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Matrix Resolvent and the Discrete KdV Hierarchy

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Abstract: Based on the matrix-resolvent approach, for an arbitrary solution to the discrete KdV hierarchy, we define the tau-function of the solution, and compare it with another tau-function of the solution defined via reduction of the Toda lattice hierarchy. Explicit formulae for generating series of logarithmic derivatives of the tau-functions are obtained, and applications to enumeration of ribbon graphs with even valencies and to certain special cubic Hodge integrals are considered.

Contents

Boris Dubrovin: Deceased on March 19, 2019.

1. Introduction

The discrete KdV equation (*aka the Volterra lattice equation*) is an integrable Hamiltonian equation in $(1+1)$ dimensions, i.e. one discrete space variable and one continuous time variable, which extends to a commuting system of Hamiltonian equations, called the discrete KdV integrable hierarchy. This integrable hierarchy has important applications in algebraic geometry and symplectic geometry (in particular in the theory of Riemann surfaces) (see e.g. [22]). Significance of the discrete KdV hierarchy was further pointed out by E. Witten [39] in the study of the GUE partition function with even couplings—the *matrix gravity*, and was recently addressed also in the study of the special cubic Hodge partition function [15,19,20]—the *topological gravity* in the sense of [15,20]. The explicit relationship between the two gravities, called the Hodge–GUE correspondence, has been established in [15,20]. In this paper, by using the matrix-resolvent (MR) approach recently introduced and developed in [1–3,14,18] we study the tau-structure for the discrete KdV hierarchy, and apply it to studying the above mentioned enumerative problems.

1.1. The discrete KdV hierarchy. Let $P(n)$ be the following difference operator

$$P(n) := \Lambda + w_n \Lambda^{-1}, \tag{1}$$

where Λ denotes the shift operator $\Lambda : f_n \mapsto f_{n+1}$. Introduce

$$A_\ell := \left(P^{\ell+1}\right)_+, \quad \ell \geq 0. \tag{2}$$

Here, for an operator Q of the form $Q = \sum_{k\in\mathbb{Z}} Q_k \Lambda^k$, the positive part $Q_+ := \sum_{k\geq 0} Q_k \Lambda^k$. The discrete KdV hierarchy is defined as the following system of commuting flows:

$$\frac{\partial P}{\partial s_j} = \left[A_{2j-1},\ P\right], \qquad j \geq 1. \tag{3}$$

For example, the s_1-flow reads

$$\frac{\partial w_n}{\partial s_1} = w_n\,(w_{n+1} - w_{n-1}), \tag{4}$$

which is the *discrete KdV equation*. The commutativity implies that Eq. (3) for all $j \geq 1$ can be solved together, yielding solutions of the form $w_n = w_n(\mathbf{s})$, $\mathbf{s} := (s_1, s_2, s_3, \ldots)$.

Let us introduce

$$L := P^2 = \Lambda^2 + w_{n+1} + w_n + w_n\, w_{n-1}\, \Lambda^{-2}. \tag{5}$$

Then $A_{2j-1} = \left(P^{2j}\right)_+ = \left(L^j\right)_+$.

Lemma 1. *The discrete KdV hierarchy* (3) *can be equivalently written as*

$$\frac{\partial L}{\partial s_j} = \left[A_{2j-1},\, L\right], \qquad j \geq 1. \tag{6}$$

The proof will be given in Sect. 2. For the particular case $j = 1$, we have

$$\frac{\partial(w_{n+1} + w_n)}{\partial s_1} = w_{n+2}\, w_{n+1} - w_n\, w_{n-1}, \tag{7}$$

$$\frac{\partial(w_n\, w_{n-1})}{\partial s_1} = (w_{n+1} + w_n - w_{n-1} - w_{n-2})\, w_n\, w_{n-1}. \tag{8}$$

It can be shown that Eqs. (7)–(8) are equivalent to Eq. (4); the details for this equivalence are in Sect. 2.2.

Observe that Eq. (6) are the compatibility conditions of the following scalar Lax pairs:

$$L\psi_n = \lambda\, \psi_n, \quad \text{i.e.} \quad \psi_{n+2} + (w_{n+1} + w_n - \lambda)\, \psi_n + w_n\, w_{n-1}\, \psi_{n-2} = 0, \tag{9}$$

$$\frac{\partial \psi_n}{\partial s_j} = A_{2j-1}\, \psi. \tag{10}$$

We want to write the spectral problem (9) into a matrix form. The scalar Lax operator L, defined in (5), could be viewed as a reduction of

$$\widetilde{L} = \Lambda^2 + a_1(n)\, \Lambda + a_2(n) + a_3(n)\, \Lambda^{-1} + a_4(n)\, \Lambda^{-2},$$

which is the Lax operator of a bigraded Toda hierarchy. However, observe that L contains Λ^{even} only (with even $= -2, 0, 2$). So, instead of considering a 4×4 matrix-valued Lax operator, a 2×2 matrix-valued operator will be sufficient. Indeed, introduce

$$\mathcal{L} := \begin{pmatrix} \Lambda^2 & 0 \\ 0 & \Lambda^2 \end{pmatrix} + U_n, \qquad U_n := \begin{pmatrix} w_{n+1} + w_n - \lambda & w_n\, w_{n-1} \\ -1 & 0 \end{pmatrix}. \tag{11}$$

Then the spectral problem (9) reads

$$\mathcal{L} \begin{pmatrix} \psi_n \\ \psi_{n-2} \end{pmatrix} = 0. \tag{12}$$

1.2. The MR approach to tau-functions. In this subsection, we apply the MR approach to studying further some basics in the theory of the discrete KdV hierarchy (in particular about tau-function), and will arrive at a formula for computing logarithm of the tau-function. Denote by $\mathbb{Z}[\mathbf{w}]$ the ring of polynomials with integer coefficients in the variables $\mathbf{w} := (w_{n+i})|_{i\in\mathbb{Z}}$.

Definition 1. An element $R_n \in \mathrm{Mat}\big(2, \mathbb{Z}[\mathbf{w}]((\lambda^{-1}))\big)$ is called a **matrix resolvent** (MR) of $\mathcal{L}$, if

$$R_{n+2}\, U_n - U_n\, R_n = 0. \tag{13}$$

Definition 2. The basic (matrix) resolvent R_n is defined as the MR of $\mathcal{L}$ satisfying

$$R_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}\left(\lambda^{-1}\right), \tag{14}$$

$$\operatorname{tr} R_n = 1, \quad \det R_n = 0. \tag{15}$$

The basic resolvent R_n exists and is unique. See in Sect. 3 for the proof. Write

$$R_n(\lambda) = \begin{pmatrix} 1 + \alpha_n(\lambda) & \beta_n(\lambda) \\ \gamma_n(\lambda) & -\alpha_n(\lambda) \end{pmatrix}. \tag{16}$$

Then Definition 2 for $R_n(\lambda)$ is equivalent to the following set of equations

$$\beta_n = -w_n\, w_{n-1}\, \gamma_{n+2} \tag{17}$$

$$\alpha_{n+2} + \alpha_n + 1 = (\lambda - w_{n+1} - w_n)\, \gamma_{n+2} \tag{18}$$

$$(\lambda - w_{n+1} - w_n)(\alpha_n - \alpha_{n+2}) = w_n\, w_{n-1}\, \gamma_n - w_{n+2}\, w_{n+1}\, \gamma_{n+4} \tag{19}$$

$$\alpha_n + \alpha_n^2 + \beta_n\, \gamma_n = 0 \tag{20}$$

together with Eq. (14). These equations give recursive relations and initial values for the coefficients of $\alpha_n,\ \beta_n,\ \gamma_n$ (see (60)–(62) below), which will be called the *MR recursive relations*.

Lemma 2. *For an arbitrary solution $w_n(\mathbf{s})$ to the discrete KdV hierarchy, let $R_n(\lambda)$ denote the basic matrix resolvent of $\mathcal{L}$ evaluated at $w_n = w_n(\mathbf{s})$. There exists a function $\tau_n^{\mathrm{dKdV}}(\mathbf{s})$ satisfying*

$$\sum_{i,j\geq 1} \frac{\partial^2 \log \tau_n^{\mathrm{dKdV}}(\mathbf{s})}{\partial s_i \partial s_j} \lambda^{-i-1} \mu^{-j-1} = \frac{\operatorname{tr}\left(R_n(\lambda) R_n(\mu)\right) - 1}{(\lambda - \mu)^2}, \tag{21}$$

$$\frac{1}{\lambda} + \sum_{i\geq 1} \frac{1}{\lambda^{i+1}} \frac{\partial}{\partial s_i} \log \frac{\tau_{n+2}^{\mathrm{dKdV}}}{\tau_n^{\mathrm{dKdV}}} = \left[R_{n+2}(\lambda)\right]_{21}, \tag{22}$$

$$\frac{\tau_{n+2}^{\mathrm{dKdV}}\, \tau_{n-1}^{\mathrm{dKdV}}}{\tau_{n+1}^{\mathrm{dKdV}}\, \tau_n^{\mathrm{dKdV}}} = w_n. \tag{23}$$

Moreover, the function $\tau_n^{\mathrm{dKdV}}(\mathbf{s})$ is uniquely determined by $w_n(\mathbf{s})$ up to a factor of the form

$$e^{\alpha n + \beta_0 + \sum_{k\geq 1} \beta_k s_k},$$

where $\alpha, \beta_0, \beta_1, \beta_2, \ldots$ are arbitrary constants that are independent of $n, \mathbf{s}$.

We call $\tau_n^{\mathrm{dKdV}}(\mathbf{s})$ the *tau-function* of the solution $w_n = w_n(\mathbf{s})$ to the discrete KdV hierarchy.

The matrix-resolvent method then allows to compute logarithmic derivatives of $\tau_n^{\mathrm{dKdV}}(\mathbf{s})$, which is achieved via the following proposition.

Proposition 1. *For any $k \geq 3$, the generating series of the k_{th}-order logarithmic derivatives of $\tau_n^{\mathrm{dKdV}}(\mathbf{s})$ has the following expression:*

$$\sum_{j_1,\ldots,j_k=1}^{\infty} \frac{1}{\lambda_1^{j_1+1} \cdots \lambda_k^{j_k+1}} \frac{\partial^k \log \tau_n^{\mathrm{dKdV}}(\mathbf{s})}{\partial s_{j_1} \ldots \partial s_{j_k}} = -\frac{1}{k} \sum_{\sigma\in S_k} \frac{\operatorname{tr}\left(R_n(\lambda_{\sigma_1}) \cdots R_n(\lambda_{\sigma_k})\right)}{\prod_{i=1}^k (\lambda_{\sigma_i} - \lambda_{\sigma_i+1})}, \tag{24}$$

where it is understood that $\sigma_{k+1} = \sigma_1$.

The proof of this proposition is in Sect. 3.5.

1.3. The factorization formula. In [14] we gave the definition of tau-function for the Toda lattice using the MR approach. Observe that the discrete KdV hierarchy (3) is a reduction of the Toda lattice hierarchy. Therefore, for the arbitrary solution $w_n(\mathbf{s})$ to the discrete KdV, we can also associate another tau-function $\tau_n(\mathbf{s})$ of the solution $w_n(\mathbf{s})$ obtained via the reduction (see Sect. 4.2 for the precise definition). In particular, this tau-function satisfies that

$$w_n(\mathbf{s}) = \frac{\tau_{n+1}(\mathbf{s})\,\tau_{n-1}(\mathbf{s})}{\tau_n^2(\mathbf{s})}.$$

It turns out that the $\tau_n(\mathbf{s})$ factorizes into a product of two as given by the following theorem.

Theorem 1. *There exist constants $\alpha, \beta_0, \beta_1, \beta_2, \ldots$ such that*

$$\tau_n(\mathbf{s}) = e^{\alpha n+\beta_0+\sum_{k\geq 1}\beta_k s_k}\,\tau_n^{\mathrm{dKdV}}(\mathbf{s})\,\tau_{n+1}^{\mathrm{dKdV}}(\mathbf{s}). \tag{25}$$

The proof of this theorem is in Sect. 4.

Remark. Identity (25) echoes an identity between Hankel determinants. Indeed, let $d\mu(\lambda)$ be a measure with even moments on $\mathbb{R}$. Denote $\mu_j = \int \lambda^j d\mu(\lambda)$, $j \geq 0$. ($\mu_{\mathrm{odd}} = 0$.) We know that

$$\det\big(\mu_{i+j-2}\big)_{i,j=1}^{n} = \det\big(\mu_{2i+2j-2}\big)_{i,j=1}^{[n/2]}\det\big(\mu_{2i+2j-4}\big)_{i,j=1}^{[(n+1)/2]}. \tag{26}$$

If we deform the measure $d\mu(\lambda)$ to be $d\mu(\lambda;\mathbf{t}) = e^{-\sum_{j\geq 1} t_{j-1}\lambda^j}d\mu(\lambda)$, then the LHS$\cdot(2\pi)^{-n}$ becomes a Toda tau-function (cf. the formula (3.9) of [10] and the references therein; cf. also [10,14,34]; the $(2\pi)^{-n}$ is a normalization factor for convenience that does not affect the fact that the LHS is already a Toda tau-function). If all the even Toda times are zero, then the $\mathbf{t}$-deformed measure remains even and the factorization (26) holds identically in $\mathbf{t} = (0, s_1, 0, s_2, \ldots)$. Moreover, note that the RHS of (26) with deformation consists of two determinants which can be identified with the Hankel determinants associated with certain $\mathbf{s}$-deformed measures on $\mathbb{R}_+$, where $\mathbf{s} = (s_1, s_2, \ldots)$. Then to see (25) from (26), at least for special cases, one needs to further show that each of the *two* determinants is a tau-function for the discrete KdV hierarchy. The more precise statements for a special case may be deduced from the recent arXiv preprint by Massimo Gisonni, Giulio Ruzza and Tamara Grava [26] regarding Laguerre Unitary Ensemble (LUE) with the consideration of the parameters $\alpha = -1/2$ and $\alpha = 1/2$, *respectively* in the notations of [26] (cf. also [8,9]).

