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Long-Time Asymptotics for the Integrable Nonlocal Focusing Nonlinear Schrödinger Equation for a Family of Step-Like Initial Data

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Abstract: We study the Cauchy problem for the integrable nonlocal focusing nonlinear Schrödinger (NNLS) equation $iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(-x,t) = 0$ with the step-like initial data close, in a certain sense, to the "shifted step function" $\chi_R(x) = AH(x-R)$, where $H(x)$ is the Heaviside step function, and $A > 0$ and $R > 0$ are arbitrary constants. Our main aim is to study the large-t behavior of the solution of this problem. We show that for $R \in \left(\frac{(2n-1)\pi}{2A}, \frac{(2n+1)\pi}{2A}\right)$, $n = 1, 2, \dots$, the (x,t) plane splits into $4n+2$ sectors exhibiting different asymptotic behavior. Namely, there are $2n+1$ sectors, where the solution decays to 0, whereas in the other $2n+1$ sectors (alternating with the sectors with decay), the solution approaches (different) constants along each ray $x/t = const$. Our main technical tool is the representation of the solution of the Cauchy problem in terms of the solution of an associated matrix Riemann–Hilbert problem and its subsequent asymptotic analysis following the ideas of the nonlinear steepest descent method.

1. Introduction

We consider the Cauchy problem for the integrable nonlocal focusing nonlinear Schrödinger (NNLS) equation with step-like initial data:

$$iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(-x,t) = 0, \qquad x \in \mathbb{R},\ t > 0, \tag{1.1a}$$

$$q(x,0) = q_0(x), \qquad x \in \mathbb{R}, \tag{1.1b}$$

where

$$q_0(x) \to \begin{cases} 0, & x \to -\infty, \\ A, & x \to \infty, \end{cases} \tag{1.1c}$$

with some $A > 0$. Here and below $\bar{q}$ denotes the complex conjugate of q. We assume also that the solution $q(x,t)$ satisfies the boundary conditions (1.1c) for all $t > 0$:

$$q(x,t) \to \begin{cases} 0, & x \to -\infty, \\ A, & x \to \infty, \end{cases} \tag{1.2}$$

where the convergence is sufficiently fast.

The NNLS equation (1.1a) was introduced by Ablowitz and Musslimani [4] as a reduction $r(x,t) = -\bar{q}(-x,t)$ in the coupled system of nonlinear Schrödinger equations:

$$iq_t(x,t) + q_{xx}(x,t) - 2q^2(x,t)r(x,t) = 0, \tag{1.3a}$$

$$ir_t(x,t) - r_{xx}(x,t) + 2r^2(x,t)q(x,t) = 0, \tag{1.3b}$$

(see also [24] for multidimensional versions of the NNLS) and has attracted much attention in recent years due to its distinctive properties. Particularly, the NNLS equation is PT symmetric [7], i.e., if $q(x,t)$ is its solution, so is $\bar{q}(-x,-t)$. Therefore, the NNLS equation is closely related to the PT symmetric theory, which is a field in modern physics being actively studied (see e.g. [11,14,25,33,51] and references therein). Also, the exact soliton and breather solutions of this equation have a number of unusual properties. In particular, (i) solitons can blow up in a finite time and (ii) the NNLS equation supports both dark and anti-dark soliton solutions simultaneously (see e.g. [2,5,28,40,47–49] and references therein; see also [50], where the general soliton solutions for the coupled Schrödinger equations (1.3) are presented).

Apart from deriving exact solutions of the NNLS equation, it is important, in the both mathematical and physical perspective, to consider initial value problems with general initial data. The NNLS equation is an integrable system, i.e. it is a compatibility condition of two linear equations (the so-called Lax pair, see (2.1) below) and, therefore, it can be, in principle, treated by the powerful inverse scattering transform (IST) method [1,23,41]. This method allows reducing the original nonlinear problem to a sequence of linear ones, and in this sense to find the "exact" representation of the solution. The IST method was successfully applied in [5] to the Cauchy problem for the NNLS equation on the whole line in the class of functions rapidly decaying as $x \to \pm\infty$ (see also [26], where the complete integrability of (1.1a) in this class was proved).

Although the IST method provides some sort of exact formulas for the solutions, the qualitative analysis of the Cauchy problem for the NNLS equation, particularly, the long-time asymptotics of its solution, is a challenging problem. By using the IST method, the original problem for an integrable system can be reduced to the matrix Riemann–Hilbert (RH) factorization problem in the complex plane of the spectral parameter. The jump matrix in this problem is oscillatory, which allows applying the so-called nonlinear steepest decent method [20] for studying its long-time behavior. This method was inspired by earlier works of Manakov [37] and Its [31] and finally put in a rigorous and systematic shape by Deift and Zhou [20] (see also [15–19,38] and references therein concerning the Deift and Zhou method and its extensions). The nonlinear steepest decent method consists of series of transformations of the original RH problem in such a way that for the large values of a parameter (say, the time t in the original nonlinear evolutionary equation) this problem can be solved explicitly in terms of special functions (e.g., the parabolic cylinder functions, Riemann theta functions, Painleve transcendents etc.).

Problems with step-like initial data (with different behavior at different infinities) have also been considered for a variety of integrable systems, which include the Korteweg–de

Vries equation [6,21,27,30], the focusing and defocusing NLS equations [9,12], the Toda lattice [17,46] and the modified Korteweg–de Vries equation [35] among many others. A wide range of important physical phenomena manifest themselves in the behavior of solutions of such problems for large times, e.g., collisionless and dispersive shock waves [28], rarefaction waves [29], asymptotic solitons [34], modulated waves [46], elliptic waves [12], trapped solitons [3] to name but a few. Particularly, the so-called modulated instability (Benjamin–Feir instability in the context of water waves) phenomenon, which is suggested as a possible mechanism for the generation of rogue waves [42], has been attributed in [10] to the local focusing nonlinear Schrödinger equation with nonzero boundary conditions.

In [44] we present the large-time analysis of problem (1.1)–(1.2) in the case of initial data close (in the sense of closeness of the associated spectral functions) to the "pure step function with the step located at $x = 0$": $q_0(x) = 0$ for $x < 0$ and $q_0(x) = A$ for $x > 0$. In that case it is shown that there are two regions in the half-plane $-\infty < x < \infty, t > 0$, where the solution has qualitatively different large time behavior. Namely, for $x < 0$ the solution is slowly decaying as $t \to \infty$ and is described by the Zakharov–Manakov-type formula [37], where the power decay rate depends on $\xi = \frac{x}{4t}$. On the other hand, for $x > 0$ the solution converges to a constant $c = c(\xi)$ (depended on the direction ξ), which can be described explicitly in terms of the spectral functions associated to the initial data.

Notice that this asymptotic picture is in sharp contrast with that in the case of the conventional (local) nonlinear Schrödinger equation

$$iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(x,t) = 0. \tag{1.4}$$

Indeed, in the case when the initial data decay to zero on one side (as in our problem), there are three sectors with different asymptotic behavior: the Zakharov–Manakov (decaying) sector, the plane wave sector, and the sector of modulated elliptic oscillations [12]. Moreover, if the both sides of the initial step are non-zero, then there can be five or even more asymptotic zones, see [13,15] for the focusing NLS equation and [8,22,29] for the defocusing NLS equation.

Since the NNLS equation is not translation invariant, the behavior of the solutions of problem (1.1)–(1.2) may depend significantly on the details of the shape of the initial data. The present paper aims to rigorously demonstrate this effect taking the initial data close to a "shifted step", i.e., the pure step function with the step located at $x = R$ with some $R > 0$:

$$q_0(x) = \begin{cases} 0, & x < R, \\ A, & x > R. \end{cases} \tag{1.5}$$

"Close" means that the spectral functions associated with the initial data have properties similar to those in the case of the pure shifted step initial data (1.5), see Assumption A in Sect. 2 below.

The paper is organized as follows. In Sect. 2 we briefly present the formalism of the inverse scattering transform method in the form of the associated Riemann–Hilbert factorization problem, developed in details in [44], and discuss the properties of the spectral functions associated to the initial data (1.5). Then, in Sect. 3, we perform the asymptotic analysis of problem (1.1)–(1.2) in the case of initial data close (in the sense mentioned above) to the "shifted step" (1.5) following the ideas of the nonlinear steepest decent method [20] (see also [36]) for studying the large-time behavior of solutions of integrable nonlinear PDEs and taking into account specific characteristics (of the jump and residue conditions) of the basic Riemann–Hilbert problem. The behavior of the solution of problem (1.1)–(1.2) at the edges of the sectors specified in Sect. 3 (as sectors with qualitatively different large time behavior of the solution) is discussed in Sect. 4.

2. Inverse Scattering Transform and the Riemann–Hilbert Problem

2.1. Direct scattering. The focusing NNLS equation (1.1a) is a compatibility condition of two linear differential equations (the Lax pair) [4]

$$\begin{cases} \Phi_x + ik\sigma_3\Phi = U(x,t)\Phi, \\ \Phi_t + 2ik^2\sigma_3\Phi = V(x,t,k)\Phi, \end{cases} \tag{2.1}$$

where $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\Phi(x,t,k)$ is a 2×2 matrix-valued function, $k \in \mathbb{C}$ is an auxiliary (spectral) parameter, and the matrix coefficients $U(x,t)$ and $V(x,t,k)$ are given in terms of $q(x,t)$:

$$U(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -\bar{q}(-x,t) & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \tag{2.2}$$

where $V_{11} = -V_{22} = iq(x,t)\bar{q}(-x,t)$, $V_{12} = 2kq(x,t) + iq_x(x,t)$, and $V_{21} = -2k\bar{q}(-x,t) + i(\bar{q}(-x,t))_x$.

Assuming that there exists $q(x,t)$ satisfying (1.1) and (1.2), we define the 2×2-valued functions $\Psi_j(x,t,k)$, $j = 1, 2$, $-\infty < x < \infty$, $0 \le t < \infty$ as the solutions of the linear Volterra integral equations [44]:

$$\begin{aligned} \Psi_1(x,t,k) = N_-(k) + \int_{-\infty}^{x} G_-(x,y,k)\,(U(y,t) \\ -U_-)\,\Psi_1(y,t,k)e^{ik(x-y)\sigma_3}\,dy,\ k \in (\mathbb{C}^+, \mathbb{C}^-), \end{aligned} \tag{2.3a}$$

$$\begin{aligned} \Psi_2(x,t,k) = N_+(k) - \int_{x}^{\infty} G_+(x,y,k)\,(U(y,t) \\ -U_+)\,\Psi_2(y,t,k)e^{ik(x-y)\sigma_3}\,dy,\ k \in (\mathbb{C}^-, \mathbb{C}^+), \end{aligned} \tag{2.3b}$$

where

$$N_+(k) = \begin{pmatrix} 1 & \frac{A}{2ik} \\ 0 & 1 \end{pmatrix},\ N_-(k) = \begin{pmatrix} 1 & 0 \\ \frac{A}{2ik} & 1 \end{pmatrix},\ G_\pm(x,y,k) = \Phi_\pm(x,t,k)[\Phi_\pm(y,t,k)]^{-1},$$

with $\Phi_\pm(x,t,k) = N_\pm(k)e^{-(ikx+2ik^2t)\sigma_3}$, $U_+ = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$, $U_- = \begin{pmatrix} 0 & 0 \\ -A & 0 \end{pmatrix}$. Here $k \in (\mathbb{C}^+, \mathbb{C}^-)$, where $\mathbb{C}^\pm = \{k \in \mathbb{C} \mid \pm \operatorname{Im} k > 0\}$, means that the first and the second column of a matrix can be analytically continued into respectively the upper and lower half-plane as bounded functions. Then $\Psi_j(x,t,k)e^{-(ikx+2ik^2t)\sigma_3}$, $j = 1, 2$ are the (Jost) solutions of the Lax pair (2.1) for all $k \in \mathbb{R}\setminus\{0\}$ and thus Ψ_1 and Ψ_2 are related by

$$\Psi_1(x,t,k) = \Psi_2(x,t,k)e^{-(ikx+2ik^2t)\sigma_3}S(k)e^{(ikx+2ik^2t)\sigma_3},\ k \in \mathbb{R}\setminus\{0\}, \tag{2.4}$$

where $S(k)$ is the so-called scattering matrix; it can be obtained in terms of the initial data only, evaluating (2.3) at $t = 0$: $S(k) = \Psi_2^{-1}(0,0,k)\Psi_1(0,0,k)$.

Since the matrices $U(x,t)$ and $N_\pm(k)$ satisfy the symmetries $\Lambda\overline{U(-x,t)}\Lambda^{-1} = U(x,t)$ and, respectively, $\Lambda\overline{N_-(-k)}\Lambda^{-1} = N_+(k)$ with $\Lambda = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, it follows that

$\Lambda\overline{\Psi_1(-x,t,-\bar{k})}\Lambda^{-1} = \Psi_2(x,t,k)$ for $k \in \mathbb{R} \setminus \{0\}$. In turn, this implies that the scattering matrix $S(k)$ can be written as

$$S(k) = \begin{pmatrix} a_1(k) & -\overline{b(-\bar{k})} \\ b(k) & a_2(k) \end{pmatrix}, \qquad k \in \mathbb{R} \setminus \{0\}, \tag{2.5}$$

with some $b(k)$, $a_1(k)$, and $a_2(k)$ such that $\overline{a_j(-\bar{k})} = a_j(k)$, $j = 1, 2$. Moreover, $a_1(k)$ and $a_2(k)$ have analytic continuations into the upper and lower half-planes respectively. We summarize the properties of the spectral functions in the following proposition ($\overline{\mathbb{C}^{\pm}} = \{k \in \mathbb{C} \mid \pm \operatorname{Im} k \geq 0\}$) [44]:

Proposition 1. *The spectral functions $a_j(k)$, j=1,2, and $b(k)$ have the following properties*

1. *$a_1(k)$ is analytic in $k \in \mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+} \setminus \{0\}$; $a_2(k)$ is analytic in $k \in \mathbb{C}^-$ and continuous in $\overline{\mathbb{C}^-}$.*
2. *$a_j(k) = 1 + O\left(\frac{1}{k}\right)$, $j = 1, 2$ and $b(k) = O\left(\frac{1}{k}\right)$ as $k \to \infty$ (the latter holds for $k \in \mathbb{R}$).*
3. *$\overline{a_1(-\bar{k})} = a_1(k)$, $k \in \overline{\mathbb{C}^+} \setminus \{0\}$; $\overline{a_2(-\bar{k})} = a_2(k)$, $k \in \overline{\mathbb{C}^-}$.*
4. *$a_1(k)a_2(k) + b(k)\overline{b(-\bar{k})} = 1$, $k \in \mathbb{R} \setminus \{0\}$ (follows from $\det S(k) = 1$).*
5. *As $k \to 0$, $a_1(k) = \frac{A^2 a_2(0)}{4k^2} + O\left(\frac{1}{k}\right)$ and $b(k) = \frac{A a_2(0)}{2ik} + O(1)$.*

Remark 1. Item 5 of Proposition 1 follows from the behavior of $\Psi_j(x,t,k)$ as $k \to 0$, which has an additional symmetry [44]:

$$\Psi_1^{(1)}(x,t,k) = \frac{1}{k}\begin{pmatrix} v_1(x,t) \\ v_2(x,t) \end{pmatrix} + O(1), \qquad \Psi_1^{(2)}(x,t,k) = \frac{2i}{A}\begin{pmatrix} v_1(x,t) \\ v_2(x,t) \end{pmatrix} + O(k), \tag{2.6a}$$

$$\Psi_2^{(1)}(x,t,k) = -\frac{2i}{A}\begin{pmatrix} \overline{v_2}(-x,t) \\ \overline{v_1}(-x,t) \end{pmatrix} + O(k), \quad \Psi_2^{(2)}(x,t,k) = -\frac{1}{k}\begin{pmatrix} \overline{v_2}(-x,t) \\ \overline{v_1}(-x,t) \end{pmatrix} + O(1), \tag{2.6b}$$

with some $v_j(x,t)$, $j = 1, 2$, where $\Psi_j^{(i)}(x,t,k)$ denotes the ith column of $\Psi_j(x,t,k)$.

