



# Stabilizing Relativistic Fluids on Spacetimes with Non-Accelerated Expansion

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**Abstract:** We establish global regularity and stability for the irrotational relativistic Euler equations with equation of state  $\bar{p} = K\bar{\rho}$ , where  $0 < K < 1/3$ , for small initial data in the expanding direction of FLRW spacetimes of the form  $(\mathbb{R} \times \mathbb{T}^3, -d\bar{t}^2 + \bar{r}^2\delta_{ij}dx^i dx^j)$ . This provides the first case of non-dust fluid stabilization by spacetime expansion where the expansion rate is of power law type but non-accelerated. In particular, the time integral of the inverse scale factor diverges as  $t \rightarrow \infty$ . The result implies that structure formation in cosmological evolution associated with the development of shocks in fluids necessarily requires a phase of decelerating expansion of the Universe to occur in the case that the matter is massive.

## 1. Introduction

In this article, we consider the relativistic Euler equations

$$\bar{v}^\mu \bar{\nabla}_\mu \bar{\rho} + (\bar{\rho} + \bar{p}) \bar{\nabla}_\mu \bar{v}^\mu = 0, \quad (1.1)$$

$$(\bar{\rho} + \bar{p}) \bar{v}^\mu \bar{\nabla}_\mu \bar{v}^\nu + \bar{h}^{\mu\nu} \bar{\nabla}_\mu \bar{p} = 0, \quad (1.2)$$

with a linear equation of state

$$\bar{p} = K\bar{\rho}, \quad 0 < K < 1/3, \quad (1.3)$$

on Milne-like spacetimes of the form

$$((0, \infty) \times \mathbb{T}^3, \bar{g} = -d\bar{t}^2 + \bar{r}^2\delta_{ij}dx^i dx^j),$$

where  $\bar{\nabla}_\mu$  is the Levi-Civita connection of  $\bar{g}$ , the fluid four-velocity  $\bar{v}^\mu$  in normalized according to  $\bar{g}_{\mu\nu} \bar{v}^\mu \bar{v}^\nu = -1$ , and  $\bar{h}^{\mu\nu} = \bar{g}^{\mu\nu} + \bar{v}^\mu \bar{v}^\nu$  defines a positive definite inner product on the subspace of the cotangent space that is  $\bar{g}$ -orthogonal to  $\bar{v}_\mu = \bar{g}_{\mu\nu} \bar{v}^\nu$ .

We show that the canonical homogeneous solutions (1.8) to (1.1)–(1.2) are nonlinearly stable in the expanding direction of spacetime, in the sense that sufficiently small irrotational perturbations of these solutions exist globally towards the future and remain close to the background solutions. In particular, no shocks form in the fluids.

*1.1. Fluid regularization in expanding spacetimes.* We now discuss how the main result of this paper relates to previous work on the relativistic Euler equations.

It is well known, due to the seminal work of Christodoulou [6], that there exist arbitrary small perturbations of the canonical constant solutions to the relativistic Euler equations with a relatively general equation of state on Minkowski spacetime that form shock singularities in finite time. In particular, these homogeneous fluid solutions are unstable.

In expanding spacetimes shock formation can be suppressed. The standard models in cosmology representing expanding spacetimes are derived from the FLRW class, which in the case of zero spatial curvature takes the form

$$((0, \infty) \times \mathbb{T}^3, -d\bar{t}^2 + a(\bar{t})^2 \delta_{ij} dx^i dx^j). \tag{1.4}$$

For such spacetimes there exists a dissipative effect on the fluid induced by the space-time expansion that leads to the stability of homogeneous fluid solutions and thereby global regularization of the relativistic Euler equations. This effect is referred to as *fluid stabilization*. The complementary phenomenon of shock formation fluids in finite time is associated with structure formation during cosmological evolution in the physics literature [5]. Fluid stabilization was discovered by Brauer, Rendall and Reula in the Newtonian case in particular for dust  $K = 0$  [23] and rigorously established in the scenario of relativistic self-gravitating fluids in exponentially expanding spacetimes ( $a(\bar{t}) = e^{\bar{t}}$ ) by Rodnianski and Speck in [20] for the parameter values  $0 < K < 1/3$ . Their result was later complemented by a series of works by Lübke-Valiente Kroon [15], Hadžić-Speck [10], Oliynyk [16] and Friedrich [9] where the allowable range for  $K$  was enlarged to  $0 \leq K \leq 1/3$ . See also [17] for recent results addressing the case  $K > 1/3$ .

While the aforementioned results concern the case of exponentially expanding spacetimes, in [21] Speck considered the relativistic Euler equation with linear equation of state on spacetimes with a scale factor  $a(\bar{t})$  that obeys an integrated growth condition of the form

$$\int_1^\infty a(\bar{t})^{-1} d\bar{t} < \infty$$

or a stronger condition on the behaviour of  $a(\bar{t})$  depending on the value of  $K \in (0, 1/3)$  and proved stability of homogeneous fluid solutions. In addition, he proved that for the particular case of *radiation fluids* ( $K = 1/3$ ) shocks do form if the expansion rate fails to obey integrability, i.e. if

$$\int_1^\infty a(\bar{t})^{-1} d\bar{t} = \infty.$$

In particular this implies that radiation fluid shocks would form on our Milne-like model (1.4). For dust ( $K = 0$ ) only  $a(\bar{t})^{-2}$  needs to be integrable. In Speck’s theorem an even faster expansion rate is required to stabilize *massive fluids*  $0 < K < 1/3$  compared to radiation fluids, and so it seems that stability may a priori fail to hold in general for massive non-dust fluids at the threshold where  $a(\bar{t})^{-1}$  fails to be integrable. If we restrict the scale factor to the polynomial class  $a(t) = t^q$ , then linear expansion  $q = 1$  separates

the accelerating regime  $q > 1$  from the decelerating regime  $q < 1$ . This threshold is interesting as it also appears independently in the context of stability of solutions to the Einstein equations as we discuss in the following.

*1.2. Localization of the Einstein equations.* In a seminal work on the Einstein-non-linear scalar field system, where the potential of the scalar field emulates a positive cosmological constant and thus generates spacetime expansion at an exponential rate, Ringström demonstrated that exponential expansion leads to a decoupling of regions of spacetime for late times [18]. As a consequence, to determine the future asymptotics of solutions to the Einstein equations in a small coordinate neighborhood only the initial data in a slightly larger coordinate neighborhood is required. In this sense, the Einstein equations *localize* in the presence of exponential expansion.

In a follow-up work [19], Ringström relaxed the rate of spacetime expansion to the class of power law inflation, which in our terminology corresponds to scale factors  $a(\bar{t}) = \bar{t}^q$ , where  $q > 1$  and showed that for this class the localization property still holds. We point out that the threshold  $q = 1$  is precisely the one, where  $\int_1^\infty a(\bar{t})^{-1} d\bar{t}$  diverges. However, the Milne model, which is a spacetime on that threshold, clearly does not possess this localization property. This is because the Milne model is a quotient of Minkowski spacetime and as a consequence no two regions can causally decouple. We conclude that at this threshold rate of expansion the causal structure of spacetime changes drastically and this has consequences for the treatment of the Einstein equations. Nevertheless the Milne model is stable as a solution to the Einstein equations [2,3]. The nature of the proof is however substantially different to the power law inflation scenario since the localization property does not hold.

*1.3. Fluid regularization in non-accelerated spacetimes.* We have now identified the threshold rate  $a(\bar{t}) = \bar{t}$  as one between two causally different regimes from the perspective of the Einstein equations. Furthermore the result of Speck [21] shows shock formation for radiation fluids in spacetimes with this linear rate of expansion. Thus it may seem reasonable to believe that more general (i.e. non-dust and non-radiation) fluid regularization also fails at this linear rate.

However, in the context of these considerations, the result in our present paper shows that *fluid regularization does occur* for zero-accelerated power law expansion ( $\ddot{a} = 0$  or  $q = 1$ ) as long as the relation  $0 < K < 1/3$  holds. This implies that the localization property in spacetimes with accelerated expansion (or the integrability of the inverse scale factor) is not the necessary feature of the spacetime that regularizes the fluid for  $0 < K < 1/3$ , (in contrast to the case of  $K = 1/3$  as shown by Speck).

We conclude, that the present result establishes relativistic fluid regularization for the slowest expansion rate in comparison to previous results for the regime  $0 < K < 1/3$ . The upper bound on  $K$  is sharp by Speck's result. We do not claim that the lower bound on the expansion rate is sharp. A trivial lower bound on scale factors that provide fluid regularization for  $0 < K < 1/3$  is given by Christodoulou's result with  $a(\bar{t}) = 1$ . From the physical perspective, we conclude from our result that structure formation from small perturbations requires decelerated expansion for it to occur in the class of massive fluids. This is in contrast to massless fluids (radiation) where non-accelerated expansion was sufficient for shocks to form. Thus our results indicate a substantial difference between the ability for structure formation to occur between massive and massless fluids. For cosmological evolution this implies that the era of structure formation from massive

matter and small inhomogeneities must have occurred during a phase of decelerated expansion of the universe.

*1.4. Conformal rescaling of the metric and homogeneous fluid solutions.* We consider Milne-like spacetimes of the form  $(M, \bar{g})$  where<sup>1</sup>

$$M = (0, T_0] \times \mathbb{T}^3, \quad T_0 > 0, \tag{1.5}$$

and

$$\bar{g} = \frac{1}{t^2} \left( -\frac{1}{t^2} dt^2 + \delta_{ij} dx^i dx^j \right). \tag{1.6}$$

Here,  $t = x^0$  is a time coordinate on the interval  $(0, T_0]$ ,  $(x^i)$ ,  $i = 1, 2, 3$ , denotes standard period coordinates on the 3-torus  $\mathbb{T}^3$ . In the following, we will use  $\partial_\mu$  to denote the partial derivatives with respect to the coordinates  $(x^\mu) = (t, x^i)$  and define  $\partial^i = \delta^{ij} \partial_j$ . It is important to note that, due to our conventions, the future is located in the direction of *decreasing*  $t$  and future timelike infinity is located at  $t = 0$ . Consequently, we require that

$$\bar{v}^0 < 0$$

in order to ensure that the four-velocity is future directed.

Throughout this article, unless otherwise specified, we will assume that the constant  $K$  in the linear equation of state (1.3) satisfies

$$0 < K < 1/3. \tag{1.7}$$

A straightforward calculation then shows that, for every positive constant  $c_H > 0$ , the pair

$$(\bar{v}_H^\mu, \bar{\rho}_H) = \left( -\frac{\bar{g}^{\mu\nu} \partial_\nu \bar{\phi}_H}{\bar{\zeta}_H}, \bar{\zeta}_H^{\frac{1+K}{K}} \right), \tag{1.8}$$

where

$$\bar{\phi}_H = c_H t^{-\lambda}, \quad \lambda = 1 - 3K, \tag{1.9}$$

and

$$\bar{\zeta}_H = \sqrt{-\bar{g}(d\bar{\phi}_H, d\bar{\phi}_H)}, \tag{1.10}$$

defines a homogeneous, irrotational solution to the relativistic Euler equations (1.1)–(1.2) such that the four-velocity is future pointing. Explicitly, this solution reads

$$(\bar{v}_H^\mu, \bar{\rho}_H) = \left( -t^2 \delta_0^\mu, ((1 - 3K)c_H)^{\frac{1+K}{K}} t^{3(1+K)} \right).$$

The main aim of this article is to establish the future stability of non-linear, irrotational perturbations of this homogeneous solution on the parameter range (1.7).