The next corollary follows from Proposition 1 and Theorem 1.

Corollary 1. *Fix $k \geq 2$. Let $w_n = w_n(\mathbf{s})$ be an arbitrary solution to the discrete KdV hierarchy, and τ_n the tau-function reduced from the Toda lattice hierarchy of $w_n(\mathbf{s})$. The following formula holds true:*

$$\sum_{j_1,\ldots,j_k=1}^{\infty}\frac{\frac{\partial^k \log\tau_n(\mathbf{s})}{\partial s_{j_1}\ldots\partial s_{j_k}}}{\lambda_1^{j_1+1}\cdots\lambda_k^{j_k+1}} = -\frac{1}{k}\sum_{\sigma\in S_k}\frac{(1+\Lambda)\,\mathrm{tr}\big[R_n(\lambda_{\sigma_1})\cdots R_n(\lambda_{\sigma_k})\big]}{\prod_{i=1}^{k}(\lambda_{\sigma_i}-\lambda_{\sigma_{i+1}})} - \frac{2\,\delta_{k,2}}{(\lambda_1-\lambda_2)^2}. \tag{27}$$

In practice, the two tau-functions τ_n and τ_n^{dKdV} of some solution for the discrete KdV hierarchy may both have geometric/enumerative meanings; this is the case for the Hodge–GUE (see below).

Remark. As we shall see from Sect. 4.2 that the above mentioned reduction *does not* mean that v_n^{Toda}, w_n^{Toda} (see Sect. 4.2) are independent of the even Toda times $t_0, t_2, \dots$. The reduction means the $v_n^{\mathrm{Toda}}(0, t_1, 0, t_3, \dots) \equiv 0$; but the usage of the *MR of Toda* in the way of [14] would compute also the correlators containing the correspondence to $t_0, t_2, \dots$. The introductions of the *MR of the discrete KdV hierarchy* and of the operator $1 + \Lambda$ are essential that surprisingly solve the problem in a simple form.

1.4. Application. We will first apply Corollary 1 to some counting problem. Then by using the Hodge–GUE correspondence [15,20] we compute some combinations of Hodge integrals.

I. Enumeration of ribbon graphs with even valencies. Enumeration of ribbon graphs is closely related to the random matrix theory [4,7,11,27,30,34]: e.g. to the Gaussian Unitary Ensembles (GUE) correlators; the partition function with coupling constants in a random matrix theory is often a tau-function of some integrable system. Given $k \geq 1$ and $j_1, \dots, j_k \geq 1$, denote

$$\left\langle \mathrm{tr}\, M^{2j_1} \cdots \mathrm{tr}\, M^{2j_k} \right\rangle_c := k! \sum_{0 \leq g \leq \frac{|j|}{2} - \frac{k}{2} + \frac{1}{2}} n^{2-2g-k+|j|}\, a_g(2j_1, \dots, 2j_k), \tag{28}$$

$$a_g(2j_1, \dots, 2j_k) := \sum_{\Gamma} \frac{1}{\#\,\mathrm{Sym}\,\Gamma}. \tag{29}$$

Here, $|j| = j_1 + \cdots + j_k$, and $\sum_\Gamma$ denotes summation over connected ribbon graphs Γ with labelled half edges and unlabelled vertices of genus g with k vertices of valencies $2j_1, \dots, 2j_k$, and $\#\,\mathrm{Sym}\,\Gamma$ is the order of the symmetry group of Γ generated by permuting the vertices.[1] The notation $\left\langle \mathrm{tr}\, M^{2j_1} \cdots \mathrm{tr}\, M^{2j_k} \right\rangle_c$ is borrowed from the literature of random matrices, where it is often called a connected Gaussian Unitary Ensemble (GUE) correlator. For every $k \geq 1$, denote

$$E_k(n; \lambda_1, \dots, \lambda_k) := \sum_{j_1, \dots, j_k = 1}^{\infty} \frac{\left\langle \mathrm{tr}\, M^{2j_1} \cdots \mathrm{tr}\, M^{2j_k} \right\rangle_c}{\lambda_1^{j_1+1} \cdots \lambda_k^{j_k+1}}. \tag{30}$$

Definition 3. Define a 2×2 matrix-valued series $R_n(\lambda) \in \mathrm{Mat}\big(2, \mathbb{Z}[n][[\lambda^{-1}]]\big)$ by

$$R_n(\lambda) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{j=0}^{\infty} \frac{(2j-1)!!}{\lambda^{j+1}} \begin{pmatrix} (2j+1) A_{n,j} - (n-1) B_{n,j} & (n - n^2)\, B_{n+2,j} \\ B_{n,j} & (n-1) B_{n,j} - (2j+1) A_{n,j} \end{pmatrix} \tag{31}$$

with

$$A_{n,j} := (n-1)\, {}_2F_1(-j, 2-n; 2; 2), \tag{32}$$

$$B_{n,j} := (n-1)\, {}_2F_1(1-j, 2-n; 2; 2) + (n-2)\, {}_2F_1(1-j, 3-n; 2; 2). \tag{33}$$

[1] The number $a_g(2j_1, \dots, 2j_k)$ has the alternative expression $a_g(2j_1, \dots, 2j_k) = \sum_G \frac{\prod_{\ell=1}^k (2j_\ell)}{\#\,\mathrm{Sym}\, G}$, where $\sum_G$ denotes summation over connected ribbon graphs G with unlabelled half-edges and unlabelled vertices of genus g with k vertices of valencies $2j_1, \dots, 2j_k$.

Theorem 2. *The following formulae hold true:*

$$E_1(n;\lambda) = n\sum_{j\geq 1}\frac{(2j-1)!!}{\lambda^{2j+1}}\Big({}_2F_1(-j,-n;2;2) - j\,{}_2F_1(1-j,1-n;3;2)\Big), \tag{34}$$

$$E_2(n;\lambda_1,\lambda_2) = \frac{(1+\Lambda)\left[\mathrm{tr}\left(R_n(\lambda_1)R_n(\lambda_2)\right)\right]}{(\lambda_1-\lambda_2)^2} - \frac{2}{(\lambda_1-\lambda_2)^2}, \tag{35}$$

$$E_k(n;\lambda_1,\dots,\lambda_k) = -\frac{1}{k}\sum_{\sigma\in S_k}\frac{(1+\Lambda)\left[\mathrm{tr}\left(R_n(\lambda_{\sigma_1})\cdots R_n(\lambda_{\sigma_k})\right)\right]}{\prod_{\ell=1}^{k}(\lambda_{\sigma_\ell}-\lambda_{\sigma_{\ell+1}})} \quad (k\geq 3), \tag{36}$$

where $R_n(\lambda)$ is defined in Definition 3, and it is understood that $\sigma_{k+1}=\sigma_1$.

In the above formulae

$$ {}_2F_1(a,b;c;z) = \sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\frac{z^j}{j!} = 1 + \frac{a\,b}{c}\frac{z}{1!} + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots$$

is the Gauss hypergeometric function. Recall that it truncates to a polynomial if a or b are non-positive integers. In particular,

$$n\,{}_2F_1(-j,1-n;2;2) = \sum_{i=0}^{j}2^i\binom{j}{i}\binom{n}{i+1}.$$

The proof of Theorem 2 is in Sect. 5.

II. Combinations of certain special cubic Hodge integrals. The particular solution to the discrete KdV hierarchy considered here will be actually the same as in I. Denote by $\overline{\mathcal{M}}_{g,k}$ the Deligne–Mumford moduli space of stable algebraic curves of genus g with k distinct marked points, by $\mathcal{L}_i$ the i_{th} tautological line bundle on $\overline{\mathcal{M}}_{g,k}$, and $\mathbb{E}_{g,k}$ the Hodge bundle. Denote

$$\psi_i := c_1(\mathcal{L}_i), \quad i=1,\dots,k,$$
$$\lambda_j := c_j(\mathbb{E}_{g,k}), \quad j=0,\dots,g.$$

The Hodge integrals are some rational numbers defined by

$$\int_{\overline{\mathcal{M}}_{g,k}}\psi_1^{i_1}\cdots\psi_k^{i_k}\,\lambda_1^{j_1}\cdots\lambda_g^{j_g} =: \left\langle\lambda_1^{j_1}\cdots\lambda_g^{j_g}\,\tau_{i_1}\cdots\tau_{i_k}\right\rangle_{g,k},$$
$$i_1,\dots,i_k,\ j_1,\dots,j_g\geq 0.$$

These numbers are zero unless the degree-dimension matching is satisfied

$$3g-3+k = \sum_{\ell=1}^{k}i_\ell + \sum_{\ell=1}^{g}\ell\,j_\ell. \tag{37}$$

We are particularly interested in the following *special cubic Hodge integrals*:

$$\left\langle\Omega_g\,\tau_{i_1}\cdots\tau_{i_k}\right\rangle_{g,k}, \qquad \text{with} \quad \Omega_g := \Lambda_g(-1)\,\Lambda_g(-1)\,\Lambda_g\Big(\frac{1}{2}\Big),$$

where $\Lambda_g(z) := \sum_{j=0}^{g} \lambda_j z^j$ denotes the Chern polynomial of the Hodge bundle $\mathbb{E}_{g,k}$. Significance of these Hodge integrals is manifested by the Gopakumar–Mariño–Vafa conjecture [25,33] regarding the Chern–Simons/string duality; see e.g. [37] and the references therein.

Notations. $\mathbb{Y}$ denotes the set of partitions. For a partition λ, denote by $\ell(\lambda)$ the length of λ and by $|\lambda|$ the weight of λ. Denote $m(\lambda) := \prod_{i=1}^{\infty} m_i(\lambda)$ with $m_i(\lambda)$ being the multiplicity of i in λ.

Definition 4. For given $g, k \geq 0$ and an arbitrary set of integers $i_1, \ldots, i_k \geq 0$, define

$$H_{g,i_1,\ldots,i_k} = 2^{g-1} \sum_{\lambda \in \mathbb{Y}} \frac{(-1)^{\ell(\lambda)}}{m(\lambda)!} \left\langle \Omega_g \, \tau_{\lambda+1} \, \tau_I \right\rangle_{g,\, \ell(\lambda)+k}, \tag{38}$$

where $|i| := i_1 + \cdots + i_k$, $\tau_I := \tau_{i_1} \cdots \tau_{i_k}$, and $\tau_{\lambda+1} := \tau_{\lambda_1+1} \cdots \tau_{\lambda_{\ell(\lambda)}+1}$.

It should be noted that according to (37), "$\sum_{\lambda\in\mathbb{Y}}$" in (38) is a finite sum.

The following lemma will be proved in Sect. 5.2.

Lemma 3. *The number $H_{g,i_1,\ldots,i_k}$ vanishes unless $|i| \leq 3g - 3 + k$.*

Corollary 2. *The numbers $H_{g,i_1,\ldots,i_k}$ satisfy*

(i) For $k = 0$,

$$H_{g,\emptyset} = \begin{cases} 0, & g = 0, 1, \\ \frac{1}{4g(2g-1)(2g-2)} \sum_{g_1=0}^{g} (2g_1 - 1) \binom{2g}{2g_1} \frac{E_{2g-2g_1} B_{2g_1}}{2^{2g-2g_1}}, & g \geq 2. \end{cases}$$

(ii) For $k = 1$, $\forall\, j \geq 1$,

$$\begin{aligned} &\binom{2j}{j} \sum_{g\geq 0} \epsilon^{2g-1} \sum_{0\leq i\leq 3g-3+k} j^{i+1} H_{g,i} + \frac{1}{2\epsilon} \frac{1}{1+j} \binom{2j}{j} \\ &= \epsilon^j \left[\frac{(2j+1)!!}{2j} A_{\frac12+\frac1\epsilon,\, j} + \frac{(2j-1)!!}{2j} \left(\frac12 - \frac1\epsilon \right) B_{\frac12+\frac1\epsilon,\, j} \right], \end{aligned} \tag{39}$$

where $A_{n,j}$ and $B_{n,j}$ are defined in (32)–(33).