2.2. Spectral functions for the "shifted step" initial data. In the case of pure "shifted step" initial data (1.5), the associated spectral functions can be calculated explicitly:

$$a_1(k) = 1 + \frac{A^2}{4k^2} e^{4ikR}, \tag{2.7a}$$

$$a_2(k) = 1, \tag{2.7b}$$

$$b(k) = \frac{A}{2ik} e^{2ikR}. \tag{2.7c}$$

Indeed, evaluating (2.4) for $x = -R$ and $t = 0$ it follows that the scattering matrix $S(k)$ can be determined by

$$S(k) = e^{-ikR\sigma_3}\Psi_2^{-1}(-R,0,k)\Psi_1(-R,0,k)e^{ikR\sigma_3}. \tag{2.8}$$

Taking into account (1.5), from (2.3) for $t = 0$ we have

$$\Psi_1(-R,0,k) = N_-(k), \tag{2.9a}$$

$$\Psi_2(-R,0,k) = N_+(k) - \int_{-R}^{R} G_+(-R,y,k)\begin{pmatrix}0 & -A\\ 0 & 0\end{pmatrix}\Psi_2(y,0,k)e^{-ik(R+y)\sigma_3}\,dy, \tag{2.9b}$$

where $\Psi_2(x,0,k)$ for $x \in [-R,R]$ solves the integral equation

$$\Psi_2(x,0,k) = N_+(k) - \int_{x}^{R} G_+(x,y,k)\begin{pmatrix}0 & -A\\ 0 & 0\end{pmatrix}\Psi_2(y,0,k)e^{ik(x-y)\sigma_3}\,dy,\quad x \in [-R,R]. \tag{2.10}$$

Direct calculations show that

$$G_+(x,y,k) = \begin{pmatrix} e^{-ik(x-y)} & \frac{A}{2ik}\left(e^{ik(x-y)} - e^{-ik(x-y)}\right)\\ 0 & e^{ik(x-y)}\end{pmatrix},$$

and, therefore, the solution $\Psi_2(x,0,k)$ of (2.10) is given by

$$\Psi_2(x,0,k) = \begin{pmatrix} 1 & \frac{A}{2ik}e^{2ik(R-x)}\\ 0 & 1\end{pmatrix},\quad x \in [-R,R]. \tag{2.11}$$

Substituting (2.9) and (2.11) into (2.8) gives (2.7).

The locations of zeros of $a_1(k)$ in $\overline{\mathbb{C}^+}$, which clearly depend on A and R, and the behavior of the argument of $a_1(k)$ for $k \in \mathbb{R}$ are described in the following proposition (throughout the paper, the notation $\overline{n_1, n_2}$, $n_1, n_2 \in \mathbb{Z}$, $n_1 \le n_2$ stands for the set $\{n_1, n_1+1, \dots, n_2\}$; if $n_1 > n_2$, then it is assumed to be the empty set).

Proposition 2. *(i) For* $0 < R < \frac{\pi}{2A}$, *$a_1(k)$ has one simple zero in $\overline{\mathbb{C}^+}$ at $k = ik_0$, $k_0 > 0$, where k_0 is the unique solution of the transcendental equation*

$$k = \frac{A}{2}e^{-2kR},\quad k \in \mathbb{R}. \tag{2.12}$$

Moreover, for all $\xi > 0$,

$$\int_{-\infty}^{-\xi} d\arg a_1(k) \in (-\pi,\pi). \tag{2.13}$$

(ii) For $\frac{(2n-1)\pi}{2A} < R < \frac{(2n+1)\pi}{2A}$, $n \in \mathbb{N}$, *$a_1(k)$ has the following properties:*

- *$a_1(k)$ has $2n+1$ simple zeros in $\overline{\mathbb{C}^+}$: $\{ik_0; \{p_j, -\overline{p}_j\}_{j=1}^n\}$. Here $k_0 > 0$ is the solution of* (2.12), $\{\operatorname{Re} p_j\}_{j=1}^n$ *are the ordered set of solutions of equation*

$$k = \pm\frac{A}{2}\sin(2kR)e^{2kR\cot(2kR)}, \tag{2.14}$$

considered for $k < 0$ (see also Fig. 1), and

$$\operatorname{Im} p_j = -\operatorname{Re} p_j \cot(2\operatorname{Re} p_j R),\quad j = \overline{1,n}. \tag{2.15}$$

Notice that

$$\operatorname{Re} p_j \in \left(-\frac{j\pi}{2R}, -\frac{(2j-1)\pi}{4R}\right),\quad j = \overline{1,n}. \tag{2.16}$$

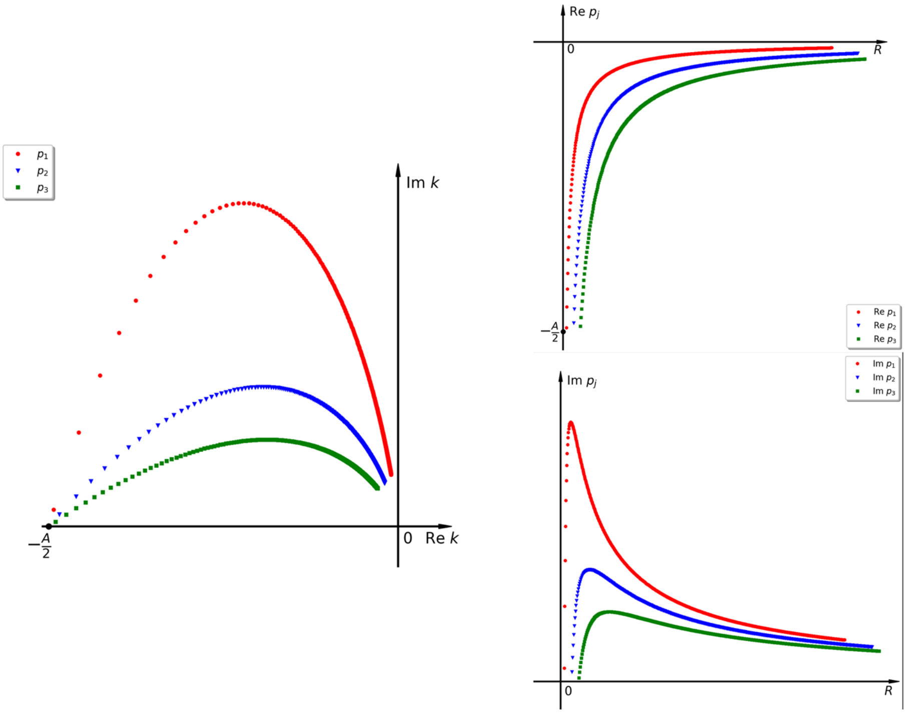

Fig. 1. "Evolution" of the zeros p_j, $j = 1, 2, 3$, as $R \to \infty$

- *Let* $\omega_0 = 0$, $\omega_j = \frac{(2j-1)\pi}{4R}$ *for* $j = \overline{1, n}$, *and* $\omega_{n+1} = \infty$. *Then*

$$\int_{-\infty}^{-\omega_{n-j+1}} d \arg a_1(k) = (2j-1)\pi, \quad j = \overline{1, n}, \tag{2.17a}$$

$$\int_{-\infty}^{-\xi} d \arg a_1(k) \in ((2j-1)\pi, (2j+1)\pi), \\ -\omega_{n-j+1} < -\xi < -\omega_{n-j}, \quad j = \overline{0, n}. \tag{2.17b}$$

(iii) If $R = \frac{(2n+1)\pi}{2A}$ *for some* $n \in \mathbb{N} \cup \{0\}$, *then* $a_1(k)$ *has* $2n + 3$ *simple zeros in* $\overline{\mathbb{C}^+}$ *at* $\{\pm\frac{A}{2}, ik_0, \{p_j, -\overline{p}_j\}_{j=1}^n\}$, *where* $k_0 > 0$ *is the solution of* (2.12), Re p_j *(*$j = \overline{1, n}$*) are the solutions of* (2.14), *and* Im p_j *are determined by* (2.15).

Proof. Observe that the equation $a_1(k) = 0$ is equivalent to the system

$$\begin{cases} k_1 = \mp\frac{A}{2}\sin(2k_1 R)e^{-2k_2 R}, \\ k_2 = \pm\frac{A}{2}\cos(2k_1 R)e^{-2k_2 R}, \end{cases} \tag{2.18}$$

where $k = k_1 + ik_2$, $k \in \overline{\mathbb{C}^+} \setminus \{0\}$. Due to the symmetry relation $a_1(k) = \overline{a_1(-\bar{k})}$ it is sufficient to consider (2.18) for $k_1 \geq 0$ only.

(i) Assuming $k_1 = 0$, the system (2.18) reduces to the equations $k_2 = \pm\frac{A}{2}e^{-2k_2 R}$ and thus $a_1(k)$ has exactly one, purely imaginary simple zero (with $k_2 > 0$) for all $R > 0$ and $A > 0$, and its imaginary part is the solution of (2.12).

(ii) Assuming $k_2 = 0$, the second equation in (2.18) implies that k_1 must be equal to $\frac{\pi+2\pi n}{4R}$ with some $n \in \mathbb{N} \cup \{0\}$. But then, from the first equation in (2.18) we conclude that $k_1 = \frac{A}{2}$. Therefore, $k = \frac{A}{2}$ is a simple zero of $a_1(k)$ if and only if there exists $n \in \mathbb{N} \cup \{0\}$ such that $\pi + 2\pi n = 2AR$.

(iii) Now, let's look at the location of zeros of $a_1(k)$ in the open quarter plane $k_1 > 0$, $k_2 > 0$. Dividing the equations in (2.18) we arrive at (cf. (2.15))

$$k_2 = -k_1 \cot(2k_1 R), \quad k_1 \neq \frac{n\pi}{4R}, \ n \in \mathbb{N}, \tag{2.19}$$

from which we conclude (cf. (2.16)) that

$$k_1 \in \left(\frac{(2n-1)\pi}{4R}, \frac{n\pi}{2R} \right), \quad n \in \mathbb{N}. \tag{2.20}$$

Substituting (2.19) into the first equation in (2.18) and taking into account the sign of $\sin(2k_1 R)$ for k_1 satisfying (2.20), we obtain an equation for k_1 in the form

$$k_1 = \frac{A}{2} \sin(2k_1 R) e^{2k_1 R \cot(2k_1 R)} \quad \text{for } k_1 \in \left(\frac{(4n-3)\pi}{4R}, \frac{(2n-1)\pi}{2R} \right), \quad n \in \mathbb{N}, \tag{2.21a}$$

or

$$k_1 = -\frac{A}{2} \sin(2k_1 R) e^{2k_1 R \cot(2k_1 R)} \quad \text{for } k_1 \in \left(\frac{(4n-1)\pi}{4R}, \frac{n\pi}{R} \right), \quad n \in \mathbb{N}. \tag{2.21b}$$

Since the r.h.s. of (2.21a) and (2.21b) monotonically decrease in the corresponding intervals for k_1, it follows that Eq. (2.21) have no solutions for $0 < R \leq \frac{\pi}{2A}$, whereas for $\frac{(2n-1)\pi}{2A} < R \leq \frac{(2n+1)\pi}{2A}$ Eq. (2.21) have n simple solutions $\{k_{1,j}\}_{j=1}^{n}$ such that $k_{1,j} \in \left(\frac{(2j-1)\pi}{4R}, \frac{j\pi}{2R} \right)$, $j = \overline{1, n}$ (cf. (2.16)).

Concerning the winding properties of $\arg a_1(k)$, (2.13) (for $0 < R < \frac{\pi}{2A}$) follows from the inequality

$$\frac{A^2}{4k_{(m)}^2} e^{4ik_{(m)}R} > -1, \quad \text{where } k_{(m)} = \frac{(1-2m)\pi}{4R}, \quad m \in \mathbb{N}, \tag{2.22}$$

whereas (2.17) (for $\frac{(2n-1)\pi}{2A} < R < \frac{(2n+1)\pi}{2A}$) follows from

$$\frac{A^2}{4k_{(m)}^2} e^{4ik_{(m)}R} > -1 \quad \text{for } k_{(m)} = \frac{(1-2m)\pi}{4R}, \quad m \in \mathbb{N}, \quad m > n, \tag{2.23a}$$

$$\frac{A^2}{4k_{(m)}^2} e^{4ik_{(m)}R} < -1 \quad \text{for } k_{(m)} = \frac{(1-2m)\pi}{4R}, \quad m \in \mathbb{N}, \quad m \leq n. \tag{2.23b}$$

□

2.3. The basic Riemann–Hilbert problem and inverse scattering. The Riemann–Hilbert formalism of the inverse scattering transform method is based on constructing a piece-wise meromorphic, 2×2-valued function in the k-complex plane, which has a prescribed jump across a contour in the complex plane and prescribed conditions at singular points (in case they are present).

The analytic properties of the Jost solutions Ψ_j suggest defining the 2×2-valued function $M(x,t,k)$, piece-wise meromorphic in $\mathbb{C}\setminus\mathbb{R}$, as follows [44]:

$$M(x,t,k)=\begin{cases}\left(\frac{\Psi_1^{(1)}(x,t,k)}{a_1(k)},\Psi_2^{(2)}(x,t,k)\right), & k\in\mathbb{C}^+\setminus\{0\},\\ \left(\Psi_2^{(1)}(x,t,k),\frac{\Psi_1^{(2)}(x,t,k)}{a_2(k)}\right), & k\in\mathbb{C}^-\setminus\{0\}.\end{cases} \tag{2.24}$$

Then the scattering relation (2.4) implies that the boundary values $M_\pm(x,t,k)=\lim_{k'\to k,k'\in\mathbb{C}^\pm} M(x,t,k')$, $k\in\mathbb{R}$ satisfy the multiplicative jump condition

$$M_+(x,t,k)=M_-(x,t,k)J(x,t,k),\qquad k\in\mathbb{R}\setminus\{0\}, \tag{2.25}$$

where

$$J(x,t,k)=\begin{pmatrix}1+r_1(k)r_2(k) & r_2(k)e^{-2ikx-4ik^2t}\\ r_1(k)e^{2ikx+4ik^2t} & 1\end{pmatrix}, \tag{2.26}$$

with the reflection coefficients r_1 and r_2 defined by

$$r_1(k):=\frac{b(k)}{a_1(k)},\quad r_2(k):=\frac{\overline{b(-k)}}{a_2(k)}. \tag{2.27}$$

Observe that by the determinant relation (see item 4 in Proposition 1) we have

$$1+r_1(k)r_2(k)=\frac{1}{a_1(k)a_2(k)}. \tag{2.28}$$

Moreover,

$$M(x,t,k)\to I,\qquad k\to\infty, \tag{2.29}$$

where I is the 2×2 identity matrix.

Taking into account the singularities of $\Psi_j(x,t,k)$, $j=1,2$ and $a_1(k)$ at $k=0$ (see Proposition 1 and Remark 1), the behavior of $M(x,t,k)$ at $k=0$ can be described as follows:

$$M(x,t,k)=\begin{pmatrix}\frac{4}{A^2a_2(0)}v_1(x,t) & -\overline{v_2}(-x,t)\\ \frac{4}{A^2a_2(0)}v_2(x,t) & -\overline{v_1}(-x,t)\end{pmatrix}(I+O(k))\begin{pmatrix}k & 0\\ 0 & \frac{1}{k}\end{pmatrix},\quad k\to i0^+, \tag{2.30a}$$

$$M(x,t,k)=\frac{2i}{A}\begin{pmatrix}-\overline{v_2}(-x,t) & \frac{v_1(x,t)}{a_2(0)}\\ -\overline{v_1}(-x,t) & \frac{v_2(x,t)}{a_2(0)}\end{pmatrix}+O(k),\quad k\to i0^-, \tag{2.30b}$$

or, in terms of M only,

$$\lim_{k\to i0^+}M(x,t,k)\begin{pmatrix}\frac{1}{k} & 0\\ 0 & k\end{pmatrix}=M(x,t,i0^-)\begin{pmatrix}0 & \frac{A}{2i}\\ -\frac{2i}{A} & 0\end{pmatrix}. \tag{2.31}$$

Now, being motivated by the properties of $a_1(k)$ and $a_2(k)$ in the case of "shifted step" initial data (see Proposition 2), we make the following additional assumptions on $a_1(k)$ and $a_2(k)$ in the case of general step-like initial data satisfying (1.1c):

Assumptions A:

(a-1) $a_1(k)$ has $2n+1, n\in\mathbb{N}\cup\{0\}$, simple zeros in $\overline{\mathbb{C}^+}\setminus\{0\}$ at $k=ik_0$ with $k_0>0$, at $\{p_j\}_{j=1}^n$ and at $\{-\overline{p}_j\}_{j=1}^n$ with $\operatorname{Im} p_j>0$ and $\operatorname{Re} p_n<\cdots<\operatorname{Re} p_1<0$.

(a-2) $a_2(k)$ has no zeros in $\overline{\mathbb{C}^-}$.

(b) There are $\omega_m>0$, $m=\overline{0,n+1}$, such that

$$\begin{aligned}&-\infty=-\omega_{n+1}<\operatorname{Re} p_n<-\omega_n<\operatorname{Re} p_{n-1}<-\omega_{n-1}<\cdots\\&\quad<\operatorname{Re} p_1<-\omega_1<\omega_0=0,\end{aligned}\tag{2.32}$$

$$\int_{-\infty}^{-\omega_{n-m+1}} d\arg\left(a_1(k)a_2(k)\right)=(2m-1)\pi,\quad m=\overline{1,n},\tag{2.33a}$$

and

$$\begin{aligned}&\int_{-\infty}^{-\xi} d\arg\left(a_1(k)a_2(k)\right)\in((2m-1)\pi,(2m+1)\pi),\\&-\omega_{n-m+1}<-\xi<-\omega_{n-m},\quad m=\overline{0,n}.\end{aligned}\tag{2.33b}$$

In accordance with this assumption, $M(x,t,k)$ satisfies the residue conditions:

$$\operatorname*{Res}_{k=ik_0} M^{(1)}(x,t,k)=\frac{\gamma_0}{\dot{a}_1(ik_0)}e^{-2k_0x-4ik_0^2t}M^{(2)}(x,t,ik_0),\quad |\gamma_0|=1,\tag{2.34a}$$

$$\operatorname*{Res}_{k=p_j} M^{(1)}(x,t,k)=\frac{\eta_j}{\dot{a}_1(p_j)}e^{2ip_jx+4ip_j^2t}M^{(2)}(x,t,p_j),\quad j=\overline{1,n},\tag{2.34b}$$

$$\operatorname*{Res}_{k=-\overline{p}_j} M^{(1)}(x,t,k)=\frac{1}{\overline{\eta}_j\dot{a}_1(-\overline{p}_j)}e^{-2i\overline{p}_jx+4i\overline{p}_j^2t}M^{(2)}(x,t,-\overline{p}_j),\quad j=\overline{1,n},\tag{2.34c}$$

where γ_0 and η_j come from the relations $\Psi_1^{(1)}(0,0,ik_0)=\gamma_0\Psi_2^{(2)}(0,0,ik_0)$ and $\Psi_1^{(1)}(0,0,p_j)=\eta_j\Psi_2^{(2)}(0,0,p_j)$ for the eigenfunctions of the first equation from the Lax pair (2.1).

