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On the Light Ray Transform of Wave Equation Solutions

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Abstract: We study the light ray transform on Minkowski space-time and its small metric perturbations acting on scalar functions which are solutions to wave equations. We show that the light ray transform uniquely determines the function in a stable way. The problem is of particular interest because of its connection to inverse problems of the Sachs–Wolfe effect in cosmology.

1. Introduction

Let $M = [t_0, t_1] \times \mathbb{R}^3$ and (t, x), $t \in [t_0, t_1]$, $x \in \mathbb{R}^3$ be the local coordinates. Let $g_M = -dt^2 + dx^2$ be the Minkowski metric on M. Consider the Lorentzian manifold (M, g_M). We denote the interior by $M^\circ = (t_0, t_1) \times \mathbb{R}^3$ and the boundaries by $\mathscr{S}_0 = \{t_0\} \times \mathbb{R}^3$ and $\mathscr{S} = \{t_1\} \times \mathbb{R}^3$. See Fig. 1.

Consider light-like geodesics on (M, g_M) which are straight lines. We parametrize the set of light rays $\mathscr{C}$ as follows: let $x_0 \in \mathscr{S}_0$ and $v \in \mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$. Then a light ray from x_0 in direction $(1, v)$ is $\gamma(\tau) = (t_0, x_0) + \tau(1, v)$, $\tau \in [0, t_1 - t_0]$. See Fig. 1. In particular, we can identify $\mathscr{C} = \mathbb{R}^3 \times \mathbb{S}^2$. The light ray transform for scalar functions on (M, g_M) is defined by

$$X_M(f)(\gamma) = \int_0^{t_1 - t_0} f(\gamma(\tau))d\tau, \quad f \in C_0^\infty(M). \tag{1.1}$$

Of course, one can regard X_M as the restriction of the light ray transform $X_{\mathbb{R}^4}$ of the Minkowski spacetime $(\mathbb{R}^4, g_M)$ acting on functions supported in M. However, it is perhaps better to think of X_M as the compact version of the transform, which is similar to the geodesic ray transform on a compact Riemannian manifold with boundary, see for instance [20].

In this work, we study X_M acting on scalar functions which are solutions to the Cauchy problem of wave equations on M. Let $c > 0$ be a constant. Denote $\Box_c = \partial_t^2 + c^2\Delta$ where

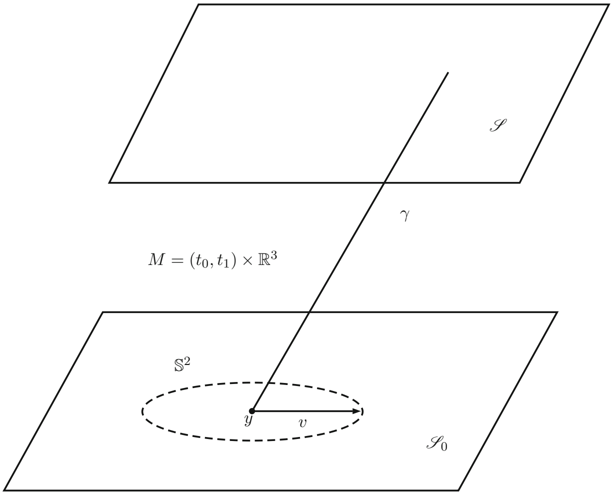

Fig. 1. The setup of the problem for the Minkowski space-time

Δ is the positive Laplacian on $\mathbb{R}^3$, namely $\Delta = \sum_{i=1}^{3} D_{x_i}^2$, $D_{x_i} = -\sqrt{-1}\frac{\partial}{\partial x_i}$. Here, c is the wave speed. On (M, g_M), $c = 1$ is the speed of light, and $\Box_c$ is the d'Alembert operator. Consider the Cauchy problem

$$\begin{aligned} \Box_c f &= 0 \quad \text{on } M^\circ \\ f = f_1, \quad \partial_t f &= f_2, \quad \text{on } \mathscr{S}_0. \end{aligned} \tag{1.2}$$

The problem we address in this paper is the determination of f or equivalently f_1, f_2 from $X_M(f)$ with the constraint (1.2). Let $\mathcal{N}^s \stackrel{\text{def}}{=} H^{s+1}_{\text{comp}}(\mathscr{S}_0) \times H^s_{\text{comp}}(\mathscr{S}_0)$. Our main result is

Theorem 1.1. *Suppose* $0 < c \leq 1$ *is constant. Assume that* $(f_1, f_2) \in \mathcal{N}^s$, $s \geq 0$, *and* f_1, f_2 *are supported in a compact set* $\mathscr{K}$ *of* $\mathscr{S}_0$. *Then* $X_M f$ *uniquely determines* f *and* f_1, f_2 *which satisfy* (1.2). *Moreover, there exists* $C > 0$ *such that*

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \leq C\|X_M f\|_{H^{s+2}(\mathscr{C})} \text{ and } \|f\|_{H^{s+1}(M)} \leq C\|X_M f\|_{H^{s+2}(\mathscr{C})}$$

where $\mathscr{C}$ *is the set of light rays on* M.

We will prove stronger versions of the theorem including lower order terms in the wave equation in Theorem 8.3 in Sect. 8. However, for ease of presentation, we use the standard wave equation on Minkowski spacetime throughout the paper until the final sections where the necessary changes are indicated.

Next, we consider the generalization of Theorem 1.1 corresponding to $c = 1$. We remark that it is not difficult to formulate the result corresponding to $c < 1$ although we do not discuss it. We consider metric perturbations $g_\delta = g_M + h$ where h satisfies assumptions (A1), (A2) in Sect. 9, which says that h is a suitably smooth small perturbation of the Minkowski spacetime. In this case, light rays may not be straight lines. Let X_δ be the light ray transform on (M, g_δ) see (9.6). Let $\Box_{g_\delta}$ be the d'Alembert operator on (M, g_δ). Consider the Cauchy problem

$$\begin{aligned} \Box_{g_\delta} f &= 0 \quad \text{on } M^\circ \\ f = f_1, \quad \partial_t f &= f_2, \quad \text{on } \mathscr{S}_0. \end{aligned} \tag{1.3}$$

Our result is

Theorem 1.2. *Consider* (M, g_δ) *satisfying assumptions (A1), (A2) to be stated in Sect. 9. Assume that* $(f_1, f_2) \in \mathcal{N}^s$, $s \geq 0$, *and* f_1, f_2 *are supported in a compact set* $\mathcal{K}$ *of* $\mathcal{S}_0$. *For* $\delta \geq 0$ *sufficiently small,* $X_\delta f$ *uniquely determines* f *and* f_1, f_2 *which satisfy* (1.3). *Moreover, there exists* $C > 0$ *such that*

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \leq C\|X_\delta f\|_{H^{s+2}(\mathscr{C}_\delta)} \text{ and } \|f\|_{H^{s+1}(M)} \leq C\|X_\delta f\|_{H^{s+2}(\mathscr{C}_\delta)}$$

where $\mathscr{C}_\delta$ *is the set of light rays on* (M, g_δ)*, see Sect. 9.*

Our motivation for this setup of the light ray transform comes from some inverse problems in cosmology. We are particularly interested in the determination of gravitational perturbations such as primordial gravitational waves from the anisotropies of the Cosmic Microwave Background (CMB), see for example [2,4,11]. Sachs and Wolfe in their 1967 paper [19] discovered the connection of the CMB anisotropy and the light ray transform of the gravitational perturbations, now called the Sachs–Wolfe effects. We discuss the background in Sects. 2 and 3. Physically, $c < 1$ and $c = 1$ in Theorem 1.1 correspond to different Universe models driven by hydrodynamical perturbations and scalar field perturbations, respectively. Moreover, Theorem 1.2 covers some cases of variable wave speeds.

The reason that we are able to get a stable determination is the restriction of singularities of f. In general, it is known that time-like singularities in f, namely all $(z, \zeta) \in T^*M$ in the wave front set $\mathrm{WF}(f)$ of f with ζ time-like, are lost after taking the light ray transform, although the light ray transform X_M is injective on $C_0^\infty(M)$. In particular, we do not expect Theorem 1.1 to hold for $c > 1$. There is a fundamental difference in our treatment between the $c < 1$ and $c = 1$ cases. The former requires a good understanding of the normal operator $X_M^* X_M$ which was considered in [12] and further generalized in [13], while the latter relies on a thorough analysis of the operator $X_M E$ where E is the fundamental solution or parametrix for the Cauchy problem.

The paper is organized as follows. In Sects. 2 and 3, we discuss the (integrated) Sachs–Wolfe effects and explain how the inverse problem is related to our theorems. In Sect. 4, we review some properties of the light ray transform. Then we consider the Cauchy problem in Sect. 5. In Sects. 6 and 7, we construct the microlocal parametrix for the light ray transform with the wave constraint for $c < 1$ and $c = 1$ respectively. We prove Theorem 1.1 and the version including lower order terms in the wave equation in Sect. 8. Finally, we address the small metric perturbations of Minkowski space-time in Sect. 9.

2. The Integrated Sachs–Wolfe Effect

Consider the flat Friedman–Lemaîte–Robertson–Walker (FLRW) model for the cosmos:

$$\mathscr{M} = (0, \infty) \times \mathbb{R}^3, \quad g_0 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$$

where (t, x), $t \in (0, \infty)$, $x \in \mathbb{R}^3$ are coordinates and $\delta_{ij} = 1$ if $i = j$ and otherwise 0. Here, the signature of g_0 is $(+, -, -, -)$ because we will refer to some results in [16] later. The factor $a(t)$ is assumed to be positive and smooth in t. It represents the rate of expansion of the Universe.

We assume that the actual cosmos is a metric perturbation $g = g_0 + \delta g$ on $\mathscr{M}$ where δg is a small perturbation compared to g_0. Here, we follow the convention of [16] that

δA denotes the perturbation of quantity A (not δ times A). We introduce the conformal time s such that $ds = a^{-1}(t)dt$. Then we get

$$g_0 = a^2(s)\left(ds^2 - \delta_{ij}dx^i dx^j\right) = a^2(s)g_M$$

where g_M is the Minkowski metric on $\mathscr{M} = (0, \infty)$ and we used $a(s)$ to denote $a(t(s))$. We write $g = a^2(s)(g_M + \delta g)$ where δg denotes the corresponding perturbation in conformal time. In the literature, the metric perturbations are classified to scalar, vector and tensor type. We consider the scalar type perturbations. In the longitudinal gauge, also called the conformal Newtonian gauge, the metric g is of the form

$$g = a^2(s)\left[(1 + 2\Phi)ds^2 - (1 - 2\Psi)dx^2\right] \tag{2.1}$$

see [16, Section 2]. Here, Φ, Ψ are scalar functions on M. We remark that there is a gauge invariant formulation of cosmological perturbations. However, in the longitudinal gauge, the gauge invariant variables are equal to Φ, Ψ, see [16]. In this work, we fix the gauge and work with Φ, Ψ for simplicity.

Consider the Cosmic Microwave Background (CMB) measurement. Our main references are [2,4,19]. Let $\mathscr{S}_0 = \{s_0\} \times \mathbb{R}^3$ be the surface of last scattering. This is the moment after which photons stopped interaction and started to travel freely in $\mathscr{M}$. Let $\mathscr{S} = \{s_1\} \times \mathbb{R}^3$ be the surface where we make observation of the photons. Let $\gamma(\tau)$ be a light ray from $\mathscr{S}_0$ to $\mathscr{S}$. It represents the trajectory of photons in $\mathscr{M}$. Explicitly, we have

$$\gamma(\tau) = (s_0, x_0) + \tau(1, v), \quad (s_0, x_0) \in \mathscr{S}_0, v \in \mathbb{S}^2, \tau \in [0, s_1 - s_0].$$

Then we consider the photon energies observed at $\mathscr{S}_0, \mathscr{S}$ denoted by $E_0 = g_0(\dot{\gamma}(s_0), \partial_s)$, $E = g_0(\dot{\gamma}(s_1), \partial_s)$. Here, the observer is represented by the flow of the vector field ∂_s. The redshift z is defined by

$$1 + z = E/E_0.$$

In [19], Sachs and Wolfe derived that to the first order linearization, $1+z$ is represented by a light ray transform of the metric perturbations, see [19, equation (39)]. In cosmological literatures, one often connects this to the CMB temperature anisotropies. Let T be the temperature observed at $\mathscr{S}$ in the isotropic background g_0. Let δT be the temperature fluctuation from the isotropic background. One can compute $\delta T/T$ in terms of the energies E_0, E. One component of $\delta T/T$ is the integrated Sachs–Wolfe (ISW) effects

$$(\frac{\delta T}{T})^{ISW}(\gamma) = \int_0^{s_1 - s_0} (\partial_s\Phi\,(\gamma(\tau)) + \partial_s\Psi\,(\gamma(\tau))\,d\tau = X_M(\partial_s\Phi + \partial_s\Psi)(\gamma) \tag{2.2}$$

see [4, Section 2.5]. Note that this quantity depends on the light ray γ which indicates the anisotropy. We remark that another component of $\delta T/T$ is the ordinary Sachs–Wolfe effect (OSW) which only involves Φ, Ψ at $\mathscr{S}_0$. The integrated Sachs–Wolfe effect can be extracted from the CMB and other astrophysical data, see for example [14].

The inverse Sachs–Wolfe problem we study is to determine Φ, Ψ on M from $(\delta T/T)^{ISW}$, which in particular includes the initial value of Φ, Ψ on $\mathscr{S}_0$. Before we proceed, we observe that there are natural obstructions to the unique determination from (2.2). If $\Phi + \Psi$ is a constant, then the integrated Sachs–Wolfe effect is always zero. So the goal is to determine Φ, Ψ up to such natural obstructions.

3. Dynamical Equations for Perturbations

For the Sachs–Wolfe problem, we should take into account that g satisfies the Einstein equations with certain source fields and initial perturbations at $\mathscr{S}_0$ from g_0. On the linearization level, this puts the perturbation δg under some wave equation constraint as we discuss in this section. The derivations of the equations for the perturbation take some amount of work and they are mostly done in the literature, see for example [2, Section 5.1] and [4]. We follow the presentation and the notations in [16, Section 4–6] closely. Instead of the gauge invariant approach, we choose to work in the longitudinal gauge for simplicity. It is not hard to transform back and forth and our analysis works for the gauge invariant formulation as well.