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<sup>1</sup> By introducing a change of time coordinate according to the formula  $\bar{t} = 1/t$ , the metric (1.6) can be brought into the more recognizable form

$$\bar{g} = -d\bar{t}^2 + \bar{t}^2 \delta_{ij} dx^i dx^j,$$

where now  $(\bar{t}, x^i) \in [1/T_0, \infty) \times \mathbb{T}^3$ . We refer to such metrics as ‘Milne-like’ since the scale factor  $\bar{t}^2$  is the same as in Milne, even though the spatial geometry  $(\mathbb{T}^3, \delta)$  is different from  $(\mathbb{H}^3, g_{\mathbb{H}^3})$  or quotients thereof, appearing in the standard Milne spacetime.

*1.5. Fuchsian approach.* To establish the global existence to the future of solutions to (1.1)–(1.2) that represent irrotational, non-linear perturbations of the homogeneous solution (1.8), we employ the Fuchsian method that was introduced in [16] and further developed in [4]. This method involves transforming the relativistic Euler equations into a Fuchsian symmetric hyperbolic equation of the form

$$B^0(U)\partial_t U + \frac{1}{t}(C^i + B^i(U))\partial_i U = \frac{1}{t}\mathcal{B}(U)\mathbb{P}U + \frac{1}{t}F(U) \quad \text{in } (0, T_0] \times \mathbb{T}^3, \quad (1.11)$$

see Sect. 3.4 for details. Equations of this type were studied in [4, 16] and the existence of solutions with uniform decay as  $t \searrow 0$  was established under certain assumptions on the system's coefficients and a small initial data assumption. The Fuchsian system that we obtain for the relativistic Euler equations in this article does not satisfy the assumptions needed to apply the existence theory from [4, 16], as we show in Sect. 4. In order to establish global existence for (1.11) we therefore generalize [4] to this extended class in Theorem 4.5. The precise statement of the global existence result can be found in Theorem 3.2. Interestingly, while we do obtain global existence, we do not get uniform decay as  $t \searrow 0$  as was the case in [4, 16]. The obstruction to decay is the singular terms  $\frac{1}{t}C^k$ , where the  $C^k$ ,  $k = 1, 2, 3$ , are constant, symmetric matrices that are not present in the Fuchsian equations considered in [4, 16].

*1.6. Outlook.* Since it is known that arbitrarily small perturbations of the homogeneous solution must form shocks in finite time [21] when  $K = 1/3$ , our results, in this sense, are sharp. It is open, whether the present result holds for spacetimes that expand slower than the Milne rate  $a(\bar{t}) = \bar{t}$  for  $0 < K < 1/3$ . Moreover, we expect that the present result can be generalized to the rotational case. Another potential generalization concerns the inclusion of gravity by coupling the Euler equations to the Einstein equations in the framework similar to [2, 7].

*1.7. Organization of the paper.* In the following section we introduce notations and setup. In Sect. 3 we bring the irrotational Euler equations into the required Fuchsian form and apply the Fuchsian global existence theorem to obtain the main theorem, Theorem 3.2. In Sect. 4 the global existence theorem for Fuchsian systems of type (1.11) is proven. The appendix contains fundamental lemmas on standard functional inequalities that are used in the paper and provided here for convenience.

## 2. Preliminaries

Before proceeding, we first fix our notation and introduce a few definitions that will be used throughout the article.

*2.1. Coordinates, indexing and derivatives.* Except for Sect. 4, we will use lower case Greek letters, e.g.  $\alpha, \beta, \gamma$ , to label spacetime coordinate indices that run from 0 to 3, while we will reserve lower case Latin letters, e.g.  $i, j, k$ , to label spatial coordinate indices that run from 1 to 3. In the appendices, we will consider general spatial dimensions, and so there, lower case Latin indices, e.g.  $i, j, k$ , will run from 1 to  $n$  and will be used to index spatial coordinate indices. Furthermore,  $x = (x^i)$  will again denote

the standard periodic coordinates, this time on the  $n$ -torus  $\mathbb{T}^n$ , and  $t$  will denote a time coordinate on intervals of  $\mathbb{R}$ . Partial derivatives with respect to these coordinates will be denoted by  $\partial_i$  and  $\partial_t$ , respectively. Additionally, in Sect. 4, we use lower case Greek letters to denote multi-indices, e.g.  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ , and we will employ the standard notation  $D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$  for spatial partial derivatives and  $Du = (\partial_j u)$  for the spatial gradient. It will be clear from context whether a Greek index is meant to be interpreted as a spacetime index or a multi-index.

2.2. *Inner-products and matrix inequalities.* Throughout this article, we use

$$(\xi|\zeta) = \xi^T \zeta, \quad \xi, \zeta \in \mathbb{R}^N,$$

to denote the Euclidean inner-product and

$$|\xi| = \sqrt{(\xi|\xi)}$$

to denote the Euclidean norm. Moreover, given matrices  $A, B \in \mathbb{M}_{N \times N}$ , we define

$$A \leq B \iff (\xi|A\xi) \leq (\xi|B\xi), \quad \forall \xi \in \mathbb{R}^N,$$

and we use

$$|A|_{\text{op}} = \sup_{|\xi|=1} |A\xi|$$

to denote the operator norm of  $A$ .

2.3. *Sobolev spaces.* The  $W^{k,p}$ ,  $k \in \mathbb{Z}_{\geq 0}$ , norm of a map  $u \in C^\infty(\mathbb{T}^N, \mathbb{R}^N)$  is defined by

$$\|u\|_{W^{k,p}} = \begin{cases} \left( \sum_{0 \leq |\alpha| \leq k} \int_{\mathbb{T}^n} |D^\alpha u|^p d^n x \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \max_{0 \leq \ell \leq k} \sup_{x \in \mathbb{T}^n} |D^\ell u(x)| & \text{if } p = \infty \end{cases}.$$

The Sobolev space  $W^{k,p}(\mathbb{T}^n, \mathbb{R}^N)$  is then defined as the completion of  $C^\infty(\mathbb{T}^n, \mathbb{R}^N)$  with respect to the norm  $\|\cdot\|_{W^{k,p}}$ . When  $N = 1$ , we will write  $W^{k,p}(\mathbb{T}^n)$  instead, and we will employ the standard notation  $L^p(\mathbb{T}^n) = W^{0,p}(\mathbb{T}^n)$ . Furthermore, when  $p = 2$ , we set  $H^k(\mathbb{T}^n, \mathbb{R}^N) = W^{k,2}(\mathbb{T}^n, \mathbb{R}^N)$ , and we use  $\langle \cdot | \cdot \rangle$  to denote that  $L^2$  inner-product on  $\mathbb{T}^n$ , that is,

$$\langle u|v \rangle = \int_{\mathbb{T}^n} (u|v) d^n x.$$

2.4. *Constants and inequalities.* We use the standard notation

$$a \lesssim b$$

for inequalities of the form

$$a \leq Cb$$

in situations where the precise value or dependence on other quantities of the constant  $C$  is not required. On the other hand, when the dependence of the constant on other inequalities needs to be specified, for example if the constant depends on the norm  $\|u\|_{L^\infty}$ , we use the notation

$$C = C(\|u\|_{L^\infty}).$$

Constants of this type will always be non-negative, non-decreasing, continuous functions of their arguments.

### 3. Irrotational Euler Equations

It is well known, see [20, §3.1] for example, that the irrotational relativistic Euler equations (1.1)–(1.2) can be formulated as a non-linear wave equation. In particular, for the linear equation of state (1.3), the irrotational relativistic Euler equations are given by

$$\bar{a}^{\alpha\beta} \bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{\phi} = 0 \tag{3.1}$$

where

$$\bar{a}^{\alpha\beta} = \bar{g}^{\alpha\beta} - \sigma \bar{v}^\alpha \bar{v}^\beta, \quad \sigma = \frac{1 - K}{K}, \tag{3.2}$$

is the *acoustic metric*. The four-velocity is determined by

$$\bar{v}^\alpha = -\frac{\bar{g}^{\alpha\beta} \partial_\beta \bar{\phi}}{\bar{\zeta}} \tag{3.3}$$

where

$$\bar{\zeta} = \sqrt{-\bar{g}(d\bar{\phi}, d\bar{\phi})} \tag{3.4}$$

is the *fluid enthalpy*. In this formulation, the proper energy density of the fluid can be recovered from the enthalpy via the formula

$$\bar{\rho} = \bar{\zeta}^{\frac{1+K}{K}}. \tag{3.5}$$

3.1. *Rescaled fluid potential.* The first step in transforming the irrotational relativistic Euler equations, given by (3.1), into Fuchsian form involves introducing a rescaled fluid potential via

$$\phi = t^\lambda \bar{\phi}. \tag{3.6}$$

We then see after a straightforward calculation involving (1.6) and (3.2)–(3.4) that the acoustic wave equation (3.1), when expressed in terms of  $\phi$ , becomes

$$\alpha^{00} t \partial_t (t \partial_t \phi) + 2\alpha^{0i} t \partial_t \partial_i \phi + \alpha^{ij} \partial_i \partial_j \phi + \beta^0 t \partial_t \phi + \beta^i \partial_i \phi + \gamma \phi = 0 \tag{3.7}$$

where

$$\alpha^{00} = -1 - \frac{\sigma}{\mu}(t\partial_t\phi - \lambda\phi)^2, \quad (3.8)$$

$$\alpha^{0i} = \frac{\sigma}{\mu}(t\partial_t\phi - \lambda\phi)\partial^i\phi, \quad (3.9)$$

$$\alpha^{ij} = \delta^{ij} - \frac{\sigma}{\mu}\partial^i\phi\partial^j\phi, \quad (3.10)$$

$$\beta^0 = \frac{(\sigma+1)\lambda(t\partial_t\phi - \lambda\phi)^2 + \lambda(\sigma-1)|D\phi|^2}{\mu}, \quad (|D\phi|^2 = \delta^{ij}\partial_i\phi\partial_j\phi), \quad (3.11)$$

$$\beta^i = -\frac{2\lambda\sigma}{\mu}(t\partial_t\phi - \lambda\phi)\partial^i\phi, \quad (3.12)$$

$$\gamma = -\frac{\lambda\sigma}{\mu}|D\phi|^2 \quad (3.13)$$

and

$$\mu = (t\partial_t\phi - \lambda\phi)^2 - |D\phi|^2. \quad (3.14)$$

For use below, we note that  $\alpha^{00}$  can be written as

$$\alpha^{00} = -(1 + \sigma) + \frac{\sigma}{\mu}|D\phi|^2 \quad (3.15)$$

and that, under the rescaling (3.6), the homogeneous solution (1.9) is transformed into the constant solution

$$\phi_H = c_H, \quad c_H \in \mathbb{R}_{>0}, \quad (3.16)$$

of (3.7).