Remark 2. If $b(k)$ allows analytical continuation into a sufficiently large band in the complex plane, the norming constants take the form:

$$\gamma_0=b(ik_0),\ \eta_j=b(p_j).$$

The matrix-valued function M defined in (2.24) can be uniquely characterized as a solution of

Basic Riemann–Hilbert Problem: Given (i) $b(k)$ for $k\in\mathbb{R}$, (ii) $a_j(k)$, $j=1,2$ having the properties of Proposition 1 and satisfying Assumptions A, with $\{ik_0,\{p_j,-\bar{p}_j\}_1^n\}$ being the zeros of $a_1(k)$ in $\mathbb{C}^+$, and (iii) γ_0 and $\{\eta_j\}_1^n$, find the 2×2-valued function $M(x,t,k)$, piece-wise meromorphic in k relative to $\mathbb{R}$ and satisfying the following conditions:

(i) Jump conditions. The boundary values $M_\pm(x,t,k)=M(x,t,k\pm i0)$, $k\in\mathbb{R}\setminus\{0\}$ satisfy the condition

$$M_+(x,t,k)=M_-(x,t,k)J(x,t,k),\qquad k\in\mathbb{R}\setminus\{0\},\tag{2.35}$$

where the jump matrix $J(x,t,k)$ is given by (2.26), with $r_j(k)$ given in terms of $b(k)$ and $a_j(k)$ by (2.27).

(ii) Normalization at $k = \infty$:

$$M(x,t,k) = I + O(k^{-1}), \qquad k \to \infty.$$

(iii) Residue conditions (2.34).

(iv) *Pseudo-residue* condition at $k = 0$: $M(x,t,k)$ satisfies (2.31).

Assume that the RH problem (i)–(iv) has a solution $M(x,t,k)$. Then the solution of the Cauchy problem (1.1), (1.2) is given in terms of the (12) and (21) entries of $M(x,t,k)$ as follows:

$$q(x,t) = 2i \lim_{k\to\infty} kM_{12}(x,t,k), \tag{2.36}$$

and

$$q(-x,t) = -2i \lim_{k\to\infty} k\overline{M_{21}(x,t,k)}. \tag{2.37}$$

Remark 3. We coin a term "*pseudo-residue*" for condition (2.31) since M hasn't, in general, an isolated singularity at $k = 0$, but after applying appropriate transformations of the basic Riemann–Hilbert problem it turns into a conventional residue condition (see the RH problem (3.17)–(3.22) below).

The solution of the RH problem is unique, if it exists. Indeed, if M and $\tilde{M}$ are two solutions, then condition (2.31) provides the boundedness of $M\tilde{M}^{-1}$ at $k = 0$ and, therefore, the applicability of standard arguments based on the Liouville theorem: $M\tilde{M}^{-1}$ turns to be a function analytic in the whole complex plane approaching I at infinity; thus it equals I identically.

Remark 4. From (2.36) and (2.37) it follows that in order to present the solution of (1.1), (1.2) for all $x \in \mathbb{R}$, it is sufficient to have the solution of the RH problem for, say, $x \geq 0$ only.

3. The Long-Time Asymptotics

In this section we study the long-time asymptotics of the solution of the Cauchy problem (1.1), (1.2). Our analysis is based on the adaptation of the nonlinear steepest-decent method [20] to the (oscillatory) RH problem (i)–(iv).

3.1. Jump factorizations. Introduce the variable $\xi := \frac{x}{4t}$ and the phase function

$$\theta(k,\xi) = 4k\xi + 2k^2. \tag{3.1}$$

In view of (2.36) and (2.37), we will study the RH problem for $\xi > 0$ only. Moreover, we will consider $x, t > 0$ in the open sectors, where $-\xi \in (-\omega_{n-m+1}, -\omega_{n-m})$ for some $m = \overline{0,n}$ (see Fig. 3), and will study the asymptotics as $t \to \infty$ while keeping ξ fixed. A special attention will be paid to the directions $\xi = \pm \operatorname{Re} p_{n-m}$, see Sect. 4.

The jump matrix (2.26) allows, similarly to [43], two triangular factorizations:

$$J(x,t,k) = \begin{pmatrix} 1 & 0 \\ \frac{r_1(k)}{1+r_1(k)r_2(k)} e^{2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 + r_1(k)r_2(k) & 0 \\ 0 & \frac{1}{1+r_1(k)r_2(k)} \end{pmatrix} \begin{pmatrix} 1 & \frac{r_2(k)}{1+r_1(k)r_2(k)} e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \tag{3.2a}$$

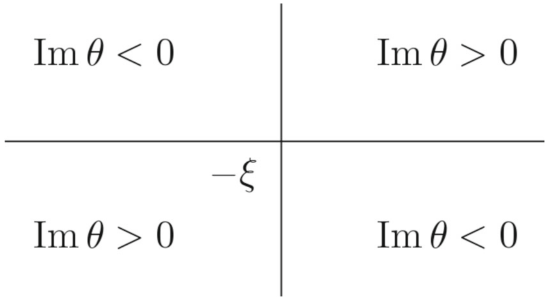

Fig. 2. Signature table

$$= \begin{pmatrix} 1 & r_2(k)e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_1(k)e^{2it\theta} & 1 \end{pmatrix}. \tag{3.2b}$$

Since the phase function $\theta(k, \xi)$ is the same as in the case of the local NLS, its signature table (see Fig. 2) suggests that we follow the standard steps [16,20] to remove the diagonal factor in (3.2a). The original RH (relative to the real axis) can then be deformed to a new one relative to a cross, where the jump matrix converges, as $t \to \infty$, to the identity matrix uniformly away from any vicinity of the stationary phase point $k = -\xi$.

In order to get rid of the diagonal factor in a factorization like (3.2a), one usually [16,20] introduces a (scalar) function $\delta(k)$ that solves the scalar RH problem with the jump condition $\delta_+(k) = \delta_-(k)(1 + r_1(k)r_2(k))$ for $k \in (-\infty, -\xi)$. In the case $1 + r_1(k)r_2(k) > 0$ for all $k \in \mathbb{R}$, as takes place for the local NLS equation, $\delta(k)$ is defined via a Cauchy integral involving $\ln(1 + r_1(k)r_2(k))$. However, in the case of the nonlocal NLS equation, the values of $1 + r_1(k)r_2(k)$ are, in general, complex, which would lead to a strong singularity of $\delta(k)$ at $k = -\xi$. In order to avoid this, we proceed as follows:

1. First, define some "partial delta functions":

$$\delta_s(k) = \delta_s(k; \omega_{n-s}, \omega_{n-s+1}) = \exp\left\{\frac{1}{2\pi i}\int_{-\omega_{n-s+1}}^{-\omega_{n-s}} \frac{\ln(1 + r_1(\zeta)r_2(\zeta))}{\zeta - k}\, d\zeta\right\},$$
$$s = \overline{0, m-1} \text{ for } m \geq 1, \tag{3.3a}$$

$$\delta_m(k) = \delta_m(k, \xi; \omega_{n-m+1}) = \exp\left\{\frac{1}{2\pi i}\int_{-\omega_{n-m+1}}^{-\xi} \frac{\ln(1 + r_1(\zeta)r_2(\zeta))}{\zeta - k}\, d\zeta\right\}, \tag{3.3b}$$

where $-\xi \in (-\omega_{n-m+1}, -\omega_{n-m})$, $m = \overline{0, n}$ and the following branches of logarithm are chosen for $s = \overline{0, m}$ (notice that since we deal with $\xi > 0$, the behavior of $r_j(k)$ at $k = 0$ does not affect $\delta_m(k)$):

$$\ln(1 + r_1(\zeta)r_2(\zeta)) = \ln|1 + r_1(\zeta)r_2(\zeta)| + i\left(\int_{-\infty}^{\zeta} d\arg(1 + r_1(z)r_2(z)) + 2\pi s\right). \tag{3.4}$$

2. Second, define

$$\delta(k, \xi) := \prod_{s=0}^{m} \delta_s(k). \tag{3.5}$$

In this way, we have that (i) $\delta(k,\xi)\to 1$ as $k\to\infty$ and (ii) it satisfies, for each $-\xi\in(-\omega_{n-m+1},-\omega_{n-m})$, $m=\overline{0,n}$ (see (2.32)), the jump condition

$$\delta_+(k,\xi)=\delta_-(k,\xi)(1+r_1(k)r_2(k)),\quad k\in(-\infty,-\xi)\setminus\{-\omega_{n-s}\}_{s=0}^{m-1}. \tag{3.6}$$

Moreover, it has particular singularities at $k=-\omega_{n-s}$, $s=\overline{0,m-1}$, and $k=-\xi$. Namely, adapting the convention that $\prod\limits_{s=m_1}^{m_2} F_s=1$ if $m_1>m_2$ and integrating by parts in (3.3), (3.5) takes the form

$$\delta(k,\xi)=(k+\xi)^{i\nu(-\xi)}\prod_{s=0}^{m-1}(k+\omega_{n-s})^{-1}\exp\left\{\sum_{s=0}^{m}\chi_s(k)\right\}, \tag{3.7}$$

with

$$\chi_s(k)=-\frac{1}{2\pi i}\int_{-\omega_{n-s+1}}^{-\omega_{n-s}}\ln(k-\zeta)\,d_\zeta\ln(1+r_1(\zeta)r_2(\zeta)),\ s=\overline{0,m-1}\text{ for }m\geq 1, \tag{3.8a}$$

$$\chi_m(k)=-\frac{1}{2\pi i}\int_{-\omega_{n-m+1}}^{-\xi}\ln(k-\zeta)\,d_\zeta\ln(1+r_1(\zeta)r_2(\zeta)), \tag{3.8b}$$

and

$$\nu(-\xi)=-\frac{1}{2\pi}\ln|1+r_1(-\xi)r_2(-\xi)|-\frac{i}{2\pi}\left(\int_{-\infty}^{-\xi}d\arg(1+r_1(\zeta)r_2(\zeta))+2\pi m\right), \tag{3.9}$$

so that (see Assumptions A(b) (2.33) and relation (2.28)) Im $\nu(-\xi)$ satisfies the inequalities

$$-\frac{1}{2}<\operatorname{Im}\nu(-\xi)<\frac{1}{2}.$$

Remark 5. In our asymptotic analysis, it is important to have Im $\nu(-\xi)\in(-\frac{1}{2},\frac{1}{2})$. This property will provide the convergence, as $t\to\infty$, of the solution of the deformed Riemann–Hilbert problem (relative to the cross centered at $k=-\xi$) to the identity matrix, see Sect. 3.2 below.

Now we define

$$\tilde{M}(x,t,k)=M(x,t,k)\delta^{-\sigma_3}(k,\xi), \tag{3.10}$$

and notice that $\tilde{M}$ satisfies the following conditions:

- normalization

$$\tilde{M}(x,t,k)\to I,\qquad k\to\infty; \tag{3.11}$$

- jump conditions

$$\tilde{M}_+(x,t,k)=\tilde{M}_-(x,t,k)\tilde{J}(x,t,k),\quad k\in\mathbb{R}\setminus\left(\{-\omega_{n-s}\}_{s=0}^{m-1}\cup\{0\}\right), \tag{3.12a}$$

where

$$\tilde{J}(x,t,k)=\begin{cases}\begin{pmatrix}1 & 0\\ \frac{r_1(k)\delta_-^{-2}(k,\xi)}{1+r_1(k)r_2(k)}e^{2it\theta} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{r_2(k)\delta_+^{2}(k,\xi)}{1+r_1(k)r_2(k)}e^{-2it\theta}\\ 0 & 1\end{pmatrix}, & k\in(-\infty,-\xi)\setminus\{-\omega_{n-s}\}_{s=0}^{m-1},\\ \begin{pmatrix}1 & r_2(k)\delta^2(k,\xi)e^{-2it\theta}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ r_1(k)\delta^{-2}(k,\xi)e^{2it\theta} & 1\end{pmatrix}, & k\in(-\xi,\infty)\setminus\{0\};\end{cases} \tag{3.12b}$$

- residue conditions at zeros of $a_1(k)$:

$$\operatorname*{Res}_{k=ik_0}\tilde{M}^{(1)}(x,t,k)=\frac{\gamma_0}{\dot{a}_1(ik_0)\delta^2(ik_0,\xi)}e^{-2k_0x-4ik_0^2t}\tilde{M}^{(2)}(x,t,ik_0),\quad |\gamma_0|=1, \tag{3.13a}$$

$$\operatorname*{Res}_{k=p_j}\tilde{M}^{(1)}(x,t,k)=\frac{\eta_j}{\dot{a}_1(p_j)\delta^2(p_j,\xi)}e^{2ip_jx+4ip_j^2t}\tilde{M}^{(2)}(x,t,p_j),\quad j=\overline{1,n}, \tag{3.13b}$$

$$\operatorname*{Res}_{k=-\overline{p}_j}\tilde{M}^{(1)}(x,t,k)=\frac{1}{\bar{\eta}_j\dot{a}_1(-\overline{p}_j)\delta^2(-\overline{p}_j,\xi)}e^{-2i\overline{p}_jx+4i\overline{p}_j^2t}\tilde{M}^{(2)}(x,t,-\overline{p}_j),\quad j=\overline{1,n}; \tag{3.13c}$$

- pseudo-residue condition at $k=0$:

$$\lim_{k\to i0^+}\tilde{M}(x,t,k)\begin{pmatrix}\frac{1}{k} & 0\\ 0 & k\end{pmatrix}=\tilde{M}(x,t,i0^-)\begin{pmatrix}0 & \frac{A}{2i}\delta^2(0,\xi)\\ -\frac{2i}{A}\delta^{-2}(0,\xi) & 0\end{pmatrix}; \tag{3.14}$$

- singularity conditions at $k=-\omega_{n-s}$, $s=\overline{0,m-1}$:

$$\tilde{M}_\pm(x,t,k)(k+\omega_{n-s})^{-\sigma_3}=O(1),\quad k\to-\omega_{n-s}. \tag{3.15}$$

Conditions (3.11)–(3.15) constitutes the Riemann–Hilbert problem, whose solution is (by the Liouville theorem) unique, if it exists.

3.2. The RH problem deformations. In order to turn oscillations to exponential decay in the Riemann–Hilbert problem (3.11)–(3.15), we "deform" the contour off the real axis. When doing it, we assume that the reflection coefficients $r_j(k)$, $j=1,2$, can be analytically continued into the whole complex plane. This takes place, for example, if $q_0(x)$ is a local (finitely supported) perturbation of the pure step initial data (1.5). Otherwise, it is possible to approximate $r_j(k)$ and $\frac{r_j(k)}{1+r_1(k)r_2(k)}$ by some rational functions with well-controlled errors (see [16]). The modern way to deal with non-analytic reflection coefficients is to use the $\bar{\partial}$ generalization of the nonlinear steepest descent method (for the local NLS equation, see [38,39]).

