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Time Global Finite-Energy Weak Solutions to the Many-Body Maxwell–Pauli Equations

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Abstract: We study the quantum mechanical many-body problem of $N \geq 1$ non-relativistic electrons with spin interacting with their self-generated classical electromagnetic field and $K \geq 0$ static nuclei. We model the dynamics of the electrons and their self-generated electromagnetic field with the so-called many-body Maxwell–Pauli equations. Here we construct time global, finite-energy, weak solutions to the many-body Maxwell–Pauli equations under the assumption that the fine structure constant α and the nuclear charges are not too large. The particular assumptions on the size of α and the nuclear charges ensure that we have energetic stability of the many-body Pauli Hamiltonian, i.e., the ground state energy is finite and uniformly bounded below with lower bound independent of the magnetic field and the positions of the nuclei. This work serves as an initial step towards understanding the connection between the energetic stability of matter and the well-posedness of the corresponding dynamical equations.

1. The Many-Body Maxwell–Pauli Equations

The three-dimensional many-body Maxwell–Pauli (MBMP) equations are a system of nonlinear, coupled partial differential equations describing the time evolution of $N \geq 1$ non-relativistic electrons interacting with both their classical self-generated electromagnetic field and $K \geq 0$ static (infinitely heavy) nuclei. In the Coulomb gauge the MBMP equations read

$$\begin{cases} i\partial_t \psi = H(\mathbf{A})\psi, \\ \Box \mathbf{A} = 4\pi\alpha \mathcal{P} \boldsymbol{\mathcal{J}}[\psi, \mathbf{A}], \\ \operatorname{div} \mathbf{A} = 0. \end{cases} \tag{1}$$

In (1), $\psi(t) \in \bigwedge^N [L^2(\mathbb{R}^3)]^2$ is the Fermionic many-body wave function at time t of the electrons ($\bigwedge^N$ is the N-fold antisymmetric tensor product), $\mathbf{A}(t) : \mathbb{R}^3 \to \mathbb{R}^3$ is the total magnetic vector potential at time t generated by the electrons, $H(\mathbf{A})$ is the many-body Pauli Hamiltonian defined by

$$H(\mathbf{A})(t) = \sum_{j=1}^{N} \mathcal{T}_j(\mathbf{A})(t) + V(\underline{\mathbf{R}}, \mathcal{Z}), \tag{2}$$

where $\mathcal{T}_j(\mathbf{A})(t) = [\boldsymbol{\sigma}_j \cdot (\mathbf{p}_j + \mathbf{A}_j(t))]^2$ is the Pauli operator corresponding to the jth electron ($\mathcal{T}_j$ is appearing in the jth factor of the tensor product: $\mathcal{T}_j \equiv I \otimes \cdots \otimes \mathcal{T}_j \otimes \cdots \otimes I$), $\mathbf{p}_j = -i\nabla_{\mathbf{x}_j}$ is the conjugate momentum of the jth electron, $\mathbf{A}_j(t) = \mathbf{A}(t, \mathbf{x}_j)$ is the total magnetic vector potential at time t evaluated at the position $\mathbf{x}_j \in \mathbb{R}^3$ of the jth electron, $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3) \in \mathbb{R}^3 \otimes M_{2\times 2}(\mathbb{C})$ is the vector of Pauli matrices, which are 2×2 Hermitian matrices assumed to satisfy the commutation relations $[\sigma^j, \sigma^k] = 2i\epsilon_{jk\ell}\sigma^\ell$ and anticommutation relations $\left\{\sigma^j, \sigma^k\right\} = 2\delta_{jk} I$, $\underline{\mathbf{R}} = (\mathbf{R}_1, \ldots, \mathbf{R}_K)$ denotes the collection of distinct centers $\mathbf{R}_j \in \mathbb{R}^3$ of the K nuclei, $\mathcal{Z} = (Z_1, \ldots, Z_K) \in [0, \infty)^K$ denotes the collection of nuclear charges of the K nuclei, $V(\underline{\mathbf{R}}, \mathcal{Z}) : \mathbb{R}^{3N} \to \mathbb{R}$ denotes the sum total of the electron–electron, electron–nuclei, and nuclei–nuclei Coulomb potential interaction and is given by

$$V(\underline{\mathbf{R}}, \mathcal{Z})(\underline{\mathbf{x}}) = \sum_{1\le i<j\le N} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} - \sum_{i=1}^{N}\sum_{j=1}^{K} \frac{Z_j}{|\mathbf{x}_i - \mathbf{R}_j|} + \sum_{1\le i<j\le K} \frac{Z_i Z_j}{|\mathbf{R}_i - \mathbf{R}_j|}, \tag{3}$$

where $\underline{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \mathbb{R}^{3N}$ is the collection of position coordinates of the N electrons, $\mathcal{P} = \operatorname{curl}(-\Delta)^{-1}\operatorname{curl}$ is the Leray-Helmholtz projection onto divergence-free vector fields, $\Box = \alpha^2\partial_t^2 - \Delta$ is the d'Alembert wave operator, and $\boldsymbol{\mathcal{J}}[\psi, \mathbf{A}](t) = \sum_{j=1}^{N} \mathbf{J}_j[\psi, \mathbf{A}](t) : \mathbb{R}^3 \to \mathbb{R}^3$ is the total probability current density of the electrons, where $\mathbf{J}_j[\psi, \mathbf{A}]$ is the probability current density of the jth electron and is defined by

$$\mathbf{J}_j[\psi, \mathbf{A}](t)(\mathbf{x}) = -2\alpha \operatorname{Re} \int \langle \boldsymbol{\sigma}\psi_{\underline{\mathbf{z}}'_j}(t), \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}(t))\psi_{\underline{\mathbf{z}}'_j}(t)\rangle_{\mathbb{C}^2}(\mathbf{x})\mathrm{d}\underline{\mathbf{z}}'_j. \tag{4}$$

In (4), for $j \in \{1, \ldots, N\}$, $\mathbf{z}_j = (\mathbf{x}_j, s_j) \in \mathbb{R}^3 \times \{\uparrow, \downarrow\}$ is the jth electron's position coordinate and spin state, and $\psi_{\underline{\mathbf{z}}'_j} : \mathbb{R}^3 \to \mathbb{C}^2$ is the spinor defined by

$$\psi_{\underline{\mathbf{z}}'_j}(\mathbf{x}, s) = \psi(\mathbf{z}_1; \ldots; \mathbf{x}, s; \ldots; \mathbf{z}_N),$$

where $s \in \{\uparrow, \downarrow\}$, $\underline{\mathbf{z}}'_j = (\mathbf{z}_1, \ldots, \mathbf{z}_{j-1}, \mathbf{z}_{j+1}, \ldots, \mathbf{z}_N)$, and $\mathrm{d}\mathbf{z}_i \equiv \sum_{s_i\in\{\uparrow,\downarrow\}} \mathrm{d}\mathbf{x}_i$. At least formally, there are two conserved quantities associated with (1): the L^2-norm $\|\psi(t)\|_2$ and the total energy E given by

$$E[\psi, \mathbf{A}, \partial_t\mathbf{A}](t) = T_{\mathrm{P}}[\psi(t), \mathbf{A}(t)] + V[\psi(t)] + F[\mathbf{A}(t), \partial_t\mathbf{A}(t)], \tag{5}$$

where $T_{\mathrm{P}}[\psi, \mathbf{A}]$ is the total Pauli kinetic energy:

$$T_{\mathrm{P}}[\phi, \mathbf{A}] = \sum_{j=1}^{N} \|\boldsymbol{\sigma}_j \cdot (\mathbf{p}_j + \mathbf{A}_j)\phi\|_2^2, \tag{6}$$

$V[\psi]$ is the total potential energy:

$$V[\psi] = \langle \psi, V(\underline{\mathbf{R}}, \mathcal{Z})\psi\rangle_{L^2}, \tag{7}$$

and $F[\mathbf{A}, \partial_t\mathbf{A}]$ is the electromagnetic field energy:

$$F[\mathbf{A}, \partial_t\mathbf{A}](t) = \frac{1}{8\pi\alpha^2}\left(\|\mathbf{B}(t)\|_2^2 + \alpha^2\|\partial_t\mathbf{A}(t)\|_2^2\right), \tag{8}$$

where $\mathbf{B} = \operatorname{curl}\mathbf{A}$ is the magnetic field. It will be important for our study of (1) to define the absolute ground state energy associated with $E[\psi, \mathbf{A}, \partial_t\mathbf{A}]$. For this we introduce the function space

$$\mathscr{C}_N := \left\{(\psi, \mathbf{A}) \in \bigwedge^N H^1(\mathbb{R}^3; \mathbb{C}^2) \times \dot{H}^1(\mathbb{R}^3; \mathbb{R}^3) \;:\; \|\psi\|_2 = 1,\;\; \operatorname{div}\mathbf{A} = 0\right\}. \tag{9}$$

The space $\mathscr{C}_N$ should be thought of as the space of all pairs $(\psi, \mathbf{A})$ for which $E[\psi, \mathbf{A}, \mathbf{0}]$ is finite. The absolute ground state energy E_{G} is then defined as

$$E_{\mathrm{G}}(N, K, \mathcal{Z}, \alpha) = \inf\left\{E[\psi, \mathbf{A}, \mathbf{0}] \;:\; (\psi, \mathbf{A}) \in \mathscr{C}_N,\;\; \underline{\mathbf{R}}\right\}. \tag{10}$$

Units Let e_0, m, $\hbar$, and c be the electron charge, electron mass, the reduced Plank's constant, and the speed of light, respectively. The length unit is half the Bohr radius $\ell = \hbar^2/(2me_0^2)$, the energy unit is 4 Rydbergs $= 2me_0^4/\hbar^2 = 2m\alpha^2c^2$, and the time unit is $\tau = \hbar/(4\text{ Rydbergs}) = \hbar^3/(2me_0^4)$, where $\alpha = e_0^2/\hbar c$ is Sommerfeld's dimensionless fine structure constant. Note that $1/(c\tau) = \alpha/\ell$. The magnetic field $\mathbf{B} = \operatorname{curl}\mathbf{A}$ and electric field $\mathbf{E} = -\nabla\varphi - \alpha\partial_t\mathbf{A}$ are both in units of $e_0/(\alpha\ell^2)$. The field energy $F[\mathbf{A}, \partial_t\mathbf{A}]$ in these units is given by (8). Throughout the paper we will think of α as a parameter that can take any positive real value.

Our study of (1) is motivated by the results on the energetic stability of matter in magnetic fields as developed in [1–5]. In particular, J. Fröhlich, E. H. Lieb, and M. Loss in 1986 [1] introduced the *critical charge*

$$Z_c := \inf\left\{\frac{F[\mathbf{A}, \mathbf{0}]}{\langle\psi, |\cdot|^{-1}\psi\rangle_{L^2}} \;:\; (\psi, \mathbf{A}) \in \mathscr{C}_1 \;\text{ and }\; \boldsymbol{\sigma}\cdot(\mathbf{p} + \mathbf{A})\psi = 0\right\}, \tag{11}$$

and proved that

$$\inf\{E[\psi, \mathbf{A}, \mathbf{0}] \;:\; (\psi, \mathbf{A}) \in \mathscr{C}_1\} = \begin{cases}\text{finite}, & Z < Z_c\\ -\infty, & Z > Z_c,\end{cases} \tag{12}$$

where $E[\psi, \mathbf{A}, \mathbf{0}]$ is defined as the $(N = K = 1)$-case of (5). In words, the ground state energy of the single electron Pauli–Coulomb Hamiltonian, $[\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})]^2 - Z/|\mathbf{x}| + F[\mathbf{A}, \mathbf{0}]$, is uniformly bounded below *independent* of the magnetic field $\mathbf{B} = \operatorname{curl}\mathbf{A}$ when $Z < Z_c$, and has no such lower bound when $Z > Z_c$. An important observation is that $Z_c < \infty$ as there exist nontrivial finite-energy solutions $(\psi, \mathbf{A})$ to the *zero mode* equation $\boldsymbol{\sigma}\cdot(\mathbf{p} + \mathbf{A})\psi = 0$ (see, for example, [3]). We note that the result (12) is *false* if one does not include the magnetic field energy $F[\mathbf{A}, \mathbf{0}]$ in the definition of $E[\psi, \mathbf{A}, \mathbf{0}]$ (see [1] for a discussion).

More generally, E. H. Lieb, M. Loss, and J. P. Solovej in 1995 [5] proved that, if $\alpha \le 0.06$ and $\alpha^2 \max\mathcal{Z} \le 0.041$, then

$$E_{\mathrm{G}} \ge -C(\alpha, \mathcal{Z})N^{1/3}K^{2/3}, \tag{13}$$

where $C(\alpha) > 0$ is a constant depending only on α and $\mathcal{Z}$. That is, for small enough $\alpha^2 \max\mathcal{Z}$ and α, the total energy $E[\psi, \mathbf{A}, \mathbf{0}]$ associated with the many-body Pauli Hamiltonian $H(\mathbf{A})$ (see (5) and (2)) is bounded below with lower bound independent of the magnetic field $\mathbf{B} = \operatorname{curl} \mathbf{A}$ *and* the positions of the nuclei $\underline{\mathbf{R}}$. Again, the result (13) is *false* if we do not include the field energy $F[\mathbf{A}, \mathbf{0}]$ in the definition of $E[\psi, \mathbf{A}, \mathbf{0}]$. Moreover, the antisymmetry condition in the definition of $\mathscr{C}_N$ is crucial for (13), as minimizing with respect to Bosonic (i.e., completely symmetric) wavefunctions results in collapse. We note the range of $\alpha \in (0, 0.06)$ includes the actual physical value $\alpha \simeq 1/137 \simeq 0.007$, and for $\alpha \simeq 1/137$, the largest nuclear charge allowed is roughly 769. It is important to emphasize that (13) requires a bound on both $\alpha^2 \max \mathcal{Z}$ *and* α. It is known that even for the one-electron molecule (single electron, $K > 1$ nuclei) $\alpha > 3\pi/\sqrt{2}$ causes instability [2]. Optimal ranges of $\alpha^2 \max \mathcal{Z}$ and α to ensure stability is a challenging open problem.

Considering these results on energetic stability it seems natural to ask whether the existence of the ground state energy has an influence on the well-posedness of the corresponding dynamical equations. Specifically, how does the existence (or non-existence) of solutions to (1) depend on Z_c in the $(N = K = 1)$-case and, more generally, the size of $\alpha^2 \max \mathcal{Z}$ and α in the $(\max\{N, K\} > 1)$-case? The aim of this paper is to make progress on these questions by constructing finite-energy weak solutions to (1) which are time global under the under the assumption that α and $\alpha^2 \max \mathcal{Z}$ are small enough to ensure $E_{\mathrm{G}} > -\infty$.

Theorem 1 (Global Finite-Energy Weak Solutions). *Suppose α and $\alpha^2 \max \mathcal{Z}$ are sufficiently small to ensure $E_{\mathrm{G}} > -\infty$. Then, given*

$$(\psi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0) \in \bigwedge^N H^1(\mathbb{R}^3; \mathbb{C}^2) \times H^1(\mathbb{R}^3; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3)$$

with $\|\psi_0\|_2 = 1$ and $\operatorname{div} \mathbf{a}_0 = \operatorname{div} \dot{\mathbf{a}}_0 = 0$, there exists at least one finite-energy weak solution

$$(\psi, \mathbf{A}, \partial_t \mathbf{A}) \in C^{\mathrm{w}}(\mathbb{R}_+; \bigwedge^N H^1(\mathbb{R}^3; \mathbb{C}^2) \times H^1(\mathbb{R}^3; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3))$$

to (1) such that the initial conditions $(\psi(0), \mathbf{A}(0), \partial_t \mathbf{A}(0)) = (\psi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0)$ are satisfied.

If we are considering the $(N = K = 1)$-case of (1), then the hypothesis of Theorem 1 changes to $Z < Z_c$ where Z_c is the critical charge and is defined by (11). Moreover, if we are considering the $(N = 1, K = 0)$-case of (1), then no additional assumptions are needed since the total energy is always positive (there are no nuclear charges present and we do not need to assume α is sufficiently small). The reason The solution obtained in Theorem 1 is a *weak* solution, but does indeed have finite energy, i.e., $(\psi, \mathbf{A})$ belong to the class of functions $\mathscr{C}_N$ and $\partial_t \mathbf{A} \in L^2(\mathbb{R}^3; \mathbb{R}^3)$.

As of this writing, there seems to be no existence theory of solutions to (1) for any initial data (aside from the present paper), *even* in the single electron case with no nuclei. To contrast this, we point out that there is an extensive literature studying the closely related Maxwell–Schrödinger (MS) system (see, e.g., [6–19]). In the Coulomb gauge, the MS equations read

$$\begin{cases} i\partial_t \psi = (\mathbf{p} + \mathbf{A})^2 \psi, \\ \Box \mathbf{A} = -8\pi\alpha^2 \mathcal{P} \operatorname{Re} \langle \psi, (\mathbf{p} + \mathbf{A})\psi \rangle_{\mathbb{C}}, \\ \operatorname{div} \mathbf{A} = 0. \end{cases} \tag{14}$$

where $\psi : \mathbb{R}^3 \to \mathbb{C}$ is the single-particle wave function without spin. Notably, M. Nakamura and T. Wada in 2007 proved the global existence of unique smooth solutions to (14) [13,14]. In order to obtain time global solutions to the MS equations, the authors in [13] first establish local well-posedness by linearizing (14) and applying a contraction mapping argument. Using a Koch-Tzvetkov type estimate on the Schrödinger piece $e^{i\Delta t}$, the authors in [14] obtain time local solutions in Sobolev spaces of low regularity and thereby improve upon the local well-posedness theory developed in [13]. The lower regularity solutions are sufficiently close to the energy class so that, together with energy conservation, they may conclude the solutions exist for all time.

We'd like to bring attention to the fact that, to our knowledge, the only result on the MS equations with Coulomb potential interaction included is the local well-posedness of a many-body MS system [18]. The methods in this paper are immediately adaptable to show the existence of a global weak solution a many-body version of the MS equations. In this case, no assumption on the size of α or $\max \mathcal{Z}$ are needed because energetic stability always holds. However, global well-posedness of strong solutions to the MS equations with a Coulomb potential included is an open problem.