*3.2. First order formulation.* The next step in the transformation of the irrotational relativistic Euler equations into Fuchsian form involves expressing the wave equation (3.7) in first order form by introducing the variables

$$\phi_0^\ell = t\partial_t\phi - \ell\delta\phi, \quad \ell \in \mathbb{R}, \quad (3.17)$$

$$\phi_i = \partial_i\phi \quad (3.18)$$

and

$$\delta\phi = \phi - c_H. \quad (3.19)$$

A short calculation show that in terms of these variables, the wave equation (3.7) is given by

$$\begin{aligned} -\alpha^{00}\partial_t\phi_0^\ell - \frac{2}{t}\alpha^{0i}\partial_i\phi_0^\ell - \frac{1}{t}\alpha^{ij}\partial_i\phi_j &= \frac{1}{t}((\beta^0 + \ell\alpha^{00})(\phi_0^\ell + \ell\delta\phi) \\ &\quad + (\beta^i + 2\ell\alpha^{0i})\phi_i + \gamma(\delta\phi + c_H)), \\ \alpha^{ij}\partial_t\phi_j - \frac{1}{t}\alpha^{ij}\partial_j\phi_0^\ell &= \frac{\ell}{t}\alpha^{ij}\phi_j. \end{aligned}$$



We now collect together two versions of this system obtained from setting  $\ell = 0$  and  $\ell = \lambda$ . This, with the help of the identity  $\partial_j \phi_0^\lambda = \partial_j \phi_0^0 - \lambda \phi_j$ , gives rise to the following system

$$-\alpha^{00} \partial_t \phi_0^0 - \frac{2}{t} \alpha^{0i} \partial_i \phi_0^\lambda - \frac{1}{t} \alpha^{ij} \partial_i \phi_j = \frac{1}{t} (\beta^0 \phi_0^0 + (\beta^i + 2\lambda \alpha^{0i}) \phi_i + \gamma (\delta \phi + c_H)), \tag{3.20}$$

$$\alpha^{ij} \partial_t \phi_j - \frac{1}{t} \alpha^{ij} \partial_j \phi_0^0 = 0, \tag{3.21}$$

$$-\alpha^{00} \partial_t \phi_0^\lambda - \frac{2}{t} \alpha^{0i} \partial_i \phi_0^0 - \frac{1}{t} \alpha^{ij} \partial_i \phi_j = \frac{1}{t} ((\beta^0 + \lambda \alpha^{00}) (\phi_0^\lambda + \lambda \delta \phi) + \beta^i \phi_i + \gamma (\delta \phi + c_H)), \tag{3.22}$$

$$\alpha^{ij} \partial_t \phi_j - \frac{1}{t} \alpha^{ij} \partial_j \phi_0^\lambda = \frac{\lambda}{t} \alpha^{ij} \phi_j, \tag{3.23}$$

where  $\delta \phi$  can be recovered from  $\phi_0^0$  and  $\phi_0^\lambda$  via the formula

$$\delta \phi = \frac{\phi_0^0 - \phi_0^\lambda}{\lambda}. \tag{3.24}$$

Recalling that  $\lambda = 1 - 3K$  and  $\sigma = \frac{1-K}{K}$ , a short calculation using (3.8), (3.11), (3.14) and (3.18) shows that

$$\beta^0 + \lambda \alpha^{00} = -\frac{\lambda \sigma}{\mu} |\boldsymbol{\phi}|^2, \quad |\boldsymbol{\phi}|^2 = \delta^{ij} \phi_i \phi_j. \tag{3.25}$$

Setting

$$u = (\phi_0^0 \quad \phi_j \quad \phi_0^\lambda \quad \phi_j)^\text{tr}, \tag{3.26}$$

we can then write the system (3.20)–(3.23) in matrix form as

$$A^0(u) \partial_t u + \frac{1}{t} (C^k + A^k(u)) \partial_k u = \frac{1}{t} \mathcal{A}(u) \Pi u + \frac{1}{t} F(u) \tag{3.27}$$

where

$$A^0(u) = \begin{pmatrix} -\alpha^{00} & 0 & 0 & 0 \\ 0 & \alpha^{ij} & 0 & 0 \\ 0 & 0 & -\alpha^{00} & 0 \\ 0 & 0 & 0 & \alpha^{ij} \end{pmatrix}, \tag{3.28}$$

$$C^k = \begin{pmatrix} 0 & -\delta^{kj} & 0 & 0 \\ -\delta^{ik} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta^{kj} \\ 0 & 0 & -\delta^{ik} & 0 \end{pmatrix}, \tag{3.29}$$

$$A^k(u) = \begin{pmatrix} 0 & \delta^{kj} - \alpha^{kj} & -2\alpha^{0k} & 0 \\ \delta^{ik} - \alpha^{ik} & 0 & 0 & 0 \\ -2\alpha^{0k} & 0 & 0 & \delta^{kj} - \alpha^{kj} \\ 0 & 0 & \delta^{ik} - \alpha^{ik} & 0 \end{pmatrix}, \tag{3.30}$$

$$\mathcal{A}(u) = \begin{pmatrix} \beta^0 & 0 & 0 & 0 \\ 0 & \lambda\delta^{ij} & 0 & 0 \\ 0 & 0 & (1 + \sigma)\lambda & 0 \\ 0 & 0 & 0 & \lambda\alpha^{ij} \end{pmatrix}, \tag{3.31}$$

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_j^i \end{pmatrix} \tag{3.32}$$

and

$$F = \begin{pmatrix} (\beta^i + 2\lambda\alpha^{0j})\phi_j + \gamma(\delta\phi + c_H) \\ 0 \\ (\beta^0 + \lambda\alpha^{00})(\phi_0^\lambda + \lambda\delta\phi) + \beta^i\phi_i + \gamma(\delta\phi + c_H) \\ 0 \end{pmatrix}. \tag{3.33}$$

*3.3. Coefficient properties.* It is not difficult to verify, with the help of the formulas (3.8)–(3.15), (3.17)–(3.19), (3.24) and (3.25), that the coefficients of (3.27) satisfy the following:

- (i) There exists a  $R_0 > 0$  such that the matrices  $A^0(u)$ ,  $A^k(u)$  and  $\mathcal{A}(u)$ , and the source term  $F(u)$  are smooth in  $u$  for  $u \in B_{R_0}(\mathbb{R}^8)$ . Moreover, the matrices  $A^0(u)$ ,  $A^k(u)$ ,  $C^k$  and  $\Pi$  are all symmetric and  $\Pi$  defines a projection operator, that is,

$$\Pi^2 = \Pi. \tag{3.34}$$

*Remark 3.1.* In the following, we will always be able to assume that  $(u, Du) \in B_R(\mathbb{R}^8 \times \mathbb{R}^{3 \times 8})$  for any  $R \in (0, R_0]$  since ultimately we will establish  $L^\infty$  bounds on both  $u$  and  $Du$ . Moreover, any of the implied constants in the  $\lesssim$  signs will depend on  $R_0$ , which we take to be fixed, but will be independent of  $R \in (0, R_0]$ .

- (ii) The matrices  $A^0(0)$  and  $\mathcal{A}(xu)$  satisfy

$$[\Pi, A^0(u)] = [\Pi, \mathcal{A}(u)] = 0 \tag{3.35}$$

and

$$\partial_k(\Pi^\perp \mathcal{A}(u)) = 0 \tag{3.36}$$

where

$$\Pi^\perp = \mathbb{I} - \Pi. \tag{3.37}$$

Furthermore,

$$|A^0(u) - A^0(0)|_{\text{op}} + |\mathcal{A}(u) - \mathcal{A}(0)|_{\text{op}} \lesssim |\phi|^2 \tag{3.38}$$

where

$$A^0(0) = \begin{pmatrix} 1 + \sigma & 0 & 0 & 0 \\ 0 & \delta^{ij} & 0 & 0 \\ 0 & 0 & 1 + \sigma & 0 \\ 0 & 0 & 0 & \delta^{ij} \end{pmatrix} \text{ and } \mathcal{A}(0) = \lambda A^0(0). \tag{3.39}$$

We also note by (3.35) that

$$\Pi A^0(u) \Pi^\perp = \Pi^\perp A^0(u) \Pi = 0.$$

(iii) The matrices  $\mathcal{C}^k$ ,  $A^k(u)$  and the source term  $F(u)$  satisfy

$$\Pi^\perp \mathcal{C}^k \Pi^\perp = 0, \quad (3.40)$$

$$\Pi^\perp A^k(u) \Pi^\perp = 0, \quad (3.41)$$

$$|\Pi^\perp A^k(u) \Pi|_{\text{op}} \lesssim |\boldsymbol{\phi}|, \quad (3.42)$$

$$|\Pi A^k(u) \Pi|_{\text{op}} \lesssim |\boldsymbol{\phi}| \quad (3.43)$$

and

$$|F(u)| \lesssim |\boldsymbol{\phi}|^2. \quad (3.44)$$

Next, using

$$\partial_t \phi_j = \frac{1}{t} \partial_j \phi_0^0$$

and noting the time derivatives of  $\phi_0^\lambda$  and  $\phi_0^0$  can be computed from (3.27), it is not difficult to verify, with the help of (3.35)–(3.38) and (3.40)–(3.44), that

$$\left| \Pi^\perp (\partial_t (A^0(u)) + \frac{1}{t} \partial_k (A^k(u)) \Pi^\perp \right|_{\text{op}} \lesssim \frac{1}{t} (|\phi_0^0|^2 + |\boldsymbol{\phi}|^2 + |D\phi_0^0|^2 + |D\boldsymbol{\phi}|^2), \quad (3.45)$$

$$\begin{aligned} & \left| \Pi^\perp (\partial_t (A^0(u)) + \frac{1}{t} \partial_k (A^k(u)) \Pi \right|_{\text{op}} + \left| \Pi (\partial_t (A^0(u)) + \frac{1}{t} \partial_k (A^k(u)) \Pi^\perp \right|_{\text{op}} \\ & \lesssim \frac{1}{t} (|\phi_0^0| + |\boldsymbol{\phi}| + |D\phi_0^0| + |D\boldsymbol{\phi}|) \end{aligned} \quad (3.46)$$

and

$$\left| \Pi (\partial_t (A^0(u)) + \frac{1}{t} \partial_k (A^k(u)) \Pi \right|_{\text{op}} \lesssim \frac{1}{t} (|\phi_0^0| + |\boldsymbol{\phi}| + |D\phi_0^0| + |D\boldsymbol{\phi}|). \quad (3.47)$$

Additionally, it is also clear from (3.38) and (3.41)–(3.43) that

$$|\partial_j (A^0(u))|_{\text{op}} + |\partial_j (A(u))|_{\text{op}} + |\partial_j (F(u))| \lesssim |\boldsymbol{\phi}|^2 + |D\boldsymbol{\phi}|^2 \quad (3.48)$$

and

$$|\Pi^\perp \partial_j (A^k(u)) \Pi|_{\text{op}} + |\Pi \partial_j (A^k(u)) \Pi^\perp|_{\text{op}} + |\Pi \partial_j (A^k(u)) \Pi|_{\text{op}} \lesssim |\boldsymbol{\phi}| + |D\boldsymbol{\phi}|. \quad (3.49)$$

*3.4. The extended system.* As it stands, the system (3.27) is almost, but not quite, in the Fuchsian form that we require in order to apply the existence theory developed in Sect. 4. To bring it into the required form, we apply the differential operator  $A^0 \partial_l (A^0)^{-1}$  to get

$$A^0(u) \partial_t \partial_l u + \frac{1}{t} (\mathcal{C}^k + A^k(u)) \partial_k \partial_l u = \frac{1}{t} \mathcal{A} \Pi \partial_l u + \frac{1}{t} G_l(u, Du) \quad (3.50)$$

where

$$\begin{aligned} G_l(u, Du) = A^0(u) & \left[ - \left( -\partial_l (A^0(u))^{-1} (\mathcal{C}^k + A^k(u)) + (A^0(u))^{-1} \partial_l (A^k(u)) \right) \partial_k u \right. \\ & \left. + \partial_l ((A^0(u))^{-1} \mathcal{A}(u)) \Pi u + \partial_l ((A^0(u))^{-1} F(u)) \right]. \end{aligned} \quad (3.51)$$

Since all the coefficients  $A^0(u)$ ,  $A^k(u)$ ,  $\mathcal{A}(u)$  and  $F(u)$  are smooth in  $u$  for  $u \in B_{R_0}(\mathbb{R}^8)$ , it is clear that  $G_I(u, v)$  is smooth in  $(u, v)$  for  $(u, v) \in B_{R_0}(\mathbb{R}^8) \times \mathbb{R}^{3 \times 8}$ .