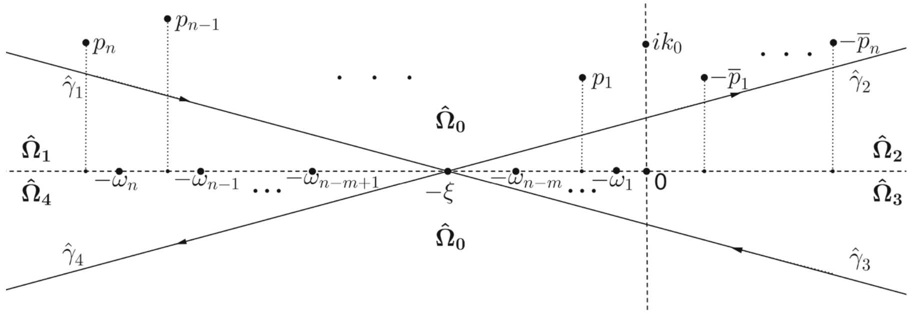

Fig. 3. The domains $\hat{\Omega}_j$, $j = 0, \dots, 4$ and the contour $\hat{\Gamma} = \hat{\gamma}_1 \cup ... \cup \hat{\gamma}_4$

Define $\hat{M}(x,t,k)$ as follows (see Fig. 3; here $\hat{\Omega}_0$ is chosen in such a way that all zeros of $a_1(k)$ are located in it):

$$\hat{M}(x,t,k) = \begin{cases} \tilde{M}(x,t,k), & k \in \hat{\Omega}_0, \\ \tilde{M}(x,t,k) \begin{pmatrix} 1 & \frac{-r_2(k)\delta^2(k,\xi)}{1+r_1(k)r_2(k)} e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\Omega}_1, \\ \tilde{M}(x,t,k) \begin{pmatrix} 1 & 0 \\ -r_1(k)\delta^{-2}(k,\xi)e^{2it\theta} & 1 \end{pmatrix}, & k \in \hat{\Omega}_2, \\ \tilde{M}(x,t,k) \begin{pmatrix} 1 & r_2(k)\delta^2(k,\xi)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\Omega}_3, \\ \tilde{M}(x,t,k) \begin{pmatrix} 1 & 0 \\ \frac{r_1(k)\delta^{-2}(k,\xi)}{1+r_1(k)r_2(k)} e^{2it\theta} & 1 \end{pmatrix}, & k \in \hat{\Omega}_4. \end{cases} \tag{3.16}$$

By (3.12b), $\hat{M}(x,t,k)$ has no jump across the real axis while having point singularities at $k = -\omega_{n-s}$, $s = \overline{0, m-1}$ and at $k = 0$. Specifically,

$$\hat{M}(x,t,k)(k+\omega_{n-s})^{-\sigma_3} = O(1), \quad k \to -\omega_{n-s}, \quad s = \overline{0, m-1}, \tag{3.17}$$

whereas the singularity condition at $k = 0$ takes the form of a conventional residue condition:

$$\operatorname*{Res}_{k=0} \hat{M}^{(2)}(x,t,k) = c_0(\xi)\hat{M}^{(1)}(x,t,0), \tag{3.18}$$

with $c_0(\xi) = \frac{A\delta^2(0,\xi)}{2i}$. Indeed, item 5 of Proposition 1 implies that

$$r_1(k) = \frac{2k}{iA} + O(k^2), \qquad k \to 0,$$

and thus, by (3.16) for $k \in \hat{\Omega}_2$, we have

$$\hat{M}^{(2)}(k) = \frac{A\delta^2(0,\xi)}{2ik}\hat{M}^{(1)}(0) + O(1), \quad k \to i0^+.$$

Similarly, by (3.16) for $k \in \hat{\Omega}_3$ and the development of $r_2(k)$

$$r_2(k) = \frac{A}{2ik} + O(1), \qquad k \to 0,$$

we have

$$\hat{M}^{(2)}(k) = \frac{A\delta^2(0,\xi)}{2ik}\hat{M}^{(1)}(0) + O(1), \quad k \to i0^-,$$

and thus (3.18) follows.

The jump conditions for $\hat{M}(x,t,k)$ are formulated on $\hat{\Gamma}$:

$$\hat{M}_+(x,t,k) = \hat{M}_-(x,t,k)\hat{J}(x,t,k), \qquad k \in \hat{\Gamma}, \tag{3.19a}$$

with the jump matrices

$$\hat{J}(x,t,k) = \begin{cases} \begin{pmatrix} 1 & \frac{r_2(k)\delta^2(k,\xi)}{1+r_1(k)r_2(k)}e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\gamma}_1, \\ \begin{pmatrix} 1 & 0 \\ r_1(k)\delta^{-2}(k,\xi)e^{2it\theta} & 1 \end{pmatrix}, & k \in \hat{\gamma}_2, \\ \begin{pmatrix} 1 & -r_2(k)\delta^2(k,\xi)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & k \in \hat{\gamma}_3, \\ \begin{pmatrix} 1 & 0 \\ \frac{-r_1(k)\delta^{-2}(k,\xi)}{1+r_1(k)r_2(k)}e^{2it\theta} & 1 \end{pmatrix}, & k \in \hat{\gamma}_4, \end{cases} \tag{3.19b}$$

Other characteristic properties of $\hat{M}(x,t,k)$ follows from their analogues for $\tilde{M}(x,t,k)$: the normalization condition

$$\hat{M}(x,t,k) \to I, \qquad k \to \infty \tag{3.20}$$

and the residue conditions

$$\operatorname*{Res}_{k=ik_0} \hat{M}^{(1)}(x,t,k) = c_1(x,t)\hat{M}^{(2)}(x,t,ik_0), \quad |\gamma_0| = 1, \tag{3.21a}$$

$$\operatorname*{Res}_{k=p_j} \hat{M}^{(1)}(x,t,k) = f_j(x,t)\hat{M}^{(2)}(x,t,p_j), \quad j = \overline{1,n}, \tag{3.21b}$$

$$\operatorname*{Res}_{k=-\overline{p}_j} \hat{M}^{(1)}(x,t,k) = \hat{f}_j(x,t)\hat{M}^{(2)}(x,t,-\overline{p}_j), \quad j = \overline{1,n}, \tag{3.21c}$$

where

$$c_1(x,t) = \frac{\gamma_0 e^{-2k_0x-4ik_0^2t}}{\dot{a}_1(ik_0)\delta^2(ik_0,\xi)}, \quad f_j(x,t) = \frac{\eta_j e^{2ip_jx+4ip_j^2t}}{\dot{a}_1(p_j)\delta^2(p_j,\xi)},$$
$$\hat{f}_j(x,t) = \frac{e^{-2i\overline{p}_jx+4i\overline{p}_j^2t}}{\bar{\eta}_j\dot{a}_1(-\overline{p}_j)\delta^2(-\overline{p}_j,\xi)}. \tag{3.22}$$

Similarly to $\tilde{M}$, $\hat{M}(x,t,k)$ can be characterized as a unique solution of the RH problem specified by conditions (3.17)–(3.22).

Proposition 3. *For any fixed* $\xi = \frac{x}{4t}$ *with* $\xi > 0$*, the solution of the Riemann–Hilbert problem* (3.17)–(3.22) *can be "reduced", as* $t \to \infty$*, to a sectionally meromorphic matrix-valued function* $M^{as}(\xi, t, k)$*, in the sense that* $q(x, t)$ *extracted from the large-k asymptotics of* $\hat{M}(x, t, k)$ *and* $M^{as}(\xi, t, k)$ *are exponentially close as* $t \to \infty$*:*

$$q(x,t) = 2i \lim_{k\to\infty} kM_{12}^{as}(\xi,t,k) + \textit{exponentially small terms}, \quad t \to \infty, \tag{3.23}$$

$$q(-x,t) = -2i \lim_{k\to\infty} k\overline{M_{21}^{as}(\xi,t,k)} + \textit{exponentially small terms}, \quad t \to \infty. \tag{3.24}$$

Here M^{as} *solves one of the following Riemann–Hilbert problems, depending on the value of* ξ*, with a single residue condition at* $k = 0$ *(to simplify the notations, we set* $\operatorname{Re} p_0 := 0$ *and* $\prod_{s=m_1}^{m_2} F_s = 1$ *if* $m_1 > m_2$*):*

(i) for $-\omega_{n-m+1} < -\xi < \operatorname{Re} p_{n-m}$*,* $m = \overline{0, n}$*,* M^{as} *solves*

$$M_+^{as}(\xi,t,k) = M_-^{as}(\xi,t,k)J^{as}(\xi,t,k), \qquad k \in \hat{\Gamma}, \tag{3.25a}$$

$$M^{as}(\xi,t,k) \to I, \qquad k \to \infty, \tag{3.25b}$$

$$\operatorname*{Res}_{k=0} M^{as\,(2)}(\xi,t,k) = c_0^{as}(\xi)M^{as\,(1)}(\xi,t,0), \tag{3.25c}$$

with

$$c_0^{as}(\xi) = \frac{A\delta^2(0,\xi)}{2i} \prod_{s=0}^{m-1} \left(\frac{\omega_{n-s}}{p_{n-s}}\right)^2, \tag{3.26}$$

and

$$J^{as}(\xi,t,k) = \left(\prod_{s=0}^{m-1} \frac{k+\omega_{n-s}}{k-p_{n-s}}\right)^{\sigma_3} \hat{J}(x,t,k) \left(\prod_{s=0}^{m-1} \frac{k+\omega_{n-s}}{k-p_{n-s}}\right)^{-\sigma_3}, \ k \in \hat{\Gamma}. \tag{3.27}$$

(ii) for $\operatorname{Re} p_{n-m} < -\xi < -\omega_{n-m}$*,* $m = \overline{0, n-1}$*,* M^{as} *solves*

$$M_+^{as}(\xi,t,k) = M_-^{as}(\xi,t,k)J^{as}(\xi,t,k), \qquad k \in \hat{\Gamma}, \tag{3.28a}$$

$$M^{as}(\xi,t,k) \to I, \qquad k \to \infty, \tag{3.28b}$$

$$\operatorname*{Res}_{k=0} M^{as\,(1)}(\xi,t,k) = c_0^{as\#}(\xi)M^{as\,(2)}(\xi,t,0), \tag{3.28c}$$

with

$$c_0^{as\#}(\xi) = \frac{2ip_{n-m}^2}{A\delta^2(0,\xi)} \prod_{s=0}^{m-1} \left(\frac{p_{n-s}}{\omega_{n-s}}\right)^2, \tag{3.29}$$

and

$$J^{as}(\xi,t,k) = \left(d(k)\prod_{s=0}^{m-1} \frac{k+\omega_{n-s}}{k-p_{n-s}}\right)^{\sigma_3} \hat{J}(x,t,k) \left(d(k)\prod_{s=0}^{m-1} \frac{k+\omega_{n-s}}{k-p_{n-s}}\right)^{-\sigma_3}, \ k \in \hat{\Gamma}, \tag{3.30}$$

with $d(k) = \frac{k}{k-p_{n-m}}$.

Proof. (i) First, consider $-\omega_{n-m+1} < -\xi < \operatorname{Re} p_{n-m}$, $m = \overline{0, n}$. Then $\hat{M}(x,t,k)$ has m singular points $k = -\omega_{n-s}$, $s = \overline{0, m-1}$, and m exponentially growing residue conditions at $k = p_{n-s}$, $s = \overline{0, m-1}$, see (3.21b). Introducing $\check{M}(x,t,k)$ by

$$\check{M}(x,t,k) := \hat{M}(x,t,k)\left(\prod_{s=0}^{m-1}\frac{k+\omega_{n-s}}{k-p_{n-s}}\right)^{-\sigma_3}, \quad k \in \mathbb{C}, \tag{3.31}$$

we obtain that $\check{M}(x,t,k)$ is, on one hand, bounded at $k = -\omega_{n-s}$, $s = \overline{0, m-1}$, and on the other hand, has exponentially decaying residue conditions at all points of the discrete spectrum: ik_0, p_j, $-\overline{p}_j$, $j = \overline{1, n}$. Direct calculations show that $\check{M}(x,t,k)$ has the residue condition at $k = 0$ and the jump across $\hat{\Gamma}$ as indicated in (3.25) and (3.26). It follows that $q(x,t)$ and $q(-x,t)$ obtained via (2.36) and (2.37) from the large-k asymptotics of $M(x,t,k)$ are exponentially close, as $t \to \infty$, to that obtained from M^{as} provided the latter exists as a solution of the RH problem (3.25).

(ii) Now consider $\operatorname{Re} p_{n-m} < -\xi < -\omega_{n-m}$, $m = \overline{0, n-1}$. Then $\hat{M}(x,t,k)$ has m singular points at $k = -\omega_{n-s}$, $s = \overline{0, m-1}$, and $m+1$ exponentially growing residue conditions at $k = p_{n-s}$, $s = \overline{0, m}$. Applying the same transformation (3.31) and ignoring the decaying residue conditions, we arrive at the Riemann–Hilbert problem with the exponentially growing residue condition at $k = p_{n-m}$ (see (3.22)):

$$\tilde{M}^{as}_{+}(\xi,t,k) = \tilde{M}^{as}_{-}(\xi,t,k)\tilde{J}^{as}(\xi,t,k), \qquad k \in \hat{\Gamma}, \tag{3.32a}$$

$$\tilde{M}^{as}(\xi,t,k) \to I, \qquad k \to \infty, \tag{3.32b}$$

$$\operatorname*{Res}_{k=p_{n-m}} \tilde{M}^{as\,(1)}(\xi,t,k) = f(x,t)\tilde{M}^{as\,(2)}(\xi,t,p_{n-m}), \tag{3.32c}$$

$$\operatorname*{Res}_{k=0} \tilde{M}^{as\,(2)}(\xi,t,k) = c_0^{as}(\xi)\tilde{M}^{as\,(1)}(\xi,t,0), \tag{3.32d}$$

where $f(x,t) = f_{n-m}(x,t)\prod_{s=0}^{m-1}\left(\frac{p_{n-m}-p_{n-s}}{p_{n-m}+\omega_{n-s}}\right)^2$, $c_0^{as}(\xi)$ is given by (3.26), and

$$\tilde{J}^{as}(x,t,k) = \left(\prod_{s=0}^{m-1}\frac{k+\omega_{n-s}}{k-p_{n-s}}\right)^{\sigma_3}\hat{J}(x,t,k)\left(\prod_{s=0}^{m-1}\frac{k+\omega_{n-s}}{k-p_{n-s}}\right)^{-\sigma_3}, \; k \in \hat{\Gamma}.$$

In order to cope with the problem of growing residue condition, we make the following transformation (recall that $d(k) = \frac{k}{k-p_{n-m}}$)

$$\hat{M}^{as}(\xi,t,k) = \tilde{M}^{as}(\xi,t,k)d^{-\sigma_3}(k). \tag{3.33}$$

Then $\hat{M}^{as}(\xi,t,k)$ solves the RH problem with exponentially decaying residue condition at $k = p_{n-m}$:

$$\hat{M}^{as}_{+}(\xi,t,k) = \hat{M}^{as}_{-}(\xi,t,k)\hat{J}^{as}(\xi,t,k), \qquad k \in \hat{\Gamma}, \tag{3.34a}$$

$$\hat{M}^{as}(\xi,t,k) \to I, \qquad k \to \infty, \tag{3.34b}$$

$$\operatorname*{Res}_{k=p_{n-m}} \hat{M}^{as\,(1)}(\xi,t,k) = \frac{p_{n-m}^2}{f(x,t)}\hat{M}^{as\,(2)}(\xi,t,p_{n-m}), \tag{3.34c}$$

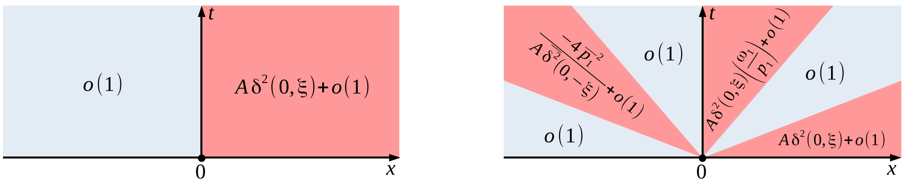

Fig. 4. Asymptotic behavior of the solution for $n = 0$ and $n = 1$

$$\operatorname*{Res}_{k=0} \hat{M}^{as\,(2)}(\xi, t, k) = \frac{p_{n-m}^2}{c_0^{as}(\xi)} \hat{M}^{as\,(1)}(\xi, t, 0), \tag{3.34d}$$

where $\hat{J}^{as}(\xi, t, k)$ is given by (3.30). Ignoring the decaying residue condition at $k = p_{n-m}$, we arrive at the Riemann–Hilbert problem (3.28). Therefore, as in case (i), (3.23) and (3.24) hold provided (3.28) has a solution M^{as}. The existence of solutions of the RH problems (3.25) and (3.28) can be justified for t sufficiently large, see the proof of Theorem 1 below. □

Proposition 3 suggests the rough (up to terms of order $o(1)$) asymptotics of $q(x, t)$:

Proposition 4. *As $t \to \infty$, $q(x, t)$ has the following asymptotics (see Fig. 4):*

$$q(x,t) = \begin{cases} A\delta^2(0,\xi) \prod_{s=0}^{m-1} \left(\frac{\omega_{n-s}}{p_{n-s}}\right)^2 + o(1), & -\operatorname{Re} p_{n-m} < \xi < \omega_{n-m+1}, \\ o(1), & -\omega_{n-m+1} < \xi < \operatorname{Re} p_{n-m} \text{ or } \omega_{n-m} < \xi < -\operatorname{Re} p_{n-m}, \\ \frac{-4\overline{p}_{n-m}^2}{A\overline{\delta^2}(0,-\xi)} \prod_{s=0}^{m-1} \left(\frac{\overline{p}_{n-s}}{\overline{\omega}_{n-s}}\right)^2 + o(1), & \operatorname{Re} p_{n-m} < \xi < -\omega_{n-m}, \end{cases} \tag{3.35}$$

where $m = \overline{0, n}$.

Remark 6. The asymptotic formulas (3.35) hold in the case of the "shifted step" initial value (1.5) with $n = 0$ for $0 < R < \frac{\pi}{2A}$, and with the corresponding value of $n \in \mathbb{N}$ for $\frac{(2n-1)\pi}{2A} < R < \frac{(2n+1)\pi}{2A}$.