Let $R^{\mu}{}_{\nu}$ be the Ricci curvature tensor and R the scalar curvature on $(\mathscr{M}, g)$ (in conformal time). Let $T^{\mu}{}_{\nu}$ denote the stress-energy tensor of certain source fields. The Einstein equations are

$$G^{\mu}{}_{\nu} = 8\pi G T^{\mu}{}_{\nu}, \quad G^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R$$

where G is Newton's gravitational constant. We assume that $T^{\mu}{}_{\nu} = {}^{(0)}T^{\mu}{}_{\nu} + \delta T^{\mu}{}_{\nu}$ where ${}^{(0)}T$ denotes the stress-energy tensor of the background field and δT denotes the perturbation. We also have $g = a^2(g_M + \delta g)$. Then we can write $G^{\mu}{}_{\nu} = {}^{(0)}G^{\mu}{}_{\nu} + \delta G^{\mu}{}_{\nu} + \cdots$. From the asymptotic expansion, one finds that the Einstein tensor for the background metric g_M are

$${}^{(0)}G_0{}^0 = 3a^{-2}H^2, \quad {}^{(0)}G^0{}_i = 0, \quad {}^{(0)}G^i{}_j = a^{-2}(2H' + H^2)\delta^i{}_j, \tag{3.1}$$

where $i, j = 1, 2, 3$, $H(s) = \partial_s a(s)/a(s)$, see [16, equation (4.2)]. Here, $H' = \partial_s H$ denotes the derivative in the conformal time variable. We emphasize that we work with a flat Universe and we get the equation ${}^{(0)}G^{\mu}{}_{\nu} = 8\pi G {}^{(0)}T^{\mu}{}_{\nu}$.

For the first order perturbation term, we get $\delta G^{\mu}{}_{\nu} = 8\pi G \delta T^{\mu}{}_{\nu}$. After lengthy calculations, one obtains (see [16, equation (4.15)]) the following equations for Φ, Ψ

$$\begin{aligned}
&-3H\left(H\Phi + \Psi'\right) + \Delta\Psi = 4\pi G a^2 \delta T^0{}_0 \\
&\partial_i(H\Phi + \Psi') = 4\pi G a^2 \delta T^0{}_i \\
&[(2H' + H^2)\Phi + H\Phi' + \Psi'' + 2H\Psi' + \frac{1}{2}\Delta(\Phi - \Psi)]\delta^i{}_j \\
&\quad - \frac{1}{2}\delta^{ik}(\Phi - \Psi)_{|kj} = -4\pi G a^2 \delta T^i{}_j,
\end{aligned} \tag{3.2}$$

where $i, j = 1, 2, 3$, ∂_i denotes the ith component of the covariant derivative with respect to the background metric g_M, Δ denotes the standard Laplacian on $\mathbb{R}^3$, and as in (3.1), prime denotes ∂_s derivative.

Now we need to specify the source field. We consider two important examples: the perfect fluid and the scalar field. We first consider Universe dominated by perfect fluid sources. Let u be the four fluid velocity of a fluid source. The stress-energy tensor for a perfect fluid is

$$T^{\alpha}{}_{\beta} = (\epsilon + p)u^{\alpha}u_{\beta} - p\delta^{\alpha}{}_{\beta}$$

see [16, equation (5.2)], Here, ϵ is the energy density and p is the pressure of the fluid. We assume that $\epsilon = \epsilon_0 + \delta\epsilon$, $p = p_0 + \delta p$ where 0 denotes the quantity for the background and δ denotes the perturbations. For fluid source, from (3.2) one deduces that the perturbations $\Phi = \Psi$. In the case of adiabatic perturbations, Φ satisfies the following equation, called Bardeen's equation

$$\Phi'' + 3H(1 + c_s^2)\Phi' - c_s^2 \Delta\Phi + [2H' + (1 + 3c_s^2)H^2]\Phi = 0, \tag{3.3}$$

see [16, equation (5.22)]. In general, the right hand side of the equation is a non-zero term related to the entropy perturbations. The fluid velocity u also satisfies a wave equation with speed c_s, see [16, equation (5.25)]. Here, $c_s < 1$ is the speed of sound. Prescribing Cauchy data of Φ at $\mathscr{S}_0$, one can solve the Cauchy problem of (3.3) to get Φ in $\mathscr{M}$. We formulate the inverse Sachs–Wolfe problem in this case as

Problem 3.1. *Determining Φ from (2.2) where Φ satisfies the Cauchy problem of (3.3).*

Commuting equation (3.3) with ∂_s, we see that $\partial_s\Phi$ also satisfies a wave equation. Hence, we arrived at the model problem we proposed in the introduction.

Next, let's consider Universe governed by a scalar field ϕ. The stress energy tensor is

$$T^\mu{}_\nu = \nabla^\mu\phi\nabla_\nu\phi - [\frac{1}{2}\nabla^\alpha\phi\nabla_\alpha\phi - V(\phi)]\delta^\mu{}_\nu$$

see [16, equation (6.2)]. Here, V is the potential function for the scalar field ϕ. The field itself satisfies the Klein-Gordon equation $\Box\phi + \partial_\phi V(\phi) = 0$. Now assume that $\phi = \phi_0 + \delta\phi$ where ϕ_0 is the scalar field which drives the background model and $\delta\phi$ denotes the perturbation. Then we can split $T^\mu{}_\nu = {}^{(0)}T^\mu{}_\nu + \delta T^\mu{}_\nu$. Again, one finds that $\Phi = \Psi$ and it satisfies the equation

$$\Phi'' + 2(H - \phi_0''/\phi_0')\Phi' - \Delta\Phi + 2(H' - H\phi_0''/\phi_0)\Phi = 0 \tag{3.4}$$

see [16, equation (6.48)]. This is a damped wave equation with wave speed $c = 1$. We can formulate the inverse Sachs–Wolfe problem in this case as

Problem 3.2. *Determining Φ from (2.2) in which Φ satisfies the Cauchy problem of (3.4).*

Again, we arrived at the model problem in the introduction with $c = 1$. We do not need it but record that the scalar field perturbation also satisfies a wave equation, see [16, equation (6.47)].

Applying our main result of the paper, in particular Theorem 8.3 which allows lower order terms in the wave equation, we obtain the following result.

Corollary 3.3. *For the inverse Sachs–Wolfe effect Problems 3.1 and 3.2, one can uniquely determine Φ in $\mathscr{M}$ (and the initial conditions at $\mathscr{S}_0$) in the longitudinal gauge up to a constant in a stable way.*

4. The Light Ray Transform on Functions

We recall some facts about the light ray transform on scalar functions. Consider the Lorentzian manifold (M, g_M) and hereafter we change the signature of g_M to $(-, +, +, +)$. For $(t, x) \in M^\circ$, $t \in (t_0, t_1)$, $x \in \mathbb{R}^3$, we use $\Xi = (\tau, \xi)$, $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^3$ for the coordinate in $T_{(t,x)}M^\circ$ so that tangent vectors are represented by $\tau\partial_t + \sum_{j=1}^3 \xi^j\partial_{x^j}$. We divide the tangent vectors in $T_{(t,x)}M^\circ$ into time-like vectors $\Omega^-_{(t,x)}M^\circ = \{\Xi \in \mathbb{R}^4 : g_M(\Xi, \Xi) = -\tau^2 + |\xi|^2 < 0\}$, space-like vectors $\Omega^+_{(t,x)}M^\circ = \{\Xi \in \mathbb{R}^4 : g_M(\Xi, \Xi) >$

$0\}$ and light-like vectors $L_{(t,x)}M^\circ = \{\Xi \in \mathbb{R}^4 : g_M(\Xi, \Xi) = 0\}$. We denote the corresponding fiber bundles by $\Omega^- M^\circ, \Omega^+ M^\circ, LM^\circ$. The cotangent vectors can be classified similarly using the dual metric g_M^* on T^*M°. The corresponding bundles are denoted by $\Omega^{*,-}M^\circ, \Omega^{*,+}M^\circ, L^*M^\circ$.

From now on, *without loss of generality, we take $t_0 = 0$ in M*, which amounts to a translation in the t variable. Let $\mathscr{C}$ be the set of light rays on (M, g_M). As M has a global coordinate system, we can parametrize $\mathscr{C}$ as follows. Let $y \in \mathbb{R}^3, v \in \mathbb{S}^2 \overset{\text{def}}{=} \{z \in \mathbb{R}^3 : |z| = 1\}$ with $|\cdot|$ the Euclidean norm. We denote $\theta = (1, v)$ so that θ is a (future pointing) light-like vector. Then all the light rays are given by $\gamma_{y,v}(\tau) = (\tau, y+\tau v)$, $\tau \in (0, t_1)$, $(y, v) \in \mathbb{R}^3 \times \mathbb{S}^2$. Thus, we can identify $\mathscr{C} = \mathbb{R}^3 \times \mathbb{S}^2$. For $f \in C_0^\infty(M^\circ)$ and $y \in \mathbb{R}^3, v \in \mathbb{S}^2$, we have

$$\begin{aligned} X_M f(y, v) &= \int_0^{t_1} f(\tau, y + \tau v) d\tau \\ &= (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{t_1} e^{i((y-x)\cdot\eta + tv\cdot\eta)} f(t, x) dt dx d\eta. \end{aligned} \tag{4.1}$$

The Schwartz kernel of X_M is δ_Z the delta distribution on $\mathscr{C} \times M^\circ$ supported on the point-line relation Z defined by

$$Z = \{(\gamma, q) \in \mathscr{C} \times M^\circ : q \in \gamma\} = \{(y, v, (t, x)) \in \mathbb{R}^3 \times \mathbb{S}^2 \times M^\circ : x = y + tv\}.$$

We know (see e.g. [12]) that X_M is an Fourier integral operator of order $-3/4$ associated with the canonical relation $(N^*Z)'$, where N^*Z denotes the conormal bundle of Z minus the zero section. Hence $X_M : \mathcal{E}'(M^\circ) \to \mathcal{D}'(\mathscr{C})$ is continuous. Here, $\mathcal{D}'(M^\circ), \mathcal{E}'(M^\circ)$ denotes the space of distributions and compactly supported distributions on M°.

It is known that on $\mathbb{R}^4$, the light ray transform is injective on $C_0^\infty(\mathbb{R}^4)$, see [10,17], but not injective on $\mathcal{S}(\mathbb{R}^4)$ (Schwartz functions on $\mathbb{R}^4$). It is proved in [10, Corollary 7] that the kernel of the transform consists of $\mathcal{S}(\mathbb{R}^4)$ functions whose Fourier transforms are supported in the time-like cone. One can obtain analogous results for X_M. The point is that after taking the light ray transform, time-like singularities in the functions are lost.

To see the difference in the treatment between space-like and light-like singularities, consider the normal operator $X_M^* X_M$. For the light ray transform on $\mathbb{R}^4$, the Schwartz kernel of the normal operator can be computed explicitly using Fourier transforms, see [17]. Let's look at the microlocal structure. The canonical relation $C = N^*Z'$ is

$$\begin{aligned} C =&\{((y, v, \eta, w); (t, x, \tau, \xi)) \in (T^*\mathscr{C}\backslash 0) \times (T^*M^\circ\backslash 0) : y = x - tv, \quad \eta = \xi, \\ w =&t\xi|_{T_v\mathbb{S}^2}, \quad \tau = -\xi \cdot v, \quad y \in \mathbb{R}^3, v \in \mathbb{S}^2, \eta \in \mathbb{R}^3, (t, x) \in M^\circ\}, \end{aligned} \tag{4.2}$$

see [12, equation (39)]. In the expression of w, ξ is regarded as a co-tangent vector to $T_v\mathbb{S}^2$. If $\Xi = (\tau, \xi)$ is light-like, then $\xi|_{T_v\mathbb{S}^2} = 0$, see [12, Lemma 10.1]. We look at the double fibration picture

$$\begin{tikzcd} & C \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ T^*M & & T^*\mathscr{C} \end{tikzcd}$$

If ρ is an injective immersion, the double fibration satisfies the Bolker condition, and the normal operator $X_M^* \circ X_M$ belongs to the clean intersection calculus so that the normal operator is a pseudo-differential operator, see [6]. As shown in [12, Lemma 10.1], ρ fails to be injective on the set $\mathcal{L} \cap C$ where

$$\mathcal{L} = \{(y, v, \eta, w; t, x, \Xi) \in (T^*\mathscr{C}\backslash 0) \times (T^*M^\circ\backslash 0) : \ \Xi \text{ is light-like}\}.$$

In particular, the normal operator is an elliptic pseudo-differential operator when restricted to space-like directions, see [17] and [12]. In general, it is proved in [23] that the Schwartz kernel of the normal operator $X_M^* X_M$ is a paired Lagrangian distribution and a parametrix can be constructed within the framework of [5]. However, the picture near light-like directions is still not so clear. We remark that Guillemin [7] considered the structure of $X_M X_M^*$ for 2 + 1 dimensional Minkowski spacetime.

5. Solution of the Cauchy Problem

We find a representation of the solution of the Cauchy problem in this section. Consider

$$\begin{aligned} \Box_c u &= 0, \quad \text{on } M^\circ = (t_0, t_1) \times \mathbb{R}^3 \\ u &= f_1, \quad \partial_t u = f_2, \ \text{on } \mathscr{S}_0 = \{t_0\} \times \mathbb{R}^3. \end{aligned} \tag{5.1}$$

The fundamental solution can be written down quite explicitly. However, it will be more convenient to look at its microlocal structure. For (5.1), all we need is the Fourier transform, see for example Trèves [21, Chapter VI, Section 1]. For general strictly hyperbolic equations, Duistermaat-Hörmander (see [3, Chaper 5]) constructed a parametrix for the Cauchy problem. So one can find a parametrix for (5.1) even when the equation contains lower order terms which will be used in Sect. 8.