Multiplying (3.51) by  $\Pi^\perp$ , we obtain

$$\begin{aligned}
 \Pi^\perp G_I &= A^0 \left[ - \left( -\partial_l (A^0)^{-1} (\Pi^\perp C^k + \Pi^\perp A^k) + (A^0)^{-1} \partial_l (\Pi^\perp A^k) \right) \partial_k u \right. \\
 &\quad \left. + \partial_l ((A^0)^{-1} \mathcal{A}) \Pi^\perp \Pi u + \Pi^\perp \partial_l ((A^0)^{-1} F) \right] \tag{by((3.35))\&((3.37))} \\
 &= A^0 \left[ - \left( -\partial_l (A^0)^{-1} (\Pi^\perp C^k + \Pi^\perp A^k) + (A^0)^{-1} \partial_l (\Pi^\perp A^k \Pi^\perp) \right. \right. \\
 &\quad \left. \left. + (A^0)^{-1} \partial_l (\Pi^\perp A^k \Pi) \right) \partial_k u + \Pi^\perp \partial_l ((A^0)^{-1} F) \right] \tag{by((3.37))and\Pi^\perp \Pi = 0} \\
 &= A^0 \left[ - \left( -\partial_l (A^0)^{-1} (\Pi^\perp C^k + \Pi^\perp A^k) \partial_k u + (A^0)^{-1} \partial_l (\Pi^\perp A^k \Pi) \partial_k \Pi u \right) \right. \\
 &\quad \left. + \Pi^\perp \partial_l ((A^0)^{-1} F) \right]. \tag{by((3.34))and((3.41))}
 \end{aligned} \tag{3.52}$$

Since we are assuming that  $(u, Du) \in B_R(\mathbb{R}^8 \times \mathbb{R}^{3 \times 8})$ ,  $R \in (0, R_0]$ , we see from (3.26), (3.32), (3.48), (3.49), (3.51) and (3.52) that

$$|\Pi^\perp G_I(u, Du)| \lesssim |\phi|^2 + |D\phi|^2 + |D\phi_0^0|^2 \tag{3.53}$$

and

$$|\Pi G_I(u, Du)| \lesssim |Du|(|\phi| + |D\phi|). \tag{3.54}$$

The *extended system* is then defined by combining (3.27) and (3.50) into the following system

$$B^0(U) \partial_t U + \frac{1}{t} (C^k + B^k(U)) \partial_k U = \frac{1}{t} \mathcal{B} \mathbb{P} U + \frac{1}{t} H(U) \tag{3.55}$$

where

$$U = \begin{pmatrix} u \\ Du \end{pmatrix}, \tag{3.56}$$

$$B^0(U) = \begin{pmatrix} A^0(u) & 0 \\ 0 & A^0(u) \end{pmatrix}, \tag{3.57}$$

$$C^k = \begin{pmatrix} C^k & 0 \\ 0 & C^k \end{pmatrix}, \tag{3.58}$$

$$B^k(U) = \begin{pmatrix} A^k(u) & 0 \\ 0 & A^k(u) \end{pmatrix}, \tag{3.59}$$

$$\mathcal{B}(U) = \begin{pmatrix} \mathcal{A}(u) & 0 \\ 0 & \mathcal{A}(u) \end{pmatrix}, \tag{3.60}$$

$$\mathbb{P} = \begin{pmatrix} \Pi & 0 \\ 0 & \Pi \end{pmatrix}, \tag{3.61}$$

$$H(U) = \begin{pmatrix} F(u) \\ G(u, Du) \end{pmatrix} \tag{3.62}$$

and we have set

$$G(u, Du) = (G_I(u, Du)).$$

The point of the extended system is that it is now in Fuchsian form and its coefficients satisfy the assumptions needed apply the existence theory developed in Sect. 4. To see that the coefficients do in fact satisfy the required assumptions, we observe, with the help of (3.26), (3.32), (3.39) and (3.57)–(3.62), that (3.38), (3.41)–(3.44), (3.45)–(3.47), and (3.53)–(3.54) imply that

$$\left| \mathbb{P}^\perp \left( \partial_t(B^0(U)) + \frac{1}{t} \partial_k(B^k(U)) \right) \mathbb{P}^\perp \right|_{\text{op}} \lesssim \frac{1}{t} |\mathbb{P}U|^2, \tag{3.63}$$

$$\left| \mathbb{P}^\perp \left( \partial_t(B^0(U)) + \frac{1}{t} \partial_k(B^k(U)) \right) \mathbb{P} \right|_{\text{op}} + \left| \mathbb{P} \left( \partial_t(B^0(U)) + \frac{1}{t} \partial_k(B^k(U)) \right) \mathbb{P}^\perp \right|_{\text{op}} \lesssim \frac{1}{t} |\mathbb{P}U|, \tag{3.64}$$

$$\left| \mathbb{P} \left( \partial_t(B^0(U)) + \frac{1}{t} \partial_k(B^k(U)) \right) \mathbb{P} \right|_{\text{op}} \lesssim \frac{1}{t} |\mathbb{P}U|, \tag{3.65}$$

$$\mathbb{P}^\perp B^k(U) \mathbb{P}^\perp = 0, \tag{3.66}$$

$$|\mathbb{P}^\perp B^k(U) \mathbb{P}|_{\text{op}} + |\mathbb{P} B^k(U) \mathbb{P}^\perp|_{\text{op}} \lesssim |\mathbb{P}U|, \tag{3.67}$$

$$|\mathbb{P} B^k(U) \mathbb{P}|_{\text{op}} \lesssim |\mathbb{P}U|, \tag{3.68}$$

$$|\mathbb{P}^\perp H(U)| \lesssim |\mathbb{P}U|^2, \tag{3.69}$$

$$|\mathbb{P} H(U)| \lesssim |U| |\mathbb{P}U|, \tag{3.70}$$

$$|B^0(u) - B^0(0)|_{\text{op}} + |\mathcal{B}(u) - \mathcal{B}(0)|_{\text{op}} \lesssim |\mathbb{P}U|^2 \tag{3.71}$$

and

$$\mathbb{P} B^0(u) \mathbb{P}^\perp = \mathbb{P}^\perp B^0(u) \mathbb{P} = 0, \tag{3.72}$$

where

$$B^0(0) = \begin{pmatrix} A^0(0) & 0 \\ 0 & A^0(0) \end{pmatrix} \quad \text{and} \quad \mathcal{B}(0) = \lambda B^0(0). \tag{3.73}$$

and

$$\mathbb{P}^\perp = \mathbb{I} - \mathbb{P}. \tag{3.74}$$

We see also from (3.34)–(3.36) and (3.60)–(3.61) that

$$[\mathbb{P}, \mathcal{B}(U)] = 0, \tag{3.75}$$

$$\partial_k(\mathbb{P}^\perp \mathcal{B}(U)) = 0 \tag{3.76}$$

and

$$\mathbb{P}^2 = \mathbb{P}. \tag{3.77}$$

*3.5. Global existence.* We are now ready to state and prove the main result of this article that guarantees the future stability of irrotational, non-linear perturbations of the homogeneous solutions (1.8) to the relativistic Euler equations (1.1)–(1.2) on Milne-like spacetimes of the form (1.5)–(1.6).

**Theorem 3.2.** *Suppose  $T_0 > 0$ ,  $k \in \mathbb{Z}_{>3/2+2}$ ,  $0 < K < 1/3$ ,  $c_H > 0$  and*

$$(\tilde{\phi}_0, \tilde{\phi}_1) \in H^{k+2}(\mathbb{T}^3) \times H^{k+1}(\mathbb{T}^3).$$

*Then there exists a  $\delta > 0$  such that if*

$$\|\tilde{\phi}_0 - c_H\|_{H^{k+2}} + T_0\|\tilde{\phi}_1\|_{H^{k+1}} \leq \delta,$$

*then there exists a*

$$\phi \in \bigcap_{\ell=0}^{k+2} C^\ell((0, T_0], H^{k+2-\ell}(\mathbb{T}^3))$$

*such that  $\bar{\phi} = t^{3K-1}\phi$  defines a unique classical solution of the wave equation (3.1) on  $(0, T_0] \times \mathbb{T}^3$  satisfying the initial conditions*

$$(\bar{\phi}|_{t=T_0}, \partial_t \bar{\phi}|_{t=T_0}) = (T_0^{3K-1}\tilde{\phi}_0, T_0^{3K-2}(T_0\tilde{\phi}_1 + (3K - 1)\tilde{\phi}_0)),$$

*which by (3.3) and (3.5), determines a (unique) irrotational solution of the relativistic Euler equations (1.1)–(1.2) on  $(0, T_0] \times \mathbb{T}^3$ . Moreover,  $\phi$  is bounded by*

$$\begin{aligned} & \|\phi(t) - c_H\|_{H^{k+2}}^2 + \|t\partial_t\phi(t)\|_{H^{k+1}}^2 + \int_t^{T_0} \frac{1}{\tau} \left( \|\tau\partial_\tau\phi(\tau)\|_{H^{k+1}}^2 + \|D\phi(\tau)\|_{H^{k+1}}^2 \right) d\tau \\ & \lesssim (\|\tilde{\phi}_0 - c_H\|_{H^{k+2}}^2 + \|\tilde{\phi}_1\|_{H^{k+1}}^2) \end{aligned}$$

*for all  $t \in (0, T_0]$ .*

*Proof.* First, we fix  $T_0 > 0$ ,  $k \in \mathbb{Z}_{>3/2+2}$ ,  $c_H > 0$ ,  $0 < K < 1/3$ ,  $\delta > 0$  and choose  $(\tilde{\phi}_0, \tilde{\phi}_1) \in H^{k+2}(\mathbb{T}^3) \times H^{k+1}(\mathbb{T}^3)$  so that

$$\|\tilde{\phi}_0 - c_H\|_{H^{k+2}} + T_0\|\tilde{\phi}_1\|_{H^{k+1}} \leq \delta.$$

We then know from standard existence and uniqueness theory for wave equations that there exists a unique solution

$$\phi \in \bigcap_{\ell=0}^{k+2} C^\ell((T^*, T_0], H^{k+2-\ell}(\mathbb{T}^3))$$

to the wave equation (3.7) for some maximal time  $T^* \in (0, T_0)$  that satisfies the the initial conditions

$$(\bar{\phi}|_{t=T_0}, \partial_t \bar{\phi}|_{t=T_0}) = (T_0^{3K-1}\tilde{\phi}_0, T_0^{3K-2}(T_0\tilde{\phi}_1 + (3K - 1)\tilde{\phi}_0)).$$

Furthermore, we know, from the calculations carried out in Sect. 3, that

$$U = (u \ Du)^{\text{tr}},$$

where

$$u = (t\partial_t\phi \ \partial_j\phi \ t\partial_t\phi - \lambda(\phi - c_H) \ \partial_j\phi)^{\text{tr}}, \quad \lambda = 1 - 3K,$$

defines a solution of (3.55) on the time interval  $(T^*, T_0]$ . It is also clear from (3.63)–(3.77), and the symmetry of the matrices  $B^0(U)$  and  $B^k(U)$  that, after the simple time transformation  $t \mapsto -t$  and for  $U$  satisfying  $|U| \leq R$  with  $R$  chosen sufficiently small,

all of the assumptions from Sect. 4.1 will be satisfied for any choice of  $\kappa \in (0, \lambda)$ , and  $\lambda_a > 0$ ,  $a = 1, 2$ , and  $\beta_a > 0$ ,  $a = 1, 2, 3, 4$ , chosen as small as we like. Since

$$\|U(0)\|_{H^k} \lesssim \|\tilde{\phi}_0 - c_H\|_{H^{k+2}} + T_0 \|\tilde{\phi}_1\|_{H^{k+1}} \leq \delta,$$

it then follows from Theorem 4.5 that, for  $\delta > 0$  chosen sufficiently small, the maximal time of existence is  $T^* = 0$  and  $U$  will satisfy an energy estimate of the form

$$\|U(t)\|_{H^k}^2 + \int_t^{T_0} \frac{1}{\tau} \|\mathbb{P}U(\tau)\|_{H^k}^2 d\tau \leq C(\delta, \delta^{-1}) \|U(T_0)\|_{H^k}^2 \quad (3.78)$$

for all  $t \in (0, T_0]$ . In particular this shows that the solution  $\phi$  exists on the time interval  $(0, T_0]$ . Additionally, since

$$\|\mathbb{P}U\|_{H^k}^2 \approx \|t\partial_t\phi\|_{H^{k+1}}^2 + \|D\phi\|_{H^{k+1}}^2 \quad \text{and} \quad \|U\|_{H^k}^2 \approx \|\phi - c_H\|_{H^{k+2}}^2 + \|t\partial_t\phi\|_{H^{k+1}}^2,$$

the stated bound satisfied by  $\phi$  is a direct consequence of the energy estimate (3.78). This complete the proof of the theorem.  $\square$