Remark 7. The spectral functions associated to the "shifted step" initial value (1.5) with $0 < R < \frac{\pi}{2A}$ satisfy conditions of the Theorem 1 in [44]. The rough asymptotics, as well as the precise one (see Theorem 1 below) are consistent with that obtained in [44].

Remark 8. The ordering of $\operatorname{Re} p_j$ and $-\omega_j$, $j = \overline{1, n}$ in (2.32) is crucial for our analysis. Indeed, let $n = 1$ and assume that $-\omega_1 < \operatorname{Re} p_1 < 0$. Then, applying (3.31) for $-\omega_1 < -\xi < \operatorname{Re} p_1$, we (asymptotically) arrive at the following Riemann–Hilbert problem:

$$\tilde{M}_+^{as}(x, t, k) = \tilde{M}_-^{as}(x, t, k)\tilde{J}^{as}(x, t, k), \qquad k \in \hat{\Gamma}, \tag{3.36a}$$

$$\tilde{M}^{as}(x, t, k) \to I, \qquad k \to \infty, \tag{3.36b}$$

$$\operatorname*{Res}_{k=p_1} \tilde{M}^{as\,(2)}(x, t, k) = f_1^{-1}(x, t)(p_1 + \omega_1)^2 \tilde{M}^{as\,(1)}(x, t, p_1), \tag{3.36c}$$

$$\operatorname*{Res}_{k=0}\tilde{M}^{as\,(2)}(x,t,k)=c_0\frac{\omega_1^2}{p_1^2}\tilde{M}^{as\,(1)}(x,t,0), \tag{3.36d}$$

where $\tilde{J}^{as}(x,t,k)=\left(\frac{k+\omega_1}{k-p_1}\right)^{\sigma_3}\hat{J}(x,t,k)\left(\frac{k+\omega_1}{k-p_1}\right)^{-\sigma_3}$, $k\in\hat{\Gamma}$, and $f_1^{-1}(x,t)$ is exponentially growing. Since the residue conditions (3.36c) and (3.36d) are formulated for the same column, we cannot proceed as in the proof of Proposition 3 above.

Applying the nonlinear steepest descent method [16,20] we are able to justify the existence of solutions of the RH problems (3.25) and (3.28) and to make the asymptotics presented in (3.35) more precise.

Theorem 1. *Consider the Cauchy problem* (1.1) *and assume that the initial value* $q_0(x)$ *converges to its boundary values fast enough and that associated spectral functions* $a_j(k)$, $j=1,2$ *satisfy Assumptions A. Assuming that the solution* $q(x,t)$ *of* (1.1) *exists, it has the following long-time asymptotics, uniformly in* ξ *in compact subsets of the corresponding intervals (to simplify the notations, we set* $\operatorname{Re}p_0:=0$ *and* $\prod_{s=m_1}^{m_2}F_s=1$ *if* $m_1>m_2$*):*

(i) *for* $-\omega_{n-m+1}<-\xi<\operatorname{Re}p_{n-m}$, $m=\overline{0,n}$ *we have three types of asymptotics, depending on the value of* $\operatorname{Im}\nu(-\xi)$*:*
1) if $\operatorname{Im}\nu(-\xi)\in\left(-\frac{1}{2},-\frac{1}{6}\right]$, *then*

$$q(x,t)=A\delta^2(0,\xi)\prod_{s=0}^{m-1}\left(\frac{\omega_{n-s}}{p_{n-s}}\right)^2$$
$$+t^{-\frac{1}{2}-\operatorname{Im}\nu(-\xi)}\alpha_1(\xi)\exp\{-4it\xi^2+i\operatorname{Re}\nu(-\xi)\ln t\}+R_1(\xi,t).$$

2) if $\operatorname{Im}\nu(-\xi)\in\left(-\frac{1}{6},\frac{1}{6}\right)$, *then*

$$q(x,t)=A\delta^2(0,\xi)\prod_{s=0}^{m-1}\left(\frac{\omega_{n-s}}{p_{n-s}}\right)^2$$
$$+t^{-\frac{1}{2}-\operatorname{Im}\nu(-\xi)}\alpha_1(\xi)\exp\{-4it\xi^2+i\operatorname{Re}\nu(-\xi)\ln t\}$$
$$+t^{-\frac{1}{2}+\operatorname{Im}\nu(-\xi)}\alpha_2(\xi)\exp\{4it\xi^2-i\operatorname{Re}\nu(-\xi)\ln t\}+R_3(\xi,t).$$

3) if $\operatorname{Im}\nu(-\xi)\in\left[\frac{1}{6},\frac{1}{2}\right)$, *then*

$$q(x,t)=A\delta^2(0,\xi)\prod_{s=0}^{m-1}\left(\frac{\omega_{n-s}}{p_{n-s}}\right)^2$$
$$+t^{-\frac{1}{2}+\operatorname{Im}\nu(-\xi)}\alpha_2(\xi)\exp\{4it\xi^2-i\operatorname{Re}\nu(-\xi)\ln t\}+R_2(\xi,t).$$

(ii) *for* $-\operatorname{Re}p_{n-m}<-\xi<\omega_{n-m+1}$, $m=\overline{0,n}$*:*

$$q(x,t)=t^{-\frac{1}{2}-\operatorname{Im}\nu(\xi)}\alpha_3(\xi)\exp\{4it\xi^2-i\operatorname{Re}\nu(\xi)\ln t\}+R_2(-\xi,t)$$

(iii) *for* $\operatorname{Re}p_{n-m}<-\xi<-\omega_{n-m}$, $m=\overline{0,n-1}$*:*

$$q(x,t)=t^{-\frac{1}{2}+\operatorname{Im}\nu(-\xi)}\alpha_4(\xi)\exp\{4it\xi^2-i\operatorname{Re}\nu(-\xi)\ln t\}+R_2(\xi,t)$$

(iv) *for* $\omega_{n-m} < -\xi < -\operatorname{Re} p_{n-m}$, $m = \overline{0, n-1}$ *we have three types of asymptotics, depending on the value of* $\operatorname{Im}\nu(\xi)$:
1) if $\operatorname{Im}\nu(\xi) \in \left(-\frac{1}{2}, -\frac{1}{6}\right]$, *then*

$$q(x,t) = \frac{-4\overline{p}_{n-m}^2}{A\overline{\delta^2}(0,-\xi)} \prod_{s=0}^{m-1} \left(\frac{\overline{p}_{n-s}}{\overline{\omega}_{n-s}}\right)^2 + t^{-\frac{1}{2}-\operatorname{Im}\nu(\xi)}\alpha_5(\xi)\exp\{4it\xi^2 - i\operatorname{Re}\nu(\xi)\ln t\} + R_1(-\xi,t),$$

2) if $\operatorname{Im}\nu(\xi) \in \left(-\frac{1}{6}, \frac{1}{6}\right)$, *then*

$$q(x,t) = \frac{-4\overline{p}_{n-m}^2}{A\overline{\delta^2}(0,-\xi)} \prod_{s=0}^{m-1} \left(\frac{\overline{p}_{n-s}}{\overline{\omega}_{n-s}}\right)^2 + t^{-\frac{1}{2}-\operatorname{Im}\nu(\xi)}\alpha_5(\xi)\exp\{4it\xi^2 - i\operatorname{Re}\nu(\xi)\ln t\} + t^{-\frac{1}{2}+\operatorname{Im}\nu(\xi)}\alpha_6(\xi)\exp\{-4it\xi^2 + i\operatorname{Re}\nu(\xi)\ln t\} + R_3(-\xi,t).$$

3) if $\operatorname{Im}\nu(\xi) \in \left[\frac{1}{6}, \frac{1}{2}\right)$, *then*

$$q(x,t) = \frac{-4\overline{p}_{n-m}^2}{A\overline{\delta^2}(0,-\xi)} \prod_{s=0}^{m-1} \left(\frac{\overline{p}_{n-s}}{\overline{\omega}_{n-s}}\right)^2 + t^{-\frac{1}{2}+\operatorname{Im}\nu(\xi)}\alpha_6(\xi)\exp\{-4it\xi^2 + i\operatorname{Re}\nu(\xi)\ln t\} + R_2(-\xi,t).$$

Here

$$\delta(k,\xi) = (k+\xi)^{i\nu(-\xi)} \prod_{s=0}^{m-1}(k+\omega_{n-s})^{-1}\exp\left\{\sum_{s=0}^{m}\chi_s(k)\right\}, \tag{3.37}$$

and

$$\nu(-\xi) = -\frac{1}{2\pi}\ln|1+r_1(-\xi)r_2(-\xi)| - \frac{i}{2\pi}\left(\int_{-\infty}^{-\xi} d\arg(1+r_1(\zeta)r_2(\zeta)) + 2\pi m\right), \tag{3.38}$$

with

$$\chi_s(k) = -\frac{1}{2\pi i}\int_{-\omega_{n-s+1}}^{-\omega_{n-s}} \ln(k-\zeta)\, d_\zeta \ln(1+r_1(\zeta)r_2(\zeta)), \quad s = \overline{0, m-1}, \tag{3.39a}$$

$$\chi_m(k) = -\frac{1}{2\pi i}\int_{-\omega_{n-m+1}}^{-\xi} \ln(k-\zeta)\, d_\zeta \ln(1+r_1(\zeta)r_2(\zeta)). \tag{3.39b}$$

The constants $\alpha_j(\xi)$, $j = \overline{1,6}$ *are as follows:*

$$\alpha_1(\xi) = \frac{\sqrt{\pi}(c_0^{as}(\xi))^2 \prod_{s=0}^{m-1}(\xi+p_{n-s})^2}{\xi^2 r_2(-\xi)\Gamma(i\nu(-\xi))}\exp\left\{-\frac{\pi}{2}\nu(-\xi) + \frac{3\pi i}{4} - 2\sum_{s=0}^{m}\chi_s(-\xi) + 3i\nu(-\xi)\ln 2\right\},$$

$$\alpha_2(\xi)=\frac{\sqrt{\pi}\prod_{s=0}^{m-1}(\xi+p_{n-s})^{-2}}{r_1(-\xi)\Gamma(-i\nu(-\xi))}\exp\left\{-\frac{\pi}{2}\nu(-\xi)+\frac{\pi i}{4}+2\sum_{s=0}^{m}\chi_s(-\xi)-3i\nu(-\xi)\ln 2\right\},$$

$$\alpha_3(\xi)=\frac{\sqrt{\pi}\prod_{s=0}^{m-1}(\overline{p}_{n-s}-\xi)^{2}}{\overline{r_2}(\xi)\Gamma(-i\overline{\nu(\xi)})}\exp\left\{-\frac{\pi}{2}\overline{\nu(\xi)}+\frac{\pi i}{4}-2\sum_{s=0}^{m}\overline{\chi_s(\xi)}-3i\overline{\nu(\xi)}\ln 2\right\},$$

$$\alpha_4(\xi)=\frac{\sqrt{\pi}\xi^2\prod_{s=0}^{m}(\xi+p_{n-s})^{-2}}{r_1(-\xi)\Gamma(-i\nu(-\xi))}\exp$$
$$\left\{-\frac{\pi}{2}\nu(-\xi)+\frac{\pi i}{4}+2\sum_{s=0}^{m}\chi_s(-\xi)-3i\nu(-\xi)\ln 2\right\},$$

$$\alpha_5(\xi)=\frac{\sqrt{\pi}\prod_{s=0}^{m}(\overline{p}_{n-s}-\xi)^{2}}{\xi^2\overline{r_2}(\xi)\Gamma(-i\overline{\nu(\xi)})}\exp\left\{-\frac{\pi}{2}\overline{\nu(\xi)}+\frac{\pi i}{4}-2\sum_{s=0}^{m}\overline{\chi_s(\xi)}-3i\overline{\nu(\xi)}\ln 2\right\},$$

$$\alpha_6(\xi)=\frac{\sqrt{\pi}\left(\overline{c_0^{as\#}(-\xi)}\right)^2\prod_{s=0}^{m}(\overline{p}_{n-s}-\xi)^{-2}}{\overline{r_1}(\xi)\Gamma(i\overline{\nu(\xi)})}\exp$$
$$\left\{-\frac{\pi}{2}\overline{\nu(\xi)}+\frac{3\pi i}{4}+2\sum_{s=0}^{m}\overline{\chi_s(\xi)}+3i\overline{\nu(\xi)}\ln 2\right\},$$

where

$$c_0^{as}(\xi)=\frac{A\delta^2(0,\xi)}{2i}\prod_{s=0}^{m-1}\left(\frac{\omega_{n-s}}{p_{n-s}}\right)^2,\quad c_0^{as\#}(\xi)=\frac{2ip_{n-m}^2}{A\delta^2(0,\xi)}\prod_{s=0}^{m-1}\left(\frac{p_{n-s}}{\omega_{n-s}}\right)^2.$$

Finally, the remainders $R_j(\xi,t)$, $j=\overline{1,3}$ are as follows:

$$R_1(\xi,t)=\begin{cases}O\left(t^{-1}\right), & \operatorname{Im}\nu(-\xi)>0,\\ O\left(t^{-1}\ln t\right), & \operatorname{Im}\nu(-\xi)=0,\\ O\left(t^{-1+2|\operatorname{Im}\nu(-\xi)|}\right), & \operatorname{Im}\nu(-\xi)<0,\end{cases}\tag{3.40}$$

$$R_2(\xi,t)=\begin{cases}O\left(t^{-1+2|\operatorname{Im}\nu(-\xi)|}\right), & \operatorname{Im}\nu(-\xi)>0,\\ O\left(t^{-1}\ln t\right), & \operatorname{Im}\nu(-\xi)=0,\\ O\left(t^{-1}\right), & \operatorname{Im}\nu(-\xi)<0,\end{cases}\tag{3.41}$$

and

$$R_3(\xi,t)=R_1(\xi,t)+R_2(\xi,t)=\begin{cases}O\left(t^{-1+2|\operatorname{Im}\nu(-\xi)|}\right), & \operatorname{Im}\nu(-\xi)\neq 0,\\ O\left(t^{-1}\ln t\right), & \operatorname{Im}\nu(-\xi)=0.\end{cases}\tag{3.42}$$

Sketch of proof of Theorem 1. We apply the nonlinear steepest descent method to the Riemann–Hilbert problems (3.25) and (3.28). The implementation of the method is close to that presented in [43], so here we briefly describe the main steps of the proof, paying attention to its peculiarities due to Assumptions A and referring the reader to [43] for details.

We begin with the asymptotics for the Riemann–Hilbert problem (3.25), the analysis for (3.28) being similar (see also Remark 9). First, we reformulate (3.25) in such a way that instead of the residue condition we have the jump across a small counterclockwise oriented circle S_0 centered at $k = 0$:

$$\check{M}^{as}(\xi,t,k) = \begin{cases} M^{as}(\xi,t,k)\begin{pmatrix}1 & -\frac{c_0^{as}(\xi)}{k}\\ 0 & 1\end{pmatrix}, & k \text{ inside } S_0,\\ M^{as}(\xi,t,k), & \text{otherwise}\,. \end{cases}$$

Then $\check{M}^{as}(\xi,t,k)$ solves the Riemann–Hilbert problem

$$\check{M}^{as}_+(\xi,t,k) = \check{M}^{as}_-(\xi,t,k)\check{J}^{as}(\xi,t,k), \qquad k \in \hat{\Gamma}\cup S_0, \tag{3.43a}$$

$$\check{M}^{as}(\xi,t,k) \to I, \qquad k\to\infty, \tag{3.43b}$$

with

$$\check{J}^{as}(\xi,t,k) = \begin{cases} J^{as}(\xi,t,k), & k\in\hat{\Gamma},\\ \begin{pmatrix}1 & -\frac{c_0^{as}(\xi)}{k}\\ 0 & 1\end{pmatrix}, & k \in S_0. \end{cases} \tag{3.44}$$

Introduce the rescaled variable z by

$$k = \frac{z}{\sqrt{8t}} - \xi, \tag{3.45}$$

so that

$$e^{2it\theta} = e^{\frac{iz^2}{2}-4it\xi^2}.$$

Introduce the "local parametrix" $\check{m}_0^{as}(\xi,t,k)$ as the solution of a RH problem with the "simplified" jump matrix $J^{as}(\xi,t,k)$ in the sense that in its construction, $r_j(k)$, $j=1,2$ are replaced by the constants $r_j(-\xi)$ and $\delta(k,\xi;\{\omega_{n-s}\}_{s=0}^{m-1})$ is replaced by (cf. (3.7))

$$\delta \simeq \left(\frac{z}{\sqrt{8t}}\right)^{i\nu(-\xi)}\prod_{s=0}^{m-1}(\omega_{n-s}-\xi)^{-1}\exp\left\{\sum_{s=0}^{m}\chi_s(-\xi)\right\}.$$

Such RH problem can be solved explicitly in terms of the parabolic cylinder functions [31,43].

