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The Initial Boundary Value Problem for the Einstein Equations with Totally Geodesic Timelike Boundary

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Abstract: We prove the well-posedness of the initial boundary value problem for the Einstein equations with sole boundary condition the requirement that the timelike boundary is totally geodesic. This provides the first well-posedness result for this specific geometric boundary condition and the first setting for which geometric uniqueness in the original sense of Friedrich holds for the initial boundary value problem. Our proof relies on the ADM system for the Einstein vacuum equations, formulated with respect to a parallelly propagated orthonormal frame along timelike geodesics. As an independent result, we first establish the well-posedness in this gauge of the Cauchy problem for the Einstein equations, including the propagation of constraints. More precisely, we show that by appropriately modifying the evolution equations, using the constraint equations, we can derive a first order symmetric hyperbolic system for the connection coefficients of the orthonormal frame. The propagation of the constraints then relies on the derivation of a hyperbolic system involving the connection, suitably modified Riemann and Ricci curvature tensors and the torsion of the connection. In particular, the connection is shown to agree with the Levi-Civita connection at the same time as the validity of the constraints. In the case of the initial boundary value problem with totally geodesic boundary, we then verify that the vanishing of the second fundamental form of the boundary leads to homogeneous boundary conditions for our modified ADM system, as well as for the hyperbolic system used in the propagation of the constraints. An additional analytical difficulty arises from a loss of control on the normal derivatives to the boundary of the solution. To resolve this issue, we work with an anisotropic scale of Sobolev spaces and exploit the specific structure of the equations.

Contents

1. Introduction

This article establishes the well-posedness of the initial boundary value problem (IBVP) for the Einstein vacuum equations

$$\mathrm{Ric}(\mathbf{g}) = 0, \tag{1.1}$$

in the specific case of a *totally geodesic* timelike boundary.

1.1. The initial boundary value problem in general relativity. In the standard formulation of the Cauchy problem for the Einstein vacuum equations, given a Riemannian manifold (Σ, h) and a symmetric 2-tensor k satisfying the constraints equations

$$R - |k|^2 + (\mathrm{tr}k)^2 = 0, \tag{1.2}$$
$$\mathrm{div}k - \mathrm{dtr}k = 0, \tag{1.3}$$

where R is the scalar curvature of the Riemannian metric h and all operators are taken with respect to h, the goal is to construct a Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ solution to the Einstein equations, together with an embedding of Σ into $\mathcal{M}$ such that (h, k) coincides with the first and second fundamental form of the embedding. For the IBVP, we now require that Σ is a manifold with boundary $\mathcal{S}$. We consider an additional manifold $\mathcal{B} = \mathbb{R} \times \mathcal{S}$, a section $\mathcal{S}_0 = \{0\} \times \mathcal{S}$ of $\mathcal{B}$ which is identified with the boundary $\mathcal{S}$ of Σ via a diffeomorphim ψ_{s,s_0}, and a set of functions $\mathcal{BC}$ on $\mathcal{B}$ representing source terms for the chosen boundary conditions. On top of the constraint equations, the initial and boundary data must also now verify the so-called corner or compatibility conditions, a set of equations involving $h, k, \mathcal{BC}$, their derivatives at all orders on $\mathcal{S}$ and $\mathcal{S}_0$, as well as a given real function ω on $\mathcal{S}$, which will eventually represent the angle between the initial slice and the timelike boundary.

A solution to the IBVP is then a Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ with a timelike boundary $\mathcal{T}$, an embbeding ψ_t of a neigbhoorhood of $\mathcal{S}_0 \subset \mathcal{B}$ into $\mathcal{T}$, such that the boundary data $\mathcal{BC}$ can be identified with the corresponding data on $\mathcal{T}$, and an embedding ψ_i of Σ into $\mathcal{M}$ respecting the initial data, with $\psi_i(\mathcal{S}) = \psi_t(\mathcal{S}_0)$, such that $\psi_t^{-1} \circ [\psi_i]_{|\mathcal{S}} = \psi_{s,s_0}$ and the angle between $\mathcal{T}$ and the future unit normal to $\psi_i(\Sigma)$ is $\omega \circ \left[\psi_i^{-1}\right]_{\psi_i(\mathcal{S})}$.

There is a priori a large freedom in the choice of boundary conditions. The sources $\mathcal{BC}$ could correspond to the values of tensor fields encoding the geometry of $\mathcal{T}$, for instance, the first or second fundamental forms of $\mathcal{T}$, its conformal geometry, some curvature invariants or they could correspond to components of geometric tensor fields in some gauge and boundary conditions for the gauge itself.

The IBVP is related to many important aspects of general relativity and the Einstein equations, such as numerical relativity, the construction of asymptotically Anti-de-Sitter spacetimes, timelike hypersurfaces emerging as the boundaries of the support of massive matter fields or the study of gravitational waves in a cavity and their nonlinear interactions. This problem was first addressed for the Einstein equations in the seminal work of Friedrich-Nagy [13], as well as by Friedrich [10] in the related Anti-de-Sitter setting.[1] Well-posedness of the IBVP has since been obtained in generalized wave coordinates, see [17] or the recent [1],[2] and for various first and second order systems derived from the ADM formulation of the Einstein equations, see for instance [9,19] and previous work in numerics [2,14]. We refer to [20] for an extensive review of the subject.

1.2. Geometric uniqueness. One of the remaining outstanding issues, concerning the study of the Einstein equations in the presence of a timelike boundary, is the geometric uniqueness problem of Friedrich [12]. Apart from the construction of asymptotically Anti-de-Sitter spacetimes [10], where the timelike boundary is a conformal boundary at spacelike infinity, all results establishing well-posedness, for some formulations of the IBVP, impose certain gauge conditions on the boundary, and the boundary data depend on these choices. In particular, given a solution to the Einstein equations with a timelike boundary, different gauge choices will lead to different boundary data, in each of the formulations for which well-posedness is known. On the other hand, if we had been given the different boundary data a priori, we would not know that these lead to the same solution. The situation is thus different from the usual initial value problem, for which only isometric data lead to isometric solutions, which one then regards as the same solution.

In the Anti-de-Sitter setting, this problem admits one solution: in [10], Friedrich proved that one can take the conformal metric of the boundary as boundary data, which is a geometric condition independent of any gauge. Even in the Anti-de-Sitter setting, it is actually possible to formulate other boundary conditions, such as dissipative boundary conditions, for which one knows how to prove well-posedness, however, with a formulation of the boundary conditions that is gauge dependent and thus, such that we do not know whether geometric uniqueness holds or not.

[1] See also [4,7] for extensions and other proofs of well-posedness in the Anti-de-Sitter case.

[2] To be more precise, the boundary data in [1] relies on an auxiliary wave map equation akin to generalized wave coordinates. This introduces a geometric framework to address the IBVP, albeit for the Einstein equations coupled to the auxiliary wave map equation.

1.3. The IBVP with totally geodesic boundary. Our main result shows the local well-posedness of the IBVP with totally geodesic boundary, analogous to the classical result [5] for the initial value problem. Recall that in the absence of a timelike boundary, $(\mathcal{M}, g)$ is an extension of $(\mathcal{M}'', g'')$, if there exists an isometric embedding $\psi : \mathcal{M}'' \to \mathcal{M}$, preserving orientation, and such that $\psi \circ \psi_i'' = \psi_i$, where $\psi_i'' : \Sigma \to \mathcal{M}''$ is the embedding of the initial hypersurface into $\mathcal{M}''$. For the IBVP, we require in addition that $\psi \circ \psi_t'' = \psi_t$, where $\psi_t'' : \mathcal{B} \to \mathcal{M}''$ is the embedding of the timelike boundary $\mathcal{B}$ into $\mathcal{M}''$. More precisely, the statement of our theorem is the following:

Theorem 1.1. *Let (Σ, h, k) be a smooth initial data set for the Einstein vacuum equations such that Σ is a 3-manifold with boundary $\partial\Sigma = \mathcal{S}$. For a smooth function ω defined on $\mathcal{S}$, we assume that the corner conditions corresponding to the totally geodesic boundary condition with respect to ω hold on $\mathcal{S}$. Then, there exists a smooth Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ solution to the Einstein vacuum equations with boundary $\partial\mathcal{M} = \widehat{\Sigma} \cup \mathcal{T}$ such that*

1. *there exists an embedding ψ_i of Σ onto $\widehat{\Sigma}$ with (h, k) coinciding with the first and second fundamental form of the embedding,*
2. *$\mathcal{T} \cap \widehat{\Sigma} = \psi_i(\mathcal{S})$ and $\mathcal{T}$ is a timelike hypersurface emanating from $\psi_i(\mathcal{S})$ at an angle $\omega \circ [\psi_i^{-1}]_{|\psi_i(\mathcal{S})}$ relative to the future unit normal of $\widehat{\Sigma}$,*
3. *$\mathcal{T}$ is totally geodesic, i.e. it has vanishing second fundamental form χ,*
4. *geometric uniqueness holds: given any other solution $(\mathcal{M}', \mathbf{g}')$ verifying 1, 2 and 3, $(\mathcal{M}, \mathbf{g})$ and $(\mathcal{M}', \mathbf{g}')$ are both extensions of yet another solution $(\mathcal{M}'', \mathbf{g}'')$ verifying 1, 2 and 3.*

Remark 1.2. The function ω corresponds to the angle of the (hyperbolic) rotation that takes the future unit normal of $\widehat{\Sigma}$ on $\psi_i(\mathcal{S})$ to the future unit normal of $\psi_i(\mathcal{S})$ within $\mathcal{T}$, see (3.2) and Fig. 2.

In the specific case of a totally geodesic boundary, the corner conditions mentioned in the theorem can then be written purely in terms of (h, k), ω and their derivatives at all orders on $\mathcal{S}$. These are the conditions that would be satisfied at the intersection of a totally geodesic timelike boundary and a spacelike hypersurface of a solution to the vacuum Einstein equations. In Lemma 3.1, we write the zeroth order condition with an angle explictly. We do not write the higher order conditions explicitly, but they can be obtained from our choice of boundary conditions, the zeroth order conditions and the Einstein equations. We do state the first order condition in the simple case of an orthogonal slice in Lemma 3.4.

For the proof of Theorem 1.1, we will only consider the case of the initial hypersurface intersecting $\mathcal{T}$ orthogonally, since a standard argument allows one to select such an orthogonal slice in the domain of dependence region of (Σ, h, k), by appealing to the classical initial boundary value problem, see Sect. 3.2.

Remark 1.3. The geometric uniqueness is a direct consequence of our choice of geometric boundary conditions. Although totally geodesic boundaries are of course quite special, this result provides the first setting in which geometric uniqueness holds for the Einstein vacuum equations with zero cosmological constant $\Lambda = 0$. Note also that here, since we prescribe homogeneous boundary conditions, we did not introduce an abstract embedding of $\mathcal{S} \times \mathbb{R}$ into the spacetime, since the prescribed value of χ is identical on each point of the boundary.

Remark 1.4. The above theorem is obtained using a system of reduced equations based on the ADM system in a geodesic gauge. For the reduced equations, due to the presence of a boundary and our choice of boundary conditions, we prove local well-posedness in a scale of anisotropic Sobolev spaces, see Definition 3.6 and Proposition 3.10. Indeed, the boundary conditions can a priori only be commuted by tangential derivatives to the boundary. Thus, our Sobolev spaces distinguish between derivatives tangential and normal to the boundary. In view of this, the normal derivatives cannot be estimated using commutation and standard energy estimates, but instead, are recovered from the equations directly, which allow to rewrite normal derivatives in terms of tangential ones. However, the structure of the equations plays an essential role here, since some components do not have any normal derivatives appearing in the equations. The anisotropic Sobolev spaces (3.9) provide a solution to this analytical problem. Such issues have been investigated for more general first order symmetric hyperbolic systems, already in [21], where similar anisotropic spaces are used to study the local well-posedness of the IBVP with characteristic boundaries of constant multiplicity (cf. Remark 3.8). Nevertheless, we include a treatment of the reduced IBVP in our specific setting, for the sake of completeness, see the proof of Proposition 3.10.

Remark 1.5. Since the reduced system is solved in (anisotropic) Sobolev spaces, one can obtain a similar statement assuming only that the initial data lie in a standard H^s space, $s \geq 7$, with corner conditions satisfied up to the corresponding finite order.

Remark 1.6. We note that, importantly, our choice of boundary conditions for the Einstein equations translates to admissible boundary conditions both for the reduced system of evolution equations that we use to construct a solution (see Lemma 3.3) and for the hyperbolic system that allows a posteriori to prove the propagation of constraints (see Lemma 4.7) and recover the Einstein equations. More precisely, $\chi \equiv 0$ on the boundary implies the validity of the momentum constraint, which translates to homogeneous boundary conditions for certain Ricci components. Note that we are not referring to (1.3) here, but to the analogous constraint equations where k is replaced by χ and div and tr are the divergence and trace with respect to the induced metric on the boundary.

Remark 1.7. In order to construct initial data sets to which local well-posedness theorems of the IBVP can be applied, including Theorem 1.1, one needs to solve the constraint equations with boundary. However, the corner conditions already mentioned impose various relations between h and k and their derivatives at all orders on $\mathcal{S}$. For these reasons, it is unclear how the known methods for solving the constraint equations can be adapted to construct solutions in this setting.

On the other hand, it is not hard to construct a special family of initial data verifying the assumptions of Theorem 1.1. Such data can for instance be constructed by considering solutions to the Einstein equations admitting a spacelike Killing vector field, which is also hypersurface orthogonal. Explicit examples are given by the Schwarzschild solution

$$\mathbf{g}_{Schw} = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where the level sets of ϕ in the exterior region or the level sets of t in the interior of the black hole are totally geodesic hypersurfaces.

Remark 1.8. If one thinks of $\chi \equiv 0$ as the vanishing of the Lie derivative of the solution in the normal direction to the boundary, our boundary conditions could be interpreted as homogeneous Neumann boundary conditions, and, in this respect, a natural direction for possible extensions of the present result would be to consider inhomogeneous Neumann type boundary conditions, for instance by prescribing a non-zero χ. However, there seem to be nontrivial obstructions for such type of results to hold, both analytic, due to various losses of derivatives, and geometric, since geodesics of the boundary are no longer geodesics of the Lorentzian manifold. On a more physical point of view, note that if one thinks of $\chi \equiv 0$ as a form of homogeneous Neumann boundary conditions, our setting is applicable to the study of gravitational waves in a cavity.

Remark 1.9. Recall that if $\psi : (\mathcal{M}, \mathbf{g}) \to (\mathcal{M}, \mathbf{g})$ is an isometry of a Riemanian or Lorentzian manifold, then every connected component of the set of fixed points $\{p \in \mathcal{M} : \psi(p) = p\}$ is totally geodesic [16]. This suggests[3] another possible proof of Theorem 1.1, at least in the case where the initial data intersect the boundary orthogonally, based on extending the initial data via reflection, then solving the regular Cauchy problem for the extended data and finally checking that the resulting spacetime enjoys a discrete isometry. Of course, this approach is clearly not generalizable to other kind of boundary conditions, while the proof of this paper may serve as a basis for further applications in the subject.

1.4. The hyperbolicity of the ADM system in a geodesic gauge. As already explained, our choice of evolution equations is based on the ADM formulation of the Einstein equations. This formalism and its many variants are widely used in the study of the Einstein equations, by theoretical or numerical means. They are based on a 3+1 splitting of the underlying Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ through a choice of time function t and the foliation induced by its level sets Σ_t. The main dynamical variables are the first and second fundamental forms (g, K) of each Σ_t, satisfying, together with the lapse and shift of the foliation, a system of partial differential equations, which is first order in the time derivative. This system is generally underdetermined due to the geometric invariance of the equations. In order to render it well-determined, one naturally needs to make additional gauge choices, leading to a reduced system of equations. In full generality, they are many possible such choices, see for example [11] and the references therein. For rigorous studies of the well-posedness problem we refer the reader to [3,19].

In this paper, we consider the reduced ADM system for the Einstein vacuum equations, obtained by writing the equations in an orthonormal frame $\{e_\mu\}_{\mu=0}^3$, which is parallelly propagated with respect to a family of timelike geodesics. In this setting, the lapse of the foliation is fixed to 1, while the shift vector field is set to zero, and the spacetime metric takes the form

$$\mathbf{g} = -dt^2 + g_{pq} dx^p dx^q, \tag{1.4}$$

where (x^1, x^2, x^3) are t-transported coordinates, with respect to which the orthonormal frame is expressed via

$$e_0 = \partial_t, \quad e_i = f_i{}^p \partial_p, \quad \partial_p = f^b{}_p e_b, \quad i, j = 1, 2, 3, \tag{1.5}$$

where $f_i{}^p f^b{}_p = \delta_i^b$, $f^b{}_p f_b{}^q = \delta_p^q$. Here, and throughout the text, the Einstein summation is used for the Latin indices that range in 1,2,3.

In the classical ADM formalism, the main evolution equations are first order equations (in ∂_t) for g_{pq}, $\partial_t g_{pq}$; the second variable corresponding to the second fundamental form of Σ_t. When expressed in terms of the previous orthonormal frame, the components g_{pq}

[3] We would like to thank M.T. Anderson and E. Witten for this suggestion.

correspond to the frame coefficients $f_i{}^p$, while the second fundamental form is now evaluated against the spatial frame components e_i:

$$K_{ij} := \mathbf{g}(\mathbf{D}_{e_i}e_0, e_j) = K_{ji}, \tag{1.6}$$

where $\mathbf{D}$ is the Levi-Civita connection of $\mathbf{g}$. In our framework, K_{ij}, $f_i{}^p$ satisfy the evolution equations (2.6), (2.8). The right-hand-side of (2.6) contains up to two spatial derivatives of $f_i{}^p$, encoded in the Ricci tensor of g. However, we find it analytically convenient to expand this term using the spatial connection coefficients of the frame:

$$\Gamma_{ijb} := \mathbf{g}(\mathbf{D}_{e_i}e_j, e_b) = g(D_{e_i}e_j, e_b) = -\Gamma_{ibj}, \tag{1.7}$$

where D is the Levi-Civita connection of g. These then satisfy the propagation equation (2.7).

At first glance, the system (2.6)–(2.8) does not seem to be eligible for an energy estimate, due to the first term in the right-hand-side of (2.6) that renders the system non-symmetric and could lead to a loss of derivatives. This is a well-known problem of the ADM system. One remedy is to consider a harmonic gauge [3] on the slices Σ_t, which would eliminate this bad term. The adoption of such gauges introduces new variables to the system (lapse, shift vector field) that satisfy elliptic equations. Another argument was given in [18], where the authors made use of the momentum constraint (1.3), in order to eliminate any such bad terms in the energy estimates by integrating by parts. For this, they used a CMC foliation, a transported coordinate system (t, x_1, x_2, x_3) and the associated Christoffel symbols, to derive a priori energy estimates, assuming the existence of a spacetime solution verifying the constraints, instead of working with a system for which well-posedness holds, as we do in this paper.

In contrast, the ADM system can be transformed into a second order system of equations for the second fundamental form of the time slices, expressed in terms of transported coordinates (t, x^1, x^2, x^3). This was first derived in [6], where the authors demonstrated its hyperbolicity under the gauge assumption $\Box_g t = 0$. It turns out that the second order system for K is also hyperbolic in normal transported coordinates (1.4), without any additional gauge assumptions, see the framework presented in [8] with an application to asymptotically Kasner-like singularities. Recently, we also used the aforementioned second order system for K (see [9]) to analyse the initial boundary value problem for the Einstein vacuum equations in the maximal gauge.

In the present study, we carry out the analysis in the geodesic gauge presented above, circumventing the apparent loss of derivatives issue (see Lemma 2.5) by making use of both the Hamiltonian and momentum constraints. In [14,19], both the Hamiltonian and momentum constraints were already used to modify the ADM system and obtain well-posedness of the equations in coordinate-based gauges. The orthonormal frame that we consider in the present article seems to simplify the analysis of the boundary conditions in our setting. We thus prove that by modifying (2.6)–(2.7), adding appropriate multiples of (2.19)–(2.20), one obtains a first order symmetric hyperbolic system for the unknowns, see (2.19)–(2.20), which is suitable for a local existence argument. In order to facilitate the propagation of the (anti)symmetries of K and Γ, we also (anti)symmetrize parts of the equations.

In general, once the reduced system is solved, one then recovers the Einstein equations through the Bianchi equations. For a modified system, however, the equations one solves for are not directly equivalent to the vanishing of the components of the Ricci tensor and thus this procedure becomes more complicated. See [19, Appendix A] for such an example concerning the progragation of constraints in a modified ADM setting. It is for this reason that one should make minimal modifications to the reduced equations, since any additional change could complicate even further the final system for the vanishing quantities, making it intractable via energy estimates. Nonetheless, for the modified system we consider, we are able to recover the full Einstein equations by deriving a hyperbolic system for appropriate combinations of the vanishing quantities (see Lemma 4.5). Note that since the connection is obtained by solving the modified reduced equations, it can only be shown to agree with the Levi-Civita connection at the same time as the recovery of the full Einstein equations (see Sect. 4). This issue was already present in the approach of [13] using an orthonormal frame. In particular, it is not known a priori that the torsion of the connection vanishes. Thus, the unknowns in the hyperbolic system used for the recovery of the Einstein equations are the components of the torsion, as well as the components of the Ricci and Riemann tensors, after suitable symmetrizations and modifications. The modifications involve the torsion and are similar to the modifications used in [13], where the authors study the Einstein equations at the level of the Bianchi equations, which results into a different system for the recovery of the Einstein equations.

Our result on the well-posedness of the initial value problem for the Einstein equations in the above framework can be stated as follows:

Theorem 1.10. *The initial value problem for the modified reduced system* (2.8)*,* (2.19)*,* (2.20)*, for the frame and connection coefficients is locally well-posed in* $L_t^\infty H^s(\Sigma_t)$*, for* $s \geq 3$*. Moreover, if the initial data* (Σ, h, K) *satisfy the constraint equations* (1.2)*–*(1.3)*, then the solution to* (2.8)*,* (2.19)*,* (2.20)*, with the induced initial data* (*see Sect.* 2.4)*, induces a solution of* (1.1)*. In particular, the initial value problem for the Einstein vacuum equations, cast as a modified ADM system, is locally well-posed.*

Remark 1.11. Adding the boundary conditions for the reduced system that arise from the totally geodesic condition, assuming that the corner conditions hold, and replacing the usual Sobolev spaces H^s by anisotropic Sobolev spaces B^s (see Definition 3.6), which contain half as many transversal derivatives to the boundary compared to the number of tangential derivatives in $L^2(\Sigma_t)$, the analogue of Theorem 1.10 for the initial boundary value problem holds true, see Propositions 3.10, 4.8.

Remark 1.12. Note that the geodesic gauge considered here respects the hyperbolicity of the equations. In particular, the usual finite speed of propagation and domain of dependence arguments can be proven in this gauge. Hence, in the case of the initial boundary value problem, one can localize the analysis near a point on the boundary, provided that the orthonormal frame we consider is adapted to the boundary. This requirement is verified for vanishing second fundamental form χ (Lemma 3.2).

1.5. Outline. In Sect. 2, we set up our modified version of the ADM system. We first formulate the standard ADM evolution equations in the geodesic gauge (Lemma 2.2) and then prove (in Lemma 2.5) a first order energy identity, assuming that the constraints hold. This identity leads us to the introduction of the modified evolution equations (2.19)–(2.20). The resulting system is then shown to be symmetric hyperbolic in Lemma 2.8. Although the local well-posedness of the usual initial value problem follows from standard arguments, to simplify the treatment of the IBVP, we establish localized energy estimates in Sect. 2.3 (see Proposition 2.12), using the structure of the commuted equations identified in Sect. 2.2. In Sect. 2.4, we briefly describe how to derive the initial data for the reduced system from the geometric initial data.