Let (τ, ξ), $\xi \in \mathbb{R}^3$ be the dual variables in T^*M° to (t, x), $x \in \mathbb{R}^3$. Taking the Fourier transform of (5.1) in the x variable, we get (for $t_0 = 0$)

$$\begin{aligned} &\partial_t^2 \hat{u}(t, \xi) + c^2|\xi|^2 \hat{u}(t, \xi) = 0, \\ &\hat{u}(0, \xi) = \hat{f}_1(\xi), \quad \partial_t \hat{u}(0, \xi) = \hat{f}_2(\xi). \end{aligned}$$

Solving this ODE, we get

$$\hat{u}(t, \xi) = \frac{1}{2} e^{itc|\xi|}(\hat{f}_1 + \frac{1}{ic|\xi|}\hat{f}_2) + \frac{1}{2} e^{-itc|\xi|}(\hat{f}_1 - \frac{1}{ic|\xi|}\hat{f}_2).$$

Taking the inverse Fourier transform, we get

$$\begin{aligned} u(t, x) &= (2\pi)^{-3}\frac{1}{2}\int_{\mathbb{R}^3} e^{i(x\cdot\xi + ct|\xi|)}(\hat{f}_1 + \frac{1}{ic|\xi|}\hat{f}_2) d\xi + (2\pi)^{-3}\frac{1}{2}\int_{\mathbb{R}^3} e^{i(x\cdot\xi - tc|\xi|)}(\hat{f}_1 \\ &\quad - \frac{1}{ic|\xi|}\hat{f}_2) d\xi \\ &= (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i(x\cdot\xi + ct|\xi|)}\hat{h}_1(\xi) d\xi + (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i(x\cdot\xi - tc|\xi|)}\hat{h}_2(\xi) d\xi \\ &= E_+ h_1 + E_- h_2, \end{aligned} \tag{5.2}$$

where

$$\hat{h}_1 = \frac{1}{2}(\hat{f}_1 + \frac{1}{ic|\xi|}\hat{f}_2), \quad \hat{h}_2 = \frac{1}{2}(\hat{f}_1 - \frac{1}{ic|\xi|}\hat{f}_2).$$

We see that $E_\pm$ are represented by oscillatory integrals

$$E_\pm(f) = (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i((x-y)\cdot\xi \pm ct|\xi|)} f(y) dy d\xi. \tag{5.3}$$

The phase functions are $\phi_\pm(t, x, y, \xi) = (x - y) \cdot \xi \pm ct|\xi|$ and amplitude function $a(t, x, \xi) = 1$. In Hörmander's notation, we conclude that $E_\pm \in I^{-\frac{1}{4}}(\mathbb{R}^3 \times M^\circ; (C^\pm)')$ are Fourier integral operators where the canonical relations are

$$\begin{aligned} C^\pm = \{&\big(t, x, \zeta_0, \zeta'; y, \xi\big) \in T^*M^\circ\backslash 0 \times T^*\mathbb{R}^3\backslash 0 : \\ &y = x - ct\,(\pm\xi/|\xi|)\,, \zeta' = \xi, \zeta_0 = \pm c|\xi|\}. \end{aligned} \tag{5.4}$$

It suffices to regard h_1, h_2 as the reparametrized initial conditions for the Cauchy problem and represent $u = E_+h_1 + E_-h_2$ in (5.2). Once we find h_1, h_2, we can easily find f_1, f_2 from

$$f_1 = h_1 + h_2, \quad f_2 = ic\Delta^{\frac{1}{2}}\,(h_1 - h_2)\,. \tag{5.5}$$

6. The Microlocal Inversion: $c < 1$

For $0 < c < 1$, it is important to observe that singularities (or the wave front set) of the solution u to (5.1) are all in space-like directions for (M, g_M). From the canonical relation $C^\pm$ in (5.4), we know that for u in (5.1)

$$\mathrm{WF}(u) \subset \{(t, x, \xi_0, \xi') \in T^*M^\circ\backslash 0 : \xi_0 = \pm c|\xi'|\},$$

and $|(\xi_0, \xi')|^2_{g^*_M} = -\xi_0^2 + |\xi'|^2 = (-c^2+1)|\xi'|^2 > 0$ for $c < 1$. For such (ξ_0, ξ'), the corresponding vector in TM° is time-like. So these singularities correspond to trajectories of particles moving slower than photons in (M, g_M).

Now we can use the fact that in space-like directions, the normal operator $X^*_M \circ X_M$ is actually a pseudo-differential operator as shown in [12]. The symbol of $\Box_c$ is $p_c(\xi_0, \xi') = -\xi_0^2 + c^2|\xi'|^2$. Let $\chi(t)$ be a smooth cut-off function with $\chi(t) = 1, |t| < 1$ and $\chi(t) = 0, |t| > 1/c^2$ for $c < 1$. Then we define

$$\chi_1\left(\xi_0, \xi'\right) = \chi(\frac{\xi_0^2}{c^2|\xi'|^2})$$

so $\chi_1(\xi_0, \xi') = 1$ on $\{(\xi_0, \xi') \in \mathbb{R}^4 : p_c(\xi_0, \xi') > 0\}$ and $\chi_1(\xi_0, \xi') = 0$ on $\Omega^{*,-}M^\circ$. Let $\chi_1(D)$ be the pseudo-differential operator with symbol χ_1. We have

Lemma 6.1. *$\chi_1(D)X^*_M \circ X_M\chi_1(D)$ is a pseudo-differential operator of order -1 on M°. The principal symbol at $(t, x, \xi_0, \xi') \in T^*M^\circ$ is*

$$\frac{4\pi^2}{|\xi'|}\chi_1^2\left(\xi_0, \xi'\right).$$

Proof. It follows from Theorem 2.1 of [13] that $\chi_1(D)X_M^* \circ X_M\chi_1(D)$ is a pseudo-differential operator on M° with an oscillatory integral representation. The symbol is

$$\sigma\left(t, x, \xi_0, \xi'\right) = 2\pi|\mathbb{S}^1|\chi_1^2\left(\xi_0, \xi'\right)|\xi'|^{-1} \tag{6.1}$$

We remark that the symbol is singular at $\xi = 0$ but this can be removed by introducing a smooth cut-off function supported near $\xi = 0$ and noticing that $|\xi|^{-1}$ is integrable near $\xi = 0$. Since it only changes $\chi_1(D)X_M^* \circ X_M\chi_1(D)$ by a smoothing operator, we will not show it for simplicity. □

Now we show that

Lemma 6.2. *The normal operator $E_+^*X_M^* \circ X_M E_+$, $E_-^*X_M^* \circ X_M E_-$ are elliptic pseudo-differential operators of order -1 on $\mathbb{R}^3$, and $E_+^*X_M^* \circ X_M E_-$ and $E_-^*X_M^* \circ X_M E_+$ are smoothing operators on $\mathbb{R}^3$.*

Proof. First of all, we know that $(X_M^* \circ X_M)E_+ = (\chi_1(D)X_M^* \circ X_M\chi_1(D))E_+$ modulo a smoothing operator, thus $(X_M^* \circ X_M)E_+ \in I^{-\frac{5}{4}}(M^\circ \times \mathbb{R}^3; (C^+)')$ from the composition of a pseudo-differential operator and an FIO. The principal symbol is non-vanishing. We also know that $E_+^* \in I^{-\frac{1}{4}}(M^\circ \times \mathbb{R}^3; (C^{+,-1})')$. To compose these two operators, we would like to apply the clean composition theorem [8, Theorem 25.2.3], however, the operators are not properly supported. But this can be justified using the oscillatory integral representation. We have (modulo a pseudo-differential operator of a lower order)

$$\begin{aligned}
&E_+^*\left(X_M^* \circ X_M E_+\right)f(z)\\
&= (2\pi)^{-6}\int_{\mathbb{R}^3}\int_0^{t_1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i((z-x)\cdot\eta - ct|\eta|)}e^{i((x-y)\cdot\xi + ct|\xi|)}a(\xi)f(y)dyd\xi dxdtd\eta\\
&= (2\pi)^{-6}\int_{\mathbb{R}^3}\int_0^{t_1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i(z\cdot\eta - y\cdot\xi + x(\xi-\eta) - ct|\eta| + ct|\xi|)}a(\xi)f(y)dyd\xi dxdtd\eta\\
&= (2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i(z\cdot\xi - y\cdot\xi)}t_1 a(\xi)f(y)dyd\xi.
\end{aligned}$$

This is a pseudo-differential operator of order -1 on $\mathbb{R}^3$. The same proof works for the minus sign.

To see that $E_+^*X_M^* \circ X_M E_-$ is smoothing, we just need to observe that the canonical relations C^+, C^- in (5.4) are disjoint. So a wave front analysis using e.g. [3, Theorem 1.3.7] tells that the operator is smoothing. □

We finished the proof but we mention the following alternative argument. Essentially, we want to consider the operator E_+ for fixed t, denoted by $E_+(t)$. We know that $E_+(t) : \mathcal{E}'(\mathbb{R}^3) \to \mathcal{D}'(\mathbb{R}^3)$ is a Fourier integral operator

$$E_+(t)f(x) = (2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i((x-y)\cdot\xi + ct|\xi|)}f(y)dyd\xi$$

with canonical relation $C_t = \{(y, \eta; x, \xi) \in T^*\mathbb{R}^3\backslash 0 \times T^*\mathbb{R}^3\backslash 0 : y = x + ct\xi/|\xi|, \xi = \eta\}$. Then $E_+(t) \in I^0(\mathbb{R}^3 \times \mathbb{R}^3; C_t')$ is properly supported. The canonical relation C_t is a graph of a symplectic transformation, thus the composition $E_+^*(t)E_+(t)$ is a pseudo-differential operator of order 0 on $\mathbb{R}^3$. In our case, $E_+^*(t)X_M^*X_M E_+(t)$ is a pseudo-differential operator of order -1 and the symbols are smooth in $t \in [t_0, t_1]$. Finally, integrating the symbols in t produces a symbol and we get the result.

Now we construct a parametrix for the transform.

Proposition 6.3. *For $c < 1$, there exist operators A_1, A_2 such that*

$$A_1 X_M f = f_1 + R_1 f_1 + R_1' f_2, \quad A_2 X_M f = f_2 + R_2 f_1 + R_2' f_2$$

where R_1, R_2, R_1', R_2' are smoothing operators and $A_i = \widetilde{A}_i \circ X_M^*, i = 1, 2$ in which $\widetilde{A}_i$ are Fourier integral operators.

Proof. First, we represent $f = E_+ h_1 + E_- h_2$ and write

$$X_M f = X_M E_+ h_1 + X_M E_- h_2. \tag{6.2}$$

We apply $E_+^* X_M^*$ to get

$$E_+^* X_M^* X_M f = E_+^* X_M^* X_M E_+ h_1 + E_+^* X_M^* X_M E_- h_2 = E_+^* X_M^* X_M E_+ h_1 + R_1 h_2.$$

Since $E_+^* X_M^* X_M E_+$ is an elliptic pseudo-differential operator of order -1, we can find a parametrix B_+ which is a pseudo-differential operator of order 1 on $\mathbb{R}^3$ and

$$B_+ \circ E_+^* X_M^* X_M f = h_1 + R_1 h_1 + R_1' h_2$$

where R_1, R_1' are smoothing. We repeat the argument for the minus sign. Apply $E_-^* X_M^*$ to (6.2), we get

$$E_-^* X_M^* X_M f = E_-^* X_M^* X_M E_+ h_1 + E_-^* X_M^* X_M E_- h_2 = E_-^* X_M^* X_M E_- h_2 + R_2 h_2.$$

Apply the parametrix B_- for $E_-^* X_M^* X_M E_-$ and we get

$$B_- \circ E_-^* X_M^* X_M f = h_2 + R_2 h_1 + R_2' h_2.$$

Finally, we get

$$f_1 + R_1 f_1 + R_2 f_2 = \left(B_+ \circ E_+^* + B_- \circ E_-^*\right) X_M^* X_M f$$
$$\text{and } f_2 + R_1' f_1 + R_2' f_2 = ic\Delta^{\frac{1}{2}} \left(B_+ \circ E_+^* + B_- \circ E_-^*\right) X_M^* X_M f$$

as claimed. We set $\widetilde{A}_1 = B_+ \circ E_+^* + B_- \circ E_-^*$ which is a sum of two FIOs in $I^{3/4}(M^\circ \times \mathbb{R}^3; (C^{+,-1})')$ and $I^{3/4}(M^\circ \times \mathbb{R}^3; (C^{-,-1})')$, and $\widetilde{A}_2 = ic\Delta^{\frac{1}{2}}(B_+ \circ E_+^* + B_- \circ E_-^*)$ which is a sum of two FIOs in $I^{7/4}(M^\circ \times \mathbb{R}^3; (C^{+,-1})')$ and $I^{7/4}(M^\circ \times \mathbb{R}^3; (C^{-,-1})')$. This completes the proof. □

For convenience, we formulate a microlocal inversion result for determining f.

Corollary 6.4. *For $c < 1$, there exist operators A such that*

$$A X_M f = f + R_1 f_1 + R_2 f_2,$$

where R_1, R_2 are smoothing operators.

Proof. Again, we simply solve the wave equation (5.1) using the parametrix. In fact, it is easier to use h_1, h_2.

$$\begin{aligned} f &= E_+ h_1 + E_- h_2 = E_+ B_+ \circ E_+^* X_M^* X_M f + E_- B_- \circ E_-^* X_M^* X_M f + \widetilde{R}_1 h_1 + \widetilde{R}_2 h_2 \\ &= \left(E_+ B_+ \circ E_+^* + E_- B_- \circ E_-^*\right) X_M^* X_M f + R_1 f_1 + R_2 f_2 \end{aligned}$$

as claimed, where $\widetilde{R}_1, \widetilde{R}_2, R_1, R_2$ are smoothing operators and $A = (E_+ B_+ \circ E_+^* + E_- B_- \circ E_-^*) X_M^*$. □

7. The Microlocal Inversion: $c = 1$

For $c = 1$, the singularities of the solutions of (5.1) are all in light-like directions. As explained in the end of Sect. 4, the Schwartz kernel of $X_M^* \circ X_M$ is more complicated and the previous argument does not work directly. We will take a different approach by considering the composition $X_M \circ E_\pm$. Let φ be a smooth function on $\mathbb{S}^2$, and I^φ be the integration operator on $C^\infty(\mathbb{R}^3 \times \mathbb{S}^2)$ defined by

$$I^\varphi f(y) = \int_{\mathbb{S}^2} \varphi(v) f(y, v) dv.$$

Then we consider the composition $I^\varphi \circ X_M \circ E_\pm$ as an operator from $C^\infty(\mathscr{S}_0)$ to $C^\infty(\mathscr{S}_0)$. For technical reasons, we introduce a smooth cut-off function. For $\epsilon > 0$ small, let $\chi_\epsilon(t)$ be a smooth cut-off function on $\mathbb{R}$ such that $\chi_\epsilon(t) = 1$ for $2\epsilon < t < t_1 - 2\epsilon$ and $\chi_\epsilon(t) = 0$ for $t < \epsilon$ and $t > t_1 - \epsilon$. We prove

Proposition 7.1. *$K_\pm \doteq I^\varphi X_M \chi_\epsilon E_\pm \in \Psi^{-1}(\mathscr{S}_0)$ are pseudo-differential operators of order -1 with complete symbol $k_\pm(\xi), \xi \in \mathbb{R}^3 \backslash 0$ and the principal symbols are given by*

$$k_{+,-1}(\xi) = 2\pi i c_\epsilon |\xi|^{-1} \varphi(-\xi/|\xi|), \quad k_{-,-1}(\xi) = -2\pi i c_\epsilon |\xi|^{-1} \varphi(\xi/|\xi|),$$
$$\textit{where } c_\epsilon = \int_0^{t_1} t^{-1} \chi_\epsilon(t) dt$$

Proof. We start with K_+. We recall from (4.1) that

$$X_M f(y, v) = (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{t_1} e^{i((y-x)\cdot\eta + tv\cdot\eta)} f(t, x) dt dx d\eta$$

and from Sect. 5 that

$$E_+(f)(t, x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i((x-z)\cdot\xi + t|\xi|)} f(z) dz d\xi.$$

Consider the oscillatory integral integral representation of the Schwartz kernel K_+

$$K_+(y, z) = (2\pi)^{-6} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{t_1} \int_{\mathbb{R}^3} e^{i((y-x)\cdot\eta + tv\cdot\eta + (x-z)\cdot\xi + t|\xi|)} \varphi(v) \chi_\epsilon(t) d\xi dt dx d\eta dv \tag{7.1}$$

In this case, the oscillatory integral can be computed explicitly. But before we proceed with the calculation, we examine the phase function

$$\phi(y, z, \xi, t; \eta, x, v) = (y - x) \cdot \eta + tv \cdot \eta + (x - z) \cdot \xi + t|\xi|$$

