Commun. Math. Phys. 378, 117–147 (2020)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-019-03664-5

Communications in Mathematical Physics

The Covariant Stone–von Neumann Theorem for Actions of Abelian Groups on C^*-Algebras of Compact Operators

Leonard Huang[1], **Lara Ismert**[2]

[1] Department of Mathematics and Statistics, University of Nevada, Reno, 1664 N. Virginia Street, Reno, NV 89557, USA. E-mail: LeonardHuang@unr.edu
[2] Mathematics Department, Embry-Riddle Aeronautical University, 3700 Willow Creek Road, Prescott, AZ 86301, USA. E-mail: Lara.Ismert@erau.edu

Received: 24 April 2019 / Accepted: 5 November 2019
Published online: 1 February 2020 –

Abstract: In this paper, we formulate and prove a version of the Stone–von Neumann Theorem for every C^*-dynamical system of the form $(G, \mathbb{K}(\mathcal{H}), \alpha)$, where G is a locally compact Hausdorff abelian group and $\mathcal{H}$ is a Hilbert space. The novelty of our work stems from our representation of the Weyl Commutation Relation on Hilbert $\mathbb{K}(\mathcal{H})$-modules, instead of just Hilbert spaces, and our introduction of two additional commutation relations, which are necessary to obtain a uniqueness theorem. Along the way, we apply one of our basic results on Hilbert C^*-modules to significantly shorten the length of Iain Raeburn's well-known proof of Takai–Takesaki Duality.

1. Introduction

One of the most famous mathematical results in quantum mechanics is the Stone–von Neumann Theorem. Informally, the theorem establishes the physical equivalence of Werner Heisenberg's matrix mechanics and Erwin Schrödinger's wave mechanics, which was seen by Heisenberg to be an outstanding problem in the early days of quantum mechanics [7]. The theorem was an attempt to prove that any pair (A, B) of self-adjoint unbounded operators on a Hilbert space $\mathcal{H}$ that satisfies the Heisenberg Commutation Relation on a common dense invariant subset D of their domain, i.e.,

$$[A|_D, B|_D] = i\hbar \cdot \mathrm{Id}_D,$$

is unitarily equivalent to a direct sum of copies of $\left(\widehat{X}, \widehat{P_\hbar}\right)$, which are self-adjoint unbounded operators on $L^2(\mathbb{R})$ defined as follows:

$$\mathrm{Dom}\left(\widehat{X}\right) = \left\{ f \in L^2(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}} |x f(x)|^2 \, \mathrm{d}\mu(x) < \infty \right\},$$

$$\forall f \in \mathrm{Dom}\left(\widehat{X}\right) : \widehat{X}(f) \stackrel{\mathrm{df}}{=} \begin{Bmatrix} \mathbb{R} \to & \mathbb{C} \\ x \mapsto & x f(x) \end{Bmatrix};$$

$$\mathrm{Dom}\big(\widehat{P_\hbar}\big) = W^{1,2}(\mathbb{R}),$$
$$\forall f \in \mathrm{Dom}\big(\widehat{P_\hbar}\big): \quad \widehat{P_\hbar}(f) \stackrel{\mathrm{df}}{=} -i\hbar \cdot f'.$$

We recall that $W^{1,2}(\mathbb{R})$ denotes the space of weakly-differentiable square-integrable functions on $\mathbb{R}$ whose weak derivative is also square-integrable. This statement about unbounded operators is not true in general, where complications arise from domain issues — a well-known counterexample involving multiplication and differentiation operators on $L^2([0, 1])$ is given in [12].

The Stone–von Neumann Theorem was first given a rigorous formulation by Marshall Stone in 1930 [19], and it was this formulation that John von Neumann proved in 1931 [20]. The exponentiated form of the Heisenberg Commutation Relation, called the *Weyl Commutation Relation*, is investigated in these papers, because it involves only one-parameter unitary groups. More precisely, a pair (R, S) of strongly-continuous one-parameter unitary groups on a Hilbert space $\mathcal{H}$ satisfies the Weyl Commutation Relation if and only if

$$\forall x, y \in \mathbb{R}: \qquad S(y)R(x) = e^{-i\hbar xy} \cdot R(x)S(y).$$

von Neumann proved that any such pair is unitarily equivalent to a direct sum of copies of (U, V), where U and V are strongly-continuous one-parameter unitary groups on $L^2(\mathbb{R})$ defined by

$$\forall x, y \in \mathbb{R},\ \forall f \in L^2(\mathbb{R}): \qquad \big[\mathsf{U}(x)\big](f) = f(\bullet + \hbar x) \qquad \text{and} \qquad \big[\mathsf{V}(y)\big](f) = e^{iy\bullet} f.$$

Basically, U acts by translations, and V acts by phase modulations.

The statement of the Stone–von Neumann Theorem has undergone major revisions in the decades since its initial formulation. George Mackey appears to have been the first to recognize its generalization to second-countable locally compact Hausdorff abelian groups, in [10]. Nowadays, his generalization is treated as part of his theory of induced representations of locally compact Hausdorff groups and is generally considered the standard modern formulation of the Stone–von Neumann Theorem, which we now state.

Theorem 1. *Let G be a locally compact Hausdorff abelian group. If R and S are strongly-continuous unitary representations of G and $\widehat{G}$, respectively, on a Hilbert space $\mathcal{H}$ that satisfy the Weyl Commutation Relation, i.e.,*

$$\forall x \in G,\ \forall \gamma \in \widehat{G}: \qquad S(\gamma)R(x) = \gamma(x) \cdot R(x)S(\gamma),$$

then $(\mathcal{H}, R, S)$ must be unitarily equivalent to a direct sum of copies of $\big(L^2(G), \mathsf{U}_G, \mathsf{V}_G\big)$, where U_G denotes the unitary representation of G on $L^2(G)$ by left translations, and V_G denotes the unitary representation of $\widehat{G}$ on $L^2(G)$ by phase modulations, i.e.,

$$\big[\mathsf{U}_G(x)\big](f) \stackrel{df}{=} f\Big(x^{-1}\bullet\Big) \qquad \textit{and} \qquad \big[\mathsf{V}_G(\gamma)\big](f) \stackrel{df}{=} \gamma f$$

for all $x \in G$, $\gamma \in \widehat{G}$, and $f \in L^2(G)$.

The work of Marc Rieffel in [16,17] has revealed that Theorem 1 is actually a statement about the Morita equivalence of the C^*-algebras $\mathbb{C}$ and $C^*(G, \mathrm{C}_0(G), \mathsf{lt})$, where lt denotes the strongly-continuous action of G on $\mathrm{C}_0(G)$ by left translations. The theorem thus acquires a more algebraic flavor. This Morita equivalence is a special case of a more general result known as *Green's Imprimitivity Theorem*, which we actually need to prove our covariant generalization of Theorem 1.

Several generalizations of the Stone–von Neumann Theorem can be found in the literature. For example, [4] extends the theorem to measurable unitary representations of G and $\widehat{G}$ on a Hilbert space, and [13] extends the theorem to Hecke pairs using the machinery of non-abelian duality. Although these generalizations are non-trivial and interesting, their use of only Hilbert-space representations is a common limiting feature.

In this paper, we provide not just another incremental generalization of the Stone–von Neumann Theorem, but a complete paradigm shift that significantly augments the theorem's range of applicability. By leaving the realm of Hilbert spaces and using representations on Hilbert C^*-modules, we show that the Stone–von Neumann Theorem is not so much about representations of locally compact Hausdorff abelian groups on Hilbert spaces as it is about representations of abelian C^*-dynamical systems on Hilbert C^*-modules — for every C^*-dynamical system of the form $(G, \mathbb{K}(\mathcal{H}), \alpha)$, where G is a locally compact Hausdorff abelian group and $\mathcal{H}$ is a non-trivial Hilbert space, our covariant generalization classifies up to unitary equivalence all quadruples (X, ρ, R, S) with the following properties:

- X is a Hilbert $\mathbb{K}(\mathcal{H})$-module.
- R and S are strongly-continuous unitary representations of G and $\widehat{G}$, respectively, on X that satisfy the Weyl Commutation Relation.
- ρ is a non-degenerate $*$-representation of $\mathbb{K}(\mathcal{H})$ on X that obeys the following commutation relations:

$$R(x)\rho(a) = \rho(\alpha_x(a))R(x) \quad \text{and} \quad S(\gamma)\rho(a) = \rho(a)S(\gamma)$$

  for all $x \in G$, $\gamma \in \widehat{G}$, and $a \in A$. These relations are also called *covariance relations*.

Using results on non-abelian duality, one could very well generalize our covariant version of the Stone–von Neumann Theorem to non-abelian C^*-dynamical systems, or even quantum-group dynamical systems, but such an undertaking would take us too far afield, so we content ourselves with presenting only the abelian case, which we feel is already a significant advance. Further generalizations will be explored in a sequel.

This paper is organized as follows:

- Section 2 is a short preliminary section that recalls some concepts and results about C^*-crossed products that we need. In particular, we show how to associate a Hilbert C^*-module to a C^*-dynamical system in a canonical way. This Hilbert C^*-module is featured in Green's Imprimitivity Theorem and is crucial to a formulation of our covariant generalization of the Stone–von Neumann Theorem.
- Section 3 introduces Heisenberg module representations and the Schrödinger module representation of an abelian C^*-dynamical system (G, A, α). These concepts allow an efficient formulation of our covariant generalization of the Stone–von Neumann Theorem. We construct an injective map from the class of all Heisenberg module representations of (G, A, α) to the class of all covariant module representations of $(G, C_0(G, A), \mathsf{lt} \otimes \alpha)$, which is a C^*-dynamical system that plays a pivotal role in Iain Raeburn's proof of Takai–Takesaki Duality.
- Section 4 provides an overview of the properties of Hilbert $\mathbb{K}(\mathcal{H})$-modules that have been established in [2,3]. Hilbert $\mathbb{K}(\mathcal{H})$-modules obviously generalize Hilbert spaces, yet they behave much like Hilbert spaces, which makes them very desirable to work with.

- Section 5 contains our main result: the Covariant Stone–von Neumann Theorem (Proposition 7).
- Section 6 explains why the Covariant Stone–von Neumann Theorem is non-trivial, i.e., it generalizes the classical Stone–von Neumann Theorem. A basic result in this section also allows us to shorten Raeburn's proof of Takai–Takesaki Duality.
- The Conclusions section describes some problems that this paper was unable to resolve. It also suggests new avenues of research that would be of interest to both mathematicians and physicists.
- Finally, an appendix contains a proof of an approximation lemma, stated in the main body of the paper, that would be considered folklore, but for which we were unable to locate an adequate reference.

We assume that the reader has a reasonable working knowledge of C^*-algebras, C^*-dynamical systems, and Hilbert C^*-modules. Throughout this paper, we adopt the following notations and conventions:

- $\mathbb{N}$ denotes the set of positive integers, and for each $n \in \mathbb{N}$, let $[n] \stackrel{\text{df}}{=} \mathbb{N}_{\leq n}$.
- For a set I, let $\text{Fin}(I)$ denote the set of finite subsets of I.
- For a locally compact Hausdorff abelian group G, let $\widehat{G}$ denote its Pontryagin dual.
- For a locally compact Hausdorff space X and a normed vector space V, let $\diamond$: $\text{C}_0(X) \times V \to \text{C}_0(X, V)$ be defined by
$$\forall f \in \text{C}_0(X),\ \forall v \in V: \qquad f \diamond v \stackrel{\text{df}}{=} \begin{Bmatrix} X \to & V \\ x \mapsto & f(x) \cdot v \end{Bmatrix}.$$
Note that $\diamond[\text{C}_c(X) \times V] \subseteq \text{C}_c(X, V)$.
- For a locally compact Hausdorff space X and a C^*-algebra A, let $\centerdot : A \times \text{C}_0(X, A) \to \text{C}_0(X, A)$ be defined by
$$\forall a \in A,\ \forall f \in \text{C}_0(X, A): \qquad a \centerdot f \stackrel{\text{df}}{=} \begin{Bmatrix} X \to & A \\ x \mapsto & af(x) \end{Bmatrix}.$$
- All Hilbert-space inner products are conjugate-linear in the first argument and linear in the second.
- For a Hilbert space $\mathcal{H}$ and vectors $v, w \in \mathcal{H}$, let $|v\rangle\langle w|$ denote the rank-one operator on $\mathcal{H}$ defined by
$$\forall x \in \mathcal{H}: \qquad (|v\rangle\langle w|)(x) \stackrel{\text{df}}{=} \langle w|x\rangle_{\mathcal{H}} \cdot v.$$
- Let $\text{Proj}_{\mathcal{H},\mathcal{K}}$ denote the orthogonal projection of a Hilbert space $\mathcal{H}$ onto a closed subspace $\mathcal{K}$.
- For a C^*-algebra A and Hilbert A-modules X and Y, the set of adjointable/compact/unitary operators from X to Y is denoted by $\mathbb{L}(\mathsf{X}, \mathsf{Y})/\mathbb{K}(\mathsf{X}, \mathsf{Y})/\mathbb{U}(\mathsf{X}, \mathsf{Y})$. If $\mathsf{X} = \mathsf{Y}$, then we write $\mathbb{L}(\mathsf{X})/\mathbb{K}(\mathsf{X})/\mathbb{U}(\mathsf{X})$.
- For a C^*-algebra A and a Hilbert C^*-module X (not necessarily over A), a $*$-representation of A on X is a C^*-homomorphism $\rho : A \to \mathbb{L}(\mathsf{X})$, which is then said to be *non-degenerate* if and only if
$$\overline{\text{Span}(\{[\rho(a)](\zeta) \mid a \in A \text{ and } \zeta \in \mathsf{X}\})}^{\mathsf{X}} = \mathsf{X}.$$

- For a locally compact Hausdorff group G and a Hilbert C^*-module X, a unitary representation of G on X is a group homomorphism R from G to the group $\mathbb{U}(\mathsf{X})$ of unitary adjointable operators on X, which is then said to be *strongly continuous* if and only if the map

$$\begin{Bmatrix} G \to & \mathsf{X} \\ x \mapsto & [R(x)](\zeta) \end{Bmatrix}$$

is continuous for each $\zeta \in \mathsf{X}$.
- For a C^*-dynamical system (G, A, α), let lt denote the left G-action on $\mathrm{C}_0(G, A)$ by left translations, and let $\mathsf{lt} \otimes \alpha$ denote the left G-action on $\mathrm{C}_0(G, A)$ by α-twisted left translations, which is defined by

$$\forall x \in G,\ \forall f \in \mathrm{C}_0(G, A): \qquad (\mathsf{lt} \otimes \alpha)_x(f) \stackrel{\mathrm{df}}{=} \begin{Bmatrix} G \to & A \\ y \mapsto & \alpha_x\left(f\left(x^{-1}y\right)\right) \end{Bmatrix}.$$

Note that $(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha)$ is also a C^*-dynamical system.