Indeed, $\check{m}_0^{as}(\xi,t,k)$ (cf. $\tilde{m}_0(x,t,k)$ in [43]) can be determined by

$$\check{m}_0^{as}(\xi,t,k) = \Delta(\xi,t)m^{\Gamma}(\xi,z(k))\Delta^{-1}(\xi,t), \tag{3.46}$$

where

$$\Delta(\xi,t) = e^{(2it\xi^2+\sum_{s=0}^{m}\chi_s(-\xi))\sigma_3}\left((8t)^{\frac{i\nu(-\xi)}{2}}\prod_{s=0}^{m-1}(\omega_{n-s}-\xi)\right)^{-\sigma_3}, \tag{3.47}$$

$m^{\Gamma}(\xi,z)$ is determined by

$$m^{\Gamma}(\xi,z)=m_0(\xi,z)D_j^{-1}(\xi,z),\qquad z\in\Omega_j,\ \ j=\overline{0,4}, \tag{3.48}$$

see Fig. 5, where

$$D_0(\xi,z)=e^{-i\frac{z^2}{4}\sigma_3}z^{i\nu(-\xi)\sigma_3},$$
$$D_1(\xi,z)=D_0(\xi,z)\begin{pmatrix}1 & \frac{r_2^{as}(-\xi)}{1+r_1^{as}(-\xi)r_2^{as}(-\xi)}\\ 0 & 1\end{pmatrix},\quad D_2(\xi,z)=D_0(\xi,z)\begin{pmatrix}1 & 0\\ r_1^{as}(-\xi) & 1\end{pmatrix},$$
$$D_3(\xi,z)=D_0(\xi,z)\begin{pmatrix}1 & -r_2^{as}(-\xi)\\ 0 & 1\end{pmatrix},\qquad D_4(\xi,z)=D_0(\xi,z)\begin{pmatrix}1 & 0\\ \frac{-r_1^{as}(-\xi)}{1+r_1^{as}(-\xi)r_2^{as}(-\xi)} & 1\end{pmatrix},$$

with

$$r_1^{as}(k)=r_1(k)\prod_{s=0}^{m-1}\left(\frac{k-p_{n-s}}{k+\omega_{n-s}}\right)^2,\quad r_2^{as}(k)=r_2(k)\prod_{s=0}^{m-1}\left(\frac{k+\omega_{n-s}}{k-p_{n-s}}\right)^2, \tag{3.49}$$

and $m_0(\xi,z)$ is the solution of the following RH problem in z-plane, relative to $\mathbb{R}$, with a constant jump matrix:

$$m_{0+}(\xi,z)=m_{0-}(\xi,z)j_0(\xi),\qquad z\in\mathbb{R}, \tag{3.50a}$$

$$m_0(\xi,z)=(I+O(1/z))\,e^{-i\frac{z^2}{4}\sigma_3}z^{i\nu(-\xi)\sigma_3},\qquad z\to\infty, \tag{3.50b}$$

where

$$j_0(\xi)=\begin{pmatrix}1+r_1^{as}(-\xi)r_2^{as}(-\xi) & r_2^{as}(-\xi)\\ r_1^{as}(-\xi) & 1\end{pmatrix}. \tag{3.51}$$

□

It is the RH problem for $m_0(\xi,z)$ that can be solved explicitly, in terms of the parabolic cylinder functions, see, e.g., Appendix A in [43]. Since we are interested what happens for large t, we actually need from $m_0(\xi,z)$ (and, correspondingly, $m^{\Gamma}(\xi,z)$) its large-z asymptotics only. The latter has the form

$$m^{\Gamma}(\xi,z)=I+\frac{i}{z}\begin{pmatrix}0 & \beta(\xi)\\ -\gamma(\xi) & 0\end{pmatrix}+O(z^{-2}),\qquad z\to\infty,$$

where

$$\beta(\xi)=\frac{\sqrt{2\pi}e^{-\frac{\pi}{2}\nu(-\xi)}e^{-\frac{3\pi i}{4}}}{r_1^{as}(-\xi)\Gamma(-i\nu(-\xi))}, \tag{3.52a}$$

$$\gamma(\xi)=\frac{\sqrt{2\pi}e^{-\frac{\pi}{2}\nu(-\xi)}e^{-\frac{\pi i}{4}}}{r_2^{as}(-\xi)\Gamma(i\nu(-\xi))}. \tag{3.52b}$$

Now, having defined the parametrix $\check{m}_0^{as}(\xi,t,k)$, we define $\check{M}^{as}(\xi,t,k)$ as follows (cf. $\hat{m}(x,t,k)$ in [43]):

$$\check{M}^{as}(\xi,t,k)=\begin{cases}\check{M}^{as}(\xi,t,k)(\check{m}_0^{as})^{-1}(\xi,t,k)V(k), & k \text{ inside } S_{-\xi},\\ \check{M}^{as}(\xi,t,k), & k \text{ inside } S_0,\\ \check{M}^{as}(\xi,t,k)V(k), & \text{otherwise},\end{cases} \tag{3.53}$$

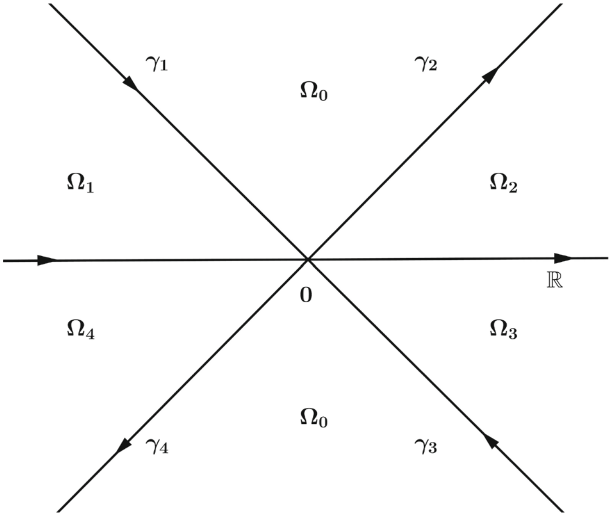

Fig. 5. Contour and domains for $m^{\Gamma}(\xi, z)$ in the z-plane

where $V(k) = \begin{pmatrix} 1 & -\frac{c_0^{as}(\xi)}{k} \\ 0 & 1 \end{pmatrix}$, and $S_{-\xi}$ is a small counterclockwise oriented circle centered at $k = -\xi$. Then the sectionally analytic matrix $\check{M}^{as}$ solves the following Riemann–Hilbert problem on the contour $\hat{\Gamma}_1 = \hat{\Gamma} \cup S_{-\xi}$:

$$\check{M}^{as}_+(\xi, t, k) = \check{M}^{as}_-(\xi, t, k)\check{J}^{as}(\xi, t, k), \qquad k \in \hat{\Gamma}_1, \tag{3.54}$$

$$\check{M}^{as}(\xi, t, k) \to I, \qquad k \to \infty, \tag{3.55}$$

with the jump matrix (cf. (3.23) in [43])

$$\check{J}^{as}(\xi, t, k) = \begin{cases} V^{-1}(k)\check{m}^{as}_{0-}(\xi, t, k)\check{J}^{as}(\xi, t, k)(\check{m}^{as}_{0+})^{-1}(\xi, t, k)V(k), & k \in \hat{\Gamma}_1, k \text{ inside } S_{-\xi}, \\ V^{-1}(k)(\check{m}^{as}_0)^{-1}(\xi, t, k)V(k), & k \in S_{-\xi}, \\ V^{-1}(k)\check{J}^{as}(\xi, t, k)V(k), & \text{otherwise}. \end{cases} \tag{3.56}$$

Observe that the solution of the original problem is given in terms of $\check{M}^{as}(\xi, t, k)$ as follows:

$$q(x, t) = 2i\left(c_0^{as}(\xi) + \lim_{k\to\infty} k\check{M}^{as}_{12}(\xi, t, k)\right), \tag{3.57}$$

and

$$q(-x, t) = -2i \lim_{k\to\infty} \overline{k\check{M}^{as}_{21}(\xi, t, k)}. \tag{3.58}$$

Notice that $(\check{m}^{as}_0)^{-1}(\xi, t, k)$ has the following large-t asymptotics:

$$(\check{m}^{as}_0)^{-1}(\xi, t, k) = \Delta(\xi, t)(m^{\Gamma})^{-1}(\xi, \sqrt{8t}(k+\xi))\Delta^{-1}(\xi, t) = I + \frac{B(\xi, t)}{\sqrt{8t}(k+\xi)} + \tilde{r}(\xi, t), \tag{3.59}$$

where the entries of $B(\xi,t)$ are as follows (cf. with (3.32) in [43]):

$$B_{11}(\xi,t)=B_{22}(\xi,t)=0, \tag{3.60a}$$

$$B_{12}(\xi,t)=-i\beta(\xi)e^{4it\xi^2+2\sum_{s=0}^{m}\chi_s(-\xi)}(8t)^{-i\nu(-\xi)}\prod_{s=0}^{m-1}(\omega_{n-s}-\xi)^{-2}, \tag{3.60b}$$

$$B_{21}(\xi,t)=i\gamma(\xi)e^{-4it\xi^2-2\sum_{s=0}^{m}\chi_s(-\xi)}(8t)^{i\nu(-\xi)}\prod_{s=0}^{m-1}(\omega_{n-s}-\xi)^{2}, \tag{3.60c}$$

and the remainder is (cf. (3.33) in [43]):

$$\tilde{r}(\xi,t)=\begin{pmatrix} O\left(t^{-1-\mathrm{Im}\,\nu(-\xi)}\right) & O\left(t^{-1+\mathrm{Im}\,\nu(-\xi)}\right)\\ O\left(t^{-1-\mathrm{Im}\,\nu(-\xi)}\right) & O\left(t^{-1+\mathrm{Im}\,\nu(-\xi)}\right)\end{pmatrix},\quad t\to\infty. \tag{3.61}$$

The jump matrix in (3.56) can be estimated similarly to [43], giving that $\|\breve{J}^{as}(\xi,t,\cdot)-I\|_{L^2(\hat{\Gamma}_1)\cap L^\infty(\hat{\Gamma}_1)}\to 0$ as $t\to\infty$, with uniform error estimates for ξ in compact subsets of the intervals indicated in the statement of Theorem 1. Particularly, this provides the unique solvability of the RH problem (3.54). Moreover, we evaluate the asymptotics of $\breve{M}^{as}(\xi,t,k)$ as $t\to\infty$ using its integral representation in terms of the solution of the singular integral equation:

$$\breve{M}^{as}(\xi,t,k)=I+\frac{1}{2\pi i}\int_{\hat{\Gamma}_1}\mu(\xi,t,s)(\breve{J}^{as}(\xi,t,s)-I)\frac{ds}{s-k}, \tag{3.62}$$

where μ solves the integral equation $\mu-C_w\mu=I$, with $w=\breve{J}^{as}-I$ and the Cauchy-type operator C_w defined as follows:

$$C_wf=(C_-fw)(k)=\frac{1}{2\pi i}\lim_{\substack{k'\to k\\ k'\in -side}}\int_{\hat{\Gamma}_1}\frac{f(s)w(s)}{s-k'}ds.$$

Since $V(k)$ is uniformly bounded on $\hat{\Gamma}_1$ and does not depend on t and x, we can proceed as in [43] and conclude that the main term in the large-t evaluation of $\breve{M}^{as}$ in (3.62) is given by the integral along the circle $S_{-\xi}$. In this way we obtain the following representation for $\breve{M}^{as}(\xi,t,k)$ (see (3.30) and (3.34) in [43]):

$$\begin{aligned}&\lim_{k\to\infty}k\left(\breve{M}^{as}(\xi,t,k)-I\right)\\ &\quad=-\frac{1}{2\pi i}\int_{S_{-\xi}}V^{-1}(k)\left((\breve{m}_0^{as})^{-1}(\xi,t,k)-I\right)V(k)\,dk+R(\xi,t).\end{aligned}$$

Taking into account (3.59) we conclude that

$$\lim_{k\to\infty}k\left(\breve{M}^{as}(\xi,t,k)-I\right)=B^{as}(\xi,t)+R(\xi,t), \tag{3.63}$$

where $R(\xi,t)=\begin{pmatrix}R_1(\xi,t) & R_1(\xi,t)+R_2(\xi,t)\\ R_1(\xi,t) & R_1(\xi,t)+R_2(\xi,t)\end{pmatrix}$ and (see (3.60))

$$B^{as}(\xi,t)=\frac{1}{\sqrt{8t}}\begin{pmatrix}\frac{c_0^{as}(\xi)}{\xi}B_{21}(\xi,t) & \frac{(c_0^{as}(\xi))^2}{\xi^2}B_{21}(\xi,t)-B_{12}(\xi,t)\\ -B_{21}(\xi,t) & -\frac{c_0^{as}(\xi)}{\xi}B_{21}(\xi,t)\end{pmatrix}. \tag{3.64}$$

Remark 9. In the analysis of the Riemann–Hilbert problem (3.28), the reflection coefficients $r_j^{as}(k)$, $j=1,2$ (see (3.49)) have the form

$$r_1^{as}(k)=r_1(k)d^{-2}(k)\prod_{s=0}^{m-1}\left(\frac{k-p_{n-s}}{k+\omega_{n-s}}\right)^2,\quad r_2^{as}(k)=r_2(k)d^{2}(k)\prod_{s=0}^{m-1}\left(\frac{k+\omega_{n-s}}{k-p_{n-s}}\right)^2,$$

where $d(k)=\frac{k}{k-p_{n-m}}$. Moreover, $V(k)=\begin{pmatrix}1 & 0\\ -\frac{c_0^{as\#}(\xi)}{k} & 1\end{pmatrix}$ in the definition of $\breve{M}^{as}(\xi,t,k)$ (see (3.53)). Therefore,

$$q(x,t)=2i\lim_{k\to\infty}k\breve{M}_{12}^{as}(\xi,t,k), \tag{3.65}$$

$$q(-x,t)=-2i\left(\overline{c_0^{as\#}(\xi)}+\lim_{k\to\infty}k\overline{\breve{M}_{21}^{as}(\xi,t,k)}\right), \tag{3.66}$$

and $B^{as}(\xi,t)$ and $R(\xi,t)$ in (3.63) are as follows:

$$B^{as}(\xi,t)=\frac{1}{\sqrt{8t}}\begin{pmatrix}-\frac{c_0^{as\#}(\xi)}{\xi}B_{12}(\xi,t) & -B_{12}(\xi,t)\\ \frac{(c_0^{as\#}(\xi))^2}{\xi^2}B_{12}(\xi,t)-B_{21}(\xi,t) & \frac{c_0^{as\#}(\xi)}{\xi}B_{12}(\xi,t)\end{pmatrix}, \tag{3.67}$$

and $R(\xi,t)=\begin{pmatrix}R_1(\xi,t)+R_2(\xi,t) & R_2(\xi,t)\\ R_1(\xi,t)+R_2(\xi,t) & R_2(\xi,t)\end{pmatrix}$.

4. Transition Regions

Theorem 1 presents the asymptotics of $q(x,t)$ along the rays $\xi=\frac{x}{4t}=const$ uniformly for all ξ in compact subsets of $\mathbb{R}\setminus\{\pm\operatorname{Re}p_m,\pm\omega_m|m=\overline{1,n}\}$, i.e., for all fixed $\xi\in\mathbb{R}$ except the boundaries of sectors with qualitatively different asymptotic behavior. Since the asymptotic regimes (decaying and non-decaying) in the adjacent sectors do not match as ξ approaches the edges of the sectors, there must be certain transition zones between the sectors. The study of transition zones is a challenging and all-important problem. These zones are a rich source of interesting nonlinear effects, such as shock waves and asymptotic solitons, among many others.

One can distinguish three types of qualitatively different transition zones:

(1) zones near the rays $\xi=\pm\operatorname{Re}p_m$, $m=\overline{1,n}$,
(2) zones near the rays $\xi=\pm\omega_m$, $m=\overline{1,n}$,
(3) a zone as ξ approaches 0.

First, we address the transition zones for $\xi=\pm\operatorname{Re}p_m$, $m=\overline{1,n}$ and show that in these zones, $q(x,t)$ asymptotically behaves as solitary kinks propagating along the rays $\xi=\pm\operatorname{Re}p_m$.