Consider ϕ in η, x, v variables. We have

$$\phi_\eta = y - x + tv, \quad \phi_x = \xi - \eta, \quad \phi_v = t\eta|_{T_v\mathbb{S}^2},$$

so the critical points are given by

$$\xi = \eta, \quad v = \pm\xi/|\xi|, \quad x = y - t\xi/|\xi|$$

Here, we remark that $t\xi|_{T_v\mathbb{S}^2} = 0$ implies that ξ is parallel to v so $v = \pm\xi/|\xi|$. Also, we have

$$\partial^2_{(\eta,x,v)}\phi = \begin{pmatrix} 0 & -\operatorname{Id} & t \\ \operatorname{Id} & 0 & 0 \\ * & 0 & * \end{pmatrix}$$

To compute $*$, we introduce local coordinates on $\mathbb{S}^2$ near the critical point. By using an orthogonal transformation, we can assume that $\xi/|\xi| = (0, 0, 1)$. We use $v = (v_1, v_2, \pm\sqrt{1 - v_1^2 - v_2^2})$ near $\pm\xi/|\xi|$ where $v_1^2 + v_2^2 < 1$. Then we have

$$\partial_v\phi = \partial_{(v_1,v_2)}(tv\cdot\eta) = t\begin{pmatrix} \eta_1 \pm \eta_3\frac{-v_1}{\sqrt{1-v_1^2-v_2^2}} \\ \eta_2 \pm \eta_3\frac{-v_2}{\sqrt{1-v_1^2-v_2^2}} \end{pmatrix}.$$

On the set of critical point, $v = \pm(0, 0, 1)$ and $\eta = (0, 0, |\xi|)$. We observe that $\partial_v\phi = 0$. Next,

$$\partial_\eta\left(\frac{\partial\phi}{\partial v}\right) = t\begin{pmatrix} 1 & 0 & \pm\frac{-v_1}{\sqrt{1-v_1^2-v_2^2}} \\ 0 & 1 & \pm\frac{-v_2}{\sqrt{1-v_1^2-v_2^2}} \end{pmatrix} \text{ and } \partial_v\left(\frac{\partial\phi}{\partial v}\right) = \pm t\eta_3\begin{pmatrix} \frac{-1+v_2^2}{(1-v_1^2-v_2^2)^{\frac{3}{2}}} & \frac{-v_1v_2}{(1-v_1^2-v_2^2)^{\frac{3}{2}}} \\ \frac{-v_1v_2}{(1-v_1^2-v_2^2)^{\frac{3}{2}}} & \frac{-1+v_1^2}{(1-v_1^2-v_2^2)^{\frac{3}{2}}} \end{pmatrix}.$$

On critical points,

$$\partial_v\left(\frac{\partial\phi}{\partial v}\right) = \pm t|\xi|\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This shows that the phase function is non-degenerate in η, x, v. We can apply stationary phase argument so the phase becomes

$$\phi(y,z,\xi,t) = (y-z)\cdot\xi + 2t|\xi| \text{ when } v = \xi/|\xi|$$
$$\phi(y,z,\xi,t) = (y-z)\cdot\xi \text{ when } v = -\xi/|\xi|$$

Finally, after integrating in t, we will get a pseudo-differential operator. This will be shown explicitly in the follows.

First, in (7.1), we integrate in x, η to get

$$K_+(y,z) = (2\pi)^{-3}\int_{\mathbb{S}^2}\int_0^{t_1}\int_{\mathbb{R}^3} e^{i(y\cdot\xi+tv\cdot\xi-z\cdot\xi+t|\xi|)}\varphi(v)\chi_\epsilon(t)d\xi dt dv$$

Consider the integral in v. For t non-zero, the v integral is non-degenerate with stationary points at $v = \pm\xi/|\xi|$. Applying stationary phase argument see e.g. [15, Lemma 1.2], we get

$$\begin{aligned} K_+(y,z) = (2\pi)^{-3}&\int_0^{t_1}\int_{\mathbb{R}^3} e^{i((y-z)\cdot\xi+2t|\xi|)}\left(\varphi(\xi/|\xi|)\right. \\ &\left.+\varphi^+(t,\xi)\right)\chi_\epsilon(t)\,(t|\xi|)^{-1}e^{-\frac{1}{2}i\pi}(2\pi)d\xi dt \\ &+(2\pi)^{-3}\int_0^{t_1}\int_{\mathbb{R}^3} e^{i(y-z)\cdot\xi}\left(\varphi(-\xi/|\xi|)\right. \\ &\left.+\varphi^-(t,\xi)\right)\chi_\epsilon(t)(t|\xi|)^{-1}e^{\frac{1}{2}i\pi}(2\pi)d\xi dt \\ =(2\pi)^{-3}&\int_{\mathbb{R}^3} e^{i(y-z)\cdot\xi}k_+(\xi)d\xi \end{aligned} \tag{7.2}$$

where $\varphi^{\pm}$ come from the stationary phase argument and they have asymptotic expansions

$$\varphi^{\pm}(t,\xi) \sim \sum_{k=1}^{\infty} a_k^{\pm}(\xi/|\xi|)(t|\xi|)^{-k} \tag{7.3}$$

in which $a_k^{\pm}$ are smooth functions on $\mathbb{S}^2$. Also,

$$\begin{aligned} k_+(\xi) &= +2\pi i|\xi|^{-1}\varphi(-\xi/|\xi|)\int_0^{t_1} t^{-1}\left(1+\varphi^-(t,\xi)\right)\chi_\epsilon(t)dt \\ &\quad - 2\pi i|\xi|^{-1}\varphi(\xi/|\xi|)\int_0^{t_1} e^{2it|\xi|}t^{-1}(1+\varphi^+(t,\xi))\chi_\epsilon(t)dt \end{aligned} \tag{7.4}$$

The second integral in t is $O(|\xi|^{-\infty})$ for $|\xi|$ large because t is away from 0 and χ_ϵ is smooth. For the first integral, the integral of each asymptotic term of φ^- in (7.3) in t is finite. Thus $k_+(\xi)$ is a symbol of order -1 and the leading order term is

$$k_{+,-1}(\xi) = 2\pi i|\xi|^{-1}\varphi(-\xi/|\xi|)\int_0^{t_1} t^{-1}\chi_\epsilon(t)dt.$$

This shows that K_+ in (7.2) is a pseudo-differential operator of order -1 on $\mathbb{R}^3$.

For K_-, the calculation is similar and we look for the symbol.

$$\begin{aligned} K_-(y,z) &= (2\pi)^{-3}\int_{t_0}^{t_1}\int_{\mathbb{R}^3} e^{i(y\cdot\xi+tv\cdot\xi-z\cdot\xi-t|\xi|)}\chi_\epsilon(t)d\xi dt dv \\ &= -i(2\pi)^{-2}\int_{t_0}^{t_1}\int_{\mathbb{R}^3} e^{i(y-z)\cdot\xi}(t|\xi|)^{-1}\left(\varphi(\xi/|\xi|)+\widetilde{\varphi}^+(t,\xi)\right)\chi_\epsilon(t)d\xi dt \\ &\quad + i(2\pi)^{-2}\int_{t_0}^{t_1}\int_{\mathbb{R}^3} e^{i((y-z)\cdot\xi-2t|\xi|)}(t|\xi|)^{-1}\left(\varphi(-\xi/|\xi|)+\widetilde{\varphi}^-(t,\xi)\right)\chi_\epsilon(t)d\xi dt \\ &= (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i(y-z)\cdot\xi}k_-(\xi)d\xi \end{aligned}$$

where $\widetilde{\varphi}^{\pm}$ have similar asymptotic expansion as (7.3), and $k_-(\xi)$ is given by

$$\begin{aligned} k_-(\xi) &= -2\pi i|\xi|^{-1}\int_0^{t_1}\left(\varphi(\xi/|\xi|)+\widetilde{\varphi}^+(t,\xi)\right)t^{-1}\chi_\epsilon(t)dt \\ &\quad + 2\pi i|\xi|^{-1}\int_0^{t_1} e^{-2it|\xi|}\left(\varphi(-\xi/|\xi|)+\widetilde{\varphi}^-(t,\xi)\right)t^{-1}\chi_\epsilon(t)dt \end{aligned} \tag{7.5}$$

This is a symbol of order -1 and the leading order term is

$$k_{-,-1}(\xi) = -2\pi i|\xi|^{-1}\varphi(\xi/|\xi|)\int_0^{t_1} t^{-1}\chi_\epsilon(t)dt$$

This completes the proof of the proposition. $\square$

Next we discuss what needs to be changed when the smooth cut-off function χ_ϵ is replaced by the characteristic function $\chi_{[\epsilon,t_1]}$ of the interval $[\epsilon, t_1]$ in $\mathbb{R}$. All the calculations in Proposition 7.1 hold up to (7.4) which is now

$$\begin{aligned} k_+(\xi) = &+2\pi i|\xi|^{-1}\varphi(-\xi/|\xi|)\int_\epsilon^{t_1} t^{-1}\left(1+\varphi^-(t,\xi)\right)dt \\ &-2\pi i|\xi|^{-1}\varphi(\xi/|\xi|)\int_\epsilon^{t_1} e^{2it|\xi|}t^{-1}\left(1+\varphi^+(t,\xi)\right)dt \end{aligned} \tag{7.6}$$

The first integral, denoted by I_1 below, still gives a symbol of order -1. For the second integral denoted by I_2 below, integration by parts gives

$$\begin{aligned} I_2(\xi) = &-2\pi i|\xi|^{-1}\varphi(\xi/|\xi|)\{\frac{1}{2i|\xi|}(e^{2it_1|\xi|}t_1^{-1}(1+\varphi^+(t_1,\xi))) \\ &-\frac{1}{2i|\xi|}e^{2i\epsilon|\xi|}\epsilon^{-1}(1+\varphi^+(\epsilon,\xi)) \\ &-\frac{1}{2i|\xi|}\int_\epsilon^{t_1} e^{2it|\xi|}\frac{d}{dt}[t^{-1}(1+\varphi^+(t,\xi))]dt\} \end{aligned}$$

We can repeat the integration by parts and get

$$I_2(\xi) = e^{2it_1|\xi|}a(\xi) + e^{2i\epsilon|\xi|}b(\xi)$$

where $a(\xi), b(\xi)$ are symbols of order -2. Using these in (7.2), we get

$$\begin{aligned} K_+(y,z) =&(2\pi)^{-3}\int_{\mathbb{R}^3} e^{i(y-z)\cdot\xi}I_1(\xi)d\xi + (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i(y-z)\cdot\xi+2it_1|\xi|}a(\xi)d\xi \\ &+(2\pi)^{-3}\int_{\mathbb{R}^3} e^{i(y-z)\cdot\xi+2i\epsilon|\xi|}b(\xi)d\xi. \end{aligned}$$

Thus, we can write $K_+ = K_+^0 + K_+^\epsilon + K_+^{t_1}$ where $K_+^0 \in \Psi^{-1}(\mathbb{R}^3)$, and $K_+^\epsilon \in I^{-2}(\mathbb{R}^3,\mathbb{R}^3;C_\epsilon)$, $K_+^{t_1} \in I^{-2}(\mathbb{R}^3,\mathbb{R}^3;C_{t_1})$ are Fourier integral operators of order -2. The canonical relation C_ϵ, C_{t_1} can be described as follows. For $\alpha\in\mathbb{R}$, we define

$$C_\alpha = \{(y,\eta,z,\zeta)\in T^*\mathbb{R}^3\backslash 0\times T^*\mathbb{R}^3\backslash 0 : y = z+2\alpha\xi/|\xi|, \xi=\eta\}.$$

We see that C_α is a graph of a canonical transformation, see [8, Section 25.3]. The same argument shows that K_- is also a sum of $K_-^0 \in \Psi^{-1}(\mathbb{R}^3)$ and $K_-^\epsilon \in I^{-2}(\mathbb{R}^3,\mathbb{R}^3;C_{-\epsilon})$, $K_-^{t_1} \in I^{-2}(\mathbb{R}^3,\mathbb{R}^3;C_{-t_1})$.

Now we are ready to obtain a parallel result of Proposition 6.3 about the microlocal inversion.

Proposition 7.2. *For $c=1$ and any $N\in\mathbb{N}$, there exist operators A_1, A_2 such that*

$$A_1X_M\chi_{[\epsilon,t_1]}f = h_1 + R_1h_1 + R_1'h_2, \quad A_2X_M\chi_{[\epsilon,t_1]}f = h_2+R_2h_1+R_2'h_2$$

where h_1, h_2 are defined in Sect. 5 and $R_1, R_1', R_2, R_2' \in I^{-N}(\mathbb{R}^3,\mathbb{R}^3;C^N_{\epsilon,t_1})$ which is the N-fold composition of elements in $I^{-1}(\mathbb{R}^3,\mathbb{R}^3;C_{\pm\epsilon})$ and $I^{-1}(\mathbb{R}^3,\mathbb{R}^3;C_{\pm t_1})$, more explicitly

$$\begin{aligned} I^{-N}(\mathbb{R}^3,\mathbb{R}^3;C^N_{\epsilon,t_1}) = \{&A_1\circ A_2\cdots A_N : A_i\in I^{-1}(\mathbb{R}^3,\mathbb{R}^3;C_\epsilon)+I^{-1}(\mathbb{R}^3,\mathbb{R}^3;C_{t_1}) \\ &+I^{-1}(\mathbb{R}^3,\mathbb{R}^3;C_{-\epsilon})+I^{-1}(\mathbb{R}^3,\mathbb{R}^3;C_{-t_1})\}. \end{aligned}$$

Proof. We divide the proof in two steps.

Step 1: Let's replace $\chi_{[\epsilon,t_1]}$ with the smooth cut-off χ_ϵ as in Proposition 7.1 and see how to get h_1, h_2 using Proposition 7.1. We write

$$X_M\chi_\epsilon f = X_M\chi_\epsilon E_+h_1 + X_M\chi_\epsilon E_-h_2.$$

Let φ be a smooth function on $\mathbb{S}^2$. Applying I^φ we get

$$I^\varphi X_M\chi_\epsilon f = I^\varphi X_M\chi_\epsilon E_+h_1 + I^\varphi X_M\chi_\epsilon E_-h_2 = K^{\varphi,+}h_1 + K^{\varphi,-}h_2 \tag{7.7}$$

where we added φ to the notation of $K_\pm$ to emphasize the dependency because we will choose different φ below.

First, let $\varphi_1 = 1$. From Proposition 7.1, we see that $K_\pm^{\varphi_1} \in \Psi^{-1}(\mathbb{R}^3)$ and the principal symbols are given by

$$k_{+,-1}^{\varphi_1}(\xi) = -k_{-,-1}^{\varphi_1}(\xi) = 2\pi i c_\epsilon|\xi|^{-1}.$$

We let Q_+^1 be a parametrix of $K_+^{\varphi_1}$ and get

$$Q_+^1 I^{\varphi_1}X_M\chi_\epsilon f = h_1 + Q_+^1K_-^{\varphi_1}h_2 + R_1h_1 \tag{7.8}$$

where R_1, R_2 are smoothing operators. From the composition of pseudo-differential operators, we know that $Q_+^1K_-^{\varphi_1} \in \Psi^0(\mathbb{R}^3)$ with principal symbol equal to -1.