Section 3 is devoted to the initial boundary value problem for the modified ADM system. First, in Sect. 3.1, we compute the zeroth compatibility conditions with a given angle ω. Then, in Sect. 3.2, we describe the procedure that allows us to work with an initial hypersurface orthogonal to the timelike boundary $\mathcal{T}$. Using that our geodesic frame is adapted to the totally geodesic boundary (Lemma 3.2), we express the vanishing of χ in terms of certain of the components K_{ij}, Γ_{ijb}, and derive the first order compatibility conditions for an orthogonal slice (see Lemmas 3.3, 3.4). Finally, in Sect. 3.4, we prove the local well-posedness of the initial boundary value problem for the modified reduced system of equations, subject to the boundary conditions induced by the vanishing of χ. The main difficulty here arises from a loss in the control of the normal derivatives to the boundary, cf. Remark 1.4, forcing us to introduce anisoptropic Sobolev spaces.

Finally, in Sect. 4, we show that once a solution to the reduced system has been obtained, our framework allows for the recovery of the Einstein vacuum equations, both for the standard Cauchy problem and in the presence of a totally geodesic boundary, thus completing the proofs of Theorems 1.1 and 1.10 (see Sect. 4.4). The starting point is to introduce the Lorentzian metric and the connection associated to a solution of the reduced equations. One easily verifies that the connection is compatible with the metric, by virtue of the propagation of the antisymmetry of the spatial connection coefficients Γ_{ijb} (see Lemma 2.7). On the other hand, the connection is not a priori torsion free and therefore, does not a priori agree with the Levi-Civita of the metric. We first derive various geometrical identities such as the Bianchi equations and the Gauss–Codazzi equations, in the presence of torsion (cf. Lemma 4.1). Since the resulting equations are not suitable to propagate the constraints, we consider modified Riemann and Ricci curvature tensors (4.10), both for the spacetime geometry and the geometry of the time slices, the modifications depending on the torsion (cf. [13, Section 6]). The symmetries of these modified curvatures are studied in Lemma 4.2 and 4.4. Then, we prove that they lead to a symmetric hyperbolic system (4.25)–(4.29) for the modified spacetime Ricci curvature components and the torsion. Finally, we show that the boundary conditions satisfied by the solution to the modified ADM system, which are in turn induced by the vanishing of χ (see Lemma 3.3), imply boundary conditions for the modified spacetime Ricci curvature (Lemma 4.7) that are suitable for an energy estimate. The final argument for the recovery of the Einstein equations, both for the Cauchy problem and in the case a totally geodesic timelike boundary, is presented in Sect. 4.3.

1.6. Notation. We use Greek letters α, β, μ, ν etc, for indices ranging from 0 to 3, Latin letters i, j, a, b, c etc, as spatial indices $1, 2, 3$, and capital letters A, B for the indices $1, 2$ (which correspond below to spacelike vector fields tangential to the boundary).

The indices p, q, r are reserved for the coordinate vector fields, while the remaining Latin and Greek letters correspond to the orthonormal frame.

Einstein's summation is used for repeated upper and lower indices, with the range of the sum being that of the specific indices. All tensors throughout the paper are evaluated against the orthonormal frame $\{e_\mu\}_0^3$. In particular, we raise and lower indices using $m_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. For example, $K_i{}^j = K_{ij}$, $e^0 = -e_0$.

2. The ADM System in a Geodesic Gauge

In this section we introduce our framework and show that the Einstein vacuum equations (EVE) reduce to a first order symmetric hyperbolic system for the connection coefficients of a parallelly propagated orthonormal frame. For completeness, we confirm the well-posedness of the initial value problem in usual H^s spaces.

2.1. The modified ADM evolution equations and their hyperbolicity. Let $(\mathcal{M}, \mathbf{g})$ be a 3 + 1-dimensional Lorentzian manifold and let Σ_0 be a Cauchy hypersurface equipped with an orthonormal frame e_1, e_2, e_3. Also, let e_0 be the future unit normal to Σ_0. We extend the frame $\{e_\mu\}_0^3$ by parallel propagation along timelike geodesics emanating from Σ_0 with initial speed e_0:

$$\mathbf{D}_{e_0} e_\mu = 0 \tag{2.1}$$

If t is the proper time parameter of the e_0 geodesics, $\{t = 0\} = \Sigma_0$, then $\mathbf{g}$ takes the form (1.4), where g is the induced metric on Σ_t, and the transition between $\{e_\mu\}_0^3$ and a transported coordinate system (t, x_1, x_2, x_3) is defined via (1.5). The connection coefficients of the orthonormal frame are K_{ij}, Γ_{ijb}, defined in (1.6), (1.7).

Our convention for the spacetime Riemann, Ricci, and scalar curvatures is

$$\mathbf{R}_{\alpha\beta\mu\nu} = \mathbf{g}((\mathbf{D}_{e_\alpha}\mathbf{D}_{e_\beta} - \mathbf{D}_{e_\beta}\mathbf{D}_{e_\alpha} - \mathbf{D}_{[e_\alpha, e_\beta]})e_\mu, e_\nu), \quad \mathbf{R}_{\beta\mu} = \mathbf{R}_{\alpha\beta\mu}{}^{\alpha}, \quad \mathbf{R} = \mathbf{R}_\mu{}^\mu \tag{2.2}$$

and similarly for the curvature tensors of g, denoted by R_{ijlb}, R_{jl}, R. We now state the well-known Gauss–Codazzi equations. We also provide a short proof for the convenience of the reader since similar computations will be used to retrieve the Einstein equtions in Sect. 4, but with a torsion.

Lemma 2.1. *With the above conventions, the Gauss and Codazzi equations for Σ_t read:*

$$\mathbf{R}_{aijb} = R_{aijb} + K_{ab}K_{ij} - K_{aj}K_{ib}, \tag{2.3}$$
$$\mathbf{R}_{0ijb} = D_j K_{bi} - D_b K_{ji}, \tag{2.4}$$

where

$$R_{aijb} = e_a\Gamma_{ijb} - e_i\Gamma_{ajb} - \Gamma_{ab}{}^c\Gamma_{ijc} + \Gamma_{ib}{}^c\Gamma_{ajc} - \Gamma_{ai}{}^c\Gamma_{cjb} + \Gamma_{ia}{}^c\Gamma_{cjb} \tag{2.5}$$

Proof. We employ the formulas

$$\mathbf{D}_{e_i} e_j = D_{e_i} e_j + K_{ij} e_0, \quad \mathbf{D}_{e_b} e_0 = K_b{}^c e_c, \quad [e_j, e_b] = D_{e_j} e_b - D_{e_b} e_j = (\Gamma_{jb}{}^c - \Gamma_{bj}{}^c) e_c$$

to compute

$$\begin{aligned}
\mathbf{R}_{aijb} &= \mathbf{g}((\mathbf{D}_{e_a}\mathbf{D}_{e_i} - \mathbf{D}_{e_i}\mathbf{D}_{e_a} - \mathbf{D}_{[e_a,e_i]})e_j, e_b)\\
&= \mathbf{g}(\mathbf{D}_{e_a}(D_{e_i}e_j + K_{ij}e_0) - \mathbf{D}_{e_i}(D_{e_a}e_j + K_{aj}e_0) - D_{[e_a,e_i]}e_j, e_b)\\
&= g(D_{e_a}D_{e_i}e_j - D_{e_i}D_{e_a}e_j - D_{[e_a,e_i]}e_j, e_b) + K_{ij}K_{ab} - K_{aj}K_{ib},\\
\mathbf{R}_{0ijb} = \mathbf{R}_{jb0i} &= \mathbf{g}((\mathbf{D}_{e_j}\mathbf{D}_{e_b} - \mathbf{D}_{e_b}\mathbf{D}_{e_j} - \mathbf{D}_{[e_j,e_b]})e_0, e_i)\\
&= \mathbf{g}(\mathbf{D}_{e_j}(K_b{}^c e_c) - \mathbf{D}_{e_b}(K_j{}^c e_c), e_i) - (\Gamma_{jb}{}^c - \Gamma_{bj}{}^c)K_{ci}\\
&= e_j K_{bi} + K_b{}^c\Gamma_{jci} - e_b K_{ji} - K_j{}^c\Gamma_{bci} - (\Gamma_{jb}{}^c - \Gamma_{bj}{}^c)K_{ci}
\end{aligned}$$

and

$$\begin{aligned}
R_{aijb} &= g((D_{e_a}D_{e_i} - D_{e_i}D_{e_a} - D_{[e_a,e_i]})e_j, e_b)\\
&= g(D_{e_a}(\Gamma_{ij}{}^c e_c), e_b) - g(D_{e_i}(\Gamma_{aj}{}^c e_c), e_b) - (\Gamma_{ai}{}^c - \Gamma_{ia}{}^c)g(D_{e_c}e_j, e_b)\\
&= e_a\Gamma_{ijb} + \Gamma_{ij}{}^c\Gamma_{acb} - e_i\Gamma_{ajb} - \Gamma_{aj}{}^c\Gamma_{icb} - (\Gamma_{ai}{}^c - \Gamma_{ia}{}^c)\Gamma_{cjb}
\end{aligned}$$

which can be seen to correspond to the asserted formulas by using the antisymmetry of Γ_{ijb} in $(j;b)$. □

Lemma 2.2. *The components K_{ij}, Γ_{ijb}, $f_i{}^p$, $f^b{}_p$ satisfy the following identities:*

$$\begin{aligned} e_0 K_{ij} + \mathrm{tr} K K_{ij} &= -R^{(S)}_{ij} + \mathbf{R}^{(S)}_{ij}, \\ &= \frac{1}{2}\Big[e_i \Gamma^b{}_{jb} - e^b \Gamma_{ijb} + \Gamma^b{}_i{}^c \Gamma_{cjb} + \Gamma^b{}_b{}^c \Gamma_{ijc} \\ &\quad + e_j \Gamma^b{}_{ib} - e^b \Gamma_{jib} + \Gamma^b{}_j{}^c \Gamma_{cib} + \Gamma^b{}_b{}^c \Gamma_{jic} \Big] + \mathbf{R}^{(S)}_{ij} \end{aligned} \tag{2.6}$$

$$\begin{aligned} e_0 \Gamma_{ijb} + K_i{}^c \Gamma_{cjb} &= D_j K_{bi} - D_b K_{ji} \\ &= e_j K_{bi} - e_b K_{ji} - \Gamma_{jb}{}^c K_{ci} - \Gamma_{ji}{}^c K_{bc} + \Gamma_{bj}{}^c K_{ci} + \Gamma_{bi}{}^c K_{jc} \end{aligned} \tag{2.7}$$

$$e_0 f_i{}^p + K_i{}^c f_c{}^p = 0 \tag{2.8}$$

$$e_0 f^b{}_p - K_c{}^b f^c{}_p = 0 \tag{2.9}$$

for all indices $i, j, b = 1, 2, 3$, where

$$R^{(S)}_{ij} := \frac{1}{2}(R_{ij} + R_{ji}), \quad \mathbf{R}^{(S)}_{ij} := \frac{1}{2}(\mathbf{R}_{ij} + \mathbf{R}_{ji}) \tag{2.10}$$

Remark 2.3. The Ricci tensor associated to the Levi-Civita connection is always symmetric, and thus $R^{(S)}_{ij} = R_{ij}$ in this case. However, in order to establish local well-posedness, we will construct the connection from modified equations below and it will no longer hold a priori that R_{ij} or K_{ij} are symmetric, unless we expand the right-hand side of (2.6) in terms of the symmetrised Ricci tensor $R^{(S)}_{ij}$. In this form, the symmetry of K_{ij} is automatically propagated, provided it is valid initially.

Proof. The propagation condition (2.1) implies the second variation equation

$$\begin{aligned} \mathbf{R}_{0i0j} &= \mathbf{g}((\mathbf{D}_{e_0}\mathbf{D}_{e_i} - \mathbf{D}_{e_i}\mathbf{D}_{e_0} - \mathbf{D}_{[e_0,e_i]})e_0, e_j) = \mathbf{g}(\mathbf{D}_{e_0}(K_i{}^c e_c) - \mathbf{D}_{(\mathbf{D}_{e_0} e_i - \mathbf{D}_{e_i} e_0)} e_0, e_j) \\ &= e_0 K_{ij} + K_i{}^b \mathbf{g}(\mathbf{D}_{e_b} e_0, e_j) = e_0 K_{ij} + K_i{}^b K_{bj}. \end{aligned} \tag{2.11}$$

Utilising (2.3) we have

$$\mathbf{R}_{0i0j} = -\mathbf{R}_{0ij0} = \mathbf{R}_{ij} - \mathbf{R}_{bij}{}^b = \mathbf{R}_{ij} - R_{ij} - \mathrm{tr} K K_{ij} + K_i{}^b K_{jb} \tag{2.12}$$

On the other hand, contracting (2.5) in $(a;b)$ gives

$$\begin{aligned} -R_{ij} &= -R_{bij}{}^b = e_i \Gamma^b{}_{jb} - e^b \Gamma_{ijb} + \Gamma^b{}_i{}^c \Gamma_{cjb} - \Gamma_i{}^{bc} \Gamma_{cjb} + \Gamma^b{}_b{}^c \Gamma_{ijc} - \Gamma_i{}^{bc} \Gamma_{bjc} \\ &= e_i \Gamma^b{}_{jb} - e^b \Gamma_{ijb} + \Gamma^b{}_i{}^c \Gamma_{cjb} + \Gamma^b{}_b{}^c \Gamma_{ijc} \\ &= e_j \Gamma^b{}_{ib} - e^b \Gamma_{jib} + \Gamma^b{}_j{}^c \Gamma_{cib} + \Gamma^b{}_b{}^c \Gamma_{jic} = -R_{ji}, \end{aligned} \tag{2.13}$$

where in the last equality we used the symmetry of the Ricci tensor of g. Combining (2.11), (2.12) and (2.13), we conclude (2.6).

By (2.1) and the Codazzi equation (2.4) it follows that

$$\begin{aligned} e_0\Gamma_{ijb} &= \mathbf{g}(\mathbf{D}_{e_0}\mathbf{D}_{e_i}e_j, e_b) = \mathbf{R}_{0ijb} + \mathbf{g}(\mathbf{D}_{e_i}\mathbf{D}_{e_0}e_j, e_b) + \mathbf{g}(\mathbf{D}_{[e_0,e_i]}e_j, e_b) \\ &= \mathbf{R}_{0ijb} + \mathbf{g}\left(\mathbf{D}_{(\mathbf{D}_{e_0}e_i - \mathbf{D}_{e_i}e_0)}e_j, e_b\right) \\ &= D_j K_{bi} - D_b K_{ji} - K_i{}^c\Gamma_{cjb}, \end{aligned} \tag{2.14}$$

which yields (2.7).

Finally, we have

$$K_i{}^c e_c = \mathbf{D}_{e_i}e_0 = \mathbf{D}_{e_i}e_0 - \mathbf{D}_{e_0}e_i = [e_i, e_0] = [f_i{}^p\partial_p, \partial_t] \quad \Rightarrow \quad K_i{}^c f_c{}^p\partial_p = -e_0 f_i{}^p\partial_p,$$

which implies (2.8). Utilising the relation $f_i{}^p f^b{}_p = \delta_i^b$, we also conclude (2.9). □

Remark 2.4. Contracting the formula (2.13) and using antisymmetry of Γ_{ijb} with respect to the last two indices, we notice that the two first order terms combine to give

$$-R = 2e^j\Gamma^b{}_{jb} + \Gamma^{bjc}\Gamma_{cjb} + \Gamma^b{}_b{}^c\Gamma^j{}_{jc}. \tag{2.15}$$

In the next lemma, we illustrate the structure of the equations (2.6)–(2.7) that we exploit in the local existence argument below, by deriving the main energy identity for K_{ij}, Γ_{ijb} (at zeroth order). For the moment, we make use of both the Hamiltonian and momentum contraints (1.2)–(1.3), i.e., the fact that we have an actual solution to (1.1).

Lemma 2.5. *Let* $\mathbf{g}$ *be a solution to the EVE. Then the variables* K_{ij}, Γ_{ijb} *satisfy the following identity:*

$$\begin{aligned} &\frac{1}{2}e_0(|K|^2) + \mathrm{tr}K|K|^2 + \frac{1}{4}e_0[\Gamma_{ijb}\Gamma^{ijb}] + \frac{1}{2}K_i{}^c\Gamma_{cjb}\Gamma^{ijb} \\ &\quad = e_j[K^{ij}\Gamma^b{}_{ib}] - e^i[\mathrm{tr}K\Gamma^b{}_{ib}] - e_b[\Gamma^{ijb}K_{ji}] + \frac{1}{2}\mathrm{tr}K\left[(\mathrm{tr}K)^2 - |K|^2 - \Gamma^{bic}\Gamma_{cib} - \Gamma^b{}_b{}^c\Gamma^i{}_{ic}\right] \\ &\qquad - \Gamma_j{}^i{}_c K^{cj}\Gamma^b{}_{ib} - \Gamma_j{}^j{}_c K^{ic}\Gamma^b{}_{ib} + K^{ij}[\Gamma^b{}_j{}^c\Gamma_{cib} + \Gamma^b{}_b{}^c\Gamma_{jic}] + \Gamma^{ijb}[\Gamma_{bj}{}^c K_{ci} + \Gamma_{bi}{}^c K_{jc}], \end{aligned} \tag{2.16}$$

where $|K|^2 = K^{ij}K_{ij}$.

Proof. Multiplying (2.7) by Γ^{ijb} and using its antisymmetry in $(j; b)$ gives the identity

$$\frac{1}{4}e_0[\Gamma_{ijb}\Gamma^{ijb}] + \frac{1}{2}K_i{}^c\Gamma_{cjb}\Gamma^{ijb} = -e_b[\Gamma^{ijb}K_{ji}] + K^{ji}e^b\Gamma_{ijb} + \Gamma^{ijb}[\Gamma_{bj}{}^c K_{ci} + \Gamma_{bi}{}^c K_{jc}]. \tag{2.17}$$

Multiplying (2.6) by K^{ij} and using its symmetry in $(i; j)$, we also have

$$\frac{1}{2}e_0(|K|^2) + \mathrm{tr}K|K|^2 = K^{ij}e_i\Gamma^b{}_{jb} - K^{ij}e^b\Gamma_{ijb} + K^{ij}[\Gamma^b{}_i{}^c\Gamma_{cjb} + \Gamma^b{}_b{}^c\Gamma_{ijc}]. \tag{2.18}$$

Notice that the second terms on the right-hand sides of (2.17) and (2.18) are exact opposites, hence, canceling out upon summation of the two identities.

We proceed by rewriting the first term on the right-hand side of (2.18), making use of both constraint equations (1.3)–(1.2) in the following manner

$$
\begin{aligned}
K^{ij}e_i\Gamma^b{}_{jb} &= e_i[K^{ij}\Gamma^b{}_{jb}] - e_i(K^{ij})\Gamma^b{}_{jb}\\
&= e_i[K^{ij}\Gamma^b{}_{jb}] - D_iK^{ij}\Gamma^b{}_{jb} - \Gamma_i{}^i{}_cK^{cj}\Gamma^b{}_{jb} - \Gamma_i{}^j{}_cK^{ic}\Gamma^b{}_{jb}\\
&= e_i[K^{ij}\Gamma^b{}_{jb}] - e^j\mathrm{tr}K\Gamma^b{}_{jb} - \Gamma_i{}^i{}_cK^{cj}\Gamma^b{}_{jb} - \Gamma_i{}^j{}_cK^{ic}\Gamma^b{}_{jb} \qquad \text{(by (1.3))}\\
&= e_i[K^{ij}\Gamma^b{}_{jb}] - e^j[\mathrm{tr}K\Gamma^b{}_{jb}] + \mathrm{tr}K e^j\Gamma^b{}_{jb} - \Gamma_i{}^i{}_cK^{cj}\Gamma^b{}_{jb} - \Gamma_i{}^j{}_cK^{ic}\Gamma^b{}_{jb}\\
&= e_i[K^{ij}\Gamma^b{}_{jb}] - e^j[\mathrm{tr}K\Gamma^b{}_{jb}]\\
&\quad - \frac{1}{2}\mathrm{tr}K\big[R+\Gamma^{bjc}\Gamma_{cjb} + \Gamma^b{}_b{}^c\Gamma^j{}_{jc}\big] - \Gamma_i{}^i{}_cK^{cj}\Gamma^b{}_{jb} - \Gamma_i{}^j{}_cK^{ic}\Gamma^b{}_{jb} \qquad \text{(by (2.15))}\\
&= e_i[K^{ij}\Gamma^b{}_{jb}] - e^j[\mathrm{tr}K\Gamma^b{}_{jb}] + \frac{1}{2}\mathrm{tr}K\big[(\mathrm{tr}K)^2 - |K|^2 - \Gamma^{bjc}\Gamma_{cjb} - \Gamma^b{}_b{}^c\Gamma^j{}_{jc}\big] \qquad \text{(by (1.2))}\\
&\quad - \Gamma_i{}^i{}_cK^{cj}\Gamma^b{}_{jb} - \Gamma_i{}^j{}_cK^{ic}\Gamma^b{}_{jb}
\end{aligned}
$$

Combining the above identities, we obtain (2.16). □

Although the differential identity (2.16) provides a way of deriving a priori estimates for K_{ij}, Γ_{ijb}, the equations (2.6)–(2.7) are still not eligible for a local existence argument, because of the heavy use of the constraint equations in the argument. Indeed, in a local existence proof via a Picard iteration scheme, the constraints are no longer valid off of the initial hypersurface Σ_0. This implies that a structure similar to the one identified in Lemma 2.5 is no longer present, which leads to a loss of derivatives.