2. Preliminaries

As C^*-crossed products will be used extensively in this paper, let us recall some concepts in this area.

Throughout this section, we fix an arbitrary C^*-dynamical system (G, A, α), with G not assumed to be abelian. We also fix a Haar measure μ on G.

Recall that the $\mathbb{C}$-vector space $\mathrm{C_c}(G, A)$ can be given a convolution $\star_{G,A,\alpha}$ and an involution $^{*_{G,A,\alpha}}$ by

$$\forall f, g \in \mathrm{C_c}(G, A): \qquad f \star_{G,A,\alpha} g \stackrel{\mathrm{df}}{=} \begin{Bmatrix} G \to & A \\ x \mapsto & \int_G f(y)\alpha_y\left(g\left(y^{-1}x\right)\right) \, \mathrm{d}\mu(y) \end{Bmatrix};$$

$$f^{*_{G,A,\alpha}} \stackrel{\mathrm{df}}{=} \begin{Bmatrix} G \to & A \\ x \mapsto & \Delta_G\left(x^{-1}\right) \cdot \alpha_x\left(f\left(x^{-1}\right)^*\right) \end{Bmatrix},$$

where Δ_G denotes the modular function of G.

Definition 1. A (G, A, α)*-covariant module representation* is a triple (X, ρ, R) with the following properties:

(1) X is a Hilbert C^*-module (not necessarily over A).
(2) ρ is a non-degenerate $*$-representation of A on X.
(3) R is a strongly-continuous unitary representation of G on X.
(4) $R(x)\rho(a) = \rho(\alpha_x(a))R(x)$ for all $x \in G$ and $a \in A$.

Covariant module representations are used in the construction of C^*-crossed products. Given a (G, A, α)-covariant module representation (X, ρ, R), we can define an algebraic $*$-homomorphism $\rho \rtimes R$, called the *integrated form* of (X, ρ, R), from the convolution $*$-algebra $\left(\mathrm{C_c}(G, A), \star_{G,A,\alpha}, ^{*_{G,A,\alpha}}\right)$ to $\mathbb{L}(\mathsf{X})$ by

$$\forall f \in \mathrm{C_c}(G, A): \qquad (\rho \rtimes R)(f) \stackrel{\mathrm{df}}{=} \int_G \rho(f(x))R(x) \, \mathrm{d}\mu(x).$$

The *full crossed product* $C^*(G, A, \alpha)$ is defined as the C^*-algebraic completion of $\left(\mathrm{C_c}(G, A), \star_{G,A,\alpha}, {}^{*_{G,A,\alpha}}\right)$ with respect to the universal norm $\|\cdot\|_{(G,A,\alpha),\mathrm{u}}$ given by

$$\|f\|_{(G,A,\alpha),\mathrm{u}} \stackrel{\mathrm{df}}{=} \sup\left(\left\{\|(\rho \rtimes R)(f)\|_{\mathbb{L}(\mathsf{X})} \,\middle|\, (\mathsf{X}, \rho, R) \text{ is a } (G, A, \alpha)\text{-covariant module representation}\right\}\right)$$

for all $f \in \mathrm{C_c}(G, A)$. This norm is well-defined as it is dominated by the L^1-norm on $\mathrm{C_c}(G, A)$.

We let $\eta_{(G,A,\alpha)}$ denote the canonical dense linear embedding of $\mathrm{C_c}(G, A)$ into $C^*(G, A, \alpha)$, and if $A = \mathbb{C}$, in which case α is necessarily trivial, we simply write η_G.

For a (G, A, α)-covariant module representation (X, ρ, R), we denote by $\overline{\rho \rtimes R}$ the extension of $\rho \rtimes R$ to a C^*-homomorphism from $C^*(G, A, \alpha)$ to $\mathbb{L}(\mathsf{X})$.

To (G, A, α), one can associate a special Hilbert A-module, denoted by $\mathsf{L}^2(G, A, \alpha)$, in a canonical manner. Observe that $\mathrm{C_c}(G, A)$ is a pre-Hilbert A-module, whose right A-action $\bullet$ and A-valued pre-inner product $[\cdot|\cdot] : \mathrm{C_c}(G, A) \times \mathrm{C_c}(G, A) \to A$ are defined as follows:

(1) $\phi \bullet a \stackrel{\mathrm{df}}{=} \left\{\begin{matrix} G \to & A \\ x \mapsto & \phi(x)\alpha_x(a) \end{matrix}\right\}$ for all $a \in A$ and $\phi \in \mathrm{C_c}(G, A)$.

(2) $[\phi|\psi] \stackrel{\mathrm{df}}{=} \int_G \alpha_{x^{-1}}\left(\phi(x)^*\psi(x)\right) \mathrm{d}\mu(x) = \left(\phi^{*_{G,A,\alpha}} \star_{G,A,\alpha} \psi\right)(e_G)$ for all $\phi, \psi \in \mathrm{C_c}(G, A)$.

Define $\mathsf{L}^2(G, A, \alpha)$ to be the Hilbert A-module obtained by completing $\mathrm{C_c}(G, A)$ with respect to the norm induced by $[\cdot|\cdot]$. Let $q_{(G,A,\alpha)} : \mathrm{C_c}(G, A) \hookrightarrow \mathsf{L}^2(G, A, \alpha)$ denote the canonical dense linear embedding, and if no confusion can arise, we will omit the subscript and simply write q.

We will use q when defining operators on $\mathsf{L}^2(G, A, \alpha)$ to remind the reader that unless $A = \mathbb{C}$, the elements of $\mathsf{L}^2(G, A, \alpha)$ are generally not functions from G to A. Having said this, we can equip $\mathsf{L}^2(G, A, \alpha)$ with the following structural data:

- A $*$-representation $\mathsf{M}^{(G,A,\alpha)}$ of A on $\mathsf{L}^2(G, A, \alpha)$ such that for all $a \in A$ and $\phi \in \mathrm{C_c}(G, A)$,
$$\left[\mathsf{M}^{(G,A,\alpha)}(a)\right](q(\phi)) = q(a \centerdot \phi).$$
- A unitary representation $\mathsf{U}^{(G,A,\alpha)}$ of G on $\mathsf{L}^2(G, A, \alpha)$ such that for all $x \in G$ and $\phi \in \mathrm{C_c}(G, A)$,
$$\left[\mathsf{U}^{(G,A,\alpha)}(x)\right](q(\phi)) = q\left((\mathsf{lt} \otimes \alpha)_x(\phi)\right).$$
- A unitary representation $\mathsf{V}^{(G,A,\alpha)}$ of G on $\mathsf{L}^2(G, A, \alpha)$ such that for all $\gamma \in \widehat{G}$ and $\phi \in \mathrm{C_c}(G, A)$,
$$\left[\mathsf{V}^{(G,A,\alpha)}(\gamma)\right](q(\phi)) = q(\gamma \cdot \phi).$$

Proving that these representations are well-defined is a routine exercise. We refer the reader to Chapter 4 of [22] for details. We will omit supscripts and simply write M, U, and V if no confusion arises from doing so.

The Hilbert A-module $\mathsf{L}^2(G, A, \alpha)$ is the linchpin of our formulation of the covariant Stone–von Neumann Theorem, and it is also the main player in Green's Imprimitivity Theorem, which we now state.

Theorem 2 (Green's Imprimitivity Theorem). *Let (G, A, α) be a C^*-dynamical system. Then $\mathsf{L}^2(G, A, \alpha)$ is a $(C^*(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha), A)$-imprimitivity bimodule with the following properties:*

- *If Ξ denotes the non-degenerate $*$-representation of $\mathrm{C}_0(G, A)$ on $\mathsf{L}^2(G, A, \alpha)$ uniquely determined by*

$$\forall g \in \mathrm{C}_0(G, A),\ \forall \phi \in \mathrm{C}_c(G, A): \qquad [\Xi(g)](q(\phi)) = q(g\phi),$$

  then $\left(\mathsf{L}^2(G, A, \alpha), \Xi, \mathsf{U}\right)$ is a $(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha)$-covariant module representation, and the left action of $C^*(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha)$ on $\mathsf{L}^2(G, A, \alpha)$ is $\Xi \rtimes \mathsf{U}$.
- *The left $C^*(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha)$-valued inner product on $\mathsf{L}^2(G, A, \alpha)$ is uniquely determined by*

$$\begin{aligned} &{}_{C^*(G,\mathrm{C}_0(G,A),\mathsf{lt}\otimes\alpha)}\langle q(\phi) | q(\psi) \rangle \\ &= \eta_{(G,\mathrm{C}_0(G,A),\mathsf{lt}\otimes\alpha)}\left(\left\{ \begin{matrix} G \to & \mathrm{C}_0(G, A) \\ x \mapsto & \Delta_G\left(x^{-1}\bullet\right) \cdot \phi(\bullet)\alpha_x\left(\psi\left(x^{-1}\bullet\right)^*\right) \end{matrix} \right\} \right) \end{aligned}$$

  for all $\phi, \psi \in \mathrm{C}_c(G, A)$.
- *The right A-action and the right A-valued inner product on $\mathsf{L}^2(G, A, \alpha)$ are precisely the ones that define $\mathsf{L}^2(G, A, \alpha)$ as a Hilbert A-module.*

Complete proofs of Green's Imprimitivity Theorem may be found in [5,22].

Remark 1. Proposition 3.8 of [15] says that

$$\overline{\Xi \rtimes \mathsf{U}} : C^*(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha) \to \mathbb{L}\left(\mathsf{L}^2(G, A, \alpha)\right)$$

is an injective C^*-homomorphism and that $\operatorname{Range}\left(\overline{\Xi \rtimes \mathsf{U}}\right) = \mathbb{K}\left(\mathsf{L}^2(G, A, \alpha)\right)$.

3. Module Representations

Throughout this section, we shall fix an arbitrary C^*-dynamical system (G, A, α) with G abelian. We shall also fix a Haar measure μ on G and a Haar measure ν on $\widehat{G}$.

Definition 2. A (G, A, α)-*Heisenberg module representation* is a quadruple (X, ρ, R, S) with the following properties:

(1) X is a full Hilbert A-module.
(2) ρ is a non-degenerate $*$-representation of A on X.
(3) R is a strongly continuous representation of G on X.
(4) S is a strongly continuous representation of $\widehat{G}$ on X.
(5) $S(\gamma)R(x) = \gamma(x) \cdot R(x)S(\gamma)$ for all $x \in G$ and $\gamma \in \widehat{G}$.
Hence, (R, S) satisfies the Weyl Commutation Relation for G on X.
(6) $R(x)\rho(a) = \rho(\alpha_x(a))R(x)$ for all $x \in G$ and $a \in A$.
Hence, (X, ρ, R) is a (G, A, α)-covariant module representation.
(7) $S(\gamma)\rho(a) = \rho(a)S(\gamma)$ for all $\gamma \in \widehat{G}$ and $a \in A$.
Hence, (X, ρ, S) is a $\left(\widehat{G}, A, \iota\right)$-covariant module representation, with ι denoting the trivial action of $\widehat{G}$ on A.

Definition 3. $\left(\mathsf{L}^2(G, A, \alpha), \mathsf{M}, \mathsf{U}, \mathsf{V}\right)$ is called the (G, A, α)-*Schrödinger module representation.*

Proposition 1. $\left(\mathsf{L}^2(G, A, \alpha), \mathsf{M}, \mathsf{U}, \mathsf{V}\right)$ *is a* (G, A, α)-*Heisenberg module representation.*

Proof. We will verify the various axioms in Definition 2.
The fullness of $\mathsf{L}^2(G, A, \alpha)$ *as a Hilbert* $A-$ *module*
By Green's Imprimitivity Theorem, $\mathsf{L}^2(G, A, \alpha)$ is a $(C^*(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha), A)$-imprimitivity bimodule, so it is a full Hilbert A-module.

For the verification of Axioms (2) through (4), we shall exploit a well-established fact that is often used to prove Green's Imprimitivity Theorem.
Fact Let $\phi \in \mathrm{C_c}(G, A)$, and let $(\phi_i)_{i\in I}$ be a net in $\mathrm{C_c}(G, A)$. Then $\lim_{i\in I} q(\phi_i) = q(\phi)$ in $\mathsf{L}^2(G, A, \alpha)$ whenever $\lim_{i\in I} \phi_i = \phi$ in the inductive limit topology on $\mathrm{C_c}(G, A)$, i.e., the following two conditions hold:

(I1) $(\phi_i)_{i\in I}$ converges uniformly to ϕ.
(I2) There exist a compact subset K of G and an index $i' \in I$ such that $\mathrm{Supp}(\phi_i) \subseteq K$ for all $i \in I_{\geq i'}$.

The non-degeneracy of M
Let $\phi \in \mathrm{C_c}(G, A)$. Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for A that is norm-bounded by 1. We claim that $\lim_{\lambda\in\Lambda} \left[\mathsf{M}(e_\lambda)\right](q(\phi)) = q(\phi)$ in $\mathsf{L}^2(G, A, \alpha)$, which is true once we can show that $\lim_{\lambda\in\Lambda} e_\lambda \centerdot \phi = \phi$ in the inductive limit topology on $\mathrm{C_c}(G, A)$.

Firstly, note that (I2) is satisfied as $\mathrm{Supp}(e_\lambda \centerdot \phi) \subseteq \mathrm{Supp}(\phi)$ for all $\lambda \in \Lambda$.