(iii) For $k \geq 2$,

$$\begin{aligned} &\epsilon^k \sum_{j_1,\ldots,j_k\geq 1} \frac{\prod_{r=1}^{k} \binom{2j_r}{j_r}}{\lambda_1^{j_1+1} \cdots \lambda_k^{j_k+1}} \sum_{g\geq 0} \epsilon^{2g-2} \sum_{\substack{i_1,\ldots,i_k\geq 0 \\ |i|\leq 3g-3+k}} \prod_{r=1}^{k} j_\ell^{i_r+1} H_{g,i_1,\ldots,i_k} \\ &= -\frac1k \sum_{\sigma\in S_k} \frac{\operatorname{tr}\left[R_{\frac12+\frac1\epsilon}\left(\frac{\lambda_{\sigma_1}}{\epsilon}\right) \cdots R_{\frac12+\frac1\epsilon}\left(\frac{\lambda_{\sigma_k}}{\epsilon}\right) \right]}{\prod_{\ell=1}^{k} (\lambda_{\sigma_\ell} - \lambda_{\sigma_{\ell+1}})} - \frac{\delta_{k,2}}{(\lambda_1-\lambda_2)^2} \\ &\quad - \delta_{k,2} \sum_{j_1,j_2\geq 1} \frac{j_1\, j_2}{j_1+j_2} \frac{\binom{2j_1}{j_1}\binom{2j_2}{j_2}}{\lambda_1^{j_1+1}\lambda_2^{j_2+1}}, \end{aligned} \tag{40}$$

where $R_n(\lambda)$ is defined as in (31).

The proof, using the Hodge–GUE correspondence and Theorem 2, will be given in Sect. 5.2. We note that the sum $\sum_{\substack{i_1,\dots,i_k\geq 0\\ |i|\leq 3g-3+k}}$ appearing in the LHS of (39), (40) has the following alternative expression, which can be deduced from "Appendix A":

$$\sum_{\substack{i_1,\dots,i_k\geq 0\\ |i|\leq 3g-3+k}} \prod_{r=1}^k j_r^{i_r+1} H_{g,i_1,\dots,i_k}$$
$$= \sum_{q\geq k} \frac{1}{(q-k)!} \int_{\overline{\mathcal{M}}_{g,q}} \Omega_{g,q} \prod_{m=k+1}^q \left(-\frac{\psi_m^2}{1-\psi_m}\right) \prod_{m=1}^k \frac{j_m}{1-j_m\psi_m}.$$

Organization of the paper. In Sect. 2, we derive several useful formulae. In Sect. 3, we study MR, and use it to describe the discrete KdV flows and the tau-structure. Section 4 is devoted to the proof of Theorem 1. Proofs of Theorem 2 and Corollary 2 are in Sect. 5.

2. Basic Formulation

In this section we will do some preparations for the later sections by reviewing the basics of the theory of the discrete KdV hierarchy.

2.1. Some useful identities. Recall that $P(n) := \Lambda + w_n\,\Lambda^{-1}$, $L = P^2$. Denote

$$P(n)^{\ell+1} =: \sum_{k\in\mathbb{Z}} A_{\ell,k}(n)\,\Lambda^k, \qquad \ell\geq -1, \tag{41}$$
$$L(n)^j =: \sum_{k\in\mathbb{Z}} m_{j,k}(n)\,\Lambda^k, \qquad j\geq 0, \tag{42}$$

where the coefficients $A_{\ell,k}(n)$ and $m_{j,k}(n)$, $k\in\mathbb{Z}$ belong to $\mathbb{Z}[\mathbf{w}]$. It is easy to see that if k is odd, or if $|k|>2j$, then $m_{j,k}\equiv 0$. It is also easy to see that

$$m_{j,k} = A_{2j-1,k}. \tag{43}$$

Lemma 4. *The following identities hold true*

$$m_{j,-2}(n) = w_n\,w_{n-1}\,m_{j,2}(n-2), \tag{44}$$
$$m_{j,0}(n) = m_{j-1,-2}(n) + m_{j-1,-2}(n+2) + (w_{n+1}+w_n)\,m_{j-1,0}(n), \tag{45}$$
$$m_{j,-2}(n) - m_{j,-2}(n-2) - (w_{n-1}+w_{n-2})\big(m_{j-1,-2}(n) - m_{j-1,-2}(n-2)\big)$$
$$+ w_{n-2}\,w_{n-3}\,m_{j-1,0}(n-4) - w_n\,w_{n-1}\,m_{j-1,0}(n) = 0. \tag{46}$$

Proof. Comparing the constant terms of the identity

$$L^j = L^{j-1}\,L = L\,L^{j-1} \tag{47}$$

we obtain that

$$m_{j,0}(n) = m_{j-1,-2}(n) + (w_{n+1}+w_n)\,m_{j-1,0}(n) + w_{n+2}\,w_{n+1}\,m_{j-1,2}(n)$$
$$= m_{j-1,-2}(n+2) + (w_{n+1}+w_n)\,m_{j-1,0}(n) + w_n\,w_{n-1}\,m_{j-1,2}(n-2).$$

This proves (44)–(45). Similarly, comparing the coefficients of Λ^{-2} of (47) we obtain

$$\begin{aligned} m_{j,-2}(n) &= m_{j-1,-4}(n) + (w_{n-1} + w_{n-2})\, m_{j-1,-2}(n) + w_n\, w_{n-1}\, m_{j-1,0}(n) \\ &= m_{j-1,-4}(n+2) + (w_{n+1} + w_n)\, m_{j-1,-2}(n) + w_n\, w_{n-1}\, m_{j-1,0}(n-2), \end{aligned}$$

which implies identity (46). The lemma is proved. □

Lemma 5. *The following identities hold true*

$$A_{\ell,-1}(n) = w_n\, A_{\ell,1}(n-1), \tag{48}$$

$$A_{\ell,0}(n) = w_{n+1}\, A_{\ell-1,1}(n) + w_n\, A_{\ell-1,1}(n-1), \tag{49}$$

$$w_n\, A_{\ell,1}(n-1) - w_{n+1}\, A_{\ell,1}(n) + w_{n+1}\, A_{\ell-1,0}(n-1) - w_n\, A_{\ell-1,0}(n-1) = 0, \tag{50}$$

$$A_{\ell,0}(n+1) - A_{\ell,0}(n) = w_{n+2}\, A_{\ell,2}(n) - w_n\, A_{\ell,2}(n-1). \tag{51}$$

Proof. Identities (48)–(50) are contained in the Lemma 2.2.1 of [14] (see the proof therein). Identity (51) follows from comparing coefficients of Λ on the both sides of the following identity:

$$P^{\ell+1} P = P P^{\ell+1}.$$

The lemma is proved. □

Taking $\ell = 2j - 1$ in identity (51) and using (43) we obtain

$$m_{j,0}(n+1) - m_{j,0}(n) = w_{n+2}\, m_{j,2}(n) - w_n\, m_{j,2}(n-1). \tag{52}$$

We call this identity the *key identity*. It should be noted that the above identities (43)–(46), (48)–(51) hold in $\mathbb{Z}[\mathbf{w}]$ absolutely (namely, the validity does not require that w_n is a solution of the discrete KdV hierarchy), because they are nothing but properties of the operators P and L.

2.2. Proof of Lemma 1. Note that this lemma means the following: if $w_n = w_n(\mathbf{s})$ satisfies (3), then it satisfies (6); *vice versa*. Firstly, let $w_n = w_n(\mathbf{s})$ be an arbitrary solution to (3), i.e.,

$$\frac{\partial P}{\partial s_j} = \left[A_{2j-1},\ P\right]$$

for all $j \geq 1$. Since $L = P^2$ we have

$$\frac{\partial L}{\partial s_j} = P \frac{\partial P}{\partial s_j} + \frac{\partial P}{\partial s_j} P = P\left[A_{2j-1},\ P\right] + \left[A_{2j-1},\ P\right] P = [A_{2j-1},\ L].$$

Secondly, let $w_n = w_n(\mathbf{s})$ be an arbitrary solution to (6), namely, it satisfies that

$$\frac{\partial (w_{n+1} + w_n)}{\partial s_j} = w_{n+2}\, w_{n+1}\, m_{j,2}(n) - w_n\, w_{n-1}\, m_{j,2}(n-2), \tag{53}$$

$$\frac{\partial (w_n w_{n-1})}{\partial s_j} = w_n\, w_{n-1} \left(m_{j,0}(n) - m_{j,0}(n-2)\right). \tag{54}$$

Identity (53) implies that

$$(\Lambda+1)\frac{\partial w_n}{\partial s_j} = w_{n+2}\,w_{n+1}\,m_{j,2}(n) - w_{n+1}\,w_n\,m_{j,2}(n-1) + w_{n+1}\,w_n\,m_{j,2}(n-1) - w_n\,w_{n-1}\,m_{j,2}(n-2)$$
$$= w_{n+1}\big(m_{j,0}(n+1) - m_{j,0}(n)\big) + w_n\big(m_{j,0}(n) - m_{j,0}(n-1)\big),$$

where we have used identity (52). Identity (54) implies that

$$w_n\frac{\partial w_{n+1}}{\partial s_j} + w_{n+1}\frac{\partial w_n}{\partial s_j} = w_{n+1}\,w_n\,\big(m_{j,0}(n+1) - m_{j,0}(n)\big) + w_{n+1}\,w_n \big(m_{j,0}(n) - m_{j,0}(n-1)\big).$$

Combining the above two identities and assuming that $w_n \not\equiv w_{n+1}$ yields

$$\frac{\partial w_n}{\partial s_j} = w_n\,\big(m_{j,0}(n) - m_{j,0}(n-1)\big) = \mathrm{Coef}_{\Lambda^{-1}}\big[A_{2j-1}, P\big]. \tag{55}$$

(One can see from (53) that solutions satisfying $w_n \equiv w_{n+1}$ are independent of $\mathbf{s}$. Therefore these trivial solutions also satisfy (3).) The proposition is proved. □

2.3. Lax pairs in matrix form. In this subsection we write the scalar Lax pairs (9)–(10) into matrix form. The following lemma plays an important role.

Lemma 6. *The wave function ψ_n satisfies that*

$$\frac{\partial \psi_n}{\partial s_j} = \lambda^j\,\psi_n + \sum_{i=1}^{j}\lambda^{j-i}\,\big(m_{i-1,-2}\,\psi_n - w_n\,w_{n-1}\,m_{i-1,0}\,\psi_{n-2}\big), \quad j\geq 1. \tag{56}$$

Proof. We have for any $j\geq 1$

$$\big(L^j\big)_+ = \big(L^{j-1}L\big)_+ = \big(L^{j-1}\big)_+ L_+ + \Big(\big(L^{j-1}\big)_- L\Big)_+ + \Big(\big(L^{j-1}\big)_+ L_-\Big)_+$$
$$= \big(L^{j-1}\big)_+ L - \Big(\big(L^{j-1}\big)_+ L_-\Big)_- + \Big(\big(L^{j-1}\big)_- L\Big)_+$$
$$= \big(L^{j-1}\big)_+ L + m_{j-1,-2} - w_n\,w_{n-1}\,m_{j-1,0}\,\Lambda^{-2}.$$

In the above derivations it is understood that $L = L(n)$ and $m_{j,k} = m_{j,k}(n)$. Therefore,

$$A_{2j-1} = \big(L^j\big)_+ = L^j + \sum_{i=1}^{j}\big(m_{i-1,-2} - w_n\,w_{n-1}\,m_{i-1,0}\,\Lambda^{-2}\big)\,L^{j-i}, \qquad \forall\, j\geq 0.$$

The lemma is proved. □

Lemma 7. *The vector-valued wave function* $\Psi_n = \left(\psi_n, \psi_{n-2}\right)^T$ *satisfies that*

$$\frac{\partial \Psi_n}{\partial s_j} = V_j(n)\, \Psi_n, \qquad j \geq 1, \tag{57}$$

where $V_j(n)$ *are the following* 2×2 *matrices*

$$V_j(n) := \begin{pmatrix} \lambda^j + \sum_{i=1}^j \lambda^{j-i}\, m_{i-1,-2}(n) & -w_n\, w_{n-1} \sum_{i=1}^j \lambda^{j-i}\, m_{i-1,0}(n) \\ \sum_{i=1}^j \lambda^{j-i} m_{i-1,0}(n-2) & m_{j,0}(n-2) - \sum_{i=1}^j \lambda^{j-i} m_{i-1,-2}(n) \end{pmatrix}. \tag{58}$$

Proof. Equation (57) follows straightforwardly from (56) and (9). □

We therefore arrive at

Proposition 2. *The discrete KdV hierarchy are the compatibility conditions of* (12) *and* (57)*:*

$$\frac{\partial U_n}{\partial s_j} = V_j(n+2)\, U_n - U_n\, V_j(n), \qquad j = 1, 2, 3, \ldots.$$

3. Tau-Structure for the Discrete KdV Hierarchy

In this section, we use the MR method to study the tau-structure of the discrete KdV hierarchy; in particular, we will prove Proposition 1. The notations about the matrix-resolvents are the same as in the Introduction.