#### 4. Fuchsian Initial Value Problems

In this section, we develop an existence theory for the initial value problem (IVP) for Fuchsian equations of the form

$$B^0(u)\partial_t u + \frac{1}{t}(C^i + B^i(u))\partial_i u = \frac{1}{t}\mathcal{B}(u)\mathbb{P}u + \frac{1}{t}F(u) \quad \text{in } [T_0, T_1) \times \mathbb{T}^n, \quad (4.1)$$

$$u = u_0 \quad \text{in } \{T_0\} \times \mathbb{T}^n, \quad (4.2)$$

where now  $T_0 < T_1 \leq 0$  and the coefficients satisfy the assumptions set out in the following section. Since these assumptions imply, in particular, that (4.1) is symmetric hyperbolic, this evolution equation enjoys the Cauchy stability property. As a consequence, the existence of solutions to (4.1)-(4.2) when  $T_1 < 0$  is guaranteed for sufficiently small initial data. Thus the main aim of this section will be to establish the existence and uniqueness of solutions for  $T_1 = 0$  under a suitable smallness assumption on the initial data.

The study of the IVP for Fuchsian equations was initiated in [16], and, there, the existence and uniqueness of solutions to (4.1)-(4.2) on intervals of the form  $[T_0, 0)$  was established under a small initial data assumption. Furthermore, decay estimates as  $t \nearrow 0$  were also obtained. This existence theory was then generalized in [4] to allow for certain coefficients to have a singular dependence on  $t$ , which significantly widened the applicability of this theory to establish global existence and decay results for systems of hyperbolic equations. For examples of global existence results for a range of different hyperbolic systems that have been established using the Fuchsian method see [4, 11–14, 16, 24].

The existence theory from [4] does not apply to the IVP (4.1)–(4.2) due to the appearance of the term  $\frac{1}{t}C^k\partial_k u$ , which does not satisfy the assumptions needed for the existence theory. In the present paper we therefore establish a complementary theorem to [4] which provides existence and uniqueness of solutions to (4.1)–(4.2) on time intervals  $[T_0, 0)$ . Interestingly, due to the term  $\frac{1}{t}C^k\partial_k u$ , these solutions do not decay uniformly as  $t \nearrow 0$ , which is a key difference compared to the Fuchsian equations considered in [4] whose solutions do decay uniformly. The precise statement of our existence result that is applicable to the IVP (4.1)–(4.2) is given below in Theorem 4.5.

*4.1. Coefficient assumptions.* We specify in the following the assumptions on the coefficients in (4.1)–(4.2).

- (i) The solution  $u(t, x)$  is a  $\mathbb{R}^N$ -valued map.  
(ii) The matrix  $\mathbb{P} \in \mathbb{M}_{N \times N}$  is a constant, symmetric projection operator, that is,

$$\mathbb{P}^2 = \mathbb{P}, \quad \mathbb{P}^{\text{tr}} = \mathbb{P}, \quad \partial_t \mathbb{P} = 0 \quad \text{and} \quad \partial_j \mathbb{P} = 0. \quad (4.3)$$

For use below, we define the *complementary projection operator* by

$$\mathbb{P}^\perp = \mathbb{I} - \mathbb{P}.$$

- (iii) The matrices  $C^k \in \mathbb{M}_{N \times N}$  are constant and symmetric, that is

$$(C^k)^{\text{tr}} = C^k, \quad \partial_t C^k = 0 \quad \text{and} \quad \partial_j C^k = 0. \quad (4.4)$$

- (iv) There exist constants  $\kappa, \gamma_1, \gamma_2 > 0$  such that the maps  $B^0, \mathcal{B} \in C^\infty(B_R(\mathbb{R}^N), \mathbb{M}_{N \times N})$  satisfy

$$\frac{1}{\gamma_1} \mathbb{I} \leq B^0(v) \leq \frac{1}{\kappa} \mathcal{B}(v) \leq \gamma_2 \mathbb{I} \quad (4.5)$$

for all  $v \in B_R(\mathbb{R}^N)$ , and the following additional properties:

$$(B^0(v))^{\text{tr}} = B^0(v), \quad (4.6)$$

$$[\mathbb{P}, \mathcal{B}(v)] = 0, \quad (4.7)$$

$$\partial_k (\mathbb{P}^\perp \mathcal{B}(v)) = 0, \quad (4.8)$$

$$|\mathbb{P}(B^0(v) - B^0(0))\mathbb{P}|_{\text{op}} + |\mathbb{P}\mathcal{B}(v) - \mathbb{P}\mathcal{B}(0)|_{\text{op}} \lesssim |\mathbb{P}v|, \quad (4.9)$$

and

$$|\mathbb{P}^\perp(B^0(v) - B^0(0))\mathbb{P}^\perp|_{\text{op}} + |\mathbb{P}^\perp B^0(v)\mathbb{P}|_{\text{op}} + |\mathbb{P}B^0(v)\mathbb{P}^\perp|_{\text{op}} \lesssim |\mathbb{P}v|^2 \quad (4.10)$$

for all  $v \in B_R(\mathbb{R}^N)$ .

It is then not difficult to see that (4.7)–(4.10) imply that

$$[\mathbb{P}^\perp, \mathcal{B}(v)] = [\mathbb{P}, \mathcal{B}(v)^{-1}] = [\mathbb{P}^\perp, \mathcal{B}(v)^{-1}] = 0, \quad (4.11)$$

$$\partial_k (\mathbb{P}^\perp \mathcal{B}(v)^{-1}) = 0, \quad (4.12)$$

$$|\mathbb{P}((B^0(v))^{-1} - (B^0(0))^{-1})\mathbb{P}|_{\text{op}} + |\mathbb{P}\mathcal{B}(v)^{-1} - \mathbb{P}\mathcal{B}(0)^{-1}|_{\text{op}} \lesssim |\mathbb{P}v| \quad (4.13)$$

and

$$|\mathbb{P}^\perp((B^0(v))^{-1} - (B^0(0))^{-1})\mathbb{P}^\perp|_{\text{op}} + |\mathbb{P}^\perp(B^0(v))^{-1}\mathbb{P}|_{\text{op}} + |\mathbb{P}(B^0(v))^{-1}\mathbb{P}^\perp|_{\text{op}} \lesssim |\mathbb{P}v|^2 \quad (4.14)$$

for all  $v \in B_R(\mathbb{R}^N)$ .



(v) There exist constants  $\lambda_a, a = 1, 2$ , such that the map  $F \in C^\infty(B_R(\mathbb{R}^N), \mathbb{R}^N)$  satisfies

$$|\mathbb{P}F(v)| \leq \lambda_1 |\mathbb{P}v| \tag{4.15}$$

and

$$|\mathbb{P}^\perp F(v)| \leq \frac{\lambda_2}{R} |\mathbb{P}v|^2 \tag{4.16}$$

for all  $v \in B_R(\mathbb{R}^N)$ .

(vi) The maps  $B^k \in C^\infty(B_R(\mathbb{R}^N), \mathbb{M}_{N \times N})$  satisfy

$$(B^k(v))^{\text{tr}} = B^k(v), \tag{4.17}$$

$$|\mathbb{P}^\perp B^k(v) \mathbb{P}|_{\text{op}} + |\mathbb{P} B^k(v) \mathbb{P}^\perp|_{\text{op}} \lesssim |\mathbb{P}v|, \tag{4.18}$$

$$|\mathbb{P}^\perp B^k(v) \mathbb{P}^\perp|_{\text{op}} \lesssim |\mathbb{P}v|^2 \tag{4.19}$$

and

$$|\mathbb{P} B^k(v) \mathbb{P}|_{\text{op}} \lesssim |v| \tag{4.20}$$

for all  $v \in B_R(\mathbb{R}^N)$ .

(vii) There exist constants  $\beta_a \geq 0, a = 1, 2, 3, 4$ , such that the map

$$\text{div} B : B_R(\mathbb{R}^N \times \mathbb{R}^{n \times N}) \longrightarrow \mathbb{M}_{N \times N}$$

defined by

$$\begin{aligned} \text{div} B(t, v, w) &= D_v B^0(v) \cdot (B^0(v))^{-1} \left( -\frac{1}{t} (C^k + B^k(v)) w_k + \frac{1}{t} \mathcal{B}(v) \mathbb{P}v + \frac{1}{t} F(v) \right) \\ &\quad + \frac{1}{t} D_v B^k(v) w_k \end{aligned} \tag{4.21}$$

satisfies

$$|\mathbb{P} \text{div} B(t, v, w) \mathbb{P}|_{\text{op}} \leq |t|^{-1} \beta_1, \tag{4.22}$$

$$|\mathbb{P} \text{div} B(t, v, w) \mathbb{P}^\perp|_{\text{op}} \leq \frac{|t|^{-1} \beta_2}{R} |\mathbb{P}v|, \tag{4.23}$$

$$|\mathbb{P}^\perp \text{div} B(t, v, w) \mathbb{P}|_{\text{op}} \leq \frac{|t|^{-1} \beta_3}{R} |\mathbb{P}v| \tag{4.24}$$

and

$$|\mathbb{P}^\perp \text{div} B(t, v, w) \mathbb{P}^\perp|_{\text{op}} \leq \frac{|t|^{-1} \beta_4}{R^2} |\mathbb{P}v|^2. \tag{4.25}$$

*Remark 4.1.* (i) For a symmetric matrix  $A \in \mathbb{M}_{N \times N}$ , we have the equality  $|\mathbb{P}^\perp A \mathbb{P}| = |\mathbb{P} A \mathbb{P}^\perp|$ . From this property, it is clear that some of the assumptions above are redundant and we can always take  $\beta_2 = \beta_3$ .

