of ratios

$$\frac{\theta_1(\tau | v_\alpha)}{\theta_1(\tau | -z + v_\alpha)}, \qquad (3.13)$$

for nonzero v_α, where

$$v_\alpha = \ln \frac{\omega^\alpha_p}{2\pi i} + \tau \ln \frac{\omega^\alpha_q}{2\pi i}, \qquad (3.14)$$

and a residue integral of the form

$$\left(\frac{2\pi \eta(q)^3}{\theta_1(q, y^{-1})}\right)^{\text{rank } \tilde{G}(w)} \prod_{\alpha \in G} \frac{\theta_1(\tau | v_\alpha)}{\theta_1(\tau | -z + v_\alpha)} \prod_{a=1}^{\text{rank } \tilde{G}(w)} du_a \ . \qquad (3.15)$$

for every vanishing v. The resulting residue integral is computed as a Jeffrey-Kirwan residue over (a cover of) the moduli space of those flat connections preserving the holonomy.

This determines the elliptic genus (for fixed bundle characteristic class w) up to an overall normalization factor, which reflects residual gauge transformations that preserve the holonomies. For theories of the form $SU(n)/\mathbb{Z}_n$, that normalization factor is computed in e.g. [46, section 2.2.1].

So far we have described how one computes contributions to the elliptic genus from bundles with different characteristic classes $w \in H^2(T^2, \Gamma)$. Finally, we will combine them, to form the elliptic genus as a function of the discrete theta angle. These different contributions are each weighted with potentially two different phases. First, there is a factor $\exp(i\theta \cdot w)$, where $\theta \in \hat{\Gamma}$ is a choice of discrete theta angle. Second, as studied

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[9] We omit the flavor holonomies ξ since they are absent in pure theories.

in detail in [5] and reviewed in Sect. 2.2, there is a factor of the form $\exp(iw \cdot t)$, where [5, equ'n (2.7)]

$$t_a = -\frac{\pi i}{|\Gamma|} \sum_{\tilde{\mu} \text{ pos}'} \alpha_{\tilde{\mu}}^a, \tag{3.16}$$

and w is encoded in w_a so that

$$t \cdot w = \sum_a t_a w_a. \tag{3.17}$$

Strictly speaking, the t_a are not uniquely defined, as there are e.g. branch cut ambiguities, but the phase factor above is well-defined, as discussed in detail in [5]. Put another way, the t_a encode a constant shift, due to quantum corrections, to the discrete theta angle θ.

Thus, if we label the contribution to the elliptic genus of a pure G/Γ gauge theory in a sector with bundles of characteristic class w by $Z(G/\Gamma, w)$, then the elliptic genus for a general characteristic class has the form

$$Z(G/\Gamma, \theta) = \sum_w \exp(iw \cdot \theta) \exp(iw \cdot t) Z(G/\Gamma, w). \tag{3.18}$$

In the next several sections we will carry out this program for several low-rank examples. Specifically, we will apply the procedure above to derive elliptic genera for $SU(2)/\mathbb{Z}_2$, $SU(3)/\mathbb{Z}_3$, $SO(4)$, $\text{Spin}(4)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $SO(5)$, and $Sp(6)/\mathbb{Z}_2$ gauge theories with various discrete theta angles. The special case of $SU(2)/\mathbb{Z}_2$ was previously discussed in [12, appendix A]; we recover their results through this systematic method. In each case, we will find that the elliptic genus vanishes unless the discrete theta angle takes the value listed in Table 3, as expected [3–5]. We will also see that the results are consistent with decomposition [13–15].

Furthermore, in each case we discuss, we will also find that the contribution to the elliptic genus from bundles with characteristic class $w \neq 0$ matches (up to a phase) the contribution from bundles of characteristic class $w = 0$. This is reminiscent of the fact that elliptic genera are independent of deformations, and so one is naturally led to wonder if there is a more elegant approach to these computations that demonstrates that contributions to the elliptic genus are (modulo an overall phase) independent of w. For example, for sigma models on Calabi–Yau manifolds, the scale r of the Calabi–Yau is a marginal parameter, so as the elliptic genus is an index, it is independent of r, and the resulting elliptic genera are necessarily independent of worldsheet instanton corrections [47,48]. In two-dimensional gauge theories, on the other hand, the gauge coupling is irrelevant[10], so this argument does not apply. In any event, we leave this question for future work.

4. Pure $SU(2)/\mathbb{Z}_2 = SO(3)$ Gauge Theories

The elliptic genus of pure $SU(2)$ gauge theory is [6]

$$\frac{1}{2} \sum_{u_* \in \mathfrak{M}_{\text{sing}}^+} \frac{i\eta(q)^3}{\theta_1(\tau| - z)} \oint_{u_*} du \, \frac{\theta_1(\tau|2u)}{\theta_1(\tau| - z + 2u)} \frac{\theta_1(\tau| - 2u)}{\theta_1(\tau| - z - 2u)}, \tag{4.1}$$

---

[10] We should be careful as terms such as 'marginal' and 'irrelevant' are not well-defined away from fixed points of renormalization group flow, but we are not aware of examples of two-dimensional (2,2) supersymmetric gauge theories in which the gauge coupling flows in the IR to a marginal operator.

where the contributing poles are located at

$$\mathfrak{M}_{\text{sing}}^{+} = \left\{ \frac{z}{2}, \ \frac{z+1}{2}, \ \frac{z+\tau}{2}, \ \frac{z+\tau+1}{2} \right\}. \tag{4.2}$$

Elliptic genera of pure $SO(3)$ gauge theories were computed in [12, appendix A]. Briefly, the authors argued that the pure $SU(2)$ and the $SO(3)_{-}$ theories have the same elliptic genus, given by

$$\frac{\theta_1(\tau|-z)}{\theta_1(\tau|-2z)} = \frac{1}{2} \frac{\theta_1(\tau|+1/2)}{\theta_1(\tau|-z+1/2)} \frac{\theta_1(\tau|+\tau/2)}{\theta_1(\tau|-z+\tau/2)} \frac{\theta_1(\tau|-(1+\tau)/2)}{\theta_1(\tau|-z-(1+\tau)/2)}, \tag{4.3}$$

while the elliptic genus of the pure $SO(3)_{+}$ theory vanishes identically. This is consistent with the results of [5], which argued that in pure $SO(3)$ gauge theories, only for the nontrivial discrete theta angle are there supersymmetric vacua, and supersymmetry is broken in the IR in $SO(3)_{+}$. It is also consistent with decomposition [13–15], which in this case can be schematically expressed as

$$SU(2) = SO(3)_{+} + SO(3)_{-}. \tag{4.4}$$

In more detail, [12, appendix A] combined the contributions of the two distinct types of $SO(3)$ bundles. The contribution to the $SO(3)$ elliptic genus from bundles of vanishing characteristic class is obtained from

$$\frac{1}{2} \sum_{u_* \in \mathfrak{M}_{\text{sing}}^{+}} \frac{i\eta(q)^3}{\theta_1(\tau|-z)} \oint_{u_*} du \frac{\theta_1(\tau|u)}{\theta_1(\tau|-z+u)} \frac{\theta_1(\tau|-u)}{\theta_1(\tau|-z-u)}, \tag{4.5}$$

with a single contributing pole located at $\mathfrak{M}_{\text{sing}}^{+} = z/2$. This results in

$$Z(SO(3)_0) = \frac{1}{2} \frac{\theta_1(\tau|-z)}{\theta_1(\tau|-2z)}, \tag{4.6}$$

which is the $SU(2)$ elliptic genus up to a factor of $1/|\Gamma| = 1/2$. As explained in Sect. 3, this factor arises from the differing character lattices of the $SU(2)$ and $SO(3)$ groups. Note that all four poles in Eq. 4.2 contribute equally to the $SU(2)$ elliptic genus, but there is only one pole for the $SO(3)$ elliptic genus. Since there are only $1/|\Gamma \times \Gamma| = 1/4$ as many poles, but each pole has a Jacobian contribution of $|\Gamma|$ relative to the $SU(2)$ poles, we arrive at the previously claimed factor of $|\Gamma|/|\Gamma \times \Gamma| = 1/|\Gamma| = 1/2$.

The contribution from bundles of nonzero characteristic class is

$$Z(SO(3)_1) = -\frac{1}{2} \frac{\theta_1(\tau|-z)}{\theta_1(\tau|-2z)}. \tag{4.7}$$

For a discrete theta angle $\theta \in \{0, \pi\}$, the possible $SO(3)$ elliptic genera are

$$Z(SO(3)) = Z(SO(3)_0) + \exp(i\theta)Z(SO(3)_1), \tag{4.8}$$

$$= \frac{1}{2} \frac{\theta_1(\tau|-z)}{\theta_1(\tau|-2z)} (1 - \exp(i\theta)). \tag{4.9}$$

When $\theta = 0$, this vanishes, and when $\theta = \pi$, this is nonzero and matches $Z(SU(2))$.