3.1. The MR recursive relations. Write

$$\alpha_n = \sum_{j\geq 0} \frac{a_{n,j}}{\lambda^{j+1}}, \qquad \gamma_n = \sum_{j\geq 0} \frac{c_{n,j}}{\lambda^{j+1}}. \tag{59}$$

Then we find that $a_{n,j},\ c_{n,j}$ satisfy

$$c_{n,j+1} = (w_{n-1} + w_{n-2})\, c_{n,j} + a_{n,j} + a_{n-2,j}, \tag{60}$$

$$a_{n,j+1} - a_{n+1,j+1} + (w_{n+1} + w_n)(a_{n+2} - a_{n,j}) + w_{n+1}\, w_n\, c_{n+4,j} - w_n\, w_{n-1}\, c_{n,j} = 0, \tag{61}$$

$$a_{n,j} = \sum_{i=0}^{j-1} \left(w_n\, w_{n-1}\, c_{n,i}\, c_{n,j-1-i} - a_{n,i}\, a_{n,j-1-i}\right) \tag{62}$$

as well as

$$a_{n,0} = 0, \qquad c_{n,0} = 1. \tag{63}$$

Lemma 8. *The basic resolvent of* $\mathcal{L}$ *exists and is unique.*

Proof. Observe that multiplying (18) and (19) gives (20). This proves existence of R_n. Uniqueness follows directly from the MR recursive relations (60)–(62), as we can solve $a_{n,j},\ c_{n,j}$ uniquely in an algebraic way for all $j \geq 1$. The lemma is proved. □

For the reader's convenience we give in below the first few terms of the basic resolvent of $\mathcal{L}$:

$$R_n(\lambda) = \begin{pmatrix} 1 + \frac{w_{n-1} w_n}{\lambda^2} + \cdots & -\frac{w_{n-1} w_n}{\lambda} - \frac{w_{n-1}(w_n + w_{n+1}) w_n}{\lambda^2} + \cdots \\ \frac{1}{\lambda} + \frac{w_{n-2} + w_{n-1}}{\lambda^2} + \cdots & -\frac{w_{n-1} w_n}{\lambda^2} + \cdots \end{pmatrix}.$$

3.2. MR and the discrete KdV flows. In this subsection we use the basic MR to express the discrete KdV flows. (We would like to mention that the materials that we give in this subsection are rather standard.) Let R_n be the basic matrix resolvent of $\mathcal{L}$.

Lemma 9. *The following formulae hold true:*

$$c_{n,j} = m_{j,0}(n-2), \tag{64}$$

$$a_{n,j} = m_{j,-2}(n). \tag{65}$$

Proof. By identifying their recursive relations as well as the initial values of the recursions. □

It follows from the above Lemma 9 that the matrices $V_j(n)$ defined in (58) have the following expressions:

$$V_j(n) = \left(\lambda^j R_n\right)_+ + \begin{pmatrix} 0 & 0 \\ 0 & c_{n,j} \end{pmatrix}, \tag{66}$$

where "+" means taking the polynomial part in λ (including the constant term).

3.3. Loop operator. Introduce a linear operator $\nabla(\lambda)$ by

$$\nabla(\lambda) := \sum_{j\geq 1} \frac{1}{\lambda^{j+1}} \frac{\partial}{\partial s_j}. \tag{67}$$

It readily follows from Eq. (66) that

$$\nabla(\mu)\, \Psi_n(\lambda) = \left[\frac{R_n(\mu)}{\mu-\lambda} + Q_n(\mu)\right] \Psi_n(\lambda),$$

where

$$Q_n(\mu) := -\frac{I}{\mu} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma_n(\mu) \end{pmatrix}.$$

Lemma 10. *The following formula holds true:*

$$\nabla(\mu)\, R_n(\lambda) = \frac{1}{\mu-\lambda}\left[R_n(\mu), R_n(\lambda)\right] + \left[Q_n(\mu), R_n(\lambda)\right]. \tag{68}$$

3.4. From MR to tau-function. The MR allows us to define tau-function of an arbitrary solution of the discrete KdV hierarchy. Recall that a family of elements $\Omega_{p;q}(n) \in \mathbb{Z}[\mathbf{w}]$, $p, q \geq 1$ are called a tau-structure of the discrete KdV hierarchy if

$$\Omega_{p;q}(n) = \Omega_{q;p}(n), \qquad \forall\, p, q \geq 1 \tag{69}$$

and for an arbitrary solution $w_n = w_n(\mathbf{s})$ of the discrete KdV hierarchy

$$\frac{\partial \Omega_{p;q}(n)}{\partial s_r} = \frac{\partial \Omega_{p;r}(n)}{\partial s_q}, \qquad \forall\, p, q, r \geq 1. \tag{70}$$

Definition 5. Define $\Omega_{i;j}(n)$, $i, j \geq 1$ via the generating series

$$\sum_{i,j\geq 1} \Omega_{i;j}(n)\, \lambda^{-i-1} \mu^{-j-1} = \frac{\operatorname{tr}\left(R_n(\lambda) R_n(\mu)\right) - 1}{(\lambda-\mu)^2}. \tag{71}$$

Lemma 11. *The $\Omega_{i;j}(n)$, $i, j \geq 1$ (71) are well-defined, and live in $\mathbb{Z}[\mathbf{w}]$. Moreover, they form a tau-structure of the discrete KdV hierarchy.*

Proof. The proof is almost identical with the one for the Toda lattice hierarchy [14] (or the one for the Drinfeld–Sokolov hierarchies [3]); details are omitted here. □

Proof of Lemma 2. By Lemma 11, it suffices to prove the compatibility between (21)–(23).

Firstly, on one hand,

$$\begin{aligned} &\sum_{i,j\geq 1} \lambda^{-i-1}\mu^{-j-1}\left[\Omega_{i;j}(n+2) - \Omega_{i;j}(n)\right] \\ &= \frac{\operatorname{tr}\left(R_{n+2}(\lambda)R_{n+2}(\mu)\right) - \operatorname{tr}\left(R_n(\lambda)R_n(\mu)\right)}{(\lambda-\mu)^2} \\ &= \frac{(1+2\alpha_n(\lambda))\,\gamma_{n+2}(\mu) - (1+2\alpha_n(\mu))\,\gamma_{n+2}(\lambda)}{\lambda-\mu} - \gamma_{n+2}(\lambda)\gamma_{n+2}(\mu). \end{aligned}$$

On the other hand,

$$\nabla(\mu)\left[R_{n+2}(\lambda)\right]_{21} = \frac{\left(1+2\alpha_{n+2}(\mu)\right)\gamma_{n+2}(\lambda) - \left(1+2\alpha_{n+2}(\lambda)\right)\gamma_{n+2}(\mu)}{\lambda-\mu} + \gamma_{n+2}(\lambda)\gamma_{n+2}(\mu).$$

Hence by using (18) we find that

$$\sum_{i,j\geq 1} \lambda^{-i-1}\mu^{-j-1}\left[\Omega_{i;j}(n+2) - \Omega_{i;j}(n)\right] = \nabla(\mu)\left[R_{n+2}(\lambda)\right]_{21}. \tag{72}$$

This proves the compatibility between (21) and (22).

Secondly, on one hand,

$$\begin{aligned} &\sum_{i,j\geq 1} \lambda^{-i-1}\mu^{-j-1}\left[\Omega_{i;j}(n+2) + \Omega_{i;j}(n-1) - \Omega_{i;j}(n+1) - \Omega_{i;j}(n)\right] \\ &= \sum_{i,j\geq 1} \lambda^{-i-1}\mu^{-j-1}\left[\Omega_{i;j}(n+2) - \Omega_{i;j}(n)\right] \\ &\quad - \sum_{i,j\geq 1} \lambda^{-i-1}\mu^{-j-1}\left[\Omega_{i;j}(n+1) - \Omega_{i;j}(n-1)\right]. \end{aligned}$$

On the other hand,

$$\nabla(\mu)\nabla(\lambda)\log w_n = \nabla(\mu)\left[\gamma_{n+2}(\lambda) - \gamma_{n+1}(\lambda)\right] = \nabla(\mu)\gamma_{n+2}(\lambda) - \nabla(\mu)\gamma_{n+1}(\lambda). \tag{73}$$

Using (72) we find

$$\sum_{i,j\geq 1} \lambda^{-i-1}\mu^{-j-1}\Big[\Omega_{i;j}(n+2)+\Omega_{i;j}(n-1)-\Omega_{i;j}(n+1)-\Omega_{i;j}(n)\Big] \\ = \nabla(\mu)\nabla(\lambda)\log w_n. \qquad (74)$$

This proves compatibility between (21) and (23). Thirdly, the following identity

$$\nabla(\lambda)\log w_n \;=\; \gamma_{n+2}(\lambda)-\gamma_{n+1}(\lambda)$$

shows the compatibility between (22) and (23). The proposition is proved. □

3.5. Generating series of multi-point correlations functions. For an arbitrary solution $w_n = w_n(\mathbf{s})$ to the discrete KdV hierarchy, let $\tau_n^{\mathrm{dKdV}} = \tau_n^{\mathrm{dKdV}}(\mathbf{s})$ denote the tau-function of this solution. The logarithmic derivatives

$$\frac{\partial^k \log \tau_n^{\mathrm{dKdV}}(\mathbf{s})}{\partial s_{j_1}\dots\partial s_{j_k}}, \quad j_1,\dots,j_k\geq 1,\; k\geq 1$$

can be called the k-point correlation functions[2] of the solution $w_n = w_n(\mathbf{s})$.

Proof of Proposition 1. The proof can be achieved by the mathematical induction, as in [1]; we hence omit the details. □

We see from Proposition 1 that the logarithmic derivatives $\frac{\partial^k \log \tau_n^{\mathrm{dKdV}}(\mathbf{s})}{\partial s_{j_1}\dots\partial s_{j_k}}$ with $k\geq 2$ all live in $\mathbb{Z}[\mathbf{w}]$, as their generating series are expressed by MR via algebraic manipulations; this simple fact agrees with footnote 2 (and can be of course deduced from other techniques).

4. Proof of Theorem 1

The goal of this section is to prove Theorem 1.

4.1. Review of the MR approach to the Toda lattice hierarchy. Denote

$$\mathcal{P} \;:=\; \Lambda + v_n^{\mathrm{Toda}} + w_n^{\mathrm{Toda}}\Lambda^{-1}, \qquad \mathcal{A}_\ell \;:=\; \left(\mathcal{P}^{\ell+1}\right)_+, \quad \ell\geq 0.$$

The Toda lattice hierarchy is defined as the following system of commuting flows

$$\frac{\partial \mathcal{P}}{\partial t_\ell} \;=\; \Big[\mathcal{A}_\ell,\, \mathcal{P}\Big], \qquad \ell\geq 0. \qquad (75)$$

[2] We can say in a more accurate sense that the logarithmic derivatives are identified with the correlation functions, where the latter are defined as abstract differential polynomials; see for example [18] for the details.

Let us briefly review part of the results of [14]. Introduce $\mathcal{U}_n = \begin{pmatrix} v_n^{\text{Toda}} - \lambda & w_n^{\text{Toda}} \\ -1 & 0 \end{pmatrix}$. The basic resolvent $\mathcal{R}_n$ associated to $\mathcal{P}^{\text{M}} := \Lambda + \mathcal{U}_n$ is defined as the unique solution in $\text{Mat}\big(2, \mathbb{Z}\big[\mathbf{v}^{\text{Toda}}, \mathbf{w}^{\text{Toda}}\big][[\lambda^{-1}]]\big)$ to the problem:

$$\mathcal{R}_{n+1}\,\mathcal{U}_n - \mathcal{U}_n\,\mathcal{R}_n = 0, \tag{76}$$

$$\mathcal{R}_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}\big(\lambda^{-1}\big), \tag{77}$$

$$\text{tr}\,\mathcal{R}_n = 1, \qquad \det \mathcal{R}_n = 0. \tag{78}$$

Write

$$\mathcal{R}_n(\lambda) = \begin{pmatrix} 1 + \mathcal{A}_n(\lambda) & \mathcal{B}_n(\lambda) \\ \mathcal{G}_n(\lambda) & -\mathcal{A}_n(\lambda) \end{pmatrix}, \qquad \mathcal{A}_n, \mathcal{B}_n, \mathcal{G}_n \in \mathbb{Z}[\mathbf{v}^{\text{Toda}}, \mathbf{w}^{\text{Toda}}][[\lambda^{-1}]]. \tag{79}$$

Then $\mathcal{A}_n, \mathcal{B}_n, \mathcal{G}_n$ satisfy that

$$\mathcal{B}_n = -w_n^{\text{Toda}}\,\mathcal{G}_{n+1} \tag{80}$$

$$\mathcal{A}_{n+1} + \mathcal{A}_n + 1 = \mathcal{G}_{n+1}\,(\lambda - v_n^{\text{Toda}}) \tag{81}$$

$$(\lambda - v_n^{\text{Toda}})(\mathcal{A}_n - \mathcal{A}_{n+1}) = w_n^{\text{Toda}}\,\mathcal{G}_n - w_{n+1}^{\text{Toda}}\,\mathcal{G}_{n+2} \tag{82}$$

$$\mathcal{A}_n + \mathcal{A}_n^2 = -\mathcal{B}_n\,\mathcal{G}_n. \tag{83}$$

The following lemma was proven in [14].