Next, we change the function φ. Ideally, we will take an odd function $\varphi(-v) = -\varphi(v)$ but then φ vanishes somewhere on $\mathbb{S}^2$ so we proceed as follows. Let $x = (x_1, x_2, x_3)$ be the coordinate for $\mathbb{R}^3$. For $\delta > 0$, let $\mathscr{U}_k = \{v : v = (x_1, x_2, x_3), \|x\| = 1, |x_k| > \delta/2\}$, $k = 1, 2, 3$. For δ sufficiently small, $\mathscr{U}_k$, $k = 1, 2, 3$ form an open covering of $\mathbb{S}^2$. Let $\chi_k(v)$, $k = 1, 2, 3$ be a partition of unity subordinated to this covering and $\chi_k(v) = 1$ on $\mathscr{V}_k = \{v : v = (x_1, x_2, x_3), \|x\| = 1, |x_k| > \delta\}$, $k = 1, 2, 3$. Here, by possibly taking δ smaller, we can assume that $\mathscr{V}_k$ also form an open covering of $\mathbb{S}^2$. For $v \in \mathbb{S}^2$, we let

$$\varphi_{2,k}(v) = \chi_k(x)x_k + 2, \quad k = 1, 2, 3$$

Then $\varphi_2(v) \neq 0$ and $\varphi_{2,k}(-v) - \varphi_2(v, k) \neq 0$ for $v \in \mathscr{U}_k$. From Proposition 7.1, we know that $K_\pm^{\varphi_{2,k}} \in \Psi^{-1}(\mathbb{R}^3)$ with principal symbols

$$k_{+,-1}^{\varphi_{2,k}}(\xi) = 2\pi i c_\epsilon|\xi|^{-1}\varphi_{2,k}(-\xi/|\xi|), \quad k_{-,-1}^{\varphi_{2,k}}(\xi) = -2\pi i c_\epsilon|\xi|^{-1}\varphi_{2,k}(\xi/|\xi|).$$

We consider $k = 1$ in the follows as the other cases are similar. Let $Q_+^{2,1}$ be a parametrix for $K_+^{\varphi_{2,1}}$. We get

$$Q_+^{2,1}I^{\varphi_{2,1}}X_M\chi_\epsilon f = h_1 + Q_+^{2,1}K_-^{\varphi_{2,1}}h_2 + R_3h_1$$

where R_3 is a smoothing operator, and $Q_+^{2,1}K_-^{\varphi_{2,1}} \in \Psi^0(\mathbb{R}^3)$ with principal symbol

$$\sigma_0\left(Q_+^{2,1}K_-^{\varphi_{2,1}}\right)(x,\xi) = -\frac{\varphi_{2,1}(\xi/|\xi|)}{\varphi_{2,1}(-\xi/|\xi|)} \neq -1 \tag{7.9}$$

when $\xi/|\xi| \in \mathscr{U}_1$. Now we consider

$$Q_+^1I^{\varphi_1}X_M\chi_\epsilon f - Q_+^{2,1}I^{\varphi_{2,1}}X_M\chi_\epsilon f = (Q_+^1K_-^{\varphi_1} - Q_+^{2,1}K_-^{\varphi_{2,1}})h_2 + R_1h_1 + R_2h_1 - R_3h_1$$

We observe that $A = Q^1_+ K^{\varphi_1}_- - Q^{2,1}_+ K^{\varphi_{2,1}}_-$ is a pseudo-differential operator of order 0 and the principal symbol does not vanish on $\mathscr{U}_1$. Let $\widetilde{\chi}_k$, $k = 1, 2, 3$ be a smooth partition of unity subordinated to $\mathscr{V}_k$. Then $\chi_1\widetilde{\chi}_1 = \widetilde{\chi}_1$. Let B_1 be a pseudo-differential operator of order 0 with principal symbol $\sigma_0(B_1)(\xi) = \widetilde{\chi}_1(\xi/|\xi|)/\sigma_0(A)(\xi)$. We can improve B_1 to a parametrix for A so that $B_1 \circ A = \widetilde{\chi}_1(D) + R_4$ with R_4 smoothing. So we get

$$B_1(Q^1_+ I^{\varphi_1} X_M \chi_\epsilon - Q^{2,1}_+ I^{\varphi_{2,1}} X_M \chi_\epsilon) f = \widetilde{\chi}_1(D) h_2 + R_3 h_2 + R_4 h_1$$

where by abusing notations, R_3, R_4 are smoothing operators. We can repeat the construction for $k = 2, 3$ to get the corresponding $B_2, B_3 \in \Psi^0(\mathbb{R}^3)$. Then we arrive at

$$\sum_{k=1}^{3} B_k(Q^1_+ I^{\varphi_1} X_M \chi_\epsilon - Q^{2,k}_+ I^{\varphi_{2,k}} X_M \chi_\epsilon) f = h_2 + R_5 h_2 + R_6 h_1 \tag{7.10}$$

with R_5, R_6 smoothing. This gives $A_2 = \sum_{k=1}^3 B_k(Q^1_+ I^{\varphi_1} - Q^{2,k}_+ I^{\varphi_{2,k}})$ so that $A_2 X_M \chi_\epsilon f = h_2 + R_6 h_1 + R_5 h_2$. For A_1, we can use (7.8) and (7.10) to get

$$Q^1_+ I^{\varphi_1} X_M \chi_\epsilon f = h_1 + Q^1_+ K^{\varphi_1,-} A_2 X_M \chi_\epsilon f + R'_5 h_1 + R'_6 h_2$$

where R'_5, R'_6 are smoothing operators. So we obtain $A_1 = Q^1_+ I^{\varphi_1} - Q^1_+ K^{\varphi_1}_- A_2$ so that $A_1 X_M \chi_\epsilon f = h_1 + R'_5 h_1 + R'_6 h_2$.

Step 2: Now we deal with the characteristic function $\chi_{[\epsilon,t_1]}$. We start with

$$X_M \chi_{[\epsilon,t_1]} f = X_M \chi_{[\epsilon,t_1]} E_+ h_1 + X_M \chi_{[\epsilon,t_1]} E_- h_2.$$

Applying I^φ, we get

$$I^\varphi X_M \chi_{[\epsilon,t_1]} f = K^\varphi_+ h_1 + K^\varphi_- h_2$$

where $K^\varphi_\pm = I^\varphi X_M \chi_{[\epsilon,t_1]} E_\pm$. According to the arguments after Proposition 7.1, we can write the above as

$$I^\varphi X_M \chi_{[\epsilon,t_1]} f = (K^{\varphi,0}_+ + K^{\varphi,\epsilon}_+ + K^{\varphi,t_1}_+) h_1 + (K^{\varphi,0}_- + K^{\varphi,\epsilon}_- + K^{\varphi,t_1}_-) h_2 \tag{7.11}$$

where $K^{\varphi,0}_\pm \in \Psi^{-1}(\mathbb{R}^3)$, $K^{\varphi,\epsilon}_\pm \in I^{-2}(\mathbb{R}^3, \mathbb{R}^3; C_{\pm\epsilon})$ and $K^{\varphi,t_1}_\pm \in I^{-2}(\mathbb{R}^3, \mathbb{R}^3; C_{\pm t_1})$. As in Step 1, we can apply pseudo-differential operators $Q^1_+, Q^{2,k}_+, k = 1, 2, 3$ to (7.11). The arguments for $K^{\varphi,0}_\pm$ are the same as before. As for $K^{\varphi,\epsilon}_\pm, K^{\varphi,t_1}_\pm$, we notice that the composition $Q^1_+ K^{\varphi,j}_\pm$, $Q^{2,k}_+ K^{\varphi,j}_\pm$, $k = 1, 2, 3$, $j = \epsilon, t_1$ are all Fourier integral operators of order -1 with canonical relation $C_{\pm\epsilon}$ or $C_{\pm t_1}$. Therefore, using the same A_1, A_2 in Step 1, we obtain

$$A_1 X_M \chi_{[\epsilon,t_1]} f = h_1 + R_1 h_1 + R'_1 h_2, \quad A_2 X_M \chi_{[\epsilon,t_1]} f = h_2 + R_2 h_1 + R'_2 h_2 \tag{7.12}$$

where $R_1, R'_1, R_2, R'_2 \in I^{-1}(\mathbb{R}^3, \mathbb{R}^3; C_\epsilon) + I^{-1}(\mathbb{R}^3, \mathbb{R}^3; C_{t_1}) + I^{-1}(\mathbb{R}^3, \mathbb{R}^3; C_{-\epsilon}) + I^{-1}(\mathbb{R}^3, \mathbb{R}^3; C_{-t_1})$.

Finally, we improve the remainder term using the Neumann series. We write (7.12) in matrix form

$$\begin{pmatrix} A_1 X_M \chi_{[\epsilon,t_1]} f \\ A_2 X_M \chi_{[\epsilon,t_1]} f \end{pmatrix} = \mathrm{Id} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + R \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad R = \begin{pmatrix} R_1 & R'_1 \\ R_2 & R'_2 \end{pmatrix}.$$

For $N \in \mathbb{N}$, we let $W = \sum_{n=0}^{N-1}(-R)^n$ and get

$$W \begin{pmatrix} A_1 X_M \chi_{[\epsilon,t_1]} f \\ A_2 X_M \chi_{[\epsilon,t_1]} f \end{pmatrix} = \mathrm{Id} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + R^N \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

Because R_1, R_1', R_2, R_2' are FIOs of the canonical graph type, we can apply the composition result in [8, Section 25.3] to conclude that the terms in R^N belongs to $I^{-N}(\mathbb{R}^3, \mathbb{R}^3; C^N_{\epsilon,t_1})$. Finally, we set

$$\begin{pmatrix} \widetilde{A}_1 \\ \widetilde{A}_2 \end{pmatrix} = W \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

Changing notations of $\widetilde{A}_1, \widetilde{A}_2$ to A_1, A_2 finishes the proof. □

8. The Stable Determination

We prove Theorem 1.1, starting with the injectivity of the light ray transform. It is known, see for instance [10,17], that the light ray transform on $\mathbb{R}^{n+1}$ is injective on C_0^∞ functions. This also holds for L^1_{comp} functions and the proof is similar, see [17].

Theorem 8.1. *Suppose $f \in L^1_{\mathrm{comp}}(\mathbb{R}^{n+1})$, $n \geq 2$ and $X_{\mathbb{R}^{n+1}} f = 0$. Then $f = 0$.*

Proof. For $f \in L^1_{\mathrm{comp}}(\mathbb{R}^{n+1})$, the Fourier transform $\hat{f}$ is analytic. Let $\theta \in \mathbb{S}^{n-1}$ and $\Theta = (1, \theta)$ be a light-like vector. Let $z = (s, y + s\theta) \in \mathbb{R}^{n+1}$, $s \in \mathbb{R}$, $y \in \mathbb{R}^n$. We parametrize the light ray transform as

$$X_{\mathbb{R}^{n+1}} f(z, \Theta) = \int_{\mathbb{R}} f(t, y + t\theta)\, dt.$$

From the standard Fourier Slice Theorem for geodesic ray transforms on $\mathbb{R}^{n+1}$, we get

$$\hat{f}(\zeta) = \int_{\Theta^\perp} e^{-iy\cdot\zeta} X_{\mathbb{R}^{n+1}} f(z, \Theta) dS_z$$

where the integration is over the hyperplane $\Theta^\perp$ perpendicular to Θ with respect to the Euclidean inner product in $\mathbb{R}^{n+1}$ and $\zeta = (\tau, \xi) \in \mathbb{R}^{n+1}$, $\xi \in \mathbb{R}^n$, $\xi \neq 0$ is perpendicular to Θ. We notice that if $|\tau| \leq |\xi|$, then there is a null vector $(1, \theta)$ which is Euclidean orthogonal to ζ. Actually, $\tau + \theta \cdot \xi = 0$ so $\theta \cdot (\xi/|\xi|) = -\tau/|\xi| \in [-1, 1]$ and we can find $\theta \in \mathbb{S}^{n-1}$. We conclude that $\hat{f}(\zeta) = 0$ for $|\tau| \leq |\xi|$. By analyticity, $\hat{f} = 0$ and thus $f = 0$. □

Corollary 8.2. *Suppose $X_M f = 0$ where f satisfies the wave equation constraint (1.2) in which $f_1 \in H^{s+1}_{\mathrm{comp}}(\mathbb{R}^3)$, $f_2 \in H^s_{\mathrm{comp}}(\mathbb{R}^3)$, $s \geq 0$ are compactly supported. Then $f = f_1 = f_2 = 0$.*

Proof. Let $K = \operatorname{supp} f_1 \cup \operatorname{supp} f_2 \subset \mathbb{R}^3$. Let $I^+_c(K)$ be the chronological future of K with respect to the Lorentzian metric induced by c. We know that there is a unique solution $f \in H^{s+1}(M)$ of (1.2). By finite speed of propagation (or strong Huygens principle), the solution f is supported in $I^+_c(K) \cap M$. Now we extend f trivially to $\widetilde{f} \in L^1_{\mathrm{comp}}(\mathbb{R}^4)$ and we regard X_M as the light ray transform $X_{\mathbb{R}^4}$ on $\mathbb{R}^4$. We still have $X_{\mathbb{R}^4} \widetilde{f} = 0$. By Theorem 8.1, we conclude that $f = 0$ on $\mathbb{R}^4$ so that $f = 0$ on M and $f_1 = f_2 = 0$ on $\mathscr{S}_0$. □

Proof of Theorem 1.1. The uniqueness part is done in Corollary 8.2. So we prove the stability estimate below. We divide the proof into three steps.