We remedy this problem by adding appropriate multiples of the constraints in the RHS of the evolution equations (2.6)–(2.7), resulting to the system:

$$
\begin{aligned}
e_0K_{ij} + \mathrm{tr}K K_{ij} &= \frac{1}{2}\Big[e_i\Gamma^b{}_{jb} - e^b\Gamma_{ijb} + \Gamma^b{}_i{}^c\Gamma_{cjb} + \Gamma^b{}_b{}^c\Gamma_{ijc} + e_j\Gamma^b{}_{ib} - e^b\Gamma_{jib} + \Gamma^b{}_j{}^c\Gamma_{cib} + \Gamma^b{}_b{}^c\Gamma_{jic}\Big]\\
&\quad - \frac{1}{2}\delta_{ij}\Big[2e^a\Gamma^b{}_{ab} + \Gamma^{bac}\Gamma_{cab} + \Gamma^b{}_b{}^c\Gamma^a{}_{ac} + |K|^2 - (\mathrm{tr}K)^2\Big] \qquad (2.19)\\
e_0\Gamma_{ijb} + K_i{}^c\Gamma_{cjb} &= e_jK_{bi} - e_bK_{ji} - \Gamma_{jb}{}^cK_{ci} - \Gamma_{ji}{}^cK_{bc} + \Gamma_{bj}{}^cK_{ci} + \Gamma_{bi}{}^cK_{jc}\\
&\quad + \delta_{ib}\Big[e^cK_{cj} - \Gamma_c{}^{cl}K_{lj} - \Gamma^c{}_j{}^lK_{cl} - e_j\mathrm{tr}K\Big]\\
&\quad - \delta_{ij}\Big[e^cK_{cb} - \Gamma_c{}^{cl}K_{lb} - \Gamma^c{}_b{}^lK_{cl} - e_b\mathrm{tr}K\Big] \qquad (2.20)
\end{aligned}
$$

Remark 2.6. Contracting (2.3) in $(a; b)$, $(i; j)$, contracting (2.4) in $(i; b)$, and utilising (2.15), we notice that the added expressions in the last lines of (2.19)–(2.20) correspond to

$$
\begin{aligned}
&-\frac{1}{2}\delta_{ij}\Big[2e^a\Gamma^b{}_{ab} + \Gamma^{bac}\Gamma_{cab} + \Gamma^b{}_b{}^c\Gamma^a{}_{ac} + |K|^2 - (\mathrm{tr}K)^2\Big]\\
&\quad = \frac{1}{2}\delta_{ij}[R - |K|^2 + (\mathrm{tr}K)^2] = \frac{1}{2}\delta_{ij}[\mathbf{R} + 2\mathbf{R}_{00}],\\
&\delta_{ib}\Big[e^cK_{cj} - \Gamma_c{}^{cl}K_{lj} - \Gamma^c{}_j{}^lK_{cl} - e_j\mathrm{tr}K\Big] - \delta_{ij}\Big[e^cK_{cb} - \Gamma_c{}^{cl}K_{lb} - \Gamma^c{}_b{}^lK_{cl} - e_b\mathrm{tr}K\Big]\\
&\quad = \delta_{ib}\Big[D^cK_{cj} - e_j\mathrm{tr}K\Big] - \delta_{ij}\Big[D^cK_{cb} - e_b\mathrm{tr}K\Big] = \delta_{ib}\mathbf{R}_{0j} - \delta_{ij}\mathbf{R}_{0b},
\end{aligned}
$$

which are indeed multiples of the Hamiltonian and momentum constraints (1.2), (1.3).

By definition of the initial data, cf. Sect. 2.4, $K_{ij} = K_{ji}$, $\Gamma_{ijb} = -\Gamma_{ibj}$, $f_i{}^pf^b{}_p = \delta_i^b$, $f^b{}_pf_b{}^q = \delta_p^q$ will be valid initially for any solution and the same can be imposed for any iterate in a Picard iteration scheme.

Lemma 2.7. *A solution $K_{ij}, \Gamma_{ijb}, f_i{}^p, f^b{}_p$ to (2.19), (2.20), (2.8), (2.9) satisfies the properties $K_{ij} = K_{ji}$, $\Gamma_{ijb} = -\Gamma_{ibj}$, $f_i{}^p f^b{}_p = \delta_i^b$, $f^b{}_p f_b{}^q = \delta_p^q$, provided they hold true initially.*

Proof. The variables $K_{ij} - K_{ji}$, $\Gamma_{ijb} + \Gamma_{ibj}$, $f_i{}^p f^b{}_p - \delta_i^b$, $f^b{}_p f_b{}^q - \delta_p^q$ satisfy the following homogeneous ODE system with trivial initial data:

$$
\begin{aligned}
&e_0(K_{ij} - K_{ji}) + \mathrm{tr}K(K_{ij} - K_{ji}) = 0\\
&e_0(\Gamma_{ijb} + \Gamma_{ibj}) + K_i{}^c(\Gamma_{cjb} + \Gamma_{cbj}) = 0\\
&e_0(f_i{}^p f^b{}_p - \delta_i^b) + K_i{}^c(f_c{}^p f^b{}_p - \delta_c^b) - K_c{}^b(f_i{}^p f^c{}_p - \delta_i^c) = 0\\
&e_0(f^b{}_p f_b{}^q - \delta_p^q) = 0
\end{aligned}
$$

This implies that they must be identically zero. □

Lemma 2.8. *The equations (2.19), (2.20), coupled to the ODE (2.8), constitute a first order symmetric hyperbolic system.*

Proof. It suffices to look at the linearised equations around zero:[4]

$$
\begin{aligned}
&e_0K_{11} = e_3\Gamma_{223} - e_2\Gamma_{323}, \quad 2e_0K_{12} = -e_3\Gamma_{123} - e_3\Gamma_{213} + e_1\Gamma_{323} + e_2\Gamma_{313},\\
&e_0K_{22} = e_3\Gamma_{113} - e_1\Gamma_{313}, \quad 2e_0K_{13} = e_3\Gamma_{212} + e_2\Gamma_{123} - e_1\Gamma_{223} - e_2\Gamma_{312},\\
&e_0K_{33} = e_2\Gamma_{112} - e_1\Gamma_{212}, \quad 2e_0K_{23} = -e_3\Gamma_{112} + e_1\Gamma_{213} - e_2\Gamma_{113} + e_1\Gamma_{312},\\
&e_0\Gamma_{113} = e_3K_{22} - e_2K_{23}, \quad e_0\Gamma_{223} = e_3K_{11} - e_1K_{13}, \quad e_0\Gamma_{123} = -e_3K_{12} + e_2K_{13},\\
&e_0\Gamma_{213} = -e_3K_{12} + e_1K_{23}, \quad e_0\Gamma_{313} = e_2K_{12} - e_1K_{22}, \quad e_0\Gamma_{323} = e_1K_{12} - e_2K_{11},\\
&e_0\Gamma_{312} = e_1K_{23} - e_2K_{13}, \quad e_0\Gamma_{112} = -e_3K_{23} + e_2K_{33}, \quad e_0\Gamma_{212} = e_3K_{13} - e_1K_{33}
\end{aligned}
\tag{2.21}
$$

As one can tediously check, (2.21) is symmetric.

Alternatively, one can verify that the principal symbol of (2.19), (2.20) is symmetric relative to the scalar product associated with the quadratic form $2K^{ij}K_{ij} + \Gamma^{ijb}\Gamma_{ijb}$,[5] since the spatial symbol of each equation reads

$$
\begin{aligned}
2(\sigma_\xi\dot{K})_{ij} &= \xi_i\dot{\Gamma}^b{}_{jb} - \xi^b\dot{\Gamma}_{ijb} + \xi_j\dot{\Gamma}^b{}_{ib} - \xi^b\dot{\Gamma}_{jib} - 2\delta_{ij}\xi^a\dot{\Gamma}^b{}_{ab}\\
(\sigma_\xi\dot{\Gamma})_{ijb} &= \xi_j\dot{K}_{bi} - \xi_b\dot{K}_{ij} + \delta_{ib}(\xi^c\dot{K}_{cj} - \xi_j\mathrm{tr}\dot{K}) - \delta_{ij}(\xi^c\dot{K}_{cb} - \xi_b\mathrm{tr}\dot{K}),
\end{aligned}
$$

and hence,

$$
2\tilde{K}^{ij}(\sigma_\xi\dot{K})_{ij} + \tilde{\Gamma}^{ijb}(\sigma_\xi\cdot\dot{\Gamma})_{ijb} = 2\dot{K}^{ij}(\sigma_\xi\tilde{K})_{ij} + \dot{\Gamma}^{ijb}(\sigma_\xi\tilde{\Gamma})_{ijb},
$$

where $\dot{K}_{ij}, \dot{\Gamma}_{ijb}, \tilde{K}_{ij}, \tilde{\Gamma}_{ijb}$ are variations of the components K_{ij}, Γ_{ijb}, having the same (anti)symmetry properties. □

[4] In fact, the system (2.21) corresponds exactly to (2.19)–(2.20) up to zeroth order terms.

[5] We are grateful to an anonymous referee for this observation.

2.2. The differentiated system. In order to derive higher order energy estimates below, we will need to work with differentiated versions of (2.19)–(2.20). Moreover, for the boundary value problem (Sect. 3), we commute the equations with components of the orthonormal frame, which enables us to use the structure identified in (2.21) to control energies that contain an appropriate number of normal derivatives to the boundary (see Proposition 3.10).

For this purpose, we consider a multi-index I and the corresponding combination of vector fields e^I among $\{e_\mu\}_0^3$. We will use the following commutation formulas to compute the differentiated equations below:

$$[e_i, e_0] = K_i{}^c e_c, \quad [e_i, e_j] = f^d{}_p(e_i f_j{}^p)e_d - f^d{}_p(e_j f_i{}^p)e_d. \tag{2.22}$$

We note that (2.22) follows by (1.5) and (2.8). It is important that we do not use relations between the orthonormal frame and its connection coefficients, as for example, $[e_i, e_j] = \Gamma_{ij}{}^c e_c - \Gamma_{ji}{}^c e_c$, to compute the commuted equations, since in a local existence argument it is not a priori known that Γ_{ijb} are the connection coefficients of e_1, e_2, e_3. The fact that the solution to the modified evolution equations (2.19)–(2.20) gives indeed the connection coefficients of the orthonormal frame $\{e_\mu\}_0^3$, with respect to the Levi-Civita connection of the metric induced by the latter, is shown in Sect. 4 together with the vanishing of the Einstein tensor.

Applying e^I to both sides of the equations (2.19), (2.20), we obtain:

$$\begin{aligned}
&e_0 e^I K_{ij} + e^I(\mathrm{tr}K K_{ij})\\
&= \frac{1}{2}\Big[e_i e^I \Gamma^b{}_{jb} - e^b e^I \Gamma_{ijb} + e_j e^I \Gamma^b{}_{ib} - e^b e^I \Gamma_{jib} - 2\delta_{ij} e^a e^I \Gamma^b{}_{ab}\Big] - [e^I, e_0]K_{ij}\\
&\quad + \frac{1}{2}\Big[[e^I, e_i]\Gamma^b{}_{jb} - [e^I, e^b]\Gamma_{ijb} + [e^I, e_j]\Gamma^b{}_{ib} - [e^I, e^b]\Gamma_{jib} - 2\delta_{ij}[e^I, e^a]\Gamma^b{}_{ab}\Big]\\
&\quad + \frac{1}{2} e^I\Big[\Gamma^b{}_i{}^c\Gamma_{cjb} + \Gamma^b{}_b{}^c\Gamma_{ijc} + \Gamma^b{}_j{}^c\Gamma_{cib} + \Gamma^b{}_b{}^c\Gamma_{jic} - \delta_{ij}\big[\Gamma^{bac}\Gamma_{cab} + \Gamma^b{}_b{}^c\Gamma^a{}_{ac} + |K|^2 - (\mathrm{tr}K)^2\big]\Big],
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
&e_0 e^I \Gamma_{ijb} + e^I(K_i{}^c\Gamma_{cjb})\\
&= e_j e^I K_{bi} - e_b e^I K_{ji} + \delta_{ib}(e^c e^I K_{cj} - e_j e^I \mathrm{tr}K) - \delta_{ij}(e^c e^I K_{cb} - e_b e^I \mathrm{tr}K) - [e^I, e_0]\Gamma_{ijb}\\
&\quad + [e^I, e_j]K_{bi} - [e^I, e_b]K_{ji} + \delta_{ib}([e^I, e^c]K_{cj} - [e^I, e_j]\mathrm{tr}K) - \delta_{ij}([e^I, e^c]K_{cb} - [e^I, e_b]\mathrm{tr}K)\\
&\quad + e^I\Big[\Gamma_{bj}{}^c K_{ci} + \Gamma_{bi}{}^c K_{jc} - \Gamma_{jb}{}^c K_{ci} - \Gamma_{ji}{}^c K_{bc} - \delta_{ib}(\Gamma_c{}^{cl}K_{lj} + \Gamma^c{}_j{}^l K_{cl}) + \delta_{ij}(\Gamma_c{}^{cl}K_{lb} + \Gamma^c{}_b{}^l K_{cl})\Big]
\end{aligned} \tag{2.24}$$

The differentiated versions of the equations (2.8), (2.9) read

$$e_0 e^I f_i{}^p + K_i{}^c e^I f_c{}^p = -\sum_{I_1\cup I_2 = I,\, |I_2|<|I|} e^{I_1} K_i{}^c e^{I_2} f_c{}^p - [e^I, e_0] f_i{}^p, \tag{2.25}$$

$$e_0 e^I f^b{}_p - K_c{}^b e^I f^c{}_p = \sum_{I_1\cup I_2 = I,\, |I_2|<|I|} e^{I_1} K_c{}^b e^{I_2} f^c{}_p - [e^I, e_0] f^b{}_p, \tag{2.26}$$

where the union of the multi-indices I_1, I_2 is unordered and can be any possible permutation of I.

Lemma 2.9. *Let K_{ij}, Γ_{ijb}, $f_i{}^p$ be either a solution to* (2.8), (2.19), (2.20) *or an iterative version of these equations, where the frame coefficients $f_i{}^j$ (and hence $e_i = f_i{}^j\partial_j$) are determined by solving* (2.8) *with K_{ij} of the previous step. In the latter case, the first* *order terms in the RHS of* (2.19)–(2.20) *should have* K_{ij}, Γ_{ijb} *of the current iterates we're solving for. Then for any a combination of derivatives* e^I, K_{ij}, Γ_{ijb} *satisfy the following identity:*

$$
\begin{aligned}
&\frac{1}{2}e_0(e^I K^{ij} e^I K_{ij}) + \frac{1}{4}e_0(e^I \Gamma^{ijb} e^I \Gamma_{ijb}) \\
&\quad = e_i[e^I K^{ij} e^I \Gamma^b{}_{jb}] - e^b[e^I K^{ij} e^I \Gamma_{ijb}] - e_j[e^I \mathrm{tr}K e^I \Gamma^{bj}{}_b] \\
&\quad\quad + e^I K^{ij}\Big[[e^I, e_i]\Gamma^b{}_{jb} - [e^I, e^b]\Gamma_{ijb} - \delta_{ij}[e^I, e^a]\Gamma^b{}_{ab}\Big] - e^I K^{ij}[e^I, e_0]K_{ij} \\
&\quad\quad + e^I \Gamma^{ijb}\Big[[e^I, e_j]K_{bi} + \delta_{ib}([e^I, e^c]K_{cj} - [e^I, e_j]\mathrm{tr}K)\Big] - \frac{1}{2}e^I \Gamma^{ijb}[e^I, e_0]\Gamma_{ijb} \\
&\quad\quad + e^I K^{ij} e^I\Big[\Gamma^b{}_i{}^c\Gamma_{cjb} + \Gamma^b{}_b{}^c\Gamma_{ijc} - \mathrm{tr}K K_{ij} - \frac{1}{2}\delta_{ij}\big[\Gamma^{bac}\Gamma_{cab} + \Gamma^b{}_b{}^c\Gamma^a{}_{ac} + |K|^2 - (\mathrm{tr}K)^2\big]\Big] \\
&\quad\quad + e^I \Gamma^{ijb} e^I\Big[\Gamma_{bj}{}^c K_{ci} + \Gamma_{bi}{}^c K_{jc} - \frac{1}{2}K_i{}^c\Gamma_{cjb} - \delta_{ib}(\Gamma_c{}^{cl}K_{lj} + \Gamma^c{}_j{}^l K_{cl})\Big] \qquad (2.27)
\end{aligned}
$$

Proof. It follows straightforwardly by multiplying (2.23)–(2.24) with $e^I K^{ij}$, $\frac{1}{2}e^I\Gamma^{ijb}$ and using Lemma 2.7. □

2.3. Local well-posedness of the reduced equations for the Cauchy problem. Since the above equations form a symmetric hyperbolic system, local well-posedness follows from standard arguments. Nonetheless, we provide details below concerning the derivation of higher order energy estimates and the domain of dependence. This will allow us to treat the boundary case by a modification of the present section.

Define the $H^s(U_t)$ spaces, $U_t \subset \Sigma_t$, as the set of functions satisfying

$$
\|u\|^2_{H^s(U_t)} := \sum_{|I|\le s}\int_{U_t}(e^I u)^2 \mathrm{vol}_{U_t} < +\infty, \qquad (2.28)
$$

where I is a multi-index consisting only of spatial indices so that e^I is a combination of I derivatives among e_1, e_2, e_3, and vol_{U_t} is the intrinsic volume form. One might need more than one orthonormal frame to cover all of $T\Sigma_t$, but we could also consider the corresponding norms restricted to the slicing U_t of a neighbourhood of a point. In the case where u depends on various spatial indices, we define its H^s norm similarly, summing as well over all indices.

Remark 2.10. The above H^s spaces are equivalent to the usual spaces defined using coordinate derivatives ∂^I, provided we have control over the transition coefficients $f_i{}^p$, $f^b{}_p$. The use of e^I vector fields is essential for the treatment of the boundary problem in the next section. For this subsection we could have used ∂^I instead.

Lemma 2.11. *Let* U_t *be an open, bounded, subset of* Σ_t *with smooth boundary. Assume the transition coefficients* $f_i{}^p$, $f^b{}_p$ *satisfy the bounds*

$$
\sum_{|I|\le 1}\sum_{i,b,p=1}^{3}\sup_{t\in[0,T]}(\|e^I f_i{}^p\|_{L^\infty(U_t)} + \|e^I f^b{}_p\|_{L^\infty(U_t)}) \le D, \qquad (2.29)
$$

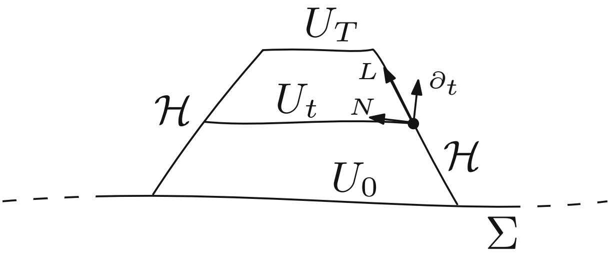

Fig. 1. Local domain of dependence

for some $T > 0$. *Then the following Sobolev inequalities hold with respect to the* H^s *spaces defined above:*

$$\|u\|_{L^\infty(U_t)} \leq C\|u\|_{H^2(U_t)}, \quad \|u\|_{L^4(U_t)} \leq C\|u\|_{H^1(U_t)} \tag{2.30}$$

for all $t \in [0, T]$, *where* $C > 0$ *depends on* U_t *and* D.

Proof. It is immediate by invoking the corresponding classical inequalities (for H^s spaces defined via coordinate derivatives) and using (2.29):

$$\begin{aligned}\|u\|^2_{L^\infty(U_t)} &\leq C\int_{U_t}(\partial^2 u)^2 + (\partial u)^2 + u^2 \mathrm{vol}_{U_t}\\ &= C\int_{U_t} f^4(e^2u)^2 + f^2(ef)^2(eu)^2 + f^2(eu)^2 + u^2\mathrm{vol}_{U_t}\\ &\leq C\int_{U_t} D^4(e^2u)^2 + D^4(eu)^2 + D^2(eu)^2 + u^2\mathrm{vol}_{U_t}\end{aligned}$$

The second inequality is derived similarly. □

Proposition 2.12. *The initial value problem for the system of reduced equations* (2.19)*,* (2.20)*,* (2.8)*,* (2.9) *is well-posed in* $L^\infty_t H^s$, $s \geq 3$, *with initial data prescribed along the Cauchy hypersurface* Σ_0.

Proof. We assume that a globally hyperbolic solution exists, in the relevant spaces, and derive a priori energy estimates below. Since the estimates for ${f_i}^p$, ${f^b}_p$ can be trivially derived using the ODEs (2.8)–(2.9), we assume we already have control over their H^s norm. Note that the assumption $s \geq 3$ is consistent with the pointwise control (2.29) of up one derivative of ${f_i}^p$, ${f^b}_p$ via (2.30).

Consider the differentiated equations (2.23)–(2.24), for a multi-index I of order $|I| \leq s$, over the future domain of dependence[6] of a neighbourhood of a point, U_0, foliated by U_t, $t \in [0, T]$, for some small $T \geq 0$, as depicted in Fig. 1.

Using Lemma 2.9, we obtain the energy inequality:

[6] Since the domain of dependence depends on the spacetime metric, in a Picard iteration the actual region of spacetime is not known until after a solution has been found, but one can enlarge slightly the domain to guarantee that in the end the resulting region includes the true domain of dependence of U_0.

$$\begin{aligned}
&\sum_{|I|\leq s}\partial_t\int_{U_t}\frac{1}{2}\sum_{i,j}(e^I K_{ij})^2+\frac{1}{4}\sum_{i,j,b}(e^I\Gamma_{ijb})^2\text{vol}_{U_t}\\
&\leq -\sum_{|I|\leq s}\int_{\partial U_t}\frac{1}{2}\sum_{i,j}(e^I K_{ij})^2+\frac{1}{4}\sum_{i,j,b}(e^I\Gamma_{ijb})^2\text{vol}_{\partial U_t}\\
&+\sum_{|I|\leq s}\int_{U_t}\left[\frac{1}{2}\sum_{i,j}(e^I K_{ij})^2+\frac{1}{4}\sum_{i,j,b}(e^I\Gamma_{ijb})^2\right]\text{tr}K\,\text{vol}_{U_t}\qquad(\partial_t\text{vol}_{U_t}=\text{tr}K\,\text{vol}_{U_t})\\
&+\sum_{|I|\leq s}\int_{U_t} e_j[e^I K^{ij}e^I\Gamma^b{}_{ib}]-e^i[e^I\text{tr}K\,e^I\Gamma^b{}_{ib}]-e_b[e^I\Gamma^{ijb}e^I K_{ji}]\text{vol}_{U_t}\\
&+\sum_{|I|\leq s}\int_{U_t}\Bigg(e^I K^{ij}\Big[[e^I,e_i]\Gamma^b{}_{jb}-[e^I,e^b]\Gamma_{ijb}-\delta_{ij}[e^I,e^a]\Gamma^b{}_{ab}\Big]-e^I K^{ij}[e^I,e_0]K_{ij}\\
&+e^I\Gamma^{ijb}\Big[[e^I,e_j]K_{bi}+\delta_{ib}([e^I,e^c]K_{cj}-[e^I,e_j]\text{tr}K)\Big]-\frac{1}{2}e^I\Gamma^{ijb}[e^I,e_0]\Gamma_{ijb}\Bigg)\text{vol}_{U_t}\\
&+C\|K\|_{H^s(U_t)}\|\Gamma\|^2_{H^s(U_t)}+C\|K\|^3_{H^s(U_t)},\qquad(\text{by }(2.30))
\end{aligned}\tag{2.31}$$

for a constant $C>0$, depending on the number of derivatives s and U_t. The first term in the RHS comes from the coarea formula, having a negative sign since the null boundary of $\{U_\tau\}_{\tau\in[0,t]}$ is ingoing. Indeed, we can write U_t as a union of an open set U_T (independent of t) and 2D surfaces constituting a variation of ∂U_t in the inward normal direction N to the surfaces. Decomposing $\partial_t=L-N$, we then notice that L commutes with the integral, while the $-N$ component gives the additional boundary term above. The last line includes all the terms corresponding to the last two lines in (2.27), which are treated by estimating the lowest order term in L^∞ and using Cauchy–Schwarz. To bound the terms in the second and third from last lines, we expand the commutators schematically in the two types of terms using (2.22):

$$\begin{aligned}
\left|\int_{U_t}e^I u*[e^I,e_0]u\text{vol}_{U_t}\right|&=\left|\sum_{|I_1|+|I_2|=|I|-1}\int_{U_t}e^I u*e^{I_1}K*e^{I_2}eu\text{vol}_{U_t}\right|\\
&\overset{(2.30)}{\leq}C\|K\|_{H^s(U_t)}\|u\|^2_{H^s(U_t)},
\end{aligned}\tag{2.32}$$

for $u=K,\Gamma$, and

$$\begin{aligned}
\left|\int_{U_t}e^I u*[e^I,e_i]v\text{vol}_{U_t}\right|&=\left|\sum_{|I_1|+|I_2|+|I_3|=|I|-1}\int_{U_t}e^I u*e^{I_1}f*e^{I_2}ef*e^{I_3}ev\text{vol}_{U_t}\right|\\
&\leq C\|u\|_{H^s(U_t)}\|f\|^2_{H^s(U_t)}\|v\|_{H^s(U_t)},\qquad(\text{by }(2.30))
\end{aligned}\tag{2.33}$$

for $(u,v)=(K,\Gamma)$ or (Γ,K), where we make use of the L^4 estimate (2.30) only in the last inequality, when $s=3$, $|I_2|=|I_3|=1$, after performing Cauchy–Schwarz twice, otherwise estimating the lowest order term in L^∞.