Lemma 12. ([14]) *For an arbitrary solution $v_n^{\text{Toda}} = v_n^{\text{Toda}}(\mathbf{t})$, $w_n^{\text{Toda}} = w_n^{\text{Toda}}(\mathbf{t})$ to the Toda lattice hierarchy there exists a function $\tau_n^{\text{Toda}}(\mathbf{t})$ such that*

$$\sum_{i,\,j\geq 0} \frac{1}{\lambda^{i+2}\mu^{j+2}} \frac{\partial^2 \log \tau_n^{\text{Toda}}(\mathbf{t})}{\partial t_i\,\partial t_j} = \frac{\text{tr}\,\mathcal{R}_n(\mathbf{t},\lambda)\mathcal{R}_n(\mathbf{t},\mu) - 1}{(\lambda-\mu)^2} \tag{84}$$

$$\frac{1}{\lambda} + \sum_{i\geq 0} \frac{1}{\lambda^{i+2}} \frac{\partial}{\partial t_i} \log \frac{\tau_{n+1}^{\text{Toda}}(\mathbf{t})}{\tau_n^{\text{Toda}}(\mathbf{t})} = [\mathcal{R}_{n+1}(\mathbf{t},\lambda)]_{21} \tag{85}$$

$$\frac{\tau_{n+1}^{\text{Toda}}(\mathbf{t})\tau_{n-1}^{\text{Toda}}(\mathbf{t})}{\tau_n^{\text{Toda}}(\mathbf{t})^2} = w_n. \tag{86}$$

The function $\tau_n^{\text{Toda}}(\mathbf{t})$ is uniquely determined by the solution $v_n^{\text{Toda}}(\mathbf{t})$, $w_n^{\text{Toda}}(\mathbf{t})$ up to

$$\tau_n^{\text{Toda}}(\mathbf{t}) \mapsto e^{a_0 + a_1 n + \sum_{j\geq 0} b_j t_j}\,\tau_n^{\text{Toda}}(\mathbf{t})$$

for some constants a_0, a_1, b_0, b_1, b_2, ….

In [14] the $\tau_n^{\text{Toda}}(\mathbf{t})$ is called the *tau-function* of the solution $v_n^{\text{Toda}}(\mathbf{t})$, $w_n^{\text{Toda}}(\mathbf{t})$ to the Toda lattice hierarchy. The logarithmic derivatives of $\tau_n^{\text{Toda}}(\mathbf{t})$

$$\frac{\partial^k \log \tau_n^{\text{Toda}}(\mathbf{t})}{\partial t_{i_1} \dots \partial t_{i_k}}, \qquad i_1, \dots, i_k \geq 0, k \geq 1$$

can be called k-point correlations functions (cf. footnote 2) of the Toda lattice hierarchy. Define

$$C_k(\lambda_1, \dots, \lambda_k; n; \mathbf{t}) := \sum_{i_1,\dots,i_k\geq 0} \frac{1}{\lambda_1^{i_1+2} \dots \lambda_k^{i_k+2}} \frac{\partial^k \log \tau_n^{\text{Toda}}(\mathbf{t})}{\partial t_{i_1} \dots \partial t_{i_k}}.$$

4.2. Reduction to the discrete KdV hierarchy. Now consider solutions to the Toda lattice hierarchy in the ring $\mathbb{C}[[t_0, t_1, \ldots]] \otimes \mathcal{V}$, where $\mathcal{V}$ is any ring of functions of n, closed under Λ and Λ^{-1}. These solutions can be specified by (i.e. are in 1–1 correspondence to) the initial value:

$$f(n) = v_n^{\mathrm{Toda}}(\mathbf{t} = \mathbf{0}), \quad g(n) = w_n^{\mathrm{Toda}}(\mathbf{t} = \mathbf{0}).$$

Let us explain how a subset of solutions to the Toda lattice hierarchy be reduced to solutions of the discrete KdV hierarchy. On one hand, let $v_n^{\mathrm{Toda}} = v_n^{\mathrm{Toda}}(\mathbf{t})$, $w_n^{\mathrm{Toda}} = w_n^{\mathrm{Toda}}(\mathbf{t})$ be an arbitrary solution in $\mathbb{C}[[t_0, t_1, \ldots]] \otimes \mathcal{V}$ of the Toda lattice hierarchy satisfying the following type of initial conditions

$$f(n) \equiv 0.$$

It follows that

$$v_n^{\mathrm{Toda}}\Big|_{t_0=t_2=t_4=\cdots=0} \equiv 0 \qquad (\forall\, n, t_1, t_3, t_5, \cdots). \tag{87}$$

This further implies that the commuting flows $\frac{\partial w_n^{\mathrm{Toda}}(\mathbf{t})}{\partial t_{2j-1}}\Big|_{t_0=t_2=t_4=\cdots=0}$ $(j \geq 1)$ are decoupled, namely, there are no v_n^{Toda}-dependence in these flows (of course when restricting to $t_0 = t_2 = t_4 = \cdots = 0$). Moreover, these flows coincide with the discrete KdV hierarchy (3). Therefore if we define

$$w_n(\mathbf{s}) \ := \ w_n^{\mathrm{Toda}}(\mathbf{t})\Big|_{t_{2i-1}=s_i,\, t_{2i-2}=0,\, i\geq 1}, \tag{88}$$

then $w_n = w_n(\mathbf{s})$ is a solution to the discrete KdV hierarchy. On the other hand, let $w_n = w_n(\mathbf{s})$ be an arbitrary solution to the discrete KdV hierarchy in the ring $\mathbb{C}[s_1, s_2, \ldots] \otimes \mathcal{V}$. Let $g(n)$ denote its initial value, i.e. $g(n) := w_n(\mathbf{s} = \mathbf{0})$. Define $v_n^{\mathrm{Toda}}(\mathbf{t})$, $w_n^{\mathrm{Toda}}(\mathbf{t})$ as the unique solution in $\mathbb{C}[[t_0, t_1, \ldots]] \otimes \mathcal{V}$ to the Toda lattice hierarchy with $(f(n) \equiv 0, g(n))$ as the initial value. Then $w_n^{\mathrm{Toda}}(\mathbf{t})|_{t_{2i-1}=s_i,\, t_{2i-2}=0,\, i\geq 1} = w_n(\mathbf{s})$.

Hence the correspondence between solutions of the discrete KdV hierarchy and a suitable subset of solutions of the Toda lattice hierarchy has been established.

For a solution $\big(v_n^{\mathrm{Toda}}(\mathbf{t}), w_n^{\mathrm{Toda}}(\mathbf{t})\big)$ to the Toda lattice hierarchy satisfying $v_n^{\mathrm{Toda}}(\mathbf{0}) \equiv 0$ $(\forall\, n)$, let $\tau_n^{\mathrm{Toda}}(\mathbf{t})$ denote the tau-function of this solution. Define $w_n(\mathbf{s})$ as in (88), and

$$\tau_n(\mathbf{s}) \ := \ \tau_n^{\mathrm{Toda}}(t_0 = 0, t_1 = s_1, t_2 = 0, t_3 = s_2, \cdots).$$

Then we know that the function $w_n = w_n(\mathbf{s})$ satisfies the discrete KdV hierarchy (3), and that

$$w_n(\mathbf{s}) \ = \ \frac{\tau_{n+1}(\mathbf{s})\, \tau_{n-1}(\mathbf{s})}{\tau_n^2(\mathbf{s})}. \tag{89}$$

As indicated above, all solutions of the discrete KdV hierarchy can be obtained from this way.

Definition 6. We call $\tau_n(\mathbf{s})$ the tau-function reduced from the Toda lattice hierarchy of the solution $w_n = w_n(\mathbf{s})$ to the discrete KdV hierarchy.

Introduce the notations:

$$
\begin{aligned}
A_n(\lambda) &:= \mathcal{A}_n(\lambda)|_{v_n^{\mathrm{Toda}}\equiv 0,\ w_n^{\mathrm{Toda}}\equiv w_n},\\
B_n(\lambda) &:= \mathcal{B}_n(\lambda)|_{v_n^{\mathrm{Toda}}\equiv 0,\ w_n^{\mathrm{Toda}}\equiv w_n},\\
G_n(\lambda) &:= \mathcal{G}_n(\lambda)|_{v_n^{\mathrm{Toda}}\equiv 0,\ w_n^{\mathrm{Toda}}\equiv w_n}.
\end{aligned}
$$

Clearly, $A_n,\ B_n,\ G_n$ belong to $\mathbb{Z}[\mathbf{w}][[\lambda^{-1}]]$. Note that definitions of $A_n(\lambda),\ B_n(\lambda)$, $G_n(\lambda)$ are in the absolute sense, namely, they do not depend on whether w_n is a solution or not.

Lemma 13. *The* $A_n(\lambda)$ *satisfies*

$$
w_{n+1}\big(A_{n+2}(\lambda)+A_{n+1}(\lambda)+1\big)-w_n\big(A_n(\lambda)+A_{n-1}(\lambda)+1\big) = \lambda^2\left(A_{n+1}(\lambda)-A_n(\lambda)\right). \tag{90}
$$

Proof. Following from (81) and (82) with $v_n^{\mathrm{Toda}}\equiv 0$. □

4.3. Proof of Theorem 1. Firstly, on one hand, it follows from the Lemma 1.2.3 of [14] that

$$
m_{j,0}(n;\mathbf{s}) = \frac{\partial}{\partial s_j}\log\frac{\tau_{n+1}(\mathbf{s})}{\tau_n(\mathbf{s})},\qquad j\geq 1. \tag{91}
$$

On the other hand, from (22) and (64) we find

$$
m_{j,0}(n;\mathbf{s}) = \frac{\partial}{\partial s_j}\log\frac{\tau_{n+2}^{\mathrm{dKdV}}(\mathbf{s})}{\tau_n^{\mathrm{dKdV}}(\mathbf{s})},\qquad j\geq 1. \tag{92}
$$

Comparing the above two expressions we find

$$
\log\frac{\tau_{n+1}(\mathbf{s})}{\tau_n(\mathbf{s})}-\log\frac{\tau_{n+2}^{\mathrm{dKdV}}(\mathbf{s})}{\tau_n^{\mathrm{dKdV}}(\mathbf{s})} = S(n), \tag{93}
$$

where $S(n)$ is some function depending only on n. Equation (93) implies that

$$
\log\tau_n(\mathbf{s}) - (\Lambda+1)\log\tau_n^{\mathrm{dKdV}}(\mathbf{s}) = \widetilde{S}(n) + f(\mathbf{s}), \tag{94}
$$

where $\widetilde{S}(n)$ is some function depending only on n, and $f(\mathbf{s})$ is some function depending only on $\mathbf{s}$.

Secondly, it follows from (23) and (89) that

$$
\frac{\tau_{n+1}(\mathbf{s})\,\tau_{n-1}(\mathbf{s})}{\tau_n^2(\mathbf{s})} = \frac{\tau_{n+2}^{\mathrm{dKdV}}(\mathbf{s})\,\tau_{n-1}^{\mathrm{dKdV}}(\mathbf{s})}{\tau_{n+1}^{\mathrm{dKdV}}(\mathbf{s})\,\tau_n^{\mathrm{dKdV}}(\mathbf{s})}. \tag{95}
$$

Substituting (94) in (95) we find that $\widetilde{S}(n)$ can only be an affine function of n, namely,

$$
\log\tau_n(\mathbf{s}) - (\Lambda+1)\log\tau_n^{\mathrm{dKdV}}(\mathbf{s}) = \alpha\,n + \alpha' + f(\mathbf{s}), \tag{96}
$$

where α,α' are some constants independent of $n,\mathbf{s}$.

Thirdly, on one hand, using (21) we find

$$\sum_{i,j\geq 1} \frac{\partial^2 \log \tau_n^{\mathrm{dKdV}}(\mathbf{s})}{\partial s_i \partial s_j} \frac{1}{\lambda^{i+1}\mu^{j+1}}$$
$$= \frac{\alpha_n(\lambda)+\alpha_n(\mu)+2\alpha_n(\lambda)\alpha_n(\mu) - w_n\, w_{n-1}\left(\gamma_n(\lambda)\gamma_{n+2}(\mu)+\gamma_n(\mu)\gamma_{n+2}(\lambda)\right)}{(\lambda-\mu)^2}.$$

Therefore,

$$\sum_{i,j\geq 1} \frac{\partial^2 \log \tau_n^{\mathrm{dKdV}}(\mathbf{s})}{\partial s_i \partial s_j} \frac{1}{\lambda^{2i+1}\mu^{2j+1}}$$
$$= \lambda\mu \frac{\alpha_n(\lambda^2)+\alpha_n(\mu^2)+2\alpha_n(\lambda^2)\alpha_n(\mu^2) - w_n\, w_{n-1}\left(\gamma_n(\lambda^2)\gamma_{n+2}(\mu^2)+\gamma_n(\mu^2)\gamma_{n+2}(\lambda^2)\right)}{(\lambda^2-\mu^2)^2}$$
$$=:\ W_2(\lambda,\mu;n,\mathbf{s}).$$

So

$$\sum_{i,j\geq 1}\left(\frac{\partial^2 \log \tau_n^{\mathrm{dKdV}}(\mathbf{s})}{\partial s_i \partial s_j} + \frac{\partial^2 \log \tau_{n+1}^{\mathrm{dKdV}}(\mathbf{s})}{\partial s_i \partial s_j}\right)\frac{1}{\lambda^{2i+1}\mu^{2j+1}}$$
$$= W_2(\lambda,\mu;n,\mathbf{s}) + W_2(\lambda,\mu;n+1,\mathbf{s}).$$

On the other hand, for $P(n) = \Lambda + w_n\Lambda^{-1}$, recall the notation

$$P(n)^{\ell+1} = \sum_{k\in\mathbb{Z}} A_{\ell,k}(n)\,\Lambda^k, \qquad \ell = -1, 0, 1, 2, \ldots.$$