Step 1: Consider $c < 1$. From Proposition 6.3, we know that there are operators A_1, A_2 such that

$$A_1 X_M f = f_1 + R_1 f_1 + R_1' f_2, \quad A_2 X_M f = f_2 + R_2 f_1 + R_2' f_2$$

and $R_i, R_i', i = 1, 2$ are all smoothing operators. We denote

$$T\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathrm{Id}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + K\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad K = \begin{pmatrix} R_1 & R_1' \\ R_2 & R_2' \end{pmatrix}.$$

We consider T acting on $\mathcal{N}^s$, $s \geq 0$. Then K is compact from $\mathcal{N}^s$ to $\mathcal{N}^{s-\rho}$, $\rho \in \mathbb{R}$. So we have the estimate

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \leq \|A_1 X_M f\|_{H^{s+1}(\mathbb{R}^3)} + \|A_2 X_M f\|_{H^s(\mathbb{R}^3)} + C_\rho \|(f_1, f_2)\|_{\mathcal{N}^{s-\rho}}$$

for some constant C_ρ. Recall from Proposition 6.3 that $A_1 = B_+ \circ (X_M \circ E_+)^* + B_-(X_M \circ E_-)^*$ and $A_2 = ic\Delta^{\frac{1}{2}}(B_+ \circ (X_M \circ E_+)^* + B_- \circ (X_M \circ E_-)^*)$. Since the normal operator $(X_M E_\pm)^* X_M E_\pm$ are pseudo-differential operators of order -1. By the L^2 estimate of pseudo-differential operators, we conclude that $X_M \circ E_\pm : H^s_{\mathrm{comp}}(\mathbb{R}^3) \to H^{s+\frac{1}{2}}_{\mathrm{loc}}(\mathscr{C})$ is bounded. Also, $(X_M \circ E_\pm)^* : H^s_{\mathrm{comp}}(\mathscr{C}) \to H^{s+\frac{1}{2}}_{\mathrm{loc}}(\mathbb{R}^3)$ is bounded. Therefore, $A_1 : H^{s+\frac{1}{2}}_{\mathrm{comp}}(\mathscr{C}) \to H^s_{\mathrm{loc}}(\mathbb{R}^3)$ and $A_2 : H^{s+\frac{1}{2}}_{\mathrm{comp}}(\mathscr{C}) \to H^{s-\frac{1}{2}}_{\mathrm{loc}}(\mathbb{R}^3)$ are bounded. For $(f_1, f_2) \in \mathcal{N}^s$, we know from (5.2) that $X_M f = X_M E_+ h_1 + X_M E_- h_2$ and $h_1, h_2 \in H^{s+1}(\mathbb{R}^3)$. Thus, $X_M f \in H^{s+3/2}(\mathscr{C})$ so we get

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \leq C\|X_M f\|_{H^{s+3/2}(\mathscr{C})} + C_\rho \|(f_1, f_2)\|_{\mathcal{N}^{s-\rho}} \tag{8.1}$$

where $C_\rho > 0$ is a constant depending on ρ. Note that the order is better than what claimed in the theorem for this case.

Step 2: Consider $c = 1$. It is convenient to work with $t_0 > 0$ which can be always arranged. For the Cauchy problem in Sect. 3 with initial condition on $t = t_0$

$$\begin{aligned} \Box f = 0, \quad &\text{on } \mathbb{R} \times \mathbb{R}^3 \\ f = f_1, \quad \partial_t f = f_2, \ &\text{on } \{t_0\} \times \mathbb{R}^3, \end{aligned} \tag{8.2}$$

it is known that

$$U(t) : (f_1, f_2) \to (f(t), \partial_t f(t)), \quad t \in \mathbb{R}$$

is bijective on $H^{s+1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$. In fact, for (8.2), $U(t)$ is a unitary operator with respect to the energy norm. We consider $(\widetilde{f_1}, \widetilde{f_2}) = U(-t_0)(f_1, f_2)$ which is the Cauchy data at $t = 0$ corresponding to (f_1, f_2) at $t = t_0$. Then we have

$$\begin{aligned} C_1(\|\widetilde{f_1}\|_{H^{s+1}(\mathbb{R}^3)} + \|\widetilde{f_2}\|_{H^s(\mathbb{R}^3)}) \leq &\|f_1\|_{H^{s+1}(\mathbb{R}^3)} + \|f_2\|_{H^s(\mathbb{R}^3)} \\ \leq &C_2(\|\widetilde{f_1}\|_{H^{s+1}(\mathbb{R}^3)} + \|\widetilde{f_2}\|_{H^s(\mathbb{R}^3)}) \end{aligned} \tag{8.3}$$

for some $C_1, C_2 > 0$, which follows from the energy estimate of the wave equation. We observe that the solution of (8.2) on $[t_0, t_1] \times \mathbb{R}^3$ can be expressed as

$$f = \chi_{[t_0,t_1]} E\left(\widetilde{f_1}, \widetilde{f_2}\right)$$

where $E(\widetilde{f}_1, \widetilde{f}_2) = E_+\widetilde{h}_1 + E_-\widetilde{h}_2$ is the solution operator for the Cauchy problem from $t = 0$ in (5.2) and $\widetilde{h}_1, \widetilde{h}_2$ correspond to $\widetilde{f}_1, \widetilde{f}_2$, see Sect. 5. Therefore, we can apply Proposition 7.2 to the operator $X_M \chi_{[t_0,t_1]} E_\pm$ with $t_0 > 0$.

From Proposition 7.2, for any $\rho \in \mathbb{N}$, there are operators A_1, A_2 such that

$$A_1 X_M \chi_{[t_0,t_1]} f = \widetilde{h}_1 + R_1\widetilde{h}_1 + R_1'\widetilde{h}_2, \quad A_2 X_M \chi_{[t_0,t_1]} f = \widetilde{h}_2 + R_2\widetilde{h}_1 + R_2'\widetilde{h}_2$$

and $R_i, R_i', i = 1, 2$ are FIOs of order $-\rho$. By the same argument in Step 1 and using Sobolev estimate of FIOs of canonical graph type, we have

$$\begin{aligned}\|\widetilde{h}_1\|_{H^s(\mathbb{R}^3)} + \|\widetilde{h}_2\|_{H^s(\mathbb{R}^3)} \leq &\|A_1 X_M f\|_{H^{s+1}(\mathbb{R}^3)} + \|A_2 X_M f\|_{H^s(\mathbb{R}^3)} \\ &+ C_\rho(\|\widetilde{h}_1\|_{H^{s-\rho}(\mathbb{R}^3)} + \|\widetilde{h}_2\|_{H^{s-\rho}(\mathbb{R}^3)})\end{aligned}$$

for some constant C_ρ. Using (5.5), we can change the estimate of $\widetilde{h}_1, \widetilde{h}_2$ to that of $\widetilde{f}_1, \widetilde{f}_2$ and get

$$\begin{aligned}\| \left(\widetilde{f}_1, \widetilde{f}_2\right) \|_{\mathcal{N}^s} \leq &\|A_1 X_M \chi_{[t_0,t_1]} f\|_{H^{s+1}(\mathbb{R}^3)} \\ &+ \|A_2 X_M \chi_{[t_0,t_1]} f\|_{H^s(\mathbb{R}^3)} + C_\rho \| \left(\widetilde{f}_1, \widetilde{f}_2\right) \|_{\mathcal{N}^{s-\rho}}\end{aligned}$$

Finally, using (8.3), we get

$$\| (f_1, f_2) \|_{\mathcal{N}^s} \leq \|A_1 X_M f\|_{H^{s+1}(\mathbb{R}^3)} + \|A_2 X_M f\|_{H^s(\mathbb{R}^3)} + C_\rho \|(f_1, f_2)\|_{\mathcal{N}^{s-\rho}}$$

Now recall from the proof of Proposition 7.2 that

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = W \begin{pmatrix} \widetilde{A}_1 \\ \widetilde{A}_2 \end{pmatrix} \text{ where } \widetilde{A}_1 = Q_+^1 I^{\varphi_1} - Q_+^1 K^{\varphi_1,-} \widetilde{A}_2,$$

$$\widetilde{A}_2 = \sum_{k=1}^{3} B_k (Q_+^1 I^{\varphi_1} - Q_+^{2,k} I^{\varphi_{2,k}})$$

in which $Q_+^1, Q_+^{2,k} \in \Psi^1(\mathbb{R}^3)$, $k = 1, 2, 3$, $B_k \in \Psi^0(\mathbb{R}^3)$, $K^{\varphi_1,-} \in \Psi^{-1}(\mathbb{R}^3)$ and $W = \sum_{n=0}^{\rho-1}(-R)^n$ with elements of R belonging to $I^{-1}(\mathbb{R}^3, \mathbb{R}^3; C_{t_0}) + I^{-1}(\mathbb{R}^3, \mathbb{R}^3; C_{t_1}) + I^{-1}(\mathbb{R}^3, \mathbb{R}^3; C_{-t_0}) + I^{-1}(\mathbb{R}^3, \mathbb{R}^3; C_{-t_1})$. Using the estimate for pseudo-differential operators and FIOs of canonical graph type, we get

$$\begin{aligned}&\|A_1 X_M f\|_{H^{s+1}(\mathbb{R}^3)} + \|A_2 X_M f\|_{H^s(\mathbb{R}^3)} \\ &\leq C\|I^{\varphi_1} X_M f\|_{H^{s+2}(\mathbb{R}^3)} + C \sum_{k=1}^{3} \|I^{\varphi_{2,k}} X_M f\|_{H^{s+1}(\mathbb{R}^3)} \leq C\|X_M f\|_{H^{s+2}}\end{aligned}$$

So in this case, we get

$$\| (f_1, f_2) \|_{\mathcal{N}^s} \leq C\|X_M f\|_{H^{s+2}(\mathscr{C})} + C_\rho \| (f_1, f_2) \|_{\mathcal{N}^{s-\rho}}, \quad \rho \in \mathbb{N}. \tag{8.4}$$

Step 3: We get rid of the last term in (8.1) and (8.4). Let $\mathscr{K}$ be a compact subset of $\mathbb{R}^3$ and denote by $\mathcal{N}^s(\mathscr{K})$ the function space consisting of $(f_1, f_2) \in \mathcal{N}^s$ supported in $\mathscr{K}$. Then the inclusion of $\mathcal{N}^s(\mathscr{K})$ into $\mathcal{N}^{s-\rho}(\mathscr{K})$, $\rho > 0$ is compact. We claim that

$$\| (f_1, f_2) \|_{\mathcal{N}^s(\mathscr{K})} \leq C\|X_M f\|_{H^{s+2}(\mathscr{C})}$$

for some $C > 0$. We argue by contradiction. Assume the estimate without the error term is not true. We can get a sequence $(f_1^{(j)}, f_2^{(j)})$, $j = 1, 2, \dots$ with unit norm in $\mathcal{N}^s(\mathscr{K})$ such that $X_M f^{(j)}$ goes to 0 in $H^{s+2}(\mathscr{C})$ as $j \to \infty$. By (8.1) (for (f_1, f_2) supported in $\mathscr{K}$), we conclude that $1 = \|(f_1^{(j)}, f_2^{(j)})\|_{\mathcal{N}^s(\mathscr{K})} \le C_\rho \|(f_1^{(j)}, f_2^{(j)})\|_{\mathcal{N}^{s-\rho}(\mathscr{K})}$. This gives a weak limit (f_1, f_2) in $\mathcal{N}^s(\mathscr{K})$ along a subsequence, which thus converges strongly in $\mathcal{N}^{s-\rho}(\mathscr{K})$. Therefore, $\|(f_1, f_2)\|_{\mathcal{N}^{s-\rho}(\mathscr{K})}$ is bounded below by $1/C_\rho$, thus non-zero. However, $X_M f = 0$ so $f = 0$ by the injectivity of X_M. So $(f_1, f_2) = 0$ a contradiction. This finishes the proof. □

Finally, we prove a stronger version of Theorem 1.1 which allows lower order terms in the wave equation. We consider differential operators of the form

$$P(x, t, D_x, \partial_t) = \partial_t^2 + c^2 \sum_{i=1}^{3} D_{x_i}^2 + P_1(x, t, iD_x, \partial_t) + P_0(x, t)$$

where P_1 is a first order differential operator with real valued smooth coefficients and P_0 is smooth. Then we consider the Cauchy problem

$$\begin{aligned} P(x, t, D_x, \partial_t) f = 0 \quad & \text{on } M^\circ \\ f = f_1, \quad \partial_t f = f_2, \quad & \text{on } \mathscr{S}_0. \end{aligned} \tag{8.5}$$

We remark that the equations for Φ in Sect. 3 are of this type. We prove

Theorem 8.3. *Under the same assumptions as in Theorem 1.1, $X_M f$ uniquely determines f and f_1, f_2 which satisfy (8.5). Moreover, there exists a $C > 0$ such that*

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \le C\|X_M f\|_{H^{s+2}(\mathscr{C})} \text{ and } \|f\|_{H^{s+1}(M)} \le C\|X_M f\|_{H^{s+2}(\mathscr{C})}$$

where $\mathscr{C}$ is the set of light rays on M.

Proof. The proof follows the same arguments as for Theorem 1.1. So we just point out what needs to be modified. When the wave equation contains lower order terms, one can construct parametrices $E_\pm$ for the Cauchy problem, see [3, Chapter 5]. These are Fourier integral operators and can be represented by oscillatory integrals. So the construction in Sect. 5 works through, and the analysis for $X_M E_\pm$ is the same as the standard wave equation case. However, we do need to justify the ellipticity of the involved operators in Lemma 6.2 and Proposition 7.1. We remark that ellipticity of the solution itself is standard, and follows simply from the principal symbol satisfying a transport equation.

We follow the parametrix construction in Trèves [21, Section 1, Chapter VI] to check this in a transparent manner.

We look for operators E_j, $j = 0, 1$ such that

$$\begin{aligned} P(x, t, D_x, \partial_t) E_j = 0 \quad & \text{on } M^\circ \\ \partial_t^k E_j = \delta_{kj}, k = 0, 1, \quad & \text{on } \mathscr{S}_0. \end{aligned}$$

Here, for $j = 0, 1$ we have

$$\begin{aligned} E_j f(x) =& (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\phi_0(x,t,\xi)} a_{j0}(x, t, \xi) \hat{f}(\xi) d\xi \\ &+ (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\phi_1(x,t,\xi)} a_{j1}(x, t, \xi) \hat{f}(\xi) d\xi + R_j(t) f(x) \end{aligned}$$

where R_j are smoothing operators, see [21, (1.37)]. The phase functions are

$$\phi_0(x,t,\xi) = x\cdot\xi + ct|\xi|, \quad \phi_1(x,t,\xi) = x\cdot\xi - ct|\xi|.$$

The amplitude can be written as $a_{jk}(x,t,\xi) = \sum_{l=0}^{\infty} a_{jkl}(x,t,\xi)$ and each a_{jkl} is homogeneous of degree $-j-l$ for $|\xi|$ large. Before we look into the structures that we need of the amplitude, we find the initial values of the leading order term a_{jk0} at $t=t_0$. They satisfy (see [21, (1.53)])

$$a_{000}(x,t,\xi) = \frac{1}{2}, \; a_{010}(x,t,\xi) = \frac{1}{2}, \; a_{100}(x,t,\xi) = \frac{1}{2ic|\xi|}, \; a_{110}(x,t,\xi) = -\frac{1}{2ic|\xi|}.$$

The amplitudes satisfy first order equations which are deduced from (see [21, (1.39)])

$$P\left(x,t,D_x+\partial_x\phi_k, \partial_t + i\partial_t\phi_k\right) a_{jk}(x,t,\xi) = 0.$$

For the leading order term, we get

$$\begin{aligned} &\partial_\tau P_2\left(x,t,\partial_x\phi_k, i\partial_t\phi_k\right)\partial_t a_{jk0} \\ &+\sum_{\nu=1}^{3} \partial_{\xi_\nu} P_2\left(x,t,\partial_x\phi_k, i\partial_t\phi_k\right) D_{x^\nu} a_{jk0} + C\left(\phi_k; x,t,\xi\right) a_{jk0} = 0 \end{aligned} \tag{8.6}$$

and the C term in this case is (the sub-principal symbol of P)

$$C\left(\phi_k; x,t,\xi\right) = P_1\left(x,t,i\partial_x\phi_k, i\partial_t\phi_k\right).$$

Note that P_1 has real valued coefficients and is homogeneous of degree one in $i\partial_x\phi_k, i\partial_t\phi_k$. Dividing by $i=\sqrt{-1}$, we see that equation (8.6) is a first order linear equation with real valued coefficients. Solving the equation amounts to solving a ODE along the integral curve and the solution a_{jk0} will be positive scalar multiples of the initial conditions hence not only non-vanishing, but is real or purely imaginary depending on its initial value.