Next, let $\epsilon > 0$. By the continuity of ϕ, there exists a $\mathrm{Supp}(\phi)$-indexed family $(\mathcal{O}_x)_{x\in\mathrm{Supp}(\phi)}$ of open subsets of G such that for each $x \in \mathrm{Supp}(\phi)$, we have $x \in \mathcal{O}_x$ and $\|\phi(x) - \phi(y)\|_A < \dfrac{\epsilon}{3}$ for all $y \in \mathcal{O}_x$.

As $\mathrm{Supp}(\phi)$ is a compact subset of G, there exist $x_1, \ldots, x_n \in \mathrm{Supp}(\phi)$ such that $K = \bigcup_{i=1}^{n} \left(K \cap \mathcal{O}_{x_i}\right)$. Find corresponding $\lambda_1, \ldots, \lambda_n \in I$ such that for any $i \in [n]$, we have $\|\phi(x_i) - (e_\lambda \centerdot \phi)(x_i)\|_A < \dfrac{\epsilon}{3}$ for all $\lambda \in \Lambda_{\geq\lambda_i}$. Then as Λ is a directed set, there exists a $\lambda' \in \Lambda$ such that $\lambda_i \leq \lambda'$ for all $i \in [n]$.

Let $y \in G$. If $y \notin \mathrm{Supp}(\phi)$, then $\|(e_\lambda \centerdot \phi)(y) - \phi(y)\|_A = 0$ for all $\lambda \in \Lambda_{\geq\lambda'}$; if $y \in \mathrm{Supp}(\phi)$, then we can find an $i \in [n]$ so that $y \in \mathcal{O}_{x_i}$, in which case we have for all $\lambda \in \Lambda_{\geq\lambda'}$ that

$$\begin{aligned}\|(e_\lambda \centerdot \phi)(y) - \phi(y)\|_A &\leq \|(e_\lambda \centerdot \phi)(y) - (e_\lambda \centerdot \phi)(x_i)\|_A + \|(e_\lambda \centerdot \phi)(x_i) - \phi(x_i)\|_A \\ &\quad + \|\phi(x_i) - \phi(y)\|_A \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon,\end{aligned}$$

where we have used the fact that $\|e_\lambda\|_A \leq 1$ for all $\lambda \in \Lambda$. Hence, $\|e_\lambda \centerdot \phi - \phi\|_\infty \leq \epsilon$ for all $\lambda \in \Lambda_{\geq\lambda'}$, and as $\epsilon > 0$ is arbitrary, we find that $(e_\lambda \centerdot \phi)_{\lambda\in\Lambda}$ converges uniformly to ϕ, which means that (I1) is satisfied.

Finally, another $\dfrac{\epsilon}{3}$-argument yields $\lim\limits_{\lambda \in \Lambda} \big[\mathsf{M}(e_\lambda)\big](\Phi) = \Phi$ for all $\Phi \in \mathsf{L}^2(G, A, \alpha)$, so M is non-degenerate.

The strong continuity of U

Let $\phi \in C_c(G, A)$, and let $(x_i)_{i \in I}$ be a net in G that converges to e_G. We claim that $\lim\limits_{i \in I} \big[\mathsf{U}(x_i)\big](q(\phi)) = q(\phi)$ in $\mathsf{L}^2(G, A, \alpha)$, which is true once we can show that $\lim\limits_{i \in I} (\mathsf{lt} \otimes \alpha)_{x_i}(\phi) = \phi$ in the inductive limit topology on $C_c(G, A)$.

Firstly, note that (I1) is satisfied as $\mathsf{lt} \otimes \alpha$ is a strongly continuous action of G on $C_0(G, A)$, which means that $\big((\mathsf{lt} \otimes \alpha)_{x_i}(\phi)\big)_{i \in I}$ converges uniformly to ϕ.

Let K be a compact neighborhood of e_G. Then there exists an $i' \in I$ such that $x_i \in K$ for all $i \in I_{\geq i'}$, which means that for all such i,

$$\operatorname{Supp}\big((\mathsf{lt} \otimes \alpha)_{x_i}(\phi)\big) = \operatorname{Supp}\left(\left\{\begin{matrix} G \to & A \\ x \mapsto & \alpha_{x_i}\left(\phi\left(x_i^{-1} y\right)\right) \end{matrix}\right\}\right) = x_i \operatorname{Supp}(\phi) \subseteq K \operatorname{Supp}(\phi).$$

As $K \operatorname{Supp}(\phi)$ is a compact subset of G, we find that (I2) is satisfied.

An $\dfrac{\epsilon}{3}$-argument now shows that $\lim\limits_{i \in I} \big[\mathsf{U}(x_i)\big](\Phi) = \Phi$ for all $\Phi \in \mathsf{L}^2(G, A, \alpha)$, so U is strongly continuous.

The strong continuity of V

Let $\phi \in C_c(G, A)$, and let $(\gamma_i)_{i \in I}$ be a net in $\widehat{G}$ that converges to $e_{\widehat{G}}$. We claim that $\lim\limits_{i \in I} \big[\mathsf{V}(\gamma_i)\big](q(\phi)) = q(\phi)$ in $\mathsf{L}^2(G, A, \alpha)$, which is true once we show that $\lim\limits_{i \in I} \gamma_i \cdot \phi = \phi$ in the inductive limit topology on $C_c(G, A)$.

Firstly, note that (I2) is satisfied as $\operatorname{Supp}(\gamma_i \phi) = \operatorname{Supp}(\phi)$ for all $i \in I$.

As the topology on $\widehat{G}$ is the compact-open topology, $(\gamma_i)_{i \in I}$ converges uniformly to 1 on $\operatorname{Supp}(\phi)$, which is a compact subset of G, so $(\gamma_i \cdot \phi)_{i \in I}$ converges uniformly to ϕ on $\operatorname{Supp}(\phi)$ and thus on all of G. Hence, (I1) is satisfied.

An $\dfrac{\epsilon}{3}$-argument now shows that $\lim\limits_{i \in I} \big[\mathsf{V}(\gamma_i)\big](\Phi) = \Phi$ for all $\Phi \in \mathsf{L}^2(G, A, \alpha)$, so V is strongly continuous.

(U, V) *satisfies the Weyl Commutation Relation for* G *on* $\mathsf{L}^2(G, A, \alpha)$

Observe for all $x \in G$, $\gamma \in \widehat{G}$, and $\phi \in C_c(G, A)$ that

$$\begin{aligned}
\big[\mathsf{V}(\gamma)\mathsf{U}(x)\big](q(\phi)) &= \big[\mathsf{V}(\gamma)\big]\big(\big[\mathsf{U}(x)\big](q(\phi))\big) \\
&= \big[\mathsf{V}(\gamma)\big]\left(q\left(\left\{\begin{matrix} G \to & A \\ y \mapsto & \alpha_x\left(\phi\left(x^{-1} y\right)\right) \end{matrix}\right\}\right)\right) \\
&= q\left(\left\{\begin{matrix} G \to & A \\ y \mapsto & \gamma(y) \cdot \alpha_x\left(\phi\left(x^{-1} y\right)\right) \end{matrix}\right\}\right) \\
&= q\left(\left\{\begin{matrix} G \to & A \\ y \mapsto & \gamma(x)\gamma\left(x^{-1} y\right) \cdot \alpha_x\left(\phi\left(x^{-1} y\right)\right) \end{matrix}\right\}\right) \\
&= q\left(\left\{\begin{matrix} G \to & A \\ y \mapsto & \gamma(x) \cdot \alpha_x\left(\gamma\left(x^{-1} y\right) \cdot \phi\left(x^{-1} y\right)\right) \end{matrix}\right\}\right)
\end{aligned}$$

$$= \gamma(x) \cdot q\left(\left\{\begin{matrix} G \to & A \\ y \mapsto & \alpha_x\Big(\gamma\big(x^{-1}y\big) \cdot \phi\big(x^{-1}y\big)\Big)\end{matrix}\right\}\right)$$

$$= \gamma(x) \cdot q\left(\left\{\begin{matrix} G \to & A \\ y \mapsto & \alpha_x\Big((\gamma \cdot \phi)\big(x^{-1}y\big)\Big)\end{matrix}\right\}\right)$$

$$= \gamma(x) \cdot \big[\mathsf{U}(x)\big](q(\gamma \cdot \phi))$$

$$= \gamma(x) \cdot \big[\mathsf{U}(x)\big]\big(\big[\mathsf{V}(\gamma)\big](q(\phi))\big)$$

$$= \gamma(x) \cdot \big[\mathsf{U}(x)\mathsf{V}(\gamma)\big](q(\phi)),$$

so by continuity, $\mathsf{V}(\gamma)\mathsf{U}(x) = \gamma(x) \cdot \mathsf{U}(x)\mathsf{V}(\gamma)$ for all $x \in G$ and $\gamma \in \widehat{G}$.

$\big(\mathsf{L}^2(G, A, \alpha), \mathsf{M}, \mathsf{U}\big)$ *is a* (G, A, α)*-covariant module representation*

Observe for all $x \in G$, $a \in A$, and $\phi \in \mathrm{C_c}(G, A)$ that

$$\begin{aligned}
\big[\mathsf{U}(x)\mathsf{M}(a)\big](q(\phi)) &= \big[\mathsf{U}(x)\big]\big(\big[\mathsf{M}(a)\big](q(\phi))\big) \\
&= \big[\mathsf{U}(x)\big]\left(q\left(\left\{\begin{matrix} G \to & A \\ y \mapsto & a\phi(y)\end{matrix}\right\}\right)\right) \\
&= q\left(\left\{\begin{matrix} G \to & A \\ y \mapsto & \alpha_x\Big(a\phi\big(x^{-1}y\big)\Big)\end{matrix}\right\}\right) \\
&= q\left(\left\{\begin{matrix} G \to & A \\ y \mapsto & \alpha_x(a)\alpha_x\Big(\phi\big(x^{-1}y\big)\Big)\end{matrix}\right\}\right) \\
&= \big[\mathsf{M}(\alpha_x(a))\big]\left(q\left(\left\{\begin{matrix} G \to & A \\ y \mapsto & \alpha_x\Big(\phi\big(x^{-1}y\big)\Big)\end{matrix}\right\}\right)\right) \\
&= \big[\mathsf{M}(\alpha_x(a))\big]\big(\big[\mathsf{U}(x)\big](q(\phi))\big) \\
&= \big[\mathsf{M}(\alpha_x(a))\mathsf{U}(x)\big](q(\phi)),
\end{aligned}$$

so by continuity, $\mathsf{U}(x)\mathsf{M}(a) = \mathsf{M}(\alpha_x(a))\mathsf{U}(x)$ for all $x \in G$ and $a \in A$.

$\big(\mathsf{L}^2(G, A, \alpha), \mathsf{M}, \mathsf{V}\big)$ *is a* $\big(\widehat{G}, A, \iota\big)$*-covariant module representation*

Observe for all $\gamma \in \widehat{G}$, $a \in A$, and $\phi \in \mathrm{C_c}(G, A)$ that

$$\begin{aligned}
\big[\mathsf{V}(\gamma)\mathsf{M}(a)\big](q(\phi)) &= \big[\mathsf{V}(\gamma)\big]\big(\big[\mathsf{M}(a)\big](q(\phi))\big) \\
&= \big[\mathsf{V}(\gamma)\big]\left(q\left(\left\{\begin{matrix} G \to & A \\ y \mapsto & a\phi(y)\end{matrix}\right\}\right)\right) \\
&= q\left(\left\{\begin{matrix} G \to & A \\ y \mapsto & \gamma(y) \cdot a\phi(y)\end{matrix}\right\}\right) \\
&= q\left(\left\{\begin{matrix} G \to & A \\ y \mapsto & a\big[(\gamma \cdot \phi)(y)\big]\end{matrix}\right\}\right) \\
&= \big[\mathsf{M}(a)\big](q(\gamma \cdot \phi)) \\
&= \big[\mathsf{M}(a)\big]\big(\big[\mathsf{V}(\gamma)\big](q(\phi))\big) \\
&= \big[\mathsf{M}(a)\mathsf{V}(\gamma)\big](q(\phi)),
\end{aligned}$$

so by continuity, $\mathsf{V}(\gamma)\mathsf{M}(a) = \mathsf{M}(a)\mathsf{V}(\gamma)$ for all $\gamma \in \widehat{G}$ and $a \in A$. □

The ultimate goal of this section is to establish the following proposition, which we presently state in an imprecise form.

Proposition 2. *There is an injective map from the class of* (G, A, α)*-Heisenberg module representations to the class of* $(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha)$*-covariant module representations.*

The proposition is imprecisely stated because we have not yet specified what the injective map is, but this will be explicated in due course.

The main tool for proving the proposition is a C^*-algebra-valued version of the Fourier transform, which we will introduce soon. In order to show that this generalized Fourier transform is well-defined, the following approximation lemma is indispensable.

Lemma 1. *Let* X *be a locally compact Hausdorff space,* V *a normed vector space, and* D *a dense subset of* V*. Then for any* $f \in \mathrm{C_c}(X, V)$ *and* $\epsilon > 0$*, there exist* $\gamma_1, \ldots, \gamma_n \in \mathrm{C_c}(X)$ *and* $v_1, \ldots, v_n \in D$ *such that*

$$\forall x \in X : \qquad \left\| f(x) - \sum_{i=1}^{n} \gamma_i(x) \cdot v_i \right\|_V < \epsilon.$$

If λ *is a regular Borel measure on* X*, then for any* $f \in \mathrm{C_c}(X, V)$ *and* $\epsilon > 0$*, there exist* $\gamma_1, \ldots, \gamma_n \in \mathrm{C_c}(X)$ *and* $v_1, \ldots, v_n \in D$ *such that*

$$\int_X \left\| f(x) - \sum_{i=1}^{n} \gamma_i(x) \cdot v_i \right\|_V \mathrm{d}\lambda(x) < \epsilon.$$

This is a folklore result that can be straightforwardly proven using partitions of unity. To avoid disrupting the flow of this paper, we will provide a proof of it in the appendix.