Consider the one-body MP equations, namely the $(N = 1, K = 0)$-case of (1), which read

$$\begin{cases} i\partial_t \psi = [\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})]^2\psi, \\ \Box \mathbf{A} = -8\pi\alpha^2 \mathcal{P}\,\mathrm{Re}\,\langle \boldsymbol{\sigma}\psi, \boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})\psi\rangle_{\mathbb{C}^2}, \\ \mathrm{div}\,\mathbf{A} = 0. \end{cases} \tag{15}$$

The difference between the magnetic Schrödinger equation (14, first equation) and Pauli equation (15, first equation) comes from the coupling between the spin of the electron and the magnetic field $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$, as seen through the identity

$$[\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})]^2 = (\mathbf{p}+\mathbf{A})^2 + \boldsymbol{\sigma}\cdot\mathbf{B}. \tag{16}$$

Similarly, the only difference between the probability current densities on the right hand sides of (14, second equation) and (15, second equation) is the inclusion of the spin current, namely $\mathrm{curl}\,\langle\psi, \boldsymbol{\sigma}\psi\rangle_{\mathbb{C}^2}$, appearing in the identity

$$\mathrm{Re}\,\langle \boldsymbol{\sigma}\psi, \boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})\psi\rangle_{\mathbb{C}^2} = \mathrm{Re}\,\langle \psi, (\mathbf{p}+\mathbf{A})\psi\rangle_{\mathbb{C}^2} + \frac{1}{2}\,\mathrm{curl}\,\langle\psi, \boldsymbol{\sigma}\psi\rangle_{\mathbb{C}^2}. \tag{17}$$

Our attempts to apply the methods found in [13] to prove just *local* existence of solutions in, for example, $H^2 \times H^2 \times H^1$ to the one-body MP equations (15) have not succeeded. These strategies appear to break down due to inclusion of the spin-magnetic field coupling $\boldsymbol{\sigma}\cdot\mathbf{B}$ and the spin current $\mathrm{curl}\,\langle\psi, \boldsymbol{\sigma}\psi\rangle_{\mathbb{C}^2}$. Indeed, it appears to be necessary to estimate $\|\langle\psi, \mathrm{curl}\,\boldsymbol{\sigma}\varphi\rangle_{\mathbb{C}^2}\|_{L^2}$ by $\|\psi\|_{H^2}\|\varphi\|_{L^2}$ for $\psi, \varphi \in H^2(\mathbb{R}^3; \mathbb{C}^2)$, and such an estimate seems impossible in general. In [13], the authors manage to make such an estimate on the similar term $\mathrm{Re}\,\langle\psi, \mathbf{p}\varphi\rangle_{\mathbb{C}}$ appearing in (14, second equation) by utilizing the projection operator $\mathcal{P}$ and observing that $\mathcal{P}(\psi\nabla\varphi) = -\mathcal{P}(\varphi\nabla\psi)$. However, the spin current is a pure curl and is thus already divergence-free. Therefore, one loses the utility of the projection operator $\mathcal{P}$ being applied to the right hand side of (15, second equation). For these reasons, local well-posedness of (15) in $H^2 \times H^2 \times H^1$ remains an open problem.

Instead of attempting to prove local well-posedness of (15) and, more generally, (1), we've turned our attention to proving the existence of global, *weak* solutions to (1). For this we combine the contraction mapping strategy in [13] with ideas from the 1995 work on the MS equations by Guo et al. [8]. In the latter article the authors consider an ε-modified version of the MS equations that read

$$\begin{cases} \partial_t \psi = -(i+\varepsilon)(\mathbf{p}+\mathbf{A})^2\psi, \\ \Box \mathbf{A} = -8\pi\alpha^2 \mathcal{P}\,\mathrm{Re}\,\langle \psi, (\mathbf{p}+\mathbf{A})\psi\rangle_{\mathbb{C}}, \\ \operatorname{div} \mathbf{A} = 0. \end{cases} \tag{18}$$

By taking advantage of the regularity-improving, dispersive properties of the heat kernel $e^{\varepsilon t\Delta}$ and the dissipative charge and energy associated with the ε-modified MS equations, the authors in [8] are able to prove the existence of low regularity time global solutions to (18). Then, by using a compactness argument to consider the $\varepsilon \to 0$ limit, the authors prove these low regularity time global solutions to (18) converge to time global finite-energy weak solutions to (14).

The consideration of [8], therefore, leads us to study our own approximate system to the MBMP equations. Referred to as the ε-modified MBMP equations, this approximate system reads

$$\begin{cases} \partial_t \phi^\varepsilon = -(i+\varepsilon)\mathcal{H}^\varepsilon(\mathbf{A}^\varepsilon)\phi^\varepsilon + \varepsilon\left(T_{\mathrm{P}}[\phi^\varepsilon, \tilde{\mathbf{A}}^\varepsilon] + V[\phi^\varepsilon]\right)\phi^\varepsilon \\ \Box \mathbf{A}^\varepsilon = 4\pi\alpha\Lambda_\varepsilon^{-1}\mathcal{P}\boldsymbol{\mathcal{J}}[\phi^\varepsilon, \tilde{\mathbf{A}}^\varepsilon] \\ \operatorname{div} \mathbf{A}^\varepsilon = 0, \quad \tilde{\mathbf{A}}^\varepsilon = \Lambda_\varepsilon^{-1}\mathbf{A}^\varepsilon, \end{cases} \tag{19}$$

where $\Lambda_\varepsilon = \sqrt{1-\varepsilon\Delta}$, $\mathcal{H}^\varepsilon(\mathbf{A}^\varepsilon)$ is the ε-modified Hamiltonian

$$\mathcal{H}^\varepsilon(\mathbf{A}^\varepsilon) = \sum_{j=1}^N \mathcal{T}_j(\tilde{\mathbf{A}}^\varepsilon) + V(\underline{\mathbf{R}}, \mathcal{Z}), \tag{20}$$

$T_{\mathrm{P}}[\phi^\varepsilon, \tilde{\mathbf{A}}^\varepsilon]$ is defined by (6) and $V[\phi^\varepsilon]$ is defined by (7). We define the total energy of the ε-modified system as

$$\mathcal{E}[\phi^\varepsilon, \mathbf{A}^\varepsilon, \partial_t\mathbf{A}^\varepsilon] = T_{\mathrm{P}}[\phi^\varepsilon, \tilde{\mathbf{A}}^\varepsilon] + V[\phi^\varepsilon] + F[\mathbf{A}^\varepsilon, \partial_t\mathbf{A}^\varepsilon]\|\phi^\varepsilon\|_2^2, \tag{21}$$

where $F[\mathbf{A}^\varepsilon, \partial_t\mathbf{A}^\varepsilon]$ is the field energy defined by (8).

For the remainder of this paper we will drop the dependence on ε when it is not needed. Note that the Pauli operators $\mathcal{T}_j$ in the definition (20) of $\mathcal{H}(\mathbf{A})$ are evaluated at the regularized vector potential $\tilde{\mathbf{A}}$, whereas the field energy F is evaluated at $(\mathbf{A}, \partial_t\mathbf{A})$. Similarly, note that the probability current density $\boldsymbol{\mathcal{J}}$ in (19) is evaluated at $\tilde{\mathbf{A}}$. These choices are made so that the total energy (21) is dissipative under the time evolution of (19) (see Theorem 3). Moreover, the choice of the right hand side of the first equation in (19) is made so that normalized wavefunctions remain normalized under the flow of (19). This point will be crucial for the application of the results concerning the stability of matter in magnetic fields to construct *global* solutions to (19).

The space of initial conditions we will consider for the ε-modified MBMP system is

$$\mathcal{X}_0^m = \Big\{(\psi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0) \in [H^m(\mathbb{R}^{3N})]^{2^N} \oplus H^m(\mathbb{R}^3;\mathbb{R}^3) \oplus H^{m-1}(\mathbb{R}^3;\mathbb{R}^3) \\ \text{s.t. } \operatorname{div}\mathbf{a}_0 = \operatorname{div}\dot{\mathbf{a}}_0 = 0\Big\}. \tag{22}$$

Combining the regularity improving estimates of the heat kernel $e^{\varepsilon t\Delta}$ (see Lemma 3) with a contraction mapping scheme similar to the one in [13], we prove the following local well-posedness result for (19).

Theorem 2 (Local Well-posedness of the ε-Modified System). *Fix $m \in [1, 2]$ and $\varepsilon > 0$. Given initial data $(\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0) \in \mathcal{X}_0^m$, there exists a maximal time interval $\mathcal{I} = [0, T_{\max})$ and a unique solution*

$$(\phi, \mathbf{A}) \in C_{\mathcal{I}}[H^m(\mathbb{R}^{3N})]^{2^N} \times [C_{\mathcal{I}} H^m(\mathbb{R}^3; \mathbb{R}^3) \cap C^1_{\mathcal{I}} H^{m-1}(\mathbb{R}^3; \mathbb{R}^3)]$$

to (19) such that the initial conditions $(\phi(0), \mathbf{A}(0), \partial_t \mathbf{A}(0)) = (\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0)$ are satisfied and the blow-up alternative holds: either $T_{\max} = \infty$ or $T_{\max} < \infty$ and

$$\limsup_{t \to T_{\max}} \|(\phi(t), \mathbf{A}(t), \partial_t \mathbf{A}(t))\|_{H^m \oplus H^m \oplus H^{m-1}} = \infty.$$

Furthermore, we can approximate lower regularity solutions by higher regularity solutions in the following sense: if $\{(\phi_0^j, \mathbf{a}_0^j, \dot{\mathbf{a}}_0^j)\}_{j \geq 1} \in \mathcal{X}_0^m$ converges, as $j \to \infty$, to $(\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0) \in \mathcal{X}_0^1$ in $H^1 \oplus H^1 \oplus L^2$, then, for each $t \in \mathcal{I}$, the sequence of solutions $\{(\phi^j(t), \mathbf{A}^j(t), \partial_t \mathbf{A}^j(t))\}_{j \geq 1}$ corresponding to the initial datum $\{(\phi_0^j, \mathbf{a}_0^j, \dot{\mathbf{a}}_0^j)\}_{j \geq 1}$ converges in $H^1 \oplus H^1 \oplus L^2$ to the solution $(\phi(t), \mathbf{A}(t), \partial_t \mathbf{A}(t))$ corresponding to the initial datum $(\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0)$.

The limited range of regularity, namely $m \in [1, 2]$, in Theorem 2 comes from controlling the Coulomb term $V(\underline{\mathbf{R}}, \mathcal{Z})\phi$ in (19) (see Lemma 6). We can, in fact, prove Theorem 2 for m up to $\frac{5}{2} - \delta$, $\delta > 0$. However, doing so seems to be an unnecessary mathematical generality and has no bearing on the validly of Theorem 1. However, we do expect this to be the maximum range of regularity for this system. Indeed, already for the Hydrogen ground state $\psi_0(\mathbf{x}) \propto e^{-|\mathbf{x}|/2}$ one has $\|\psi_0\|_{s,2} < \infty$ if and only if $s < 5/2$.

With Theorem 2 at our disposal, we would then like to consider the limit $\varepsilon \to 0$ of the low regularity ($m = 1$) solutions to (19). However, one potential obstruction to considering the $\varepsilon \to 0$ limit is that the local time interval of existence $[0, T_{\max})$ in Theorem 2 might shrink to zero as $\varepsilon \to 0$. It is therefore necessary to prove that the low regularity $H^1 \oplus H^1 \oplus L^2$-local solutions to (19) are, in fact, global. A key ingredient that allows us to extend from local to global solutions is to prove apriori ε, t-independent bounds in $H^1 \oplus H^1 \oplus L^2$ on solutions $(\phi^\varepsilon, \mathbf{A}^\varepsilon)$ to (19). Our proof of such uniform bounds uses energy dissipation together with the fact that the Coulomb energy $V[\phi^\varepsilon(t)]$ along a solution $(\phi^\varepsilon, \mathbf{A}^\varepsilon)$ is bounded, with upper bound independent of ε and t. This latter fact is crucial for our proof strategy and only true when the energy $\mathcal{E}$ is uniformly bounded below. From the results on the stability of matter in magnetic fields we know a uniform lower bound on $\mathcal{E}$ requires sufficiently small α and $\alpha^2 \max \mathcal{Z}$. We express the fact $V[\,\cdot\,]$ is a bounded functional on $\mathscr{C}_N$ when α and $\alpha^2 \max \mathcal{Z}$ are sufficiently small and that low regularity $H^1 \oplus H^1 \oplus L^2$ local solutions to (19) are global as the following Lemma and Theorem.

Lemma 1 (Uniform Bound on the Coulomb Energy). *Let $\{(\phi^n, \mathbf{A}^n)\}_{n \geq 1} \subset \mathscr{C}_N$, where $\mathscr{C}_N$ is defined by(9), and assume that $E[\phi^n, \mathbf{A}^n, \mathbf{0}] \leq E_0$ where E_0 is a constant depending on N, K, α, $\mathcal{Z}$, $\underline{\mathbf{R}}$, and $(\phi^0, \mathbf{A}^0)$, but independent of n. Assume α and $\alpha^2 \max \mathcal{Z}$ are sufficiently small to ensure $E_{\mathrm{G}} > -\infty$. Then the sequence of Coulomb energies $\{V[\phi^n]\}_{n=1}^\infty$ is uniformly bounded, $\sup_n |V[\phi^n]| < \infty$.*

Theorem 3 (Dissipation of Energy and Uniform Bounds). *Fix $\varepsilon > 0$ and $m \in [1, 2]$. Let $(\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0) \in \mathcal{X}_0^m$ with $\phi_0 \in \bigwedge^N H^m(\mathbb{R}^3; \mathbb{C}^2)$ and $\|\phi_0\|_2 = 1$. Let $(\phi, \mathbf{A}) \in C_{\mathcal{I}} H^m \times$*

$[C_{\mathcal{I}}H^m \cap C^1_{\mathcal{I}}H^{m-1}]$ *be the corresponding solution to* (19) *provided by Theorem* 2*. Then* $\phi(t)$ *remains completely antisymmetric and normalized for* $t \in \mathcal{I}$*, and, if* $m = 2$*,*

$$\mathcal{E}[\phi, \mathbf{A}, \partial_t \mathbf{A}](t) - \mathcal{E}[\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0] = -2\varepsilon \int_0^t \left[\|\mathcal{H}(\mathbf{A}(\tau))\phi(\tau)\|_2^2 - \langle \phi(\tau), \mathcal{H}(\mathbf{A}(\tau))\phi(\tau)\rangle_{L^2}^2 \right] \mathrm{d}\tau, \tag{23}$$

for all $t \in \mathcal{I}$*. Moreover, if* α *and* $\alpha^2 \max \mathcal{Z}$ *are sufficiently small to ensure* $E_{\mathrm{G}} > -\infty$*, then*

$$\|\nabla\phi(t)\|_2 \le C_1, \qquad F[\mathbf{A}, \partial_t \mathbf{A}](t) \le C_2, \qquad \|\mathbf{A}(t)\|_2 \le C_3(1+t), \tag{24}$$

for all $t \in \mathcal{I}$*, where* $C_1, C_2, C_3 > 0$ *are constants depending on* N*,* K*,* $\mathcal{Z}$*,* α*, and the initial data, but not* ε *or* t*. As a consequence, for* $m = 1$ *and for each fixed* $\varepsilon > 0$*, the solution* $(\phi, \mathbf{A})$ *exists for all* $t \in \mathbb{R}_+$*.*

We again emphasize the importance of the bounds (24). As already mentioned in the paragraph preceding Theorem 3, for each fixed $\varepsilon > 0$, it is necessary to have time-independent bounds on $(\phi(t), \mathbf{A}(t), \partial_t \mathbf{A}(t))$ in $H^1 \times H^1 \times L^2$-norm in order to apply the blow-up alternative of Theorem 2 and assert the $m = 1$ solutions of Theorem 2 exist for all time. Furthermore, in order to apply a compactness argument to take the $\varepsilon \to 0$ limit, we need ε-independent bounds on $(\phi(t), \mathbf{A}(t), \partial_t \mathbf{A}(t))$ in $H^1 \times H^1 \times L^2$-norm to apply the Banach-Alaoglu Theorem and extract a weak* converging subsequence. This weak* limit will be shown to be a finite-energy weak solution to (1), thus yielding a proof of Theorem 1. We also emphasize that the complete antisymmetry and normalization of $\phi(t)$ is crucial, as otherwise we cannot make use of the stability result (13) and use Lemma 1 to control the Coulomb energy.

This paper is organized as follows: In Sect. 2 we clarify our notation, define what we mean by a weak solution, and recall standard estimates in Sobolev spaces, including those for the heat kernel and wave equation. Section 3 is divided into two subsections: Sects. 3.1 and 3.2. In Sect. 3.1 we prove several estimates for the right hand sides of (19) in various Sobolev spaces. Such estimates are crucial to the proof of Theorem 2. In Sect. 3.2 we introduce the metric space on which the Banach fixed point theorem will be applied, and then give a proof of Theorem 2. In Sect. 4 we provide a proof that the Coulomb energy is uniformly bounded, and use this result to prove Theorem 3. Finally, Sect. 5 is devoted to completing the proof of Theorem 1.

2. Notation, Definitions, and Mathematical Preliminaries

If $a, b \in \mathbb{R}$, $a \lesssim b$ means that there exists a universal constant $C > 0$ such that $a \le Cb$. For $p \in [1, \infty)$ and $s \ge 0$, we will denote by $L^p = L^p(\mathbb{R}^d)$ the usual Lebesgue space, $W^{s,p} \equiv W^{s,p}(\mathbb{R}^d)$ the usual Sobolev space equipped with the norm $\|f\|_{s,p} \equiv \|(1-\Delta)^{\frac{s}{2}} f\|_{L^p}$, and $\dot{W}^{s,p} \equiv \dot{W}^{s,p}(\mathbb{R}^d)$ the homogeneous Sobolev space equipped with the seminorm $\|f\|_{\dot{W}^{s,p}} \equiv \|(-\Delta)^{\frac{s}{2}} f\|_p$. When $s = 0$, we simply write $\|f\|_p$ and when $p = 2$ we will use the notation $H^s \equiv W^{s,2}$, $\dot{H}^s \equiv \dot{W}^{s,2}$. The negative index Sobolev spaces $H^{-s}(\mathbb{R}^d) \equiv (H^s(\mathbb{R}^d))^*$, for $s > 0$, are equipped with the usual norm $\|f\|_{-s,2} = \sup\{\|f\eta\|_1 : \|\eta\|_{s,2} = 1\}$.

Let $\mathcal{I} \subset \mathbb{R}$ be a (possibly infinite) time interval and $(X, \|\cdot\|_X)$ be a reflexive Banach space. Then $C_{\mathcal{I}}X \equiv C(\mathcal{I}; X)$, $C^1_{\mathcal{I}}X \equiv C^1(\mathcal{I}; X)$, and $C^{\mathrm{w}}_{\mathcal{I}}X \equiv C^{\mathrm{w}}(\mathcal{I}; X)$ denote the space of strongly continuous, strongly continuously differentiable, and weakly continuous mappings from $\mathcal{I}$ to X, respectively. For $p \in [1, \infty]$, $L^p_{\mathcal{I}} X \equiv L^p(\mathcal{I}; X)$ denotes the space of strongly Lebesgue measurable functions $g : \mathcal{I} \to X$ with the property that

$$\|g\|_{L^p_{\mathcal{I}} X} = \begin{cases} \left(\int_{\mathcal{I}} \|g(s)\|^p_X ds\right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty \\ \operatorname{ess\,sup}_{s \in \mathcal{I}} \|g(s)\|_X & \text{for } \quad p = \infty \end{cases}$$

is finite. Moreover, $W^{1,p}_{\mathcal{I}} X = W^{1,p}(\mathcal{I}; X)$ denotes the space of all $L^p_{\mathcal{I}} X$-functions whose first distributional time derivative is in $L^p_{\mathcal{I}} X$. Often $\mathcal{I} = [0, T]$ for some $T > 0$ and in this case we will usually write $L^p_T X$ and likewise for $C_T X$, etc.

For us $\mathscr{D}'(\mathbb{R}_+; X)$ denotes the space of distributions from $\mathbb{R}_+ = [0, \infty)$ to X. That is, $\mathscr{D}'(\mathbb{R}_+; X)$ is the set of strongly continuous linear maps from $C^\infty_c(\mathbb{R}_+)$ to X, where $C^\infty_c(\mathbb{R}_+)$ is equipped with uniform convergence on compact sets. When $g \in L^1_{\text{loc}}(\mathbb{R}_+; X)$ we denote the corresponding distribution in $\mathscr{D}'(\mathbb{R}_+; X)$ defined via the formula

$$C^\infty_c(\mathbb{R}_+) \ni \phi \mapsto \int_{\mathbb{R}_+} g(s)\phi(s)\mathrm{d}s \in X$$

by the same symbol.