For later use, the elliptic genus, given up to numerical factors we will describe later, is

$$\frac{\theta_1(\tau\,|+1/2)}{\theta_1(\tau\,|-z+1/2)}\frac{\theta_1(\tau\,|+\tau/2)}{\theta_1(\tau\,|-z+\tau/2)}\frac{\theta_1(\tau\,|-(1+\tau)/2)}{\theta_1(\tau\,|-z-(1+\tau)/2)} \qquad (4.10)$$

can be derived directly from thinking about the contribution of $Z_{1-\text{loop}}$ in the sector with $w_2 \neq 0$, in the notation of [6]. Briefly, for $w_2 \neq 0$, the moduli space of flat connections is a point, so that one does not integrate over a space of u's. Instead, the u's are fixed, with holonomies about the T^2 which can be taken to be

$$\text{diag}(-1,-1,+1), \quad \text{diag}(+1,-1,-1). \qquad (4.11)$$

An $SO(3)$ bundle with these holonomies cannot be lifted to an $SU(2)$ bundle. A heuristic way to see this is to observe that the lifts of the holonomies to $SU(2)$ are given in Eq. 4.19 and they anticommute. We can also see this more formally by computing the second Stiefel–Whitney class w_2, which gives the obstruction to lifting, in this case, an $SO(3)$ bundle to an $SU(2)$ bundle. With the holonomies above, we can describe this bundle as

$$L_1 \oplus L_2 \oplus L_3, \qquad (4.12)$$

where L_1 and L_2 each have nontrivial monodromy about a single S^1 on T^2, and $L_3 = L_1 \otimes L_2$. Thus, for example,

$$w(L_1) \;=\; 1+J_1, \quad w(L_2) \;=\; 1+J_2, \quad w(L_3) \;=\; 1+J_1+J_2, \qquad (4.13)$$

where J_1, J_2 generate $H^1(T^2,\mathbb{Z}_2) = (\mathbb{Z}_2)^2$, and in this case give w_1 of L_1, L_2, respectively. Thus,

$$w(L_1 \oplus L_2 \oplus L_3) \;=\; w(L_1)w(L_2)w(L_3) \;=\; 1+J_1J_2+\cdots, \qquad (4.14)$$

hence

$$w_2(L_1 \oplus L_2 \oplus L_3) \;=\; J_1J_2, \qquad (4.15)$$

and in particular is nonzero. Thus, indeed, this $SO(3)$ bundle has nonzero w_2, and can not be lifted to an $SU(2)$ bundle.

Returning to the computation of the elliptic genus for the pure $SO(3)$ gauge theory in a sector in which $w_2 \neq 0$, in terms of holonomies encoded in the parameter u, it can be written

$$\prod_{\text{roots }\alpha} \frac{\theta_1(\tau\,|\alpha\cdot u)}{\theta_1(\tau\,|-z+\alpha\cdot u)} \;=\; \frac{\theta_1(\tau\,|u)}{\theta_1(\tau\,|-z+u)}\frac{\theta_1(\tau\,|0)}{\theta_1(\tau\,|-z)}\frac{\theta_1(\tau\,|-u)}{\theta_1(\tau\,|-z-u)}, \qquad (4.16)$$

corresponding to the three generators of the Lie algebra of $SO(3)$. The three boundary conditions correspond to values of u as follows:

u	(U_1, U_2)
0	$(+1,+1)$
$1/2$	$(-1,+1)$
$\tau/2$	$(+1,-1)$
$(1+\tau)/2$	$(-1,-1)$

Plugging in the single holonomy, we find that the elliptic genus for the $w_2 \neq 0$ sector of the pure $SO(3)$ gauge theory is proportional to

$$\frac{\theta_1(\tau\,|+1/2)}{\theta_1(\tau\,|-z+1/2)}\frac{\theta_1(\tau\,|+\tau/2)}{\theta_1(\tau\,|-z+\tau/2)}\frac{\theta_1(\tau\,|-(1+\tau)/2)}{\theta_1(\tau\,|-z-(1+\tau)/2)}, \qquad (4.17)$$

confirming the results of [12, appendix A] up to numerical factors we will describe momentarily.

So far, we have discussed known results for $SU(2)$ elliptic genera, and also used a trick to compute the $SO(3)$ elliptic genus in a sector where the characteristic class is nontrivial. Let us now repeat the computation systematically using the method of Sect. 3, which we will apply to other examples.

Following the method of Sect. 3, we compute the contribution to the elliptic genus from $SO(3)$ bundles of vanishing characteristic (Stiefel–Whitney) class w_2. As discussed there, the contribution in this sector is the same as that of a pure $SU(2)$ theory, albeit with a constant factor of $1/|\Gamma| = 1/2$ from Sect. 3:

$$Z(SO(3), w_2 = 0) = \frac{1}{2} \frac{\theta_1(\tau | - z)}{\theta_1(\tau | - 2z)}. \tag{4.18}$$

Next, we compute the contribution from $SO(3)$ bundles of nontrivial characteristic class. As in Sect. 3, we define this sector through holonomies lifted to $SU(2)$, where they anticommute. Specifically, consider the $SU(2)$ matrices

$$p = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \tag{4.19}$$

It is easy to verify that $pq = -qp$. Viewing p and q as holonomies, they define a flat $SU(2)/\mathbb{Z}_2 = SO(3)$ bundle with nontrivial characteristic class. Under the adjoint action of p and q, the Pauli sigma matrices are diagonal:

$$p\sigma_1 p^{-1} = -\sigma_1, \quad p\sigma_2 p^{-1} = +\sigma_2, \quad p\sigma_3 p^{-1} = -\sigma_3, \tag{4.20}$$
$$q\sigma_1 q^{-1} = -\sigma_1, \quad q\sigma_2 q^{-1} = -\sigma_2, \quad q\sigma_3 q^{-1} = +\sigma_3. \tag{4.21}$$

From Table 3, we see that the moduli space of flat $SO(3)$ connections with nontrivial characteristic class is a point. We compute the contribution to the elliptic genus for this nontrivial characteristic class by applying Eq. 3.15 with the phases listed above to get the the product of theta functions in Eq. 4.17 up to a constant factor.

Finally, to derive the elliptic genus for bundles of nonzero second Stiefel–Whitney class, we need to add a suitable numerical factor, corresponding to dividing out by the number of residual gauge transformations which preserve the holonomies. From [46, section 2.2.1] for this case, we multiply the theta function product (4.17) by a factor of $1/|W|$, where $W = \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, we have that

$$Z(SO(3), w_2 \neq 0) = \frac{1}{4} \frac{\theta_1(\tau | + 1/2)}{\theta_1(\tau | - z + 1/2)} \frac{\theta_1(\tau | + \tau/2)}{\theta_1(\tau | - z + \tau/2)} \frac{\theta_1(\tau | - (1 + \tau)/2)}{\theta_1(\tau | - z - (1 + \tau)/2)},$$
$$= \frac{1}{2} \frac{\theta_1(\tau | - z)}{\theta_1(\tau | - 2z)}. \tag{4.22}$$

Now, let us assemble these contributions. For a discrete theta angle θ,

$$Z(SO(3), \theta) = Z(SO(3), w_2 = 0) + \exp(iw \cdot t) \exp(iw \cdot \theta) Z(SO(3), w_2 \neq 0). \tag{4.23}$$

As computed in [5, section 3.1], $t = -\pi i$, hence

$$\exp(iw \cdot t) = -1, \tag{4.24}$$

and trivially $\exp(i w \cdot \theta) = \exp(i\theta)$, hence

$$Z(SO(3), \theta) = Z(SO(3), w_2 = 0) - \exp(i\theta)Z(SO(3), w_2 \neq 0), \quad (4.25)$$

$$= \frac{1}{2}\frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} (1 - \exp(i\theta)), \quad (4.26)$$

which duplicates the $SO(3)$ elliptic genus as a function of θ computed in [12, appendix A].

5. Pure $SU(3)/\mathbb{Z}_3$ Gauge Theories

In this section, we apply the method of Sect. 3 to compute the elliptic genus of a pure supersymmetric $SU(3)/\mathbb{Z}_3$ as a function of the discrete theta angle. First, for a vanishing characteristic class, from Eq. (3.2), the elliptic genus of the pure $SU(3)/\mathbb{Z}_3$ gauge theory is the same as the elliptic genus of the pure $SU(3)$ gauge theory, up to a factor of $1/|\Gamma|$:

$$Z(SU(3)/\mathbb{Z}_3, w = 0) = \frac{1}{3}Z(SU(3)) = \frac{1}{3}\frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 3z)}. \quad (5.1)$$

Next, we consider the elliptic genus of a pure $SU(3)/\mathbb{Z}_3$ gauge theory with a nontrivial characteristic class. We can describe an $SU(3)/\mathbb{Z}_3$ bundle with nonzero $w \in H^2(T^2, \mathbb{Z}_3)$ as two holonomies p and q in $SU(3)$ such that

$$pq = wqp \quad (5.2)$$

for $w = \exp(2\pi i k/3)$ with $k = \pm 1$. To that end, consider the $SU(3)$ matrices

$$p = \begin{bmatrix} w & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & w^{-1} \end{bmatrix}, \quad q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad (5.3)$$

then, using $w^3 = 1$, one can verify that

$$pq = wqp. \quad (5.4)$$

Taking linear combinations of the Lie algebra generators λ_a (in the three-dimensional adjoint representation) to solve

$$p\lambda_a p^{-1} = \omega_p^a \lambda_a, \quad q\lambda_a q^{-1} = \omega_q^a \lambda_a, \quad (5.5)$$

we find that

$$(\omega_p, \omega_q) \in \{(1, w), (1, w^2), (w^2, 1), (w^2, w^2), (w^2, w), (w, 1), (w, w^2), (w, w)\}. \quad (5.6)$$

In particular, the dimension of this component of the moduli space of flat $SU(3)/\mathbb{Z}_3$ connections is zero, as can be confirmed from Table 3.