Definition 4. The A-valued *generalized Fourier transform* for G is the map $\mathcal{F} : \mathrm{C_c}\big(\widehat{G}, A\big) \to \mathrm{C}_0(G, A)$ defined by

$$\forall f \in \mathrm{C_c}\big(\widehat{G}, A\big) : \qquad \mathcal{F}(f) \stackrel{\mathrm{df}}{=} \left\{ \begin{matrix} G \to & A \\ x \mapsto & \displaystyle\int_{\widehat{G}} \widehat{x}(\gamma) \cdot f(\gamma) \, \mathrm{d}\nu(\gamma) \end{matrix} \right\}.$$

We proceed to demonstrate the consistency of this definition.

When $A \neq \mathbb{C}$, it is not at all obvious why the image of $\mathcal{F}$ should be in $\mathrm{C}_0(G, A)$. To see this, let us pick $f \in \mathrm{C_c}\big(\widehat{G}, A\big)$. For every $x \in G$, the integrand of $\displaystyle\int_{\widehat{G}} \widehat{x}(\gamma) \cdot f(\gamma) \, \mathrm{d}\nu(\gamma)$ belongs to $\mathrm{C_c}\big(\widehat{G}, A\big)$, so the integral exists. Furthermore, for all $x \in G$,

$$\left\| \int_{\widehat{G}} \widehat{x}(\gamma) \cdot f(\gamma) \, \mathrm{d}\nu(\gamma) \right\|_A \leq \int_{\widehat{G}} |\widehat{x}(\gamma)| \|f(\gamma)\|_A \, \mathrm{d}\nu(\gamma) = \int_{\widehat{G}} \|f(\gamma)\|_A \, \mathrm{d}\nu(\gamma) = \|f\|_1,$$

so $\mathcal{F}(f)$ is a function from G to A that is pointwise-bounded by $\|f\|_1$. To check that it is also continuous, fix $x \in G$ and $\epsilon > 0$. As $\mathrm{Supp}(f)$ is a compact subset of $\widehat{G}$ with respect to the compact-open topology on $\mathrm{C}(G)$, the Arzelà-Ascoli Theorem says that $\mathrm{Supp}(f)$ is an equicontinuous subset of $\mathrm{C}(G)$, so there exists an open neighborhood U of x in G such that for all $y \in U$ and $\gamma \in \mathrm{Supp}(f)$,

$$|\gamma(x) - \gamma(y)| < \frac{\epsilon}{1 + \|f\|_1}.$$

Consequently, for all $y \in U$,

$$\begin{aligned}\|[\mathcal{F}(f)](x) - [\mathcal{F}(f)](y)\|_A &= \left\| \int_{\widehat{G}} \widehat{x}(\gamma) \cdot f(\gamma)\, \mathrm{d}\nu(\gamma) - \int_{\widehat{G}} \widehat{y}(\gamma) \cdot f(\gamma)\, \mathrm{d}\nu(\gamma) \right\|_A \\ &= \left\| \int_{\widehat{G}} \big[\gamma(x) - \gamma(y)\big] \cdot f(\gamma)\, \mathrm{d}\nu(\gamma) \right\|_A \\ &\leq \int_{\widehat{G}} |\gamma(x) - \gamma(y)| \|f(\gamma)\|_A \, \mathrm{d}\nu(\gamma) \\ &= \int_{\mathrm{Supp}(f)} |\gamma(x) - \gamma(y)| \|f(\gamma)\|_A \, \mathrm{d}\nu(\gamma) \\ &\leq \int_{\mathrm{Supp}(f)} \frac{\epsilon}{1 + \|f\|_1} \cdot \|f(\gamma)\|_A \, \mathrm{d}\nu(\gamma) \\ &= \frac{\epsilon}{1 + \|f\|_1} \cdot \|f\|_1 \\ &< \epsilon.\end{aligned}$$

As $x \in G$ is arbitrary, this proves that $\mathcal{F}(f)$ is continuous, so the image of $\mathcal{F}$ is contained in $\mathrm{C_b}(G, A)$.

Now, given an $f \in \mathrm{C_c}\big(\widehat{G}\big)$ and an $a \in A$, we have for all $x \in G$ that

$$\begin{aligned}[\mathcal{F}(f \diamond a)](x) &= \int_{\widehat{G}} \widehat{x}(\gamma) \cdot (f \diamond a)(\gamma)\, \mathrm{d}\nu(\gamma) \\ &= \int_{\widehat{G}} \widehat{x}(\gamma) \cdot \big[f(\gamma) \cdot a\big]\, \mathrm{d}\nu(\gamma) \\ &= \left[\int_{\widehat{G}} \widehat{x}(\gamma) f(\gamma)\, \mathrm{d}\nu(\gamma)\right] \cdot a \\ &= \widehat{f}(x) \cdot a. \qquad \big(\text{Here, } \widehat{f} \text{ denotes the Fourier transform of } f.\big)\end{aligned}$$

As we already know that $\widehat{f} \in \mathrm{C_0}(G)$, we get $\mathcal{F}(f \diamond a) = \widehat{f} \diamond a \in \mathrm{C_0}(G, A)$. Hence, as $f \in \mathrm{C_c}\big(\widehat{G}\big)$ and $a \in A$ are arbitrary, we obtain

$$\mathcal{F}\big[\mathrm{Span}\big(\mathrm{C_c}\big(\widehat{G}\big) \diamond A\big)\big] \subseteq \mathrm{Span}(\mathrm{C_0}(G) \diamond A) \subseteq \mathrm{C_0}(G, A).$$

Let $f \in \mathrm{C_c}\big(\widehat{G}, A\big)$. Lemma 1 makes it possible to obtain a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Span}\big(\mathrm{C_c}\big(\widehat{G}\big) \diamond A\big)$ such that $\lim_{n\to\infty} \|f - f_n\|_1 = 0$. Then because $\|\mathcal{F}(f - f_n)\|_\infty \leq \|f - f_n\|_1$ for all $n \in \mathbb{N}$, we get

$$\lim_{n\to\infty} \|\mathcal{F}(f) - \mathcal{F}(f_n)\|_\infty = \lim_{n\to\infty} \|\mathcal{F}(f - f_n)\|_\infty = 0.$$

However, as seen above, $\mathcal{F}(f_n) \in \mathrm{C_0}(G, A)$ for all $n \in \mathbb{N}$, so because $\mathrm{C_0}(G, A)$ is complete with respect to the supremum norm, it follows that $\mathcal{F}(f) \in \mathrm{C_0}(G, A)$. As $f \in \mathrm{C_c}\big(\widehat{G}, A\big)$ is arbitrary, we have proven that $\mathcal{F}$ maps $\mathrm{C_c}\big(\widehat{G}, A\big)$ to $\mathrm{C_0}(G, A)$.

Knowing now that $\mathcal{F} : \mathrm{C_c}\big(\widehat{G}, A\big) \to \mathrm{C_0}(G, A)$ is well-defined, our next step is to show the following.

Proposition 3. *$\mathcal{F}$ extends to a C^*-isomorphism $\overline{\mathcal{F}} : C^*\big(\widehat{G}, A, \iota\big) \to \mathrm{C_0}(G, A)$.*

Proof. It is routine to check that $\mathcal{F} : \left(\mathrm{C_c}\left(\widehat{G}, A\right), \star_{\widehat{G},A,\iota}, {}^{*_{\star_{\widehat{G},A,\iota}}} \right) \to \mathrm{C}_0(G, A)$ is a $*$-homomorphism. As we know that $\mathcal{F}$ is contractive with respect to $\|\cdot\|_{\nu,1}$, the theory of C^*-crossed products says that $\mathcal{F}$ extends to a C^*-homomorphism $\overline{\mathcal{F}} : C^*\left(\widehat{G}, A, \iota\right) \to \mathrm{C}_0(G, A)$.

By Lemma 2.73 of [22] and the theory of C^*-tensor products, we have the series of C^*-isomorphisms

$$C^*\left(\widehat{G}, A, \iota\right) \cong C^*\left(\widehat{G}\right) \otimes A \cong \mathrm{C}_0(G) \otimes A \cong \mathrm{C}_0(G, A),$$

which are implemented as follows: For all $f_1, \ldots, f_n \in \mathrm{C_c}\left(\widehat{G}\right)$ and $a_1, \ldots, a_n \in A$,

$$\eta_{(\widehat{G},A,\iota)}\left(\sum_{i=1}^{n} f_i \diamond a_i \right) \mapsto \sum_{i=1}^{n} \eta_{\widehat{G}}(f_i) \odot a_i \mapsto \sum_{i=1}^{n} \widehat{f_i} \odot a_i \mapsto \sum_{i=1}^{n} \widehat{f_i} \diamond a_i .$$

However, $\mathcal{F}\left(\sum_{i=1}^{n} f_i \diamond a_i \right) = \sum_{i=1}^{n} \widehat{f_i} \diamond a_i$, so $\overline{\mathcal{F}}$ agrees with some C^*-isomorphism from $C^*\left(\widehat{G}, A, \iota\right)$ to $\mathrm{C}_0(G, A)$ on a dense subset. It is therefore precisely that C^*-isomorphism. □

Definition 5. For a $\left(\widehat{G}, A, \iota\right)$-covariant module representation (X, ρ, S), let

$$\pi^{\rho,S} \overset{\text{df}}{=} \overline{\rho \rtimes S} \circ \overline{\mathcal{F}}^{-1} : \mathrm{C}_0(G, A) \to \mathbb{L}(\mathsf{X}),$$

which is a non-degenerate $*$-representation of $\mathrm{C}_0(G, A)$ on X.

Finally, let us tackle the main objective of this section.

Proof of Proposition 2. We divide the proof into two parts.

Defining the desired class map

Observe for all $x, y \in G$ and $f \in \mathrm{C_c}\left(\widehat{G}, A\right)$ that

$$\begin{aligned}
\left[(\mathsf{lt} \otimes \alpha)_x(\mathcal{F}(f))\right](y) &= \alpha_x\left([\mathcal{F}(f)]\left(x^{-1}y\right) \right) \\
&= \alpha_x\left(\int_{\widehat{G}} \widehat{\left(x^{-1}y\right)}(\gamma) \cdot f(\gamma) \, \mathrm{d}\nu(\gamma) \right) \\
&= \int_{\widehat{G}} \widehat{\left(x^{-1}y\right)}(\gamma) \cdot \alpha_x(f(\gamma)) \, \mathrm{d}\nu(\gamma) \\
&= \int_{\widehat{G}} \widehat{y}(\gamma)\widehat{x^{-1}}(\gamma) \cdot \alpha_x(f(\gamma)) \, \mathrm{d}\nu(\gamma) \\
&= \int_{\widehat{G}} \widehat{y}(\gamma) \cdot \left[\widehat{x^{-1}} \cdot (\alpha_x \circ f) \right](\gamma) \, \mathrm{d}\nu(\gamma) \\
&= \left[\mathcal{F}\left(\widehat{x^{-1}} \cdot (\alpha_x \circ f) \right) \right](y).
\end{aligned}$$

Hence, $(\mathsf{lt} \otimes \alpha)_x(\mathcal{F}(f)) = \mathcal{F}\left(\widehat{x^{-1}} \cdot (\alpha_x \circ f) \right)$ for all $x \in G$ and $f \in \mathrm{C_c}\left(\widehat{G}, A\right)$.

Given a (G, A, α)-Heisenberg module representation (X, ρ, R, S), we will exploit the computation above to show that $\left(\mathsf{X}, \pi^{\rho,S}, R\right)$ is a $(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha)$-covariant module representation.

Firstly, ρ is a non-degenerate $*$-representation of A on X, so $\rho \rtimes S$ is a non-degenerate $*$-representation of $C^*\big(\widehat{G}, A, \iota\big)$ on X, which, in turn, means that $\pi^{\rho,S}$ is a non-degenerate $*$-representation of $\mathrm{C}_0(G, A)$ on X. Secondly, we have for all $x \in G$ and $f \in \mathrm{C_c}\big(\widehat{G}, A\big)$ that

$$\begin{aligned}
R(x)\pi^{\rho,S}(\mathcal{F}(f)) &= R(x)\left[\int_{\widehat{G}} \rho(f(\gamma))S(\gamma)\, \mathrm{d}\nu(\gamma)\right] \\
&= \int_{\widehat{G}} R(x)\rho(f(\gamma))S(\gamma)\, \mathrm{d}\nu(\gamma) \\
&= \int_{\widehat{G}} \rho(\alpha_x(f(\gamma)))R(x)S(\gamma)\, \mathrm{d}\nu(\gamma) \\
&= \int_{\widehat{G}} \gamma\left(x^{-1}\right) \cdot \rho(\alpha_x(f(\gamma)))S(\gamma)R(x)\, \mathrm{d}\nu(\gamma) \\
&= \left[\int_{\widehat{G}} \gamma\left(x^{-1}\right) \cdot \rho(\alpha_x(f(\gamma)))S(\gamma)\, \mathrm{d}\nu(\gamma)\right]R(x) \\
&= \left[\int_{\widehat{G}} \rho\left(\gamma\left(x^{-1}\right) \cdot \alpha_x(f(\gamma))\right)S(\gamma)\, \mathrm{d}\nu(\gamma)\right]R(x) \\
&= \left[\int_{\widehat{G}} \rho\left(\widehat{x^{-1}}(\gamma) \cdot (\alpha_x \circ f)(\gamma)\right)S(\gamma)\, \mathrm{d}\nu(\gamma)\right]R(x) \\
&= \left[\int_{\widehat{G}} \rho\left(\left[\widehat{x^{-1}} \cdot (\alpha_x \circ f)\right](\gamma)\right)S(\gamma)\, \mathrm{d}\nu(\gamma)\right]R(x) \\
&= \pi^{\rho,S}\left(\mathcal{F}\left(\widehat{x^{-1}} \cdot (\alpha_x \circ f)\right)\right)R(x) \\
&= \pi^{\rho,S}\big((\mathsf{lt} \otimes \alpha)_x(\mathcal{F}(f))\big)R(x).
\end{aligned}$$

As the image of $\mathcal{F}$ is dense in $\mathrm{C}_0(G, A)$, it follows from continuity that for all $x \in G$ and $f \in \mathrm{C}_0(G, A)$,

$$R(x)\pi^{\rho,S}(f) = \pi^{\rho,S}\big((\mathsf{lt} \otimes \alpha)_x(f)\big)R(x).$$

Hence, $\big(\mathsf{X}, \pi^{\rho,S}, R\big)$ is a $(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha)$-covariant module representation.