Very often X is either $\mathbb{R}^n$, $\mathbb{C}^m$, a Sobolev space $W^{m,p}$, or a direct sum of Sobolev spaces $W^{m_1,p_1} \oplus W^{m_2,p_2}$. For this reason we introduce some special notations we employ in this setting. First of all, when $X = \mathbb{R}^n$ or $\mathbb{C}^m$ we simply write $\|\cdot\|_X = |\cdot|$ where $|\cdot|$ is the usual Euclidean distance on $\mathbb{R}^n$ or $\mathbb{C}^m$. More importantly, we will abbreviate $\|\cdot\|_p$, $\|\cdot\|_{m,p}$, $\|\cdot\|_{q;m,p}$, and $\|\cdot\|_{q_1;m_1,p_1 \oplus q_2;m_2,p_2}$ for the norms on L^p, $W^{m,p}$, $L^q W^{m,p}$, and $L^{q_1} W^{m_1,p_1} \oplus L^{q_2} W^{m_2,p_2}$, respectively. This notation comes with the understanding that $\|\cdot\|_p \equiv \|\cdot\|_{0,p} \equiv \|\cdot\|_{0;0,p}$ and $\|\cdot\|_{q;p} \equiv \|\cdot\|_{q;0,p}$.

When considering vector fields $\mathbf{A} \in L^p(\mathbb{R}^3; \mathbb{R}^3)$, $\mathbf{A} = (A^1, A^2, A^3)$, we write

$$\|\mathbf{A}\|^p_p = \sum_{j=1}^3 \|A^j\|^p_p = \sum_{j=1}^3 \int_{\mathbb{R}^3} |A^j(\mathbf{x})|^p \mathrm{d}\mathbf{x}.$$

Likewise, the L^p-norm of gradients of vector fields is defined by

$$\|\nabla \mathbf{A}\|^p_p = \sum_{j=1}^3 \|\nabla A^j\|^p_p = \sum_{i,j=1}^3 \int_{\mathbb{R}^3} |\partial_{x^i} A^j(\mathbf{x})|^p \mathrm{d}\mathbf{x}.$$

We will frequently use the identity $\|\operatorname{curl} \mathbf{A}\|_2 = \|\nabla \mathbf{A}\|_2$, when $\operatorname{div} \mathbf{A} = 0$ and $\mathbf{A} \in \dot{H}^1(\mathbb{R}^3; \mathbb{R}^3)$. When discussing many-body wave functions, we always consider $H^s(\mathbb{R}^{3N}; \mathbb{C}^{2^N}) \simeq [H^s(\mathbb{R}^{3N})]^{2^N} \equiv \bigotimes^N [H^s(\mathbb{R}^3)]^2$ through the canonical isomorphism, and we recall that $\bigwedge^N [H^s(\mathbb{R}^3)]^2$ denotes the closed subspace of $\bigotimes^N [H^s(\mathbb{R}^3)]^2$ consisting of completely antisymmetric many-body wave functions. Similar to vector fields, the L^p-norm of a many-body wave function ψ is defined as

$$\|\psi\|^p_p = \int |\psi(\underline{\mathbf{z}})|^p \mathrm{d}\underline{\mathbf{z}} \equiv \sum_{s_1=1}^2 \cdots \sum_{s_N=1}^2 \int_{\mathbb{R}^{3N}} |\psi(\mathbf{x}_1, s_1; \ldots; \mathbf{x}_N, s_N)|^p \mathrm{d}\underline{\mathbf{x}}.$$

By a *weak* solution to (1) we mean a distributional solution $(\psi, \mathbf{A})$ in the space $\mathscr{D}'(\mathbb{R}_+; [H^{-1}(\mathbb{R}^{3N})]^{2^N}) \times \mathscr{D}'(\mathbb{R}_+; H^{-1}(\mathbb{R}^3; \mathbb{R}^3))$. In particular, the solution $(\psi, \mathbf{A}, \partial_t \mathbf{A})$ in Theorem 1 satisfies

$$
\begin{aligned}
& i \int_0^\infty \langle \xi, \psi(s) \partial_t f(s) \rangle_{L^2} \mathrm{d}s + \int_0^\infty \sum_{i=1}^N \langle \nabla_{\mathbf{x}_i} \xi, f(s) \nabla_{\mathbf{x}_i} \psi(s) \rangle_{L^2} \mathrm{d}s \\
& \quad = - \int_0^\infty \langle \xi, f(s) [\mathscr{L}(\mathbf{A}(s)) - V(\underline{R}, \mathcal{Z})] \psi(s) \rangle_{L^2} \mathrm{d}s, \\
& \int_0^\infty \sum_{k=1}^3 \langle \partial_k \eta, f(s) \partial_k \mathbf{A}(s) \rangle_{L^2} \mathrm{d}s - \alpha^2 \int_0^\infty \langle \eta, \partial_t f(s) \partial_t \mathbf{A}(s) \rangle_{L^2} \mathrm{d}s \\
& \quad = 4\pi \alpha \int_0^\infty \langle \eta, f(s) \mathcal{P} \boldsymbol{\mathcal{J}}[\psi(s), \mathbf{A}(s)] \rangle_{L^2} \mathrm{d}s,
\end{aligned}
$$

for all $f \in C_c^\infty(\mathbb{R}_+)$, $\xi \in [H^1(\mathbb{R}^{3N})]^{2^N}$, and $\eta \in H^1(\mathbb{R}^3; \mathbb{R}^3)$, where $\mathscr{L}(\mathbf{A}) = \sum_{j=1}^N \mathcal{L}_j(\mathbf{A})$ and $\mathcal{L}_j(\mathbf{A}) = 2\mathbf{A}_j \cdot \mathbf{p}_j + |\mathbf{A}_j|^2 + \boldsymbol{\sigma}_j \cdot \mathbf{B}_j$. The solutions $(\phi, \mathbf{A}) \in C_{\mathcal{I}} [H^m(\mathbb{R}^{3N})]^{2^N} \times [C_{\mathcal{I}} H^m(\mathbb{R}^3; \mathbb{R}^3) \cap C^1_{\mathcal{I}} H^{m-1}(\mathbb{R}^3; \mathbb{R}^3)]$, where $\mathcal{I} = [0, T]$, constructed in Theorem 2 are considered to satisfy the integrated versions of (19):

$$
\begin{cases}
\phi(t) = e^{(i+\varepsilon)t\Delta} \phi_0 + \int_0^t e^{(i+\varepsilon)(t-\tau)\Delta} f[\phi(\tau), \tilde{\mathbf{A}}(\tau)] \mathrm{d}\tau \\
\mathbf{A}(t) = \dot{\mathfrak{s}}(t/\alpha) \mathbf{a}_0 + \alpha\, \mathfrak{s}(t/\alpha) \dot{\mathbf{a}}_0 + 4\pi \int_0^t \mathfrak{s}((t-\tau)/\alpha) \Lambda_\varepsilon^{-1} \mathcal{P} \boldsymbol{\mathcal{J}}[\phi(\tau), \tilde{\mathbf{A}}(\tau)] \mathrm{d}\tau,
\end{cases}
$$

where

$$
f[\phi, \tilde{\mathbf{A}}] = -(i + \varepsilon) \left(\mathscr{L}(\tilde{\mathbf{A}}) + V(\underline{\mathbf{R}}, \mathcal{Z}) \right) \phi + \varepsilon \left(T_{\mathrm{P}}[\phi, \tilde{\mathbf{A}}] + V[\phi] \right) \phi,
$$

and $e^{(i+\varepsilon)t\Delta}$, $\mathfrak{s}(t) = \sin\left(\sqrt{-\Delta} t\right)/\sqrt{-\Delta}$, and $\dot{\mathfrak{s}}(t) = \cos\left(\sqrt{-\Delta} t\right)$ are all defined by their Fourier multipliers (or, equivalently, as convolutions against the respective kernels). In particular, $(\phi, \mathbf{A})$ satisfy (19) pointwise a.e. when $m = 2$.

Throughout the paper (and, in particular, Sect. 3) we will make repeated use of Sobolev inequalities, dispersive estimates for the heat kernel, the Strichartz estimate for the wave equation, and the Kato-Ponce commutator estimate. The Sobolev inequalities are completely standard, but they are worth recalling here. Let $1 \le p \le q$, $s \ge 0$. If $sp < d$ and $f \in W^{s,p}(\mathbb{R}^d)$, then

$$
\|f\|_q \lesssim \|f\|_{s,p} \quad \text{when} \quad p \le q \le \frac{dp}{d - sp}.
$$

The other valuable estimates mentioned above are listed as a series of lemmas below.

Lemma 2 (Generalized Kato-Ponce inequality). *Suppose* $1 < p < \infty$, $s \ge 0$, $\alpha \ge 0$, $\beta \ge 0$ *and* $\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{p}$ *with* $i = 1, 2$, $1 < q_1 \le \infty$, $1 < p_2 \le \infty$. *If* $\phi \in W^{s+\alpha, p_1} \cap W^{-\beta, p_2}$ *and* $\psi \in W^{s+\beta, q_2} \cap W^{-\alpha, q_1}$, *then*

$$
\begin{aligned}
\|(1-\Delta)^{\frac{s}{2}} (\phi\psi)\|_p \lesssim\; & \|(1-\Delta)^{\frac{s+\alpha}{2}} \phi\|_{p_1} \|(1-\Delta)^{-\frac{\alpha}{2}} \psi\|_{q_1} \\
& + \|(1-\Delta)^{-\frac{\beta}{2}} \phi\|_{p_2} \|(1-\Delta)^{\frac{s+\beta}{2}} \psi\|_{q_2}.
\end{aligned}
$$

The same conclusion holds for $(1-\Delta)^{\frac{s}{2}}$ *replaced with* $(-\Delta)^{\frac{s}{2}}$.

Proof. See [20, Theorem 2]. □

Lemma 3 (Dispersive Estimates for the Heat Kernel). *For any* $m \geq 0$, $1 < r \leq p \leq \infty$, *and* $f \in L^r(\mathbb{R}^d)$ *we have*

$$\|e^{t\Delta} f\|_{m,p} \lesssim t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} \left(1 + t^{-\frac{m}{2}}\right) \|f\|_r$$

Proof. This is a standard result and a proof can be found in [21, Chapter 2, Equation 2.15]. □

Lemma 4 (Energy Estimate for the Wave Equation). *Let* $k \in \{0, 1\}$ *and* $\mathcal{I} = [0, T]$ *for some* $T > 0$. *Then for* $m \in \mathbb{R}$, $(\mathbf{a}_0, \dot{\mathbf{a}}_0) \in H^m(\mathbb{R}^3; \mathbb{R}^3) \times H^{m-1}(\mathbb{R}^3; \mathbb{R}^3)$ *and* $\mathbf{F} \in L^1_{\mathcal{I}} H^{m-1}(\mathbb{R}^3; \mathbb{R}^3)$ *the function*

$$\mathbf{K}(t) = \dot{\mathfrak{s}}(t/\alpha)\mathbf{a}_0 + \alpha\, \mathfrak{s}(t/\alpha)\dot{\mathbf{a}}_0 + \frac{1}{\alpha}\int_0^t \mathfrak{s}((t-\tau)/\alpha)\mathbf{F}(\tau)\mathrm{d}\tau,$$

where $\dot{\mathfrak{s}}(t) = \cos(\sqrt{-\Delta}t) : H^m \to H^m$ *and* $\mathfrak{s}(t) = \frac{\sin(\sqrt{-\Delta}t)}{\sqrt{-\Delta}} : H^{m-1} \to H^m$ *are defined a Fourier multipliers for* $t \in \mathbb{R}$, *is contained in* $C_{\mathcal{I}} H^m(\mathbb{R}^3; \mathbb{R}^3) \cap C^1_{\mathcal{I}} H^{m-1}(\mathbb{R}^3; \mathbb{R}^3)$ *and satisfies the energy estimate*

$$\max_{k\in\{0,1\}} \|\partial_t^k \mathbf{K}\|_{\infty;m-k,2} \lesssim \|(\mathbf{a}_0, \dot{\mathbf{a}}_0)\|_{m,2\oplus m-1,2} + \|\mathbf{F}\|_{1;m-1,2}.$$

Proof. This lemma is stated as a special case of [22, Theorem 2.6]. For original proofs, see [23,24]. □

3. Local Well-Posedness of the ε-Modified System: The Contraction Mapping Argument

3.1. Technical estimates. This section is devoted to the derivation of several estimates, stated as Lemmas 5–8, for the right hand side of (19) in various Sobolev spaces. To obtain such estimates we will repeatedly make use of Lemmas 2 and 3. The estimates will be crucial for our proof of Theorem 2. Some remarks on a particular notation used in this subsection are in order. We will denote $\Lambda = \sqrt{1-\Delta}$ and $\dot{\Lambda} = \sqrt{-\Delta}$ and, for $k \in \{1, \ldots, N\}$, we will use the notation $\Lambda_k^s = (1 - \Delta_{\mathbf{x}_k})^{s/2}$ (likewise for $\dot{\Lambda}_k^s$), where $\Delta_{\mathbf{x}_k}$ is the Laplacian acting on the kth electron coordinates, $\Delta_{\mathbf{x}_k} = \sum_{j=1}^3 \partial^2_{x_k^j}$. We emphasize that if no subscript k is present on Λ, then the Laplacian in the definition of Λ is taken to be the full Laplacian acting on all the coordinates in the given context. The Λ_k notation should not be confused with the Λ_ε notation introduced for the ε-modified system (19).

Lemma 5 (Estimates for the Pauli Term). *Let* $m \in [1, \infty)$ *and* $N \geq 1$. *For all* $(\phi, \mathbf{A}) \in [H^m(\mathbb{R}^{3N})]^{2^N} \times H^m(\mathbb{R}^3; \mathbb{R}^3)$, *with* $\operatorname{div} \mathbf{A} = 0$ *and* $\mathbf{B} = \operatorname{curl} \mathbf{A}$, *and for each* $j \in \{1, \ldots, N\}$, *the operator* $\mathcal{L}_j(\mathbf{A})$ *given by*

$$\mathcal{L}_j(\mathbf{A}) = 2\mathbf{A}_j \cdot \mathbf{p}_j + |\mathbf{A}_j|^2 + \boldsymbol{\sigma}_j \cdot \mathbf{B}_j \tag{25}$$

satisfies the estimates

$$\|\mathcal{L}_j(\mathbf{A})\phi\|_{m-1,\frac{3}{2}} \lesssim (1 + \|\mathbf{A}\|_{m,2})\|\mathbf{A}\|_{m,2}\|\phi\|_{m,2}, \tag{26}$$

and

$$\|e^{t\Delta}\mathcal{L}_j(\mathbf{A})\phi\|_{m,2} \lesssim t^{-\frac{1}{4}}\left[1+t^{-\frac{1}{2}}\right]\left(1+\|\mathbf{A}\|_{m,2}\right)\|\mathbf{A}\|_{m,2}\|\phi\|_{m,2}, \tag{27}$$

for all $t > 0$. *Furthermore, for* $(\phi, \mathbf{A}), (\phi', \mathbf{A}') \in [H^1(\mathbb{R}^{3N})]^{2^N} \times H^1(\mathbb{R}^3; \mathbb{R}^3)$, *with* $\operatorname{div}\mathbf{A} = \operatorname{div}\mathbf{A}' = 0$, *and each* $j \in \{1, \ldots, N\}$, *we have, for all* $t > 0$,

$$\begin{aligned}&\|e^{t\Delta}\left[\mathcal{L}_j(\mathbf{A})\phi - \mathcal{L}_j(\mathbf{A}')\phi'\right]\|_{1,2}\\ &\lesssim t^{-\frac{1}{4}}\left(1+t^{-\frac{1}{2}}\right)[(1+\|\mathbf{A}\|_{1,2}+\|\mathbf{A}'\|_{1,2})\|\phi'\|_{1,2}+(1+\|\mathbf{A}\|_{1,2})\|\mathbf{A}\|_{1,2}]\\ &\quad\times\max\{\|\phi-\phi'\|_{1,2}, \|\mathbf{A}-\mathbf{A}'\|_{1,2}\}.\end{aligned} \tag{28}$$

Proof. To show (26) it suffices to consider the case $N = 1$, as the general case follows in a similar fashion. We use Lemma 2 and the Sobolev inequality $H^1(\mathbb{R}^3) \subset L^r(\mathbb{R}^3)$, $2 \le r \le 6$, to prove (26)

$$\begin{aligned}\|\mathcal{L}(\mathbf{A})f\|_{m-1,\frac{3}{2}} &\lesssim \|\Lambda^{m-1}\mathbf{A}\|_6\|\mathbf{p}f\|_2 + \|\mathbf{A}\|_6\|\Lambda^{m-1}\mathbf{p}f\|_2 + \|\Lambda^{m-1}\mathbf{A}\|_6\|\mathbf{A}f\|_2\\ &\quad+\|\mathbf{A}\|_6\|\Lambda^{m-1}(\mathbf{A}f)\|_2 + \|\Lambda^{m-1}\mathbf{B}\|_2\|f\|_6 + \|\mathbf{B}\|_2\|\Lambda^{m-1}f\|_6\\ &\lesssim \|\mathbf{A}\|_{m,2}\left(2\|f\|_{m,2}+\|\mathbf{A}\|_6\|f\|_3+\|f\|_{1,2}\right)\\ &\quad+\|\mathbf{A}\|_{1,2}\left(\|f\|_{m,2}+\|\Lambda^{m-1}\mathbf{A}\|_6\|f\|_3+\|\mathbf{A}\|_6\|\Lambda^{m-1}f\|_3\right)\\ &\lesssim (1+\|\mathbf{A}\|_{m,2})\|\mathbf{A}\|_{m,2}\|f\|_{m,2}.\end{aligned}$$

To prove (27), fix $j \in \{1, \ldots, N\}$, and note that

$$\|e^{t\Delta}\mathcal{L}_j(\mathbf{A})\phi\|_{m,2} \lesssim \sum_{k=1}^{N}\|\Lambda_k^m e^{t\Delta}\mathcal{L}_j(\mathbf{A})\phi\|_2. \tag{29}$$

We separate into two cases: (a) $k \neq j$ and (b) $k = j$. For case (a) we use Lemma 3 and (26) to find

$$\begin{aligned}\|\Lambda_k^m e^{t\Delta}\mathcal{L}_j(\mathbf{A})\phi\|_2 &\le \|\Lambda_k e^{t\Delta_{\mathbf{x}_k}}\mathcal{L}_j(\mathbf{A})\Lambda_k^{m-1}\phi\|_2\\ &\lesssim t^{-\frac{1}{4}}\left[1+t^{-\frac{1}{2}}\right]\|\mathcal{L}_j(\mathbf{A})\Lambda_k^{m-1}\phi\|_{\frac{3}{2}}\\ &\lesssim t^{-\frac{1}{4}}\left[1+t^{-\frac{1}{2}}\right](1+\|\mathbf{A}\|_{1,2})\|\mathbf{A}\|_{1,2}\|\Lambda_k^{m-1}\phi\|_{1,2}\\ &\lesssim t^{-\frac{1}{4}}\left[1+t^{-\frac{1}{2}}\right](1+\|\mathbf{A}\|_{1,2})\|\mathbf{A}\|_{1,2}\|\phi\|_{m,2}.\end{aligned} \tag{30}$$

For case (b) we use Lemmas 2 and 3, and the estimate (26), to find

$$\begin{aligned}\|\Lambda_j^m e^{t\Delta}\mathcal{L}_j(\mathbf{A})\phi\|_2 &= \|\Lambda_j e^{t\Delta_{\mathbf{x}_j}}\Lambda_j^{m-1}(\mathcal{L}_j(\mathbf{A})\phi)\|_2\\ &\lesssim t^{-\frac{1}{4}}\left[1+t^{-\frac{1}{2}}\right]\|\Lambda_j^{m-1}(\mathcal{L}_j(\mathbf{A})\phi)\|_{\frac{3}{2}}\\ &\lesssim t^{-\frac{1}{4}}\left[1+t^{-\frac{1}{2}}\right](1+\|\mathbf{A}\|_{m,2})\|\mathbf{A}\|_{m,2}\|\phi\|_{m,2}.\end{aligned} \tag{31}$$

Combining (29) through (31) we arrive at (28).