Using Eq. 3.15 with these phases, we find that the elliptic genus of the pure $SU(3)/\mathbb{Z}_3$ gauge theory with nontrivial bundle is

$$\frac{1}{|W|} \frac{\theta_1(\tau \,|\, \tau k/3)}{\theta_1(\tau| - z + \tau k/3)} \frac{\theta_1(\tau| - \tau k/3)}{\theta_1(\tau| - z - \tau k/3)} \frac{\theta_1(\tau| - k/3)}{\theta_1(\tau| - z - k/3)} \frac{\theta_1(\tau| - k/3 - \tau k/3)}{\theta_1(\tau| - z - k/3 - \tau k/3)}$$

$$\cdot \frac{\theta_1(\tau| - k/3 + \tau k/3)}{\theta_1(\tau| - z - k/3 + \tau k/3)} \frac{\theta_1(\tau|k/3)}{\theta_1(\tau| - z + k/3)} \frac{\theta_1(\tau|k/3 - \tau k/3)}{\theta_1(\tau| - z + k/3 - \tau k/3)}$$

$$\cdot \frac{\theta_1(\tau|k/3 + \tau k/3)}{\theta_1(\tau| - z + k/3 + \tau k/3)}, \quad (5.7)$$

where W is the unbroken gauge symmetry of the pair (p, q), which for this case is [46, section 2.2.1] $W = \mathbb{Z}_3 \times \mathbb{Z}_3$, hence $|W| = 9$. Recall that $w = \exp(2\pi i k/3)$ for $k = \pm 1$ (corresponding to the two nontrivial possible values of the characteristic class), Note that this expression is symmetric under $k \leftrightarrow -k$.

For $k = 1$, the product above can be written more succinctly as

$$\frac{1}{9} \prod_{j,\ell=-1}^{1} \left[\frac{\theta_1(\tau|j/3 + \ell\tau/3)}{\theta_1(\tau|j/3 + \ell\tau/3 - z)} \frac{\theta_1(\tau|-z)}{\theta_1(\tau|0)} \right], \tag{5.8}$$

where in the product one should omit the case $j = k = 0$. One can show that[11]

$$\prod_{j,\ell=-1}^{1} \left[\frac{\theta_1(\tau|j/3 + \ell\tau/3)}{\theta_1(\tau|j/3 + \ell\tau/3 - z)} \frac{\theta_1(\tau|-z)}{\theta_1(\tau|0)} \right] = 3 \frac{\theta_1(\tau|-z)}{\theta_1(\tau|-2z)} \frac{\theta_1(\tau|-2z)}{\theta_1(\tau|-3z)}, \tag{5.9}$$

where $y = \exp(2\pi i z)$.

Now, let us assemble these pieces to build the elliptic genus of the pure $SU(3)/\mathbb{Z}_3$ theory with discrete theta angle $\theta \in \{0, 2\pi/3, 4\pi/3\}$. From [5, section 3.2], the quantum correction is given by

$$t_a = \frac{2\pi i}{3} m_a, \tag{5.10}$$

where

$$\sum_a m_a \equiv 0 \mod 3. \tag{5.11}$$

Without loss of generality, we can choose $m_1 = 0 = m_2$, hence the phase factor

$$\exp(i w \cdot t) = +1, \tag{5.12}$$

and so the elliptic genus can be written as a function of $\theta \in \{0, 2\pi/3, 4\pi/3\}$ as

$$Z(SU(3)/\mathbb{Z}_3, \theta) = Z(SU(3)/\mathbb{Z}_3, w = 0) + \exp(i\theta) Z(SU(3)/\mathbb{Z}_3, w = 1)$$
$$+ \exp(-i\theta) Z(SU(3)/\mathbb{Z}_3, w = 2), \tag{5.13}$$
$$= \frac{1}{3} \frac{\theta_1(\tau|-z)}{\theta_1(\tau|-3z)} (1 + \exp(i\theta) + \exp(-i\theta)). \tag{5.14}$$

As a consistency check, the reader should note that for $\theta \neq 0$, the expression above for the elliptic genus vanishes, whereas for $\theta = 0$, it matches that of the pure $SU(3)$ gauge theory. This is consistent with the computation in [5, section 3.2] that supersymmetry is only unbroken in a supersymmetric pure $SU(3)/\mathbb{Z}_3$ gauge theory when $\theta = 0$.

Furthermore,

$$\sum_{\theta=0,\pm 2\pi/3} \frac{1}{3} \frac{\theta_1(\tau|-z)}{\theta_1(\tau|-3z)} (1 + \exp(i\theta) + \exp(-i\theta)) = \frac{\theta_1(\tau|-z)}{\theta_1(\tau|-3z)} = Z(SU(3)). \tag{5.15}$$

This matches the prediction of decomposition [13–15], which in this case schematically says that

$$SU(3) = (SU(3)/\mathbb{Z}_3)_{\theta=0} + (SU(3)/\mathbb{Z}_3)_{\theta=2\pi/3} + (SU(3)/\mathbb{Z}_3)_{\theta=4\pi/3}. \tag{5.16}$$

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[11] A careful reader will observe that if we had instead chosen $k = 1, 2$, we would have crossed a branch cut, which can generate factors such as y^3. We note that fact here, but it will not play a role in our further computations.

6. Pure $SO(4)$ Gauge Theories

Let us now turn to the elliptic genera of pure $SO(4)$ gauge theories. These can be derived from the results above for pure $SO(3)$ gauge theories.

First, consider a pure $SO(4)$ theory in the sector in which w_2 vanishes (so that all bundles can be lifted to Spin(4) bundles). Now, Spin(4) $= SU(2) \times SU(2)$, so the elliptic genus in this sector is the product of elliptic genera corresponding to two pure $SU(2)$ gauge theories. Thus, as explained in Sect. 3, the elliptic genus of a pure $SO(4)$ gauge theory in a sector with $w_2 = 0$ is

$$\frac{1}{2} \left(\frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \right)^2, \tag{6.1}$$

taking into account the constant factor of $1/|\Gamma|$ from Sect. 3. This is consistent with the prediction (2.4) since there are two Casimirs each of the form Tr Σ^2.

Now, let us turn to the sector in which $w_2 \neq 0$. Here, we can apply the same analysis as in the case of the analogous $SO(3)$ sectors. A set of holonomies describing such $SO(4)$ bundles are given by

$$\text{diag}(+1, -1, -1, +1), \quad \text{diag}(+1, +1, -1, -1). \tag{6.2}$$

It is straightforward to check that these holonomies describe an $SO(4)$ bundle with nonzero w_2, and from Table 3, the moduli space of flat $SO(4)$ connections with nonzero w_2 is a point. These holonomies emerge as a special case of the results in [9, equ'n (3.2)]. We can think of these holonomies as describing transformations under one of the two factors in $SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2$. Now, the nonzero roots of $SO(4)$ can be expressed as

$$\pm u_1 \pm u_2, \tag{6.3}$$

where $u_{1,2}$ couple to Cartan holonomies. If one of the two $SU(2)$ factors has trivial holonomy, then we can set $u_1 = 0$, in which case, these roots become two copies of the roots of $SO(3)$. Using previous results for $SO(3)$ holonomies and elliptic genera, we immediately have that the $SO(4)$ elliptic genus for $w_2 \neq 0$ is proportional to

$$\left[\frac{\theta_1(\tau| + 1/2)}{\theta_1(\tau| - z + 1/2)} \frac{\theta_1(\tau| + \tau/2)}{\theta_1(\tau| - z + \tau/2)} \frac{\theta_1(\tau| - (1 + \tau)/2)}{\theta_1(\tau| - z - (1 + \tau)/2)} \right]^2 = \left[2 \frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \right]^2, \tag{6.4}$$

which from Eq. (4.3) is proportional to the elliptic genus for pure $SO(4)$ gauge theories with vanishing w_2.

Now, let us assemble these contributions. In principle, for discrete theta angle $\theta \in \{0, \pi\}$,

$$Z(SO(4), \theta) = Z(SO(4), w = 0) + \exp(iw \cdot t) \exp(iw \cdot \theta) Z(SO(4), w \neq 0). \tag{6.5}$$

As computed in [5], $t_a = i\pi m_a$ where

$$\sum_a m_a \equiv 1 \mod 2, \tag{6.6}$$

hence

$$\exp(iw \cdot t) = -1. \tag{6.7}$$

Thus, the elliptic genus is given by

$$Z(SO(4), \theta) = \frac{1}{2} \left(\frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \right)^2 (1 - \exp(i\theta)). \tag{6.8}$$

As a consistency check, note that $Z(SO(4), \theta)$ vanishes for $\theta = 0$, which is consistent with the result [3, section 13.1] that supersymmetry is broken in this theory for $\theta = 0$.