We can therefore define a map from the class of (G, A, α)-Heisenberg module representations to the class of $(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha)$-covariant module representation according to the rule

$$(\mathsf{X}, \rho, R, S) \mapsto \left(\mathsf{X}, \pi^{\rho,S}, R\right).$$

Injectivity of the class map

Let $(\mathsf{X}_1, \rho_1, R_1, S_1)$ and $(\mathsf{X}_2, \rho_2, R_2, S_2)$ be (G, A, α)-Heisenberg module representations such that

$$\left(\mathsf{X}_1, \pi^{\rho_1,S_1}, R_1\right) = \left(\mathsf{X}_2, \pi^{\rho_2,S_2}, R_2\right).$$

Clearly, $\mathsf{X}_1 = \mathsf{X}_2$, $R_1 = R_2$, and $\pi^{\rho_1,S_1} = \pi^{\rho_2,S_2}$. Hence,

$$\overline{\rho_1 \rtimes S_1} = \pi^{\rho_1,S_1} \circ \overline{\mathcal{F}} = \pi^{\rho_2,S_2} \circ \overline{\mathcal{F}} = \overline{\rho_2 \rtimes S_2},$$

which yields $\rho_1 \rtimes S_1 = \rho_2 \rtimes S_2$. By Proposition 2.39 of [22], $\rho_1 = \rho_2$ and $S_1 = S_2$. Therefore, the proposed class map is indeed injective. □

Actually, one can show that the image of the class map above is the class of $(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha)$-covariant module representations whose underlying Hilbert C^*-module is a full Hilbert A-module. However, we will have no need of this fact.

4. Hilbert $\mathbb{K}(\mathcal{H})$-Modules

In this section, we take a brief excursion into Hilbert $\mathbb{K}(\mathcal{H})$-modules. The initial material can be found in [2,3], but we have decided to supply our own proofs, some of which are simpler than the original ones.

Throughout this section, we shall fix a non-trivial Hilbert space $\mathcal{H}$.

Lemma 2. *Let P be a rank-one projection on $\mathcal{H}$. Then there exists a unique positive linear functional f on $\mathbb{K}(\mathcal{H})$ such that $PSP = f(S) \cdot P$ for all $S \in \mathbb{K}(\mathcal{H})$.*

Proof. This lemma is a simple exercise left to the reader. □

Let $\mathcal{P}_1$ denote the set of all rank-one projections on $\mathcal{H}$. Lemma 2 says that there exists a $\mathcal{P}_1$-indexed family $(f_P)_{P \in \mathcal{P}_1}$ of linear functionals on $\mathcal{H}$ such that

$$\forall P \in \mathcal{P}_1, \ \forall S \in \mathbb{K}(\mathcal{H}) : \qquad PSP = f_P(S) \cdot P.$$

These linear functionals play a pivotal role in the next result.

Theorem 3. *Let X be a non-trivial Hilbert $\mathbb{K}(\mathcal{H})$-module and P a rank-one projection on $\mathcal{H}$. Then $\mathsf{X} \bullet P$ is a non-trivial closed subspace of $\mathsf{X} \bullet P$ that has the structure of a Hilbert space, whose inner product $\langle \cdot | \cdot \rangle_{\mathsf{X} \bullet P}$ is given by*

$$\forall \zeta, \eta \in \mathsf{X} \bullet P : \qquad \langle \zeta | \eta \rangle_{\mathsf{X} \bullet P} \stackrel{df}{=} f_P\big(\langle \zeta | \eta \rangle_{\mathsf{X}}\big).$$

Furthermore, the norm on $\mathsf{X} \bullet P$ induced by $\langle \cdot | \cdot \rangle_{\mathsf{X} \bullet P}$ coincides with the restriction of $\|\cdot\|_{\mathsf{X}}$ to $\mathsf{X} \bullet P$.

Proof. It is clear that $\mathsf{X} \bullet P$ is a subspace of X. As X is non-trivial, $\overline{\mathrm{Span}(\langle \mathsf{X} | \mathsf{X} \rangle_{\mathsf{X}})}^{\mathbb{K}(\mathcal{H})}$ is a non-trivial ideal of $\mathbb{K}(\mathcal{H})$, but as $\mathbb{K}(\mathcal{H})$ is a simple C^*-algebra, we have $\overline{\mathrm{Span}(\langle \mathsf{X} | \mathsf{X} \rangle_{\mathsf{X}})}^{\mathbb{K}(\mathcal{H})} = \mathbb{K}(\mathcal{H})$. Hence,

$$\begin{aligned} P &\in P\mathbb{K}(\mathcal{H})P \\ &= P\overline{\mathrm{Span}(\langle \mathsf{X} | \mathsf{X} \rangle_{\mathsf{X}})}^{\mathbb{K}(\mathcal{H})} P \\ &\subseteq \overline{P \ \mathrm{Span}(\langle \mathsf{X} | \mathsf{X} \rangle_{\mathsf{X}}) P}^{\mathbb{K}(\mathcal{H})} \\ &= \overline{\mathrm{Span}(\langle \mathsf{X} \bullet P | \mathsf{X} \bullet P \rangle_{\mathsf{X}})}^{\mathbb{K}(\mathcal{H})}, \end{aligned}$$

which implies that $\mathsf{X} \bullet P$ is a non-trivial subspace of X.

To see that $\mathsf{X} \bullet P$ is a closed subspace of X, suppose that $(\zeta_n)_{n \in \mathbb{N}}$ is a sequence in $\mathsf{X} \bullet P$ that converges to some $\eta \in \mathsf{X}$. Then because $\zeta_n \bullet P = \zeta_n$ for all $n \in \mathbb{N}$, we have

$$\eta = \lim_{n \to \infty} \zeta_n = \lim_{n \to \infty} \zeta_n \bullet P = \left(\lim_{n \to \infty} \zeta_n \right) \bullet P = \eta \bullet P.$$

Hence, $\eta \in \mathsf{X} \bullet P$, which proves that $\mathsf{X} \bullet P$ is a non-trivial closed subspace of X.

Clearly, $\langle\cdot|\cdot\rangle_{\mathsf{X}\bullet P}$ is a sesquilinear form on $\mathsf{X}\bullet P$, so it remains to see that it is positive definite and complete. Let $\zeta\in\mathsf{X}\bullet P$. Then $\langle\zeta|\zeta\rangle_{\mathsf{X}}$ is positive in $\mathbb{K}(\mathcal{H})$, so by Lemma 2, $\langle\zeta|\zeta\rangle_{\mathsf{X}\bullet P}=f_P(\langle\zeta|\zeta\rangle_{\mathsf{X}})\geq 0$. Furthermore,

$$\begin{aligned}\langle\zeta|\zeta\rangle_{\mathsf{X}\bullet P}=f_P(\langle\zeta|\zeta\rangle_{\mathsf{X}})=0 &\iff \langle\zeta|\zeta\rangle_{\mathsf{X}}=P\langle\zeta|\zeta\rangle_{\mathsf{X}}P=f_P(\langle\zeta|\zeta\rangle_{\mathsf{X}})\cdot P=0_{\mathbb{K}(\mathcal{H})}\\ &\iff \zeta=0_{\mathsf{X}}.\end{aligned}$$

A short computation now shows that $\|\zeta\|_{\mathsf{X}\bullet P}=\|\zeta\|_{\mathsf{X}}$ for all $\zeta\in\mathsf{X}\bullet P$. As $\mathsf{X}\bullet P$ is a closed subspace of X, it is a Banach space with respect to the restriction of $\|\cdot\|_{\mathsf{X}}$ to $\mathsf{X}\bullet P$ and is thus a Banach space with respect to $\|\cdot\|_{\mathsf{X}\bullet P}$. Therefore, $\mathsf{X}\bullet P$ equipped with $\langle\cdot|\cdot\rangle_{\mathsf{X}\bullet P}$ is a Hilbert space, and the Hilbert-space norm on $\mathsf{X}\bullet P$ is precisely the restriction of $\|\cdot\|_{\mathsf{X}}$ to $\mathsf{X}\bullet P$. □

Theorem 4. *Let X be a Hilbert $\mathbb{K}(\mathcal{H})$-module, Y a $\mathbb{K}(\mathcal{H})$-submodule of X that is not necessarily closed, and P a rank-one projection on $\mathcal{H}$. Then the closed $\mathbb{K}(\mathcal{H})$-linear span of $\mathsf{Y}\bullet P$ in X is the closure $\overline{\mathsf{Y}}^{\mathsf{X}}$ of Y in X.*

Proof. As Hilbert C^*-modules are automatically non-degenerate, we have $\overline{\mathsf{Y}}^{\mathsf{X}}=\overline{\mathrm{Span}(\mathsf{Y}\bullet\mathbb{K}(\mathcal{H}))}^{\mathsf{X}}$. As $\mathbb{K}(\mathcal{H})$ is simple, the closure of the two-sided ideal $I\overset{\mathrm{df}}{=}\mathrm{Span}(\mathbb{K}(\mathcal{H})P\mathbb{K}(\mathcal{H}))$ of $\mathbb{K}(\mathcal{H})$ is all of $\mathbb{K}(\mathcal{H})$. Hence,

$$\overline{\mathrm{Span}(\mathsf{Y}\bullet\mathbb{K}(\mathcal{H}))}^{\mathsf{X}}=\overline{\mathrm{Span}(\mathsf{Y}\bullet I)}^{\mathsf{X}}.$$

Given $\zeta_1,\ldots,\zeta_n\in\mathsf{X}$ and $S_1,\ldots,S_n,T_1,\ldots,T_n\in\mathbb{K}(\mathcal{H})$, observe that

$$\sum_{i=1}^{n}\zeta_i\bullet(S_iPT_i)=\sum_{i=1}^{n}((\zeta_i\bullet S_i)\bullet P)\bullet T_i,$$

so

$$\overline{\mathsf{Y}}^{\mathsf{X}}=\overline{\mathrm{Span}(\mathsf{Y}\bullet I)}^{\mathsf{X}}=\overline{\mathrm{Span}(((\mathsf{Y}\bullet\mathbb{K}(\mathcal{H}))\bullet P)\bullet\mathbb{K}(\mathcal{H}))}^{\mathsf{X}}=\overline{\mathrm{Span}((\mathsf{Y}\bullet P)\bullet\mathbb{K}(\mathcal{H}))}^{\mathsf{X}},$$

which proves that $\overline{\mathsf{Y}}^{\mathsf{X}}$ is the closure of the $\mathbb{K}(\mathcal{H})$-linear span of $\mathsf{Y}\bullet P$ in X. □

The next theorem is the main result of [3], and it explains why Hilbert $\mathbb{K}(\mathcal{H})$-modules behave like Hilbert spaces. It says that the C^*-algebra of adjointable operators on a Hilbert $\mathbb{K}(\mathcal{H})$-module X is isomorphic to the C^*-algebra of bounded operators on the Hilbert space $\mathsf{X}\bullet P$, for any rank-one projection P on $\mathcal{H}$. At first sight, this seems rather astonishing because $\mathsf{X}\bullet P$ is generally a much smaller space that X itself, and it would be hard to imagine why it should have much to say about X. However, having seen in Theorem 4 that $\mathsf{X}\bullet P$ generates a dense submodule of X, one can start to understand why the theorem holds.

The proof given in [3] relies on concepts from an earlier paper [2], but the proof that we give here is very direct and only depends on the previous definitions and results of this section.

Theorem 5. *Let X be a Hilbert $\mathbb{K}(\mathcal{H})$-module and P a rank-one projection on $\mathcal{H}$. Then $\mathsf{X}\bullet P$ is an invariant subspace for each $T\in\mathbb{L}(\mathsf{X})$, and the map*

$$\left\{\begin{matrix}\mathbb{L}(\mathsf{X}) & \to & \mathbb{B}(\mathsf{X}\bullet P)\\ T & \mapsto & T|_{\mathsf{X}\bullet P}\end{matrix}\right\}$$

is a C^*-isomorphism, where $\mathsf{X}\bullet P$ is viewed as a Hilbert space. Furthermore, the restriction of this map to $\mathbb{K}(\mathsf{X})$ yields a C^*-isomorphism from $\mathbb{K}(\mathsf{X})$ to $\mathbb{K}(\mathsf{X}\bullet P)$.

Proof. If X is trivial, then the theorem is clearly true, so let us assume henceforth that X is non-trivial.

For each $T \in \mathbb{L}(\mathsf{X})$, the $\mathbb{K}(\mathcal{H})$-linearity of T implies that $\mathsf{X} \bullet P$ is an invariant subspace of T.

It is easy to check that $\left\{ \begin{matrix} \mathbb{L}(\mathsf{X}) & \to & \mathbb{B}(\mathsf{X} \bullet P) \\ T & \mapsto & T|_{\mathsf{X} \bullet P} \end{matrix} \right\}$ is at least a C^*-homomorphism. To see that it is injective, let $S, T \in \mathbb{L}(\mathsf{X})$ satisfy $S|_{\mathsf{X} \bullet P} = T|_{\mathsf{X} \bullet P}$. Then by the $\mathbb{K}(\mathcal{H})$-linearity and continuity of both S and T, they must agree on the closed $\mathbb{K}(\mathcal{H})$-linear span of $\mathsf{X} \bullet P$, which is equal to X by Theorem 4. Hence, $S = T$.