To prove (29) we write

$$\mathcal{L}_j(\mathbf{A})\phi - \mathcal{L}_j(\mathbf{A}')\phi' = L_{1,j}[\phi - \phi', \mathbf{A}] + L_{2,j}[\phi, \mathbf{A} - \mathbf{A}']$$

where

$$\begin{aligned} L_{1,j}[\phi - \phi', \mathbf{A}] &= 2\mathbf{A}_j \cdot \mathbf{p}_j(\phi - \phi') + |\mathbf{A}_j|^2(\phi - \phi') + \boldsymbol{\sigma}_j \cdot \mathbf{B}_j(\phi - \phi'), \\ L_{2,j}[\phi', \mathbf{A} - \mathbf{A}'] &= 2(\mathbf{A}_j - \mathbf{A}'_j) \cdot \mathbf{p}_j\phi' + (|\mathbf{A}|_j^2 - |\mathbf{A}'|_j^2)\phi' + \boldsymbol{\sigma}_j \cdot (\mathbf{B}_j - \mathbf{B}'_j)\phi'. \end{aligned}$$

Using Hölder's inequality and the Sobolev inequality $H^1(\mathbb{R}^3) \subset L^r(\mathbb{R}^3)$, $2 \leq r \leq 6$, to find

$$\begin{aligned} \|L_{1,j}[\phi - \phi', \mathbf{A}]\|_{\frac{3}{2}} &\lesssim \|\mathbf{A}\|_6\|\phi - \phi'\|_{1,2} + \|\mathbf{A}\|_4^2\|\phi - \phi'\|_6 + \|\mathbf{B}\|_2\|(\phi - \phi')\|_6 \\ &\lesssim \left(2 + \|\mathbf{A}\|_{1,2}\right)\|\mathbf{A}\|_{1,2}\|(\phi - \phi')\|_{1,2}. \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\|L_{2,j}[\phi', \mathbf{A} - \mathbf{A}']\|_{\frac{3}{2}} \\ &\quad\lesssim \|\mathbf{A} - \mathbf{A}'\|_6\|\phi'\|_{1,2} + \|\mathbf{A} - \mathbf{A}'\|_3(\|\mathbf{A}\|_6 + \|\mathbf{A}'\|_6)\|\phi'\|_6 + \|\mathbf{B} - \mathbf{B}'\|_2\|\phi'\|_6 \\ &\quad\lesssim \left(2 + (\|\mathbf{A}\|_{1,2} + \|\mathbf{A}'\|_{1,2})\right)\|\phi'\|_{1,2}\|\mathbf{A} - \mathbf{A}'\|_{1,2}. \end{aligned} \tag{33}$$

Lemma 3 gives

$$\|e^{t\Delta}\left[\mathcal{L}_j(\mathbf{A})\phi - \mathcal{L}_j(\mathbf{A}')\phi'\right]\|_{1,2} \lesssim t^{-\frac{1}{4}}\left[1 + t^{-\frac{1}{2}}\right]\|\mathcal{L}_j(\mathbf{A})\phi - \mathcal{L}_j(\mathbf{A}')\phi'\|_{\frac{3}{2}},$$

which, together with (32) and (33) allows us to conclude (28). □

Lemma 6 (Estimates for the Coulomb Term). *Fix $m \in [1, 2]$ and let $N, K \geq 1$, $\mathcal{Z} \in [0, \infty)^K$, and $\underline{\mathbf{R}} = (\mathbf{R}_1, \ldots, \mathbf{R}_K) \in \mathbb{R}^{3K}$, with $\mathbf{R}_i \neq \mathbf{R}_j$ for all $i \neq j$. Then, for all $\phi \in H^m(\mathbb{R}^{3N}; \mathbb{C})$, the operator $V(\underline{\mathbf{R}}, \mathcal{Z})$ given by (3), satisfies the estimate*

$$\|e^{t\Delta}V(\underline{\mathbf{R}}, \mathcal{Z})\phi\|_{m,2} \lesssim \left[1 + \left(1 + t^{-\frac{1}{2}}\right)\left(t^{-\frac{9}{20}} + t^{-\frac{1}{4}}\right)\right]\|\phi\|_{m,2}, \tag{34}$$

for all $t > 0$.

Proof. To prove (34) we need to first prove the following inequalities. Let $v : \mathbb{R}^3 \to \mathbb{R}^3$ be the function $v(\mathbf{x}) = |\mathbf{x}|^{-1}$. Then, for all $\psi \in H^m(\mathbb{R}^3; \mathbb{C})$, we have

$$\|v\psi\|_{\frac{3}{2}} \lesssim \|\psi\|_{1,2} \tag{35}$$

and

$$\|v\psi\|_{m-1,\frac{5}{4}} \lesssim \|\psi\|_{m,2}. \tag{36}$$

Moreover, for all $\psi \in H^m(\mathbb{R}^6; \mathbb{C})$, we have

$$\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\left|\frac{\psi(\mathbf{x}_1, \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|}\right|^{\frac{3}{2}}\mathrm{d}\mathbf{x}_1\right)^{\frac{4}{3}}\mathrm{d}\mathbf{x}_2 \lesssim \|\Lambda_1\psi\|_2^2 \tag{37}$$

and

$$\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\left|\Lambda_1^{m-1}\frac{\psi(\mathbf{x}_1,\mathbf{x}_2)}{|\mathbf{x}_1-\mathbf{x}_2|}\right|^{\frac{5}{4}}\mathrm{d}\mathbf{x}_1\right)^{\frac{8}{5}}\mathrm{d}\mathbf{x}_2 \lesssim \|\Lambda_1^m\psi\|_2^2. \tag{38}$$

Let B_1 denote the unit ball in $\mathbb{R}^3$, and $B_1^c = \mathbb{R}^3\backslash B_1$. Using Hölder's inequality we find

$$\begin{aligned}\|v\psi\|_{\frac{3}{2}}^{\frac{3}{2}} &= \int_{B_1}\frac{|\psi(\mathbf{x})|^{\frac{3}{2}}}{|\mathbf{x}|^{\frac{3}{2}}}\mathrm{d}\mathbf{x} + \int_{B_1^c}\frac{|\psi(\mathbf{x})|^{\frac{3}{2}}}{|\mathbf{x}|^{\frac{3}{2}}}\mathrm{d}\mathbf{x}\\ &\le \left(\int_{B_1}|\mathbf{x}|^{-2}\mathrm{d}\mathbf{x}\right)^{\frac{3}{4}}\|\psi\|_{L^6(B_1)}^{\frac{3}{2}} + \left(\int_{B_1^c}|\mathbf{x}|^{-6}\mathrm{d}\mathbf{x}\right)^{\frac{1}{4}}\|\psi\|_{L^2(B_1^c)}^{\frac{3}{2}}\\ &\lesssim \|\psi\|_6^{\frac{3}{2}} + \|\psi\|_2^{\frac{3}{2}}.\end{aligned} \tag{39}$$

The estimate (39) and the Sobolev inequality $\|\psi\|_6 \lesssim \|\nabla\psi\|_2$ imply (35).

For estimate (36) we focus on the case $m = 2$, as the $m = 1$ case is proved in the same way as (35) and then general case $m \in (1, 2)$ will follow similarly. Below we will make use of the homogeneous Sobolev space $\dot{W}^{1,\frac{5}{4}}(\mathbb{R}^3)$ defined through the seminorm $\|f\|_{\dot{W}^{1,5/4}(\mathbb{R}^3)} = \|\dot{\Lambda} f\|_{\frac{5}{4}}$. As before, we write

$$\|v\psi\|_{\dot{W}^{1,\frac{5}{4}}(\mathbb{R}^3)}^{\frac{5}{4}} = \|v\psi\|_{\dot{W}^{1,\frac{5}{4}}(B_1)}^{\frac{5}{4}} + \|v\psi\|_{\dot{W}^{1,\frac{5}{4}}(B_1^c)}^{\frac{5}{4}}. \tag{40}$$

We argue, separately, that both terms on the right hand side of (37) are bounded by $\|\psi\|_{2,2}$. For this it will be useful to remind ourselves of the identity $\dot{\Lambda}|\mathbf{x}|^{-1} = C|\mathbf{x}|^{-2}$ where C is a nonessential constant. To show

$$\|v\psi\|_{\dot{W}^{1,\frac{5}{4}}(B_1^c)} \lesssim \|\psi\|_{2,2} \tag{41}$$

we use Lemma 2 to find

$$\begin{aligned}\|v\psi\|_{\dot{W}^{1,\frac{5}{4}}(B_1^c)} &\lesssim \|\dot{\Lambda} v\|_{L^{\frac{10}{3}}(B_1^c)}\|\psi\|_{L^2(B_1^c)} + \|v\|_{L^{\frac{10}{3}}(B_1^c)}\|\dot{\Lambda}\psi\|_{L^2(B_1^c)}\\ &\lesssim \|v^2\|_{L^{\frac{10}{3}}(B_1^c)}\|\psi\|_2 + \|v\|_{L^{\frac{10}{3}}(B_1^c)}\|\psi\|_{1,2}.\end{aligned} \tag{42}$$

Since $\||\cdot|^{-k}\|_{L^{\frac{10}{3}}(B_1^c)} < \infty$ for $k \in \{1, 2\}$, (42) implies (41).

Showing the inequality

$$\|v\psi\|_{\dot{W}^{1,\frac{5}{4}}(B_1)} \lesssim \|\psi\|_{2,2}. \tag{43}$$

follows in a similar fashion. Indeed, using Lemma 2 we find

$$\begin{aligned}\|v\psi\|_{\dot{W}^{1,\frac{5}{4}}(B_1)} &\lesssim \|\dot{\Lambda} v\|_{L^{\frac{5}{4}}(B_1)}\|\psi\|_{L^\infty(B_1)} + \|v\|_{L^{\frac{30}{19}}(B_1)}\|\dot{\Lambda}\psi\|_{L^6(B_1)}\\ &\lesssim \|v^2\|_{L^{\frac{5}{4}}(B_1)}\|\psi\|_{L^\infty(B_1)} + \|v\|_{L^{\frac{30}{19}}(B_1)}\|\Lambda^2\psi\|_2.\end{aligned} \tag{44}$$

Estimate (44), together with the Sobolev inequality $\|\psi\|_\infty \lesssim \|\psi\|_{2,2}$ and the observation that $\max\{\|v^2\|_{L^{\frac{5}{4}}(B_1)}, \|v\|_{L^{\frac{30}{19}}(B_1)}\} < \infty$, implies (43). With (40), (41), and (43) we are able to conclude $\|v\psi\|_{\dot{W}^{1,\frac{5}{4}}} \lesssim \|\psi\|_{2,2}$.

Proving (37) is similar to showing (35). Indeed, using Hölder's inequality and the Sobolev inequality $\|f\|_6 \lesssim \|\nabla f\|_2$ we find

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\left|\frac{\psi(\mathbf{x}_1,\mathbf{x}_2)}{|\mathbf{x}_1-\mathbf{x}_2|}\right|^{\frac{3}{2}}\mathrm{d}\mathbf{x}_1\right)^{\frac{4}{3}}\mathrm{d}\mathbf{x}_2\\
&=\int_{\mathbb{R}^3}\left(\int_{B_1}\left|\frac{\psi(\mathbf{y}+\mathbf{x}_2,\mathbf{x}_2)}{|\mathbf{y}|}\right|^{\frac{3}{2}}\mathrm{d}\mathbf{y}+\int_{B_1^c}\left|\frac{\psi(\mathbf{y}+\mathbf{x}_2,\mathbf{x}_2)}{|\mathbf{y}|}\right|^{\frac{3}{2}}\mathrm{d}\mathbf{y}\right)^{\frac{4}{3}}\mathrm{d}\mathbf{x}_2\\
&\lesssim\int_{\mathbb{R}^3}\left(\left(\int_{\mathbb{R}^3}|\mathbf{p}_1\psi(\mathbf{x}_1,\mathbf{x}_2)|^2\mathrm{d}\mathbf{x}_1\right)^{\frac{3}{4}}+\left(\int_{\mathbb{R}^3}|\psi(\mathbf{x}_1,\mathbf{x}_2)|^2\mathrm{d}\mathbf{x}_1\right)^{\frac{3}{4}}\right)^{\frac{4}{3}}\mathrm{d}\mathbf{x}_2\\
&\lesssim\|\Lambda_1\psi\|_2^2.
\end{aligned}$$

To show estimate (38) one combines the strategy used to show (36) and (37).

With estimates (35) through (38) at our disposal we may prove (34). We split $V(\underline{\mathbf{R}},\mathcal{Z})$ into three pieces: $V(\underline{\mathbf{R}},\mathcal{Z})=\sum_{n=1}^{3}V_n(\underline{\mathbf{R}},\mathcal{Z})$ where

$$\begin{aligned}
V_1(\underline{\mathbf{R}},\mathcal{Z})&=\sum_{1\le i<j\le N}\frac{1}{|\mathbf{x}_i-\mathbf{x}_j|},\\
V_2(\underline{\mathbf{R}},\mathcal{Z})&=-\sum_{i=1}^{N}\sum_{j=1}^{K}\frac{Z_j}{|\mathbf{x}_i-\mathbf{R}_j|},\\
V_3(\underline{\mathbf{R}},\mathcal{Z})&=\sum_{1\le i<j\le K}\frac{Z_iZ_j}{|\mathbf{R}_i-\mathbf{R}_j|}.
\end{aligned}$$

We show (34) with $V(\underline{\mathbf{R}},\mathcal{Z})$ replaced by $V_n(\underline{\mathbf{R}},\mathcal{Z})$, $n=1,2,3$. The estimate is trivial for $V_3(\underline{\mathbf{R}},\mathcal{Z})$ since $\underline{\mathbf{R}}$ is fixed. Indeed, we find

$$\|e^{t\Delta}V_3(\underline{\mathbf{R}},\mathcal{Z})\phi\|_{m,2}\le\left(\sum_{i,j=1}^{K}\frac{Z_iZ_j}{|\mathbf{R}_i-\mathbf{R}_j|}\right)\|e^{t\Delta}\phi\|_{m,2}\lesssim\|\phi\|_{m,2}. \tag{45}$$

For $V_2(\underline{\mathbf{R}},\mathcal{Z})$, the desired estimate is equivalent to controlling $\|e^{t\Delta}|\mathbf{x}_i|^{-1}\phi\|_{m,2}$ by $\|\phi\|_{m,2}$ for each $i=1,\dots,N$. For this, fix $i\in\{1,\dots,N\}$ and note that

$$\|e^{t\Delta}|\mathbf{x}_i|^{-1}\phi\|_{m,2}\lesssim\sum_{k=1}^{N}\|\Lambda_k^m e^{t\Delta}|\mathbf{x}_i|^{-1}\phi\|_2. \tag{46}$$

To estimate the right hand side of (46) we consider two cases: (a) $k \neq i$ and (b) $k = i$. For case (a), we use Lemma 3 and the estimate (35) to find

$$\begin{aligned}
\|\Lambda_k^m e^{t\Delta}|\mathbf{x}_i|^{-1}\phi\|_2 &\leq \|e^{t\Delta_{\mathbf{x}_i}}|\mathbf{x}_i|^{-1}\Lambda_k^m e^{t\Delta_{\mathbf{x}_k}}\phi\|_2 \\
&\lesssim t^{-\frac{1}{4}}\left(\int_{\mathbb{R}^{3(N-1)}}\left(\int_{\mathbb{R}^3}\left||\mathbf{x}_i|^{-1}\Lambda_k^m e^{t\Delta_{\mathbf{x}_k}}\phi(\underline{\mathbf{x}})\right|^{\frac{3}{2}}\mathrm{d}\mathbf{x}_i\right)^{\frac{4}{3}}\mathrm{d}\underline{\mathbf{x}}_i'\right)^{\frac{1}{2}} \\
&\lesssim t^{-\frac{1}{4}}\|\Lambda_k e^{t\Delta_{\mathbf{x}_k}}\Lambda_i\Lambda_k^{m-1}\phi\|_2 \\
&\lesssim t^{-\frac{1}{4}}\left[1+t^{-\frac{1}{2}}\right]\|\Lambda_i\Lambda_k^{m-1}\phi\|_2 \\
&\lesssim t^{-\frac{1}{4}}\left[1+t^{-\frac{1}{2}}\right]\|\phi\|_{m,2}.
\end{aligned} \tag{47}$$

For case (b) the estimating is similar to that of (47). Using (36) we find

$$\begin{aligned}
\|\Lambda_i^m e^{t\Delta}|\mathbf{x}_i|^{-1}\phi\|_2 &\lesssim \|\Lambda_i e^{t\Delta_{\mathbf{x}_i}}\Lambda_i^{m-1}|\mathbf{x}_i|^{-1}\phi\|_2 \\
&\lesssim t^{-\frac{9}{20}}\left[1+t^{-\frac{1}{2}}\right]\left(\int_{\mathbb{R}^{3(N-1)}}\left(\int_{\mathbb{R}^3}\left|\Lambda_i^{m-1}\frac{\phi(\underline{\mathbf{x}})}{|\mathbf{x}_i|}\right|^{\frac{5}{4}}\mathrm{d}\mathbf{x}_i\right)^{\frac{8}{5}}\mathrm{d}\underline{\mathbf{x}}_i'\right)^{\frac{1}{2}} \\
&\lesssim t^{-\frac{9}{20}}\left[1+t^{-\frac{1}{2}}\right]\|\Lambda_i^m\phi\|_2 \\
&\lesssim t^{-\frac{9}{20}}\left[1+t^{-\frac{1}{2}}\right]\|\phi\|_{m,2}.
\end{aligned} \tag{48}$$

Combining estimates (47) and (48) we arrive at

$$\begin{aligned}
\|e^{t\Delta}V_2(\underline{\mathbf{R}},\mathcal{Z})\phi\|_{m,2} &\leq \sum_{i=1}^{N}\sum_{j=1}^{K} Z_j\|e^{t\Delta}|\mathbf{x}_i-\mathbf{R}_j|^{-1}\phi\|_{m,2} \\
&\lesssim \left(1+t^{-\frac{1}{2}}\right)\left(t^{-\frac{9}{20}}+t^{-\frac{1}{4}}\right)\|\phi\|_{m,2}.
\end{aligned} \tag{49}$$

Finally we need to control $\|e^{t\Delta}|\mathbf{x}_i-\mathbf{x}_j|^{-1}\phi\|_{m,2}$ by $\|\phi\|_{m,2}$ for each $i, j = 1, \ldots, N$ with $i \neq j$. The estimates involved are similar to those involved with controlling $\|e^{t\Delta}V_2(\underline{\mathbf{R}},\mathcal{Z})\phi\|_{m,2}$, and thus we choose to be brief with the computations. Fix $(i, j) \in \{1, \ldots, N\}^2$ with $i \neq j$. Note that

$$\|e^{t\Delta}|\mathbf{x}_i-\mathbf{x}_j|^{-1}\phi\|_{m,2} \lesssim \sum_{k=1}^{N}\|\Lambda_k^m e^{t\Delta}|\mathbf{x}_i-\mathbf{x}_j|^{-1}\phi\|_2. \tag{50}$$