As another consistency check, note that

$$\sum_{\theta=0,\pi} \frac{1}{2} \left(\frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \right)^2 (1 - \exp(i\theta)) = \left(\frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \right)^2, \tag{6.9}$$

the elliptic genus of the pure Spin(4) theory. This confirms the prediction of decomposition [13–15] in this case, which schematically says

$$\text{Spin}(4) = SO(4)_{\theta=0} + SO(4)_{\theta=\pi}. \tag{6.10}$$

7. Pure Spin(4)/($\mathbb{Z}_2 \times \mathbb{Z}_2$) Gauge Theories

The group Spin(4) $= SU(2) \times SU(2)$, so the analysis of this group will be closely related to the analysis of $SU(2)$. We can describe the Lie algebra of Spin(4) in terms of block-diagonal matrices and we can describe sectors with nontrivial characteristic classes by taking holonomies to be of the form

$$\text{diag}(p, 1), \quad \text{diag}(q, 1) \tag{7.1}$$

for one $\mathbb{Z}_2$ and

$$\text{diag}(1, p), \quad \text{diag}(1, q) \tag{7.2}$$

for the other $\mathbb{Z}_2$. Proceeding in a simple generalization of the analysis for a single copy of $SU(2)$, we find results for elliptic genera as follows:

1. Vanishing characteristic class. In this case, the elliptic genus is a product of two copies of the $SU(2)$ elliptic genus (divided by a factor of $|\mathbb{Z}_2 \times \mathbb{Z}_2| = 4$):

$$\frac{1}{4} (Z(SU(2)))^2 = \frac{1}{4} \left(\frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \right)^2. \tag{7.3}$$

2. Nontrivial characteristic class in one $\mathbb{Z}_2$. Here, if we let $Z(SO(3)_1)$ denote the elliptic genus of a single $SO(3)$ theory with nontrivial characteristic class, then the elliptic genus is given by

$$\frac{1}{2} Z(SU(2)) Z(SO(3)_1) = \frac{1}{4} \left(\frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \right)^2 \tag{7.4}$$

(up to a phase).
3. Nontrivial characteristic classes in both $\mathbb{Z}_2$'s. Here, the elliptic genus is given by

$$(Z(SO(3)_1))^2 = \left(\frac{1}{2} \frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \right)^2 \tag{7.5}$$

(up to a phase).

In the expressions above, we have used that

$$Z(SU(2)) \;=\; \frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \tag{7.6}$$

and

$$Z(SO(3)_1) \;=\; \frac{1}{2} \frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \tag{7.7}$$

up to a phase, matching [12].

Now, let us assemble these results. In principle, a sector of bundles of nontrivial characteristic class should be weighted by factors $\exp(iw \cdot t)$ and $\exp(i\theta)$, for θ a discrete theta angle, and using results in [5], one can derive both phases for each sector. However, in this case there is a faster way, as the gauge group can equivalently be written as $SO(3) \times SO(3)$, so we can reuse the results of [12, appendix A] to immediately write the elliptic genus of a pure $\mathrm{Spin}(4)/\mathbb{Z}_2 \times \mathbb{Z}_2$ gauge theory with discrete theta angles (θ_1, θ_2), $\theta_i \in \{0, \pi\}$ as

$$\left[\frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \right]^2 \left(\frac{1 - \exp(i\theta_1)}{2} \right) \left(\frac{1 - \exp(i\theta_2)}{2} \right). \tag{7.8}$$

In particular, note that

$$\sum_{\theta_1, \theta_2 \in \{0,\pi\}} \left[\frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \right]^2 \left(\frac{1 - \exp(i\theta_1)}{2} \right) \left(\frac{1 - \exp(i\theta_2)}{2} \right) = \left[\frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \right]^2, \tag{7.9}$$

and so we see that the elliptic genus of the pure $\mathrm{Spin}(4)$ theory matches that of the sum of the elliptic genera of pure $\mathrm{Spin}(4)/\mathbb{Z}_2 \times \mathbb{Z}_2$ theories with the various possible discrete theta angles, as expected from decomposition [13–15] of two-dimensional theories with a $B(\mathbb{Z}_2 \times \mathbb{Z}_2)$ symmetry.

8. Pure $SO(5)$ Gauge Theories

Now, let us turn to elliptic genera for pure $SO(5)$ gauge theories. From Eq. (2.4) and the fact that there are two operators, $\mathrm{tr}\, \Sigma^2$ and $\mathrm{tr}\, \Sigma^4$, of R-charges 4 and 8, one expects that the elliptic genus of the pure $\mathrm{Spin}(5)$ theory and that of a pure $SO(5)$ theory for one value of the discrete theta angle is

$$\frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \frac{\theta_1(\tau| - 3z)}{\theta_1(\tau| - 4z)}, \tag{8.1}$$

as discussed in Sect. 2. This will also be derived by a direct residue computation in [11].

For bundles with vanishing w_2, from Eq. (3.2), the contribution to the elliptic genus of the pure $SO(5)$ gauge theory is $1/2$ of the elliptic genus of the pure $\mathrm{Spin}(5)$ theory

$$\frac{1}{2} \frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \frac{\theta_1(\tau| - 3z)}{\theta_1(\tau| - 4z)}, \tag{8.2}$$

as discussed in Sect. 3.

Table 4. List of eigenvalues of $SO(5)$ under the adjoint action of p, q

ω_p	ω_q	θ argument
-1	-1	$-(1+\tau)/2$
-1	$+1$	$1/2$
$+1$	-1	$\tau/2$
$+1$	$+1$	0
$-\exp(2\pi i\lambda_1)$	$-\exp(2\pi i\lambda_2)$	$-(1+\tau)/2+u$
$+\exp(-2\pi i\lambda_1)$	$-\exp(-2\pi i\lambda_2)$	$\tau/2-u$
$-\exp(-2\pi i\lambda_1)$	$-\exp(-2\pi i\lambda_2)$	$-(1+\tau)/2-u$
$\exp(2\pi i\lambda_1)$	$-\exp(2\pi i\lambda_2)$	$\tau/2+u$
$-\exp(2\pi i\lambda_1)$	$+\exp(2\pi i\lambda_2)$	$1/2+u$
$-\exp(-2\pi i\lambda_1)$	$+\exp(-2\pi i\lambda_2)$	$1/2-u$

Next, let us consider the case of nonzero w_2, which we analyze following the pattern of Sect. 3. Following [9, equ'n (3.3)], we can express the holonomies p, q in the form

$$p = \text{diag}\left(\exp(2\pi i\lambda_1\sigma_2), -1, -1, +1\right), \quad q = \text{diag}\left(\exp(2\pi i\lambda_2\sigma_2), +1, -1, -1\right).$$
(8.3)

Since we have already descended to $SO(5)$ matrices, and are not working in Spin(5), these matrices commute. Then, we diagonalize, finding a basis T^α of the Lie algebra such that

$$pT^\alpha p^{-1} = \omega_p^\alpha T^\alpha, \quad qT^\alpha q^{-1} = \omega_q^\alpha T^\alpha.$$
(8.4)

Doing so, we find the eigenvalues $\omega_{p,q}^\alpha$, which we list in Table 4. In each case, the θ argument is computed as

$$\frac{\ln \omega_p^\alpha}{2\pi i} + \tau \frac{\ln \omega_q^\alpha}{2\pi i},$$
(8.5)

and $u = \lambda_1 + \tau\lambda_2$. The number of eigenvalues $(\omega_p, \omega_q) = (1, 1)$ gives the dimension of the residue integral, as it reflects moduli of flat connections that are not constrained by the holonomies p, q.

Alternatively, one could think of Table 4 in terms of a (maximal-rank) $SO(2) \times SO(3)$ subgroup of $SO(5)$. The weights of the nonzero roots of $SO(5)$ are

$$\alpha \cdot u \in \{\pm u_1 \pm u_2, \pm u_1, \pm u_2\},$$
(8.6)

In principle, for nonzero holonomies, the product over roots is of the same form as in the case $w_2 = 0$, except that the values of one of the u_i are constrained (to match those of $SU(2)$, while the other is unconstrained. Thinking of the roots above in this fashion can also be used to generate Table 4.