(ii) For solutions  $u(t, x)$  of (4.1), we have

$$\begin{aligned} \operatorname{div} B(t, u(t, x), Du(t, x)) &= \partial_t (B^0(u(t, x))) + \partial_k \left( \frac{1}{t} (C^k + B^k(u(t, x))) \right) \\ &= \partial_t (B^0(u(t, x))) + \frac{1}{t} \partial_k (B^k(u(t, x))). \end{aligned}$$

Furthermore, for the proof of Theorem 4.5, it will be clear that we only require that the estimates (4.22)–(4.25) hold for solutions where  $w_i = \partial_i u$ . This is important when we want to consider the spatially differentiated version of (4.1) together with (4.1), which will yield an equation of the same form as (4.1) for the variables  $(u, w = Du)$ . By considering this extended system, we will be able to relax the assumptions (4.22)–(4.25) to

$$\begin{aligned} |\mathbb{P} \operatorname{div} B(t, v, w)|_{\text{op}} &\leq |t|^{-1} \beta_1, \\ |\mathbb{P} \operatorname{div} B(t, v, w) \mathbb{P}^\perp|_{\text{op}} &\leq \frac{|t|^{-1} \beta_2}{R} (|\mathbb{P}v| + |\mathbb{P}w|), \\ |\mathbb{P}^\perp \operatorname{div} B(t, v, w) \mathbb{P}|_{\text{op}} &\leq \frac{|t|^{-1} \beta_3}{R} (|\mathbb{P}v| + |\mathbb{P}w|) \end{aligned}$$

and

$$|\mathbb{P}^\perp \operatorname{div} B(t, v, w) \mathbb{P}^\perp|_{\text{op}} \leq \frac{|t|^{-1} \beta_4}{R^2} (|\mathbb{P}v|^2 + |\mathbb{P}w|^2).$$

4.2. *Preliminary estimates.* Before proceeding with the statement and proof of Theorem 4.5, we first establish some estimates that will be used in the proof. For a given  $k \in \mathbb{Z}_{>n/2+1}$ , we let  $C_{\text{Sob}} > 0$  be the constant from Sobolev’s inequality, that is,

$$\max \{ \|Du(t)\|_{L^\infty}, \|u(t)\|_{L^\infty} \} \leq C_{\text{Sob}} \|u(t)\|_{H^k}. \tag{4.26}$$

**Proposition 4.2.** *Suppose  $k \in \mathbb{Z}_{>n/2+1}$ ,  $u \in B_{C_{\text{Sob}}^{-1}R} (H^k(\mathbb{T}^n, \mathbb{R}^N))$ ,  $v \in L^2(\mathbb{T}^n, \mathbb{R}^N)$ ,  $F = F(t, u(x))$  and  $\operatorname{div} B = \operatorname{div} B(t, u(x), Du(x))$ . Then*

$$|\langle u | F \rangle| \leq (\lambda_1 + \lambda_2) \|\mathbb{P}u\|_{L^2}^2$$

and

$$|\langle v | \operatorname{div} Bv \rangle| \leq |t|^{-1} \left( \beta_1 \|\mathbb{P}v\|_{L^2}^2 + \frac{\beta_2 + \beta_3}{R} \| |v| |\mathbb{P}v| |\mathbb{P}u| \|_{L^1} + \frac{\beta_4}{R^2} \| |v|^2 |\mathbb{P}u|^2 \|_{L^1} \right).$$

*Proof.* The estimates

$$|\langle u | \mathbb{P}^\perp F \rangle| \leq \lambda_2 \|\mathbb{P}u\|_{L^2}^2 \tag{4.27}$$

and

$$|\langle v | \operatorname{div} Bv \rangle| \leq |t|^{-1} \left( \beta_1 \|\mathbb{P}v\|_{L^2}^2 + \frac{\beta_2 + \beta_3}{R} \| |v| |\mathbb{P}v| |\mathbb{P}u| \|_{L^1} + \frac{\beta_4}{R^2} \| |v|^2 |\mathbb{P}u|^2 \|_{L^1} \right)$$

are a direct consequence of Proposition 3.4 of [4] and the assumptions (4.15)–(4.16) and (4.22)–(4.25). We see also from (4.3) and (4.15) that

$$|\langle u | \mathbb{P}F \rangle| = |\langle \mathbb{P}u | \mathbb{P}F \rangle| \leq \lambda_1 \|\mathbb{P}u\|_{L^2}^2.$$

So from this and (4.27), we find, with the help of the triangle inequality, that

$$\begin{aligned} |\langle u|F\rangle| &= |\langle u|\mathbb{P}F + \mathbb{P}^\perp F\rangle| = |\langle u|\mathbb{P}F\rangle + \langle u|\mathbb{P}^\perp F\rangle| \leq |\langle u|\mathbb{P}F\rangle| + |\langle u|\mathbb{P}^\perp F\rangle| \\ &\leq (\lambda_1 + \lambda_2)\|\mathbb{P}u\|_{L^2}^2, \end{aligned}$$

which completes the proof.  $\square$

**Proposition 4.3.** *Suppose  $k \in \mathbb{Z}_{>n/2+2}$ ,  $1 \leq |\alpha| \leq k$ ,  $v \in L^2(\mathbb{T}^n, \mathbb{R}^N)$ ,  $u \in B_{C_{Sob}^{-1}R}^{-1}(H^k(\mathbb{T}^n, \mathbb{R}^N))$ ,  $\mathcal{B} = \mathcal{B}(u(x))$ ,  $B^0 = B^0(u(x))$  and  $B^k = B^k(u(x))$ . Then*

$$\begin{aligned} |\langle v|\mathcal{B}D^\alpha(\mathcal{B}^{-1}F)\rangle| + |\langle v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}B^0](B^0)^{-1}F\rangle| &\leq \Xi, \\ |\langle v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}B^0](B^0)^{-1}\mathcal{B}\mathbb{P}u\rangle| &\leq \Xi, \\ |\langle v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}B^k]\partial_k u\rangle| + |\langle v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}B^0](B^0)^{-1}B^k\partial_k u\rangle| &\leq \Xi \end{aligned}$$

and

$$|\langle v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}C^k]\partial_k u\rangle| + |\langle v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}B^0](B^0)^{-1}C^k\partial_k u\rangle| \leq \Xi$$

where

$$\Xi = C(\|u\|_{H^k})\left(\|\mathbb{P}v\|_{L^2}\|\mathbb{P}u\|_{H^{k-1}} + \|v\|_{L^2}\|\mathbb{P}u\|_{H^k}^2 + \|\mathbb{P}v\|_{L^2}\|u\|_{H^k}\|\mathbb{P}u\|_{H^k}\right).$$

*Proof.* Since the first three estimates follow directly from Proposition 3.6 of [4] and the coefficient assumptions from Sect. 4.1, in particular, (ii),(iv), (v) and (vi), we only need to establish the last estimate. To do so, we assume that  $1 \leq |\alpha| \leq k$ , and we observe from (4.3), (4.4), (4.7) and (4.11)–(4.12) that

$$\begin{aligned} \langle v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}C^k]\partial_k u\rangle &= \langle \mathbb{P}v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}C^k]\partial_k u\rangle + \langle \mathbb{P}^\perp v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}C^k]\partial_k u\rangle \\ &= \langle \mathbb{P}v|\mathcal{B}[D^\alpha, \mathbb{P}\mathcal{B}^{-1}C^k]\partial_k u\rangle + \langle \mathbb{P}^\perp v|\mathcal{B}[D^\alpha, \mathbb{P}^\perp\mathcal{B}^{-1}C^k]\partial_k u\rangle \\ &= \langle \mathbb{P}v|\mathcal{B}[D^\alpha, \mathbb{P}\mathcal{B}^{-1}C^k]\partial_k u\rangle. \end{aligned}$$

Applying the Cauchy-Schwartz inequality gives

$$|\langle v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}C^k]\partial_k u\rangle| \leq \|\mathbb{P}v\|_{L^2}\|\mathcal{B}[D^\alpha, \mathbb{P}\mathcal{B}^{-1}C^k]\partial_k u\|_{L^2}.$$

With the help of the Hölder's inequality, see Theorem A.1, the commutator estimates from Theorem A.3, and Sobolev's inequality, see Theorem A.2, we then get

$$|\langle v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}C^k]\partial_k u\rangle| \leq \|\mathbb{P}v\|_{L^2}\|\mathcal{B}\|_{H^k}\|D\mathbb{P}\mathcal{B}^{-1}\|_{H^{k-1}}\|u\|_{H^k}. \quad (4.28)$$

But from (4.13), Sobolev's inequality and the Moser estimates from Theorem A.4, we know that  $\|D\mathbb{P}\mathcal{B}^{-1}\|_{H^{k-1}} \leq C(\|u\|_{H^k})\|\mathbb{P}u\|_{H^k}$ , and hence, we conclude from (4.28) that

$$|\langle v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}C^k]\partial_k u\rangle| \leq C(\|u\|_{H^k})\|\mathbb{P}v\|_{L^2}\|\mathbb{P}u\|_{H^k}\|u\|_{H^k}. \quad (4.29)$$

Next, by (4.3), (4.7) and (4.11), we have

$$\begin{aligned} \langle v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}B^0](B^0)^{-1}C^k\partial_k u\rangle &= \langle \mathbb{P}v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}B^0](B^0)^{-1}C^k\partial_k u\rangle + \langle \mathbb{P}^\perp v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}B^0](B^0)^{-1}C^k\partial_k u\rangle \\ &= \langle \mathbb{P}v|\mathcal{B}[D^\alpha, \mathcal{B}^{-1}B^0](B^0)^{-1}C^k\partial_k u\rangle + \langle \mathbb{P}^\perp v|\mathcal{B}[D^\alpha, \mathbb{P}^\perp\mathcal{B}^{-1}B^0](B^0)^{-1}C^k\partial_k u\rangle. \end{aligned}$$

Estimating this expression as above yields

$$\begin{aligned}
 & | \langle v | \mathcal{B}[D^\alpha, \mathcal{B}^{-1} B^0](B^0)^{-1} C^k \partial_k u \rangle | \\
 & \leq C(\|u\|_{H^k}) \left( \|\mathbb{P}v\|_{L^2} \|D(\mathcal{B}^{-1} B^0)\|_{H^{k-1}} \|u\|_{H^k} + \|v\|_{L^2} \|D(\mathbb{P}^\perp \mathcal{B}^{-1} B^0)\|_{H^{k-1}} \right).
 \end{aligned}
 \tag{4.30}$$

But

$$| \mathcal{B}^{-1} B^0 - (B^{-1} B^0) |_{u=0} \lesssim |\mathbb{P}u| \quad \text{and} \quad | \mathbb{P}^\perp \mathcal{B}^{-1} B^0 - (\mathbb{P}^\perp B^{-1} B^0) |_{u=0} \lesssim |\mathbb{P}u|^2$$

by (4.7)–(4.10) and (4.13), and consequently, by Sobolev’s inequality and the Moser estimates from Theorem A.4, we see that

$$\|D(\mathcal{B}^{-1} B^0)\|_{H^{k-1}} \leq C(\|u\|_{H^k}) \|\mathbb{P}u\|_{H^k} \quad \text{and} \quad \|D(\mathbb{P}^\perp \mathcal{B}^{-1} B^0)\|_{H^{k-1}} \leq C(\|u\|_{H^k}) \|\mathbb{P}u\|_{H^k}^2.$$

Substituting these into (4.30) gives

$$| \langle v | \mathcal{B}[D^\alpha, \mathcal{B}^{-1} B^0](B^0)^{-1} C^k \partial_k u \rangle | \leq C(\|u\|_{H^k}) \left( \|\mathbb{P}v\|_{L^2} \|\mathbb{P}u\|_{H^k} \|u\|_{H^k} + \|v\|_{L^2} \|\mathbb{P}u\|_{H^k}^2 \right).
 \tag{4.31}$$

Combining the inequalities (4.29) and (4.31), we see that the final estimate in the statement of the proposition holds, which completes the proof.  $\square$

*Remark 4.4.* The structure of the term  $\Xi$  in the above proposition plays an important role in the following proof of global existence. Schematically, the second bracketed term of  $\Xi$  involves a quadratic term  $\|\mathbb{P}u\|_{H^k}^2$  which will be bounded by the energy times a small coefficient coming from  $\|v\|_{L^2}$ . A similar argument holds for the third term of  $\Xi$ . By contrast, the first term of  $\Xi$  will require a more subtle analysis using Ehrling’s lemma. This lemma will allow us to obtain a small coefficient at the expense of gaining derivatives and additional terms. Note that commutators involving the matrices  $C^k$ , which are the key new terms compared to [4], also lead to these more problematic terms in  $\Xi$ .

**4.3. Global existence.** The following theorem guarantees, under a suitable small initial data hypothesis, the existence of solutions to the the IVP (4.1)–(4.2) on the time interval  $[T_0, 0)$ . The proof is similar to the first part of the proof of Theorem 3.8 from [4] where existence is established. However, unlike Theorem 3.8 from [4], there is no corresponding uniform decay estimate as  $t \nearrow 0$  for solutions. This is due to the singular term  $\frac{1}{t} C^k \partial_k u$  that prevents solutions to (4.1)–(4.2) from satisfying an estimate analogous to the one from Proposition 3.2 of [4] unless additional assumptions on the coefficients are imposed.