Estimating the right hand side of (50) is similar to estimating the right hand side of (46). We again consider two cases: (a) $k \neq j, i$ and (b) $k = j, i$. For case (a) we use Lemma 3 and (37) to find

$$
\begin{aligned}
\|\Lambda_k^m e^{t\Delta}\frac{\phi}{|\mathbf{x}_i-\mathbf{x}_j|}\|_2 &\leq \|e^{t\Delta_{\mathbf{x}_i}}\frac{\Lambda_k^m e^{t\Delta_{\mathbf{x}_k}}\phi}{|\mathbf{x}_i-\mathbf{x}_j|}\|_2 \\
&\lesssim t^{-\frac{1}{4}}\left(\int_{\mathbb{R}^{3(N-1)}}\left(\int_{\mathbb{R}^3}\left|\frac{\Lambda_k^m e^{t\Delta_{\mathbf{x}_k}}\phi(\underline{\mathbf{x}})}{|\mathbf{x}_i-\mathbf{x}_j|}\right|^{\frac{3}{2}}\mathrm{d}\mathbf{x}_i\right)^{\frac{4}{3}}\mathrm{d}\underline{\mathbf{x}}_i'\right)^{\frac{1}{2}} \\
&\lesssim t^{-\frac{1}{4}}\|\Lambda_i\Lambda_k^m e^{t\Delta_{\mathbf{x}_k}}\phi\|_2 \\
&\lesssim t^{-\frac{1}{4}}[1+t^{-\frac{1}{2}}]\|\Lambda_i\Lambda_k^{m-1}\phi\|_2 \\
&\lesssim t^{-\frac{1}{4}}[1+t^{-\frac{1}{2}}]\|\phi\|_{m,2}.
\end{aligned} \tag{51}
$$

For case (b) the estimating is similar. We choose $k=i$, and note that the case $k=j$ is identical by symmetry. Using Lemma 3 and (38) we find

$$
\begin{aligned}
\|\Lambda_i^m e^{t\Delta}\frac{\phi}{|\mathbf{x}_i-\mathbf{x}_j|}\|_2 &\leq \|\Lambda_i e^{t\Delta_{\mathbf{x}_i}}\Lambda_i^{m-1}|\mathbf{x}_i-\mathbf{x}_j|^{-1}\phi\|_2 \\
&\lesssim t^{-\frac{9}{20}}[1+t^{-\frac{1}{2}}]\left(\int_{\mathbb{R}^{3(N-1)}}\left(\int_{\mathbb{R}^3}\left|\Lambda_i^{m-1}\frac{\phi(\underline{\mathbf{x}})}{|\mathbf{x}_i-\mathbf{x}_j|}\right|^{\frac{5}{4}}\mathrm{d}\mathbf{x}_j\right)^{\frac{8}{5}}\mathrm{d}\underline{\mathbf{x}}_i'\right)^{\frac{1}{2}} \\
&\lesssim t^{-\frac{9}{20}}[1+t^{-\frac{1}{2}}]\|\Lambda_i^m\phi\|_2 \\
&\lesssim t^{-\frac{9}{20}}[1+t^{-\frac{1}{2}}]\|\phi\|_{m,2}.
\end{aligned} \tag{52}
$$

Combining estimates (51) and (52) we arrive at

$$
\begin{aligned}
\|e^{t\Delta}V_3(\underline{\mathbf{R}},\mathcal{Z})\phi\|_{m,2} &\leq \sum_{1\leq i<j\leq N}\|e^{t\Delta}|\mathbf{x}_i-\mathbf{x}_j|^{-1}\phi\|_{m,2} \\
&\lesssim \left(1+t^{-\frac{1}{2}}\right)\left(t^{-\frac{9}{20}}+t^{-\frac{1}{4}}\right)\|\phi\|_{m,2}.
\end{aligned} \tag{53}
$$

Collecting estimates (45), (49), and (53) we arrive at (34). □

Lemma 7 (Estimates for the Energies). *Fix $\varepsilon>0$, $N,K\geq 1$, and let and $\mathcal{Z}\in[0,\infty)^K$, $\underline{\mathbf{R}}=(\mathbf{R}_1,\ldots,\mathbf{R}_K)\in\mathbb{R}^{3K}$, with $\mathbf{R}_i\neq\mathbf{R}_j$ for all $i\neq j$. For all $(\phi,\mathbf{A})\in[H^1(\mathbb{R}^{3N})]^{2^N}\times\dot{H}^1(\mathbb{R}^3;\mathbb{R}^3)$, with $\operatorname{div}\mathbf{A}=0$, the kinetic energy $T_{\mathrm{P}}=T_{\mathrm{P}}[\phi,\mathbf{A}]$, as defined in (6), and the potential energy $V=V[\phi]$, as defined in (7), satisfy the estimates*

$$
T_{\mathrm{P}}\lesssim(1+\|\nabla\mathbf{A}\|_2)^2\|\phi\|_{1,2}^2 \quad\text{and}\quad V\lesssim\|\phi\|_{1,2}^2, \tag{54}
$$

respectively. Moreover, for all $(\phi,\mathbf{A}),(\phi',\mathbf{A}')\in[H^1(\mathbb{R}^{3N})]^{2^N}\times\dot{H}^1(\mathbb{R}^3;\mathbb{R}^3)$, the difference of the total kinetic energies and potential energies $T_{\mathrm{P}}-T_{\mathrm{P}}'+V-V'\equiv T_{\mathrm{P}}[\phi,\mathbf{A}]-T_{\mathrm{P}}[\phi',\mathbf{A}']+V[\phi]-V[\phi']$ satisfies the estimate

$$
\begin{aligned}
&|T_{\mathrm{P}}-T_{\mathrm{P}}'+V-V'| \\
&\quad\lesssim\omega(\|\phi\|_{1,2},\|\phi'\|_{1,2},\|\nabla\mathbf{A}\|_2,\|\nabla\mathbf{A}'\|_2)\max\{\|\phi-\phi'\|_{1,2},\|\nabla(\mathbf{A}-\mathbf{A}')\|_2\},
\end{aligned} \tag{55}
$$

where

$$
\omega(x_1,x_2,x_3,x_4)=(1+x_2+x_3)\left[(1+x_3)x_1+(1+x_4)x_2\right]+(x_1+x_2). \tag{56}
$$

Proof. To show the first estimate in (54) it suffices to prove the $N = 1$ case, as for general $N \geq 1$ the estimating goes in a similar fashion. Using Hölder's inequality and Sobolev's inequality $H^1(\mathbb{R}^3) \subset L^r(\mathbb{R}^3)$, $1 \leq r \leq 6$, we find

$$\begin{aligned}\|\boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A})\phi\|_2 &\leq \|\mathbf{p}\phi\|_2 + \|\mathbf{A}\phi\|_2 \\ &\lesssim \|\phi\|_{1,2} + \|\mathbf{A}\|_6 \|\phi\|_3 \\ &\lesssim (1 + \|\nabla \mathbf{A}\|_2)\|\phi\|_{1,2}.\end{aligned}$$

To show the second estimate in (54), first note that

$$V[\phi] \leq \sum_{1 \leq i < j \leq N} \langle \phi, |\mathbf{x}_i - \mathbf{x}_j|^{-1}\phi\rangle_{L^2} + \left(\sum_{1 \leq i < j \leq K} \frac{Z_i Z_j}{|\mathbf{R}_i - \mathbf{R}_j|} \right) \|\phi\|_2^2. \tag{57}$$

Considering (57) we focus on controlling the electron–electron repulsion energy since the nuclei–nuclei repulsion energy is trivially bounded by $\|\phi\|_{1,2}$. The desired estimate on the electron–electron repulsion energy follows from the uncertainty principle for Hydrogen, namely $\langle \psi, |\mathbf{x}|^{-1}\psi\rangle \leq \|\psi\|_2 \|\nabla\psi\|_2$. It suffices to consider the case $N = 2$. Using Hölder's inequality and Sobolev's inequality we find

$$\begin{aligned}&\langle \phi, |\mathbf{x}_1 - \mathbf{x}_2|^{-1}\phi\rangle_{L^2} \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\phi(\mathbf{y} + \mathbf{x}_2, \mathbf{x}_2)|^2}{|\mathbf{y}|} \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}_2 \\ &\leq \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |\phi(\mathbf{y} + \mathbf{x}_2, \mathbf{x}_2)|^2 \mathrm{d}\mathbf{y} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\mathbf{p}_1\phi(\mathbf{y} + \mathbf{x}_2, \mathbf{x}_2)|^2 \mathrm{d}\mathbf{y} \right)^{\frac{1}{2}} \mathrm{d}\mathbf{x}_2 \\ &\leq \frac{1}{2}\left(\|\phi\|_2^2 + \|\mathbf{p}_1\phi\|_2^2 \right).\end{aligned} \tag{58}$$

Estimates (57) and (58) imply the second estimate in (54).

To estimate $T_{\mathrm{P}} - T'_{\mathrm{P}}$ it suffices to consider the $N = 1$ case. Write $T_{\mathrm{P}} - T'_{\mathrm{P}} = \sum_{k=1}^{6} T_k$ where

$$\begin{aligned}T_1[\phi, \phi', \mathbf{A}, \mathbf{A}'] &= \langle \boldsymbol{\sigma} \cdot \mathbf{p}(\phi - \phi'), \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A})\phi\rangle, \\ T_2[\phi, \phi', \mathbf{A}, \mathbf{A}'] &= \langle \boldsymbol{\sigma} \cdot (\mathbf{A} - \mathbf{A}')\phi', \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A})\phi\rangle, \\ T_3[\phi, \phi', \mathbf{A}, \mathbf{A}'] &= \langle \boldsymbol{\sigma} \cdot \mathbf{A}(\phi - \phi'), \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A})\phi\rangle, \\ T_4[\phi, \phi', \mathbf{A}, \mathbf{A}'] &= \langle \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}')\phi', \boldsymbol{\sigma} \cdot \mathbf{p}(\phi - \phi')\rangle, \\ T_5[\phi, \phi', \mathbf{A}, \mathbf{A}'] &= \langle \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}')\phi', \boldsymbol{\sigma} \cdot (\mathbf{A} - \mathbf{A}')\phi'\rangle, \\ T_6[\phi, \phi', \mathbf{A}, \mathbf{A}'] &= \langle \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}')\phi', \boldsymbol{\sigma} \cdot \mathbf{A}(\phi - \phi')\rangle.\end{aligned}$$

Using Cauchy–Schwartz together with first estimate in (54) we find

$$T_1[\phi, \phi', \mathbf{A}, \mathbf{A}'] \lesssim (1 + \|\nabla \mathbf{A}\|_2)\|\phi\|_{1,2}\|\phi - \phi'\|_{1,2}, \tag{59}$$

$$T_2[\phi, \phi', \mathbf{A}, \mathbf{A}'] \lesssim (1 + \|\nabla \mathbf{A}\|_2)\|\phi\|_{1,2}\|\phi'\|_{1,2}\|\nabla(\mathbf{A} - \mathbf{A}')\|_2, \tag{60}$$

$$T_3[\phi, \phi', \mathbf{A}, \mathbf{A}'] \lesssim (1 + \|\nabla \mathbf{A}\|_2)\|\phi\|_{1,2}\|\nabla \mathbf{A}\|_2\|\phi - \phi'\|_{1,2}, \tag{61}$$

$$T_4[\phi, \phi', \mathbf{A}, \mathbf{A}'] \lesssim (1 + \|\nabla \mathbf{A}'\|_2)\|\phi'\|_{1,2}\|\phi - \phi'\|_{1,2}, \tag{62}$$

$$T_5[\phi, \phi', \mathbf{A}, \mathbf{A}'] \lesssim (1 + \|\nabla \mathbf{A}'\|_2)\|\phi'\|_{1,2}^2\|\nabla(\mathbf{A} - \mathbf{A}')\|_2, \tag{63}$$

$$T_6[\phi, \phi', \mathbf{A}, \mathbf{A}'] \lesssim (1 + \|\nabla \mathbf{A}'\|_2)\|\phi'\|_{1,2}\|\nabla \mathbf{A}\|_2\|\phi - \phi'\|_{1,2}. \tag{64}$$

Collecting estimates (59) through (64) we conclude

$$|T_\mathrm{P} - T'_\mathrm{P}| \lesssim \omega_1(\|\phi\|_{1,2}, \|\phi'\|_{1,2}, \|\nabla \mathbf{A}\|_2, \|\nabla \mathbf{A}'\|_2) \\ \times \max\{\|\phi - \phi'\|_{1,2}, \|\nabla(\mathbf{A} - \mathbf{A}')\|_2\} \tag{65}$$

where ω_1 function

$$\omega_1(x, y, z, w) = (1 + y + z)\,[(1 + z)x + (1 + w)y]\,.$$

To estimate $V - V'$, write $V - V' = V_1 + V_2$ where

$$V_1[\phi, \phi'] = \langle \phi - \phi', V(\underline{\mathbf{R}}, \mathcal{Z})\phi\rangle_{L^2}, \qquad V_2[\phi, \phi'] = \langle \phi', V(\underline{\mathbf{R}}, \mathcal{Z})(\phi - \phi')\rangle_{L^2}.$$

We want to control $\max\{V_1, V_2\}$ by $\|\phi\|_{1,2}$, $\|\phi'\|_{1,2}$, and $\|\phi - \phi'\|_{1,2}$. Therefore, we show the inequality

$$|\langle h, V(\underline{\mathbf{R}}, \mathcal{Z})g\rangle| \lesssim \|h\|_{1,2}\|g\|_{1,2}, \qquad \forall h, g \in H^1(\mathbb{R}^{3N}, \mathbb{C}). \tag{66}$$

Note that

$$\langle h, V(\underline{\mathbf{R}}, \mathcal{Z})g\rangle_{L^2} = \sum_{i<j}^{N} \langle h, |\mathbf{x}_i - \mathbf{x}_j|^{-1} g\rangle_{L^2} - \sum_{i=1}^{N}\sum_{j=1}^{K} Z_j \langle h, |\mathbf{x}_i - \mathbf{R}_j|^{-1} g\rangle_{L^2} \\ + \sum_{i<j}^{K} \frac{Z_i Z_j}{|\mathbf{R}_i - \mathbf{R}_j|}\langle h, g\rangle_{L^2}. \tag{67}$$

The third term on the right hand side of (67) is bounded by $\|g\|_2\|h\|_2$ via Cauchy–Schwartz. To estimate the second term on the right hand side of (67) it suffices to consider the case $N, K = 1$ and $\mathbf{R}_1 = 0$. Indeed, in this situation $\langle h, |\mathbf{x}|^{-1} g\rangle \lesssim \sqrt{\|h\|_6\|g\|_6\|h\|_2\|g\|_2}$. This follows by writing $\langle h, |\mathbf{x}|^{-1} g\rangle$ as the sum of an integral over the ball of radius R and its complement, using Hölder's inequality, and then optimizing over R. The desired estimate (66) then follows from the Sobolev inequality. Estimating the first term on the right hand side of (67) by $\|h\|_{1,2}\|g\|_{1,2}$ follows the same proof as that of (58). Hence (66) holds, and therefore

$$|V - V'| \lesssim |V_1| + |V_2| \lesssim (\|\phi\|_{1,2} + \|\phi'\|_{1,2})\|\phi - \phi'\|_{1,2}. \tag{68}$$

Collecting estimates (65) and (68), we arrive at (55). $\square$

Lemma 8 (Estimates for the Probability Current Density). *Fix $m \in [1, \infty)$ and $N \geq 1$. For all $(\phi, \mathbf{A}) \in [H^m(\mathbb{R}^{3N})]^{2^N} \times H^m(\mathbb{R}^3; \mathbb{R}^3)$, with $\operatorname{div}\mathbf{A} = 0$, and each $j \in \{1, \ldots, N\}$, the probability current density $\boldsymbol{\mathcal{J}}[\phi, \mathbf{A}] = \sum_{j=1}^{N} \mathbf{J}_j[\psi, \mathbf{A}]$ as given by (4) is in the Sobolev space $H^{m-2}(\mathbb{R}^3; \mathbb{R}^3)$ and satisfies the estimate*

$$\|\boldsymbol{\mathcal{J}}[\phi, \mathbf{A}]\|_{m-2,2} \lesssim (1 + \|\mathbf{A}\|_{m,2})\|\phi\|_{m,2}^2. \tag{69}$$

Moreover, for $(\phi, \mathbf{A}), (\phi', \mathbf{A}') \in [H^1(\mathbb{R}^{3N})]^{2^N} \times H^1(\mathbb{R}^3; \mathbb{R}^3)$, with $\operatorname{div}\mathbf{A} = \operatorname{div}\mathbf{A}' = 0$, and each $j \in \{1, \ldots, N\}$, we have

$$\|\boldsymbol{\mathcal{J}}[\phi, \mathbf{A}] - \boldsymbol{\mathcal{J}}[\phi', \mathbf{A}']\|_{-1,2} \\ \lesssim \left\{\left[(1 + \|\mathbf{A}\|_{1,2})\|\phi\|_{1,2} + (1 + \|\mathbf{A}'\|_{1,2})\|\phi'\|_{1,2}\right] + \|\phi\|_{1,2}\|\phi'\|_{1,2}\right\} \\ \times \max\{\|\phi - \phi'\|_{1,2}\|\mathbf{A} - \mathbf{A}'\|_{1,2}\}. \tag{70}$$

Proof. To prove (69) we split into two cases: (a) $1 \le m \le 2$ and (b) $m > 2$. For (a), we specialize to $m = 1$ and note that the general case $1 \le m \le 2$ follows in a similar fashion. Since

$$\|\mathcal{J}[\phi, \mathbf{A}]\|_{-1,2} \lesssim \|\mathcal{J}[\phi, \mathbf{A}]\|_{\frac{6}{5}}$$

we need to estimate $\|\mathcal{J}[\phi, \mathbf{A}]\|_{\frac{6}{5}}$ by $(1 + \|\mathbf{A}\|_{1,2})\|\phi\|_{1,2}^2$. Using Minkowski's integral inequality, Hölder's inequality, and the Sobolev inequality $H^1(\mathbb{R}^3) \subset L^r(\mathbb{R}^3)$, $2 \le r \le 6$, we have

$$\begin{aligned}
\|\mathbf{J}_j[\phi, \mathbf{A}]\|_{\frac{6}{5}} &= \alpha \left(\int_{\mathbb{R}^3} \left| \int \langle \boldsymbol{\sigma}\phi_{\underline{\mathbf{z}}'_j}, \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}_j)\phi_{\underline{\mathbf{z}}'_j} \rangle_{\mathbb{C}^2}(\mathbf{x}_j) \mathrm{d}\underline{\mathbf{z}}'_j \right|^{\frac{6}{5}} \mathrm{d}\mathbf{x}_j \right)^{\frac{5}{6}} \\
&\le \alpha \int \left(\int_{\mathbb{R}^3} \left| \langle \boldsymbol{\sigma}\phi_{\underline{\mathbf{z}}'_j}, \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}_j)\phi_{\underline{\mathbf{z}}'_j} \rangle_{\mathbb{C}^2}(\mathbf{x}_j) \right|^{\frac{6}{5}} \mathrm{d}\mathbf{x}_j \right)^{\frac{5}{6}} \mathrm{d}\underline{\mathbf{z}}'_j \\
&\lesssim \int \left[\|\phi_{\underline{\mathbf{z}}'_j}\|_3 \|(\mathbf{p} + \mathbf{A})\phi_{\underline{\mathbf{z}}'_j}\|_2 \right] \mathrm{d}\underline{\mathbf{z}}'_j \\
&\lesssim (1 + \|\mathbf{A}\|_{1,2})\|\phi\|_{1,2}^2.
\end{aligned} \tag{71}$$

The estimate (71) thus yields $\|\mathbf{J}_j[\phi, \mathbf{A}]\|_{-1,2} \lesssim (1 + \|\mathbf{A}\|_{1,2})\|\phi\|_{1,2}^2$ For case (b), we use Minkowski's integral inequality, Lemma 2, and the Sobolev inequality to find

$$\begin{aligned}
\|\mathbf{J}_j[\phi, \mathbf{A}]\|_{m-2,2} &= \alpha \left(\int_{\mathbb{R}^3} \left| \int \Lambda_j^{m-2} \langle \boldsymbol{\sigma}\phi_{\underline{\mathbf{z}}'_j}, \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}_j)\phi_{\underline{\mathbf{z}}'_j} \rangle_{\mathbb{C}^2}(\mathbf{x}_j) \mathrm{d}\underline{\mathbf{z}}'_j \right|^2 \mathrm{d}\mathbf{x}_j \right)^{\frac{1}{2}} \\
&\le \alpha \int \left(\int_{\mathbb{R}^3} \left| \Lambda_j^{m-2} \langle \boldsymbol{\sigma}\phi_{\underline{\mathbf{z}}'_j}, \boldsymbol{\sigma} \cdot (\mathbf{p} + \mathbf{A}_j)\phi_{\underline{\mathbf{z}}'_j} \rangle_{\mathbb{C}^2}(\mathbf{x}_j) \right|^2 \mathrm{d}\mathbf{x}_j \right)^{1/2} \mathrm{d}\underline{\mathbf{z}}'_j \\
&\lesssim \int \Big[\|\phi_{\underline{\mathbf{z}}'_j}\|_{m-2,6} \|\phi_{\underline{\mathbf{z}}'_j}\|_{1,3} + \|\phi_{\underline{\mathbf{z}}'_j}\|_3 \|\phi_{\underline{\mathbf{z}}'_j}\|_{m-1,6} \\
&\quad + \|\mathbf{A}\|_{m-2,6} \|\phi_{\underline{\mathbf{z}}'_j}\|_6^2 + \|\mathbf{A}\|_6 \|\phi_{\underline{\mathbf{z}}'_j}\|_{m-2,6} \|\phi_{\underline{\mathbf{z}}'_j}\|_3 \Big] \mathrm{d}\underline{\mathbf{z}}'_j \\
&\lesssim (1 + \|\mathbf{A}\|_{m,2}) \int \|\phi_{\underline{\mathbf{z}}'_j}\|_{m,2}^2 \mathrm{d}\underline{\mathbf{z}}'_j \lesssim (1 + \|\mathbf{A}\|_{m,2})\|\phi\|_{m,2}^2.
\end{aligned} \tag{72}$$

Combining (71) and (72) we arrive at (69).