In any event, from Table 4, we read off a one-dimensional residue integral, proportional to

$$\frac{N}{2}\left(\frac{2\pi\eta(q)^3}{\theta_1(\tau|-z)}\right)\oint \frac{du}{2\pi i}$$
$$\cdot \frac{\theta_1(\tau|u+1/2)}{\theta_1(\tau|-z+u+1/2)} \frac{\theta_1(\tau|u+\tau/2)}{\theta_1(\tau|-z+u+\tau/2)} \frac{\theta_1(\tau|u-(1+\tau)/2)}{\theta_1(\tau|-z+u-(1+\tau)/2)}$$
$$\cdot \frac{\theta_1(\tau|-u+1/2)}{\theta_1(\tau|-z-u+1/2)} \frac{\theta_1(\tau|-u+\tau/2)}{\theta_1(\tau|-z-u+\tau/2)} \frac{\theta_1(\tau|-u-(1+\tau)/2)}{\theta_1(\tau|-z-u-(1+\tau)/2)},$$
(8.7)

where

$$N = \frac{\theta_1(\tau | + 1/2)}{\theta_1(\tau | - z + 1/2)} \frac{\theta_1(\tau | + \tau/2)}{\theta_1(\tau | - z + \tau/2)} \frac{\theta_1(\tau | - (1 + \tau)/2)}{\theta_1(\tau | - z - (1 + \tau)/2)}, \tag{8.8}$$

$$= 2 \frac{\theta_1(\tau | - z)}{\theta_1(\tau | - 2z)}, \tag{8.9}$$

where the second line follows from [12, equ'n (A.6)].

From Table 3, the moduli space of flat $SO(5)$ connections with nontrivial characteristic class is the same as the moduli space of flat $SU(2)$ connections, i.e., $T^2/\mathbb{Z}_2$, which is the origin of the integral above. We integrate over the covering space T^2 and add a factor of $1/2$ (given in the expression above) to take into account the fact that we are integrating over a double cover of the moduli space.

Let us now evaluate this integral. Following the Jeffrey–Kirwan residue prescription in this case, we consider residues about three of the six poles, defined by denominators with positive u coefficients. (Alternatively, we could sum only over poles with negative u coefficients, but we pick the former convention in this paper.) These poles are given by

$$u = z - 1/2, \quad z - \tau/2, \quad z + (1 + \tau)/2. \tag{8.10}$$

The fact that the integrand is symmetric under $u \mapsto -u$ reflects the Weyl group action on the moduli space of flat $SU(2)$ connections. Also note that the prescription above is summing over distinct residues which are not related by the Weyl group.

We will use the identity [6, equ'n (B.6)]

$$\theta_1'(\tau | 0) = 2\pi \eta(q)^3, \tag{8.11}$$

where the derivative is taken with respect to the second variable. As a result, and using the fact that [6, equ'n (B.4)]

$$\theta_1(\tau | z + a + b\tau) = (-)^{a+b} \exp(-2\pi i b z - i\pi b^2 \tau) \theta_1(\tau | z) \tag{8.12}$$

for $a, b \in \mathbb{Z}$, one has [6, equ'n (B.7)]

$$\frac{1}{2\pi i} \int_{u=a+b\tau} \frac{du}{\theta_1(\tau | u)} = (-)^{a+b} \frac{\exp(i\pi b^2 \tau)}{\theta_1'(\tau | 0)} = (-)^{a+b} \frac{\exp(i\pi b^2 \tau)}{2\pi \eta(q)^3}, \tag{8.13}$$

for $a, b \in \mathbb{Z}$.

From the pole at $u = z - 1/2$, we have a contribution

$$\frac{1}{2} \frac{N}{\theta_1(\tau | - z)} \theta_1(\tau | + z) \frac{\theta_1(\tau | z - 1/2 + \tau/2)}{\theta_1(\tau | - 1/2 + \tau/2)} \frac{\theta_1(\tau | z - 1/2 - (1 + \tau)/2)}{\theta_1(\tau | - 1/2 - (1 + \tau)/2)}$$
$$\cdot \frac{\theta_1(\tau | - z + 1/2)}{\theta_1(\tau | - 2z)} \frac{\theta_1(\tau | - z + 1/2 + \tau/2)}{\theta_1(\tau | - 2z + 1/2 + \tau/2)} \frac{\theta_1(\tau | - z + 1/2 - (1 + \tau)/2)}{\theta_1(\tau | - 2z + 1/2 - (1 + \tau)/2)}. \tag{8.14}$$

From the pole at $u = z - \tau/2$, we have a contribution

$$\frac{1}{2} \frac{N}{\theta_1(\tau | - z)} \theta_1(\tau | + z) \frac{\theta_1(\tau | z + 1/2 - \tau/2)}{\theta_1(\tau | + 1/2 - \tau/2)} \frac{\theta_1(\tau | z - \tau/2 - (1 + \tau)/2)}{\theta_1(\tau | - \tau/2 - (1 + \tau)/2)}$$
$$\cdot \frac{\theta_1(\tau | - z + \tau/2 + 1/2)}{\theta_1(\tau | - 2z + \tau/2 + 1/2)} \frac{\theta_1(\tau | - z + \tau)}{\theta_1(-z + \tau)} \frac{\theta_1(\tau | - z - 1/2)}{\theta_1(\tau | - 2z - 1/2)}. \tag{8.15}$$

From the pole at $u = z + (1 + \tau)/2$, we have a contribution

$$\frac{1}{2} \frac{N}{\theta_1(\tau| - z)} \theta_1(\tau| + z) \frac{\theta_1(\tau|z + 1/2 + (1 + \tau)/2)}{\theta_1(1/2 + (1 + \tau)/2)} \frac{\theta_1(\tau|z + 1/2 + \tau)}{\theta_1(\tau|1/2 + \tau)}$$

$$\cdot \frac{\theta_1(\tau| - z - \tau/2)}{\theta_1(\tau| - 2z - \tau/2)} \frac{\theta_1(\tau| - z - 1/2)}{\theta_1(\tau| - 2z - 1/2)} \frac{\theta_1(\tau| - z - 1 - \tau)}{\theta_1(\tau| - 2z - 1 - \tau)}. \tag{8.16}$$

One can verify (e.g. numerically) that the sum of these residues is

$$2 \frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \frac{\theta_1(\tau| - 3z)}{\theta_1(\tau| - 4z)}. \tag{8.17}$$

To derive $Z(SO(5), w \neq 0)$, we still need a numerical factor, $1/|W|$ for some W as in [46]. Rather than compute W directly, for the moment, we write

$$Z(SO(5), w \neq 0) = \alpha Z(SO(5), w = 0) \tag{8.18}$$

for some positive real number α, which we will compute by using known results for supersymmetry breaking.

Now, let us assemble these results into the elliptic genus for $SO(5)$ with discrete theta angle $\theta \in \{0, \pi\}$. The contribution from the sector with $w_2 = 0$ is independent of θ, and is just a factor of $1/|\Gamma|$ away from the elliptic genus of Spin(5):

$$Z(SO(5), w = 0) = \frac{1}{2} \frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \frac{\theta_1(\tau| - 3z)}{\theta_1(\tau| - 4z)}. \tag{8.19}$$

Next, we consider the contribution from the sector with $w \neq 0$. There is a factor of $\exp(i\theta)$ from the discrete theta angle $\theta \in \{0, \pi\}$. In addition, there is also a phase $\exp(iw \cdot t)$ where, from the analysis of [5],

$$t_a = i\pi m_a, \tag{8.20}$$

where

$$\sum_a m_a \equiv 1 \mod 2. \tag{8.21}$$

As a result, $\exp(iw \cdot t) = -1$.

Putting this together, we have the elliptic genus of a pure supersymmetric $SO(5)$ gauge theory as a function of discrete theta angle $\theta \in \{0, \pi\}$:

$$Z(SO(5), \theta) = Z(SO(5), w = 0) - \exp(i\theta)Z(SO(5), w \neq 0), \tag{8.22}$$

$$= \frac{1}{2} \frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \frac{\theta_1(\tau| - 3z)}{\theta_1(\tau| - 4z)} (1 - \alpha \exp(i\theta)). \tag{8.23}$$

From [3, section 13.2], we know that supersymmetry is broken in pure $SO(5)$ theories with $\theta = 0$, hence we must require that $\alpha = 1$, hence the elliptic genus of the pure $SO(5)$ theory with discrete theta angle θ is

$$Z(SO(5), \theta) = \frac{1}{2} \frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \frac{\theta_1(\tau| - 3z)}{\theta_1(\tau| - 4z)} (1 - \exp(i\theta)). \tag{8.24}$$

As a consistency check, note that α is a positive real number, as expected—phase factors have already been accounted for. As another consistency check, note that for

$\theta = \pi$, the elliptic genus of the pure $SO(5)$ gauge theory matches that of the Spin(5) theory, in agreement with expectations from [3, section 13.2].

As another consistency check, note that this implies that the elliptic genus of the pure Spin(5) theory is the sum of the elliptic genera of the pure $SO(5)$ theories with either value of θ:

$$\text{Spin}(5) = SO(5)_{\theta=0} + SO(5)_{\theta=\pi}, \qquad (8.25)$$

which is consistent with decomposition of two-dimensional theories with a $B\mathbb{Z}_2$ symmetry [13–15].