We may combine (2.31)–(2.33), integrate in $[0,t]$ and integrate by parts to obtain the overall integral inequality

$$\begin{aligned}
\frac{1}{2}\|K\|^2_{H^s(U_t)}+\frac{1}{4}\|\Gamma\|^2_{H^s(U_t)}\leq\frac{1}{2}\|K\|^2_{H^s(U_0)}+\frac{1}{4}\|\Gamma\|^2_{H^s(U_0)}+C\int_0^t\Big[\|K\|_{H^s(U_\tau)}\|\Gamma\|^2_{H^s(U_\tau)}\\
+\|K\|^3_{H^s(U_\tau)}+\|f\|^2_{H^s(U_\tau)}\|K\|_{H^s(U_\tau)}\|\Gamma\|_{H^s(U_\tau)}\Big]d\tau
\end{aligned}$$

$$-\int_0^t \sum_{|I|\le s}\int_{\partial U_t} \frac{1}{2}\sum_{i,j}(e^I K_{ij})^2 + \frac{1}{4}\sum_{i,j,b}(e^I\Gamma_{ijb})^2 \mathrm{vol}_{U_t} d\tau$$
$$-\int_0^t \sum_{|I|\le 3}\int_{\partial U_t}\left[e^I K^i{}_c e^I \Gamma^b{}_{ib} - e^I \mathrm{tr}K e^I\Gamma^b{}_{cb} - e^I\Gamma^{ij}{}_c e^I K_{ji}\right] N^c \mathrm{vol}_{U_t} d\tau \tag{2.34}$$

where we have used that the coefficients of the interior terms generated by integrating by parts the terms in the fourth line in (2.31), contain first derivatives of $f_i{}^p$ that can be estimated in L^∞.

It remains to show that the sum of all boundary terms in the last two lines of (2.33) has a favourable sign. For notational simplicity in the following computations, we assume that $e_3\big|_{\partial U_t} = -N$ and omit e^I. This is without loss of generally, since $N_{\partial U_\tau}$ can be written as a linear combination of e_1, e_2, e_3, where the sum of the squares of the coefficients is 1. Repeating the argument that follows for each component of $N_{\partial U_\tau}$, leads to the same conclusion by using Cauchy's inequality.

The integrands of the boundary terms then read:

$$\begin{aligned}
&-\frac{1}{2}K^{ij}K_{ij} - \frac{1}{4}\Gamma^{ijb}\Gamma_{ijb} + K^i{}_3\Gamma^b{}_{ib} - \mathrm{tr}K\Gamma^b{}_{3b} - \Gamma^{ij}{}_3 K_{ji}\\
&\quad = -\frac{1}{2}(K_{33})^2 - K^A{}_3 K_{A3} - \frac{1}{2}\hat{K}^{AB}\hat{K}_{AB} - \frac{1}{4}(K_C{}^C)^2 \quad (\hat{K}_{AB} := K_{AB} - \frac{1}{2}\delta_{AB}K_C{}^C)\\
&\qquad - \frac{1}{4}\Gamma^{ABC}\Gamma_{ABC} - \frac{1}{4}\Gamma_3{}^{AB}\Gamma_{3AB} - \frac{1}{2}\Gamma_{33}{}^A\Gamma_{33A} - \frac{1}{2}\Gamma^A{}_3{}^B\Gamma_{A3B}\\
&\qquad + K_3{}^A\Gamma^b{}_{Ab} - K_C{}^C\Gamma^B{}_{3B} - \Gamma^{iB}{}_3 K_{iB}\\
&\quad = -\frac{1}{2}(K_{33})^2 - K^A{}_3 K_{A3} - \frac{1}{2}\hat{K}^{AB}\hat{K}_{AB} - \frac{1}{4}(K_C{}^C)^2\\
&(\widehat{\Gamma}_{A3B} := \Gamma_{A3B} - \frac{1}{2}\delta_{AB}\Gamma_{C3}{}^C) - \frac{1}{4}\Gamma^{ABC}\Gamma_{ABC} - \frac{1}{4}\Gamma_3{}^{AB}\Gamma_{3AB}\\
&\qquad - \frac{1}{2}\Gamma_{33}{}^A\Gamma_{33A} - \frac{1}{2}\widehat{\Gamma}^A{}_3{}^B\widehat{\Gamma}_{A3B} - \frac{1}{4}(\Gamma^C{}_{3C})^2\\
&\qquad + K_3{}^A\Gamma^B{}_{AB} - K_C{}^C\Gamma^B{}_{3B} - \Gamma^{AB}{}_3 K_{AB}.
\end{aligned} \tag{2.35}$$

Rewrite the last line

$$\begin{aligned}
&K_3{}^A\Gamma^B{}_{AB} - K_C{}^C\Gamma^B{}_{3B} - \Gamma^{AB}{}_3 K_{AB}\\
&\quad = K_3{}^A\Gamma^B{}_{AB} - K_C{}^C\Gamma^B{}_{3B} + \Gamma^A{}_3{}^B K_{AB}\\
&\quad = K_3{}^A\Gamma^B{}_{AB} - \frac{1}{2}K_C{}^C\Gamma^B{}_{3B} + \widehat{\Gamma}^A{}_3{}^B\hat{K}_{AB}\\
&\quad \le K_3{}^A K_{3A} + \frac{1}{4}\Gamma^{BA}{}_B\Gamma^B{}_{AB} + \frac{1}{4}(K_C{}^C)^2 + \frac{1}{4}(\Gamma^B{}_{3B})^2 + \frac{1}{2}\widehat{\Gamma}^A{}_3{}^B\widehat{\Gamma}_{A3B} + \frac{1}{2}\hat{K}^{AB}\hat{K}_{AB}
\end{aligned} \tag{2.36}$$

Notice that $\Gamma^{BA}{}_B\Gamma^B{}_{AB} = (\Gamma_{112})^2 + (\Gamma_{221})^2$.

Thus, plugging (2.36) into (2.35), we conclude that the sum of all boundary terms has an overall negative sign. Therefore, they can be dropped in (2.34), giving

$$\begin{aligned}
\frac{1}{2}\|K\|^2_{H^s(U_t)} + \frac{1}{4}\|\Gamma\|^2_{H^s(U_t)} \le \frac{1}{2}\|K\|^2_{H^s(U_0)} + \frac{1}{4}\|\Gamma\|^2_{H^s(U_0)} + C\int_0^t\Big[\|K\|_{H^s(U_\tau)}\|\Gamma\|^2_{H^s(U_\tau)}\\
+ \|K\|^3_{H^s(U_\tau)} + \|f\|^2_{H^s(U_\tau)}\|K\|_{H^s(U_\tau)}\|\Gamma\|_{H^s(U_\tau)}\Big]d\tau
\end{aligned} \tag{2.37}$$

The preceding estimate, combined with bounds for $\|f\|^2_{H^s(U_t)}$ that are straightforwardly derived using (2.25)–(2.26), can be upgraded to a Picard iteration and a local existence result in a standard way, we omit the details. □

2.4. Initial data. Our initial data (Σ_0, g, K) are that of the EVE, i.e. the induced metric and the second fundamental form on Σ_0, verifying the constraint equations (1.2)–(1.3), with h, k replaced by g, K. Given an orthonormal frame e_1, e_2, e_3 on Σ_0 and an abstract coordinate system (x_1, x_2, x_3), the initial data for f_i^p, $f^b{}_p$, K_{ij}, Γ_{ijb} are determined in the obvious way from (1.5), (1.6), (1.7). In particular, the functions Γ_{ijb} are the connection coefficients associated to e_1, e_2, e_3, with respect to the Levi-Civita connection D of g, and they are hence anti-symmetric in the indices j, b, while $K_{ij} = K_{ji}$, $f_i{}^p f^b{}_p = \delta_i^b$, $f^b{}_p f_b{}^q = \delta_p^q$ on Σ_0.

3. Application to Totally Geodesic Boundaries

We now consider how to apply our framework to the initial boundary value problem, in the case of timelike, totally geodesic boundaries. In this case, with N being the outward unit normal to $\mathcal{T}$, the second fundamental form of the boundary

$$\chi(Y, Z) := \mathbf{g}(\mathbf{D}_Y N, Z) = \chi(Z, Y), \quad Y, Z \perp N, \tag{3.1}$$

is identically zero, $\chi \equiv 0$.

We start in Sect. 3.1 by deriving the zeroth order compability conditions in the case of an angle ω. In order to use the setting of the previous section, it will be convenient that the orthonormal frame is adapted to the boundary, so that e_0 is for instance tangent to $\mathcal{T}$. For this, we need the spacelike slice to be orthogonal to the boundary, so we briefly outline a standard reduction that allows us to do so in Sect. 3.2. We also state the first order compatibility condition in the case $\omega = 0$ and then derive the induced boundary conditions from the geometric ones in Sect. 3.3. Then, in Sect. 3.4, we prove the local well-posedness of the resulting IBVP for the reduced equations (2.8), (2.19), (2.20), see Proposition 3.10.

3.1. Zeroth order compatibility conditions with an angle. Consider a solution to the Einstein equations $(\mathcal{M}, g)$ with a totally geodesic timelike boundary $\mathcal{T}$. For a time function t and a foliation of $\mathcal{M}$ by the level sets Σ_t of t, with Σ_0 an initial orthogonal slice, we use the notation $\mathcal{S}_t$ for the cross sections $\Sigma_t \cap \mathcal{T}$ and denote $\mathcal{S}_0 = \mathcal{S}$.

Let e_0 be the normal to $\mathcal{S}$, within $\mathcal{T}$, and N be the outward unit normal to the boundary. The initial slice Σ will not in general be orthogonal to the boundary $\mathcal{T}$, but it will have a given angle $\omega : \mathcal{S} \to \mathbb{R}$, that defines the hyperbolic rotation between (e_0, N) and the pair $(n, \overline{N})$, where n is the future unit normal to Σ on $\mathcal{S}$ and $\overline{N}$ is the outward unit normal to $\mathcal{S}$, within Σ, see Fig. 2:

$$\begin{cases} e_0 = n\cosh\omega - \overline{N}\sinh\omega \\ N = -n\sinh\omega + \overline{N}\cosh\omega \end{cases}, \quad \begin{cases} n = e_0\cosh\omega + N\sinh\omega \\ \overline{N} = e_0\sinh\omega + N\cosh\omega \end{cases}. \tag{3.2}$$

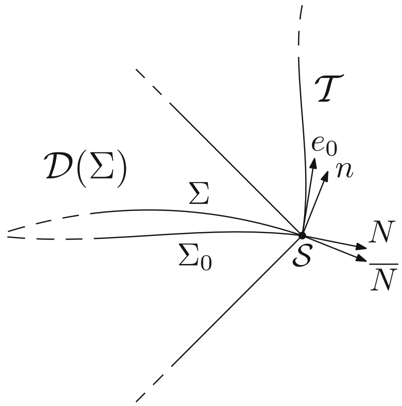

Fig. 2. The classical solution in the domain of dependence $\mathcal{D}(\Sigma)$

Lemma 3.1. (zeroth order compatibility conditions with an angle) *Let* (h, k) *denotes the first and second fundamental form of* Σ. *Then,* (Σ, h, k) *must satisfy the following zeroth order compatibility conditions:*

$$\begin{aligned} \chi(\overline{X}, \overline{Y}) = 0 \quad &\Longleftrightarrow \quad k(\overline{X}, \overline{Y}) \sinh\omega = h(\nabla_{\overline{X}}\overline{N}, \overline{Y}) \cosh\omega \\ \chi(\overline{X}, e_0) = 0 \quad &\Longleftrightarrow \quad k(\overline{X}, \overline{N}) = \overline{X}\omega \end{aligned} \tag{3.3}$$

for all $\overline{X}, \overline{Y} \in T\mathcal{S}$, *where* ∇ *is the Levi-Civita connection of* h.

Higher order compatibility conditions can be derived using the zeroth order compatibility conditions (3.3), the boundary conditions and the Einstein vacuum equations. These can be written intrinsically on Σ as relations involving only (h, k), the function ω and their derivatives on $\mathcal{S}$. We do not write the details in the general case, but in Lemma 3.4, we write the first order conditions in the particular case of an initial slice intersecting the timelike boundary orthogonally.

Proof. Recall that $k(X, Y) = \mathbf{g}(\mathbf{D}_X n, Y)$, for $X, Y \in T\Sigma$. Plugging (3.2) in the definition (3.1) of χ, we have

$$\begin{aligned} \chi(\overline{X}, \overline{Y}) &= \mathbf{g}(\mathbf{D}_{\overline{X}}(-n\sinh\omega + \overline{N}\cosh\omega), \overline{Y}) \\ &= -k(\overline{X}, \overline{Y})\sinh\omega + h(\nabla_{\overline{X}}\overline{N}, \overline{Y})\cosh\omega - \mathbf{g}(n, \overline{Y})X\sinh\omega + h(\overline{N}, \overline{Y})X\cosh\omega \\ &= -k(\overline{X}, \overline{Y})\sinh\omega + h(\nabla_{\overline{X}}\overline{N}, \overline{Y})\cosh\omega \end{aligned}$$

and

$$\begin{aligned} \chi(\overline{X}, e_0) &= \mathbf{g}(\mathbf{D}_{\overline{X}}(-n\sinh\omega + \overline{N}\cosh\omega), n\cosh\omega - \overline{N}\sinh\omega) \\ &= (\cosh\omega)^2\overline{X}\omega + k(\overline{X}, \overline{N})(\sinh\omega)^2 - k(\overline{X}, \overline{N})(\cosh\omega)^2 - (\sinh\omega)^2\overline{X}\omega \\ &= -k(\overline{X}, \overline{N}) + \overline{X}\omega, \end{aligned}$$

which proves the equivalence (3.3). $\square$

3.2. Choosing an orthogonal initial slice to the boundary. With the notations of the previous section, we may consider $\mathcal{D}(\Sigma)$ the (future and past) domain of dependence of Σ, see Fig. 2. Note that $\mathcal{D}(\Sigma)$ can be constructed by solving a pure initial value problem with the initial data (Σ, h, k). Consider an another initial hypersurface, Σ_0 contained in $\mathcal{D}(\Sigma)$ and such that $\Sigma_0 \cap \Sigma = \mathcal{S}$. Let N_{Σ_0} be the unit normal to $\mathcal{S}$ in Σ_0 and e_{Σ_0} be its future unit normal. If the hyperbolic angle between the pair $(e_{\Sigma_0}, N_{\Sigma_0})$ and $(n, \overline{N})$ is equal to the angle ω, then it follows that $(e_{\Sigma_0}, N_{\Sigma_0}) = (e_0, N)$ and that Σ_0 is orthogonal to $\mathcal{T}$.

Thus, given initial data (Σ, h, k) verifying the compatibility conditions with an angle ω, one can first solve the classical initial value problem with data (Σ, h, k) to obtain a development $\mathcal{D}(\Sigma)$ and an embedding ψ_i of Σ into $\mathcal{D}(\Sigma)$. One then chooses a new initial slice Σ_0 in $\mathcal{D}(\Sigma)$, such that $\psi_i(\Sigma) \cap \Sigma_0 = \psi_i(\mathcal{S})$ and the hyperbolic angle between $(e_{\Sigma_0}, N_{\Sigma_0})$, $(n, \overline{N})$ is equal to ω. Provided we can solve the IBVP with compatibility conditions induced from the original ones (3.3) by setting $\omega = 0$, the new slice will then be orthogonal to the boundary.

3.3. Boundary and compatibility conditions in the geodesic frame. From now on, we focus on solving the initial boundary value problem in the case of initial data with $\omega = 0$ and thus also consider a spacetime with a totally geodesic timelike boundary $\mathcal{T}$ and a foliation Σ_t orthogonal to $\mathcal{T}$.

For a spacetime with a timelike boundary $\mathcal{T}$, the e_0 geodesics relative to $\mathbf{g}$, emanating from $\mathcal{S}$, will not in general remain tangent to $\mathcal{T}$. This makes the geodesic frame (2.1), as it stands, unsuitable for studying the general boundary value problem. However, for a totally geodesic boundary, the e_0 geodesics will indeed foliate a neighbourhood of $\mathcal{T}$.

An essential ingredient in our approach is the use of an adapted frame to the boundary. The existence of such a frame, compatible with the propagation conditon (2.1), is possible thanks to the vanishing of the second fundamental form χ.

Lemma 3.2. (Adapted frame to the boundary) *Let Σ_0 be orthogonal to $\mathcal{T}$, as in Fig. 2, and let e_0 denote its future unit normal. Also, let e_1, e_2, e_3 be an orthonormal frame tangent to Σ_0, such that at the boundary $e_1, e_2 \in T\mathcal{S}$ and e_3 coincides with the outward unit normal N. Then the frame verifying (2.1) is adapted to the boundary. In particular, the e_0 curves emanating from $\mathcal{S}$ remain tangent to $\mathcal{T}$ and $e_3 = N$ on $\mathcal{T}$.*

Proof. Define the tangential, orthonormal frame $\tilde{e}_0, \tilde{e}_1, \tilde{e}_2$ on $\mathcal{T}$ by the condition:

$$\not{\nabla}_{\tilde{e}_0}\tilde{e}_0 = \not{\nabla}_{\tilde{e}_0}\tilde{e}_1 = \not{\nabla}_{\tilde{e}_0}\tilde{e}_2 = 0, \tag{3.4}$$

where $\not{\nabla}$ is the covariant connection intrinsic to $\mathcal{T}$. Also, we impose that $\tilde{e}_0 = e_0, \tilde{e}_1 = e_1, \tilde{e}_2 = e_2$ at $\mathcal{S}$.

Then, $\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, N$ satisfy

$$\mathbf{D}_{\tilde{e}_0}\tilde{e}_0 = \mathbf{D}_{\tilde{e}_0}\tilde{e}_1 = \mathbf{D}_{\tilde{e}_0}\tilde{e}_2 = \mathbf{D}_{\tilde{e}_0}N = 0, \quad \text{on } \mathcal{T}, \tag{3.5}$$

since the second fundamental form of $\mathcal{T}$ vanishes. Hence, the two set of frames $\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, N$ and e_0, e_1, e_2, e_3 satisfy the same propagation equation and have the same initial configurations at $\mathcal{S}$. We arrive at the conclusion that they must coincide. □

Lemma 3.3 (Boundary conditions for the orthogonal foliation). *For the particular geodesic frame e_0, e_1, e_2, e_3 that is adapted to the boundary, as above, the vanishing of χ induces the following boundary conditions on K_{ij}, Γ_{ijb}:*

$$K_{A3} = K_{3A} = \Gamma_{A3B} = \Gamma_{AB3} = 0, \tag{3.6}$$

satisfied on $\mathcal{T}$, for every $A, B = 1, 2$.

Proof. The conditions (3.6) follow from the relations

$$K_{A3} = K_{3A} = \mathbf{g}(\mathbf{D}_{e_A} e_0, e_3) = -\chi_{0A}, \quad \Gamma_{A3B} = -\Gamma_{AB3} = \mathbf{g}(\mathbf{D}_{e_A} e_3, e_B) = \chi_{AB}, \quad \text{on } \mathcal{T}, \tag{3.7}$$

and the vanishing of χ. □

Lemma 3.4 (Compatibility conditions for the orthogonal foliation). *The initial data of $K_{ij}, \Gamma_{ijb}, f_i{}^j$ that correspond to the orthonormal frame in Lemma 3.2, must satisfy corner conditions at $\mathcal{S}$ to all orders allowed in our energy spaces. The zeroth order conditions are the boundary conditions (3.6), which are induced by (3.3), while the first order conditions read:*

$$\begin{aligned} &e_3\Gamma^B{}_{AB} = e^B\Gamma_{3AB} - \Gamma^b{}_b{}^C\Gamma_{3AC}, \quad e_3K_{22} = -2\Gamma_{31}{}^C K_{1C}, \\ &e_3K_{11} = -2\Gamma_{32}{}^C K_{2C}, \quad e_3K_{12} = \Gamma_{31}{}^C K_{2C} + \Gamma_{32}{}^C K_{1C}. \end{aligned} \tag{3.8}$$

Remark 3.5. Note that the above conditions only involve the initial data (Σ, h, k) since the frame (e_1, e_2, e_3) is tangential to Σ and the connection coefficients are those of the Riemannian metric h. If we assume that we have a solution foliated by Σ_t, then the relations (3.8) would hold on each $\mathcal{S}_t$. We can then inductively take e_0 derivatives of (3.8), commute e_0 with e_B, e_3 and replace all e_0K, $e_0\Gamma$ derivatives using the evolution equations (2.19)–(2.20). Restricted to $\mathcal{S}_0$, the resulting equations only involve the initial data and are the higher order compatibility conditions expressed with respect to an orthonormal frame.

Proof. The zeroth order compatibility conditions are derived from the ones with an angle (3.3), using the rotation relations (3.2). We confirm that these are indeed the boundary conditions (3.6), without using the existence of a spacetime solution to the IBVP, as in the proof of Lemma 3.3, showing that they can be derived solely from the initial data on the original initial slice Σ and the knowledge of the angle ω. We compute on $\mathcal{S}$:

$$\begin{aligned} K_{A3} &= \mathbf{g}(\mathbf{D}_{e_A} e_0, e_3) = \mathbf{g}(\mathbf{D}_{e_A}(n\cosh\omega - \overline{N}\sinh\omega), -n\sinh\omega + \overline{N}\cosh\omega) \\ &= e_A\omega\sinh^2\omega + k(e_A, \overline{N})\cosh^2\omega - k(e_A, \overline{N})\sinh^2\omega - e_A\omega\cosh^2\omega \\ &= -e_A\omega + k(e_A, \overline{N}) = 0 \end{aligned}$$

and

$$\begin{aligned} \Gamma_{A3B} &= \mathbf{g}(\mathbf{D}_{e_A} e_3, e_B) = \mathbf{g}(\mathbf{D}_{e_A}(-n\sinh\omega + \overline{N}\cosh\omega), e_B) \\ &= -k(e_A, e_B)\sinh\omega + h(\nabla_{e_A}\overline{N}, e_B) = 0. \end{aligned}$$

Together with the symmetry/antisymmetry of K_{3A}, Γ_{A3B}, this proves our claim for the zeroth order corner conditions.