Finally, we can represent the solution to (8.5) as

$$f(x,t) = E_0 f_1 + E_1 f_2 = E_+ h_1 + E_- h_2$$

where

$$\begin{aligned} E_+ h &= (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i(x\cdot\xi+ct|\xi|)}\left(a_{00}(x,t,\xi) + 2ic|\xi| a_{10}(x,t,\xi)\right)\hat{h}(\xi)d\xi \\ &= (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i(x\cdot\xi+ct|\xi|)} a_+(x,t,\xi)\hat{h}(\xi)d\xi \\ E_- h &= (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i(x\cdot\xi-ct|\xi|)}\left(a_{01}(x,t,\xi) - 2ic|\xi| a_{11}(x,t,\xi)\right)\hat{h}(\xi)d\xi \\ &= (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i(x\cdot\xi-ct|\xi|)} a_-(x,t,\xi)\hat{h}(\xi)d\xi \end{aligned} \tag{8.7}$$

and

$$h_1 = f_1 + \frac{1}{2ic}\Delta^{-\frac{1}{2}} f_2, \quad h_2 = f_1 - \frac{1}{2ic}\Delta^{-\frac{1}{2}} f_2.$$

We see that the leading order terms of a_+, a_- are all positive. From these oscillatory integral representations, it is easy to see that Lemma 6.2 holds for $c < 1$. For Proposition 7.1, we see that the principal symbol of k_+ is given by

$$k_{+,-1}(x,\xi) = 2\pi i|\xi|^{-1}\varphi(-\xi/|\xi|)\int_0^{t_1} t^{-1}\chi_\epsilon(t)a_{+,0}(t,x)dt$$

where $a_{+,0}$ is the in the expansion $a_+ \sim \sum_{k=0}^\infty a_{+,k}(t,\xi)|\xi|^{-1-k}$. So $k_{+,-1}(x,\xi)$ is non-vanishing. Thus the operator $I^\varphi X_M \chi_\epsilon E_+$ is elliptic. The rest of the proof is the same as for Theorem 1.1. □

9. Small Perturbations of the Minkowski Spacetime

We consider metric perturbations $g_\delta = g_M + h$ with $h = \sum_{i,j=0}^3 h_{ij}dx^i dx^j$. We assume that

(A1) h is a symmetric two tensor smooth on M;
(A2) for $\delta > 0$ small, the seminorm $\|h_{ij}\|_{C^3} = \sup_{(t,x)\in M}\sum_{|\alpha|\le 3}|\partial^\alpha h_{ij}(t,x)| < \delta, i, j = 0, 1, 2, 3$.

Without loss of generality, we can assume that h is extended to some larger manifold $\widetilde{M} = (\widetilde{t}_0, \widetilde{t}_1)\times\mathbb{R}^3$ such that $M \subset \widetilde{M}$ and (A2) holds on $\widetilde{M}$. In this section, we study the inverse problem on (M, g_δ) for δ sufficiently small. Note that in this case, light rays may not follow straight lines and the injectivity of the light ray transform on scalar functions is not known. We will show that by using a perturbation argument on the Fourier integral operator level, one can obtain the same determination result as for the Minkowski case.

We start with the light-like geodesics on (M, g_δ) and their parametrizations. Let $\gamma(s)$ denote a light-like geodesic from $\mathscr{S}_0$. It satisfies

$$\begin{aligned} \partial_s^2\gamma^k(s) + \Gamma_{ij}^k\partial_s\gamma^i(s)\partial_s\gamma^j(s) &= 0 \\ \gamma(0) = (0, y),\ \partial_s\gamma(0) &= (\beta, v) \end{aligned} \tag{9.1}$$

where Γ_{ij}^k is the Christoffel symbol for g_δ, $v \in \mathbb{S}^2$ and β is such that $g_\delta(\beta, v) = 0$ and (β, v) future pointing. It is known, see for example [1], that (9.1) is equivalent to a first order system on T^*M. Here, M is regarded as a submanifold of $\widetilde{M}$. We use (t, x) and (τ, ξ) for the local coordinates on T^*M. Consider the Hamiltonian

$$\begin{aligned} p(t,x,\tau,\xi) &= \frac{1}{2}g_\delta^*(\tau,\xi) = \frac{1}{2}g_M^*(\tau,\xi) + H(t,x,\tau,\xi) \\ &= \frac{1}{2}\left(-|\tau|^2 + \sum_{i=1}^3|\xi_i|^2\right) + H(t,x,\tau,\xi). \end{aligned}$$

Here, $2H$ is the perturbation of the dual metric corresponding to the perturbation h. Let $\Xi = (\tau, \xi)$, then $H(t,x,\Xi) = \sum_{i,j=0,1,2,3} H_{ij}(t,x)\Xi_i\Xi_j$ is homogeneous of degree two in Ξ and the seminorm $\|H_{ij}\|_{C^3} < C\delta$ for some constants C. We denote the Hamilton vector field by H_p. Let $(t(s), x(s), \tau(s), \xi(s))$ be an integral curve of

H_p in the characteristic set $\Sigma_p = \{(t, x, \tau, \xi) \in T^*M : p(t, x, \tau, \xi) = 0\}$, called null-bicharacteristics. With $\gamma(s) = (t(s), x(s))$, (9.1) can be converted to

$$\begin{aligned}
&\frac{dt}{ds} = \frac{\partial p}{\partial \tau} = -\tau + \partial_\tau H(t, x, \tau, \xi); \quad \frac{dx_i}{ds} = \frac{\partial p}{\partial \xi_i} = \xi_i + \partial_{\xi_i} H(t, x, \tau, \xi) \\
&\frac{d\tau}{ds} = -\partial_t H(t, x, \tau, \xi); \quad \frac{d\xi_i}{ds} = -\partial_{x_i} H(t, x, \tau, \xi), \quad i = 1, 2, 3 \\
&t(0) = t_0 = 0, \quad x_i(0) = y_i, \quad \tau(0) = \tau_0, \quad \xi_i(0) = \xi_{0,i}.
\end{aligned} \tag{9.2}$$

Here, (τ_0, ξ_0) is the cotangent vector obtained from (β, v) using g_δ and we also denote it by $(\tau_0, \xi_0) = (\beta, v)^\flat$. If we consider the system for the Minkowski metric namely $H = 0$, then $\beta = 1$ and the covector $(\tau_0, \xi_0) = (-1, v)$. (9.2) becomes

$$\begin{aligned}
&\frac{dt}{ds} = -\tau, \quad \frac{dx_i}{ds} = \xi_i, \quad \frac{d\tau}{ds} = 0, \quad \frac{d\xi_i}{ds} = 0, \quad i = 1, 2, 3 \\
&t(0) = 0, \quad x_i(0) = y_i, \quad \tau(0) = -1, \quad \xi_i(0) = v_i.
\end{aligned} \tag{9.3}$$

We see that $x(s) = (s, y + sv)$, $t(s) = s$, which agrees with our parametrization used previously. Now we have the following result.

Lemma 9.1. *For $\delta > 0$ sufficiently small, the set of light rays on (M, g_δ) is given by $\mathscr{C}_\delta = \{\gamma = (t, x(t, y, v)) : (y, v) \in \mathscr{S}_0 \times \mathbb{S}^2, t \in [t_0, t_1]\}$, where x is a smooth function of t, y, v. Moreover, we have*

$$\|x(t, y, v) - (y + tv)\|_{C^2} < C\delta$$

for some constant C.

Proof. For $v \in \mathbb{S}^2$, the co-vectors $(\tau_0, \xi_0) = (\beta, v)^\flat$ are in a bounded set of $\mathbb{R}^4$. We assume that $|(\tau_0, \xi_0)| < M_1$. We also notice that τ_0 is away from zero, say $|\tau_0| > M_0 > 0$. Then we consider (τ, ξ) such that $|(\tau, \xi) - (\tau_0, \xi_0)| < M_0/2$ so that $|(\tau, \xi)| < M \doteq M_1 + M_0/2$ and $|\tau| > M_0/2$. Consider the system (9.2). Because H is homogeneous of degree two in (τ, ξ), for $|(\tau, \xi)| < M$ and for $\delta > 0$ sufficiently small, we see that $\frac{dt}{ds} \neq 0$. Therefore, we can take t as the parameter and convert (9.2) to

$$\begin{aligned}
&\frac{ds}{dt} = \frac{1}{-\tau + \partial_\tau H(t, x, \tau, \xi)}; \quad \frac{dx_i}{dt} = \frac{\xi_i + \partial_{\xi_i} H(t, x, \tau, \xi)}{-\tau + \partial_\tau H(t, x, \tau, \xi)} \\
&\frac{d\tau}{dt} = \frac{-\partial_t H(t, x, \tau, \xi)}{-\tau + \partial_\tau H(t, x, \tau, \xi)}; \quad \frac{d\xi_i}{dt} = \frac{-\partial_{x_i} H(t, x, \tau, \xi)}{-\tau + \partial_\tau H(t, x, \tau, \xi)}, \quad i = 1, 2, 3 \\
&s(0) = 0, \quad x_i(0) = y_i, \quad \tau(0) = \tau_0, \quad \xi_i(0) = \xi_{0,i}.
\end{aligned} \tag{9.4}$$

The system corresponding to (9.3) is

$$\begin{aligned}
&\frac{ds}{dt} = \frac{1}{-\tau}; \quad \frac{dx_i}{dt} = \frac{\xi_i}{-\tau}, \quad \frac{d\tau}{dt} = 0; \quad \frac{d\xi_i}{dt} = 0, \quad i = 1, 2, 3 \\
&s(0) = 0, \quad x_i(0) = y_i, \quad \tau(0) = -1, \quad \xi_i(0) = v.
\end{aligned} \tag{9.5}$$

Let $(\widetilde{t}, \widetilde{x}, \widetilde{\tau}, \widetilde{\xi})$ be the solution of (9.5) and (t, x, τ, ξ) satisfy (9.4). Then let $u = (t - \widetilde{t}, x - \widetilde{x}, \tau - \widetilde{\tau}, \xi - \widetilde{\xi})$. We see that u satisfies the system

$$\begin{aligned}
&\frac{du}{ds} = F(u) \\
&u(0) = u_0,
\end{aligned}$$

where F is smooth and $|F(u)| < C\delta$, $|u_0| < C\delta$ for generic constant C. Now it follows from standard ODE theorems, see for instance [9, Theorem 1.2.3] that for δ sufficiently small, there is a unique C^∞ solution u on $[t_0, t_1]$ and $|u| \leq C\delta$. Higher order estimates can be obtained similarly. This finishes the proof. □

Now we consider the light ray transform X_δ on (M, g_δ). The parametrization of the light rays is not unique, although all choices give rise to equivalent analysis for our purpose. Perhaps the most natural parameterization is to use the cosphere bundle on $\mathscr{S}_0$ of the induced metric. Let $\bar{g}_\delta$ be the induced Riemannian metric of g_δ on $\mathscr{S}_0$. For $y \in \mathscr{S}_0$, let $\mathbb{S}^2_{\delta,y} = \{v \in T\mathscr{S}_0 : \bar{g}_\delta(v, v) = 1\}$. For $v \in \mathbb{S}^2_{\delta,y}$, there is a unique future pointing light-like vector (v_0, v) at y. In particular, v_0 is close to 1 for δ small. Then the light ray from $(0, y)$ in direction (v_0, v) is parametrized by $\gamma_{y,v}(s) = \exp_{(0,y)} s(v_0, v)$, $s \in [0, s_1]$ where s is the affine parameter such that $\gamma_{y,v}(0) = (0, y) \in \mathscr{S}_0$ and $\gamma_{y,v}(s_1) \in \mathscr{S}_1$. In this parametrization, we can write

$$X_\delta f(y, v) = \int_0^{s_1} f\left(\gamma_{y,v}(s)\right) ds. \tag{9.6}$$

Now we can identify $\mathbb{S}^2_{\delta,y}$ with $\mathbb{S}^2_y$ via a diffeomorphism. By the above Lemma 9.1, s is a smooth function of y, t and $v \in \mathbb{S}^2$ so we can use t variable to parametrize the light rays. We have

$$X_\delta f(y, v) = \int_0^{t_1} w(y, v, t) f(t, x(t, y, v))dt, \quad y \in \mathscr{S}_0, v \in \mathbb{S}^2,$$

where w is a weight coming from the change of variables. In fact, w is smooth and close to 1 for δ sufficiently small. w only mildly affects the argument, changing the elliptic principal symbol of the final operator $X_\delta \circ E_+$ in (9.14), thus maintaining ellipticity. For simplicity, we will ignore it in the follows and take

$$\begin{aligned} X_\delta f(y, v) &= \int_0^{t_1} f(t, x(t, y, v))dt \\ &= (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{t_1} e^{i((x(t,y,v)-z)\cdot\eta)} f(t, z)dtdzd\eta. \end{aligned} \tag{9.7}$$

This is the parametrization of X_δ we work with in the rest of this section. The Schwartz kernel of X_δ is the delta distribution on $\mathscr{C} \times M^\circ$ supported on the point-line relation Z_δ defined by

$$Z_\delta = \left\{(\gamma, q) \in \mathscr{C} \times M^\circ : q \in \gamma\right\} = \left\{(y, v, (t, x)) \in \mathbb{R}^3 \times \mathbb{S}^2 \times M^\circ : x = x(t, y, v)\right\}.$$

Next, let $\square_{g_\delta}$ be the d'Alembert operator on (M, g_δ) and we consider the second order operator

$$P_\delta\left(x, t, D_x, \partial_t\right) = \square_{g_\delta} + P_1\left(x, t, iD_x, \partial_t\right) + P_0(x, t) \tag{9.8}$$

where P_1 is a first order differential operator with real valued smooth coefficients and P_0 is smooth. Then we consider the Cauchy problem

$$\begin{aligned} P_\delta\left(x, t, D_x, \partial_t\right) f &= 0 \quad \text{on } M^\circ \\ f = f_1, \quad \partial_t f &= f_2, \quad \text{on } \mathscr{S}_0. \end{aligned} \tag{9.9}$$

We remark that for sufficiently small metric perturbations, the operators $\Box_{g_\delta}$ and P_δ are both strictly hyperbolic with respect to $\mathscr{S}_0$. Therefore, as in previous sections, the parametrix construction of Duistermaat-Hörmander can be applied. In general, the parametrix does not have a global oscillatory integral representation on M. However, we show below that for sufficiently small perturbations of the Minkowski spacetime, this is possible.