Theorem 2. *Fix an integer m such that $0 \le m \le n-1$. Then, under assumptions of Theorem 1, the solution $q(x,t)$ of problem (1.1), (1.2) has the following asymptotics along the rays $\xi = \pm\operatorname{Re} p_{n-m}$:*

$$q(x,t) = \begin{cases} \dfrac{2ip_{n-m}^2 c_0^{as}(-\operatorname{Re} p_{n-m})}{p_{n-m}^2 + c_0^{as}(-\operatorname{Re} p_{n-m}) f_{n-m}^{as}(x',t)} + F_{1,n-m}(x',t), & t\to\infty, \quad x = -4\operatorname{Re} p_{n-m}t + x', \\ \dfrac{-2i\overline{p}_{n-m}^2 \overline{f_{n-m}^{as}}(x',t)}{\overline{p}_{n-m}^2 + \overline{c_0^{as}}(-\operatorname{Re} p_{n-m}) \overline{f_{n-m}^{as}}(x',t)} + F_{2,n-m}(x',t), & t\to\infty, \quad x = 4\operatorname{Re} p_{n-m}t - x', \end{cases} \tag{4.1}$$

where $c_0^{as}(\xi)$ is given by (3.26), $f_{n-m}^{as}(x',t)$ is given by

$$f_{n-m}^{as}(x',t) = \frac{\eta_{n-m}\exp\{2ip_{n-m}x' - 4it(\operatorname{Re}^2 p_{n-m} + \operatorname{Im}^2 p_{n-m})\}}{\dot{a}_1(p_{n-m})\delta^2(p_{n-m}, -\operatorname{Re} p_{n-m})} \prod_{s=0}^{m-1}\left(\frac{p_{n-m}-p_{n-s}}{p_{n-m}+\omega_{n-s}}\right)^2, \tag{4.2}$$

and $F_{j,n-m}(x',t)$, $j = 1,2$ are decaying terms. The asymptotics (4.1) holds for all $x'\in\mathbb{R}$ and $t \gg 0$ such that

$$\|(x',t) - (\tilde{x}', t_j)\|_{\mathbb{R}^2} \ge \varepsilon' \quad \text{for all } j\in\mathbb{Z}, \tag{4.3}$$

with any fixed $\varepsilon' > 0$, where $(\tilde{x}', t_j)$, $j\in\mathbb{Z}$ are the zeros of the denominators of the principal terms in (4.1):

$$\tilde{x}' = \frac{\ln|C_{n-m}|}{2\operatorname{Im} p_{n-m}}, \quad t_j = \frac{\operatorname{Re} p_{n-m}\ln|C_{n-m}| + \phi_{n-m}\operatorname{Im} p_{n-m}}{4|p_{n-m}|^2\operatorname{Im} p_{n-m}} + \frac{\pi j}{2|p_{n-m}|^2}, \tag{4.4}$$

with $\phi_{n-m} = \arg C_{n-m}$ and

$$C_{n-m} = \frac{-\eta_{n-m}c_0^{as}(-\operatorname{Re} p_{n-m})}{p_{n-m}^2\dot{a}_1(p_{n-m})\delta^2(p_{n-m}, -\operatorname{Re} p_{n-m})}\prod_{s=0}^{m-1}\left(\frac{p_{n-m}-p_{n-s}}{p_{n-m}+\omega_{n-s}}\right)^2 \equiv |C_{n-m}|e^{i\phi_{n-m}}.$$

More precisely, the decaying terms $F_{j,n-m}(x',t)$, $j = 1,2$ have the following form depending on the value of $\operatorname{Im}\nu(\operatorname{Re} p_{m-m})$:

1) if $\operatorname{Im}\nu(\operatorname{Re} p_{m-m}) \in \left(-\frac{1}{2}, -\frac{1}{6}\right]$, then

$$F_{1,n-m}(x',t) = \frac{t^{-\frac{1}{2}-\operatorname{Im}\nu(\operatorname{Re} p_{m-m})}}{(p_{n-m}^2 + c_0^{as}(-\operatorname{Re} p_{n-m}) f_{n-m}^{as}(x',t))^2} \times \alpha_7\exp\{-4it\operatorname{Re}^2 p_{m-m} + i\operatorname{Re}\nu(\operatorname{Re} p_{m-m})\ln t\} + \frac{R_1(\operatorname{Re} p_{n-m}, t)}{\varepsilon'^2},$$

$$F_{2,n-m}(x',t) = t^{-\frac{1}{2}-\operatorname{Im}\nu(\operatorname{Re} p_{m-m})} \frac{(i\overline{p}_{n-m}^2\operatorname{Im} p_{n-m} + \operatorname{Re} p_{n-m}\cdot\overline{c_0^{as}}(-\operatorname{Re} p_{n-m})\overline{f_{n-m}^{as}}(x',t))^2}{(\overline{p}_{n-m}^2 + \overline{c_0^{as}}(-\operatorname{Re} p_{n-m})\overline{f_{n-m}^{as}}(x',t))^2} \times \alpha_9\exp\{4it\operatorname{Re}^2 p_{m-m} - i\operatorname{Re}\nu(\operatorname{Re} p_{m-m})\ln t\} + \frac{R_1(\operatorname{Re} p_{n-m}, t)}{\varepsilon'^2}.$$

2) if $\operatorname{Im}\nu(\operatorname{Re} p_{m-m}) \in \left(-\frac{1}{6}, \frac{1}{6}\right)$*, then*

$$F_{1,n-m}(x',t) = \frac{t^{-\frac{1}{2}-\operatorname{Im}\nu(\operatorname{Re} p_{m-m})}}{(p_{n-m}^2 + c_0^{as}(-\operatorname{Re} p_{n-m}) f_{n-m}^{as}(x',t))^2} \alpha_7 \exp\{-4it \operatorname{Re}^2 p_{m-m}$$
$$+ i \operatorname{Re}\nu(\operatorname{Re} p_{m-m}) \ln t\}$$
$$+ t^{-\frac{1}{2}+\operatorname{Im}\nu(\operatorname{Re} p_{m-m})} \frac{(p_{n-m}^2 \operatorname{Re} p_{n-m} - i \operatorname{Im} p_{n-m} \cdot c_0^{as}(-\operatorname{Re} p_{n-m}) f_{n-m}^{as}(x',t))^2}{(p_{n-m}^2 + c_0^{as}(-\operatorname{Re} p_{n-m}) f_{n-m}^{as}(x',t))^2}$$
$$\times \alpha_8 \exp\{4it \operatorname{Re}^2 p_{m-m} - i \operatorname{Re}\nu(\operatorname{Re} p_{m-m}) \ln t\} + \frac{R_3(\operatorname{Re} p_{n-m}, t)}{\varepsilon'^2}.$$

$$F_{2,n-m}(x',t) = t^{-\frac{1}{2}-\operatorname{Im}\nu(\operatorname{Re} p_{m-m})}$$
$$\frac{(i\overline{p}_{n-m}^2 \operatorname{Im} p_{n-m} + \operatorname{Re} p_{n-m} \cdot \overline{c_0^{as}}(-\operatorname{Re} p_{n-m}) \overline{f_{n-m}^{as}}(x',t))^2}{(\overline{p}_{n-m}^2 + \overline{c_0^{as}}(-\operatorname{Re} p_{n-m}) \overline{f_{n-m}^{as}}(x',t))^2}$$
$$\times \alpha_9 \exp\{4it \operatorname{Re}^2 p_{m-m} - i \operatorname{Re}\nu(\operatorname{Re} p_{m-m}) \ln t\}$$
$$+ t^{-\frac{1}{2}+\operatorname{Im}\nu(\operatorname{Re} p_{m-m})} \frac{\overline{f_{n-m}^{as}}^2(x',t)}{(\overline{p}_{n-m}^2 + \overline{c_0^{as}}(-\operatorname{Re} p_{n-m}) \overline{f_{n-m}^{as}}(x',t))^2}$$
$$\times \alpha_{10} \exp\{-4it \operatorname{Re}^2 p_{m-m} + i \operatorname{Re}\nu(\operatorname{Re} p_{m-m}) \ln t\} + \frac{R_3(\operatorname{Re} p_{n-m}, t)}{\varepsilon'^2}.$$

3) if $\operatorname{Im}\nu(\operatorname{Re} p_{m-m}) \in \left[\frac{1}{6}, \frac{1}{2}\right)$*, then*

$$F_{1,n-m}(x',t) = t^{-\frac{1}{2}+\operatorname{Im}\nu(\operatorname{Re} p_{m-m})}$$
$$\frac{(p_{n-m}^2 \operatorname{Re} p_{n-m} - i \operatorname{Im} p_{n-m} \cdot c_0^{as}(-\operatorname{Re} p_{n-m}) f_{n-m}^{as}(x',t))^2}{(p_{n-m}^2 + c_0^{as}(-\operatorname{Re} p_{n-m}) f_{n-m}^{as}(x',t))^2}$$
$$\times \alpha_8 \exp\{4it \operatorname{Re}^2 p_{m-m} - i \operatorname{Re}\nu(\operatorname{Re} p_{m-m}) \ln t\} + \frac{R_2(\operatorname{Re} p_{n-m}, t)}{\varepsilon'^2},$$

$$F_{2,n-m}(x',t) = t^{-\frac{1}{2}+\operatorname{Im}\nu(\operatorname{Re} p_{m-m})} \frac{\overline{f_{n-m}^{as}}^2(x',t)}{(\overline{p}_{n-m}^2 + \overline{c_0^{as}}(-\operatorname{Re} p_{n-m}) \overline{f_{n-m}^{as}}(x',t))^2}$$
$$\times \alpha_{10} \exp\{-4it \operatorname{Re}^2 p_{m-m} + i \operatorname{Re}\nu(\operatorname{Re} p_{m-m}) \ln t\} + \frac{R_2(\operatorname{Re} p_{n-m}, t)}{\varepsilon'^2}.$$

with

$$\alpha_7 = \frac{\sqrt{\pi}(c_0^{as}(-\operatorname{Re} p_{n-m}))^2 p_{n-m}^4 \prod_{s=0}^{m-1} (p_{n-s} - \operatorname{Re} p_{n-m})^2}{\operatorname{Re}^2 p_{n-m} r_2(\operatorname{Re} p_{n-m}) \Gamma(i\nu(\operatorname{Re} p_{n-m}))}$$
$$\times \exp\left\{-\frac{\pi}{2}\nu(\operatorname{Re} p_{n-m}) + \frac{3\pi i}{4} - 2\sum_{s=0}^{m} \chi_s(\operatorname{Re} p_{n-m}) + 3i\nu(\operatorname{Re} p_{n-m}) \ln 2\right\},$$

$$\alpha_8 = \frac{\sqrt{\pi} \prod_{s=0}^{m-1} (p_{n-s} - \operatorname{Re} p_{n-m})^{-2}}{\operatorname{Re}^2 p_{n-m} r_1(\operatorname{Re} p_{n-m}) \Gamma(-i\nu(\operatorname{Re} p_{n-m}))}$$

$$\times\exp\left\{-\frac{\pi}{2}\nu(\mathrm{Re}\,p_{n-m})+\frac{\pi i}{4}+2\sum_{s=0}^{m}\chi_s(\mathrm{Re}\,p_{n-m})-3i\nu(\mathrm{Re}\,p_{n-m})\ln 2\right\},$$

$$\alpha_9=\frac{\sqrt{\pi}\prod\limits_{s=0}^{m-1}(\overline{p_{n-s}}-\mathrm{Re}\,p_{n-m})^2}{\mathrm{Im}^2\,p_{n-m}\overline{r_2}(\mathrm{Re}\,p_{n-m})\Gamma(-i\overline{\nu}(\mathrm{Re}\,p_{n-m}))}$$
$$\times\exp\left\{-\frac{\pi}{2}\overline{\nu}(\mathrm{Re}\,p_{n-m})-\frac{3\pi i}{4}-2\sum_{s=0}^{m}\overline{\chi_s}(\mathrm{Re}\,p_{n-m})-3i\overline{\nu}(\mathrm{Re}\,p_{n-m})\ln 2\right\}, \tag{4.5}$$

$$\alpha_{10}=\frac{\sqrt{\pi}\,\overline{p_{n-m}}^4\prod\limits_{s=0}^{m-1}(\overline{p_{n-s}}-\mathrm{Re}\,p_{n-m})^{-2}}{\mathrm{Im}^2\,p_{n-m}\overline{r_1}(\mathrm{Re}\,p_{n-m})\Gamma(i\overline{\nu}(\mathrm{Re}\,p_{n-m}))}$$
$$\times\exp\left\{-\frac{\pi}{2}\overline{\nu}(\mathrm{Re}\,p_{n-m})-\frac{\pi i}{4}+2\sum_{s=0}^{m}\overline{\chi_s}(\mathrm{Re}\,p_{n-m})+3i\overline{\nu}(\mathrm{Re}\,p_{n-m})\ln 2\right\}, \tag{4.6}$$

and $R_j(\mathrm{Re}\,p_{n-m},t)$*,* $j=1,2,3$ *are given by* (3.40)*,* (3.41) *and* (3.42)*.*

Proof. Similarly to Item (i) in Proposition 3, it can be shown that along the ray $\xi=-\mathrm{Re}\,p_{n-m}$ (see Fig. 3), the long-time behavior of $q(x,t)$ can be described in terms of the solutions of the RH problem (cf. (3.25))

$$M_+^{as}(\xi,t,k)=M_-^{as}(\xi,t,k)J^{as}(\xi,t,k),\qquad k\in\hat{\Gamma}, \tag{4.7a}$$
$$M^{as}(\xi,t,k)\to I,\qquad k\to\infty, \tag{4.7b}$$
$$\operatorname*{Res}_{k=0}M^{as\,(2)}(\xi,t,k)=c_0^{as}(\xi)M^{as\,(1)}(\xi,t,0), \tag{4.7c}$$
$$\operatorname*{Res}_{k=p_{n-m}}M^{as\,(1)}(\xi,t,k)=f_{n-m}^{as}(x',t)M^{as\,(2)}(\xi,t,p_{n-m}), \tag{4.7d}$$

where $\xi=-\mathrm{Re}\,p_{n-m}$, $J^{as}(\xi,t,k)$ is given by (3.27) and $f_{n-m}^{as}(x',t)\equiv f_{n-m}(x,t)\prod\limits_{s=0}^{m-1}\left(\frac{p_{n-m}-p_{n-s}}{p_{n-m}+\omega_{n-s}}\right)^2$ with $x=-4t\,\mathrm{Re}\,p_{n-m}+x'$ is given by (4.2). Notice that the shift $x'\in\mathbb{R}$ doesn't affect the value of the slow variable $\xi\equiv\frac{x}{4t}\sim-\mathrm{Re}\,p_{n-m}$ as $t\to\infty$.

Using the Blaschke-Potapov factors (see, e.g., [23]) one can show that the RH problem

$$\operatorname*{Res}_{k=0}\acute{M}^{as\,(2)}(\xi,t,k)=c_0^{as}(\xi)\acute{M}^{as\,(1)}(\xi,t,0), \tag{4.8a}$$
$$\operatorname*{Res}_{k=p_{n-m}}\acute{M}^{as\,(1)}(\xi,t,k)=f_{n-m}^{as}(x',t)\acute{M}^{as\,(2)}(\xi,t,p_{n-m}), \tag{4.8b}$$
$$\acute{M}^{as}(\xi,t,k)\to I,\quad k\to\infty, \tag{4.8c}$$

can be solved explicitly for all $x'\in\mathbb{R}$ and $t>0$ away from any fixed vicinities of the singular points $\tilde{x}'$, t_j, $j\in\mathbb{Z}$, see (4.4), by (we drop the arguments of the respective functions)

$$
\acute{M}^{as}(x',\xi,t,k)=\begin{pmatrix}\frac{(p_{n-m}^2+c_0^{as}f_{n-m}^{as})k-p_{n-m}^3}{(p_{n-m}^2+c_0^{as}f_{n-m}^{as})(k-p_{n-m})} & \frac{c_0^{as}p_{n-m}^2}{(p_{n-m}^2+c_0^{as}f_{n-m}^{as})k}\\ \frac{p_{n-m}^2f_{n-m}^{as}}{(p_{n-m}^2+c_0^{as}f_{n-m}^{as})(k-p_{n-m})} & \frac{(p_{n-m}^2+c_0^{as}f_{n-m}^{as})k-p_{n-m}c_0^{as}f_{n-m}^{as}}{(p_{n-m}^2+c_0^{as}f_{n-m}^{as})k}\end{pmatrix}, \tag{4.9}
$$

where x' and t satisfy condition (4.3).

Notice that $\acute{M}^{as}(x',\xi,t,k)=O(\varepsilon'^{-1})$ as $(x',t)\to(\tilde{x}',t_j)$ and $\acute{M}^{as}(x',\xi,t,k)=O(1)$ as $x'\to\pm\infty$, which implies that the asymptotics in Theorem 2 are uniform w.r.t. x' outside any (fixed) neighborhood of $x'=\tilde{x}'$.