Restricting (2.19), (2.20), for $i = 3$, $j = A$ and $i = A$, $j = 3$, $b = B$ respectively, to the intersection $\mathcal{S}$ and utilising (3.6), we obtain the equations:

$$\begin{aligned} 0 = e_0K_{3A} + \mathrm{tr}K\, K_{3A} = \frac{1}{2}\Big[&e_3\Gamma^b{}_{Ab} - e^b\Gamma_{3Ab} + \Gamma^b{}_3{}^c\Gamma_{cAb} + \Gamma^b{}_b{}^c\Gamma_{3Ac} \\ &+ e_A\Gamma^b{}_{3b} - e^b\Gamma_{A3b} + \Gamma^b{}_A{}^c\Gamma_{c3b} + \Gamma^b{}_b{}^c\Gamma_{A3c} \Big] \end{aligned}$$

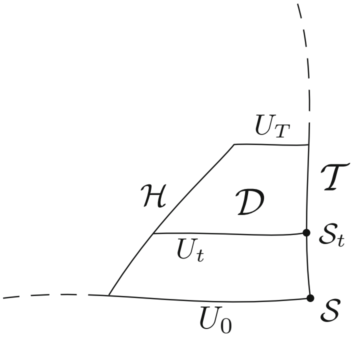

Fig. 3. A local domain of dependence region near the boundary

$$
\begin{aligned}
&= \frac{1}{2}\Big[e_3\Gamma^B{}_{AB} - e^B\Gamma_{3AB} + \Gamma^b{}_b{}^C\Gamma_{3AC}\Big]\\
0 = e_0\Gamma_{A3B} + K_A{}^c\Gamma_{c3B} &= e_3K_{BA} - e_BK_{3A} - \Gamma_{3B}{}^cK_{cA} - \Gamma_{3A}{}^cK_{Bc} + \Gamma_{B3}{}^cK_{cA} + \Gamma_{BA}{}^cK_{3c}\\
&\quad + \delta_{AB}\Big[e^cK_{c3} - \Gamma_c{}^{cb}K_{b3} - \Gamma^c{}_3{}^bK_{cb} - e_3\mathrm{tr}K\Big]\\
&= e_3K_{BA} - \Gamma_{3B}{}^CK_{CA} - \Gamma_{3A}{}^CK_{BC} - \delta_{AB}e_3K_C{}^C
\end{aligned}
$$

which give the conditions (3.8). □

3.4. Local well-posedness for the initial boundary value problem. In the following, we consider a Lorentzian manifold with boundary of the form $(\mathcal{D}, \mathbf{g})$ with $\mathcal{D}$ foliated by spacelike hypersurfaces $U_t \subset \Sigma_t$, $\mathcal{D} = \bigcup_{t\in[0,T]} U_t$, such that $\partial\mathcal{D} = U_0 \cup \mathcal{T} \cup \mathcal{H} \cup U_T$, with $\mathcal{T}$ timelike and $\mathcal{H}$ ingoing null, $T > 0$, as depicted in Fig. 3. The U_t are moreover assumed to be orthogonal to $\mathcal{T}$.

We assume that $(\mathcal{D}, \mathbf{g})$ is globally hyperbolic in the sense of a Lorentzian manifold with timelike boundary [15]. In particular, we assume that $\mathcal{D}$ is such that given $p \in \mathcal{D}$, $J^-(p) \cap J^+(U_0 \cup \mathcal{T})$ is compact, so that all the computations below are well defined. We will prove high order energy estimates on $\mathbf{g}$ assuming it solves the Einstein equations with the corresponding initial and boundary data. These a priori estimates can then be upgraded via a Picard iteration to obtain the existence of a solution.

We denote by $\underline{e}$ any of the derivatives tangential to the boundary e_0, e_1, e_2. We consider the following modified Sobolev spaces, denoted B^s, which, for a given s, contains $[\frac{s}{2}]$ normal derivatives compared to s tangential derivatives, $[\frac{s}{2}]$ being the integer part of $\frac{s}{2}$.

Definition 3.6. Let $s \in \mathbb{N}$. For any $u \in L^\infty_t L^2(U_t)$, we consider the following energy norm on each slice U_t,

$$
\|u\|_{B^s(U_t)} := \sum_{|I_1|+2|I_2|\le s} \|\underline{e}^{I_1} e_3^{I_2} u\|_{L^2(U_t)},
$$

and the corresponding energy space

$$
B^s(U_t) = \left\{ u \in L^\infty\big([0,T]; L^2(U_t)\big) \;:\; \sup_{t\in[0,T]} \text{ess} \sum_{|I_1|+2|I_2|\le s} \|\underline{e}^{I_1} e_3^{I_2} u\|_{L^2(U_t)} < +\infty \right\}. \quad (3.9)
$$

Remark 3.7. The need for the B^s spaces is dictated by the form (2.21) of the modified ADM system, which only allows the control of roughly half the number of normal derivatives compared with the number of tangential derivatives in L^2. Here is also where the definition of the norms with respect to e_μ vector fields becomes particularly useful. Note that we have also included time derivatives in the norms.

Remark 3.8. The reduced equations (2.19), (2.20), coupled to the ODE (2.8), and subject to the boundary conditions (3.6), fit in the context of more general characteristic symmetric hyperbolic systems that have been studied in the literature [21]. In the language of [21]:

– the space $H^m_*(\Omega)$ coincides with $B^s(U_t)$, for $e_3 = \partial_1$,
– A_ν has a kernel of constant rank 4, equal to the number of variables in (2.21) whose evolution equation does not contain e_3 derivatives,
– and $M(x)$ is the (constant) projection matrix on the latter variables, satisfying $\langle A_\nu u, u\rangle = 0$ for all $u \in \ker M(x)$.

Hence, the local well-posedness theorems in [21] apply to our specific case as well. For the convenience of the reader, we derive the local well-posedness estimates for the present problem in Proposition 3.10.

Lemma 3.9. *Let $s \geq 6$ and let*

$$\|v\|_{B^s(U_t)} \leq C_0, \quad v = K_{ij}, \Gamma_{ijb}, f_i{}^p, f^b{}_p,$$

for all $0 \leq t \leq T$. Changing the order of the tangential and normal derivatives in the definition (3.9) gives an equivalent norm, up to a constant depending on C_0. More precisely, the following inequality holds true

$$\|e^I u\|_{L^2(U_t)} \leq C \sum_{|I_1|\leq r} \sum_{|I_2|\leq m} \|\underline{e}^{I_1} e_3^{I_2} u\|_{L^2(U_t)}, \tag{3.10}$$

for any e^I that consists of r tangential and m e_3 derivatives, $r + 2m \leq s$. The constant C is of the form $C = D + TC_0^2$, where $D > 0$ depends only on initial norms.

Proof. We argue by induction in $|I| = r + m$. According to (2.22), a commutation between a tangential and a normal derivative in $e^I u$ gives terms of the form

$$e^{J_1}[\underline{e}, e_3]e^{J_2}u = e^{J_1}(K * ee^{J_2}u) + e^{J_1}(f * ef * ee^{J_2}u), \quad |J_1| + |J_2| = |I| - 2$$

There are two distinct cases for bounding the L^2 norm of the previous RHS.

- The derivatives are relatively equally distributed among the corresponding factors, in which case we can bound their L^2 norms using the L^4 estimate in (2.30) and the inductive step.
- Most derivatives hit one of the factors, in which we can apply (2.30) to the lowest order factor and use the inductive step.

The assumption $s \geq 6$ allows for a pointwise bound on the factor ef via (2.30). The dependence of the constant C in D, C_0 comes from the use of the following basic estimate:

$$\|e^{I'} v\|^2_{L^2(U_t)} \overset{C-S}{\leq} \|e^{I'} v\|^2_{L^2(U_0)} + \int_0^t 2\|e^{I'} v\|_{L^2(U_\tau)} \|e_0 e^{I'} v\|_{L^2(U_\tau)} d\tau \leq D + TC_0^2$$

to the terms $e^{I'} v$, $v = (K_{ij}, \Gamma_{ijb}, f_i{}^j, f^b{}_j)$, that have one less than maximum number of tangential derivatives, ie. for $e^{I'}$ containing r' tangential and m' e_3 derivatives with $r' + 2m' \leq s - 1$. □

Proposition 3.10. *Under the boundary and compatibility conditions* (3.6), (3.8)*, as well as their higher order analogues up to order $s-1$ (see Remark* 3.5*), the system* (2.19), (2.20), (2.8), (2.9) *has a locally well-posed initial boundary value problem in $L^\infty_t B^s(U_t)$, for $s \geq 7$.*

Proof. We prove an energy estimate in the space $L^\infty([0,T];B^s)$, for $s \geq 7$ and T sufficiently small. To this end, we proceed by a bootstrap argument and assume that we have a smooth solution u on $[0,T]$, for some $T>0$, satisfying

$$\|u\|_{B^s(U_t)} \leq C_0, \tag{3.11}$$

for all $0 \leq t \leq T$. The slices U_t correspond now to the neighbourhood of a point at the boundary $\Sigma_0 \cap \mathcal{T}$. We will upgrade this type of non-quantitative estimate into a quantitative one depending only on the initial data. This is the kind of estimate that is required to then prove existence and uniqueness of solutions via a Picard iteration scheme (cf. also [21]). As for the solution to the linear problem required at each step in the iteration, this follows by a duality argument, using the symmetry of the system (2.21), combined with the fact that the boundary terms in a usual energy argument for (2.19)–(2.20) vanish by virtue of the boundary conditions (3.6), see (3.12) below. This implies that the dual system has the same form, satisfying the same homogeneous Dirichlet boundary conditions, as in (3.6), for the corresponding dual variables.

Step 1: Estimates for tangential derivatives. Consider I a multi-index of order $|I| \leq s$, such that $e^I = \underline{e}^I$ does not contain any e_3 derivative. We repeat the energy argument of Proposition 2.12, where we use the differentiated equations (2.23)–(2.24) for $e^I = \underline{e}^I$. Note that by the above bootstrap assumption and since $s \geq 7$, the Sobolev inequalities used to control the L^∞, L^4 norms of certain terms are still applicable. In particular, the error terms generated by the various integration by parts (due to e_i not being Killing) can be controlled in this way and then absorbed by choosing T sufficiently small depending only on the norm of the initial data.

We examine now the arising $\mathcal{T}$-boundary terms in the energy argument. Going back to the fourth line of (2.31), we notice that these terms are

$$\begin{aligned}&\int_{\mathcal{S}_t} \underline{e}^I K_3{}^j \underline{e}^I \Gamma^b{}_{jb} - \underline{e}^I \mathrm{tr}K \underline{e}^I \Gamma^b{}_{3b} - \underline{e}^I \Gamma^{ij}{}_3 \underline{e}^I K_{ij} \mathrm{vol}_{\mathcal{S}_t} \\ &\quad = \int_{\mathcal{S}_t} \underline{e}^I K_{33} \underline{e}^I \Gamma^B{}_{3B} + \underline{e}^I K_3{}^A \underline{e}^I \Gamma^B{}_{AB} - \underline{e}^I \mathrm{tr}K \underline{e}^I \Gamma^B{}_{3B} - \underline{e}^I \Gamma^{AB}{}_3 \underline{e}^I K_{AB} \mathrm{vol}_{\mathcal{S}_t},\end{aligned} \tag{3.12}$$

for $|I| \leq s$. Since $\underline{e}^I$ is tangential, we infer by (3.6) that all boundary terms vanish.

Since we commute only with tangential derivatives, the error terms corresponding to (2.32)–(2.33) take the form

$$\begin{aligned}&\sum_{|I_1|+|I_2|=|I|-1} \int_{U_t} \underline{e}^I K * \underline{e}^{I_1} K * \underline{e}^{I_2} e K \mathrm{vol}_{U_t} \\ &\quad + \sum_{|I_1|+|I_2|+|I_3|=|I|-1} \int_{U_t} \underline{e}^I K * \underline{e}^{I_1} f * \underline{e}^{I_2} e f * \underline{e}^{I_3} e \Gamma \mathrm{vol}_{U_t}\end{aligned}$$

Thus, they contain at most one e_3 derivative and can be handled using (2.30) (since we include at least three normal derivatives in our B^s norms, $s \geq 7$) and the bootstrap assumption. We make use of the L^4 estimate only for the second term, in the case $|I_1| = 0$, where both $e = e_3$.

Step 2: Consequences of the bootstrap assumption. First, we note that a standard energy argument for (2.25)–(2.26) gives the desired estimate for the part of the norm involving ${f_i}^j$, ${f^b}_j$, making use only of the bootstrap assumption:

$$\sum_{i,b,p=1}^{3} \sup_{t\in[0,T]} (\|\underline{e}^{I_1} e_3^{I_2} {f_i}^p\|_{L^2(U_t)} + \|\underline{e}^{I_1} e_3^{I_2} {f^b}_p\|_{L^2(U_t)}) \leq D_f, \tag{3.13}$$

where D_f depends on initial norms and $C_0^2 T$. The latter can be made smaller than a universal constant by taking $T > 0$ sufficiently small.

Moreover, any term having a less than top order number of tangential derivatives can also be bounded in L^2 by using only the bootstrap assumption in the following manner:

$$\begin{aligned}
\|\underline{e}^{I_1'} e_3^{I_2} u\|^2_{L^2(U_t)} &\leq \|\underline{e}^{I_1'} e_3^{I_2} u\|^2_{L^2(U_0)} + \int_0^t 2\|\underline{e}^{I_1'} e_3^{I_2} u\|_{L^2(U_\tau)} \|\partial_\tau \underline{e}^{I_1'} e_3^{I_2} u\|_{L^2(U_\tau)} d\tau \\
&\leq \|\underline{e}^{I_1'} e_3^{I_2} u\|^2_{L^2(U_0)} + T C_0^2 \qquad (|I_1'| < s - 2|I_2|) \\
&\leq D_{\text{low}}
\end{aligned} \tag{3.14}$$

for some $T > 0$ sufficiently small, where D_{low} denotes a constant depending on the initial L^2 norms of $\partial^{I_1'} \partial^{I_2} u$, $\partial_t \partial^{I_1'} \partial^{I_2} u$.

Thus, matters are reduced to estimating the top order K_{ij}, Γ_{ijb} terms.

Step 3: Induction for the normal derivatives. To complete the energy argument, we must estimate the norms $\|\underline{e}^{I_1} e_3^{I_2} u\|_{L^2}$, with $|I_1| + 2|I_2| = s$, $u = K, \Gamma$.

We proceed by induction in $|I_2|$. *Step 1* above shows in particular that one can deal with $|I_2| = 0$. Let $m \leq [\frac{s}{2}] - 1$ and assume that, for all multi-indices I_1 and I_2 verifying $|I_1| + 2|I_2| = s$, $|I_2| \leq m$, we have a bound of the form

$$\sup_{t\in[0,T]} \|\underline{e}^{I_1} e_3^{I_2} u\|_{L^2(U_t)} \leq D_m, \tag{3.15}$$

for some $T > 0$ sufficiently small, where D_m depends only on the B^s-norm of the initial data of $(K_{ij}, \Gamma_{ijb}, {f_i}^j, {f^b}_j)$, the number of derivatives s, m, and is independent of C_0. We will derive (3.15) for $|I_2| = m + 1$.

Let us split the variables K_{ij}, Γ_{ijb} into two sets:

1. the good set

$$\mathcal{G} = \{K_{11}, K_{12}, K_{22}, K_{31}, K_{32}, \Gamma_{112}, \Gamma_{212}, \Gamma_{113}, \Gamma_{223}, \Gamma_{123} + \Gamma_{213}\}, \tag{3.16}$$

2. the bad set

$$\mathcal{B} = \{K_{33}, \Gamma_{313}, \Gamma_{323}, \Gamma_{312}, \Gamma_{123} - \Gamma_{213}\}. \tag{3.17}$$

According to the composition of the system (2.19)–(2.20) identified in (2.21), we have the following two schematic types of equations

$$e_0 \Psi_{\mathcal{G}} = e_3 \Psi_{\mathcal{G}} + e_A \Psi_{\mathcal{B}} + \Psi * \Psi, \tag{3.18}$$

$$e_0 \Psi_{\mathcal{B}} = e_A \Psi_{\mathcal{G}} + \Psi * \Psi, \tag{3.19}$$

satisfied by $\Psi_{\mathcal{G}} \in \mathcal{G}, \Psi_{\mathcal{B}} \in \mathcal{B}, \Psi \in \mathcal{G} \cup \mathcal{B}$, where we note that $e_3\Psi_{\mathcal{G}}$ corresponds to a single representative of the good set, whereas $e_A\Psi_{\mathcal{B}}, e_A\Psi_{\mathcal{G}}$ could be an algebraic combination of more than one terms from the corresponding sets.

Starting with (3.18), we differentiate the equation in $\underline{e}^{I_1}e_3^{I_2'}$, $|I_1| + 2|I_2'| = s - 1$, $|I_2'| = m$:

$$\underline{e}^{I_1}e_3^{I_2'}e_3\Psi_{\mathcal{G}} = \underline{e}^{I_1}e_3^{I_2'}e_0\Psi_{\mathcal{G}} - \underline{e}^{I_1}e_3^{I_2'}e_A\Psi_{\mathcal{B}} - \underline{e}^{I_1}e_3^{I_2'}(\Psi * \Psi) \tag{3.20}$$

Note that the derivatives acting on $\Psi_{\mathcal{G}}$ in the LHS contain one more tangential derivative than what we need for the desired estimate on $\Psi_{\mathcal{G}}$. It is necessary to include this extra tangential derivative in order to infer the bound on $\Psi_{\mathcal{B}}$ directly below.

The first two terms in the RHS of (3.20) are at the level of the inductive assumption (3.15). To bound the L^2 norm of the third term we employ (2.30) as follows:

$$\begin{aligned}\|\underline{e}^{I_1}e_3^{I_2'}(\Psi * \Psi)\|_{L^2(U_t)} &\leq \|\Psi\underline{e}^{I_1}e_3^{I_2'}\Psi\|_{L^2(U_t)} + \sum_{|J_2|,|L_2|<m}\|\underline{e}^{J_1}e_3^{J_2}\Psi * \underline{e}^{L_1}e_3^{L_2}\Psi\|_{L^2(U_t)}\\ &\overset{C-S}{\leq} \|\Psi\|_{H^2(U_t)}\|\underline{e}^{I_1}e_3^{I_2'}\Psi\|_{L^2(U_t)} + \sum_{|J_2|,|L_2|<m}\|\underline{e}^{J_1}e_3^{J_2}\Psi\|_{H^1(U_t)}\|\underline{e}^{L_1}e_3^{L_2}\Psi\|_{H^1(U_t)}\\ &\leq D_{\text{low}}^2 \qquad (\text{by } \textit{Step 2})\end{aligned} \tag{3.21}$$

This implies an L^2 estimate for $\underline{e}^{I_1}e_3^{I_2}\Psi_{\mathcal{G}}$, $|I_1| + 2|I_2| = s + 1$, $|I_2| = m + 1$, in accordance with (3.15) (choosing $D_{m+1} \geq D_{\text{low}}^2 + D_m$). As we remarked above, the previous term contains one tangential derivative more than required. The desired estimate for $|I_1| + 2|I_2| = s$, $|I_2| = m + 1$ is in fact simpler.

For $\Psi_{\mathcal{B}}$ we apply $\underline{e}^{I_1}e_3^{I_2}$, $|I_1| + 2|I_2| = s$, $|I_2| = m + 1$ to (3.19):

$$\underline{e}^{I_1}e_3^{I_2}e_0\Psi_{\mathcal{B}} = \underline{e}^{I_1}e_3^{I_2}e_A\Psi_{\mathcal{G}} + \underline{e}^{I_1}e_3^{I_2}(\Psi * \Psi), \tag{3.22}$$

The first term in the RHS has just being controlled in L^2 (cf. Lemma 3.9). The argument for the second term is as in (3.21), only now the final RHS becomes

$$\|\underline{e}^{I_1}e_3^{I_2}(\Psi * \Psi)\|_{L^2(U_t)} \leq D_{\text{low}}^2 + D_{\text{low}}\|\underline{e}^{I_1}e_3^{I_2}\Psi\|_{L^2(U_t)} \leq D_{\text{low}}D_{m+1} + D_{\text{low}}\|\underline{e}^{I_1}e_3^{I_2}\Psi_{\mathcal{B}}\|_{L^2(U_t)} \tag{3.23}$$

Thus, combining with Lemma 3.9, we have the bound

$$\|e_0\underline{e}^{I_1}e_3^{I_2}\Psi_{\mathcal{B}}\|_{L^2(U_t)} \leq D_{\text{low}}D_{m+1} + D_{\text{low}}\|\underline{e}^{I_1}e_3^{I_2}\Psi_{\mathcal{B}}\|_{L^2(U_t)} \tag{3.24}$$

The first line in (3.14) then gives the estimate

$$\sum_{\Psi_{\mathcal{B}}\in\mathcal{B}}\|\underline{e}^{I_1}e_3^{I_2}\Psi_{\mathcal{B}}\|^2_{L^2(U_t)} \leq D + \sum_{\Psi_{\mathcal{B}}\in\mathcal{B}}\int_0^t\|\underline{e}^{I_1}e_3^{I_2}\Psi_{\mathcal{B}}\|_{L^2(U_\tau)}\big(D_{\text{low}}D_{m+1} + D_{\text{low}}\|\underline{e}^{I_1}e_3^{I_2}\Psi_{\mathcal{B}}\|_{L^2(U_\tau)}\big)d\tau \tag{3.25}$$

Employing Gronwall's inequality, we obtain the desired estimate for $\Psi_{\mathcal{B}}$ by taking $T > 0$ sufficiently small. This completes the proof of the proposition. □

4. A Solution to the EVE

In this section we show that the solution of the modified reduced system, with initial data as in Sect. 2.4, either for the standard Cauchy problem or for the boundary value problem, subject to the conditions in Lemmas 3.3, 3.4, is in fact a solution to the EVE, see Proposition 4.8 and the conclusion in Sect. 4.3. Thus, completing the proofs of Theorems 1.1, 1.10.

4.1. The geometry of a solution to the reduced equations. Having solved (2.19), (2.20), (2.8), (2.9) for K_{ij}, Γ_{ijb}, $f_i{}^p$, $f^b{}_p$, we declare that $e_0 = \partial_t$, together with e_1, e_2, e_3 given by (1.5), constitute an orthonormal frame. This completely determines the spacetime metric $\mathbf{g}$, which splits in the form (1.4). We then need to verify that the variables K_{ij}, Γ_{ijb} are indeed the second fundamental form of the t-slices and spatial connection coefficients of the orthonormal frame we have just defined, with respect to the Levi-Civita connection $\mathbf{D}$ of $\mathbf{g}$. In fact, this must be derived at the same time with the vanishing of the spacetime Ricci tensor, confirming that the solution of the reduced system is in fact a solution of the EVE.

For this purpose, we define the connection $\widetilde{\mathbf{D}}$ by the relations

$$\widetilde{\mathbf{D}}_{e_0} e_\mu = 0, \quad \widetilde{\mathbf{D}}_{e_i} e_0 = K_i{}^j e_j, \quad \widetilde{\mathbf{D}}_{e_i} e_j = \Gamma_{ij}{}^b e_b + K_{ij} e_0 \tag{4.1}$$

and denote the projection of $\widetilde{\mathbf{D}}$ onto the span of e_1, e_2, e_3 by $\widetilde{D}$. Let

$$\widetilde{\mathbf{R}}_{\alpha\beta\mu}{}^\nu e_\nu := (\widetilde{\mathbf{D}}_{e_\alpha}\widetilde{\mathbf{D}}_{e_\beta} - \widetilde{\mathbf{D}}_{e_\beta}\widetilde{\mathbf{D}}_{e_\alpha} - \widetilde{\mathbf{D}}_{[e_\alpha, e_\beta]})e_\mu = -\widetilde{\mathbf{R}}_{\beta\alpha\mu}{}^\nu e_\nu, \quad \widetilde{\mathbf{R}}_{\beta\mu} = \widetilde{\mathbf{R}}_{\alpha\beta\mu}{}^\alpha, \quad \widetilde{\mathbf{R}} = \widetilde{\mathbf{R}}_\mu{}^\mu \tag{4.2}$$

be the Riemann, Ricci, and scalar curvatures of $\widetilde{\mathbf{D}}$; the curvatures $\widetilde{R}_{aijb}$, $\widetilde{R}_{ij}$, $\widetilde{R}$ associated to $\widetilde{D}$ are defined similarly. For notational simplicity, we will also use in certain places below the convention $\Gamma_{\alpha\beta\nu} := \mathbf{g}(\widetilde{\mathbf{D}}_{e_\alpha} e_\beta, e_\nu) = -\Gamma_{\alpha\nu\beta}$, despite the fact that we have used Γ so far to denote only spatial connection coefficients. In particular, with this convention $\Gamma_{i0j} = -\Gamma_{ij0} = K_{ij}$, $\Gamma_{0\alpha\beta} = 0$.