Arguing (70) in similar to the case $m = 1$ in proving (69). Specifically, we need to estimate $\mathbf{J}_j[\phi, \mathbf{A}] - \mathbf{J}_j[\phi', \mathbf{A}']$ in $L^{\frac{6}{5}}$-norm. We write

$$\mathbf{J}_j[\phi, \mathbf{A}] - \mathbf{J}_j[\phi', \mathbf{A}'] = -\alpha \operatorname{Re} \sum_{\alpha=1}^{4} \mathbf{F}_j^{\alpha}[\phi, \phi', \mathbf{A}, \mathbf{A}'] \tag{73}$$

where

$$\mathbf{F}^1_j[\phi,\phi',\mathbf{A},\mathbf{A}'](\mathbf{x}_j) = \int \langle \boldsymbol{\sigma}\left(\phi_{\underline{\mathbf{z}}'_j} - \phi'_{\underline{\mathbf{z}}'_j}\right), \boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A}_j)\phi_{\underline{\mathbf{z}}'_j}\rangle_{\mathbb{C}^2}(\mathbf{x}_j)\mathrm{d}\underline{\mathbf{z}}'_j,$$
$$\mathbf{F}^2_j[\phi,\phi',\mathbf{A},\mathbf{A}'](\mathbf{x}_j) = \int \langle \boldsymbol{\sigma}\phi'_{\underline{\mathbf{z}}'_j}, \boldsymbol{\sigma}\cdot\mathbf{p}\left(\phi_{\underline{\mathbf{z}}'_j} - \phi'_{\underline{\mathbf{z}}'_j}\right)\rangle_{\mathbb{C}^2}(\mathbf{x}_j)\mathrm{d}\underline{\mathbf{z}}'_j,$$
$$\mathbf{F}^3_j[\phi,\phi',\mathbf{A},\mathbf{A}'](\mathbf{x}_j) = \int \langle \boldsymbol{\sigma}\phi'_{\underline{\mathbf{z}}'_j}, \boldsymbol{\sigma}\cdot\left(\mathbf{A}_j - \mathbf{A}'_j\right)\phi_{\underline{\mathbf{z}}'_j}\rangle_{\mathbb{C}^2}(\mathbf{x}_j)\mathrm{d}\underline{\mathbf{z}}'_j,$$
$$\mathbf{F}^4_j[\phi,\phi',\mathbf{A},\mathbf{A}'](\mathbf{x}_j) = \int \langle \boldsymbol{\sigma}\phi'_{\underline{\mathbf{z}}'_j}, \boldsymbol{\sigma}\cdot\mathbf{A}'_j\left(\phi_{\underline{\mathbf{z}}'_j} - \phi'_{\underline{\mathbf{z}}'_j}\right)\rangle_{\mathbb{C}^2}(\mathbf{x}_j)\mathrm{d}\underline{\mathbf{z}}'_j.$$

Estimating $\mathbf{F}^\alpha_j$, for $\alpha = 1,\ldots,4$, in $L^{\frac{6}{5}}$-norm is straightforward and involves the same strategy used to show (71). We find

$$\|\mathbf{F}^1_j[\phi,\phi',\mathbf{A},\mathbf{A}']\|_{\frac{6}{5}} \lesssim (1+\|\mathbf{A}\|_{1,2})\|\phi\|_{1,2}\|\phi-\phi'\|_{1,2}. \tag{74}$$
$$\|\mathbf{F}^2_j[\phi,\phi',\mathbf{A},\mathbf{A}']\|_{\frac{6}{5}} \lesssim \|\phi'\|_{1,2}\|\phi-\phi'\|_{1,2}, \tag{75}$$
$$\|\mathbf{F}^3_j[\phi,\phi',\mathbf{A},\mathbf{A}']\|_{\frac{6}{5}} \lesssim \|\phi\|_{1,2}\|\phi'\|_{1,2}\|\mathbf{A}-\mathbf{A}'\|_{1,2}, \tag{76}$$
$$\|\mathbf{F}^4_j[\phi,\phi',\mathbf{A},\mathbf{A}']\|_{\frac{6}{5}} \lesssim \|\mathbf{A}'\|_{1,2}\|\phi'\|_{1,2}\|\phi-\phi'\|_{1,2}. \tag{77}$$

Estimates (74) through (77) imply (70). □

3.2. Metric space, linearization, and proof of theorem 2. Let $N, K \geq 1$, $m \in [1,\infty)$, $\varepsilon > 0$, and $(\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0) \in \mathcal{X}^m_0$, where $\mathcal{X}^m_0$ is defined by (22), and let $\mathcal{Z}$ and $\underline{\mathbf{R}}$ be defined as in Sect. 1. Given $T, R \in (0,\infty)$, consider the (T, R)-dependent space

$$\begin{aligned}\mathcal{X}^m_T(R) =&\{(\phi,\mathbf{A}) \in L^\infty_T[H^m(\mathbb{R}^{3N})]^{2^N} \times [L^\infty_T H^m(\mathbb{R}^3;\mathbb{R}^3) \cap W^{1,\infty}_T H^{m-1}(\mathbb{R}^3;\mathbb{R}^3)] \\ &\text{s.t. } \max\{\|\phi\|_{\infty;m,2}, \|\mathbf{A}\|_{\infty;m,2}, \|\partial_t\mathbf{A}\|_{\infty;m-1,2}\} \leq R,\ \operatorname{div}\mathbf{A} = 0\}.\end{aligned}$$

Recall, for $(\phi,\mathbf{A}) \in \mathcal{X}^m_T$, we denote the magnetic field $\mathbf{B} = \operatorname{curl}\mathbf{A}$ and the regularized vector potential $\tilde{\mathbf{A}} = \Lambda_\varepsilon^{-1}\mathbf{A}$. When the radius $R > 0$ is understood we will simply write $\mathcal{X}^m_T$ for $\mathcal{X}^m_T(R)$. Consider the mapping

$$\Psi : \mathcal{X}^m_T \ni (\phi,\mathbf{A}) \mapsto (\xi,\mathbf{K})$$

where

$$\xi(t) = e^{(i+\varepsilon)t\Delta}\phi_0 + \int_0^t e^{(i+\varepsilon)(t-\tau)\Delta} f[\phi(\tau),\tilde{\mathbf{A}}(\tau)]\mathrm{d}\tau, \tag{78}$$

with

$$f[\phi,\tilde{\mathbf{A}}] = \left[-(i+\varepsilon)\left(\mathscr{L}(\tilde{\mathbf{A}}) + V(\underline{\mathbf{R}},\mathcal{Z})\right) + \varepsilon\left(T_{\mathrm{P}}[\phi,\tilde{\mathbf{A}}] + V[\phi]\right)\right]\phi, \tag{79}$$

and

$$\mathbf{K}(t) = \dot{\mathfrak{s}}(t/\alpha)\mathbf{a}_0 + \alpha\,\mathfrak{s}(t/\alpha)\dot{\mathbf{a}}_0 + 4\pi\int_0^t \mathfrak{s}((t-\tau)/\alpha)\Lambda_\varepsilon^{-1}\mathcal{P}\boldsymbol{\mathcal{J}}[\phi,\tilde{\mathbf{A}}](\tau)\mathrm{d}\tau \tag{80}$$

In (78)–(80), $T_{\mathrm{P}}[\phi, \tilde{\mathbf{A}}]$ is given by (6), $V[\phi]$ by (7), $\mathscr{L}(\tilde{\mathbf{A}}) = \sum_{j=1}^{N} \mathcal{L}_j(\tilde{\mathbf{A}})$ where $\mathcal{L}_j(\tilde{\mathbf{A}})$ is given by (25), $\dot{\mathfrak{s}}$ and $\mathfrak{s}$ are defined in Lemma 4, and $\boldsymbol{\mathcal{J}}[\phi, \tilde{\mathbf{A}}] = \sum_{j=1}^{N} \mathbf{J}_j[\phi, \tilde{\mathbf{A}}]$ given by (4). In other words, Ψ maps $(\phi, \mathbf{A}) \in \mathcal{X}_T^m$ into the solution of the linearized system

$$\begin{cases} \partial_t \xi - (i + \varepsilon) \sum_{j=1}^{N} \Delta_{\mathbf{x}_j} \xi = f[\phi, \tilde{\mathbf{A}}], \\ \Box \mathbf{K} = 4\pi\alpha \Lambda_\varepsilon^{-1} \mathcal{P} \boldsymbol{\mathcal{J}}[\phi, \tilde{\mathbf{A}}], \\ \xi(0) = \phi_0, \quad \mathbf{K}(0) = \mathbf{a}_0, \quad \partial_t \mathbf{K}(0) = \dot{\mathbf{a}}_0. \end{cases}$$

At this point we observe that a fixed point of Ψ would give us a proof of the first part of Theorem 2. Hence the strategy is to equip $\mathcal{X}_T^m$ with an appropriate metric, prove that, for small enough $T > 0$, Ψ is a contraction on $\mathcal{X}_T^m$ with respect to that metric, and thereby prove that Ψ has a fixed point via the Banach fixed point theorem. We equip $\mathcal{X}_T^m$ with the metric

$$d((\phi, \mathbf{A}), (\phi', \mathbf{A}')) = \max\{\|\phi - \phi'\|_{\infty;1,2}, \|\mathbf{A} - \mathbf{A}'\|_{\infty;1,2}, \|\partial_t \mathbf{A} - \partial_t \mathbf{A}'\|_{\infty;2}\}. \quad (81)$$

Standard functional analysis arguments show that $(\mathcal{X}_T^m, d)$ is a complete metric space.

Proof of Theorem 2. Fix $\varepsilon > 0, m \in [1, 2]$, and let $(\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0) \in \mathcal{X}_0^m$. The first task is to demonstrate that we can make Ψ map $\mathcal{X}_T^m$ into itself by choosing R and T appropriately. Indeed, we claim that there exists $R, T_* > 0$ such that for all $T \in (0, T_*]$ the function Ψ maps $\mathcal{X}_T^m$ into itself, where the time $T_* > 0$ depends on ε, m, N, K, α, $\mathcal{Z}$, $\underline{\mathbf{R}}$, and $\|(\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0)\|_{m,2\oplus m,2\oplus m-1,2}$. To this end, let $(\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0) \in \mathcal{X}_0^m$ and $(\phi, \mathbf{A}) \in \mathcal{X}_T^m$, and consider $\Psi(\phi, \mathbf{A}) = (\xi, \mathbf{K})$.

Observe that $\mathbf{K}$ is divergence-free using the formula (80). Fix $j \in \{1, \ldots, N\}$ and note that

$$\|\Lambda_\varepsilon^{-1} \mathcal{P} \boldsymbol{\mathcal{J}}[\phi, \tilde{\mathbf{A}}]\|_{m-1,2} \leq \frac{1}{\varepsilon} \|\mathcal{P} \boldsymbol{\mathcal{J}}[\phi, \tilde{\mathbf{A}}]\|_{\dot{H}^{m-2}} \lesssim \frac{1}{\varepsilon} \|\boldsymbol{\mathcal{J}}[\phi, \tilde{\mathbf{A}}]\|_{m-2,2},$$

where we've used the boundedness of $\mathcal{P} : H^{m-2} \to H^{m-2}$. Therefore estimate (69) of Lemma 8 gives us

$$\|\Lambda_\varepsilon^{-1} \mathcal{P} \boldsymbol{\mathcal{J}}[\phi, \tilde{\mathbf{A}}](t)\|_{m-1,2} \lesssim (1 + R) R^2, \qquad \forall t \in [0, T], \quad (82)$$

and thus $\Lambda_\varepsilon^{-1} \mathcal{P} \boldsymbol{\mathcal{J}}[\phi, \tilde{\mathbf{A}}] \in L_T^1 H^{m-1}$. With the previous conclusion we've satisfied the hypotheses in Lemma 4 and, as a consequence, we have $\mathbf{K} \in C_T H^m \cap C_T^1 H^{m-1}$ and

$$\max_{k \in \{0,1\}} \|\partial_t^k \mathbf{K}\|_{\infty;m-k,2} \lesssim \|(\mathbf{a}_0, \dot{\mathbf{a}}_0)\|_{m,2\oplus m-1,2} + \|\Lambda_\varepsilon^{-1} \mathcal{P} \boldsymbol{\mathcal{J}}[\phi, \tilde{\mathbf{A}}]\|_{1;m-1,2} \quad (83)$$

Combining (82) with (83), we conclude the existence of a constant $C_1 > 0$ depending on ε, m, N, and α such that

$$\max_{k \in \{0,1\}} \|\partial_t^k \mathbf{K}\|_{\infty;m-k,2} \leq C_1 \left[\|(\mathbf{a}_0, \dot{\mathbf{a}}_0)\|_{m,2\oplus m-1,2} + T(1 + R) R^2 \right]. \quad (84)$$

We turn to estimating $\|\xi(t)\|_{m,2}$. To estimate $\|\xi(t)\|_{m,2}$, we take the H^m-norm of the defining formula (78) for $\xi(t)$ and apply (54), (28), (34) of Lemmas 7, 5, and 6 respectively. This yields

$$\begin{aligned}\|\xi(t)\|_{m,2} &\lesssim \|\phi_0\|_{m,2} + \int_0^t \Big(\Big| T_{\mathrm{P}}[\phi, \tilde{\mathbf{A}}] + V[\phi] \Big| \, \|\phi\|_{m,2} + \|e^{(i+\varepsilon)(t-\tau)\Delta}[\mathscr{L}(\tilde{\mathbf{A}})\phi]\|_{m,2} \\ &\quad + \|e^{(i+\varepsilon)(t-\tau)\Delta} V(\underline{\mathbf{R}}, \mathcal{Z})\phi\|_{m,2} \Big) \mathrm{d}\tau \\ &\lesssim \|\phi_0\|_{m,2} + \int_0^t \Big[(1 + \|\tilde{\mathbf{A}}\|_{1,2})^2 + 1 \Big] \|\phi\|_{1,2}^2 \|\phi\|_{m,2} \mathrm{d}\tau \\ &\quad + \int_0^t (t-\tau)^{-\frac{1}{4}} \Big[1 + (t-\tau)^{-\frac{1}{2}} \Big] \Big(1 + \|\tilde{\mathbf{A}}\|_{m,2} \Big) \|\tilde{\mathbf{A}}\|_{m,2} \|\phi\|_{m,2} \mathrm{d}\tau \\ &\quad + \int_0^t \Big\{ 1 + \Big(1 + (t-\tau)^{-\frac{1}{2}} \Big) \Big((t-\tau)^{-\frac{9}{20}} + (t-\tau)^{-\frac{1}{4}} \Big) \Big\} \|\phi\|_{m,2} \mathrm{d}\tau. \end{aligned} \tag{85}$$

The last estimate (85) allow us to conclude the existence of a constant $C_2 > 0$, depending on $\varepsilon, m, N, K, \alpha, \underline{\mathbf{R}}$, and $\mathcal{Z}$, such that

$$\begin{aligned}\|\xi\|_{\infty;m,2} \le C_2 \Big[&\|\phi_0\|_{m,2} + T \Big(2 + 2R + R^2 \Big) R^3 + \Big(T^{\frac{3}{4}} + T^{\frac{1}{4}} \Big) (1+R) R^2 \\ &+ \Big(T + T^{\frac{3}{4}} + T^{\frac{11}{20}} + T^{\frac{1}{4}} + T^{\frac{1}{20}} \Big) R \Big]. \end{aligned} \tag{86}$$

Considering estimates (84) and (86) choose $R > 0$ such that

$$\|(\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0)\|_{m,2\oplus m,2\oplus m-1,2} \le \frac{R}{2\max\{C_1, C_2\}}, \tag{87}$$

and choose $T_* > 0$ such that

$$\begin{aligned} &T_*(1 + 3R + 2R^2 + R^3)R + (T_*^{\frac{3}{4}} + T_*^{\frac{1}{4}})(1+R)R \\ &\quad + (T_* + T_*^{\frac{3}{4}} + T_*^{\frac{11}{20}} + T_*^{\frac{1}{4}} + T_*^{\frac{1}{20}})] \le \frac{1}{2\max\{C_1, C_2\}}. \end{aligned} \tag{88}$$

Equations (87) and (88) ensure that Ψ maps $\mathcal{X}_T^m$ into itself for each $T \in (0, T_*]$.