9. Pure $Sp(6)/\mathbb{Z}_2$ Gauge Theories

We now turn to pure $Sp(6)/\mathbb{Z}_2$ gauge theories (in conventions in which $Sp(2) = SU(2)$). Since $Sp(2) = SU(2)$ and $Sp(4) = \text{Spin}(5)$, the first interesting case amongst $Sp(2k)/\mathbb{Z}_2$ is $Sp(6)/\mathbb{Z}_2$.

As before, for bundles of vanishing characteristic class, from Eq. (3.2), the elliptic genus matches that of the pure $Sp(6)$ gauge theory, up to the factor $1/|\Gamma|$:

$$Z(Sp(6)/\mathbb{Z}_2, w = 0) = \frac{1}{2} Z(Sp(6)) = \frac{1}{2} \frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \frac{\theta_1(\tau| - 3z)}{\theta_1(\tau| - 4z)} \frac{\theta_1(\tau| - 5z)}{\theta_1(\tau| - 6z)}, \quad (9.1)$$

as discussed in Sect. 3.

To describe a nontrivial bundle, we give two anticommuting holonomies in $Sp(2k)$, which following [49, section 4.1], [50, equ'n (8)] we can take to be

$$p = \text{diag}\left(\lambda_1, -\lambda_1, i, -i, -\lambda_1^{-1}, \lambda_1^{-1}\right), \qquad (9.2)$$

$$q = \text{diag}\left(\begin{bmatrix} 0 & -\lambda_2 \\ -\lambda_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\lambda_2^{-1} \\ -\lambda_2^{-1} & 0 \end{bmatrix}\right), \qquad (9.3)$$

and where we take the symplectic form to be

$$\Omega = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (9.4)$$

so that

$$p^T \Omega p = \Omega, \quad q^T \Omega q = \Omega. \qquad (9.5)$$

Following the procedure of Sect. 3, we diagonalize a basis of the Lie algebra [12] with respect to the diagonal action of p, q above. The eigenvalues $\omega_{p,q}^\alpha$ of the adjoint action are given in Table 5.

In Table 5,

$$u = \frac{\ln \lambda_1}{2\pi i} + \tau \frac{\ln \lambda_2}{2\pi i}, \qquad (9.6)$$

---

[12] In case the reader finds it helpful, the Lie algebra with the symplectic form given in Eq. (9.4) is described in detail in [51, Chapter 30].

Table 5. Table of eigenvalues of the adjoint action of the holonomy matrices

ω_p	ω_q	θ argument
λ_1^{-2}	$-\lambda_2^{-2}$	$\tau/2 - 2u$
λ_1^{-2}	λ_2^{-2}	$-2u$
$-\lambda_1^{-2}$	λ_2^{-2}	$1/2 - 2u$
$-i\lambda_1^{-1}$	$-i\lambda_2^{-1}$	$3/4 + 3\tau/4 - u$
$i\lambda_1^{-1}$	$-i\lambda_2^{-1}$	$1/4 + 3\tau/4 - u$
$-i\lambda_1^{-1}$	$i\lambda_2^{-1}$	$3/4 + \tau/4 - u$
$i\lambda_1^{-1}$	$i\lambda_2^{-1}$	$1/4 + \tau/4 - u$
-1	-1	$-(1+\tau)/2$
-1	-1	$-(1+\tau)/2$
1	-1	$\tau/2$
1	-1	$\tau/2$
-1	1	$1/2$
-1	1	$1/2$
1	1	0
$-i\lambda_1$	$-i\lambda_2$	$3/4 + 3\tau/4 + u$
$i\lambda_1$	$-i\lambda_2$	$1/4 + 3\tau/4 + u$
$-i\lambda_1$	$i\lambda_2$	$3/4 + \tau/4 + u$
$i\lambda_1$	$i\lambda_2$	$1/4 + \tau/4 + u$
λ_1^2	$-\lambda_2^2$	$\tau/2 + 2u$
λ_1^2	λ_2^2	$2u$
$-\lambda_1^2$	λ_2^2	$1/2 + 2u$

and the θ coefficient is

$$\frac{\ln \omega_p^\alpha}{2\pi i} + \tau \frac{\ln \omega_q^\alpha}{2\pi i}. \tag{9.7}$$

As a simple consistency check, note that the number of entries, 21, is the same as the dimension of $Sp(6)$.

The fact that there is only one entry in Table 5 with p, q eigenvalues $(1, 1)$ means that the elliptic genus will be computed by a one-dimensional residue integral. From Table 3, we see that the moduli space of flat $Sp(6)/\mathbb{Z}_2$ connections with nontrivial characteristic class is the same as the moduli space of $SU(2)$ connections—indeed, one-dimensional. The moduli space is $T^2/\mathbb{Z}_2$, but we will integrate over the T^2 cover, quotienting by a factor of 2 to reflect that fact.

Putting this together, the elliptic genus of a pure $Sp(6)/\mathbb{Z}_2$ gauge theory with bundles of nontrivial characteristic class is proportional to

$$\frac{N}{2} \left(\frac{2\pi \eta(q)^3}{\theta_1(\tau| - z)} \right) \oint \frac{du}{2\pi i}$$
$$\cdot \frac{\theta_1(\tau|3/4 + 3\tau/4 + u)}{\theta_1(-z + 3/4 + 3\tau/4 + u)} \frac{\theta_1(\tau|1/4 + 3\tau/4 + u)}{\theta_1(\tau| - z + 1/4 + 3\tau/4 + u)} \frac{\theta_1(\tau|3/4 + \tau/4 + u)}{\theta_1(\tau| - z + 3/4 + \tau/4 + u)}$$
$$\cdot \frac{\theta_1(\tau|1/4 + \tau/4 + u)}{\theta_1(\tau| - z + 1/4 + \tau/4 + u)} \frac{\theta_1(\tau|\tau/2 + 2u)}{\theta_1(\tau| - z + \tau/2 + 2u)} \frac{\theta_1(\tau|2u)}{\theta_1(\tau| - z + 2u)} \frac{\theta_1(\tau|1/2 + 2u)}{\theta_1(-z + 1/2 + 2u)}$$
$$\cdot \frac{\theta_1(\tau|3/4 + 3\tau/4 - u)}{\theta_1(-z + 3/4 + 3\tau/4 - u)} \frac{\theta_1(\tau|1/4 + 3\tau/4 - u)}{\theta_1(\tau| - z + 1/4 + 3\tau/4 - u)} \frac{\theta_1(\tau|3/4 + \tau/4 - u)}{\theta_1(\tau| - z + 3/4 + \tau/4 - u)}$$
$$\cdot \frac{\theta_1(\tau|1/4 + \tau/4 - u)}{\theta_1(\tau| - z + 1/4 + \tau/4 - u)} \frac{\theta_1(\tau|\tau/2 - 2u)}{\theta_1(\tau| - z + \tau/2 - 2u)} \frac{\theta_1(\tau| - 2u)}{\theta_1(\tau| - z - 2u)} \frac{\theta_1(\tau|1/2 - 2u)}{\theta_1(-z + 1/2 - 2u)},$$
$$\tag{9.8}$$

where

$$N = \left[\frac{\theta_1(\tau|-(1+\tau)/2)}{\theta_1(\tau|-z-(1+\tau)/2)} \frac{\theta_1(\tau|\tau/2)}{\theta_1(\tau|-z+\tau/2)} \frac{\theta_1(\tau|1/2)}{\theta_1(\tau|-z+1/2)} \right]^2, \quad (9.9)$$

$$= \left[2\frac{\theta_1(\tau|-z)}{\theta_1(\tau|-2z)} \right]^2, \quad (9.10)$$

using [12, appendix A]. The overall factor of $1/2$ is due to the fact that we are integrating over the double-cover T^2 of the moduli space of flat connections.

The reader will note that the expression above is symmetric under $u \leftrightarrow -u$. This reflects the Weyl group action on the moduli space of flat $SU(2)$ connections, whose double-cover we are integrating over in the expression above.

Following the Jeffrey–Kirwan residue prescription, we will take poles of denominators in which u appears with a positive coefficient. (Alternatively, we could equivalently take poles in which u appears with negative coefficient, but we will use the positive coefficient prescription in this paper.) In passing, note that none of these poles are related by the Weyl group action to one another.

Four of the poles are at

$$u = z - 3/4 - 3\tau/4, \quad z - 1/4 - 3\tau/4, \quad z - 3/4 - \tau/4, \quad z - 1/4 - \tau/4. \quad (9.11)$$

To find all of the remaining poles, one must take into account the periodicities of the theta function. Taking those into account, we find four poles at

$$2u = z - \tau/2 + \{0, 1, \tau, 1+\tau\}, \text{ or } u = z/2 - \tau/4 + \{0, 1/2, \tau/2, (1+\tau)/2)\}, \quad (9.12)$$

another four poles at

$$u = z/2 + \{0, 1/2, \tau/2, (1+\tau)/2)\}, \quad (9.13)$$

and another four at

$$u = z/2 - 1/4 + \{0, 1/2, \tau/2, (1+\tau)/2)\}, \quad (9.14)$$

for a total of 16 residues that must be summed over.