Define the torsion of $\widetilde{\mathbf{D}}$:

$$C_{\alpha\mu\nu} = \mathbf{g}([e_\alpha, e_\mu] - \widetilde{\mathbf{D}}_{e_\alpha} e_\mu + \widetilde{\mathbf{D}}_{e_\mu} e_\alpha, e_\nu) = -C_{\mu\alpha\nu} \tag{4.3}$$

Note that $\widetilde{\mathbf{D}}$ is not a priori torsion-free, however, it annihilates the metric $\mathbf{g}$.

The next lemma is a list of standard identities for the torsion and curvature tensors of $\widetilde{\mathbf{D}}$, which are the same for any affine connection compatible with $\mathbf{g}$. We include a proof for the sake of completeness.

Lemma 4.1. *The connection $\widetilde{\mathbf{D}}$ is compatible with* $\mathbf{g}$, $\widetilde{\mathbf{D}}\mathbf{g} = 0$, *while its curvature and torsion tensors satisfy:*

$$C_{ijb} = f^b{}_p e_i f_j{}^p - f^b{}_p e_j f_i{}^p - \Gamma_{ijb} + \Gamma_{jib} = -C_{jib}, \quad C_{\alpha\beta 0} = C_{0ij} = C_{i0j} = 0, \tag{4.4}$$

$$\begin{aligned} 0 = {} & \widetilde{\mathbf{R}}_{\alpha\beta\mu\nu} + \widetilde{\mathbf{R}}_{\beta\mu\alpha\nu} + \widetilde{\mathbf{R}}_{\mu\alpha\beta\nu} + \widetilde{\mathbf{D}}_\mu C_{\alpha\beta\nu} + \widetilde{\mathbf{D}}_\alpha C_{\beta\mu\nu} + \widetilde{\mathbf{D}}_\beta C_{\mu\alpha\nu} \\ & + C_{\alpha\beta}{}^l C_{l\mu\nu} + C_{\mu\alpha}{}^l C_{l\beta\nu} + C_{\beta\mu}{}^l C_{l\alpha\nu} \end{aligned} \tag{4.5}$$

$$\widetilde{\mathbf{R}}_{\alpha\beta\mu\nu} = -\widetilde{\mathbf{R}}_{\alpha\beta\nu\mu}, \quad \widetilde{R}_{aijb} = -\widetilde{R}_{aibj} \tag{4.6}$$

$$0 = \widetilde{\boldsymbol{D}}_\mu \widetilde{\boldsymbol{R}}_{\alpha\beta\gamma\delta} + \widetilde{\boldsymbol{D}}_\alpha \widetilde{\boldsymbol{R}}_{\beta\mu\gamma\delta} + \widetilde{\boldsymbol{D}}_\beta \widetilde{\boldsymbol{R}}_{\mu\alpha\gamma\delta} - C_{\mu\alpha l}\widetilde{\boldsymbol{R}}^l{}_{\beta\gamma\delta} - C_{\alpha\beta l}\widetilde{\boldsymbol{R}}^l{}_{\mu\gamma\delta} - C_{\beta\mu l}\widetilde{\boldsymbol{R}}^l{}_{\alpha\gamma\delta} \tag{4.7}$$

Moreover, the Gauss and Codazzi equations in Lemma 2.1 become:

$$\widetilde{\boldsymbol{R}}_{aijb} = \widetilde{R}_{aijb} + K_{ab}K_{ij} - K_{aj}K_{ib}, \tag{4.8}$$

$$\widetilde{\boldsymbol{R}}_{jb0i} = \widetilde{D}_j K_{bi} - \widetilde{D}_b K_{ji} - C_{jb}{}^l K_{li}. \tag{4.9}$$

Proof. The compatibility of $\widetilde{\mathbf{D}}$ with $\mathbf{g}$ is equivalent to:

$$(\widetilde{\mathbf{D}}_\alpha \mathbf{g})_{\mu\nu} = 0 \quad\Longleftrightarrow\quad \mathbf{g}(\widetilde{\mathbf{D}}_{e_\alpha} e_\mu, e_\nu) + \mathbf{g}(e_\mu, \widetilde{\mathbf{D}}_{e_\alpha} e_\nu) = 0$$

Hence, it follows from the antisymmetry of $\Gamma_{ijb} = -\Gamma_{ibj}$, see Lemma 2.7, and the definition (4.1). Therefore, $\widetilde{D}$ is also compatible with g. By definition (4.2), this also implies the antisymmetry of the curvatures $\widetilde{\mathbf{R}}_{\alpha\beta\mu\nu}, \widetilde{R}_{aijb}$ with respect to the last two indices.

By (2.22) and (4.1) we derive the identities

$$[e_0, e_i] - \widetilde{\mathbf{D}}_{e_0} e_i + \widetilde{\mathbf{D}}_{e_i} e_0 = -K_i{}^c e_c + K_i{}^c e_c = 0,$$
$$[e_i, e_j] - \widetilde{\mathbf{D}}_{e_i} e_j + \widetilde{\mathbf{D}}_{e_j} e_i = f^b{}_p e_i f_j{}^p e_b - f^b{}_p e_j f_i{}^p e_b - (\Gamma_{ij}{}^b - \Gamma_{ji}{}^b) e_b + (K_{ji} - K_{ij}) e_0,$$

which yield (4.4), thanks to the symmetry of K_{ij} (Lemma 2.7).

Next, we derive the first Bianchi identity (4.5) using the definitions (4.2), (4.3):

$$\begin{aligned}
&\widetilde{\mathbf{R}}_{\alpha\beta\mu\nu} + \widetilde{\mathbf{R}}_{\beta\mu\alpha\nu} + \widetilde{\mathbf{R}}_{\mu\alpha\beta\nu} \\
&= \mathbf{g}((\widetilde{\mathbf{D}}_{e_\alpha}\widetilde{\mathbf{D}}_{e_\beta} - \widetilde{\mathbf{D}}_{e_\beta}\widetilde{\mathbf{D}}_{e_\alpha} - \widetilde{\mathbf{D}}_{[e_\alpha,e_\beta]})e_\mu, e_\nu) + \mathbf{g}((\widetilde{\mathbf{D}}_{e_\beta}\widetilde{\mathbf{D}}_{e_\mu} - \widetilde{\mathbf{D}}_{e_\mu}\widetilde{\mathbf{D}}_{e_\beta} - \widetilde{\mathbf{D}}_{[e_\beta,e_\mu]})e_\alpha, e_\nu) \\
&\quad + \mathbf{g}((\widetilde{\mathbf{D}}_{e_\mu}\widetilde{\mathbf{D}}_{e_\alpha} - \widetilde{\mathbf{D}}_{e_\alpha}\widetilde{\mathbf{D}}_{e_\mu} - \widetilde{\mathbf{D}}_{[e_\mu,e_\alpha]})e_\beta, e_\nu) \\
&= \mathbf{g}(\widetilde{\mathbf{D}}_{e_\alpha}([e_\beta, e_\mu] - C_{\beta\mu}{}^l e_l), e_\nu) + \mathbf{g}(\widetilde{\mathbf{D}}_{e_\beta}([e_\mu, e_\alpha] - C_{\mu\alpha}{}^l e_l), e_\nu) + \mathbf{g}(\widetilde{\mathbf{D}}_{e_\mu}([e_\alpha, e_\beta] - C_{\alpha\beta}{}^l e_l), e_\nu) \\
&\quad - \mathbf{g}(\widetilde{\mathbf{D}}_{[e_\alpha,e_\beta]}e_\mu, e_\nu) - \mathbf{g}(\widetilde{\mathbf{D}}_{[e_\beta,e_\mu]}e_\alpha, e_\nu) - \mathbf{g}(\widetilde{\mathbf{D}}_{[e_\mu,e_\alpha]}e_\beta, e_\nu) \\
&= \mathbf{g}(\widetilde{\mathbf{D}}_{e_\alpha}([e_\beta, e_\mu]) - \widetilde{\mathbf{D}}_{[e_\beta,e_\mu]}e_\alpha, e_\nu) + \mathbf{g}(\widetilde{\mathbf{D}}_{e_\beta}([e_\mu, e_\alpha]) - \widetilde{\mathbf{D}}_{[e_\mu,e_\alpha]}e_\beta, e_\nu) + \mathbf{g}(\widetilde{\mathbf{D}}_{e_\mu}([e_\alpha, e_\beta]) - \widetilde{\mathbf{D}}_{[e_\alpha,e_\beta]}e_\mu, e_\nu) \\
&\quad - \mathbf{g}(\widetilde{\mathbf{D}}_{e_\alpha}(C_{\beta\mu}{}^l e_l), e_\nu) - \mathbf{g}(\widetilde{\mathbf{D}}_{e_\beta}(C_{\mu\alpha}{}^l e_l), e_\nu) - \mathbf{g}(\widetilde{\mathbf{D}}_{e_\mu}(C_{\alpha\beta}{}^l e_l), e_\nu) \\
&= [e_\mu, [e_\alpha, e_\beta]] + [e_\alpha, [e_\beta, e_\mu]] + [e_\beta, [e_\mu, e_\alpha]] - C_{\mu l\nu}\mathbf{g}(e^l, [e_\alpha, e_\beta]) - C_{\alpha l\nu}\mathbf{g}(e^l, [e_\beta, e_\mu]) \\
&\quad - C_{\beta l\nu}\mathbf{g}(e^l, [e_\mu, e_\alpha]) - e_\alpha C_{\beta\mu\nu} - C_{\beta\mu}{}^l\Gamma_{\alpha l\nu} - e_\beta C_{\mu\alpha\nu} - C_{\mu\alpha}{}^l\Gamma_{\beta l\nu} - e_\mu C_{\alpha\beta\nu} - C_{\alpha\beta}{}^l\Gamma_{\mu l\nu} \\
&= -C_{\mu l\nu}C_{\alpha\beta}{}^l - C_{\mu l\nu}(\Gamma_{\alpha\beta}{}^l - \Gamma_{\beta\alpha}{}^l) - C_{\alpha l\nu}C_{\beta\mu}{}^l - C_{\alpha l\nu}(\Gamma_{\beta\mu}{}^l - \Gamma_{\mu\beta}{}^l) - C_{\beta l\nu}C_{\mu\alpha}{}^l \qquad \text{(Jacobi's identity)} \\
&\quad - C_{\beta l\nu}(\Gamma_{\mu\alpha}{}^l - \Gamma_{\alpha\mu}{}^l) - e_\alpha C_{\beta\mu\nu} - C_{\beta\mu}{}^l\Gamma_{\alpha l\nu} - e_\beta C_{\mu\alpha\nu} - C_{\mu\alpha}{}^l\Gamma_{\beta l\nu} - e_\mu C_{\alpha\beta\nu} - C_{\alpha\beta}{}^l\Gamma_{\mu l\nu}.
\end{aligned}$$

The last RHS can be seen to correspond to the torsion terms in (4.5) by using the antisymmetries of $\Gamma_{\alpha\beta\nu}, C_{\alpha\beta\nu}$ in the last two and first two indices respectively.

On the other hand, we have

$$\begin{aligned}
&\widetilde{\mathbf{D}}_\mu \widetilde{\mathbf{R}}(e_\alpha, e_\beta) + \widetilde{\mathbf{D}}_\alpha \widetilde{\mathbf{R}}(e_\beta, e_\mu) + \widetilde{\mathbf{D}}_\beta \widetilde{\mathbf{R}}(e_\mu, e_\alpha) \\
&\quad = \widetilde{\mathbf{D}}_{e_\mu}(\widetilde{\mathbf{D}}_{e_\alpha}\widetilde{\mathbf{D}}_{e_\beta} - \widetilde{\mathbf{D}}_{e_\beta}\widetilde{\mathbf{D}}_{e_\alpha} - \widetilde{\mathbf{D}}_{[e_\alpha,e_\beta]}) - (\widetilde{\mathbf{D}}_{e_\mu}e_\alpha)^\nu(\widetilde{\mathbf{D}}_{e_\nu}\widetilde{\mathbf{D}}_{e_\beta} - \widetilde{\mathbf{D}}_{e_\beta}\widetilde{\mathbf{D}}_{e_\nu} - \widetilde{\mathbf{D}}_{[e_\nu,e_\beta]}) \\
&\qquad - (\widetilde{\mathbf{D}}_{e_\mu}e_\beta)^\nu(\widetilde{\mathbf{D}}_{e_\alpha}\widetilde{\mathbf{D}}_{e_\nu} - \widetilde{\mathbf{D}}_{e_\nu}\widetilde{\mathbf{D}}_{e_\alpha} - \widetilde{\mathbf{D}}_{[e_\alpha,e_\nu]}) \\
&\qquad + \widetilde{\mathbf{D}}_{e_\alpha}(\widetilde{\mathbf{D}}_{e_\beta}\widetilde{\mathbf{D}}_{e_\mu} - \widetilde{\mathbf{D}}_{e_\mu}\widetilde{\mathbf{D}}_{e_\beta} - \widetilde{\mathbf{D}}_{[e_\beta,e_\mu]}) - (\widetilde{\mathbf{D}}_{e_\alpha}e_\beta)^\nu(\widetilde{\mathbf{D}}_{e_\nu}\widetilde{\mathbf{D}}_{e_\mu} - \widetilde{\mathbf{D}}_{e_\mu}\widetilde{\mathbf{D}}_{e_\nu} - \widetilde{\mathbf{D}}_{[e_\nu,e_\mu]}) \\
&\qquad - (\widetilde{\mathbf{D}}_{e_\alpha}e_\mu)^\nu(\widetilde{\mathbf{D}}_{e_\beta}\widetilde{\mathbf{D}}_{e_\nu} - \widetilde{\mathbf{D}}_{e_\nu}\widetilde{\mathbf{D}}_{e_\beta} - \widetilde{\mathbf{D}}_{[e_\beta,e_\nu]}) \\
&\qquad + \widetilde{\mathbf{D}}_{e_\beta}(\widetilde{\mathbf{D}}_{e_\mu}\widetilde{\mathbf{D}}_{e_\alpha} - \widetilde{\mathbf{D}}_{e_\alpha}\widetilde{\mathbf{D}}_{e_\mu} - \widetilde{\mathbf{D}}_{[e_\mu,e_\alpha]}) - (\widetilde{\mathbf{D}}_{e_\beta}e_\mu)^\nu(\widetilde{\mathbf{D}}_{e_\nu}\widetilde{\mathbf{D}}_{e_\alpha} - \widetilde{\mathbf{D}}_{e_\alpha}\widetilde{\mathbf{D}}_{e_\nu} - \widetilde{\mathbf{D}}_{[e_\nu,e_\alpha]}) \\
&\qquad - (\widetilde{\mathbf{D}}_{e_\beta}e_\alpha)^\nu(\widetilde{\mathbf{D}}_{e_\mu}\widetilde{\mathbf{D}}_{e_\nu} - \widetilde{\mathbf{D}}_{e_\nu}\widetilde{\mathbf{D}}_{e_\mu} - \widetilde{\mathbf{D}}_{[e_\mu,e_\nu]})
\end{aligned}$$

$$\begin{aligned}
&= [\widetilde{\mathbf{D}}_{e_\mu}, \widetilde{\mathbf{D}}_{e_\alpha}]\widetilde{\mathbf{D}}_{e_\beta} + [\widetilde{\mathbf{D}}_{e_\beta}, \widetilde{\mathbf{D}}_{e_\mu}]\widetilde{\mathbf{D}}_{e_\alpha} + [\widetilde{\mathbf{D}}_{e_\alpha}, \widetilde{\mathbf{D}}_{e_\beta}]\widetilde{\mathbf{D}}_{e_\mu} - (\widetilde{\mathbf{D}}_{e_\mu} e_\alpha - \widetilde{\mathbf{D}}_{e_\alpha} e_\mu)^\nu \widetilde{\mathbf{R}}(e_\nu, e_\beta)\\
&\quad - (\widetilde{\mathbf{D}}_{e_\alpha} e_\beta - \widetilde{\mathbf{D}}_{e_\beta} e_\alpha)^\nu \widetilde{\mathbf{R}}(e_\nu, e_\mu) - (\widetilde{\mathbf{D}}_{e_\beta} e_\mu - \widetilde{\mathbf{D}}_{e_\mu} e_\alpha)^\nu \widetilde{\mathbf{R}}(e_\nu, e_\alpha)\\
&\quad - [\widetilde{\mathbf{D}}_{e_\mu}, \widetilde{\mathbf{D}}_{[e_\alpha, e_\beta]}] - [\widetilde{\mathbf{D}}_{e_\alpha}, \widetilde{\mathbf{D}}_{[e_\beta, e_\mu]}] - [\widetilde{\mathbf{D}}_{e_\beta}, \widetilde{\mathbf{D}}_{[e_\mu, e_\alpha]}]\\
&\quad - \widetilde{\mathbf{D}}_{[e_\alpha, e_\beta]}\widetilde{\mathbf{D}}_{e_\mu} - \widetilde{\mathbf{D}}_{[e_\beta, e_\mu]}\widetilde{\mathbf{D}}_{e_\alpha} - \widetilde{\mathbf{D}}_{[e_\mu, e_\alpha]}\widetilde{\mathbf{D}}_{e_\beta}\\
&\quad + \widetilde{\mathbf{D}}_{[e_\mu, [e_\alpha, e_\beta]]} + \widetilde{\mathbf{D}}_{[e_\alpha, [e_\beta, e_\mu]]} + \widetilde{\mathbf{D}}_{[e_\beta, [e_\mu, e_\alpha]]} \qquad \text{(adding zero by Jacobi's identity)}\\
&= \widetilde{\mathbf{R}}(e_\mu, e_\alpha)\widetilde{\mathbf{D}}_{e_\beta} + \widetilde{\mathbf{R}}(e_\alpha, e_\beta)\widetilde{\mathbf{D}}_{e_\mu} + \widetilde{\mathbf{R}}(e_\beta, e_\mu)\widetilde{\mathbf{D}}_{e_\alpha} + ([e_\alpha, e_\beta] - \widetilde{\mathbf{D}}_{e_\alpha} e_\beta + \widetilde{\mathbf{D}}_{e_\beta} e_\alpha)^\nu \widetilde{\mathbf{R}}(e_\nu, e_\mu)\\
&\quad + ([e_\beta, e_\mu] - \widetilde{\mathbf{D}}_{e_\beta} e_\mu + \widetilde{\mathbf{D}}_{e_\mu} e_\beta)^\nu \widetilde{\mathbf{R}}(e_\nu, e_\alpha) + ([e_\mu, e_\alpha] - \widetilde{\mathbf{D}}_{e_\mu} e_\alpha + \widetilde{\mathbf{D}}_{e_\alpha} e_\mu)^\nu \widetilde{\mathbf{R}}(e_\nu, e_\beta)
\end{aligned}$$

The second Bianchi identity follows by applying the preceding expression to e_γ, taking the inner product with e_δ and utilising the idenitty $[e_\alpha, e_\beta] = C_{\alpha\beta}{}^l e_l + \widetilde{\mathbf{D}}_{e_\alpha} e_\beta - \widetilde{\mathbf{D}}_{e_\beta} e_\alpha$.

Finally, for the Gauss and Codazzi equations (4.8)–(4.9), we repeat the steps in the proof of Lemma 2.1, making use of the formula $[e_j, e_b] = C_{jb}{}^l e_l + \widetilde{\mathbf{D}}_{e_j} e_b - \widetilde{\mathbf{D}}_{e_b} e_j$, without identifying $\widetilde{\mathbf{R}}_{0ijb}, \widetilde{\mathbf{R}}_{jb0i}$ (which uses the torsion free property of the connection). This completes the proof of the lemma. □

4.2. Modified curvature and propagation equations for vanishing quantities. An essential step in proving the vanishing of $C_{ijb}, \widetilde{\mathbf{R}}_{\beta\mu}$ is the derivation of propagation equations for them, using the reduced equations (2.19)–(2.20) and the Bianchi identities for the curvature of $\widetilde{\mathbf{D}}$ in Lemma 4.1. If these equations are suitable for an energy argument, then we can infer the vanishing of the relevant variables from their vanishing on the initial hypersurface.

However, for the particular curvature of $\widetilde{\mathbf{D}}$ we have defined, this system would fail to be hyperbolic, hence, obstructing us from deriving energy estimates. Indeed, this can be seen by examining the system of evolution equations in Lemma 4.5, which we derive below for the modified curvature (4.10). The first order system (4.25)–(4.28) is in fact symmetric hyperbolic, but if we were to replace $\widehat{\mathbf{R}}$ by $\widetilde{\mathbf{R}}$ this would fail to be the case, due to the additional first order C_{ijb} terms with no particular structure.