Using Lemma 12 we have

$$C_2(\lambda,\mu;n;\mathbf{t})$$
$$= \frac{A_n(\lambda)+A_n(\mu)+2A_n(\lambda)A_n(\mu) - w_n\left(G_{n+1}(\lambda)G_n(\mu)+G_{n+1}(\mu)G_n(\lambda)\right)}{(\lambda-\mu)^2},$$

where

$$A_n(\lambda) = \sum_{\ell\geq 0}\frac{A_{\ell-1,-1}(n)}{\lambda^{\ell+1}}, \qquad G_n(\lambda) = \sum_{\ell\geq 0}\frac{A_{\ell-1,0}(n-1)}{\lambda^{\ell+1}}.$$

Taking

$$t_{2i-2} = 0,\ \ t_{2i-1} = s_i\ \ (i\geq 1)$$

we have

$$G_n(\lambda) = \sum_{j\geq 0}\frac{A_{2j-1,0}(n-1)}{\lambda^{2j+1}} = \sum_{j\geq 0}\frac{m_{j,0}(n-1)}{\lambda^{2j+1}} = \sum_{j\geq 0}\frac{c_{n+1,j}}{\lambda^{2j+1}} = \lambda\,\gamma_{n+1}(\lambda^2). \tag{97}$$

It follows from (81), (97), and respectively (18), that

$$A_n(\lambda) = (\Lambda+1)^{-1}\left(\lambda^2\gamma_{n+2}(\lambda^2)-1\right) = \lambda^2\,(\Lambda+1)^{-1}\,\gamma_{n+2}(\lambda^2) - \frac{1}{2}, \tag{98}$$
$$\alpha_n(\lambda) = (\Lambda^2+1)^{-1}\left((\lambda - w_{n+1} - w_n)\,\gamma_{n+2}(\lambda)\right) - \frac{1}{2}. \tag{99}$$

Lemma 14. *The following identities hold true:*

$$\gamma_n(\lambda^2) = \frac{A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1}{\lambda^2}, \tag{100}$$

$$G_n(\lambda) = \frac{A_n(\lambda) + A_{n-1}(\lambda) + 1}{\lambda}, \tag{101}$$

$$\alpha_n(\lambda^2) = A_{n-1}(\lambda) - \frac{w_{n-1}}{\lambda^2}\left(A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1\right). \tag{102}$$

Proof. Identities (100), (101) are easy consequences of (98), (97).

Note that identity (52) implies that

$$\begin{aligned}\alpha_n(\lambda^2) &= \frac{1}{2}\Big(w_{n-1}\,\gamma_{n+1}(\lambda^2) - w_{n-2}\,\gamma_{n-1}(\lambda^2) + (\lambda^2 - 2w_{n-1})\,\gamma_n(\lambda^2) - 1\Big)\\ &= \frac{1}{2}\Big(w_{n-1}\,\frac{A_n(\lambda) + A_{n-1}(\lambda) + 1}{\lambda^2} - w_{n-2}\,\frac{A_{n-2}(\lambda) + A_{n-3}(\lambda) + 1}{\lambda^2}\\ &\quad + (\lambda^2 - 2w_{n-1})\,\frac{A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1}{\lambda^2} - 1\Big).\end{aligned}$$

Applying Lemma 13 in this identity yields

$$\begin{aligned}\alpha_n(\lambda^2) &= \frac{1}{2\lambda^2}\Big(\lambda^2(A_{n-1}(\lambda) - A_{n-2}(\lambda)) + (\lambda^2 - 2w_{n-1})\,(A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1) - \lambda^2\Big)\\ &= \frac{1}{\lambda^2}\Big(\lambda^2 A_{n-1}(\lambda) - w_{n-1}\,(A_{n-1}(\lambda) + A_{n-2}(\lambda) + 1)\Big).\end{aligned}$$

The lemma is proved. □

Observe that $C_2(\lambda, \mu; n, \mathbf{s})$ satisfies the parity symmetries

$$C_2(\lambda, \mu) = C_2(-\lambda, -\mu), \qquad C_2(\lambda, -\mu) = C_2(-\lambda, \mu).$$

So

$$\sum_{i,j\geq 1} \frac{\partial^2 \log \tau_n(\mathbf{t})}{\partial t_{2i-1}\partial t_{2j-1}}\,\frac{1}{\lambda^{2i+1}\mu^{2j+1}} = \frac{C_2(\lambda,\mu) - C_2(-\lambda,\mu)}{2}. \tag{103}$$

Lemma 15. *The following identity hold true:*

$$\begin{aligned}&\lambda\mu\,\frac{\alpha_n(\lambda^2) + \alpha_n(\mu^2) + 2\alpha_n(\lambda^2)\alpha_n(\mu^2) - w_n\,w_{n-1}\left(\gamma_n(\lambda^2)\gamma_{n+2}(\mu^2) + \gamma_n(\mu^2)\gamma_{n+2}(\lambda^2)\right)}{(\lambda^2 - \mu^2)^2}\\ &\quad + \lambda\mu\,\frac{\alpha_{n+1}(\lambda^2) + \alpha_{n+1}(\mu^2) + 2\alpha_{n+1}(\lambda^2)\alpha_{n+1}(\mu^2) - w_{n+1}\,w_n\left(\gamma_{n+1}(\lambda^2)\gamma_{n+3}(\mu^2) + \gamma_{n+1}(\mu^2)\gamma_{n+3}(\lambda^2)\right)}{(\lambda^2 - \mu^2)^2}\\ &= \frac{A_n(\lambda) + A_n(\mu) + 2A_n(\lambda)A_n(\mu) - w_n\big(G_{n+1}(\lambda)G_n(\mu) + G_{n+1}(\mu)G_n(\lambda)\big)}{2(\lambda - \mu)^2}\\ &\quad - \frac{A_n(-\lambda) + A_n(\mu) + 2A_n(-\lambda)A_n(\mu) - w_n\big(G_{n+1}(-\lambda)G_n(\mu) + G_{n+1}(\mu)G_n(-\lambda)\big)}{2(\lambda + \mu)^2}.\end{aligned} \tag{104}$$

Proof. Applying (100)–(102) and the parity symmetry

$$A_n(-\lambda) \;=\; A_n(\lambda)$$

we find that it suffices to prove the following equality

$$\begin{aligned}
&-\lambda\mu \;+\; 2\,\lambda\mu\left[A_{n-1}(\lambda) - \frac{w_{n-1}}{\lambda^2}\,\big(A_{n-1}(\lambda)+A_{n-2}(\lambda)+1\big)+\frac{1}{2}\right]\\
&\qquad\qquad\qquad\left[A_{n-1}(\mu) - \frac{w_{n-1}}{\mu^2}\,\big(A_{n-1}(\mu)+A_{n-2}(\mu)+1\big)+\frac{1}{2}\right]\\
&-\frac{w_n\,w_{n-1}}{\lambda\mu}\,\Big[\big(A_{n-1}(\lambda)+A_{n-2}(\lambda)+1\big)\big(A_{n+1}(\mu)+A_n(\mu)+1\big)\\
&\qquad\qquad+\big(A_{n-1}(\mu)+A_{n-2}(\mu)+1\big)\big(A_{n+1}(\lambda)+A_n(\lambda)+1\big)\Big]\\
&-\frac{w_{n+1}\,w_n}{\lambda\mu}\,\Big[\big(A_n(\lambda)+A_{n-1}(\lambda)+1\big)\big(A_{n+2}(\mu)+A_{n+1}(\mu)+1\big)\\
&\qquad\qquad+\big(A_n(\mu)+A_{n-1}(\mu)+1\big)\big(A_{n+2}(\lambda)+A_{n+1}(\lambda)+1\big)\Big]\\
&+2\,\lambda\mu\left[A_n(\lambda) - \frac{w_n}{\lambda^2}\,\big(A_n(\lambda)+A_{n-1}(\lambda)+1\big)+\frac{1}{2}\right]\\
&\qquad\qquad\qquad\left[A_n(\mu) - \frac{w_n}{\mu^2}\,\big(A_n(\mu)+A_{n-1}(\mu)+1\big)+\frac{1}{2}\right]\\
&=\;\frac{(\lambda+\mu)^2}{2}\Big[A_n(\lambda)+A_n(\mu)+2A_n(\lambda)A_n(\mu)\\
&\qquad\qquad-\frac{w_n}{\lambda\mu}\Big(\big(A_{n+1}(\lambda)+A_n(\lambda)+1\big)\big(A_n(\mu)+A_{n-1}(\mu)+1\big)\\
&\qquad\qquad+\big(A_{n+1}(\mu)+A_n(\mu)+1\big)\big(A_n(\lambda)+A_{n-1}(\lambda)+1\big)\Big)\Big]\\
&\quad-\frac{(\lambda-\mu)^2}{2}\Big[A_n(\lambda)+A_n(\mu)+2A_n(\lambda)A_n(\mu)\\
&\qquad\qquad+\frac{w_n}{\lambda\mu}\Big(\big(A_{n+1}(\lambda)+A_n(\lambda)+1\big)\big(A_n(\mu)+A_{n-1}(\mu)+1\big)\\
&\qquad\qquad+\big(A_{n+1}(\mu)+A_n(\mu)+1\big)\big(A_n(\lambda)+A_{n-1}(\lambda)+1\big)\Big)\Big].
\end{aligned}$$

Noting that

$$\begin{aligned}
&\lambda\mu\cdot \mathrm{lhs} \;=\\
&-\lambda^2\mu^2 \;+\; 2\Big[\lambda^2 A_{n-1}(\lambda) - w_{n-1}\,\big(A_{n-1}(\lambda)+A_{n-2}(\lambda)+1\big)+\frac{\lambda^2}{2}\Big]\\
&\Big[\mu^2 A_{n-1}(\mu) - w_{n-1}\,\big(A_{n-1}(\mu)+A_{n-2}(\mu)+1\big)+\frac{\mu^2}{2}\Big]\\
&-w_{n-1}\,\Big[\big(A_{n-1}(\lambda)+A_{n-2}(\lambda)+1\big)\Big(\mu^2\big(A_n(\mu)-A_{n-1}(\mu)\big)+w_{n-1}\big(A_{n-1}(\mu)+A_{n-2}(\mu)+1\big)\Big)\\
&+\big(A_{n-1}(\mu)+A_{n-2}(\mu)+1\big)\Big(\lambda^2\big(A_n(\lambda)-A_{n-1}(\lambda)\big)+w_{n-1}\big(A_{n-1}(\lambda)+A_{n-2}(\lambda)+1\big)\Big)\Big]\\
&-w_n\,\Big[\big(A_n(\lambda)+A_{n-1}(\lambda)+1\big)\Big(\mu^2\big(A_{n+1}(\mu)-A_n(\mu)\big)+w_n\big(A_n(\mu)+A_{n-1}(\mu)+1\big)\Big)\\
&+\big(A_n(\mu)+A_{n-1}(\mu)+1\big)\Big(\lambda^2\big(A_{n+1}(\lambda)-A_n(\lambda)\big)+w_n\big(A_n(\lambda)+A_{n-1}(\lambda)+1\big)\Big)\Big]
\end{aligned}$$

$$+2\Big[\lambda^2 A_n(\lambda) - w_n\big(A_n(\lambda)+A_{n-1}(\lambda)+1\big)+\frac{\lambda^2}{2}\Big]\Big[\mu^2 A_n(\mu) - w_n\big(A_n(\mu)+A_{n-1}(\mu)+1\big)+\frac{\mu^2}{2}\Big]$$

and that

$$\begin{aligned}\lambda\mu\cdot\mathrm{rhs} &= 2\,\lambda^2\mu^2\Big(A_n(\lambda)+A_n(\mu)+2A_n(\lambda)A_n(\mu)\Big)\\ &\quad + w_n\,(\lambda^2+\mu^2)\Big(\big(A_{n+1}(\lambda)+A_n(\lambda)+1\big)\big(A_n(\mu)+A_{n-1}(\mu)+1\big)\\ &\quad + \big(A_{n+1}(\mu)+A_n(\mu)+1\big)\big(A_n(\lambda)+A_{n-1}(\lambda)+1\big)\Big),\end{aligned}$$

we find

$$\begin{aligned}&\lambda\mu\cdot(\mathrm{lhs}-\mathrm{rhs})\\ &= \lambda^2\mu^2\big(2A_{n-1}(\lambda)A_{n-1}(\mu)-2A_n(\lambda)A_n(\mu)+A_{n-1}(\lambda)+A_{n-1}(\mu)-A_n(\lambda)-A_n(\mu)\big)\\ &\quad -\mu^2(A_{n-1}(\mu)+A_n(\mu)+1)\big(w_{n-1}(A_{n-2}(\lambda)+A_{n-1}(\lambda)+1)-w_n(A_n(\lambda)+A_{n+1}(\lambda)+1)\big)\\ &\quad -\lambda^2\big(A_{n-1}(\lambda)+A_n(\lambda)+1\big)\big(w_{n-1}(A_{n-2}(\mu)+A_{n-1}(\mu)+1)-w_n(A_n(\mu)+A_{n+1}(\mu)+1)\big)\\ &= \lambda^2\mu^2\big(2A_{n-1}(\lambda)A_{n-1}(\mu)-2A_n(\lambda)A_n(\mu)+A_{n-1}(\lambda)+A_{n-1}(\mu)-A_n(\lambda)-A_n(\mu)\big)\\ &\quad +\lambda^2\mu^2(A_{n-1}(\mu)+A_n(\mu)+1)(A_n(\lambda)-A_{n-1}(\lambda))\\ &\quad +\lambda^2\mu^2(A_{n-1}(\lambda)+A_n(\lambda)+1)(A_n(\mu)-A_{n-1}(\mu)) \;=\; 0,\end{aligned}$$

where Lemma 13 is used. The lemma is proved. □

End of proof of Theorem 1. It follows from Lemma 15 that

$$\frac{\partial^2 \log \tau_n(\mathbf{s})}{\partial s_i \partial s_j} \;=\; \frac{\partial^2}{\partial s_i \partial s_j}(\Lambda+1)\,\log \tau_n^{\mathrm{dKdV}}(\mathbf{s}).$$

Combining with (96) we find that

$$f(\mathbf{s}) \;=\; \beta_0 + \sum_{k\ge 1} \beta_k s_k,$$

where $\beta_0, \beta_1, \beta_2, \cdots$ are constants (independent of n). The theorem is proved. □

5. Proofs of Theorem 2 and Corollary 2

In this section, using Proposition 1, Corollary 1 and Theorem 1, we are going to prove Theorem 2 and Corollary 2.