We illustrate a few examples of these residues here, to illustrate the complexity of the computation. The residue at $u = z - 3/4 - 3\tau/4$ is given by

$$\frac{1}{2} \frac{N}{\theta_1(\tau|-z)} \theta_1(\tau|+z) \frac{\theta_1(\tau|z-1/2)}{\theta_1(\tau|-1/2)} \frac{\theta_1(\tau|z-\tau/2)}{\theta_1(\tau|-\tau/2)} \frac{\theta_1(\tau|z-(1+\tau)/2)}{\theta_1(\tau|-(1+\tau)/2)}$$
$$\cdot \frac{\theta_1(\tau|2z-3/2-\tau)}{\theta_1(\tau|z-3/2-\tau)} \frac{\theta_1(\tau|2z-3/2-3\tau/2)}{\theta_1(\tau|z-3/2-3\tau/2)} \frac{\theta_1(\tau|2z-1-3\tau/4)}{\theta_1(\tau|z-3\tau/4)}$$
$$\frac{\theta_1(\tau|-2z+3/2+2\tau)}{\theta_1(\tau|-3z+3/2+2\tau)} \frac{\theta_1(\tau|-2z+3/2+3\tau/2)}{\theta_1(\tau|-3z+3/2+3\tau/2)} \frac{\theta_1(\tau|-2z+2+3\tau/2)}{\theta_1(\tau|-3z+2+3\tau/2)}$$
$$\cdot \frac{\theta_1(\tau|-z+3/2+3\tau/2)}{\theta_1(\tau|-2z+3/2+3\tau/2)} \frac{\theta_1(\tau|-z+1+3\tau/2)}{\theta_1(\tau|-2z+1+3\tau/2)} \frac{\theta_1(\tau|-z+3/2+\tau)}{\theta_1(\tau|-2z+3/2+\tau)}$$
$$\cdot \frac{\theta_1(\tau|-z+1+\tau)}{\theta_1(\tau|-2z+1+\tau)}. \quad (9.15)$$

Similarly, the residue at $u = z/2 - \tau/4$ is given by

$$
\frac{1}{4} \frac{N}{\theta_1(\tau|-z)} \frac{\theta_1(\tau|z/2+3/4+\tau/2)}{\theta_1(\tau|-z/2+3/4+\tau/2)} \frac{\theta_1(\tau|z/2+1/4+\tau/2)}{\theta_1(\tau|-z/2+1/4+\tau/2)} \frac{\theta_1(\tau|z/2+3/4)}{\theta_1(\tau|-z/2+3/4)}
$$

$$
\cdot \frac{\theta_1(\tau|z/2+1/4)}{\theta_1(\tau|-z/2+1/4)} \theta_1(+z) \frac{\theta_1(\tau|z/2-\tau/2)}{\theta_1(\tau|-\tau/2)} \frac{\theta_1(\tau|z+1/2-\tau/2)}{\theta_1(\tau|1/2-\tau/2)}
$$

$$
\cdot \frac{\theta_1(\tau|-z+\tau)}{\theta_1(\tau|-2z+\tau)} \frac{\theta_1(\tau|-z+\tau/2)}{\theta_1(\tau|-2z+\tau/2)} \frac{\theta_1(\tau|-z+1/2+\tau/2)}{\theta_1(\tau|-2z+1/2+\tau/2)} \frac{\theta_1(\tau|-z/2+3/4+\tau)}{\theta_1(\tau|-3z/2+3/4+\tau)}
$$

$$
\cdot \frac{\theta_1(\tau|-z/2+1/4+\tau)}{\theta_1(\tau|-3z/2+1/4+\tau)} \frac{\theta_1(\tau|-z/2+3/4+\tau/2)}{\theta_1(\tau|-3z/2+3/4+\tau/2)} \frac{\theta_1(\tau|-z/2+1/4+\tau/2)}{\theta_1(\tau|-3z/2+1/4+\tau/2)}.
$$

$$(9.16)$$

A leading factor of $1/2$ in the second residue is due to the fact that the pole arises from a theta function denominator that depends upon $2u$ not u. An overall factor of $1/2$ in both residues is due to the fact that we are integrating over T^2 and not $T^2/\mathbb{Z}_2$. For reasons of brevity, we do not list the other fourteen residues here, though they are straightforward to compute.

One can verify numerically that the sum of the residues above, the integral (9.8) equals

$$
8 \frac{\theta_1(\tau|-z)}{\theta_1(\tau|-2z)} \frac{\theta_1(\tau|-3z)}{\theta_1(\tau|-4z)} \frac{\theta_1(\tau|-5z)}{\theta_1(\tau|-6z)}.
$$

$$(9.17)$$

The product of theta functions above should be proportional to the elliptic genus of the pure $Sp(6)/\mathbb{Z}_2$ theory with nonzero characteristic class. The proportionality factor should be a real number of the form $1/|W|$ for W a finite subgroup of the gauge group that preserves the holonomies. For the moment, we will write

$$
Z(Sp(6)/\mathbb{Z}_2, w \neq 0) = \alpha Z(Sp(6)/\mathbb{Z}_2, w = 0), \tag{9.18}
$$

for some positive real number α. We will compute this factor indirectly, using known results for supersymmetry breaking for various discrete theta angles.

Finally, we need to weight the $w \neq 0$ contribution with relevant phases. There is a factor $\exp(i\theta)$ arising from the discrete theta angle $\theta \in \{0, \pi\}$. In addition, there is potentially a factor of $\exp(iw \cdot t)$. From [5, section 5],

$$
t_a = \pi i m_a \tag{9.19}
$$

where

$$
\sum_a m_a \equiv 0 \mod 2 \tag{9.20}
$$

(for $Sp(6)/\mathbb{Z}_2$), so without loss of generality we can take all $m_a = 0$, hence $\exp(iw \cdot t) = +1$.

Now, putting this together, combining the result for the elliptic genus in the sector with $w = 0$ with the result above, determined up to a proportionality factor, for $w \neq 0$, we have that the elliptic genus of a pure $Sp(6)/\mathbb{Z}_2$ gauge theory with discrete theta angle $\theta \in \{0, \pi\}$ is given by

$$
Z(Sp(6)/\mathbb{Z}_2, \theta) = Z(Sp(6)/\mathbb{Z}_2, w = 0) + \alpha \exp(i\theta) Z(Sp(6)/\mathbb{Z}_2, w \neq 0), \tag{9.21}
$$

$$
= \frac{1}{2} \frac{\theta_1(\tau|-z)}{\theta_1(\tau|-2z)} \frac{\theta_1(\tau|-3z)}{\theta_1(\tau|-4z)} \frac{\theta_1(\tau|-5z)}{\theta_1(\tau|-6z)} (1 + \alpha \exp(i\theta)). \tag{9.22}
$$

It was argued in [5, section 5] that a pure $Sp(6)/\mathbb{Z}_2$ gauge theory has supersymmetric vacua only if the discrete theta angle $\theta = 0$, hence for $\theta = \pi$, supersymmetry is broken, and the elliptic genus should vanish. Imposing this as a constraint, we find that $\alpha = +1$, hence the elliptic genus of a pure $Sp(6)/\mathbb{Z}_2$ gauge theory as a function of discrete theta angle θ is

$$Z(Sp(6)/\mathbb{Z}_2, \theta) = \frac{1}{2} \frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \frac{\theta_1(\tau| - 3z)}{\theta_1(\tau| - 4z)} \frac{\theta_1(\tau| - 5z)}{\theta_1(\tau| - 6z)} (1 + \exp(i\theta)). \quad (9.23)$$

As a consistency check, note that α is real and positive, as expected—phase factors have already been accounted for. As another consistency check, note that for $\theta = 0$, the elliptic genus of the pure $Sp(6)/\mathbb{Z}_2$ gauge theory matches that of the pure $Sp(6)$ gauge theory, in agreement with expectations from [5, section 5].

As a further consistency check, it is straightforward to see that this result is consistent with decomposition [13–15]:

$$\sum_{\theta=0,\pi} \frac{1}{2} \frac{\theta_1(\tau| - z)}{\theta_1(\tau| - 2z)} \frac{\theta_1(\tau| - 3z)}{\theta_1(\tau| - 4z)} \frac{\theta_1(\tau| - 5z)}{\theta_1(\tau| - 6z)} (1 + \exp(i\theta)) = Z(Sp(6)), \quad (9.24)$$

consistent with the expectation

$$Sp(6) = (Sp(6)/\mathbb{Z}_2)_{\theta=0} + (Sp(6)/\mathbb{Z}_2)_{\theta=\pi} \quad (9.25)$$

(expressed schematically).

10. Predictions for General Cases

So far, we have performed direct computations to compute elliptic genera of pure gauge theories with semisimple, non-simply-connected gauge groups in some low rank cases. Next, we are going to make a proposal for all cases, utilizing (a) our knowledge of the contribution from $w = 0$, (b) supersymmetry breaking for most discrete theta angles, and (c) decomposition. These three constraints form sufficiently many algebraic equations to enable us to solve algebraically for the elliptic genera.