For this purpose, we consider the modified curvature:

$$\widehat{\mathbf{R}}_{\alpha\beta\mu}{}^\nu e_\nu := (\widetilde{\mathbf{D}}_{e_\alpha}\widetilde{\mathbf{D}}_{e_\beta} - \widetilde{\mathbf{D}}_{e_\beta}\widetilde{\mathbf{D}}_{e_\alpha} - \widetilde{\mathbf{D}}_{\widetilde{\mathbf{D}}_{e_\alpha} e_\beta - \widetilde{\mathbf{D}}_{e_\beta} e_\alpha}) e_\mu = (\widetilde{\mathbf{R}}_{\alpha\beta\mu}{}^\nu + C_{\alpha\beta}{}^\lambda \Gamma_{\lambda\mu}{}^\nu) e_\nu \tag{4.10}$$

Note that $\widehat{\mathbf{R}}_{\alpha\beta\mu\nu}$ is not tensorial with respect to its third index μ. We also define $\widehat{\mathbf{R}}_{\beta\mu} = \widehat{\mathbf{R}}_{\alpha\beta\mu}{}^\alpha$, $\widehat{\mathbf{R}} = \widehat{\mathbf{R}}_\mu{}^\mu$ and similarly for the modified curvatures $\widehat{R}_{aijb}, \widehat{R}_{ij}, \widehat{R}$ of $\widetilde{D}$. Then we have the following identities, which are immediate consequences of Lemma 4.1 and (4.10):

Lemma 4.2. *The curvatures $\widehat{\boldsymbol{R}}_{\alpha\beta\mu\nu}, \widehat{R}_{aijb}$ satisfy the identites:*

$$\widehat{\boldsymbol{R}}_{\alpha\beta\mu\nu} = -\widehat{\boldsymbol{R}}_{\beta\alpha\mu\nu} = -\widehat{\boldsymbol{R}}_{\alpha\beta\nu\mu}, \quad \widehat{R}_{aijb} = -\widehat{R}_{iajb} = -\widehat{R}_{aibj} \tag{4.11}$$

$$\begin{aligned}
0 =& \widehat{\boldsymbol{R}}_{\alpha\beta\mu\nu} + \widehat{\boldsymbol{R}}_{\beta\mu\alpha\nu} + \widehat{\boldsymbol{R}}_{\mu\alpha\beta\nu} + \widetilde{\boldsymbol{D}}_\mu C_{\alpha\beta\nu} + \widetilde{\boldsymbol{D}}_\alpha C_{\beta\mu\nu} + \widetilde{\boldsymbol{D}}_\beta C_{\mu\alpha\nu}\\
&+ C_{\alpha\beta}{}^l C_{l\mu\nu} + C_{\mu\alpha}{}^l C_{l\beta\nu} + C_{\beta\mu}{}^l C_{l\alpha\nu} - C_{\alpha\beta}{}^\lambda \Gamma_{\lambda\mu\nu} - C_{\beta\mu}{}^\lambda \Gamma_{\lambda\alpha\nu} - C_{\mu\alpha}{}^\lambda \Gamma_{\lambda\beta\nu}
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
\widehat{\boldsymbol{R}}_{\alpha\beta} - \widehat{\boldsymbol{R}}_{\beta\alpha} =& -\widetilde{\boldsymbol{D}}^\mu C_{\alpha\beta\mu} - \widetilde{\boldsymbol{D}}_\alpha C_{\beta\mu}{}^\mu - \widetilde{\boldsymbol{D}}_\beta C_{\mu\alpha}{}^\mu\\
&- C_{\alpha\beta}{}^l C_{l\mu}{}^\mu - C_{\mu\alpha}{}^l C_{l\beta}{}^\mu - C_{\beta\mu}{}^l C_{l\alpha}{}^\mu + C_{\beta\mu}{}^\lambda \Gamma_{\lambda\alpha}{}^\mu + C_{\mu\alpha}{}^\lambda \Gamma_{\lambda\beta}{}^\mu
\end{aligned} \tag{4.13}$$

$$0 = \widetilde{\boldsymbol{D}}_\mu \widehat{\boldsymbol{R}}_{\alpha\beta\gamma\delta} + \widetilde{\boldsymbol{D}}_\alpha \widehat{\boldsymbol{R}}_{\beta\mu\gamma\delta} + \widetilde{\boldsymbol{D}}_\beta \widehat{\boldsymbol{R}}_{\mu\alpha\gamma\delta} - C_{\mu\alpha l}(\widehat{\boldsymbol{R}}^l{}_{\beta\gamma\delta} - C^l{}_{\beta\nu}\Gamma^\nu{}_{\gamma\delta})$$

$$
\begin{aligned}
&- C_{\alpha\beta l}(\widehat{\boldsymbol{R}}^{l}{}_{\mu\gamma\delta} - C^{l}{}_{\mu\nu}\Gamma^{\nu}{}_{\gamma\delta}) - C_{\beta\mu l}(\widehat{\boldsymbol{R}}^{l}{}_{\alpha\gamma\delta} - C^{l}{}_{\alpha\nu}\Gamma^{\nu}{}_{\gamma\delta})\\
&+ \Big[\widehat{\boldsymbol{R}}_{\alpha\beta\mu\nu} + \widehat{\boldsymbol{R}}_{\beta\mu\alpha\nu} + \widehat{\boldsymbol{R}}_{\mu\alpha\beta\nu} + C_{\alpha\beta}{}^{l}C_{l\mu\nu} + C_{\mu\alpha}{}^{l}C_{l\beta\nu} + C_{\beta\mu}{}^{l}C_{l\alpha\nu} \qquad (4.14)\\
&-C_{\alpha\beta}{}^{\lambda}\Gamma_{\lambda\mu\nu} - C_{\beta\mu}{}^{\lambda}\Gamma_{\lambda\alpha\nu} - C_{\mu\alpha}{}^{\lambda}\Gamma_{\lambda\beta\nu}\Big]\Gamma_{\nu\gamma\delta}\\
&- C_{\alpha\beta}{}^{\nu}\widetilde{\boldsymbol{D}}_{\mu}\Gamma_{\nu\gamma\delta} - C_{\beta\mu}{}^{\nu}\widetilde{\boldsymbol{D}}_{\alpha}\Gamma_{\nu\gamma\delta} - C_{\mu\alpha}{}^{\nu}\widetilde{\boldsymbol{D}}_{\beta}\Gamma_{\nu\gamma\delta} \qquad (4.15)
\end{aligned}
$$

$$
\widehat{\boldsymbol{R}}_{aijb} = \widehat{R}_{aijb} + K_{ab}K_{ij} - K_{aj}K_{ib}, \qquad (4.16)
$$

$$
\widehat{\boldsymbol{R}}_{jb0i} = \widetilde{D}_{j}K_{bi} - \widetilde{D}_{b}K_{ji}, \qquad (4.17)
$$

$$
\widehat{\boldsymbol{R}}_{b0} = \widetilde{D}^{i}K_{bi} - \widetilde{D}_{b}\mathrm{tr}K \qquad (4.18)
$$

$$
\widehat{R}_{aijb} = e_{a}\Gamma_{ijb} - e_{i}\Gamma_{ajb} - \Gamma_{ab}{}^{c}\Gamma_{ijc} + \Gamma_{ib}{}^{c}\Gamma_{ajc} - \Gamma_{ai}{}^{c}\Gamma_{cjb} + \Gamma_{ia}{}^{c}\Gamma_{cjb} \qquad (4.19)
$$

where everything is interpreted tensorially, e.g., $\widetilde{\boldsymbol{D}}_{\mu}\Gamma_{\nu\gamma\delta} := e_{\mu}\Gamma_{\nu\gamma\delta} - \Gamma_{\mu\nu}{}^{\lambda}\Gamma_{\lambda\gamma\delta} - \Gamma_{\mu\gamma}{}^{\lambda}\Gamma_{\nu\lambda\delta} - \Gamma_{\mu\delta}{}^{\lambda}\Gamma_{\nu\gamma\lambda}$.

Proof. The antisymmetries (4.11) follow from the definition (4.10), the antisymmetries (4.2), (4.6) of $\widetilde{\mathbf{R}}_{\alpha\beta\mu\nu}$ and that of $C_{\alpha\beta\mu}$ in $(\alpha;\beta)$. Also, plugging (4.10) into (4.5) gives (4.12), while contracting (4.12) with respect to $(\mu;\nu)$ gives (4.13). Moreover, (4.16)–(4.17) follow from (4.8)–(4.9) by plugging in the definition (4.10) and recalling that $C_{ai0} = 0$, see (4.4). Contracting (4.17) also gives (4.18). The computation of the curvature formula (4.19) is straightforward, using the definition of $\widehat{R}_{aijb}$, analogous to (4.10), cf. the proof of Lemma 2.1.]

For the less obvious Bianchi-type identity (4.14), we plug (4.10) into (4.7) and treat all the terms tensorially. Although $\widetilde{\mathbf{R}}_{\alpha\beta\gamma\delta}$ is not a tensor in γ, its difference from $C_{\alpha\beta}{}^{\nu}\Gamma_{\nu\gamma\delta}$ is. Therefore, we deduce

$$
\begin{aligned}
0 &= \widetilde{\mathbf{D}}_{\mu}(\widehat{\mathbf{R}}_{\alpha\beta\gamma\delta} - C_{\alpha\beta}{}^{\nu}\Gamma_{\nu\gamma\delta}) + \widetilde{\mathbf{D}}_{\alpha}(\widehat{\mathbf{R}}_{\beta\mu\gamma\delta} - C_{\beta\mu}{}^{\nu}\Gamma_{\nu\gamma\delta}) + \widetilde{\mathbf{D}}_{\beta}(\widehat{\mathbf{R}}_{\mu\alpha\gamma\delta} - C_{\mu\alpha}{}^{\nu}\Gamma_{\nu\gamma\delta})\\
&\quad - C_{\mu\alpha l}(\widehat{\mathbf{R}}^{l}{}_{\beta\gamma\delta} - C^{l}{}_{\beta\nu}\Gamma^{\nu}{}_{\gamma\delta}) - C_{\alpha\beta l}(\widehat{\mathbf{R}}^{l}{}_{\mu\gamma\delta} - C^{l}{}_{\mu\nu}\Gamma^{\nu}{}_{\gamma\delta}) - C_{\beta\mu l}(\widehat{\mathbf{R}}^{l}{}_{\alpha\gamma\delta} - C^{l}{}_{\alpha\nu}\Gamma^{\nu}{}_{\gamma\delta})\\
&= \widetilde{\mathbf{D}}_{\mu}\widehat{\mathbf{R}}_{\alpha\beta\gamma\delta} + \widetilde{\mathbf{D}}_{\alpha}\widehat{\mathbf{R}}_{\beta\mu\gamma\delta} + \widetilde{\mathbf{D}}_{\beta}\widehat{\mathbf{R}}_{\mu\alpha\gamma\delta}\\
&\quad - C_{\mu\alpha l}(\widehat{\mathbf{R}}^{l}{}_{\beta\gamma\delta} - C^{l}{}_{\beta\nu}\Gamma^{\nu}{}_{\gamma\delta}) - C_{\alpha\beta l}(\widehat{\mathbf{R}}^{l}{}_{\mu\gamma\delta} - C^{l}{}_{\mu\nu}\Gamma^{\nu}{}_{\gamma\delta}) - C_{\beta\mu l}(\widehat{\mathbf{R}}^{l}{}_{\alpha\gamma\delta} - C^{l}{}_{\alpha\nu}\Gamma^{\nu}{}_{\gamma\delta})\\
&\quad -(\widetilde{\mathbf{D}}_{\mu}C_{\alpha\beta}{}^{\nu} + \widetilde{\mathbf{D}}_{\alpha}C_{\beta\mu}{}^{\nu} + \widetilde{\mathbf{D}}_{\beta}C_{\mu\alpha}{}^{\nu})\Gamma_{\nu\gamma\delta} - C_{\alpha\beta}{}^{\nu}\widetilde{\mathbf{D}}_{\mu}\Gamma_{\nu\gamma\delta} - C_{\beta\mu}{}^{\nu}\widetilde{\mathbf{D}}_{\alpha}\Gamma_{\nu\gamma\delta} - C_{\mu\alpha}{}^{\nu}\widetilde{\mathbf{D}}_{\beta}\Gamma_{\nu\gamma\delta}
\end{aligned}
$$

On the other hand, we employ the first Bianchi identity (4.12) to write

$$
\begin{aligned}
&- \widetilde{\mathbf{D}}_{\mu}C_{\alpha\beta}{}^{\nu} - \widetilde{\mathbf{D}}_{\alpha}C_{\beta\mu}{}^{\nu} - \widetilde{\mathbf{D}}_{\beta}C_{\mu\alpha}{}^{\nu}\\
&= \widehat{\mathbf{R}}_{\alpha\beta\mu\nu} + \widehat{\mathbf{R}}_{\beta\mu\alpha\nu} + \widehat{\mathbf{R}}_{\mu\alpha\beta\nu} + C_{\alpha\beta}{}^{l}C_{l\mu\nu} + C_{\mu\alpha}{}^{l}C_{l\beta\nu} + C_{\beta\mu}{}^{l}C_{l\alpha\nu}\\
&\quad -C_{\alpha\beta}{}^{\lambda}\Gamma_{\lambda\mu\nu} - C_{\beta\mu}{}^{\lambda}\Gamma_{\lambda\alpha\nu} - C_{\mu\alpha}{}^{\lambda}\Gamma_{\lambda\beta\nu}
\end{aligned}
$$

which completes the proof of the lemma. □

Remark 4.3. It is important that (4.14) does not contain any spatial derivatives of C_{ijb}, which could lead to a non-symmetric system for the vanishing variables, cf. Lemma 4.5. We were able to replace such terms by using the first Bianchi identity (4.12). In turn, we must express the cyclic curvature sum in (4.14) solely by Ricci terms.

Lemma 4.4. *The cyclic sum* $\widehat{\boldsymbol{R}}_{(\alpha\beta\mu)\nu} := \widehat{\boldsymbol{R}}_{\alpha\beta\mu\nu} + \widehat{\boldsymbol{R}}_{\beta\mu\alpha\nu} + \widehat{\boldsymbol{R}}_{\mu\alpha\beta\nu}$ *satisfies:*

$$
\begin{aligned}
\widehat{\boldsymbol{R}}_{(abi)j} &= (\widehat{\boldsymbol{R}}_{ia} - \widehat{\boldsymbol{R}}_{ai})\mathbf{g}_{bj} + (\widehat{\boldsymbol{R}}_{bi} - \widehat{\boldsymbol{R}}_{ib})\mathbf{g}_{aj} + (\widehat{\boldsymbol{R}}_{ab} - \widehat{\boldsymbol{R}}_{ba})\mathbf{g}_{ij},\\
\widehat{\boldsymbol{R}}_{(0bi)0} &= \widehat{\boldsymbol{R}}_{(abi)0} = 0, \quad \widehat{\boldsymbol{R}}_{(0bi)j} = -\delta_{ij}\widehat{\boldsymbol{R}}_{b0} + \delta_{bj}\widehat{\boldsymbol{R}}_{i0}
\end{aligned}
\qquad (4.20)
$$

Proof. For the first identity, we notice that if either of a, b, i coincide, both sides are trivially zero. In the case where a, b, i are all distinct, j must coincide with one of them (since Σ_t is 3-dimensional), say $j = a$. Then we have

$$\widehat{\mathbf{R}}_{bi} - \widehat{\mathbf{R}}_{ib} = \widehat{\mathbf{R}}_{\lambda bi}{}^{\lambda} - \widehat{\mathbf{R}}_{\lambda ib}{}^{\lambda} \overset{(4.11)}{=} \widehat{\mathbf{R}}_{abia} + \widehat{\mathbf{R}}_{iaba} + \widehat{\mathbf{R}}_{0b0i} - \widehat{\mathbf{R}}_{0i0b} = \widehat{\mathbf{R}}_{(abi)a} + \widehat{\mathbf{R}}_{0b0i} - \widehat{\mathbf{R}}_{0i0b} \tag{4.21}$$

On the other hand, using the symmetry of K_{bi} it holds

$$\widehat{\mathbf{R}}_{0b0i} = \mathbf{g}((\widetilde{\mathbf{D}}_{e_0}\widetilde{\mathbf{D}}_{e_b} - \widetilde{\mathbf{D}}_{e_b}\widetilde{\mathbf{D}}_{e_0} - \widetilde{\mathbf{D}}_{\widetilde{\mathbf{D}}_{e_0}e_b - \widetilde{\mathbf{D}}_{e_b}e_0})e_0, e_i) = e_0 K_{bi} + K_b{}^l K_{li} = \widehat{\mathbf{R}}_{0i0b}, \tag{4.22}$$

Combining (4.21)–(4.22) yeilds the first identity in (4.20). Also, (4.22) implies the first part of the second identity in (4.20) regarding $\widehat{\mathbf{R}}_{(0bi)0} = 0$.

Next, we employ (4.17) to infer:

$$\begin{aligned}
\widehat{\mathbf{R}}_{(abi)0} &= -\widehat{\mathbf{R}}_{ab0i} - \widehat{\mathbf{R}}_{bi0a} - \widehat{\mathbf{R}}_{ia0b} \\
&= -\widetilde{D}_a K_{bi} + \widetilde{D}_b K_{ai} - \widetilde{D}_b K_{ia} + \widetilde{D}_i K_{ba} - \widetilde{D}_i K_{ab} + \widetilde{D}_a K_{ib} \\
&= 0
\end{aligned}$$

To prove the last identity in (4.20) we utilise the reduced equation (2.20), which we rewrite in a more covariant way using (4.18):

$$e_0\Gamma_{ijb} + K_i{}^c\Gamma_{cjb} = \widetilde{D}_j K_{bi} - \widetilde{D}_b K_{ji} + \delta_{ib}\widehat{\mathbf{R}}_{j0} - \delta_{ij}\widehat{\mathbf{R}}_{b0} \tag{4.23}$$

Appealing to the symmetry of K once more, we compute:

$$\begin{aligned}
\widehat{\mathbf{R}}_{(0bi)j} &= \widehat{\mathbf{R}}_{0bij} + \widehat{\mathbf{R}}_{bi0j} - \widehat{\mathbf{R}}_{0ibj} \\
&= \mathbf{g}((\widetilde{\mathbf{D}}_{e_0}\widetilde{\mathbf{D}}_{e_b} - \widetilde{\mathbf{D}}_{e_b}\widetilde{\mathbf{D}}_{e_0} - \widetilde{\mathbf{D}}_{\widetilde{\mathbf{D}}_{e_0}e_b - \widetilde{\mathbf{D}}_{e_b}e_0})e_i, e_j) \\
&\quad + \widetilde{D}_b K_{ij} - \widetilde{D}_i K_{bj} - \mathbf{g}((\widetilde{\mathbf{D}}_{e_0}\widetilde{\mathbf{D}}_{e_i} - \widetilde{\mathbf{D}}_{e_i}\widetilde{\mathbf{D}}_{e_0} - \widetilde{\mathbf{D}}_{\widetilde{\mathbf{D}}_{e_0}e_i - \widetilde{\mathbf{D}}_{e_i}e_0})e_b, e_j) \quad \text{(by (4.17))} \\
&= e_0\Gamma_{bij} + K_b{}^c\Gamma_{cij} - e_0\Gamma_{ibj} - K_i{}^c\Gamma_{cbj} + \widetilde{D}_b K_{ij} - \widetilde{D}_i K_{bj} \\
&= \widetilde{D}_i K_{jb} - \widetilde{D}_b K_{ji} - \delta_{ij}\widehat{\mathbf{R}}_{b0} + \delta_{bj}\widehat{\mathbf{R}}_{i0} + \widetilde{D}_b K_{ij} - \widetilde{D}_i K_{bj} \quad \text{(by (4.23))} \\
&= -\delta_{ij}\widehat{\mathbf{R}}_{b0} + \delta_{bj}\widehat{\mathbf{R}}_{i0},
\end{aligned}$$

as asserted. □

Recall that we symmetrized the RHS of (2.6), such that the symmetry of K_{ij} is automatically propagated off of the initial hypersurface. Consequently, we must treat the symmetrized and antisymmetrized Ricci tensors as different variables:

$$\widehat{\mathbf{R}}^{(S)}_{ij} = \frac{1}{2}(\widehat{\mathbf{R}}_{ij} + \widehat{\mathbf{R}}_{ji}) = \widehat{\mathbf{R}}^{(S)}_{ji}, \quad \widehat{\mathbf{R}}^{(A)}_{ij} = \frac{1}{2}(\widehat{\mathbf{R}}_{ij} - \widehat{\mathbf{R}}_{ji}) = -\widehat{\mathbf{R}}^{(A)}_{ji}. \tag{4.24}$$

With the above lemmas at our disposal, we derive the following propagation equations for the variables that should vanish.

Lemma 4.5. *The variables $\widehat{\boldsymbol{R}}_{\beta\mu}, C_{ijb}$ satisfy the following system of equations:*

$$e_0C_{ijb} = K_b{}^lC_{ijl} - K_i{}^lC_{ljb} - K_j{}^lC_{ilb} - \delta_{ib}\widehat{\boldsymbol{R}}_{j0} + \delta_{jb}\widehat{\boldsymbol{R}}_{i0}, \tag{4.25}$$

$$e_0\widehat{\boldsymbol{R}}_{i0} = e_i\widehat{\boldsymbol{R}}_{00} + e^a\widehat{\mathbf{R}}^{(A)}_{ia} - \Gamma^a{}_i{}^\beta\widehat{\mathbf{R}}_{\beta a} - \Gamma_a{}^{a\beta}\widehat{\mathbf{R}}_{i\beta} - L_i(C,\widehat{\mathbf{R}}), \quad \widehat{\boldsymbol{R}}_{0i} = -\widehat{\boldsymbol{R}}_{i0}, \tag{4.26}$$

$$e_0\widehat{\boldsymbol{R}}_{00} = e^a\widehat{\boldsymbol{R}}_{a0} - \Gamma_a{}^{ab}\widehat{\boldsymbol{R}}_{b0} + L_0(C,\widehat{\mathbf{R}}), \tag{4.27}$$

$$e_0\widehat{\boldsymbol{R}}^{(A)}_{ij} = \frac{1}{2}(e_j\widehat{\boldsymbol{R}}_{i0} - e_i\widehat{\boldsymbol{R}}_{j0}) - K_i{}^l\widehat{\boldsymbol{R}}^{(A)}_{lj} - K_j{}^l\widehat{\boldsymbol{R}}^{(A)}_{il} - \frac{1}{2}K^{bl}\widehat{\boldsymbol{R}}_{(ijl)b} + \frac{1}{2}K^{bl}\widehat{\mathbf{R}}_{(ijb)l} + M_{ij}(C,\widehat{\mathbf{R}}), \tag{4.28}$$

$$\widehat{\boldsymbol{R}}^{(S)}_{ij} = -\delta_{ij}\widehat{\boldsymbol{R}}_{00}, \tag{4.29}$$

where

$$\begin{aligned} 2L_\mu(C,\widehat{\mathbf{R}}) = &- C_{\mu\alpha l}(\widehat{\boldsymbol{R}}^l{}_{\beta\gamma\delta} - C^l{}_{\beta\nu}\Gamma^\nu{}_{\gamma\delta}) - C_{\alpha\beta l}(\widehat{\boldsymbol{R}}^l{}_{\mu\gamma\delta} - C^l{}_{\mu\nu}\Gamma^\nu{}_{\gamma\delta}) - C_{\beta\mu l}(\widehat{\boldsymbol{R}}^l{}_{\alpha\gamma\delta} - C^l{}_{\alpha\nu}\Gamma^\nu{}_{\gamma\delta}) \\ &+ \Big[\widehat{\boldsymbol{R}}_{\alpha\beta\mu\nu} + \widehat{\boldsymbol{R}}_{\beta\mu\alpha\nu} + \widehat{\boldsymbol{R}}_{\mu\alpha\beta\nu} + C_{\alpha\beta}{}^lC_{l\mu\nu} + C_{\mu\alpha}{}^lC_{l\beta\nu} + C_{\beta\mu}{}^lC_{l\alpha\nu} - C_{\alpha\beta}{}^\lambda\Gamma_{\lambda\mu\nu} \\ &- C_{\beta\mu}{}^\lambda\Gamma_{\lambda\alpha\nu} - C_{\mu\alpha}{}^\lambda\Gamma_{\lambda\beta\nu}\Big]\Gamma_{\nu\gamma\delta} - C_{\alpha\beta}{}^\nu\widetilde{\boldsymbol{D}}_\mu\Gamma_{\nu\gamma\delta} - C_{\beta\mu}{}^\nu\widetilde{\boldsymbol{D}}_\alpha\Gamma_{\nu\gamma\delta} - C_{\mu\alpha}{}^\nu\widetilde{\boldsymbol{D}}_\beta\Gamma_{\nu\gamma\delta}, \end{aligned} \tag{4.30}$$

$$\begin{aligned} M_{ij}(C,\widehat{\mathbf{R}}) = &-\frac{1}{2}e_0L_{ij}(C) - \frac{1}{2}K_i{}^lL_{lj}(C) - \frac{1}{2}K_j{}^lL_{il}(C) - \frac{1}{2}K^{bl}L_{ijlb}(C) + \frac{1}{2}K^{bl}L_{ijbl}(C) \\ &- \frac{1}{2}\Big[C_{ijl}e^bK_b{}^l + C_{jbl}e_iK^{bl} + C_{bil}e_jK^{bl} - C_{ljb}e^bK_i{}^l - C_{ilb}e^bK_j{}^l - C_{lb}{}^be_iK_j{}^l \\ &- C_{jlb}e_iK^{bl} - C_{lib}e_jK^{bl} - C_{bl}{}^be_jK_i{}^l\Big], \end{aligned} \tag{4.31}$$

$$\begin{aligned} L_{ij}(C) = &\widetilde{D}^bC_{ijb} + \widetilde{D}_iC_{jb}{}^b + \widetilde{D}_jC_{bi}{}^b - e^bC_{ijb} - e_iC_{jb}{}^b - e_jC_{bi}{}^b \\ &+ C_{ij}{}^lC_{lb}{}^b + C_{bi}{}^lC_{lj}{}^b + C_{jb}{}^lC_{li}{}^b - C_{jb}{}^l\Gamma_{li}{}^b - C_{bi}{}^l\Gamma_{lj}{}^b, \end{aligned} \tag{4.32}$$

$$\begin{aligned} L_{ijbl}(C) = &\widetilde{D}_bC_{ijl} + \widetilde{D}_iC_{jbl} + \widetilde{D}_jC_{bil} - e_bC_{ijl} - e_iC_{jbl} - e_jC_{bil} \\ &- C_{ij}{}^dC_{dbl} - C_{bi}{}^dC_{djl} - C_{jb}{}^dC_{dil} + C_{ij}{}^d\Gamma_{dbl} + C_{jb}{}^d\Gamma_{dil} + C_{bi}{}^d\Gamma_{djl}. \end{aligned} \tag{4.33}$$

Remark 4.6. The system (4.25)–(4.28) constitutes a (linear, homogeneous) first order symmetric hyperbolic system for the variables $C_{ijb}, \widehat{\mathbf{R}}_{i0}, \widehat{\mathbf{R}}_{00}, \widehat{\mathbf{R}}^{(A)}_{ij}$. Indeed, we notice that $L_\mu(C,\widehat{R}), M_{ij}(C,\widehat{\mathbf{R}}), \widehat{\mathbf{R}}_{(ijb)l}$ can be viewed, by virtue of Lemma 4.4 and (4.4), (4.25), as linear expressions in the unknowns, with coefficients depending on the solution $K_{ij}, \Gamma_{ijb}, f_i{}^p, f^b{}_p$ to the reduced equations (2.8), (2.9), (2.19), (2.20) and their first derivatives.