We claim that one may further choose a $0 < T_{**} < T_*$ so that Ψ becomes a contraction on $(\mathcal{X}_T^m, d)$ for any $T \in (0, T_{**}]$. Indeed, fix $T \in (0, T_*]$ and consider two $(\phi, \mathbf{A}), (\phi', \mathbf{A}') \in \mathcal{X}_T^m$ and write $\Psi(\phi, \mathbf{A}) = (\xi, \mathbf{K})$ and $\Psi(\phi', \mathbf{A}') = (\xi', \mathbf{K}')$. Noting (78), (80), $(\xi(0), \mathbf{K}(0), \partial_t \mathbf{K}(0)) = (\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0)$, we observe that the difference $\xi - \xi'$ satisfies

$$(\xi - \xi')(t) = \int_0^t e^{(i+\varepsilon)(t-\tau)\Delta} \Big(f[\phi(\tau), \tilde{\mathbf{A}}(\tau)] - f[\phi'(\tau), \tilde{\mathbf{A}}'(\tau)] \Big) \mathrm{d}\tau \tag{89}$$

and that the difference $\mathbf{K} - \mathbf{K}'$ satisfies

$$(\mathbf{K} - \mathbf{K}')(t) = 4\pi \int_0^t \mathfrak{s}((t-\tau)/\alpha) \Lambda_\varepsilon^{-1} \mathcal{P} \Big(\boldsymbol{\mathcal{J}}[\phi, \tilde{\mathbf{A}}] - \boldsymbol{\mathcal{J}}[\phi', \tilde{\mathbf{A}}'] \Big)(\tau) \mathrm{d}\tau. \tag{90}$$

We need to control $d((\xi, \mathbf{K}), (\xi', \mathbf{K}'))$ by $d((\phi, \mathbf{A}), (\phi', \mathbf{A}'))$ to ultimately argue that Ψ can be turned into a contraction. Estimating $\|\mathbf{K} - \mathbf{K}'\|_{\infty;1,2}$ and $\|\partial_t(\mathbf{K} - \mathbf{K}')\|_{\infty;2}$ is a straightforward application of the energy estimate of Lemma 4 and estimate (70) of Lemma 8. We find

$$\max_{k=0,1} \|\partial_t^k \left(\mathbf{K}-\mathbf{K}'\right)\|_{\infty;1-k,2} \lesssim \|\boldsymbol{\mathcal{J}}[\phi,\tilde{\mathbf{A}}]-\boldsymbol{\mathcal{J}}[\phi',\tilde{\mathbf{A}}']\|_{1;-1,2} \lesssim TR\,[2+3R]\,d((\phi,\mathbf{A}),(\phi',\mathbf{A}')). \quad (91)$$

To estimate $\|\xi-\xi'\|_{\infty;1,2}$ we start with the formula (89) for $\xi-\xi'$ and use the triangle inequality to find

$$\|(\xi-\xi')(t)\|_{1,2} \lesssim \int_0^t \|e^{(i+\varepsilon)(t-\tau)\Delta}\left(f[\phi(\tau),\tilde{\mathbf{A}}(\tau)]-f[\phi'(\tau),\tilde{\mathbf{A}}'(\tau)]\right)\|_{1,2}\mathrm{d}\tau. \quad (92)$$

Using the same strategy that yielded (85) and then (86), we apply (55), (28), (34) of Lemmas 7, 5, and 6, respectively, to find

$$\|(\xi-\xi')\|_{\infty;1,2} \lesssim \{T(4+8R+6R^2+R^3)R+(T^{\frac{3}{4}}+T^{\frac{1}{4}})(2+3R)R + T + T^{\frac{3}{4}} + T^{\frac{11}{20}} + T^{\frac{1}{4}} + T^{\frac{1}{20}}\}d((\phi,\mathbf{A}),(\phi',\mathbf{A}')). \quad (93)$$

Combining estimates (91) through (93) we find

$$d((\xi,\mathbf{K}),(\xi',\mathbf{K}')) \le Cg(T,R)d((\psi,\mathbf{A}),(\psi',\mathbf{A}')), \quad (94)$$

where $C>0$ is a constant depending on ε, N, K, α, $\underline{\mathbf{R}}$, and $\mathcal{Z}$, and

$$g(T,R) = T(6+11R+6R^2+R^3)R+(T^{\frac{3}{4}}+T^{\frac{1}{4}})(2+3R)R + T + T^{\frac{3}{4}} + T^{\frac{11}{20}} + T^{\frac{1}{4}} + T^{\frac{1}{20}}. \quad (95)$$

Choosing $0<T_{**}<T_*$ so that $g(T_{**},R)=\frac{1}{2C}$ ensures that Ψ, for example, satisfies

$$d(\Psi(\psi,\mathbf{A}),\Psi(\psi',\mathbf{A}')) \le \frac{1}{2}d((\psi,\mathbf{A}),(\psi',\mathbf{A}')).$$

Consequently, Ψ is a contraction mapping on $(\mathcal{X}_T^m,d)$ for each $T\in(0,T_{**}]$.

Then, for $R>0$ satisfying (87) and for each $T\in(0,T_{**}]$, the Banach fixed point theorem allows us to conclude the existence a unique $(\phi,\mathbf{A})\in\mathcal{X}_T^m(R)$ that satisfies $\Psi(\phi,\mathbf{A})=(\phi,\mathbf{A})$. Using the same estimates at produced the estimate (93), we can show $\phi\in C_T H^m$. Moreover, $\mathbf{A}\in C_T H^m\cap C_T^1 H^{m-1}$ by Lemma 4. In other words, the pair $(\phi,\mathbf{A})\in C_T H^m\times[C_T H^m\cap C_T^1 H^{m-1}]$ satisfies the equations

$$\begin{cases} \partial_t\phi = -(i+\varepsilon)\mathcal{H}(\mathbf{A})\phi+\varepsilon\phi\left(T_{\mathrm{P}}[\phi,\tilde{\mathbf{A}}]+V[\phi]\right), \\ \Box\mathbf{A} = 4\pi\alpha\Lambda_\varepsilon^{-1}\mathcal{P}\boldsymbol{\mathcal{J}}[\phi,\tilde{\mathbf{A}}], \\ \operatorname{div}\mathbf{A}=0, \\ (\phi,\mathbf{A},\partial_t\mathbf{A})|_{t=0}=(\phi_0,\mathbf{a}_0,\dot{\mathbf{a}}_0). \end{cases} \quad (96)$$

A straightforward argument shows the uniqueness of $(\phi,\mathbf{A})\in C_{T_{**}}H^m\times[C_{T_{**}}H^m\cap C_{T_{**}}^1 H^{m-1}]$ solving the initial value problem (96). Similarly, a straightforward continuation argument proves the blow-up alternative holds. So far we have the existence of a maximal time interval $\mathcal{I}=[0,T_{\max})$ for which we have a unique solution

$$(\phi,\mathbf{A})\in C_{\mathcal{I}}[H^m(\mathbb{R}^{3N})]^{2^N}\times[C_{\mathcal{I}}H^m(\mathbb{R}^3;\mathbb{R}^3)\cap C_{\mathcal{I}}^1 H^{m-1}(\mathbb{R}^3;\mathbb{R}^3)]$$

to (96), and such that the blow-up alternative holds. This gives us the first portion of Theorem 2. What is left to show is the approximation portion of Theorem 2.

Let $(\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0) \in \mathcal{X}_0^1$. Choose $R, T > 0$ so that

$$d(\Psi(\phi, \mathbf{A}), \Psi(\phi', \mathbf{A}')) \leq \frac{1}{2} d((\phi, \mathbf{A}), (\phi, \mathbf{A})),$$

for all $(\phi, \mathbf{A}), (\phi', \mathbf{A}') \in \mathcal{X}_T^1(R)$, and let $(\phi, \mathbf{A}) \in \mathcal{X}_T^1(R)$ denoted the corresponding unique fixed point of Ψ. Consider a sequence of initial data $\{(\phi_0^j, \mathbf{a}_0^j, \mathbf{a}_1^j)\}_{j\geq 1} \subset \mathcal{X}_0^m$ and let $\{(\phi^j, \mathbf{A}^j)\}_{j\geq 1} \subset C_T H^m \times [C_T H^m \cap C_T^1 H^{m-1}]$ denote the corresponding sequence of solutions. Suppose that

$$\|(\phi_0 - \phi_0^j, \mathbf{a}_0 - \mathbf{a}_0^j, \dot{\mathbf{a}}_0 - \mathbf{a}_1^j)\|_{1,2\oplus 1,2\oplus 2} \xrightarrow{j\to\infty} 0.$$

Observe that if j is sufficiently large then (87) holds with $(\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0)$ replaced by $(\phi_0^j, \mathbf{a}_0^j, \dot{\mathbf{a}}_0^j)$, and therefore $(\phi^j, \mathbf{A}^j) \in \mathcal{X}_T^m(R)$ when j is sufficiently large. Using identical estimates that yielded (91) and (93), we have the estimate

$$\begin{aligned} d((\phi, \mathbf{A}), (\phi^j, \mathbf{A}^j)) \leq C_1 \|(\phi_0 - \psi_0^j, \mathbf{a}_0 - \mathbf{a}_0^j, \dot{\mathbf{a}}_0 - \mathbf{a}_1^j)\|_{1,2\oplus 1,2\oplus 2} \\ + C_2 g(T, R) d((\phi, \mathbf{A}), (\phi^j, \mathbf{A}^j)), \end{aligned}$$

where the function g is defined by (95) and C_2 is the *same* constant appearing in (94). Since T was chosen so that $g(T, R) = 1/(2C_2)$, we conclude

$$d((\phi, \mathbf{A}), (\phi^j, \mathbf{A}^j)) \xrightarrow{j\to\infty} 0$$

on the time interval $[0, T]$.

Consider as initial data $(\phi(T), \mathbf{A}(T), \partial_t \mathbf{A}(T)) \in \mathcal{X}_0^1$ and

$$\{(\phi^j(T), \mathbf{A}^j(T), \partial_t \mathbf{A}^j(T))\}_{j\geq 1} \subset \mathcal{X}_0^m.$$

By the preceding arguments,

$$\|((\phi - \phi^j)(T), (\mathbf{A} - \mathbf{A}^j)(T), (\partial_t \mathbf{A} - \partial_t \mathbf{A}^j)(T))\|_{1,2\oplus 1,2\oplus 2} \xrightarrow{j\to\infty} 0.$$

Choose $R', T' > 0$ so that

$$d(\Psi(\phi, \mathbf{A}), \Psi(\phi', \mathbf{A}')) \leq \frac{1}{2} d((\phi, \mathbf{A}), (\phi, \mathbf{A})),$$

for all $(\phi, \mathbf{A}), (\phi', \mathbf{A}') \in \mathcal{X}_{T'}^1(R')$. Using the same notation, let $(\phi, \mathbf{A}) \in \mathcal{X}_{T'}^1(R')$ denoted the corresponding unique fixed point of Ψ and let $\{(\phi^j, \mathbf{A}^j)\}_{j\geq 1} \subset C_{T'} H^m \times [C_{T'} H^m \cap C_{T'}^1 H^{m-1}]$ denote the sequence of solutions corresponding to the initial data $\{(\phi^j(T), \mathbf{A}^j(T), \partial_t \mathbf{A}^j(T))\}_{j\geq 1}$ in $\mathcal{X}_0^m$. As before, if j is sufficiently large, then $(\phi^j, \mathbf{A}^j) \in \mathcal{X}_{T'}^m(R')$. By the same reasoning as before, we can conclude $d((\phi, \mathbf{A}), (\phi^j, \mathbf{A}^j)) \longrightarrow 0$ as $j \longrightarrow \infty$ on the time interval $[0, T']$ with $T' > T$. We can repeat this argument ad infinitum and conclude the desired convergence at each $t \in \mathcal{I} = [0, T_{\max})$. □

4. Bound on the Coulomb Energy and Energy Dissipation

In this section we prove Lemma 1, namely, that the Coulomb energy functional is uniformly bounded on $\mathscr{C}_N$, and the L^2-norm conservation and energy dissipation for the ε-modified system (19) as stated in Theorem 3. As we emphasized in the introduction, the crucial result that is needed to derive the uniform bounds in Theorem 3 is the uniform bound on the Coulomb energy $V[\phi]$ as stated in Lemma 1. Such a bound is a direct consequence of the energetic stability estimates as given by (12) and (13).

Proof of Lemma 1. Fix $(\phi, \mathbf{A}) \in \mathscr{C}_N$. Throughout we abuse notation and abbreviate $E[\phi, \mathbf{A}] = E[\phi, \mathbf{A}, \mathbf{0}]$, where $E[\phi, \mathbf{A}, \mathbf{0}]$ is given by (5), and $F[\mathbf{A}, \mathbf{0}] = F[\mathbf{A}]$. We claim the uniform lower bound $E \geq E_{\mathrm{G}}(\alpha)$ implies

$$(V[\phi] + F[\mathbf{A}])^2 \leq 4|E_{\mathrm{G}}(\alpha)| T_{\mathrm{P}}[\phi, \mathbf{A}]. \tag{97}$$

Indeed, for $\lambda > 0$, consider the scaling $\phi_\lambda(\underline{\mathbf{z}}) = \lambda^{3N/2}\phi(\lambda\underline{\mathbf{z}})$ and $\mathbf{A}_\lambda(\mathbf{y}) = \lambda\mathbf{A}(\lambda\mathbf{y})$. Under this scaling

$$T_{\mathrm{P}}[\phi_\lambda, \mathbf{A}_\lambda] + V[\phi_\lambda] + F[\mathbf{A}_\lambda] = \lambda^2 T_{\mathrm{P}}[\phi, \mathbf{A}] + \lambda\,(V[\phi] + F[\mathbf{A}]) \geq E_{\mathrm{G}}(\alpha)$$

Minimizing over λ in the previous expression yields (97).

Let $\{(\phi^n, \mathbf{A}^n)\}_{n\geq 1} \subset \mathscr{C}_N$ be a sequence such that $E_n = T_n + V_n + F_n \leq E_0(\alpha)$ where $E_n \equiv E[\phi^n, \mathbf{A}^n]$, $T_n \equiv T_{\mathrm{P}}[\phi^n, \mathbf{A}^n]$, $V_n \equiv V[\phi^n]$, and $F_n \equiv F[\mathbf{A}^n]$. Suppose, to the contrary, that $|V_n| \to \infty$ as $n \to \infty$. The condition $E_n \leq E_0(\alpha)$ implies that we necessarily have $V_n \to -\infty$. Set $\lambda_n = 1/|V_n|$ and note $\lambda_n \to 0$ as $n \to \infty$. Consider the scaling $\Phi^n(\underline{\mathbf{z}}) = \lambda_n^{3N/2}\phi^n(\lambda_n\underline{\mathbf{z}})$ and $\mathbf{a}_n(\mathbf{y}) = \lambda_n\mathbf{A}^n(\lambda_n\mathbf{y})$. Moreover, from $E_{\mathrm{G}}(\alpha) \leq E_n \leq E_0(\alpha)$ we have

$$E_{\mathrm{G}}(\alpha)\lambda_n \leq \frac{t_n}{\lambda_n} - 1 + \frac{1}{\alpha^2} f_n \leq E_0(\alpha)\lambda_n \tag{98}$$

where $t_n = T[\Phi^n, \mathbf{a}_n] = \lambda_n^2 T_n$ and $f_n = \|\mathbf{a}_n\|_2^2/(8\pi) = \lambda_n \alpha^2 F_n$.

Pick ν with $\alpha > \nu$ and note that we have $E_{\mathrm{G}}(\nu) > -\infty$. As before,

$$E_{\mathrm{G}}(\nu)\lambda_n \leq \frac{t_n}{\lambda_n} - 1 + \frac{1}{\nu^2} f_n \leq E_0(\nu)\lambda_n. \tag{99}$$

Subtracting (99) from (98) we conclude

$$(E_{\mathrm{G}}(\alpha) - E_0(\nu))\lambda_n \leq \left(\frac{1}{\nu^2} - \frac{1}{\alpha^2}\right) f_n \leq (E_0(\alpha) - E_{\mathrm{G}}(\nu))\lambda_n, \tag{100}$$

and thus $f_n \to 0$ as $n \to \infty$. Feeding this back into (98) we conclude $\lim_{n\to\infty}(t_n/\lambda_n) = 1$. Moreover, (97) implies

$$\left(\frac{f_n}{\alpha^2} - 1\right)^2 \leq 4E_{\mathrm{G}}(\alpha)t_n,$$

and as a consequence

$$\liminf_{n\to\infty} t_n \geq \frac{1}{4E_{\mathrm{G}}(\alpha)}. \tag{101}$$

However, (101) implies

$$\lim_{n\to\infty} \frac{t_n}{\lambda_n} = \infty.$$

□

For the proof of Theorem 3 it will be useful to recall that if ϕ is of a definite symmetry type (e.g., ϕ is completely antisymmetric, as will be the case), then the kinetic energy $T_{\mathrm{P}}[\phi, \mathbf{A}]$, as defined in (6), of the state $(\phi, \mathbf{A})$ reduces to $T_{\mathrm{P}}[\phi, \mathbf{A}] = N\|\boldsymbol{\sigma}_1\cdot(\mathbf{p}_1+\mathbf{A}_1)\phi\|_2^2$. Likewise, the total probability current density $\mathcal{J}[\phi, \mathbf{A}] = \sum_{j=1}^N \mathbf{J}_j[\phi, \mathbf{A}]$, as defined in (4), will reduce to

$$\mathcal{J}[\phi, \mathbf{A}] = -\alpha N \operatorname{Re} \int \langle \boldsymbol{\sigma}\psi_{\underline{\mathbf{z}}_1'}, \boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})\psi_{\underline{\mathbf{z}}_1'}\rangle_{\mathbb{C}^2} \mathrm{d}\underline{\mathbf{z}}_1'.$$

Proof of Theorem 3. Fix $\varepsilon > 0$ and $m \in [1, 2]$. Let $(\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0) \in \mathcal{X}_0^m$ with $\phi_0 \in \bigwedge^N H^m(\mathbb{R}^3; \mathbb{C}^2)$ and $\|\phi_0\|_2 = 1$. Let $(\phi, \mathbf{A})$ be the corresponding solution on $\mathcal{I}$ to (19) as given by Theorem 2. It is straightforward to verify that $\partial_t\phi(t) \in H^{-m}$ since $\mathcal{H}(\mathbf{A}(t))\psi(t) \in H^{-m}$ for each $t \in \mathcal{I}$. Therefore, it makes sense to compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_2^2 = 2\operatorname{Re}\langle\partial_t\phi, \phi\rangle_{H^{-m},H^m} = 2\varepsilon(\|\phi\|_2^2 - 1)\langle\mathcal{H}(\mathbf{A})\phi, \phi\rangle_{H^{-m},H^m}. \tag{102}$$

Since $\|\phi_0\|_2 = 1$, (102) implies $\|\phi(t)\|_2 = 1$.

Consider the case $m = 2$. In this case, $\mathcal{H}(\mathbf{A}(t))\phi(t) \in L^2$ for each $t \in \mathcal{I}$ and, hence, we may take the time-derivative of the total energy $\mathcal{E} = \mathcal{E}[\phi, \mathbf{A}, \partial_t\mathbf{A}]$, as defined in (21), to find

$$\begin{aligned}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} &= 2\operatorname{Re}\langle\partial_t\phi, \mathcal{H}(\mathbf{A})\phi\rangle_{L^2} + 2N\operatorname{Re}\langle\boldsymbol{\sigma}\cdot(\mathbf{p}+\tilde{\mathbf{A}})\phi, (\boldsymbol{\sigma}\cdot\partial_t\tilde{\mathbf{A}})\phi\rangle_{L^2} + \partial_t F[\mathbf{A}, \partial_t\mathbf{A}]\\ &= -2\varepsilon(\|\mathcal{H}(\mathbf{A})\phi\|_2^2 - \langle\phi, \mathcal{H}(\mathbf{A})\phi\rangle_{L^2}^2)\\ &\quad + 2N\operatorname{Re}\langle\boldsymbol{\sigma}\cdot(\mathbf{p}+\tilde{\mathbf{A}})\phi, (\boldsymbol{\sigma}\cdot\partial_t\tilde{\mathbf{A}})\phi\rangle_{L^2} + \partial_t F[\mathbf{A}, \partial_t\mathbf{A}].\end{aligned} \tag{103}$$

Using that $\mathbf{A}$ satisfies the wave equation (19, second equation) we can show that the last two terms in (103) cancel each other. From (19),

$$\begin{aligned}\partial_t F[\mathbf{A}, \partial_t\mathbf{A}] &= \frac{1}{4\pi\alpha^2}\langle\Box\mathbf{A}, \partial_t\mathbf{A}\rangle_{L^2}\\ &= \frac{1}{\alpha}\langle\Lambda_\varepsilon^{-1}\mathcal{P}\mathcal{J}[\phi, \tilde{\mathbf{A}}], \partial_t\mathbf{A}\rangle_{L^2}\\ &= -2N\langle\operatorname{Re}\int\langle\boldsymbol{\sigma}\phi_{\underline{\mathbf{z}}_1'}, \boldsymbol{\sigma}\cdot(\mathbf{p}+\tilde{\mathbf{A}})\phi_{\underline{\mathbf{z}}_1'}\rangle_{\mathbb{C}^2}\mathrm{d}\underline{\mathbf{z}}_1', \partial_t\tilde{\mathbf{A}}\rangle_{L^2}\\ &= -2N\operatorname{Re}\langle\boldsymbol{\sigma}\cdot(\mathbf{p}+\tilde{\mathbf{A}})\phi, (\boldsymbol{\sigma}\cdot\partial_t\tilde{\mathbf{A}})\phi\rangle_{L^2}.\end{aligned} \tag{104}$$

Plugging (104) into (103) we arrive at

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -2\varepsilon(\|\mathcal{H}(\mathbf{A})\phi\|_2^2 - \langle\phi, \mathcal{H}(\mathbf{A})\phi\rangle_{L^2}^2),$$

which upon integrating yields (23).