We illustrate the method using the pure $SU(4)/\mathbb{Z}_4$ gauge theory as an example. First, we know that

$$Z(SU(4)/\mathbb{Z}_4, w = 0) = \frac{1}{4} Z(SU(4)). \quad (10.1)$$

Given the results for low-rank cases, let us assume that

$$Z(SU(4)/\mathbb{Z}_4, w \neq 0) \propto Z(SU(4)/\mathbb{Z}_4, w = 0), \quad (10.2)$$

so we can write

$$Z(SU(4)/\mathbb{Z}_4, \theta) = \frac{1}{4} Z(SU(4)) (1 + \alpha_1 \exp(i\theta) + \alpha_2 \exp(2i\theta) + \alpha_3 \exp(3i\theta)), \quad (10.3)$$

for $\theta \in \{0, \pi/2, \pi, 3\pi/2\}$. From Table 2, we see that supersymmetry is broken unless $\theta = \pi$, which gives the constraints

$$1 + \alpha_1 + \alpha_2 + \alpha_3 = 0, \quad (10.4)$$
$$1 + i\alpha_1 - \alpha_2 - i\alpha_3 = 0, \quad (10.5)$$
$$1 - i\alpha_1 - \alpha_2 + i\alpha_3 = 0, \quad (10.6)$$

for $\theta = 0, \pi/2, 3\pi/2$, respectively, and from decomposition, since the elliptic genera vanish for $\theta \neq \pi$, the elliptic genus at $\theta = \pi$ must match that of $SU(4)$, hence

$$1 - \alpha_1 + \alpha_2 - \alpha_3 = 4. \tag{10.7}$$

These are four linear algebraic equations in three unknowns, which happen to admit a unique solution:

$$\alpha_1 = \alpha_3 = -1, \quad \alpha_2 = +1. \tag{10.8}$$

Putting this together, we have that

$$Z(SU(4)/\mathbb{Z}_4, \theta) = \frac{1}{4} \frac{\theta_1(\tau | - z)}{\theta_1(\tau | - 4z)} (1 - \exp(i\theta) + \exp(2i\theta) - \exp(3i\theta)). \tag{10.9}$$

We have used our knowledge of supersymmetry breaking and decomposition, and only assumed that the contributions from sectors of various characteristic classes are proportional to one another. One can check that the resulting phase factors, derived algebraically, are consistent with those described in Sect. 3.

Proceeding in this fashion, using our knowledge of supersymmetry breaking and decomposition, elliptic genera are straightforward to predict for all other cases. We summarize the results below.

First, for $SU(k)/\mathbb{Z}_k$, for k odd, supersymmetry is unbroken only for $\theta = 0$ (from Table 2), and we predict the elliptic genus

$$Z(SU(k)/\mathbb{Z}_k, \theta) = \frac{1}{k} \frac{\theta_1(\tau | - z)}{\theta_1(\tau | - kz)} \sum_{m=0}^{k-1} \exp(im\theta), \tag{10.10}$$

for $\theta \in \{0, 2\pi/k, 4\pi/k, \cdots, 2(k-1)\pi/k\}$. For k even, supersymmetry is unbroken only for $\theta = \pi$ and we predict the elliptic genus

$$Z(SU(k)/\mathbb{Z}_k, \theta) = \frac{1}{k} \frac{\theta_1(\tau | - z)}{\theta_1(\tau | - kz)} \sum_{m=0}^{k-1} (-)^m \exp(im\theta). \tag{10.11}$$

Proceeding similarly, for $\mathrm{Spin}(2k+1)/\mathbb{Z}_2$, we predict the elliptic genus

$$Z(\mathrm{Spin}(2k+1)/\mathbb{Z}_2, \theta) = \frac{1}{2} Z(\mathrm{Spin}(2k+1)) (1 - \exp(i\theta)), \tag{10.12}$$

where $Z(\mathrm{Spin}(2k+1))$ denotes the elliptic genus of the pure $\mathrm{Spin}(2k+1)$ gauge theory, as given in Sect. 2, and for $\theta \in \{0, \pi\}$.

For $\mathrm{Spin}(4k)/\mathbb{Z}_2 \times \mathbb{Z}_2$, we predict the elliptic genus

$$Z(\mathrm{Spin}(4k)/\mathbb{Z}_2 \times \mathbb{Z}_2, \theta_1, \theta_2)$$
$$= \frac{1}{4} Z(\mathrm{Spin}(4k)) \left(1 + (-)^k \exp(i\theta_1)\right) \left(1 + (-)^k \exp(i\theta_2)\right), \tag{10.13}$$

for $\theta_{1,2} \in \{0, \pi\}$.

For $\mathrm{Spin}(4k+2)/\mathbb{Z}_4$, we predict the elliptic genus

$$Z(\mathrm{Spin}(4k+2)/\mathbb{Z}_4, \theta) = \frac{1}{4} Z(\mathrm{Spin}(4k+2)) \sum_{m=0}^{3} (-)^{km} \exp(im\theta), \tag{10.14}$$

for $\theta \in \{0, \pi\}$.

For $Sp(2k)/\mathbb{Z}_2$, we predict the elliptic genus

$$Z(Sp(2k)/\mathbb{Z}_2, \theta) = \frac{1}{2} Z(Sp(2k)) \left(1 + (-)^m \exp(i\theta)\right), \qquad (10.15)$$

for $\theta \in \{0, \pi\}$, where

$$m = \begin{cases} k/2 & k \text{ even}, \\ (k+1)/2 & k \text{ odd}. \end{cases} \qquad (10.16)$$

For $E_6/\mathbb{Z}_3$, we predict the elliptic genus

$$Z(E_6/\mathbb{Z}_3, \theta) = \frac{1}{3} Z(E_6) \left(1 + \exp(i\theta) + \exp(2i\theta)\right), \qquad (10.17)$$

for $\theta \in \{0, 2\pi/3, 4\pi/3\}$.

For $E_7/\mathbb{Z}_2$, we predict the elliptic genus

$$Z(E_7/\mathbb{Z}_2, \theta) = \frac{1}{2} Z(E_7) \left(1 - \exp(i\theta)\right), \qquad (10.18)$$

for $\theta \in \{0, \pi\}$.

As a consistency check, note that the elliptic genus of $SU(2)/\mathbb{Z}_2$ matches that of $\mathrm{Spin}(3)/\mathbb{Z}_2$, the elliptic genus of $SU(4)/\mathbb{Z}_4$ matches that of $\mathrm{Spin}(6)/\mathbb{Z}_4$, and the elliptic genus of $Sp(4)/\mathbb{Z}_2$ matches that of $\mathrm{Spin}(5)/\mathbb{Z}_2$, as expected since the Lie groups are the same.

In each case, the elliptic genus vanishes for discrete theta angles θ for which supersymmetry is broken in the IR (from Table 2), and decomposition [13–15] is obeyed:

$$Z(G) = \sum_\theta Z(G/\Gamma, \theta). \qquad (10.19)$$

11. Conclusions

In this paper we have described a systematic method to compute elliptic genera of pure two-dimensional (2,2) supersymmetric G/Γ gauge theories with various discrete theta angles. Our results agree with previous computations of elliptic genera of pure $SO(3)$ gauge theories, and we also derived the elliptic genera of pure $SU(3)/\mathbb{Z}_3$, $SO(4)$, $\mathrm{Spin}(4)/\mathbb{Z}_2 \times \mathbb{Z}_2$, $SO(5)$ and $Sp(6)/\mathbb{Z}_2$ gauge theories. In each case, the results are consistent with predictions of supersymmetry breaking for certain discrete theta angles in [3–5], and the resulting elliptic genera are also consistent with expectations from decomposition [13–15] of two-dimensional gauge theories with finite global one-form symmetries. Finally, we applied these two criteria to make predictions for elliptic genera of higher-rank cases.

Pure two-dimensional (2,2) supersymmetric gauge theories have also been extensively studied by lattice simulations [52–56]. Our results also provide new analytic results that can be used to test and callibrate future lattice studies of pure two-dimensional supersymmetric gauge theories. They also suggest new avenues for research such as varying the global structure of the gauge group and including discrete theta angles.

Gauge theories correspond to sigma models on stacks [18–20], and the elliptic genera we have computed in this paper should correspond to elliptic genera of the classifying stacks BG [21,22].

The sensitivity of the elliptic genus to the global structure of the gauge group makes it a powerful tool to investigate of two-dimensional dualities. The elliptic genus has already been used to test several of Hori's proposed dualities [32] in [12, 30, 31]. Looking forward, we expect the elliptic genus of G/Γ gauge theories will be useful to establish new dualities and will help with exploring the dynamics of two-dimensional supersymmetric gauge theories [13].

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[13] That said, elliptic genera should be applied with care. For example, we have seen earlier in this paper that the $SU(2)$ elliptic genus matches that of $SO(3)_-$. However, these two theories are not dual to one another. Instead, the $SU(2)$ theory is a sum of the two $SO(3)$ theories, with each value of the discrete theta angle. Because supersymmetry is broken in the $SO(3)_+$ theory, the elliptic genus only receives contributions from the $SO(3)_-$ theory. We see that relying solely upon the equality of elliptic genera can be misleading in trying to find dualities.

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