Proof. We compute (4.25) by directly differentiating (4.4) and using the commutation formula (2.22), the evolution equations (2.8), (2.9), (4.23), Lemma 2.7, along with (4.18):

$$\begin{aligned} e_0C_{ijb} &= e_0(f^b{}_pe_if_j{}^p - f^b{}_pe_jf_i{}^p - \Gamma_{ijb} + \Gamma_{jib}) \\ &= K_c{}^bf^c{}_pe_if_j{}^p - f^b{}_pe_i(K_j{}^cf_c{}^p) - f^b{}_pK_i{}^ce_cf_j{}^p - K_c{}^bf^c{}_pe_jf_i{}^p \\ &\quad + f^b{}_pe_j(K_i{}^cf_c{}^p) + f^b{}_pK_j{}^ce_cf_i{}^p \\ &\quad - \Big[-K_i{}^l\Gamma_{ljb} + \widetilde{D}_jK_{bi} - \widetilde{D}_bK_{ji} + \delta_{ib}\widehat{\mathbf{R}}_{j0} - \delta_{ij}\widehat{\mathbf{R}}_{b0}\Big] \\ &\quad + \Big[-K_j{}^l\Gamma_{lib} + \widetilde{D}_iK_{bj} - \widetilde{D}_bK_{ij} + \delta_{jb}\widehat{\mathbf{R}}_{i0} - \delta_{ij}\widehat{\mathbf{R}}_{b0}\Big] \end{aligned}$$

$$= K_b{}^l C_{ijl} - K_i{}^l C_{ljb} - K_j{}^l C_{ilb} - \delta_{ib}\widehat{\mathbf{R}}_{j0} + \delta_{jb}\widehat{\mathbf{R}}_{i0} \tag{4.34}$$

Moreover, a direct computation shows that

$$\begin{aligned}\widehat{\mathbf{R}}_{0b} = -\widehat{\mathbf{R}}_{0ib}{}^i &= -\,\mathbf{g}((\widetilde{\mathbf{D}}_{e_0}\widetilde{\mathbf{D}}_{e_i} - \widetilde{\mathbf{D}}_{e_i}\widetilde{\mathbf{D}}_{e_0} - \widetilde{\mathbf{D}}_{\widetilde{\mathbf{D}}_{e_0}e_i - \widetilde{\mathbf{D}}_{e_i}e_0})e_b, e^i)\\ &= -\,e_0\Gamma_{ib}{}^i - K_i{}^c\Gamma_{cb}{}^i \overset{(4.23),(4.18)}{=} -\widehat{\mathbf{R}}_{b0}\end{aligned} \tag{4.35}$$

Also, contracting (4.16) in $(a; b)$ and $(i; j)$ we obtain

$$\widehat{\mathbf{R}} + 2\widehat{\mathbf{R}}_{00} = \widehat{R} - |K|^2 + (\mathrm{tr}K)^2, \tag{4.36}$$

while

$$\begin{aligned}\widehat{\mathbf{R}}_{0i0j} &= -\widehat{\mathbf{R}}_{0ij0} = \widehat{\mathbf{R}}_{ij} - \widehat{\mathbf{R}}_{bij}{}^b \overset{(4.16)}{=} \widehat{\mathbf{R}}_{ij} - \widehat{R}_{ij} - \mathrm{tr}K\,K_{ij} + K_i{}^b K_{jb} \quad (\text{by } (4.22))\\ \widehat{\mathbf{R}}_{0i0j} &= e_0 K_{ij} + K_i{}^b K_{bj}\\ &\Rightarrow e_0 K_{ij} + \mathrm{tr}K\,K_{ij} = -\widehat{R}_{ij} + \widehat{\mathbf{R}}_{ij}\end{aligned} \tag{4.37}$$

Contracting (4.19) and using the antisymmetry of Γ_{ijb} (see Lemma 2.7), the spatial Ricci tensor in the preceding RHS expands to

$$-\widehat{R}_{ij} := -\,\widehat{R}_{bij}{}^b = e_i\Gamma^b{}_{jb} - e^b\Gamma_{ijb} + \Gamma^b{}_i{}^c\Gamma_{cjb} + \Gamma^b{}_b{}^c\Gamma_{ijc} \tag{4.38}$$

By the symmetry of K_{ij} we also have

$$e_0 K_{ij} + \mathrm{tr}K\,K_{ij} = -\widehat{R}^{(S)}_{ij} + \widehat{\mathbf{R}}^{(S)}_{ij}. \tag{4.39}$$

Due to (4.38), we find that the reduced equation (2.19) corresponds to (cf. (2.15) and Remark 2.6)

$$e_0 K_{ij} + \mathrm{tr}K\,K_{ij} = -\,\widehat{R}^{(S)}_{ij} + \frac{1}{2}\delta_{ij}[\widehat{R} - |K|^2 + (\mathrm{tr}K)^2] \tag{4.40}$$

Combining (4.36)–(4.40)we deduce the identities:

$$\frac{1}{2}(\widehat{\mathbf{R}}_{ij} + \widehat{\mathbf{R}}_{ji}) = \frac{1}{2}\delta_{ij}[\widehat{\mathbf{R}} + 2\widehat{\mathbf{R}}_{00}] \tag{4.41}$$

Contracting indices in (4.41) gives

$$\widehat{\mathbf{R}} + \widehat{\mathbf{R}}_{00} = \frac{3}{2}[\widehat{\mathbf{R}} + 2\widehat{\mathbf{R}}_{00}] \quad \Rightarrow \quad \widehat{\mathbf{R}} = -4\widehat{\mathbf{R}}_{00} \quad \Rightarrow \quad \frac{1}{2}(\widehat{\mathbf{R}}_{ij} + \widehat{\mathbf{R}}_{ji}) = -\delta_{ij}\widehat{\mathbf{R}}_{00}, \tag{4.42}$$

which confirms (4.29).

Next, we contract the second Bianchi identity (4.14) in the indices $(\alpha; \delta)$ and $(\beta; \gamma)$ to obtain:

$$\widetilde{\mathbf{D}}^\alpha\widehat{\mathbf{R}}_{\mu\alpha} = \frac{1}{2}\widetilde{\mathbf{D}}_\mu\widehat{\mathbf{R}} + L_\mu(C, \widehat{\mathbf{R}}), \tag{4.43}$$

where $L_\mu(C, \widehat{\mathbf{R}})$ is given by (4.30). Hence, for $\mu = i = 1, 2, 3$, we deduce the equation

$$e_0\widehat{\mathbf{R}}_{i0} \overset{(4.1)}{=} \widetilde{\mathbf{D}}_0\widehat{\mathbf{R}}_{i0} = -\frac{1}{2}e_i\widehat{\mathbf{R}} + \widetilde{\mathbf{D}}^a\widehat{\mathbf{R}}_{ia} - L_i(C,\widehat{\mathbf{R}})$$
$$\overset{(4.42)}{=} 2e_i\widehat{\mathbf{R}}_{00} + e^a\widehat{\mathbf{R}}^{(S)}_{ia} + e^a\widehat{\mathbf{R}}^{(A)}_{ia} - \Gamma^a{}_i{}^\beta\widehat{\mathbf{R}}_{\beta a} - \Gamma_a{}^{a\beta}\widehat{\mathbf{R}}_{i\beta} - L_i(C,\widehat{\mathbf{R}})$$
$$\overset{(4.29)}{=} e_i\widehat{\mathbf{R}}_{00} + e^a\widehat{\mathbf{R}}^{(A)}_{ia} - \Gamma^a{}_i{}^\beta\widehat{\mathbf{R}}_{\beta a} - \Gamma_a{}^{a\beta}\widehat{\mathbf{R}}_{i\beta} - L_i(C,\widehat{\mathbf{R}})$$

which proves (4.26).

Employing the identity (4.43) once more, for $\mu = 0$, we have

$$e_0\widehat{\mathbf{R}}_{00} \overset{(4.1)}{=} \widetilde{\mathbf{D}}_0\widehat{\mathbf{R}}_{00} = -\frac{1}{2}e_0\widehat{\mathbf{R}} + \widetilde{\mathbf{D}}^a\widehat{\mathbf{R}}_{0a} - L_0(C,\widehat{\mathbf{R}})$$
$$\overset{(4.42)}{=} 2e_0\widehat{\mathbf{R}}_{00} + e^a\widehat{\mathbf{R}}_{0a} - K^{ab}\widehat{\mathbf{R}}^{(S)}_{ba} - \mathrm{tr}K\widehat{\mathbf{R}}_{00} - \Gamma_a{}^{ab}\widehat{\mathbf{R}}_{0b} - L_0(C,\widehat{\mathbf{R}})$$

Solving for $e_0\widetilde{\mathbf{R}}_{00}$ and using (4.29), (4.35), we obtain (4.27).

Going back to (4.13), we put $\alpha = i, \beta = j$ and use (4.4) to keep only the spatial part of the identity. Differentiating both sides in e_0 and using the commutation formula (2.22) we compute:

$$\begin{aligned}-2e_0\widehat{\mathbf{R}}^{(A)}_{ij} &= e_0\Big[\widetilde{D}^bC_{ijb} + \widetilde{D}_iC_{jb}{}^b + \widetilde{D}_jC_{bi}{}^b + C_{ij}{}^lC_{lb}{}^b + C_{bi}{}^lC_{lj}{}^b + C_{jb}{}^lC_{li}{}^b - C_{jb}{}^l\Gamma_{li}{}^b - C_{bi}{}^l\Gamma_{lj}{}^b\Big]\\ &= e^be_0C_{ijb} + e_ie_0C_{jb}{}^b + e_je_0C_{bi}{}^b - K^{bl}e_lC_{ijb} - K_i{}^le_lC_{jb}{}^b - K_j{}^le_lC_{bi}{}^b + e_0L_{ij}(C)\end{aligned} \tag{4.44}$$

where $L_{ij}(C)$ is given by (4.32). We rewrite the second line in (4.44) by plugging in (4.25):

$$\begin{aligned}&e^be_0C_{ijb} + e_ie_0C_{jb}{}^b + e_je_0C_{bi}{}^b - K^{bl}e_lC_{ijb} - K_i{}^le_lC_{jb}{}^b - K_j{}^le_lC_{bi}{}^b\\ &= e^b\Big[K_b{}^lC_{ijl} - K_i{}^lC_{ljb} - K_j{}^lC_{ilb} - \delta_{ib}\widehat{\mathbf{R}}_{j0} + \delta_{jb}\widehat{\mathbf{R}}_{i0}\Big] + e_i\Big[K^{bl}C_{jbl} - K_j{}^lC_{lb}{}^b - K^{bl}C_{jlb} - \widehat{\mathbf{R}}_{j0} + 3\widehat{\mathbf{R}}_{j0}\Big]\\ &\quad + e_j\Big[K^{bl}C_{bil} - K^{bl}C_{lib} - K_i{}^lC_{bl}{}^b - 3\widehat{\mathbf{R}}_{i0} + \widehat{\mathbf{R}}_{i0}\Big] - K^{bl}e_lC_{ijb} - K_i{}^le_lC_{jb}{}^b - K_j{}^le_lC_{bi}{}^b\\ &= e_i\widehat{\mathbf{R}}_{j0} - e_j\widehat{\mathbf{R}}_{i0} + K^{bl}(e_bC_{ijl} + e_iC_{jbl} + e_jC_{bil}) - K_i{}^l(e^bC_{ljb} + e_jC_{bl}{}^b + e_lC_{jb}{}^b)\\ &\quad - K_j{}^l(e^bC_{ilb} + e_iC_{lb}{}^b + e_lC_{bi}{}^b) - K^{bl}(e_iC_{jlb} + e_jC_{lib} + e_lC_{ijb}) + C_{ijl}e^bK_b{}^l + C_{jbl}e_iK^{bl} + C_{bil}e_jK^{bl}\\ &\quad - C_{ljb}e^bK_i{}^l - C_{ilb}e^bK_j{}^l - C_{lb}{}^be_iK_j{}^l - C_{jlb}e_iK^{bl} - C_{lib}e_jK^{bl} - C_{bl}{}^be_jK_i{}^l\end{aligned} \tag{4.45}$$

On the other hand, from (4.13) and the first Bianchi identity (4.12), the spatial derivatives of C_{ijb} in (4.45) can be replaced by

$$\begin{aligned}&K^{bl}(e_bC_{ijl} + e_iC_{jbl} + e_jC_{bil}) - K_i{}^l(e^bC_{ljb} + e_jC_{bl}{}^b + e_lC_{jb}{}^b) - K_j{}^l(e^bC_{ilb} + e_iC_{lb}{}^b + e_lC_{bi}{}^b)\\ &\quad - K^{bl}(e_iC_{jlb} + e_jC_{lib} + e_lC_{ijb})\\ &\quad = 2K_i{}^l\widehat{\mathbf{R}}^{(A)}_{lj} + 2K_j{}^l\widehat{\mathbf{R}}^{(A)}_{il} + K^{bl}\widehat{\mathbf{R}}_{(ijl)b} - K^{bl}\widehat{\mathbf{R}}_{(ijb)l} + K_i{}^lL_{lj}(C) + K_j{}^lL_{il}(C) + K^{bl}[L_{ijlb}(C) - L_{ijbl}(C)],\end{aligned} \tag{4.46}$$

where L_{ijbl} is given by (4.33).

Summarizing (4.44)–(4.33) gives (4.28) and completes the proof of the lemma. □

In the presence of a timelike, totally geodesic, boundary, the boundary conditions (3.6) yield boundary conditions for certain components of the modified Ricci curvature $\widehat{R}_{\alpha\beta}$. In particular, we have:

Lemma 4.7. *The spacetime metric* $\mathbf{g}$ *induced by the solution to the boundary problem for* (2.8)*,* (2.19)*,* (2.20)*, subject to* (3.6)*, as described above, satisfies:*

$$\widehat{\boldsymbol{R}}_{03} = \widehat{\boldsymbol{R}}_{30} = 0, \quad \widehat{\boldsymbol{R}}^{(A)}_{B3} = \widehat{\boldsymbol{R}}^{(A)}_{3B} = 0, \quad on\ \mathcal{T}. \tag{4.47}$$

Proof. All subsequent computations are restricted to the boundary $\mathcal{T}$. The first boundary condition follows by setting $b = 3$ in (4.35) and using the boundary condition (3.6):

$$\widehat{\mathbf{R}}_{03} = -\widehat{\mathbf{R}}_{30} = -e_0\Gamma_{i3}{}^i - K_i{}^c\Gamma_{c3}{}^i = -e_0\Gamma_{B3}{}^B - K_{A3}\Gamma_{33}{}^A - K_A{}^B\Gamma_{B3}{}^A = 0.$$

For the second boundary condition, we first notice that by (4.29) it holds

$$\widehat{\mathbf{R}}_{B3} = -\widehat{\mathbf{R}}_{3B} \quad \Rightarrow \quad \widehat{\mathbf{R}}^{(A)}_{B3} = -\widehat{\mathbf{R}}^{(A)}_{3B} = \widehat{\mathbf{R}}_{B3}.$$

Contracting (4.16) in $(a; b)$ and setting $i = B$, $j = 3$, we obtain

$$\begin{aligned}
\widehat{\mathbf{R}}_{B3} - \widehat{\mathbf{R}}_{0B03} &= \widehat{R}_{B3} + \mathrm{tr}K\,K_{B3} - K_3{}^a K_{Ba} \qquad (\text{by } (4.38), i = B, j = 3)\\
&= e^b\Gamma_{B3b} - e_B\Gamma^b{}_{3b} + \Gamma^b{}_B{}^c\Gamma_{c3b} + \Gamma^b{}_b{}^c\Gamma_{B3c}\\
&\quad + \mathrm{tr}K\,K_{B3} - K_3{}^a K_{Ba} \qquad (\text{by } (4.22))\\
\widehat{\mathbf{R}}_{B3} &= e^C\Gamma_{B3C} - e_B\Gamma^C{}_{3C} + \Gamma^C{}_B{}^D\Gamma_{D3C} + \Gamma^C{}_{B3}\Gamma_{33C} + \Gamma^a{}_a{}^C\Gamma_{B3C}\\
&\quad + \mathrm{tr}K\,K_{B3} - K_3{}^a K_{Ba} + e_0 K_{B3} + K_B{}^A K_{3A} + K_{B3}K_{33}
\end{aligned}$$

Every term in the preceding RHS vanished by virtue of the boundary condition (3.6), which implies the vanishing of $\widehat{\mathbf{R}}_{B3}$ and hence that of $\widehat{\mathbf{R}}^{(A)}_{B3}$. □

4.3. Final step. The equations (4.25)–(4.29) constitute a *linear* first order *symmetric* hyperbolic system (see also Remark 4.6) for the variables $\widehat{\mathbf{R}}_{\mu 0}, \widehat{\mathbf{R}}^{(A)}_{ij}, C_{ijb}$, which in the presence of a timelike boundary also satisfy the conditions (4.47). As an immediate implication, we conclude that the solution $K_{ij}, \Gamma_{ijb}, f_i{}^p, f^b{}_p$ to the reduced equations (2.19), (2.20), (2.8), (2.9), is indeed a solution to the EVE. More precisely, we have:

Proposition 4.8. *Consider a solution to the reduced equations* (2.19)*,* (2.20)*,* (2.8)*,* (2.9)*, such that*

1. $K_{ij}, \Gamma_{ijb}, f_i{}^p, f^b{}_p \in L^\infty_t H^s$, $s \geq 3$, *for the classical Cauchy problem;*
2. $K_{ij}, \Gamma_{ijb}, f_i{}^p, f^b{}_p \in L^\infty_t B^s$, $s \geq 7$, *subject to* (3.6)*, for the boundary value problem.*

Then the variables $\widehat{\boldsymbol{R}}_{\mu\nu}, C_{ijb}$ *vanish. In particular,* $\widetilde{\boldsymbol{D}}$ *is the Levi-Civita connection* $\mathbf{D}$ *of* $\mathbf{g}$. *Moreover,* $\mathbf{g}$ *satisfies the EVE and in the case* 2. *the boundary is totally geodesic.*

Proof. The coefficients in (4.25)–(4.29) depend on $K_{ij}, \Gamma_{ijb}, f_i{}^p, f^b{}_p$ and their first spatial derivatives. Hence, they are bounded, provided up to three of their spatial derivatives are bounded in L^2. This is consistent with the spaces $L^\infty_t H^s$, for $s \geq 3$, and $L^\infty_t B^s$, for $s \geq 7$.

In the absence of a boundary, the symmetry of the system (4.25)–(4.29) implies uniqueness of solutions (via a standard energy estimate). Since C_{ijb} vanishes on the initial hypersurface, we have that $\widetilde{\mathbf{D}} = \mathbf{D}$. By virtue of (4.13) (for $\alpha = i, \beta = j$) and (4.4), we have that $\widehat{\mathbf{R}}^{(A)}_{ij}\big|_{\Sigma_0} = 0$. Also, the validity of the constraints, together with the formula (4.29), implies $\widehat{\mathbf{R}}_{\mu 0}\big|_{\Sigma_0} = 0$, see (4.18), (4.36). Hence, $\widehat{\mathbf{R}}_{\mu 0}, \widehat{\mathbf{R}}^{(A)}_{ij}, C_{ijb}$ vanish everywhere and $\widetilde{\mathbf{D}} = \mathbf{D}$. By (4.29), $\widehat{\mathbf{R}}^{(S)}_{ij} = 0$, and hence, $\widehat{\mathbf{R}}_{\mu\nu} = \mathbf{R}_{\mu\nu} = 0$.

In the presence of a timelike boundary, we notice that in a typical L^2-energy estimate for (4.25)–(4.28), the arising $\mathcal{T}$-boundary terms equal

$$\int_{\mathcal{S}_t} \widehat{\mathbf{R}}_{00}\widehat{\mathbf{R}}_{30} + \widehat{\mathbf{R}}^{(A)}_{i3}\widehat{\mathbf{R}}^{i}{}_{0}\mathrm{vol}_{\mathcal{S}_t} = \int_{\mathcal{S}_t} \widehat{\mathbf{R}}_{00}\widehat{\mathbf{R}}_{30} + \widehat{\mathbf{R}}^{(A)}_{B3}\widehat{\mathbf{R}}^{B}{}_{0} + \widehat{\mathbf{R}}^{(A)}_{33}\widehat{\mathbf{R}}_{30}\mathrm{vol}_{\mathcal{S}_t} \overset{(4.47)}{=} 0. \tag{4.48}$$

Therefore, an energy estimate closes and the previous argument applies. Since $\widetilde{\mathbf{D}} = \mathbf{D}$ is the actual Levi-Civita connection of $\mathbf{g}$, the variables K_{ij}, Γ_{ijb} are the true connection coefficients of the orthonormal frame $\{e_\mu\}_0^3$, given by (1.6), (1.7). Hence, the geometric formulas (3.7) are valid, where χ_{0A}, χ_{AB} are the components of the actual second fundamental form χ of $\mathcal{T}$, which vanish by virtue of the condition (3.6). The component $\chi_{00} = \mathbf{g}(\mathbf{D}_{e_0}e_3, e_0) = -\mathbf{g}(e_3, \mathbf{D}_{e_0}e_0)$ vanishes, since e_0 is geodesic. We conclude that $\chi \equiv 0$, i.e., $\mathcal{T}$ is totally geodesic. □

4.4. Proof of Theorems 1.1, 1.10. It is a combination of Propositions 2.12, 3.10, 4.8. We note in particular that geometric uniqueness is immediate from the homogeneity of our boundary conditions. After setting up the geodesic gauge in any vacuum spacetime with totally geodesic timelike boundary, the relevant connection coefficients will vanish, in which case the uniqueness statement for the reduced system of equations applies to solutions with the same initial data.

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