**Theorem 4.5.** *Suppose  $k \in \mathbb{Z}_{>n/2+2}$ ,  $u_0 \in H^k(\mathbb{T}^n)$ , assumptions (i)–(vii) from Sect. 4.1 are fulfilled, and the constants  $\kappa$ ,  $\gamma_1$ ,  $\lambda_1$ ,  $\lambda_2$ ,  $\beta_1$ ,  $\beta_2$ ,  $\beta_3$ , and  $\beta_4$  from Sect. 4.1 satisfy*

$$\kappa > \frac{1}{2} \gamma_1 \left( \sum_{a=1}^4 \beta_{2a} + 2(\lambda_1 + \lambda_2) \right).$$

*Then there exists a  $\delta > 0$  such that if  $\|u_0\|_{H^k} < \delta$ , then there exists a unique solution*

$$u \in C^0([T_0, 0), H^k(\mathbb{T}^n, \mathbb{R}^N)) \cap C^1([T_0, 0), H^{k-1}(\mathbb{T}^n, \mathbb{R}^N))$$

of the IVP (4.1)–(4.2) that satisfies the energy estimate

$$\|u(t)\|_{H^k}^2 - \int_{T_0}^t \frac{1}{\tau} \|\mathbb{P}u(\tau)\|_{H^k}^2 d\tau \leq C(\delta, \delta^{-1}) \|u_0\|_{H^k}^2.$$

*Proof.* Fixing  $k \in \mathbb{Z}_{>n/2+2}$ , we obtain from standard local-in-time existence and uniqueness results for symmetric hyperbolic equations, e.g. [22, Ch.16 §1], the existence of a unique solution  $u \in C^0([T_0, T^*], H^k) \cap C^1([T_0, T^*], H^{k-1})$  to the IVP (4.1)–(4.2) for some maximal time  $T^* \in (T_0, 0]$ . Then taking  $R > 0$  to be as in Sect. 4.1, we choose initial data such that

$$\|u(T_0)\|_{H^k} < \delta$$

where

$$\delta \in (0, \frac{1}{4}\mathcal{R}) \quad \text{and} \quad \mathcal{R} = \min\left\{\frac{3R}{4C_{\text{Sob}}}, \frac{3R}{4}\right\}.$$

Then either  $\|u(t)\|_{H^k} < \mathcal{R}$  for all  $t \in [T_0, T^*]$  or there exists a first time  $T_* \in [T_0, T^*]$  such that

$$\|u(T_*)\|_{H^k} = \mathcal{R} \leq \frac{3}{4}R.$$

If the first case holds, we set  $T_* = T^*$ , and in either case, we observe by (4.26) that

$$\max\{\|Du(t)\|_{L^\infty}, \|u(t)\|_{L^\infty}, \|u(t)\|_{H^k}\} \leq \frac{3}{4}R, \quad T_0 \leq t < T_*. \quad (4.32)$$

Next, applying the differential operator  $\mathcal{B}D^\alpha \mathcal{B}^{-1}$ , where  $|\alpha| \leq k$ , to (4.1) on the left yields

$$\begin{aligned} B^0 \partial_t D^\alpha u + \frac{1}{t}(C^i + B^i) \partial_i D^\alpha u &= \frac{1}{t} \mathcal{B}D^\alpha \mathbb{P}u - \mathcal{B}[D^\alpha, \mathcal{B}^{-1}B^0] \partial_t u \\ &\quad - \frac{1}{t} \mathcal{B}[D^\alpha, \mathcal{B}^{-1}(C^i + B^i)] \partial_i u + \frac{1}{t} \mathcal{B}D^\alpha (\mathcal{B}^{-1}F). \end{aligned}$$

Using (4.1) to replace  $\partial_t u$ , we see that the above equation is equivalent to

$$\begin{aligned} &B^0 \partial_t D^\alpha u + \frac{1}{t}(C^i + B^i) \partial_i D^\alpha u \\ &= \frac{1}{t} \left[ \mathcal{B} \mathbb{P}D^\alpha u - \mathcal{B}[D^\alpha, \mathcal{B}^{-1}B^0](B^0)^{-1} \mathcal{B} \mathbb{P}u \right] + \frac{1}{t} \mathcal{B}[D^\alpha, \mathcal{B}^{-1}B^0](B^0)^{-1} (C^i + B^i) \partial_i u \\ &\quad - \frac{1}{t} \mathcal{B}[D^\alpha, \mathcal{B}^{-1}(C^i + B^i)] \partial_i u - \frac{1}{t} \mathcal{B}[D^\alpha, \mathcal{B}^{-1}B^0](B^0)^{-1} F + \frac{1}{t} \mathcal{B}D^\alpha (\mathcal{B}^{-1}F). \end{aligned} \quad (4.33)$$

In the following we will use (4.33) to derive energy estimates that are well-behaved in the limit  $t \nearrow 0$ . These energy estimates will be expressed in terms of the energy norms defined by

$$|||u|||_s^2 = \sum_{\ell=0}^s \langle D^\ell u | B^0 D^\ell u \rangle.$$

By (4.5), we note that the energy  $||| \cdot |||_s^2$  and Sobolev  $\| \cdot \|_{H^k}$  norms are equivalent since they satisfy

$$\frac{1}{\sqrt{\gamma_1}} \| \cdot \|_{H^s} \leq ||| \cdot |||_s \leq \sqrt{\gamma_2} \| \cdot \|_{H^s}.$$

We will employ this equivalence below without comment.

$L^2$ -energy estimate: To obtain a  $L^2$ -energy estimate for  $u$ , we set  $\alpha = 0$  in (4.33) and then employ the usual energy identity that holds for symmetric hyperbolic equations to get

$$\frac{1}{2} \partial_t \langle u | B^0 u \rangle = \frac{1}{t} \langle u | \mathcal{B} \mathbb{P} u \rangle + \frac{1}{2} \langle u | \operatorname{div} B u \rangle + \frac{1}{t} \langle u | F \rangle, \tag{4.34}$$

where

$$\operatorname{div} B = \partial_t (B^0(u)) + \frac{1}{t} \partial_k (B^k(u)).$$

Since  $t < 0$ , we have that

$$\frac{2}{t} \langle v | \mathcal{B} \mathbb{P} v \rangle = \frac{2}{t} \langle \mathbb{P} v | \mathcal{B} \mathbb{P} v \rangle \leq \frac{2\kappa}{t} \| \mathbb{P} v \|_0^2 \tag{4.35}$$

for any  $v \in L^2(\mathbb{R}^N)$  by (4.3), (4.5) and (4.7). From this inequality and (4.32), Proposition 4.2, and the energy identity (4.34), we deduce, with the help of Hölder’s inequality, the  $L^2$ -energy estimate

$$\partial_t \| \| u \| \|_0^2 \leq \frac{\rho_0}{t} \| \| \mathbb{P} u \| \|_0^2, \quad T_0 \leq t < T_*, \tag{4.36}$$

where

$$\rho_0 = 2\kappa - \gamma_1 \left[ \sum_{a=1}^4 \beta_a + 2(\lambda_1 + \lambda_2) \right] > 0. \tag{4.37}$$

$H^k$ -energy estimate: Applying the  $L^2$ -energy identity, i.e. (4.34), to (4.33) gives

$$\frac{1}{2} \partial_t \langle D^\alpha u | B^0 D^\alpha u \rangle = \frac{1}{t} \langle D^\alpha u | \mathcal{B} \mathbb{P} D^\alpha u \rangle + \frac{1}{2} \langle D^\alpha u | \operatorname{div} B D^\alpha u \rangle + \langle D^\alpha u | G^\alpha \rangle, \quad 0 \leq |\alpha| \leq k, \tag{4.38}$$

where

$$G^\alpha = \frac{1}{t} \left( -\mathcal{B}[D^\alpha, \mathcal{B}^{-1} B^0](B^0)^{-1} \mathcal{B} \mathbb{P} u + \mathcal{B}[D^\alpha, \mathcal{B}^{-1} B^0](B^0)^{-1} (C^i + B^i) \partial_i u - \mathcal{B}[D^\alpha, \mathcal{B}^{-1} (C^i + B^i)] \partial_i u - \mathcal{B}[D^\alpha, \mathcal{B}^{-1} B^0](B^0)^{-1} F + \mathcal{B} D^\alpha (\mathcal{B}^{-1} F) \right).$$

From (4.38), we obtain, with the help of (4.32), (4.35), Proposition 4.2 and Hölder’s inequality, the estimate

$$\begin{aligned} \partial_t \| \| D^\alpha u \| \|_0^2 &\leq \frac{2\kappa - \gamma_1 \beta_1}{t} \| \| D^\alpha \mathbb{P} u \| \|_0^2 \\ &\quad - \frac{\gamma_1 (\beta_2 + \beta_3 + \beta_4)}{t} \| \| \mathbb{P} u \| \|_k \| \| \mathbb{P} u \| \|_{k-1} + 2 \langle D^\alpha u | G^\alpha \rangle, \quad T_0 \leq t < T_*. \end{aligned}$$

Using Proposition 4.3, we bound the last term in the above inequality by

$$\langle D^\alpha u | G^\alpha \rangle \leq -\frac{1}{t} C(\| u \|_{H^k}) (\| \mathbb{P} u \|_{H^k} \| \mathbb{P} u \|_{H^{k-1}} + \| u \|_{H^k} \| \mathbb{P} u \|_{H^k}^2),$$

and so, we have<sup>2</sup>

$$\begin{aligned} \partial_t |||D^\alpha u|||_0^2 &\leq \frac{2\kappa - \gamma_1 \beta_1}{t} |||D^\alpha \mathbb{P}u|||_0^2 \\ &\quad - \frac{1}{t} C(|||u|||_k) (|||\mathbb{P}u|||_k |||\mathbb{P}u|||_{k-1} + |||u|||_k |||\mathbb{P}u|||_k^2), \quad T_0 \leq t < T_*. \end{aligned}$$

Summing this over  $\alpha$  for  $0 \leq |\alpha| \leq k$ , we obtain, after an application of Young’s inequality and Ehrling’s lemma (Lemma A.5), the  $H^k$  energy estimate

$$\partial_t |||u|||_k^2 \leq \frac{2\kappa - \gamma_1 \beta_1 - C(|||u|||_k)(\epsilon + |||u|||_k)}{t} |||\mathbb{P}u|||_k^2 - \frac{1}{t} c(|||u|||_k, \epsilon^{-1}) |||\mathbb{P}u|||_0^2, \tag{4.39}$$

which holds for any  $\epsilon > 0$  and  $t \in [T_0, T_*)$ .