Surjectivity is trickier to prove. Let $L \in \mathbb{B}(\mathsf{X} \bullet P)$, and let $(\varepsilon_i)_{i \in I}$ be an orthonormal basis of $\mathsf{X} \bullet P$. For each $J \in \operatorname{Fin}(I)$, let $T_J \stackrel{\text{df}}{=} \sum_{i \in J} \Theta_{L(\varepsilon_i), \varepsilon_i} \in \mathbb{K}(\mathsf{X})$ and $L_J \stackrel{\text{df}}{=} \sum_{i \in J} |L(\varepsilon_i)\rangle\langle\varepsilon_i| \in \mathbb{K}(\mathsf{X} \bullet P)$; then for all $\zeta \in \mathsf{X} \bullet P$,

$$\begin{aligned} T_J(\zeta) &= \sum_{i \in J} \Theta_{L(\varepsilon_i), \varepsilon_i}(\zeta) \\ &= \sum_{i \in J} L(\varepsilon_i) \bullet \langle \varepsilon_i | \zeta \rangle_{\mathsf{X}} \\ &= \sum_{i \in J} L(\varepsilon_i) \bullet (P \langle \varepsilon_i | \zeta \rangle_{\mathsf{X}} P) \\ &= \sum_{i \in J} L(\varepsilon_i) \bullet (\langle \varepsilon_i | \zeta \rangle_{\mathsf{X} \bullet P} \cdot P) \\ &= \sum_{i \in J} \langle \varepsilon_i | \zeta \rangle_{\mathsf{X} \bullet P} \cdot L(\varepsilon_i) \\ &= \sum_{i \in J} |L(\varepsilon_i)\rangle\langle\varepsilon_i|(\zeta) \\ &= L_J(\zeta), \end{aligned}$$

which implies that $T_J|_{\mathsf{X} \bullet P} = L_J$. Next, for all $\zeta \in \mathsf{X} \bullet P$,

$$L(\zeta) = L\left(\sum_{i \in I} \langle \varepsilon_i | \zeta \rangle_{\mathsf{X} \bullet P} \cdot \varepsilon_i \right) = \sum_{i \in I} \langle \varepsilon_i | \zeta \rangle_{\mathsf{X} \bullet P} \cdot L(\varepsilon_i) = \sum_{i \in I} |L(\varepsilon_i)\rangle\langle\varepsilon_i|(\zeta),$$

so $(L_J)_{J \in \operatorname{Fin}(I)}$ is a net (partially ordered by $\subseteq$) that strongly converges to L. Also, for all $J \in \operatorname{Fin}(I)$,

$$\begin{aligned} \|L_J\|_{\mathbb{B}(\mathsf{X} \bullet P)} &= \sup\left(\left\{ \|L_J(\zeta)\|_{\mathsf{X} \bullet P} \;\middle|\; \zeta \in \mathsf{X} \bullet P \text{ and } \|\zeta\|_{\mathsf{X} \bullet P} \leq 1 \right\} \right) \\ &= \sup\left(\left\{ \|L_J(\zeta)\|_{\mathsf{X} \bullet P} \;\middle|\; \zeta \in \operatorname{Span}\left(\{\varepsilon_i\}_{i \in J}\right) \text{ and } \|\zeta\|_{\mathsf{X} \bullet P} \leq 1 \right\} \right) \\ &= \sup\left(\left\{ \|L(\zeta)\|_{\mathsf{X} \bullet P} \;\middle|\; \zeta \in \operatorname{Span}\left(\{\varepsilon_i\}_{i \in J}\right) \text{ and } \|\zeta\|_{\mathsf{X} \bullet P} \leq 1 \right\} \right) \\ &\leq \sup\left(\left\{ \|L(\zeta)\|_{\mathsf{X} \bullet P} \;\middle|\; \zeta \in \mathsf{X} \bullet P \text{ and } \|\zeta\|_{\mathsf{X} \bullet P} \leq 1 \right\} \right) \\ &= \|L\|_{\mathbb{B}(\mathsf{X} \bullet P)}, \end{aligned}$$

which gives, by the first part, $\|T_J\|_{\mathbb{L}(\mathsf{X})} = \|L_J\|_{\mathbb{B}(\mathsf{X} \bullet P)} \leq \|L\|_{\mathbb{B}(\mathsf{X} \bullet P)}$. Notice now that $(T_J)_{J \in \operatorname{Fin}(I)}$ is a norm-bounded net in $\mathbb{L}(\mathsf{X})$ that strongly converges on the $\mathbb{K}(\mathcal{H})$-linear span of $\mathsf{X} \bullet P$, which is dense in X. By an $\dfrac{\epsilon}{3}$-argument, $(T_J)_{J \in \mathrm{Fin}(I)}$ strongly converges everywhere to some $T \in \mathbb{B}(X)$. Likewise, $\left(T_J^*\right)_{J \in \mathrm{Fin}(I)}$ strongly converges everywhere to some $S \in \mathbb{B}(\mathsf{X})$. As $\langle T_J(\zeta) | \eta \rangle_{\mathsf{X}} = \left\langle \zeta \middle| T_J^*(\eta) \right\rangle_{\mathsf{X}}$ for all $\zeta, \eta \in \mathsf{X}$ and $J \in \mathrm{Fin}(I)$, taking limits yields $\langle T(\zeta) | \eta \rangle_{\mathsf{X}} = \langle \zeta | S(\eta) \rangle_{\mathsf{X}}$. Therefore, $T \in \mathbb{L}(\mathsf{X})$ and $T|_{\mathsf{X} \bullet P} = L$, which finishes our proof that the restriction map is a C^*-isomorphism from $\mathbb{L}(\mathsf{X})$ to $\mathbb{B}(\mathsf{X} \bullet P)$.

For the final part of the proof, observe for all $\zeta, \eta \in \mathsf{X} \bullet P$, $S, T \in \mathbb{K}(\mathcal{H})$, and $\xi \in \mathsf{X} \bullet P$ that

$$\begin{aligned}
\Theta_{\zeta \bullet S, \eta \bullet T}(\xi) &= (\zeta \bullet S) \bullet \langle \eta \bullet T | \xi \rangle_{\mathsf{X}} \\
&= (\zeta \bullet S) \bullet \langle \eta \bullet PT | \xi \rangle_{\mathsf{X}} \\
&= (\zeta \bullet S) \bullet \left(T^* P \langle \eta | \xi \rangle_{\mathsf{X}}\right) \\
&= \left(\zeta \bullet ST^* P\right) \bullet \langle \eta | \xi \rangle_{\mathsf{X}} \\
&= \left(\zeta \bullet ST^* P\right) \bullet \left(P \langle \eta | \xi \rangle_{\mathsf{X}} P\right) \\
&= \left(\zeta \bullet ST^* P\right) \bullet \left(\langle \eta | \xi \rangle_{\mathsf{X} \bullet P} \cdot P\right) \\
&= \langle \eta | \xi \rangle_{\mathsf{X} \bullet P} \cdot \left(\zeta \bullet ST^* P\right) \\
&= \left|\zeta \bullet ST^* P\right\rangle \langle \eta | (\xi),
\end{aligned}$$

which means that $\Theta_{\zeta \bullet S, \eta \bullet T}|_{\mathsf{X} \bullet P} \in \mathbb{K}(\mathsf{X} \bullet P)$. Let $\zeta, \eta \in \mathsf{X}$. Then by Theorem 4, we can find

$$\zeta_1, \ldots, \zeta_m, \eta_1, \ldots, \eta_n \in \mathsf{X} \bullet P \quad \text{and} \quad S_1, \ldots, S_m, T_1, \ldots, T_n \in \mathbb{K}(\mathcal{H})$$

so that $\sum_{i=1}^{m} \zeta_i \bullet S_i$ and $\sum_{j=1}^{n} \eta_j \bullet T_j$ are arbitrarily close to ζ and η, respectively, which ensures that

$$\Theta_{\sum_{i=1}^m \zeta_i \bullet S_i, \sum_{j=1}^n \eta_j \bullet T_j} = \sum_{i=1}^{m} \sum_{j=1}^{n} \Theta_{\zeta_i \bullet S_i, \eta_j \bullet T_j}$$

is arbitrarily close to $\Theta_{\zeta,\eta}$ in $\mathbb{K}(\mathsf{X})$. Hence, by the continuity of the restriction C^*-isomorphism,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \Theta_{\zeta_i \bullet S_i, \eta_j \bullet T_j}|_{\mathsf{X} \bullet P} = \sum_{i=1}^{m} \sum_{j=1}^{n} \left|\zeta_i \bullet S_i T_j^* P\right\rangle \langle \eta_j|$$

is arbitrarily close to $\Theta_{\zeta,\eta}|_{\mathsf{X} \bullet P}$, which says that $\Theta_{\zeta,\eta}|_{\mathsf{X} \bullet P} \in \mathbb{K}(\mathsf{X} \bullet P)$. Therefore, the image of $\mathbb{K}(\mathsf{X})$ under the restriction C^*-isomorphism is a non-trivial ideal in $\mathbb{K}(\mathsf{X} \bullet P)$. As $\mathbb{K}(\mathsf{X} \bullet P)$ is a simple C^*-algebra, this image is precisely $\mathbb{K}(\mathsf{X} \bullet P)$. The proof is now complete. □

Using Theorem 5, we can show that every closed submodule of a Hilbert $\mathbb{K}(\mathcal{H})$-module has an orthogonal complement. The complementability of Hilbert $\mathbb{K}(\mathcal{H})$-modules has been known for a long while [11], but Theorem 5 appears to provide an expedient proof.

Theorem 6. *Let* Y *be a closed submodule of a Hilbert* $\mathbb{K}(\mathcal{H})$*-module* X*. Then* $\mathsf{X} = \mathsf{Y} \oplus \mathsf{Y}^{\perp}$.

Proof. Let P be a rank-one projection on $\mathcal{H}$. Then $\mathsf{Y} \bullet P$ is a closed subspace of $\mathsf{X} \bullet P$. By Theorem 5, there is a projection $Q \in \mathbb{L}(\mathsf{X})$ such that $Q|_{\mathsf{X} \bullet P} = \mathrm{Proj}_{\mathsf{X} \bullet P, \mathsf{Y} \bullet P}$. If we can show that $\mathrm{Range}(Q) = \mathsf{Y}$, then we are done, for a closed submodule of a Hilbert C^*-module is complementable if it is the range of an adjointable operator (Corollary 15.3.9 of [21]). Indeed, as Q is a projection, $\mathrm{Range}(Q)$ is a closed submodule of X, and

$$\begin{aligned}\mathrm{Range}(Q) \bullet P &= \{Q(\zeta) \bullet P \mid \zeta \in \mathsf{X}\} = \{Q(\zeta \bullet P) \mid \zeta \in \mathsf{X}\} = \mathrm{Range}(Q|_{\mathsf{X} \bullet P}) \\ &= \mathrm{Range}\big(\mathrm{Proj}_{\mathsf{X} \bullet P, \mathsf{Y} \bullet P}\big) = \mathsf{Y} \bullet P,\end{aligned}$$

so $\mathrm{Range}(Q) = \mathsf{Y}$ by Theorem 4. □

Lemma 3. *Let* X *be a non-trivial Hilbert* $\mathbb{K}(\mathcal{H})$*-module and* P *a rank-one projection on* $\mathcal{H}$*. Then there is a* $\zeta \in \mathsf{X}$ *such that* $\langle \zeta | \zeta \rangle_{\mathsf{X}} = P$*.*

Proof. By Theorem 3, we can find a non-zero $\eta \in \mathsf{X} \bullet P$, so $\langle \eta | \eta \rangle_{\mathsf{X} \bullet P} > 0$. As

$$\langle \eta | \eta \rangle_{\mathsf{X}} = \langle \eta \bullet P | \eta \bullet P \rangle_{\mathsf{X}} = P \langle \eta | \eta \rangle_{\mathsf{X}} P = \langle \eta | \eta \rangle_{\mathsf{X} \bullet P} \cdot P,$$

we see that $\zeta \overset{\mathrm{df}}{=} \dfrac{1}{\sqrt{\langle \eta | \eta \rangle_{\mathsf{X} \bullet P}}} \cdot \eta$ satisfies $\langle \zeta | \zeta \rangle_{\mathsf{X}} = P$. □

Definition 6. Let A be a non-trivial C^*-algebra and X a non-trivial Hilbert A-module. We say that $\mathbb{K}(\mathsf{X})$ *acts irreducibly* on X if and only if the only closed $\mathbb{K}(\mathsf{X})$-invariant A-submodules of X are $\{0_{\mathsf{X}}\}$ and X.

Proposition 4. *Let* X *be a non-trivial Hilbert* $\mathbb{K}(\mathcal{H})$*-module. Then* $\mathbb{K}(\mathsf{X})$ *acts irreducibly on* X*.*

Proof. As X is a full Hilbert $\mathbb{K}(\mathcal{H})$-module by the simplicity of $\mathbb{K}(\mathcal{H})$, Proposition 3.8 of [15] implies that X is a $(\mathbb{K}(\mathsf{X}), \mathbb{K}(\mathcal{H}))$-imprimitivity bimodule. By the Rieffel Correspondence Theorem (Theorem 3.22 of [15]), there exists a lattice isomorphism between

- The lattice of closed two-sided ideals of $\mathbb{K}(\mathcal{H})$, which includes $\{0_{\mathbb{K}(\mathcal{H})}\}$ and $\mathbb{K}(\mathcal{H})$, and
- The lattice of closed $\mathbb{K}(\mathsf{X})$-invariant $\mathbb{K}(\mathcal{H})$-submodules of X, which includes $\{0_{\mathsf{X}}\}$ and X.

As $\mathbb{K}(\mathcal{H})$ is simple, the only closed $\mathbb{K}(\mathsf{X})$-invariant $\mathbb{K}(\mathcal{H})$-submodules of X are therefore $\{0_{\mathsf{X}}\}$ and X. □

The next result appears to be new, and it serves as the main bridge between the theory of Hilbert $\mathbb{K}(\mathcal{H})$-modules and the main result of this paper, which will be presented in the next section. The proof that we offer here is an adaptation of Arveson's proof of Theorem 1.4.4 of [1].