Continue assuming $m = 2$. Suppose α and $\alpha^2 \max \mathcal{Z}$ are sufficiently small to ensure $E_{\mathrm{G}} > -\infty$. To prove the bounds (24), we first verify that hypothesis of Lemma 1. For the moment we include the ε and t dependence of ϕ and $\mathbf{A}$ for clarity. By previous results $\|\phi^{\varepsilon}(t)\|_2 = 1$ (this, in fact, holds for any $m \in [1, 2]$). Moreover, we note that

$$F[\tilde{\mathbf{A}}^{\varepsilon}, \mathbf{0}] \leq F[\mathbf{A}^{\varepsilon}, \mathbf{0}] \leq F[\mathbf{A}^{\varepsilon}, \partial_t \mathbf{A}^{\varepsilon}],$$

and $\langle \phi^{\varepsilon}, \mathcal{H}^{\varepsilon}(\mathbf{A}^{\varepsilon})\phi^{\varepsilon}\rangle_{L^2}^2 \leq \|\mathcal{H}^{\varepsilon}(\mathbf{A}^{\varepsilon})\phi^{\varepsilon}\|_2^2$ by Cauchy–Schwartz. Therefore, from the dissipation of energy (23), we arrive at

$$E_{\mathrm{G}} \leq T_{\mathrm{P}}[\phi^{\varepsilon}(t), \tilde{\mathbf{A}}^{\varepsilon}(t)] - V[\phi^{\varepsilon}(t)] + F[\tilde{\mathbf{A}}^{\varepsilon}(t), \mathbf{0}] \leq \mathcal{E}[\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0].$$

Consequently, Lemma 1 tells us that

$$|V[\phi^{\varepsilon}(t)]| = |\langle \phi^{\varepsilon}(t), V(\underline{\mathbf{R}}, \mathcal{Z})\phi^{\varepsilon}(t)\rangle_{L^2}| \leq C \tag{105}$$

where C is a finite constant depending on α, $\mathcal{Z}$, N, K, and the initial data $(\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0)$, but *independent* of ε and t. Proceeding we will drop the ε and t dependence.

The bound (105) immediately gives us the second estimate in (24). Indeed, using the bound on the Coulomb energy we find

$$F[\mathbf{A}, \partial_t \mathbf{A}] \leq |\mathcal{E}[\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0]| + |V[\phi]| \leq C_2,$$

where $C_2 = |\mathcal{E}[\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0]| + C$. This, in turn, yields the third estimate in (24) by differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{A}\|_2^2 = 2\langle \mathbf{A}, \partial_t \mathbf{A}\rangle_{L^2} \leq 2\|\mathbf{A}\|_2 \|\partial_t \mathbf{A}\|_2 \leq 2\|\mathbf{A}\|_2 \sqrt{C_2}.$$

Hence,

$$\|\mathbf{A}\|_2 \leq C_3(1 + t),$$

where $C_3 = \max\{\|\mathbf{a}_0\|_2, \sqrt{C_2}\}$. Deriving the first estimate in (24) requires a more careful analysis. Consider $\delta > 0$ to be specified later. First, note that

$$\|\mathbf{p}\phi\|_2 = \sqrt{N}\|\boldsymbol{\sigma}_1 \cdot \mathbf{p}_1 \phi\|_2 \leq \sqrt{N}\left(\|\boldsymbol{\sigma}_1 \cdot (\mathbf{p}_1 + \tilde{\mathbf{A}}_1)\phi\|_2 + \|\tilde{\mathbf{A}}_1 \phi\|_2\right). \tag{106}$$

The first term on the right hand side of (106) can be bounded in the same way as the field energy $F[\mathbf{A}, \partial_t \mathbf{A}]$. Indeed, using the dissipation of energy (23) and the fact that $|V[\phi]|$ is uniformly bounded, we have

$$\sqrt{N}\|\boldsymbol{\sigma}_1 \cdot (\mathbf{p}_1 + \tilde{\mathbf{A}}_1)\phi\|_2 = \sqrt{T_{\mathrm{P}}[\phi, \tilde{\mathbf{A}}]} \leq \sqrt{|\mathcal{E}[\phi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0]| + |V[\phi]|} \leq \sqrt{C_2}. \tag{107}$$

To estimate $\|\tilde{\mathbf{A}}_1 \phi\|_2$ we rely on the Gagliardo–Nirenberg inequality

$$\|f\|_3 \leq c\|f\|_2^{1/2}\|\mathbf{p}f\|_2^{1/2}, \qquad \forall f \in H^1(\mathbb{R}^3; \mathbb{C}),$$

where $c > 0$ is some universal constant. Using this inequality, together with the Young's inequality for products: $2ab \leq (\delta^{-1}a)^2 + (\delta b)^2$, we find

$$\begin{aligned}\|\tilde{\mathbf{A}}_1 \phi\|_2 &\leq \|\tilde{\mathbf{A}}\|_6 \left[\int \left(\int |\phi_{\underline{\mathbf{z}}_1'}(\mathbf{z}_1)|^3 \mathrm{d}\mathbf{z}_1\right)^{2/3} \mathrm{d}\underline{\mathbf{z}}_1'\right]^{1/2} \\ &\leq \sqrt{4\pi\alpha^2 S_3^{-1} C_2}\left[\delta^{-1}\|\phi\|_2 + \delta\|\mathbf{p}_1\|_2\right],\end{aligned} \tag{108}$$

where S_3 is the sharp constant in Sobolev's inequality on $\mathbb{R}^3$: $S_3\|f\|_6^2 \leq \|\nabla f\|_2^2$. Choosing δ so that

$$\sqrt{16\pi\alpha^2 S_3^{-1} C_2 \delta} = \frac{1}{2}$$

we can feed (107) and (108) back into (106) and arrive at a uniform bound on $\|\mathbf{p}\phi\|_2$. Summarizing, we've derived the bounds (24) for $m = 2$. That these uniform estimates in (24) hold for $1 \leq m < 2$ follows immediately from the convergence result in Theorem 2. The last claim of Theorem 3 follows immediately from the uniform estimates in the energy class (24) and the blow-up alternative in Theorem 2. □

5. Proof of Theorem 1

The proof of Theorem 1 below follows the proof of Theorem 4.1 of [8] with minor modifications.

Proof of Theorem 1. Consider

$$(\psi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0) \in \bigwedge^N H^1(\mathbb{R}^3; \mathbb{C}^2) \times H^1(\mathbb{R}^3; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3),$$

with $\|\psi_0\|_2 = 1$ and $\operatorname{div} \mathbf{a}_0 = \operatorname{div} \dot{\mathbf{a}}_0 = 0$. Let $\{\varepsilon_n\}_{n\geq 1} \subset \mathbb{R}_+$ with $\varepsilon_n \to 0$. Combining Theorems 2 and 3, there exists a sequence of solutions

$$\{(\phi^n, \mathbf{A}^n)\}_{n\geq 1} \subset C(\mathbb{R}_+; \bigwedge^N H^1(\mathbb{R}^3; \mathbb{C}^2)) \times [C(\mathbb{R}_+; H^1(\mathbb{R}^3; \mathbb{R}^3)) \cap C^1(\mathbb{R}_+; L^2(\mathbb{R}^3; \mathbb{R}^3))]$$

of the modified equations

$$\begin{cases} \partial_t \phi^n - (i + \varepsilon_n) \sum_{j=1}^N \Delta_{\mathbf{x}_j} \phi^n \\ \qquad = \varepsilon_n (T_n + V_n)\phi^n - (i + \varepsilon_n) \left(\mathscr{L}(\tilde{\mathbf{A}}^n) - V(\underline{\mathbf{R}}, \mathcal{Z}) \right) \phi^n, \\ \Box \mathbf{A}^n = 4\pi\alpha \Lambda_{\varepsilon_n}^{-1} \mathcal{P} \boldsymbol{\mathcal{J}}[\phi^n, \tilde{\mathbf{A}}^n], \\ \operatorname{div} \mathbf{A}^n = 0, \quad \tilde{\mathbf{A}}^n = \Lambda_{\varepsilon_n}^{-1} \mathbf{A}^n, \\ (\phi^n(0), \mathbf{A}^n(0), \partial_t \mathbf{A}^n(0)) = (\psi_0, \mathbf{a}_0, \dot{\mathbf{a}}_0) \end{cases} \tag{109}$$

where $T_n = T_{\mathrm{P}}[\phi^n, \mathbf{A}^n]$, $V_n = V[\phi^n]$, and $\mathscr{L}(\tilde{\mathbf{A}}^n) = \sum_{j=1}^N \mathcal{L}_j(\tilde{\mathbf{A}}^n)$ is given by (25). Moreover, the bounds

$$\|\nabla \phi^n(t)\|_2 \leq C_1, \qquad F[\mathbf{A}^n, \partial_t \mathbf{A}^n](t) \leq C_2, \qquad \|\mathbf{A}^n(t)\|_2 \leq C_3(1 + t) \tag{110}$$

are satisfied. The estimates (26) and (35) of Lemmas 5 and 6, respectively, yield

$$\|[\mathscr{L}(\tilde{\mathbf{A}}^n) - V(\underline{\mathbf{R}}, \mathcal{Z})]\phi^n\|_{\frac{3}{2}} \lesssim (1 + \|\mathbf{A}^n\|_{1,2})\|\mathbf{A}^n\|_{1,2}\|\phi^n\|_{1,2} + \|\phi^n\|_{1,2}. \tag{111}$$

Furthermore, in the same way we estimated (71), we have

$$\|\boldsymbol{\mathcal{J}}[\phi^n, \tilde{\mathbf{A}}^n]\|_{\frac{3}{2}} \lesssim (1 + \|\mathbf{A}^n\|_{1,2})\|\phi^n\|_{1,2}. \tag{112}$$

The bounds (110) allow us to apply the Banach-Alaoglu Theorem, and, thus, we may extract a subsequence, still denoted by $\{(\phi^n, \mathbf{A}^n)\}_{n\geq 1}$, such that

$$\mathbf{A}^n \xrightarrow{\text{w}^*} \mathbf{A} \quad \text{in} \quad L^\infty([0,T]; H^1), \tag{113}$$

$$\partial_t \mathbf{A}^n \xrightarrow{\text{w}^*} \partial_t \mathbf{A} \quad \text{in} \quad L^\infty(\mathbb{R}_+; L^2) \tag{114}$$

$$\phi^n \xrightarrow{\text{w}^*} \psi \quad \text{in} \quad L^\infty(\mathbb{R}_+; H^1), \tag{115}$$

$$\mathcal{J}[\phi^n, \tilde{\mathbf{A}}^n] \xrightarrow{\text{w}^*} \beta \quad \text{in} \quad L^\infty([0,T]; L^{\frac{3}{2}}) \tag{116}$$

$$[\mathscr{L}(\tilde{\mathbf{A}}^n) - V(\underline{\mathbf{R}}, \mathcal{Z})]\phi^n \xrightarrow{\text{w}^*} \gamma \quad \text{in} \quad L^\infty([0,T]; L^{\frac{3}{2}}), \tag{117}$$

for all $0 < T < \infty$. Passing to the limit in (109), and using (113) through (117), we find

$$\begin{cases} \partial_t \psi - i \sum_{j=1}^N \Delta_{\mathbf{x}_j} \psi = -i\gamma, \\ \Box \mathbf{A} = 4\pi\alpha \mathcal{P} \beta, \\ \operatorname{div} \mathbf{A} = 0 \end{cases} \tag{118}$$

as equations in $\mathscr{D}'(\mathbb{R}_+; \bigwedge^N H^{-1}(\mathbb{R}^3; \mathbb{C}^2) \times \mathscr{D}'(\mathbb{R}_+; H^{-1}(\mathbb{R}^3; \mathbb{R}^3))$. We note that in passing to the limit we've used Theorem 1 and the dissipation of energy (23) to ensure $|T_n + V_n| \nrightarrow \infty$ as $\varepsilon_n \to 0$. Now, $\partial_t \mathbf{A} \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3; \mathbb{R}^3))$, $\partial_t^2 \mathbf{A} \in L^\infty(\mathbb{R}_+; H^{-1}(\mathbb{R}^3; \mathbb{R}^3))$, and $\partial_t \psi \in L^\infty(\mathbb{R}_+; H^{-1}(\mathbb{R}^{3N}; \mathbb{C}^{2^N}))$ by (118). Thus

$$(\psi, \mathbf{A}, \partial_t \mathbf{A}) \in L^\infty_{\text{loc}}(\mathbb{R}_+; H^1 \oplus H^1 \oplus L^2) \cap C(\mathbb{R}_+; H^{-1} \oplus L^2 \oplus H^{-1}),$$

and this implies the weak continuity $(\psi, \mathbf{A}, \partial_t \mathbf{A}) \in C^{\text{w}}(\mathbb{R}_+; H^1 \oplus H^1 \oplus L^2)$.

Next we show that $\gamma = [\mathscr{L}(\mathbf{A}) - V(\underline{\mathbf{R}}, \mathcal{Z})]\psi$ and $\beta = \mathcal{J}[\psi, \mathbf{A}]$. Let $I \subset \mathbb{R}_+$ be a bounded interval and $\Omega \subset \mathbb{R}^3$, $S \subset \mathbb{R}^{3N}$ be bounded and open, and assume $\partial\Omega$, ∂S are both C^1. It suffices to show that γ and β coincide with $[\mathscr{L}(\mathbf{A}) - V(\underline{\mathbf{R}}, \mathcal{Z})]\psi$ and $\mathcal{J}[\psi, \mathbf{A}]$ on $I \times S$ and $I \times \Omega$, respectively. Now, by (110), $\{(\mathbf{A}^n, \partial_t \mathbf{A}^n)\}_{n\geq 1}$ is a bounded sequence in $L^4(I; H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3))$. From the Rellich-Kondrachov Theorem we have $H^1(\Omega; \mathbb{R}^3) \hookrightarrow L^4(\Omega; \mathbb{R}^3) \subset L^2(\Omega; \mathbb{R}^3)$ and, hence, the Aubin-Lions Lemma (see [25, Theorem 1.20]) guarantees that there is a subsequence of $\{\mathbf{A}^n\}_{n\geq 1}$, still denoted by $\{\mathbf{A}^n\}_{n\geq 1}$, such that

$$\mathbf{A}^n \xrightarrow{n\to\infty} \mathbf{A} \quad \text{in} \quad L^4(I \times \Omega) \tag{119}$$

Further, note that $\{\partial_t \phi^n\}_{n\geq 1}$ is bounded in $L^\infty(I; H^{-1}(S; \mathbb{C}^{2^N}))$ using (109). This implies that $\{(\phi^n, \partial_t \phi^n)\}_{n\geq 1}$ is bounded in

$$L^2(I; H^1(S; \mathbb{C}^{2^N}) \times H^{-1}(S; \mathbb{C}^{2^N})).$$

Again applying the Aubin-Lions Lemma, we conclude

$$\phi^n \xrightarrow{n\to\infty} \psi \quad \text{in} \quad L^4(I \times S) \tag{120}$$

From (113), (115), (119), and (120) it is straightforward to show that

$$\Lambda_{\varepsilon_n}^{-1} \mathcal{J}[\phi^n, \tilde{\mathbf{A}}^n] \rightharpoonup \mathcal{J}[\psi, \mathbf{A}] \quad \text{in} \quad L^{\frac{4}{3}}(I \times \Omega),$$

$$[\mathscr{L}(\tilde{\mathbf{A}}^n) - V(\underline{\mathbf{R}}, \mathcal{Z})]\phi^n \rightharpoonup [\mathscr{L}(\mathbf{A}) - V(\underline{\mathbf{R}}, \mathcal{Z})]\psi \quad \text{in} \quad L^{\frac{4}{3}}(I \times S).$$

Moreover (116) through (117) imply

$$\Lambda_{\varepsilon_n}^{-1}\mathbf{J}[\phi^n, \tilde{\mathbf{A}}^n] \rightharpoonup \beta \quad \text{in} \quad L^{\frac{4}{3}}(I\times\Omega),$$
$$[\mathscr{L}(\tilde{\mathbf{A}}^n) - V(\underline{\mathbf{R}}, \mathcal{Z})]\phi^n \rightharpoonup \gamma \quad \text{in} \quad L^{\frac{4}{3}}(I\times S).$$

Since weak limits are unique we conclude $\gamma = [\mathscr{L}(\mathbf{A}) - V(\underline{\mathbf{R}}, \mathcal{Z})]\psi$ and $\beta = \boldsymbol{\mathcal{J}}[\psi, \mathbf{A}]$ on $I\times\Omega$ and $I\times S$, respectively.

It remains to show that $(\psi, \mathbf{A}, \partial_t\mathbf{A})$ satisfies the initial conditions in (109). Since

$$(\mathbf{A}^n, \partial_t\mathbf{A}^n) \in L^2([0,T]; H^1(\mathbb{R}^3;\mathbb{R}^3)\times L^2(\mathbb{R}^3;\mathbb{R}^3)),$$

we may integrate by parts to find

$$\int_0^T \langle \mathbf{A}^n(s)\partial_t f(s) + \partial_t\mathbf{A}^n(s) f(s), \phi\rangle_{H^1,H^{-1}} ds = -\langle \mathbf{a}_0, \phi\rangle_{H^1,H^{-1}}$$

for all $\phi \in L^2$ and $f \in C^\infty(\mathbb{R})$ with $f(0) = 1$ and $f(T) = 0$. Passing to the limit $\varepsilon_n \to 0$ and using (114) and (115) we find

$$\int_0^T \{\mathbf{A}(s)\partial_t f(s) + \partial_t\mathbf{A}(s) f(s)\}\, ds = -\mathbf{a}_0$$

in $L^2(\mathbb{R}^3)$, which implies that $\mathbf{A}(0) = \mathbf{a}_0$. Likewise,

$$\begin{aligned} &-\langle \dot{\mathbf{a}}_0, \eta\rangle_{H^{-1},H^1} \\ &= \int_0^T \langle \partial_t\mathbf{A}^n(s)\partial_t f(s) + (\Delta\mathbf{A}^n(s) + 4\pi\alpha\Lambda_{\varepsilon_n}^{-1}\mathcal{P}\boldsymbol{\mathcal{J}}[\phi^n(s), \tilde{\mathbf{A}}_n(s)]) f(s), \eta\rangle_{H^{-1},H^1} ds \end{aligned}$$

for all $\eta \in H^1$ and $f \in C^\infty(\mathbb{R})$ with $f(0) = 1$ and $f(T) = 0$. Again, passing to the limit as $n \to \infty$ and using (115) and (118), we arrive at

$$\int_0^T \left\{\partial_t\mathbf{A}(s)\partial_t f(s) + \partial_t^2\mathbf{A}(s) f(s)\right\} ds = -\dot{\mathbf{a}}_0$$

in H^{-1}, which implies $\partial_t\mathbf{A}(0) = \dot{\mathbf{a}}_0$. An identical argument implies that $\phi(0) = \phi_0$. □

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