5.1. Ribbon graphs with even valencies. In this subsection, we first prove Theorem 2, then we give a further study to the modified GUE partition function with even couplings.

Proof of Theorem 2. Define $\mathcal{F}_n(\mathbf{s})$ and $Z_n(\mathbf{s})$ by

$$\begin{aligned}\mathcal{F}_n(\mathbf{s}) \;&:=\; \frac{n^2}{2}\Big(\log n - \frac{3}{2}\Big) - \frac{1}{12}\log n + \sum_{g\ge 2}\frac{B_{2g}}{4g(g-1)\,n^{2g-2}}\\ &\quad + \sum_{k\ge 0}\frac{1}{k!}\sum_{j_1,\dots,j_k\ge 1}\big\langle \mathrm{tr}\, M^{2j_1}\cdots \mathrm{tr}\, M^{2j_k}\big\rangle_c\, s_{j_1}\cdots s_{j_k},\\ Z_n(\mathbf{s}) \;&:=\; e^{\mathcal{F}_n(\mathbf{s})}.\end{aligned} \tag{105}$$

Here B_m denotes the m^{th} Bernoulli number. Then $Z_n(\mathbf{s})$ is a particular tau-function (of the discrete KdV hierarchy) reduced from the Toda lattice hierarchy [14]. The initial value of $w_n(\mathbf{s}) := \frac{Z_{n+1}(\mathbf{s})\, Z_{n-1}(\mathbf{s})}{Z_n(\mathbf{s})^2}$ is given by $w_n(\mathbf{s}=\mathbf{0}) = n$. The theorem then follows from Lemma 14, Corollary 1, as well as the Theorem 1.1.1 of [14]. □

Define a formal series $Z(x, \mathbf{s}; \epsilon)$ by

$$\log Z(x, \mathbf{s}; \epsilon) := \frac{x^2}{2\epsilon^2}\Big(\log x - \frac{3}{2}\Big) - \frac{1}{12}\log x + \sum_{g\geq 2} \epsilon^{2g-2} \frac{B_{2g}}{4g(g-1)x^{2g-2}} + \sum_{g\geq 0} \epsilon^{2g-2} \sum_{k\geq 1} \sum_{\substack{j_1,\dots,j_k\geq 1\\ |j|\geq 2g-2+k}} a_g(2j_1,\dots,2j_k)\, s_{j_1}\cdots s_{j_k}\, x^{2-2g-k+|j|}. \tag{106}$$

Here, x is the t'Hooft coupling constant [28,29]. Recall that we could view $Z(x, \mathbf{s}; \epsilon)$ as a tau-function *reduced from the Toda lattice* of the discrete KdV hierarchy under the identification $n = x/\epsilon$ as well as the flow rescalings $\partial_{s_j} \mapsto \epsilon\, \partial_{s_j}$. More precisely, define

$$w(x, \mathbf{s}; \epsilon) := \frac{Z(x+\epsilon, \mathbf{s}; \epsilon)\, Z(x-\epsilon, \mathbf{s}; \epsilon)}{Z(x, \mathbf{s}; \epsilon)^2},$$

then $w(x, \mathbf{s}; \epsilon)$ is a particular solution to the discrete KdV hierarchy:

$$\epsilon \frac{\partial L}{\partial s_j} = \big[A_{2j-1},\, L\big]$$

with $L := \Lambda^2 + w(x+\epsilon) + w(x) + w(x)\, w(x-\epsilon)\, \Lambda^{-2}$, $A_{2j-1} := L^j$, $\Lambda := e^{\epsilon\partial_x}$. Validity of these statements can be found in the Appendix of [14]. The initial data of this solution is given by

$$w(x, \mathbf{0}; \epsilon) \equiv x = n\,\epsilon. \tag{107}$$

Let $Z^{\text{dKdV}}(x, \mathbf{s}; \epsilon)$ be the tau-function of the solution $w(x, \mathbf{s}; \epsilon)$. The following corollary follows from Theorem 1.

Corollary 3. *There exist constants* $\alpha, \beta_0, \beta_1, \beta_2, \cdots$ *such that*

$$Z(x, \mathbf{s}; \epsilon) = e^{\alpha x + \beta_0 + \sum_{k\geq 1}\beta_j s_j}\, Z^{\text{dKdV}}(x, \mathbf{s}; \epsilon)\, Z^{\text{dKdV}}(x+\epsilon, \mathbf{s}; \epsilon). \tag{108}$$

Note that the constants $\alpha, \beta_0, \beta_1, \beta_2, \cdots$ right above now *can* depend on ϵ. In what follows, we fix the ambiguities simply by requiring $Z^{\text{dKdV}}(x, \mathbf{s}; \epsilon)$ to be the unique function satisfying

$$Z(x, \mathbf{s}; \epsilon) = Z^{\text{dKdV}}(x, \mathbf{s}; \epsilon)\, Z^{\text{dKdV}}(x+\epsilon, \mathbf{s}; \epsilon). \tag{109}$$

Remark. The following formal series of $\mathbf{s}$

$$Z^{\text{dKdV}}\Big(x + \frac{\epsilon}{2}, \mathbf{s}; \epsilon\Big) =: \widetilde{Z}(x, \mathbf{s}; \epsilon) \tag{110}$$

was introduced in [20] by Si-Qi Liu, Youjin Zhang and the authors of the present paper, called the *modified GUE partition function with even couplings*, which plays an important role in a proof of the Hodge–GUE correspondence [20]. Moreover, Liu, Zhang and the authors derived the *Dubrovin–Zhang loop equation* for $\log \widetilde{Z}$ from the corresponding Virasoro constraints, which also provides an algorithm for computing the modified GUE correlators of an arbitrary genus [20]. Very recently, Jian Zhou [42] derived the *topological recursion of Chekhov–Eynard–Orantin type* for the modified GUE correlators from the Virasoro constraints constructed in [20]; moreover, as a consequence of the topological recursion, an interesting formula between intersection numbers and k-point functions of modified GUE correlators was obtained by Zhou [42] (see the Theorem 3 in [42] for the details); it remains an open question to match the formula of Zhou with another interesting formula obtained by Gaëtan Borot and Elba Garcia-Failde [5] (see the Corollary 12.3 of [5]) as a consequence of the Hodge–GUE correspondence (or with a slightly different but equivalent consequence like (121) in below), which may lead to a new proof of the Hodge–GUE correspondence. Last but not least, as a corollary of Theorem 2, let us give a third algorithm of computing the modified GUE correlators based on the following *full genera* formulae:

$$\epsilon^2 \sum_{j_1, j_2 \geq 0} \frac{\langle \phi_{j_1} \phi_{j_2} \rangle (x;\epsilon)}{\lambda_1^{j_1+1} \lambda_2^{j_2+1}} = \frac{\operatorname{tr}\left[R_{\frac{x}{\epsilon}+\frac{1}{2}}(\frac{\lambda_1}{\epsilon}) R_{\frac{x}{\epsilon}+\frac{1}{2}}(\frac{\lambda_2}{\epsilon})\right]}{(\lambda_1-\lambda_2)^2} - \frac{1}{(\lambda_1-\lambda_2)^2}, \tag{111}$$

$$\epsilon^k \sum_{j_1,\dots,j_k \geq 0} \frac{\langle \phi_{j_1} \cdots \phi_{j_k} \rangle (x;\epsilon)}{\lambda_1^{j_1+1} \cdots \lambda_k^{j_k+1}} = -\frac{1}{k} \sum_{\sigma \in S_k} \frac{\operatorname{tr}\left[R_{\frac{x}{\epsilon}+\frac{1}{2}}(\frac{\lambda_{\sigma_1}}{\epsilon}) \cdots R_{\frac{x}{\epsilon}+\frac{1}{2}}(\frac{\lambda_{\sigma_k}}{\epsilon})\right]}{\prod_{\ell=1}^k (\lambda_{\sigma_\ell} - \lambda_{\sigma_{\ell+1}})} \quad (k \geq 3), \tag{112}$$

where $\langle \phi_{j_1} \cdots \phi_{j_k} \rangle (x;\epsilon)$ denote the modified GUE correlators with even couplings, defined by

$$\langle \phi_{j_1} \cdots \phi_{j_k} \rangle (x;\epsilon) := \frac{\partial^k \log \widetilde{Z}}{\partial s_{j_1} \dots \partial s_{j_k}} (x, \mathbf{s}=\mathbf{0};\epsilon), \tag{113}$$

and $R_n(\lambda)$ is defined in Definition 3. We notice that the reason that one can talk about "genus" for $\log \widetilde{Z}$ is because $\log \widetilde{Z}$ is even in ϵ and so are $\langle \phi_{j_1} \cdots \phi_{j_k} \rangle (x;\epsilon)$. A concrete algorithm using the formulae of the form (111)–(112) for calculating the corresponding correlators including certain large genus asymptotics is given in [16].

Remark. Very recently it was shown in [26] that $Z^{\mathrm{dKdV}}(x, \mathbf{s};\epsilon)$ and $Z^{\mathrm{dKdV}}(x+\epsilon, \mathbf{s};\epsilon)$ are identified with the LUE partition functions with $\alpha = -1/2$ and $\alpha = 1/2$, respectively. One can obtain their k-point series by putting $x \to x \mp \frac{\epsilon}{2}$ in (111)–(112). An interesting genus expansion for $Z^{\mathrm{dKdV}}(x, \mathbf{s};\epsilon)$ was discovered in [9]. The interplay between Z^{dKdV} and $\widetilde{Z}$ suggests a Hurwitz/Hodge correspondence that deserves a further study.

Using the definitions of $Z(x, \mathbf{s};\epsilon)$ and $\widetilde{Z}(x, \mathbf{s};\epsilon)$ and using the expansion

$$\frac{2}{e^z + e^{-z}} =: \sum_{k \geq 0} \frac{E_k}{k!} z^k,$$

with E_k, $k \geq 0$ being the Euler numbers, we have the following formula:

$$\begin{aligned}
&\log \widetilde{Z}(x, \mathbf{s}; \epsilon) \\
&= \Big(\frac{1}{4}\log x - \frac{3}{8}\Big)\frac{x^2}{\epsilon^2} - \frac{5}{48}\log x \\
&\quad + \sum_{g\geq 2} \frac{\epsilon^{2g-2}}{4g(2g-1)(2g-2)x^{2g-2}} \sum_{g'=0}^{g} (2g'-1)\binom{2g}{2g'} \frac{E_{2g-2g'}B_{2g'}}{2^{2g-2g'}} \\
&\quad + \sum_{h\geq 0} \epsilon^{2h-2} \sum_{\substack{g,r\geq 0 \\ g+r=h}} \sum_{\substack{k\geq 1 \\ j_1,\dots,j_k\geq 1}} \binom{2-2g-k+|j|}{2r} \frac{E_{2r}}{2^{2r}} a_g(2j_1,\dots,2j_k)\, s_{j_1}\cdots s_{j_k}\, x^{2-2h-k+|j|}.
\end{aligned}$$

In other words, the modified GUE correlators with even couplings (113) have the expressions:

$$\begin{aligned}
&\langle \phi_{j_1}\cdots\phi_{j_k}\rangle(x;\epsilon) \\
&= k! \sum_{h\geq 0} \epsilon^{2h-2} x^{2-2h-k+|j|} \sum_{\substack{g,r\geq 0 \\ g+r=h}} \binom{2-2g-k+|j|}{2r} \frac{E_{2r}}{2^{2r}} a_g(2j_1,\dots,2j_k),
\end{aligned} \tag{114}$$

where $k \geq 1$ and $j_1, \dots, j_k \geq 1$. It should be noted that the $\langle \phi_{j_1}\cdots\phi_{j_k}\rangle(x;\epsilon)$ with $k \geq 1$, $j_1, \dots, j_k \geq 1$ is a polynomial of x.

5.2. Combinations of certain special cubic Hodge integrals. Based on the Hodge–GUE correspondence and using Theorem 2, we compute in this subsection combinations of certain special cubic Hodge integrals. More precisely, we will prove Corollary 2.