Global existence on  $[T_0, 0) \times \mathbb{T}^n$  : Initially, we have  $|||u(T_0)|||_k \leq \sqrt{\gamma_2} \|u(T_0)\|_{H^k} < \delta\sqrt{\gamma_2}$ , and so, we can, for  $\delta$  satisfying

$$0 < \delta \leq \min \left\{ \frac{\mathcal{R}}{2\sqrt{\gamma_1 \gamma_2}}, \frac{\mathcal{R}}{4} \right\}, \tag{4.40}$$

define  $T_\delta \in (T_0, T_*)$  to be the first time such that  $|||u(T_\delta)|||_k = 2\delta\sqrt{\gamma_2}$ , or if such a time does not exist, set  $T_\delta = T^*$ . In either case, the inequality

$$|||u(t)|||_k \leq 2\delta\sqrt{\gamma_2}, \quad T_0 \leq t < T_\delta,$$

holds, which in turn, implies that

$$|||u(t)|||_{H^k} \leq \sqrt{\gamma_1} |||u(t)|||_k \leq 2\delta\sqrt{\gamma_1 \gamma_2} \leq \mathcal{R}, \quad T_0 \leq t < T_\delta \leq T_* \leq T^*.$$

To proceed, we fix  $\epsilon$  by setting  $\epsilon = \delta\sqrt{\gamma_2}$ . Substituting this into (4.39) gives

$$\partial_t |||u|||_k^2 \leq \frac{\rho_k}{t} |||\mathbb{P}u|||_k^2 - \frac{1}{t} c(\delta, \delta^{-1}) |||\mathbb{P}u|||_0^2, \quad T_0 \leq t < T_\delta, \tag{4.41}$$

where

$$\rho_k = 2\kappa - \gamma_1 \beta_1 - C(\delta)\delta.$$

But  $\lim_{\delta \searrow 0} C(\delta)\delta = 0$  and  $2\kappa - \gamma_1 \beta_1 > 0$  by assumption, and consequently, we have

$$\rho_k > 0 \tag{4.42}$$

provided  $\delta > 0$  is chosen small enough. Furthermore, since  $\rho_0 > 0$  by (4.37), we can add  $\rho_0^{-1} c(\delta, \delta^{-1})$  times (4.36) to (4.41) to obtain the energy estimate

$$\partial_t (|||u|||_k^2 + \rho_0^{-1} c(\delta, \delta^{-1}) |||u|||_0^2) \leq \frac{\rho_k}{t} |||\mathbb{P}u|||_k^2, \quad T_0 \leq t < T_\delta.$$

Setting

$$E_k(t) = |||u(t)|||_k^2 + \rho_0^{-1} c(\delta, \delta^{-1}) |||u(t)|||_0^2 - \int_{T_0}^t \frac{\rho_k}{\tau} |||\mathbb{P}u(\tau)|||_k^2 d\tau, \tag{4.43}$$

---

<sup>2</sup> The constant  $C(|||u|||_k)$  implicitly depends on the various constants, e.g.  $\gamma_1, \gamma_2, \beta_1$ , etc., that were introduced in the assumption in Sect. 4.1.

we can write this energy estimate as

$$\partial_t E_k \leq 0, \quad T_0 \leq t < T_\delta.$$

Integrating in time gives

$$E_k(t) \leq E_k(0), \quad T_0 \leq t < T_\delta. \tag{4.44}$$

With  $\delta$  fixed so that (4.40) and (4.42) hold, we choose  $\delta_0 \in (0, \delta)$  and assume that the initial data is chosen so that  $\|u(T_0)\|_{H^k} \leq \delta_0$ . Then (4.44) implies that

$$\|u(t)\|_k \leq \sqrt{1 + \rho_0^{-1}c(\delta, \delta^{-1})}\delta_0, \quad T_0 \leq t < T_\delta. \tag{4.45}$$

By further shrinking  $\delta_0 > 0$ , if necessary, so that  $0 < \sqrt{1 + \rho_0^{-1}c(\delta, \delta^{-1})}\delta_0 < \delta\sqrt{\gamma_2}$  also holds, we deduce from (4.45) that  $\|u(t)\|_k < \delta\sqrt{\gamma_2}$  for  $T_0 \leq t < T_\delta$ . From the definition of  $T_\delta$  and the maximality of  $T^*$ , we conclude that  $T_\delta = T_* = T^* = 0$ . Thus, we have established the global existence of solutions on  $[T_0, 0) \times \mathbb{T}^n$ . Moreover, from (4.43), (4.44) and the equivalence of the norms  $\|\cdot\|_{H^k}$  and  $\|\cdot\|_k$ , we see immediately that the energy estimate

$$\|u(t)\|_{H^k}^2 - \int_{T_0}^t \frac{1}{\tau} \|\mathbb{P}u(\tau)\|_{H^k}^2 d\tau \leq C(\delta, \delta^{-1})\|u(T_0)\|_{H^k}^2$$

holds for  $T_0 \leq t < 0$ . □

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### A. Calculus Inequalities

In this appendix, we collect, for the convenience of the reader, a number of calculus inequalities that we employ. The proof of the following inequalities are well known and may be found, for example, in the books [1], [8] and [22].

**Theorem A.1.** (Hölder’s inequality) *If  $0 < p, q, r \leq \infty$  satisfy  $1/p + 1/q = 1/r$ , then*

$$\|uv\|_{L^r} \leq \|u\|_{L^p} \|v\|_{L^q}$$

for all  $u \in L^p(\mathbb{T}^n)$  and  $v \in L^q(\mathbb{T}^n)$ .



**Theorem A.2.** (Sobolev’s inequality) *Suppose  $1 \leq p < \infty$  and  $s \in \mathbb{Z}_{>n/p}$ . Then*

$$\|u\|_{L^\infty} \lesssim \|u\|_{W^{s,p}}$$

for all  $u \in W^{s,p}(\mathbb{T}^n)$ .

**Theorem A.3.** (Product and commutator estimates)

(i) *Suppose  $1 \leq p_1, p_2, q_1, q_2 \leq \infty, s \in \mathbb{Z}_{\geq 1}, |\alpha| = s$  and*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r}.$$

Then

$$\|D^\alpha(uv)\|_{L^r} \lesssim \|u\|_{W^{s,p_1}} \|v\|_{L^{q_1}} + \|u\|_{L^{p_2}} \|v\|_{W^{s,q_2}}$$

and

$$\|[D^\alpha, u]v\|_{L^r} \lesssim \|Du\|_{L^{p_1}} \|v\|_{W^{s-1,q_1}} + \|Du\|_{W^{s-1,p_2}} \|v\|_{L^{q_2}}$$

for all  $u, v \in C^\infty(\mathbb{T}^n)$ .

(ii) *Suppose  $s_1, s_2, s_3 \in \mathbb{Z}_{\geq 0}, s_1, s_2 \geq s_3, 1 \leq p \leq \infty$ , and  $s_1 + s_2 - s_3 > n/p$ . Then*

$$\|uv\|_{W^{s_3,p}} \lesssim \|u\|_{W^{s_1,p}} \|v\|_{W^{s_2,p}}$$

for all  $u \in W^{s_1,p}(\mathbb{T}^n)$  and  $v \in W^{s_2,p}(\mathbb{T}^n)$ .

**Theorem A.4.** (Moser’s estimates) *Suppose  $1 \leq p \leq \infty, s \in \mathbb{Z}_{\geq 1}, 0 \leq k \leq s, |\alpha| = k$  and  $f \in C^s(U)$ , where  $U$  is open and bounded in  $\mathbb{R}$  and contains 0, and  $f(0) = 0$ . Then*

$$\|D^\alpha f(u)\|_{L^p} \leq C(\|f\|_{C^s(\bar{U})})(1 + \|u\|_{L^\infty}^{s-1})\|u\|_{W^{s,p}}$$

for all  $u \in C^0(\mathbb{T}^n) \cap L^\infty(\mathbb{T}^n) \cap W^{s,p}(\mathbb{T}^n)$  with  $u(x) \in U$  for all  $x \in \mathbb{T}^n$ .

**Lemma A.5.** (Ehrling’s lemma) *Suppose  $1 \leq p < \infty, s_0, s, s_1 \in \mathbb{Z}_{\geq 0}$ , and  $s_0 < s < s_1$ . Then for any  $\epsilon > 0$  there exists a constant  $C = C(\epsilon^{-1})$  such that*

$$\|u\|_{W^{s,p}} \leq \epsilon \|u\|_{W^{s_1,p}} + C \|u\|_{W^{s_0,p}}$$

for all  $u \in W^{s_1,p}(\mathbb{T}^n)$ .

## References

1. Adams, R.A., Fournier, J.: Sobolev Spaces, 2nd edn. Academic Press, Cambridge (2003)
2. Andersson, L., Fajman, D.: Nonlinear stability of the Milne model with matter. Commun. Math. Phys. **378**, 261–298 (2020)
3. Andersson, L., Moncrief, V.: Einstein spaces as attractors for the Einstein flow. J. Differ. Geom. **89**(1), 1–47 (2011)
4. Beyer, F., Oliynyk, T.A., Olvera-SantaMaría, J.A.: The Fuchsian approach to global existence for hyperbolic equations. Commun. Part. Diff. Eq. (accepted), preprint [[arXiv:1907.04071](https://arxiv.org/abs/1907.04071)]
5. Bildhauer, S., Buchert, T., Kasai, M.: Solutions in Newtonian cosmology: the Pancake theory with cosmological constant. Astron. Astrophys. **263**, 23–29 (1992)
6. Christodoulou, D.: The Formation of Shocks in 3-Dimensional Fluids. EMS, Switzerland (2007)
7. Fajman, D., Wyatt, Z.: Attractors of the Einstein-Klein Gordon system. Comm. Partial Differential Equations (2020)
8. Friedman, A.: Partial Differential Equations. Krieger Publishing Company, Malabar, USA (1976)

9. Friedrich, H.: Sharp asymptotics for Einstein- $\lambda$ -dust flows. *Commun. Math. Phys.* **350**, 803–844 (2017)
10. Hadžić, M., Speck, J.: The global future stability of the FLRW solutions to the Dust-Einstein system with a positive cosmological constant. *J. Hyper. Differ. Equ.* **12**, 87–188 (2015)
11. LeFloch, P.G., Wei, C.: The global nonlinear stability of self-gravitating irrotational Chaplygin fluids in a FRW geometry. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*. (2020)
12. Liu, C., Oliynyk, T.A.: Cosmological Newtonian limits on large spacetime scales. *Commun. Math. Phys.* **364**, 1195–1304 (2018)
13. Liu, C., Oliynyk, T.A.: Newtonian limits of isolated cosmological systems on long time scales. *Annales Henri Poincaré* **19**, 2157–2243 (2018)
14. Liu, C., Wei, C.: Future stability of the FLRW spacetime for a large class of perfect fluids. preprint [[arXiv:1810.11788](https://arxiv.org/abs/1810.11788)] (2019)
15. Lübbe, C., Kroon, J.A.V.: A conformal approach for the analysis of the non-linear stability of radiation cosmologies. *Ann Phys* **328**, 1–25 (2013)
16. Oliynyk, T.A.: Future stability of the FLRW fluid solutions in the presence of a positive cosmological constant. *Commun. Math. Phys.* **346** (2016), 293–312; see the preprint [[arXiv:1505.00857](https://arxiv.org/abs/1505.00857)] for a corrected version
17. Oliynyk, T.A.: Future global stability for relativistic perfect fluids with linear equations of state  $p = K\rho$  where  $1/3$ . preprint [[arXiv:2002.12526](https://arxiv.org/abs/2002.12526)] (2020)
18. Ringström, H.: Future stability of the Einstein-non-linear scalar field system. *Invent. Math.* **173**, 123–208 (2008)
19. Ringström, H.: Power law inflation. *Commun. Math. Phys.* **290**(1), 155–218 (2009)
20. Rodnianski, I., Speck, J.: The stability of the irrotational Euler–Einstein system with a positive cosmological constant. *J. Eur. Math. Soc.* **15**, 2369–2462 (2013)
21. Speck, J.: The stabilizing effect of spacetime expansion on relativistic fluids with sharp results for the radiation equation of state. *Arch. Rat. Mech.* **210**, 535–579 (2013)
22. Taylor, M.E.: *Partial Differential Equations III: Nonlinear Equations*. Springer, Berlin (1996)
23. Rendall, A.D., Brauer, U., Reula, O.: The cosmic no-hair theorem and the non-linear stability of homogeneous Newtonian cosmological models. *Class. Quant. Grav* **11**, 2283–2296 (1994)
24. Wei, C.: Stabilizing effect of the power law inflation on isentropic relativistic fluids. *J. Differ. Equ.* **265**, 3441–3463 (2018)

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