Since $\acute{M}^{as}$ solves RH problem (4.8), $\check{M}^{as}(x',\xi,t,k)$ defined by

$$
\check{M}^{as}(x',\xi,t,k)=\begin{cases}M^{as}(\xi,t,k)(\acute{M}^{as})^{-1}(x',\xi,t,k), & k \text{ outside } S_{-\xi},\\ M^{as}(\xi,t,k), & \text{otherwise},\end{cases}
$$

where $S_{-\xi}$ is a small counterclockwise oriented circle, satisfies the following RH problem on $\hat{\Gamma}_1=\hat{\Gamma}\cup S_{-\xi}$:

$$
\check{M}_+^{as}(x',\xi,t,k)=\check{M}_-^{as}(x',\xi,t,k)\check{J}^{as}(x',\xi,t,k), \qquad k\in\hat{\Gamma}_1, \tag{4.10a}
$$

$$
\check{M}^{as}(x',\xi,t,k)\to I, \qquad k\to\infty, \tag{4.10b}
$$

with

$$
\check{J}^{as}(x',\xi,t,k)=\begin{cases}\acute{M}^{as}(x',\xi,t,k)J^{as}(\xi,t,k)(\acute{M}^{as})^{-1}(x',\xi,t,k), & k\in\hat{\Gamma}_1, k \text{ outside } S_{-\xi},\\ (\acute{M}^{as})^{-1}(x',\xi,t,k) & k\in S_{-\xi},\\ J^{as}(\xi,t,k), & k\in\hat{\Gamma}_1, k \text{ inside } S_{-\xi}.\end{cases} \tag{4.11}
$$

Notice that $q(x,t)$ is given in terms of $\check{M}^{as}$ as follows (recall that $-\xi=\operatorname{Re}p_{n-m}$):

$$
q(x,t)=\frac{2ip_{n-m}^2c_0^{as}(-\operatorname{Re}p_{n-m})}{p_{n-m}^2+c_0^{as}(-\operatorname{Re}p_{n-m})f_{n-m}^{as}(x',t)}+2i\lim_{k\to\infty}k\check{M}_{12}^{as}(x',\xi,t,k), \tag{4.12}
$$

$$
q(-x,t)=\frac{-2i\bar{p}_{n-m}^2\overline{f_{n-m}^{as}}(x',t)}{\bar{p}_{n-m}^2+\overline{c_0^{as}}(-\operatorname{Re}p_{n-m})\overline{f_{n-m}^{as}}(x',t)}-2i\lim_{k\to\infty}\overline{k\check{M}_{21}^{as}(x',\xi,t,k)}, \tag{4.13}
$$

for all $x'\in\mathbb{R}$, $t>0$ satisfying (4.3). Constructing $\check{m}_0^{as}(\xi,t,k)$ as in the proof of Theorem 1 (see (3.46)), we define $\breve{M}^{as}(x',\xi,t,k)$ by

$$
\breve{M}^{as}(x',\xi,t,k)=\begin{cases}\check{M}^{as}(x',\xi,t,k)(\check{m}_0^{as})^{-1}(\xi,t,k)\acute{M}^{as}(x',\xi,t,k), & k \text{ inside } S_{-\xi},\\ \check{M}^{as}(x',\xi,t,k), & \text{otherwise}.\end{cases} \tag{4.14}
$$

Then $\breve{M}^{as}(x',\xi,t,k)$ solves the following RH problem on the contour $\hat{\Gamma}_1$:

$$
\breve{M}_+^{as}(x',\xi,t,k)=\breve{M}_-^{as}(x',\xi,t,k)\breve{J}^{as}(x',\xi,t,k), \qquad k\in\hat{\Gamma}_1, \tag{4.15a}
$$

$$
\breve{M}^{as}(x',\xi,t,k)\to I, \qquad k\to\infty, \tag{4.15b}
$$

with the jump matrix (we drop the arguments of $\acute{M}^{as}(x', \xi, t, k)$)

$$\check{J}^{as}(\xi,t,k)=\begin{cases}(\acute{M}^{as})^{-1}\check{m}_{0-}^{as}(\xi,t,k)\check{J}^{as}(x',\xi,t,k)(\check{m}_{0+}^{as})^{-1}(\xi,t,k)\acute{M}^{as}, & k\in\hat{\Gamma}_1, k \text{ inside } S_{-\xi},\\ (\acute{M}^{as})^{-1}(\check{m}_{0}^{as})^{-1}(\xi,t,k)\acute{M}^{as}, & k\in S_{-\xi},\\ \check{J}^{as}(\xi,t,k), & k\in\hat{\Gamma}_1, k \text{ outside } S_{-\xi}.\end{cases} \tag{4.16}$$

Taking into account that $\acute{M}^{as}$ is bounded inside $S_{-\xi}$ and arguing as in the proof of Theorem 1 we have

$$\lim_{k\to\infty} k\left(\check{M}^{as}(\xi,t,k)-I\right)=-\frac{1}{2\pi i}\int_{S_{-\xi}}(\acute{M}^{as})^{-1}\left((\check{m}_0^{as})^{-1}(\xi,t,k)-I\right)\acute{M}^{as}\,dk + \frac{R(\xi,t)}{\varepsilon'^2},$$

and thus

$$\lim_{k\to\infty} k\left(\check{M}^{as}(x',\xi,t,k)-I\right)=\acute{B}^{as}(x',\xi,t)+\frac{R(\xi,t)}{\varepsilon'^2}, \tag{4.17}$$

where $R(\xi,t)=\begin{pmatrix}R_1(\xi,t) & R_1(\xi,t)+R_2(\xi,t)\\ R_1(\xi,t) & R_1(\xi,t)+R_2(\xi,t)\end{pmatrix}$ and (see (3.60))

$$\acute{B}^{as}(x',\xi,t)=\frac{1}{\sqrt{8t}}\begin{pmatrix}\acute{M}_{11}^{as}\acute{M}_{12}^{as}B_{21}-\acute{M}_{21}^{as}\acute{M}_{22}^{as}B_{12} & (\acute{M}_{12}^{as})^2B_{21}-(\acute{M}_{22}^{as})^2B_{12}\\ -(\acute{M}_{11}^{as})^2B_{21}+(\acute{M}_{21}^{as})^2B_{12} & -\acute{M}_{11}^{as}\acute{M}_{12}^{as}B_{21}+\acute{M}_{21}^{as}\acute{M}_{22}^{as}B_{12}\end{pmatrix} \tag{4.18}$$

(here $\acute{M}_{ij}^{as}=\acute{M}_{ij}^{as}(x',\xi,t,-\xi)$). □

Remark 10. As $x'\to\pm\infty$, the principal terms in (4.1) match the asymptotics (3.35) in the adjacent sectors. Moreover, they resemble singular, kink-type exact solutions of the focusing NNLS equation [44], which correspond to the reflectionless potentials:

$$q_{A,\phi}(x,t)=\frac{A}{1-e^{-Ax-iA^2t+i\phi}},\quad \phi\in\mathbb{R}. \tag{4.19}$$

On the other hand, the principal terms in (4.1) satisfy the NNLS equation (1.1a) if and only if

$$p_{n-m}\in i\mathbb{R},\ \ |p_{n-m}|=\left|\frac{A\delta^2(0,\xi)}{2}\right|=\left|\frac{\eta_{n-m}\prod_{s=0}^{m-1}\left(\frac{p_{n-m}-p_{n-s}}{p_{n-m}+\omega_{n-s}}\right)^2}{\dot{a}_1(p_{n-m})\delta^2(p_{n-m},-\operatorname{Re}p_{n-m})}\right|;$$

in this case, the leading terms in (4.1) coincide with (4.19) with some $A,\phi\in\mathbb{R}$.

Describing the transition zones of type (2) and (3) are open, challenging problems, which will be considered elsewhere. Here we notice that for the transition zones of type (2), we face the problem that $\nu(\xi)$ takes the value $\frac{1}{2}$, namely, $|\operatorname{Im}\nu(-\omega_m)|=1/2$, which causes problems with the solvability of the asymptotic RH problem of type (3.25) or (3.28), since the norm of the corresponding Cauchy-type operator does not decay as $t\to\infty$.

In the case when ξ approaches 0 (the transition region of type (3)), we encounter another difficulty: the slow variable ξ and the singularity of the Riemann–Hilbert problem (see (2.30) and (3.18)) merge (cf. [32], where the problem for the classical focusing nonlinear Schrödinger equation with singularities on the continuous spectrum is considered). We partially address the latter problem in [45], for initial data close to (1.5) with $R = 0$, where we present the asymptotics along the curves $s = \frac{x^{2-\alpha}}{4t} = const$, with any fixed $s \in (0, \infty)$ and $\alpha \in (0, 1)$. In this case, the asymptotics inside the parabola $t = \frac{x^2}{4s}$ is an open problem as well.

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References

1. Ablowitz, M.J., Segur, H.: Solitons and Inverse Scattering Transform. SIAM, Philadelphia (1981)
2. Ablowitz, M.J., Feng, B.-F., Luo, X.-D., Musslimani, Z.H.: General soliton solution to a nonlocal nonlinear Schrödinger equation with zero and nonzero boundary conditions. Nonlinearity **31**, 5385 (2018)
3. Ablowitz, M.J., Luo, X.-D., Cole, J.: Solitons, the Korteweg–de Vries equation with step boundary values, and pseudo-embedded eigenvalues. J. Math. Phys. **59**, 091406 (2018)
4. Ablowitz, M.J., Musslimani, Z.H.: Integrable nonlocal nonlinear Schrödinger equation. Phys. Rev. Lett. **110**, 064105 (2013)
5. Ablowitz, M.J., Musslimani, Z.H.: Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation. Nonlinearity **29**, 915–946 (2016)
6. Andreiev, K., Egorova, I., Lange, T.L., Teschl, G.: Rarefaction waves of the Korteweg–de Vries equation via nonlinear steepest descent. J. Differ. Equ. **261**, 5371–5410 (2016)
7. Bender, C.M., Boettcher, S.: Real spectra in non-Hermitian Hamiltonians having P-T symmetry. Phys. Rev. Lett. **80**, 5243 (1998)
8. Bikbaev, R.F.: On the shock waves in one-dimensional models with the cubic non-linearity. Theor. Math. Phys. **97**(2), 191–212 (1993)
9. Biondini, G., Fagerstrom, E., Prinari, B.: Inverse scattering transform for the defocusing nonlinear Schrödinger equation with fully asymmetric non-zero boundary conditions. Physica D **333**, 117–136 (2016)
10. Biondini, G., Kovacic, G.: Inverse scattering transform for the focusing nonlinear Schrödinger equation with nonzero boundary conditions. J. Math. Phys. **55**, 031506 (2014)
11. Bludov, Yu., Konotop, V., Malomed, B.: Stable dark solitons in PT-symmetric dual-core waveguides. Phys. Rev. A **87**, 013816 (2013)
12. Boutet de Monvel, A., Kotlyarov, V.P., Shepelsky, D.: Focusing NLS equation: long-time dynamics of step-like initial data. Int. Math. Res. Not. **7**, 1613–1653 (2011)
13. Boutet de Monvel, A., Lenells, J., Shepelsky, D.: The focusing NLS equation with step-like oscillating background: scenarios of long-time asymptotics. arXiv:2003.08862 (2020)
14. Brody, D.C.: PT-symmetry, indefinite metric, and nonlinear quantum mechanics. J. Phys. A: Math. Theor. **50**, 485202 (2017)
15. Buckingham, R., Venakides, S.: Long-time asymptotics of the nonlinear Schrödinger equation shock problem. Commun. Pure Appl. Math. **60**, 1349–1414 (2007)
16. Deift, P.A., Its, A.R., Zhou, X.: Long-time asymptotics for integrable nonlinear wave equations. In: Fokas, A.S., Zakharov, V.E. (eds.) Important Developments in Soliton Theory 1980–1990, pp. 181–204. Springer, New York (1993)
17. Deift, P., Kamvissis, S., Kriecherbauer, T., Zhou, X.: The Toda rarefaction problem. Commun. Pure Appl. Math. **XLIX**, 35–83 (1996)
18. Deift, P.A., Venakides, S., Zhou, X.: The collisionless shock region for the long-time behavior of solutions of the KdV equation. Commun. Pure Appl. Math. **47**(2), 199–206 (1994)
19. Deift, P.A., Venakides, S., Zhou, X.: New results in small dispersion KdV by an extension of the steepest descent method for Riemann–Hilbert problems. Int. Math. Res. Not. **6**, 286–99 (1997)

20. Deift, P.A., Zhou, X.: A steepest descend method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation. Ann. Math. **137**(2), 295–368 (1993)
21. Egorova, I., Gladka, Z., Kotlyarov, V., Teschl, G.: Long-time asymptotics for the Korteweg–de Vries equation with steplike initial data. Nonlinearity **26**, 1839–1864 (2013)
22. El, G.A., Geogjaev, V.V., Gurevich, A.V., Krylov, A.L.: Decay of an initial discontinuity in the defocusing NLS hydrodynamics. Physica D **87**, 186–192 (1995)
23. Faddeev, L.D., Takhtajan, L.A.: Hamiltonian Methods in the Theory of Solitons. Springer Series in Soviet Mathematics. Springer, Berlin (1987)
24. Fokas, A.S.: Integrable multidimensional versions of the nonlocal nonlinear Schrödinger equation. Nonlinearity **29**, 319–324 (2016)
25. Gadzhimuradov, T., Agalarov, A.: Towards a gauge-equivalent magnetic structure of the nonlocal nonlinear Schrödinger equation. Phys. Rev. A **93**, 062124 (2016)
26. Gerdjikov, V.S., Saxena, A.: Complete integrability of nonlocal nonlinear Schrödinger equation. J. Math. Phys. **58**, 013502 (2017)
27. Gurevich, A.V., Pitaevskii, L.P.: Nonstationary structure of a collisionless shock wave. Zh. Eksp. Teor. Fiz. **65**, 590–604 (1973)
28. Gürses, M., Pekcan, A.: Nonlocal nonlinear Schrödinger equations and their soliton solutions. J. Math. Phys. **59**, 051501 (2018)
29. Jenkins, R.: Regularization of a sharp shock by the defocusing nonlinear Schrödinger equation. Nonlinearity **28**, 2131–2180 (2015)
30. Hruslov, E.J.: Asymptotics of the solution of the Cauchy problem for the Korteweg–de Vries equation with initial data of step type. Math. USSR-Sb. **28**, 229–248 (1976)
31. Its, A.R.: Asymptotic behavior of the solutions to the nonlinear Schrödinger equation, and isomonodromic deformations of systems of linear differential equations. Doklady Akad. Nauk SSSR **261**(1), 14–18 (1981)
32. Kamvissis, S.: Long time behavior for the focusing nonlinear Schroedinger equation with real spectral singularities. Commun. Math. Phys. **180**(2), 325–342 (1996)
33. Konotop, V.V., Yang, J., Zezyulin, D.A.: Nonlinear waves in PT-symmetric systems. Rev. Mod. Phys. **88**, 035002 (2016)
34. Kotlyarov, V.P., Khruslov, E.Y.: Solitons of the nonlinear Schrödinger equation, which are generated by the continuous spectrum. Teor. Mat. Fiz. **68**(2), 172–86 (1986)
35. Kotlyarov, V.P., Minakov, A.M.: Riemann–Hilbert problem to the modified Korteveg–deVries equation: long-time dynamics of the step-like initial data. J. Math. Phys. **51**, 093506 (2010)
36. Lenells, J.: The nonlinear steepest descent method for Riemann–Hilbert problems of low regularity. Indiana Univ. Math. **66**, 1287–1332 (2017)
37. Manakov, S.V.: Nonlinear Fraunhofer diffraction. Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki. Pis'ma v Redaktsiyu **65**, 10 (1973)
38. McLaughlin, K.T.-R., Miller, P.D.: The $\bar{\partial}$ steepest descent method and the asymptotic behavior of polynomials orthogonal on the unit circle with fixed and exponentially varying nonanalytic weights. Int. Math. Res. Pap. **48673**, 177 (2006)
39. McLaughlin, K.T.-R.,Miller, P.D.: The $\bar{\partial}$ steepest descent method for orthogonal polynomials on the real line with varying weights. Int. Math. Res. Not. **2008**, rnn075 (2008)
40. Michor, J., Sakhnovich, A.L.: GBDT and algebro-geometric approaches to explicit solutions and wave functions for nonlocal NLS. J. Phys. A: Math. Theor. **52**, 025201 (2018)
41. Novikov, S., Manakov, S.V., Pitaevskii, L.P., Zakharov, V.E.: Theory of Solitons: The Inverse Scattering Method. New York Consultants Bureau (1984)
42. Onorato, M., Osborne, A.R., Serio, M.: Modulational instability in crossing sea states: a possible mechanism for the formation of freak waves. Phys. Rev. Lett. **96**, 014503 (2006)
43. Rybalko, Y., Shepelsky, D.: Long-time asymptotics for the integrable nonlocal nonlinear Schrödinger equation. J. Math. Phys. **60**, 031504 (2019)
44. Rybalko, Y., Shepelsky, D.: Long-time asymptotics for the integrable nonlocal nonlinear Schrödinger equation with step-like initial data. J. Differ. Equ. **270**, 694–724 (2021)
45. Rybalko, Ya., Shepelsky, D.: Curved wedges in the long-time asymptotics for the integrable nonlocal nonlinear Schrödinger equation. arXiv:2004.05987
46. Venakides, S., Deift, P., Oba, R.: The Toda shock problem. Commun. Pure Appl. Math. **44**, 1171–1242 (1991)
47. Vinayagam, P.S., Radha, R., AlKhawaja, U., Ling, L.: New classes of solutions in the coupled PT symmetric nonlocal nonlinear Schrödinger equations with four wave mixing. Commun. Nonlinear Sci. Numer. Simulat. **59**, 387–395 (2018)
48. Sarma, A., Miri, M., Musslimani, Z., Christodoulides, D.: Continuous and discrete Schrödinger systems with parity-time-symmetric nonlinearities. Phys. Rev. E **89**, 052918 (2014)
49. Yang, J.: General N-solitons and their dynamics in several nonlocal nonlinear Schrödinger equations. Phys. Lett. A **383**(4), 328–337 (2019)

50. Yang, J.: Nonlinear Waves in Integrable and Nonintegrable Systems. SIAM, Philadelphia (2010)
51. Znojil, M., Borisov, D.I.: Two patterns of PT-symmetry breakdown in a non-numerical six-state simulation. Ann. Phys. NY **394**, 40–49 (2018)

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