The parametrix construction is the same as in the previous section. We look for operators E_j, $j = 0, 1$ such that

$$
\begin{aligned}
P_\delta\,(x, t, D_x, \partial_t)\,E_j &= 0 \quad \text{on } M^\circ \\
\partial_t^k E_j &= \delta_{kj}, k = 0, 1, \quad \text{on } \mathscr{S}_0.
\end{aligned}
$$

For $j = 0, 1$ we have

$$
\begin{aligned}
E_j f(x) = (2\pi)^{-3} &\int_{\mathbb{R}^3} e^{i\phi_+(x,t,\xi)} a_{j,+}(x, t, \xi) \hat{f}(\xi) d\xi \\
&+ (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\phi_-(x,t,\xi)} a_{j,-}(x, t, \xi) \hat{f}(\xi) d\xi + R_j(t) f(x)
\end{aligned}
$$

where R_j are smoothing operators, see [21, (1.37)]. We follow Trèves [21] to find the phase functions $\phi(t, x, \xi)$ for $(t, x) \in (t_0, t_1) \times \mathbb{R}^3$, $\xi \in \mathbb{R}^3$. The phase function should satisfy the eikonal equation

$$
p(\nabla\phi) = -|\partial_t\phi|^2 + |\partial_x\phi|^2 + H\,(\partial_t\phi, \partial_x\phi) = 0.
$$

By the strict hyperbolicity, there are two solutions for $\partial_t\phi$ denoted by $\partial_t\phi = \lambda_\pm(t, x, \partial_x\phi)$ and $\lambda_\pm$ are smooth functions and homogeneous of degree one in $\partial_x\phi$. We take initial conditions $\partial_t\phi = x \cdot \xi, \xi \in \mathbb{R}^3$ at $t = 0$. Below, we consider λ_+. The treatment for λ_- is identical. We consider the Hamilton-Jacobi equation

$$
\begin{aligned}
&\frac{dx}{dt} = -\partial_\eta\lambda_+(t, x, \eta), \quad \frac{d\eta}{dt} = \partial_x\lambda_+(t, x, \eta) \\
&x(0) = y, \quad \eta(0) = \xi, \quad y \in \mathbb{R}^3, \xi \in \mathbb{R}^3\backslash 0.
\end{aligned}
\tag{9.10}
$$

We denote the solution by $x(t, y, \xi), \xi(t, y, \xi)$. Then the phase function is

$$
\phi_+(t, x, \xi) = x \cdot \xi + \int_0^t \lambda_+(s, x, \eta(s, y, \xi)) ds. \tag{9.11}
$$

Here, one can express y in terms of x, see [21, Section 2, Chapter VI] for more details. For the Minkowski spacetime, we know $\lambda_+ = -|\xi|$ so that (9.10) becomes

$$
\begin{aligned}
&\frac{dx}{dt} = \xi/|\xi|, \quad \frac{d\eta}{dt} = 0 \\
&x(0) = y, \quad \eta(0) = \xi.
\end{aligned}
\tag{9.12}
$$

The solution is simply $x(t) = y + t\xi/|\xi|$, $\eta(t) = \xi$ and the phase function is $\phi_0(t, x, \xi) = x \cdot \xi + t|\xi|$. Using the same argument as for Lemma 9.1, we get

Lemma 9.2. *For $\delta > 0$ sufficiently small, there is a unique smooth solution $(x(t, y, \xi), \eta(t, y, \xi))$ to (9.10) for $t \in [t_0, t_1]$, $y \in \mathbb{R}^3$, $\xi \in \mathbb{R}^3 \backslash 0$, and they satisfy*

$$\|x\,(t, y, \xi) - (y - t\xi/|\xi|)\,\|_{C^2} < C\delta, \quad \|\eta(t, y, \xi)/|\xi| - \xi/|\xi|\|_{C^2} < C\delta$$

for some constant $C > 0$. It follows that the phase function ϕ_+ in (9.11) is also smooth and satisfies

$$\|\phi_+\,(t, x, \xi) - (x \cdot \xi + t|\xi|)\,\|_{C^2} < C\delta|\xi|.$$

We remark that similar argument was used in [18] for a backscattering problem. Using this lemma, we can represent the solution to (9.9) as

$$f(x, t) = E_0 f_1 + E_1 f_2 = E_+ h_1 + E_- h_2$$

where

$$\begin{aligned} E_+ h &= (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\phi_+(t,x,\xi)} a_+(x, t, \xi)\hat{h}(\xi) d\xi \\ E_- h &= (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\phi_-(t,x,\xi)} a_-(x, t, \xi)\hat{h}(\xi) d\xi. \end{aligned} \tag{9.13}$$

The $a_\pm$ and h_1, h_2 are the same as in (8.7).

With these preparations, we now state and prove our main result in this section.

Theorem 9.3. *Consider (M, g_δ) which satisfy the assumptions (A1), (A2) in the beginning of this section. Assume that $(f_1, f_2) \in \mathcal{N}^s$, $s \geq 0$, and f_1, f_2 are supported in a compact set $\mathscr{K}$ of $\mathscr{S}_0$. For $\delta \geq 0$ sufficiently small, $X_\delta f$ uniquely determines f and f_1, f_2 which satisfy (9.9). Moreover, there exists $C > 0$ such that*

$$\|\,(f_1, f_2)\,\|_{\mathcal{N}^s} \leq C\|X_\delta f\|_{H^{s+2}(\mathscr{C}_\delta)} \text{ and } \|f\|_{H^{s+1}(M)} \leq C\|X_\delta f\|_{H^{s+2}(\mathscr{C}_\delta)}$$

where $\mathscr{C}_\delta$ is the set of light rays on (M, g_δ).

Proof. We examine the arguments in Sects. 7 and 8 and point out what needs to be modified. We consider the composition of X_δ and E_+ defined in (9.13). We have

$$X_\delta f(y, v) = (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{t_1} e^{i((x(t,y,v)-x')\cdot\eta)} f(t, x') dt dx' d\eta$$

and

$$E_+(f)(t, x') = (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i\left(\phi_+(t,x',\xi)-z\cdot\xi\right)} a_+(t, x', \xi) f(z) dz d\xi.$$

Consider the integral operator I^φ defined in Sect. 7. Using the oscillatory integral representations, we have

$$\begin{aligned} &I^\varphi X_\delta \chi_\epsilon E_+ f(y, v) \\ &= (2\pi)^{-6} \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^{t_1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i\left(x(t,y,v)\cdot\eta - x'\cdot\eta + \phi_+(t,x',\xi) - z\cdot\xi\right)} \\ &\quad \varphi(v) a(t, x', \xi)\chi_\epsilon(t) f(z) dz d\xi dt dx' d\eta dv \end{aligned} \tag{9.14}$$

We write the phase function as $\Phi = \phi + \psi$ in which

$$\phi\left(y, z; \xi, \eta, x', t, v\right) = (y - x') \cdot \eta + tv \cdot \eta + (x' - z) \cdot \xi + t|\xi|$$

and ψ is a smooth function and homogeneous of degree one in ξ, η. In particular, Φ is a small perturbation of ϕ. As in Proposition 7.1, we first consider the integration in x', η, v in (9.14). As shown in Proposition 7.1, the phase function ϕ in these variables is non-degenerate. Since ψ is a small perturbation of ϕ, for δ sufficiently small, we see that Φ in x', η, v variables is also non-degenerate. Note that

$$\partial_{x'}\Phi = -\eta + \partial_{x'}\phi_+(t, x', \xi), \quad \partial_\eta \Phi = x(t, y, v) - x', \quad \partial_v \Phi = \partial_v x(t, y, v) \cdot \eta \tag{9.15}$$

For the stationary points, we see that $x' = x(t, y, v)$ so (t, x) is on the light ray from $(0, y)$ in direction $(1, v)$. Let τ satisfy $p_\delta(t, x, \tau, \eta) = 0$. From $\eta = \partial_{x'}\phi_+(t, x', \xi)$ we see that (t, x', τ, η) is on the bicharactersitics from (y, ξ). Since there is no conjugate points, we get $v = \pm\xi/|\xi|$. Thus at the stationary points, the phase function becomes

$$\Phi(y, z, t, \xi) = \phi_+(t, x(t, y, \pm\xi/|\xi|), \xi) - z \cdot \xi$$

After integrating in x', η, v, the Schwartz kernel becomes

$$\begin{aligned} I^\varphi X_\delta \chi_\epsilon E_+(y, z) = (2\pi)^{-3} \int_0^{t_1} \int_{\mathbb{R}^3} e^{i(\phi_+(t, x(t, y, \xi/|\xi|), \xi) - z\cdot\xi)} k_+^\delta(t, \xi) d\xi dt \\ + (2\pi)^{-3} \int_0^{t_1} \int_{\mathbb{R}^3} e^{i(\phi_+(t, x(t, y, -\xi/|\xi|), \xi) - z\cdot\xi)} k_-^\delta(t, \xi) d\xi dt \end{aligned} \tag{9.16}$$

where $k_\pm^\delta$ are small perturbations of $k_\pm$ in (7.4) and (7.5) of Proposition 7.1. Finally, we integrate in t. For the second integral in (9.16), the phase function is a small perturbation of

$$(y - z) \cdot \xi + 2t|\xi|$$

thus as in Proposition 7.1, the integral is $O(|\xi|^{-\infty})$. For the first integral of (9.16), we need to examine the phase function at the stationary points. Using (9.11), we get

$$\Phi(y, z, t, \xi) = x \cdot \xi + \int_0^t \lambda_+(s, x, \xi) ds - z \cdot \xi$$

where $x = x(t, y, \xi/|\xi|)$. Taking ξ derivative, we get

$$\begin{aligned} \partial_\xi \Phi(y, z, t, \xi) &= (x - z) + \partial_\xi x \cdot \xi + \int_0^t \left(\partial_\eta \lambda_+(s, x(s, y, \xi/|\xi|), \xi) \partial_\xi x + \partial_\eta \lambda_+\right) ds \\ &= (x - z) + \partial_\xi x \cdot \xi + \int_0^t -\frac{dx}{ds}(s, x, \xi) ds \end{aligned}$$

where we used the stationary point condition (9.15) and $\partial_\eta \lambda_+ = -dx/dt$. Note that $\partial_\xi x$ is the Jacobi field, and because $x(t, y, \xi/|\xi|)$ is a light-like geodesic, $\partial_\xi x \cdot \xi = 0$, see Lemma 3.1 and Lemma 3.4 of [13]. Therefore, $\partial_\xi \Phi(y, z, t, \xi) = (x - z)$ and $\Phi(y, z, t, \xi) = (y - z) \cdot \xi + \widetilde{\Phi}(y, z, t)$ where $\widetilde{\Phi}$ is small. Finally, integrating in t of the first integral of (9.16) gives a pseudo-differential operator of order -1 and the principal symbol $k_{+,-1}^\delta$ is a small perturbation of $k_{+,-1}(\xi)$ in Proposition 7.1. This implies that Proposition 7.1 hold for the small perturbations.

To see that the analogous result of Proposition 7.2 holds for small perturbations, it suffices to examine the kernel (9.16) in which χ_ϵ is replaced by $\chi_{[\epsilon,t_1]}$

$$I^\varphi X_\delta \chi_{[\epsilon,t_1]} E_+(y,z) = (2\pi)^{-3} \int_\epsilon^{t_1} \int_{\mathbb{R}^3} e^{i(\phi_+(t,x(t,y,\xi/|\xi|),\xi)-z\cdot\xi)} k_+^\delta(t,\xi) d\xi dt$$
$$+ (2\pi)^{-3} \int_\epsilon^{t_1} \int_{\mathbb{R}^3} e^{i(\phi_+(t,x(t,y,-\xi/|\xi|),\xi)-z\cdot\xi)} k_-^\delta(t,\xi) d\xi dt \tag{9.17}$$

The first integral still gives a pseudo-differential operator as shown above. For the second integral, integration by parts in t gives an oscillatory integral of the form

$$\int_{\mathbb{R}^3} e^{i(\phi_+(\epsilon,x(\epsilon,y,-\xi/|\xi|),\xi)-z\cdot\xi)} a(\xi) d\xi + \int_{\mathbb{R}^3} e^{i(\phi_+(t_1,x(t_1,y,-\xi/|\xi|),\xi)-z\cdot\xi)} b(\xi) d\xi \tag{9.18}$$

where a, b are symbols of order -2. Here, we used that ϕ_+ is homogeneous of degree one in ξ. To see that these are FIOs of canonical graph type, we use the characterization in [8, page 26] which says that an oscillatory integral with phase $\phi(x,\eta) - x\cdot\eta$ is an FIO whose canonical relation is a canonical graph if and only if det $\frac{\partial^2\phi}{\partial x\partial\eta} \neq 0$. Since $\phi_+(\epsilon, x(\epsilon, y, -\xi/|\xi|), \xi)$ is a small perturbation of $y\cdot\xi + 2\epsilon|\xi|$ and det $\frac{\partial^2}{\partial y\partial\xi}(y\cdot\xi + 2\epsilon|\xi|) = -1 \neq 0$, we conclude that for δ sufficiently small, the first integral in (9.18) gives an FIO of canonical graph type. The same is true for the second integral. Thus Proposition 7.2 holds for small perturbations.

Now, the proof of Theorem 1.1 in Sect. 8 go through line by line, except the injectivity of X_δ. In particular, we have the estimate as (8.1)

$$\|(f_1, f_2)\|_{\mathcal{N}^s} \leq C\|X_\delta f\|_{H^{s+2}(\mathscr{C}_\delta)} + C_\rho\|(f_1, f_2)\|_{\mathcal{N}^{s-\rho}}$$

where C_ρ is a constant depending on ρ. To get rid of the last term, we use the following argument, see [22, Section 2.7]. Notice that given s, ρ and for some fixed small δ_0, if we consider all metric g such that $\|g - g_M\|_{C^3} \leq \delta_0$, then the above estimate is uniform (a fixed constant C_ρ works for all such metrics) by the uniformity of the construction. Now suppose there is no δ such that for all metrics within δ of the Minkowski metric g_M (in the Fréchet space sense) the transform is injective. Let $F^j = (f_1^j, f_2^j)$, $j = 1, 2, \dots$ be such that the corresponding f^j is in the null-space of $X_{g_j} = X_j$ and $\|F^j\|_{\mathcal{N}^s} = 1$, with g_j within $1/j$ of the Minkowski metric. By the above inequality, $1 \leq C_\rho\|F^j\|_{\mathcal{N}^{s-\rho}}$. Now, F^j has a $\mathcal{N}^s$-weakly convergent subsequence, not shown in notation, to some $F \in \mathcal{N}^s$, which thus strongly converges in $\mathcal{N}^{s-\rho}$. By the above inequality, $F \neq 0$. But $0 = X_j f$ converges to $X_M f$ e.g. in the sense of distributions. So $X_M f = 0$ which by the injectivity of X_M, implies that $f = 0$. So we get $F = 0$ a contradiction. This shows the injectivity of X_δ and finishes the proof of Theorem 9.3. □

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