The cubic Hodge free energy associated with $\Lambda_g(-1)\,\Lambda_g(-1)\,\Lambda_g\big(\frac{1}{2}\big)$ is defined by

$$\mathcal{H}(\mathbf{t};\epsilon) = \sum_{g\geq 0} \epsilon^{2g-2} \sum_{k\geq 0} \frac{1}{k!} \sum_{i_1,\dots,i_k\geq 0} t_{i_1}\cdots t_{i_k} \int_{\overline{\mathcal{M}}_{g,k}} \Lambda_g(-1)\,\Lambda_g(-1)\,\Lambda_g\Big(\frac{1}{2}\Big)\,\psi_1^{i_1}\cdots\psi_k^{i_k}.$$

Here, $\mathbf{t} = (t_0, t_1, \dots)$. (Warning: Avoid from confusing with the variables t_ℓ, $\ell \geq 0$ of the Toda lattice hierarchy used in Sect. 4.) The *Hodge–GUE correspondence* connects $\mathcal{H}(\mathbf{t};\epsilon)$ with the GUE partition function with even couplings, which is given by the following theorem.

Theorem A. ([15,20]) The following identity holds true:

$$\begin{aligned}
&\log Z(x,\mathbf{s};\epsilon) + \epsilon^{-2}\Bigg(-\frac{1}{2}\sum_{j_1,j_2\geq 1}\frac{j_1 j_2}{j_1+j_2}\bar{s}_{j_1}\bar{s}_{j_2} + \sum_{j\geq 1}\frac{j}{1+j}\bar{s}_j - x\sum_{j\geq 1}\bar{s}_j - \frac{1}{4} + x\Bigg) \\
&\qquad = \mathcal{H}\Big(\mathbf{t}\Big(x-\frac{\epsilon}{2},\mathbf{s}\Big);\sqrt{2}\epsilon\Big) + \mathcal{H}\Big(\mathbf{t}\Big(x+\frac{\epsilon}{2},\mathbf{s}\Big);\sqrt{2}\epsilon\Big),
\end{aligned} \tag{115}$$

where $\bar{s}_j := \binom{2j}{j} s_j$ and

$$t_i(x,\mathbf{s}) := \sum_{j\geq 1} j^{i+1}\bar{s}_j - 1 + \delta_{i,1} + x\,\delta_{i,0}, \qquad i \geq 0. \tag{116}$$

Recall from (110) that the modified GUE partition function with even couplings $\widetilde{Z}$ is defined as the unique series of $x-1$ and $\mathbf{s}$ satisfying

$$Z(x, \mathbf{s}; \epsilon) \;=\; \widetilde{Z}\Big(x - \frac{\epsilon}{2}, \mathbf{s}; \epsilon\Big)\, \widetilde{Z}\Big(x + \frac{\epsilon}{2}, \mathbf{s}; \epsilon\Big). \tag{117}$$

Combining (117) with (115) we obtain the following corollary.

Corollary 4. *The following formula holds true:*

$$\log \widetilde{Z}\big(x, \mathbf{s}; \epsilon\big) \;=\; \mathcal{H}\big(\mathbf{t}(x, \mathbf{s}); \sqrt{2}\epsilon\big) + \frac{1}{4\epsilon^2} \sum_{j_1, j_2 \geq 1} \frac{j_1\, j_2}{j_1 + j_2}\, \bar{s}_{j_1}\, \bar{s}_{j_2} + \frac{x}{2\epsilon^2}\Big(\sum_{j\geq 1} \bar{s}_j - 1\Big)$$
$$- \frac{1}{2\epsilon^2} \sum_{j\geq 1} \frac{j}{1+j}\, \bar{s}_j + \frac{1}{8\epsilon^2}. \tag{118}$$

Denote $\Omega_g := \Lambda_g(-1)\, \Lambda_g(-1)\, \Lambda_g\big(\frac{1}{2}\big)$ as in the introduction, and write

$$\Omega_g =: \sum_{d\geq 0} \Omega_g^{[d]}, \qquad \Omega_g^{[d]} \in H^{2d}(\overline{\mathcal{M}}_{g,k}).$$

It might be helpful to notice that for $g = 1$, deg $\Omega_1 \leq 1$; for $g \geq 2$, deg $\Omega_g \leq 3g - 3$. Motivated by Theorem A, let us consider the following combination of Hodge integrals. For any given $k \geq 0$, $i_1, \ldots, i_k \geq 0$, define $H_{i_1,\ldots,i_k}(x; \epsilon) \in \epsilon^{-2}\mathbb{Q}[[x-1, \epsilon^2]]$ by

$$H_{i_1,\ldots,i_k}(x; \epsilon) \;:=\; 2^{g-1} \sum_{g=0}^{\infty} \epsilon^{2g-2} \sum_{d=0}^{3g} \sum_{\lambda\in\mathbb{Y}} \frac{(-1)^{\ell(\lambda)}}{m(\lambda)!} \Big\langle \Omega_g^{[d]}\, e^{(x-1)\tau_0}\, \tau_{\lambda+1}\, \tau_{i_1} \cdots \tau_{i_k} \Big\rangle_g. \tag{119}$$

Note that in the notation $\langle \ldots \rangle_g$, we omit the index m from $\langle \ldots \rangle_{g,m}$. For such an abbreviation, m should be recovered from counting the number of τ's in "…". Therefore, for each fixed g, d and for each monomial in the Taylor expansion $e^{(x-1)\tau_0} = \sum_{r=0}^{\infty} \frac{1}{r!}(x-1)^r$, the above summation over partitions $\sum_{\lambda\in\mathbb{Y}}$ is a finite sum, i.e., the degree-dimension matching $|\lambda| = 3g - 3 + k + r - d - |i|$ has to be hold. Lemma 3 also easily follows from this constrain with $r = 0$ taken. The numbers $H_{g,i_1,\ldots,i_k}$ defined by (38) and the formal series $H_{i_1,\ldots,i_k}(x; \epsilon)$ are clearly related by

$$H_{i_1,\ldots,i_k}(x = 1; \epsilon) \;=\; \sum_{g=0}^{\infty} \epsilon^{2g-2} H_{g,i_1,\ldots,i_k}. \tag{120}$$

Proposition 3. *For any $k \geq 0$ and $j_1, \ldots, j_k \geq 1$, the following formula holds true:*

$$\phi_{j_1,\ldots,j_k}(x; \epsilon) \;=\; \prod_{\ell=1}^{k} \binom{2j_\ell}{j_\ell} \sum_{i_1,\ldots,i_k\geq 0} \prod_{\ell=1}^{k} j_\ell^{i_\ell+1}\, H_{i_1,\ldots,i_k}\big(x; \epsilon\big)$$
$$+ \frac{\delta_{k,2}}{2\epsilon^2} \frac{j_1\, j_2}{j_1 + j_2} \binom{2j_1}{j_1}\binom{2j_2}{j_2} - \frac{\delta_{k,1}}{2\epsilon^2} \binom{2j_1}{j_1}\Big(\frac{j_1}{1+j_1} - x\Big). \tag{121}$$

Proof. Note that the $\mathcal{H}(\mathbf{t};\sqrt{2}\epsilon)$ has the expression

$$\mathcal{H}(\mathbf{t};\sqrt{2}\epsilon)=\sum_g 2^{g-1}\epsilon^{2g-2}\sum_{m_0,m_1,m_2,\cdots}\int_{\overline{\mathcal{M}}_{g,\sum_i m_i}}\Omega_g \cdot\prod_{s=1}^{m_0}\psi_s^0\prod_{s=m_0+1}^{m_0+m_1}\psi_s^1\prod_{s=m_0+m_1+1}^{m_0+m_1+m_2}\psi_s^2\cdots\cdot\prod_{i=0}^{\infty}\frac{t_i^{m_i}}{m_i!}.$$

Formula (121) is then proved by substituting (116) and by using Corollary 4. □

Proof of Corollary 2. Note that the $k=0$ case is already given in [20]. By taking $x=1$ in (121) and using Theorem 2 we find (40). Formula (39) is then implied in a standard way by using the following linear equation (proven in [20])

$$\sum_{k\geq 1}ks_k\frac{\partial\widetilde{Z}}{\partial s_k}+\left(\frac{x^2}{4\epsilon^2}-\frac{1}{16}\right)\widetilde{Z}=\frac{1}{2}\frac{\partial\widetilde{Z}}{\partial s_1}\tag{122}$$

and the fact that

$$a_{n,j}=\frac{\partial^2\log\tau_n^{\mathrm{dKdV}}}{\partial s_1\partial s_j},\qquad j\geq 1.\tag{123}$$

Here $\widetilde{Z}=\widetilde{Z}(x,\mathbf{s};\epsilon)$ denotes the modified GUE partition function with even couplings. Note that the fact (123) can be obtained by taking the coefficients of λ^{-1} on the both sides of (21). The corollary is proved.

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Appendix A. On Consequence of the Hodge–GUE Correspondence

In this appendix, we derive a consequence of the Hodge–GUE correspondence that has a similar flavour to formula (121). Note that

$$\begin{aligned}&\mathcal{H}(\mathbf{t}(x,\mathbf{s});\sqrt{2}\epsilon)\\&=\sum_{g\geq 0}2^{g-1}\epsilon^{2g-2}\sum_{k\geq 0}\frac{1}{k!}\int_{\overline{\mathcal{M}}_{g,k}}\Omega_{g,k}\prod_{m=1}^{k}\left(\sum_{i_m\geq 0}t_{i_m}(x,\mathbf{s})\psi_m^{i_m}\right)\\&=\sum_{g\geq 0}2^{g-1}\epsilon^{2g-2}\sum_{k\geq 0}\frac{1}{k!}\int_{\overline{\mathcal{M}}_{g,k}}\Omega_{g,k}\sum_{l=0}^{k}\binom{k}{l}\prod_{m=l+1}^{k}\left((x-1)-\frac{\psi_m^2}{1-\psi_m}\right)\\&\quad\cdot\sum_{p_1,\dots,p_l}\prod_{m=1}^{l}\frac{p_m\bar{s}_{p_m}}{1-p_m\psi_m}.\end{aligned}$$

Then by comparing the coefficients of $s_{p_1} \dots s_{p_l}$ of the both sides of (118) we get

$$\langle \sigma_{p_1} \dots \sigma_{p_l} \rangle_g(x) = \sum_{k \geq l} \frac{1}{(k-l)!} \int_{\overline{\mathcal{M}}_{g,k}} \Omega_{g,k} \prod_{m=l+1}^{k} \left((x-1) - \frac{\psi_m^2}{1-\psi_m} \right) \prod_{m=1}^{l} \frac{p_m \binom{2p_m}{p_m}}{1 - p_m \psi_m} + \delta_{g,0}\delta_{l,2} \frac{p_1 p_2}{p_1+p_2} \binom{2p_1}{p_1}\binom{2p_2}{p_2} + \frac{1}{2}\delta_{g,0}\delta_{l,1} \binom{2p_1}{p_2} \left(\frac{p_1}{1+p_1} - x \right). \tag{124}$$

Here $\langle \sigma_{p_1} \dots \sigma_{p_l} \rangle(x;\epsilon) =: \sum_{g \geq 0} \epsilon^{2g-2} \langle \sigma_{p_1} \dots \sigma_{p_l} \rangle_g(x)$, and $\langle \sigma_{p_1} \dots \sigma_{p_l} \rangle(x;\epsilon)$ are the modified GUE correlators with even couplings defined in (113). Taking $x = 1$ we find

$$\langle \sigma_{p_1} \dots \sigma_{p_l} \rangle_g|_{x=1} = \sum_{k \geq l} \frac{1}{(k-l)!} \int_{\overline{\mathcal{M}}_{g,k}} \Omega_{g,k} \prod_{m=l+1}^{k} \left(-\frac{\psi_m^2}{1-\psi_m} \right) \prod_{m=1}^{l} \frac{p_m \binom{2p_m}{p_m}}{1 - p_m \psi_m}. \tag{125}$$

A further consideration to (125) was given in [5].
Combining (124) with (115) we find for any fixed $l \geq 1$, $p_1, \dots, p_l \geq 1$ the following identities:

$$l!\, x^{2-2g-l+|j|} \sum_{\substack{g_1, r \geq 0 \\ g_1 + r = g}} \binom{2-2g_1-l+|j|}{2r} \frac{E_{2r}}{2^{2r}} a_{g_1}(2p_1, \dots, 2p_l) = \sum_{k \geq l} \frac{1}{(k-l)!} \int_{\overline{\mathcal{M}}_{g,k}} \Omega_{g,k} \prod_{m=l+1}^{k} \left((x-1) - \frac{\psi_m^2}{1-\psi_m} \right) \prod_{m=1}^{l} \frac{p_m \binom{2p_m}{p_m}}{1 - p_m \psi_m} + \delta_{g,0}\delta_{l,2} \frac{p_1 p_2}{p_1+p_2} \binom{2p_1}{p_1}\binom{2p_2}{p_2} + \frac{1}{2}\delta_{g,0}\delta_{l,1} \binom{2p_1}{p_2} \left(\frac{p_1}{1+p_1} - x \right), \quad g \geq 0. \tag{126}$$

Note that for any $g \geq 0$, the RHS is a priori a power series of $x - 1$, but the LHS shows that it is actually a monomial of x and so is also a polynomial of $x - 1$. This subset of the identities deserve a further investigation. Moreover, the LHS vanishes when g is sufficiently large, an so is the RHS; this provides another subset of the identities for the cubic Hodge integrals.

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