Proposition 5. *Let* X *and* Y *be non-trivial Hilbert* $\mathbb{K}(\mathcal{H})$*-modules. If* Φ *is a non-degenerate* $*$*-representation of* $\mathbb{K}(\mathsf{X})$ *on* Y*, then* (Y, Φ) *is unitarily equivalent to a direct sum of copies of* $\big(\mathsf{X}, i_{\mathbb{K}(\mathsf{X}) \hookrightarrow \mathbb{L}(\mathsf{X})}\big)$*.*

Proof. Fix a rank-one projection P on $\mathcal{H}$, and consider

$$\Psi : \mathbb{K}(\mathsf{X} \bullet P) \xrightarrow{\cong} \mathbb{K}(\mathsf{X}) \xrightarrow{\Phi} \mathbb{L}(\mathsf{Y}) \xrightarrow{\cong} \mathbb{B}(\mathsf{Y} \bullet P),$$

where $\mathbb{K}(\mathsf{X} \bullet P) \xrightarrow{\cong} \mathbb{K}(\mathsf{X})$ and $\mathbb{L}(\mathsf{Y}) \xrightarrow{\cong} \mathbb{B}(\mathsf{Y} \bullet P)$ come from Theorem 5. By definition, the non-degeneracy of (Φ, Y) means that

$$\mathsf{Y} = \overline{\operatorname{Span}(\{[\Phi(T)](\zeta) \mid T \in \mathbb{K}(\mathsf{X}) \text{ and } \zeta \in \mathsf{Y}\})}^{\mathsf{Y}},$$

which yields

$$\begin{aligned}
\mathsf{Y} \bullet P &= \overline{\operatorname{Span}(\{[\Phi(T)](\zeta) \mid T \in \mathbb{K}(\mathsf{X}) \text{ and } \zeta \in \mathsf{Y}\})}^{\mathsf{Y}} \bullet P \\
&\subseteq \overline{\operatorname{Span}(\{[\Phi(T)](\zeta) \mid T \in \mathbb{K}(\mathsf{X}) \text{ and } \zeta \in \mathsf{Y}\}) \bullet P}^{\mathsf{Y}} \\
&= \overline{\operatorname{Span}(\{[\Phi(T)](\zeta \bullet P) \mid T \in \mathbb{K}(\mathsf{X}) \text{ and } \zeta \in \mathsf{Y}\})}^{\mathsf{Y}} \\
&= \overline{\operatorname{Span}(\{[\Phi(T)|_{\mathsf{Y}\bullet P}](\eta) \mid T \in \mathbb{K}(\mathsf{X}) \text{ and } \eta \in \mathsf{Y} \bullet P\})}^{\mathsf{Y}} \\
&= \overline{\operatorname{Span}(\{[\Psi(S)](\eta) \mid S \in \mathbb{K}(\mathsf{X} \bullet P) \text{ and } \eta \in \mathsf{Y} \bullet P\})}^{\mathsf{Y}} \\
&= \overline{\operatorname{Span}(\{[\Psi(S)](\eta) \mid S \in \mathbb{K}(\mathsf{X} \bullet P) \text{ and } \eta \in \mathsf{Y} \bullet P\})}^{\mathsf{Y}\bullet P} \qquad \text{(By Theorem 3.)} \\
&\subseteq \mathsf{Y} \bullet P.
\end{aligned}$$

Hence, Ψ is a non-degenerate $*$-representation of $\mathbb{K}(\mathsf{X} \bullet P)$ on the Hilbert space $\mathsf{Y} \bullet P$, and so the first part of Arveson's proof says that there is a rank-one projection $Q \in \mathbb{K}(\mathsf{X} \bullet P)$ such that $\Psi(Q) \neq 0_{\mathbb{K}(\mathsf{Y}\bullet P)}$.

By Theorem 5, there is a projection $E \in \mathbb{K}(\mathsf{X})$ such that $Q = E|_{\mathsf{X}\bullet P}$. As $\Psi(Q) \neq 0_{\mathbb{K}(\mathsf{Y}\bullet P)}$, it must be that $\Phi(E) \neq 0_{\mathbb{L}(\mathsf{Y})}$, so $E \neq 0_{\mathbb{K}(\mathsf{X})}$. By Lemma 2, there exists a linear functional $f_P : \mathbb{K}(\mathsf{X} \bullet P) \to \mathbb{C}$ satisfying

$$\forall S \in \mathbb{K}(\mathsf{X} \bullet P) : \qquad QSQ = f_P(S) \cdot Q.$$

Define a linear functional $g : \mathbb{K}(\mathsf{X}) \to \mathbb{C}$ by $g(T) \overset{\text{df}}{=} f_P(T|_{\mathsf{X}\bullet P})$ for all $T \in \mathbb{K}(\mathsf{X})$; then

$$\begin{aligned}
ETE|_{\mathsf{X}\bullet P} &= (E|_{\mathsf{X}\bullet P})(T|_{\mathsf{X}\bullet P})(E|_{\mathsf{X}\bullet P}) = Q(T|_{\mathsf{X}\bullet P})Q = f_P(T|_{\mathsf{X}\bullet P}) \cdot Q \\
&= f_P(T|_{\mathsf{X}\bullet P}) \cdot E|_{\mathsf{X}\bullet P} = [g(T) \cdot E]|_{\mathsf{X}\bullet P}.
\end{aligned}$$

By Theorem 5 again, we may conclude that $ETE = g(T) \cdot E$ for all $T \in \mathbb{K}(\mathsf{X})$.

Consider the $\mathbb{K}(\mathcal{H})$-submodule $E[\mathsf{X}]$ of X, which is non-trivial as $E \neq 0_{\mathbb{K}(\mathsf{X})}$, and closed as E is a projection. Similarly, $\Phi(E)[\mathsf{Y}]$ is a non-trivial closed $\mathbb{K}(\mathcal{H})$-submodule of Y. Hence, by Lemma 3, there exist $\zeta \in E[\mathsf{X}]$ and $\eta \in \Phi(E)[\mathsf{Y}]$ such that $\langle \zeta | \zeta \rangle_{\mathsf{X}} = P = \langle \eta | \eta \rangle_{\mathsf{Y}}$. We now claim that the map

$$\begin{aligned}
U &: \operatorname{Span}(\{T(\zeta \bullet A) \in \mathsf{X} \mid T \in \mathbb{K}(\mathsf{X}),\ A \in \mathbb{K}(\mathcal{H})\}) \\
&\to \operatorname{Span}(\{[\Phi(T)](\eta \bullet A) \in \mathsf{Y} \mid T \in \mathbb{K}(\mathsf{X}),\ A \in \mathbb{K}(\mathcal{H})\})
\end{aligned}$$

defined by

$$\sum_{i=1}^{n} T_i(\zeta \bullet A_i) \mapsto \sum_{i=1}^{n} [\Phi(T_i)](\eta \bullet A_i)$$

for all $T_1, \ldots, T_n \in \mathbb{K}(\mathsf{X})$ and $A_1, \ldots, A_n \in \mathbb{K}(\mathcal{H})$ is well-defined by virtue of being an isometry. Indeed,

$$
\begin{aligned}
&\left\| \left\langle \sum_{i=1}^{n} [\Phi(T_i)](\eta \bullet A_i) \middle| \sum_{j=1}^{n} [\Phi(T_j)](\eta \bullet A_j) \right\rangle_{\mathsf{Y}} \right\|_{\mathbb{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \langle [\Phi(T_i)](\eta \bullet A_i) | [\Phi(T_j)](\eta \bullet A_j) \rangle_{\mathsf{Y}} \right\|_{\mathbb{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \langle [\Phi(T_i)]([\Phi(E)](\eta) \bullet A_i) | [\Phi(T_j)]([\Phi(E)](\eta) \bullet A_j) \rangle_{\mathsf{Y}} \right\|_{\mathbb{K}(\mathcal{H})} \\
&\quad \big(\text{As } \eta \in [\Phi(E)][\mathsf{Y}].\big) \\
&= \left\| \sum_{i,j=1}^{n} \langle [\Phi(T_i E)](\eta \bullet A_i) | [\Phi(T_j E)](\eta \bullet A_j) \rangle_{\mathsf{Y}} \right\|_{\mathbb{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \langle [\Phi(T_j E)^* \Phi(T_i E)](\eta \bullet A_i) | \eta \bullet A_j \rangle_{\mathsf{Y}} \right\|_{\mathbb{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \left\langle \left[\Phi\left(E T_j^* T_i E\right)\right](\eta \bullet A_i) \middle| \eta \bullet A_j \right\rangle_{\mathsf{Y}} \right\|_{\mathbb{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \left\langle \left[\Phi\left(g\left(T_j^* T_i\right) \cdot E\right)\right](\eta \bullet A_i) \middle| \eta \bullet A_j \right\rangle_{\mathsf{Y}} \right\|_{\mathbb{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \overline{g\left(T_j^* T_i\right)} \cdot \langle [\Phi(E)](\eta \bullet A_i) | \eta \bullet A_j \rangle_{\mathsf{Y}} \right\|_{\mathbb{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \overline{g\left(T_j^* T_i\right)} \cdot \langle \eta \bullet A_i | \eta \bullet A_j \rangle_{\mathsf{Y}} \right\|_{\mathbb{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \overline{g\left(T_j^* T_i\right)} \cdot A_i^* \langle \eta | \eta \rangle_{\mathsf{Y}} A_j \right\|_{\mathbb{K}(\mathcal{H})} \\
&= \left\| \sum_{i,j=1}^{n} \overline{g\left(T_j^* T_i\right)} \cdot A_i^* P A_j \right\|_{\mathbb{K}(\mathcal{H})}, \qquad \big(\text{As } \langle \eta | \eta \rangle_{\mathsf{Y}} = P.\big)
\end{aligned}
$$

and a near-identical computation using $\langle \zeta | \zeta \rangle_{\mathsf{X}} = P$ also yields

$$
\left\| \left\langle \sum_{i=1}^{n} T_i(\zeta \bullet A_i) \middle| \sum_{j=1}^{n} T_j(\zeta \bullet A_j) \right\rangle_{\mathsf{Y}} \right\|_{\mathbb{K}(\mathcal{H})} = \left\| \sum_{i,j=1}^{n} \overline{g\left(T_j^* T_i\right)} \cdot A_i^* P A_j \right\|_{\mathbb{K}(\mathcal{H})}.
$$

Therefore, U is a surjective isometry. By continuity, U extends to a surjective isometry $U : \mathsf{X}' \to \mathsf{Y}'$, where

$$\mathsf{X}' \stackrel{\text{df}}{=} \overline{\operatorname{Span}(\{T(\zeta \bullet A) \mid T \in \mathbb{K}(\mathsf{X}),\ A \in \mathbb{K}(\mathcal{H})\})}^{\mathsf{X}}$$

and

$$\mathsf{Y}' \stackrel{\text{df}}{=} \overline{\operatorname{Span}(\{[\Phi(T)](\eta \bullet A) \mid T \in \mathbb{K}(\mathsf{X})\ A \in \mathbb{K}(\mathcal{H})\})}^{\mathsf{Y}}.$$

Note that X' is a $\mathbb{K}(\mathsf{X})$-invariant closed submodule of X and is non-trivial as $\zeta \in \mathsf{X}'$. Also, Y' is a $\Phi\big[\mathbb{K}(\mathsf{X})\big]$-invariant closed submodule of Y and is non-trivial as $\eta \in \mathsf{Y}'$. Hence, $\mathsf{X}' = \mathsf{X}$ by Proposition 4, so $U : \mathsf{X} \to \mathsf{Y}'$ is a surjective isometry that, moreover, is $\mathbb{K}(\mathcal{H})$-linear. We may thus apply Theorem 3.5(i) of [9] to deduce that $U \in \mathbb{U}(\mathsf{X}, \mathsf{Y}')$.

Next, we claim that $UT = \Phi(T)|_{\mathsf{Y}'}U$ for all $T \in \mathbb{K}(\mathsf{X})$. Fix $T \in \mathbb{K}(\mathsf{X})$. Then for all $T_1, \ldots, T_n \in \mathbb{K}(\mathsf{X})$ and $A_1, \ldots, A_n \in \mathbb{K}(\mathcal{H})$, we have

$$\begin{aligned}(UT)\left(\sum_{i=1}^{n} T_i(\xi \bullet A_i)\right) &= U\left(\sum_{i=1}^{n} TT_i(\xi \bullet A_i)\right) \\ &= \sum_{i=1}^{n}[\Phi(TT_i)](\eta \bullet A_i) \\ &= \sum_{i=1}^{n}[\Phi(T)\Phi(T_i)](\eta \bullet A_i) \\ &= [\Phi(T)]\left(\sum_{i=1}^{n}[\Phi(T_i)](\eta \bullet A_i)\right) \\ &= [\Phi(T)U]\left(\sum_{i=1}^{n} T_i(\xi \bullet A_i)\right).\end{aligned}$$

By the density of $\operatorname{Span}(\{T(\xi \bullet A) \mid T \in \mathbb{K}(\mathsf{X}),\ a \in \mathbb{K}(\mathcal{H})\})$ in X, we obtain $UT = \Phi(T)|_{\mathsf{Y}'}U$ as expected.

Now, define a poset $(\mathfrak{P}, \sqsubseteq)$ with the following properties:

- $\mathcal{S}$ is an element of $\mathfrak{P}$ if and only if the following hold:
  - $\mathcal{S}$ consists of pairs of the form (Z, V), where Z is a non-trivial $\Phi\big[\mathbb{K}(\mathsf{X})\big]$-invariant closed submodule of Y, and $V \in \mathbb{U}(\mathsf{X}, \mathsf{Z})$ with $T = V^{-1}[\Phi(T)|_{\mathsf{Z}}]V$ for all $T \in \mathbb{K}(\mathcal{H})$.
  - If (Z_1, V_1) and (Z_2, V_2) are distinct elements of $\mathcal{S}$, then $\mathsf{Z}_1 \perp \mathsf{Z}_2$.
- For all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{P}$, we have $\mathcal{S}_1 \sqsubseteq \mathcal{S}_2$ if and only if $\mathcal{S}_1 \subseteq \mathcal{S}_2$.

If $\mathfrak{C}$ is a chain in $(\mathfrak{P}, \sqsubseteq)$, then $\bigcup \mathfrak{C}$ is an upper bound for $\mathfrak{C}$ in $(\mathfrak{P}, \sqsubseteq)$, so by Zorn's Lemma, there exists a maximal element $\mathcal{M}$ of $(\mathfrak{P}, \sqsubseteq)$. We claim that $\mathsf{Y} = \bigoplus_{(\mathsf{Z},V)\in\mathcal{M}} \mathsf{Z}$, where the direct sum is internal. If this were not true, then $\bigoplus_{(\mathsf{Z},V)\in\mathcal{M}} \mathsf{Z} \subsetneq \mathsf{Y}$. Letting

$\mathsf{Z}' \overset{\text{df}}{=} \left[\bigoplus_{(\mathsf{Z},V)\in\mathcal{M}} \mathsf{Z} \right]^{\perp}$, Theorem 6 says that Z' is a non-trivial closed submodule of Y. A routine verification reveals that $\Phi(T)\big[\mathsf{Z}'\big] \subseteq \mathsf{Z}'$ for each $T \in \mathbb{K}(\mathcal{H})$ and that

$$\begin{Bmatrix} \mathbb{K}(\mathsf{X}) & \to & \mathbb{L}\big(\mathsf{Z}'\big) \\ T & \mapsto & \Phi(T)|_{\mathsf{Z}'} \end{Bmatrix}$$

is a non-degenerate $*$-representation of $\mathbb{K}(\mathsf{X})$ on Z'. We may thus apply the first part of the proof to Z' to obtain $\big(\mathsf{Z}'', W\big)$, where

- Z'' is a non-trivial $\Phi\big[\mathbb{K}(\mathsf{X})\big]$-invariant closed submodule of Z' (and hence of Y), and
- $W \in \mathbb{U}\big(\mathsf{X}, \mathsf{Z}''\big)$ with $T = W^{-1}[\Phi(T)|_{\mathsf{Z}''}]W$ for all $T \in \mathbb{K}(\mathcal{H})$.

As $\mathcal{M} \subsetneq \big\{\big(\mathsf{Z}'', W\big)\big\} \cup \mathcal{M} \in \mathfrak{P}$, this contradicts the maximality of $\mathcal{M}$. Therefore, $\mathsf{Y} = \bigoplus_{(\mathsf{Z},V)\in\mathcal{M}} \mathsf{Z}$ indeed, so

$$\begin{aligned} \forall T \in \mathbb{K}(\mathcal{H}): \qquad \Phi(T) &= \oplus_{(\mathsf{Z},V)\in\mathcal{M}} \Phi(T)|_{\mathsf{Z}} = \oplus_{(\mathsf{Z},V)\in\mathcal{M}} VTV^{-1} \\ &= \big[\oplus_{(\mathsf{Z},V)\in\mathcal{M}} V\big] T \Big[\oplus_{(\mathsf{Z},V)\in\mathcal{M}} V^{-1}\Big]. \end{aligned}$$

The proof is finally complete. □

5. The Covariant Stone–von Neumann Theorem

In [16], Marc Rieffel used a special instance of Green's Imprimitivity Theorem to prove the classical Stone–von Neumann Theorem. According to him, the classical Stone–von Neumann Theorem is a statement about the Morita equivalence of the C^*-algebra $\mathbb{C}$ with the crossed product $C^*(G, C_0(G), \mathsf{lt})$. This gives us a more algebraic way of seeing things, and it is precisely this point of view that guided our search for the covariant Stone–von Neumann Theorem in the beginning. As we are dealing with Hilbert C^*-modules instead of just Hilbert spaces, we will require the full strength of Green's Imprimitivity Theorem, as can be seen in our proof of the next result.

For the remainder of this section, we shall fix arbitrary C^*-dynamical systems (G, A, α) and (G, A, β) with G abelian. We shall also fix a Haar measure μ on G and a Haar measure ν on $\widehat{G}$.

Proposition 6. *Recalling the (G, A, α)-Schrödinger module representation* $\big(\mathsf{L}^2(G, A, \alpha), \mathsf{M}, \mathsf{U}, \mathsf{V}\big)$,

$$\overline{\pi^{\mathsf{M},\mathsf{V}} \rtimes \mathsf{U}} : C^*(G, C_0(G, A), \mathsf{lt} \otimes \alpha) \to \mathbb{L}\Big(\mathsf{L}^2(G, A, \alpha)\Big)$$

is then an injective C^*-homomorphism such that $\operatorname{Range}\big(\overline{\pi^{\mathsf{M},\mathsf{V}} \rtimes \mathsf{U}}\big) = \mathbb{K}\big(\mathsf{L}^2(G, A, \alpha)\big)$.

Proof. It suffices by Remark 1 to show that $\pi^{\mathsf{M},\mathsf{V}} = \Xi$. Let $f \in C_c\big(\widehat{G}, A\big)$ and $\phi \in C_c(G, A)$. Then

$$\Big[\pi^{\mathsf{M},\mathsf{V}}(\mathcal{F}(f))\Big](q(\phi)) = \big[(\mathsf{M} \rtimes \mathsf{V})(f)\big](q(\phi))$$

$$= \int_{\widehat{G}} \big[\mathsf{M}(f(\gamma)) \circ \mathsf{V}(\gamma)\big](q(\phi))\, \mathrm{d}\nu(\gamma)$$
$$= \int_{\widehat{G}} q(f(\gamma)(\gamma \cdot \phi))\, \mathrm{d}\nu(\gamma).$$

The last integral looks like it should be $q(\mathcal{F}(f)\phi)$, and indeed it is, but we have to exercise some caution in justifying our guess. By Fubini's Theorem, we have for all $\psi \in \mathrm{C_c}(G, A)$ that

$$\left\langle q(\psi) \middle| \int_{\widehat{G}} q(f(\gamma)(\gamma \cdot \phi))\, \mathrm{d}\nu(\gamma) \right\rangle_{\mathsf{L}^2(G,A,\alpha)}$$
$$= \int_{\widehat{G}} \langle q(\psi) | q(f(\gamma)(\gamma \cdot \phi))\rangle_{\mathsf{L}^2(G,A,\alpha)}\, \mathrm{d}\nu(\gamma)$$
$$= \int_{\widehat{G}} \left[\int_G \alpha_{x^{-1}}\big(\psi(x)^* f(\gamma)\big[\gamma(x) \cdot \phi(x)\big]\big)\, \mathrm{d}\mu(x) \right] \mathrm{d}\nu(\gamma)$$
$$= \int_{\widehat{G}} \left[\int_G \alpha_{x^{-1}}\big(\psi(x)^* \big[\gamma(x) \cdot f(\gamma)\phi(x)\big]\big)\, \mathrm{d}\mu(x) \right] \mathrm{d}\nu(\gamma)$$
$$= \int_G \left[\int_{\widehat{G}} \alpha_{x^{-1}}\big(\psi(x)^* \big[\gamma(x) \cdot f(\gamma)\phi(x)\big]\big)\, \mathrm{d}\nu(\gamma) \right] \mathrm{d}\mu(x)$$
$$= \int_G \alpha_{x^{-1}}\left(\psi(x)^* \left[\int_{\widehat{G}} \gamma(x) \cdot f(\gamma)\, \mathrm{d}\nu(\gamma) \right] \phi(x) \right) \mathrm{d}\mu(x)$$
$$= \int_G \alpha_{x^{-1}}\big(\psi(x)^* [\mathcal{F}(f)](x) \phi(x)\big)\, \mathrm{d}\mu(x)$$
$$= \langle q(\psi) | q(\mathcal{F}(f)\phi)\rangle_{\mathsf{L}^2(G,A,\alpha)}.$$

This establishes the validity of our guess. Hence, $\pi^{\mathsf{M},\mathsf{V}}(\mathcal{F}(f)) = \Xi(\mathcal{F}(f))$ for all $f \in \mathrm{C_c}\big(\widehat{G}, A\big)$, and as the range of $\mathcal{F}$ is dense in $\mathrm{C_0}(G, A)$, we conclude that $\pi^{\mathsf{M},\mathsf{V}} = \Xi$. □

Definition 7. Let (X, ρ, R, S) be a (G, A, α)-Heisenberg module representation and $(\mathsf{Y}, \sigma, T, U)$ a (G, A, β)-Heisenberg module representation. We say that (X, ρ, R, S) is *unitarily equivalent* to $(\mathsf{Y}, \sigma, T, U)$ if and only if there exists a $W \in \mathbb{U}(\mathsf{X}, \mathsf{Y})$ such that

$$WR(x)W^* = T(x), \qquad WS(\gamma)W^* = U(\gamma), \qquad W\rho(a)W^* = \sigma(a)$$

for all $x \in G$, $\gamma \in \widehat{G}$, and $a \in A$, in which case we write $(\mathsf{X}, \rho, R, S) \sim_W (\mathsf{Y}, \sigma, T, U)$.

Definition 8. We say that (G, A, α) has the *von Neumann Uniqueness Property (vNUP)* if and only if any (G, A, α)-Heisenberg module representation is unitarily equivalent to a direct sum of copies of the (G, A, α)-Schrödinger module representation.

We are now ready to prove the main result of this paper.

Proposition 7 (The Covariant Stone–von Neumann Theorem). *(G, A, α) has the vNUP if $A = \mathbb{K}(\mathcal{H})$ for some non-trivial Hilbert space $\mathcal{H}$.*

Proof. Let (X, ρ, R, S) be a $(G, \mathbb{K}(\mathcal{H}), \alpha)$-Heisenberg module representation. We know from Proposition 2 that $\left(\mathsf{X}, \pi^{\rho,S}, R\right)$ is a $(G, \mathrm{C}_0(G, \mathbb{K}(\mathcal{H})), \mathsf{lt} \otimes \alpha)$-covariant module representation. Consequently, $\overline{\pi^{\rho,S} \rtimes R}$ is a non-degenerate $*$-representation of $C^*(G, \mathrm{C}_0(G, \mathbb{K}(\mathcal{H})), \mathsf{lt} \otimes \alpha)$ on X. However, Proposition 6 says that

$$\overline{\pi^{\mathsf{M},\mathsf{V}} \rtimes \mathsf{U}} : C^*(G, \mathrm{C}_0(G, \mathbb{K}(\mathcal{H})), \mathsf{lt} \otimes \alpha) \to \mathbb{K}\Big(\mathsf{L}^2(G, \mathbb{K}(\mathcal{H}), \alpha)\Big)$$

is a C^*-isomorphism, so it follows from Proposition 5 that

$$\left(\mathsf{X}, \overline{\pi^{\rho,S} \rtimes R} \circ \left(\overline{\pi^{\mathsf{M},\mathsf{V}} \rtimes \mathsf{U}}\right)^{-1}\right) \sim_W \bigoplus_{i \in I} \Big(\mathsf{L}^2(G, \mathbb{K}(\mathcal{H}), \alpha), i_{\mathbb{K}(\mathsf{L}^2(G,\mathbb{K}(\mathcal{H}),\alpha)) \hookrightarrow \mathbb{L}(\mathsf{L}^2(G,\mathbb{K}(\mathcal{H}),\alpha))}\Big)$$

for some index set I and some $W \in \mathbb{U}\left(\mathsf{X}, \bigoplus_{i \in I} \mathsf{L}^2(G, \mathbb{K}(\mathcal{H}), \alpha)\right)$. We thus have

$$\left[\overline{\pi^{\rho,S} \rtimes R} \circ \left(\overline{\pi^{\mathsf{M},\mathsf{V}} \rtimes \mathsf{U}}\right)^{-1}\right](T) = W^*(\oplus_{i \in I} T) W,$$

for all $T \in \mathbb{K}\big(\mathsf{L}^2(G, \mathbb{K}(\mathcal{H}), \alpha)\big)$, or equivalently,

$$\overline{\pi^{\rho,S} \rtimes R}(F) = W^*\Big[\oplus_{i \in I} \overline{\pi^{\mathsf{M},\mathsf{V}} \rtimes \mathsf{U}}(F)\Big] W$$

for all $F \in C^*(G, \mathrm{C}_0(G, \mathbb{K}(\mathcal{H})), \mathsf{lt} \otimes \alpha)$. It follows from Proposition 2.39 of [22] that

$$R(x) = W^*\big[\oplus_{i \in I} \mathsf{U}(x)\big] W \quad \text{and} \quad \pi^{\rho,S}(f) = W^*\Big[\oplus_{i \in I} \pi^{\mathsf{M},\mathsf{V}}(f)\Big] W$$

for all $x \in G$ and $f \in \mathrm{C}_0(G, \mathbb{K}(\mathcal{H}))$. However, as

$$\pi^{\rho,S} = \overline{\rho \rtimes S} \circ \overline{\mathcal{F}}^{-1} \quad \text{and} \quad \pi^{\mathsf{M},\mathsf{V}} = \overline{\mathsf{M} \rtimes \mathsf{V}} \circ \overline{\mathcal{F}}^{-1},$$

we find that

$$\overline{\rho \rtimes S}(f) = W^*\Big[\oplus_{i \in I} \overline{\mathsf{M} \rtimes \mathsf{V}}(f)\Big] W$$

for all $f \in C^*\big(\widehat{G}, \mathbb{K}(\mathcal{H}), \iota\big)$. Another application of Proposition 2.39 of [22] yields

$$S(\gamma) = W^*\big[\oplus_{i \in I} \mathsf{V}(\gamma)\big] W \quad \text{and} \quad \rho(A) = W^*\big[\oplus_{i \in I} \mathsf{M}(a)\big] W$$

for all $\gamma \in \widehat{G}$ and $a \in \mathbb{K}(\mathcal{H})$. The covariant Stone–von Neumann Theorem is hereby established. □

Our method of proof in no way depended on the classical Stone–von Neumann Theorem, so it is a proper generalization in every way, as expressed by the corollary below.

Corollary 1. *The classical Stone–von Neumann Theorem is precisely the case when $\mathcal{H} = \mathbb{C}$ (any strongly-continuous action of a locally compact Hausdorff group on $\mathbb{C}$ is necessarily trivial).*

6. The Non-Triviality of the Covariant Stone–von Neumann Theorem

One may now ask, "Does the covariant Stone–von Neumann Theorem really say anything new? Is there a unitary transformation that reduces it to the case of the trivial action of G on $\mathbb{K}(\mathcal{H})$?" The following result makes this question an extremely valid one.

Proposition 8. *Let (G, A, α) be a C^*-dynamical system with G not assumed to be abelian. Then there is a Hilbert A-module isomorphism $\Omega : \mathsf{L}^2(G, A, \alpha) \to \mathsf{L}^2(G, A, \iota)$ that satisfies*

$$\forall \phi \in \mathrm{C_c}(G, A) : \qquad \Omega\big(q_{(G,A,\alpha)}(\phi)\big) = q_{(G,A,\iota)}\left(\left\{\begin{matrix} G \to & A \\ x \mapsto & \alpha_{x^{-1}}(f(x)) \end{matrix}\right\}\right).$$

Proof. This is an easy verification that we leave to the reader. □

Even though $\mathsf{L}^2(G, A, \alpha)$ is isomorphic to $\mathsf{L}^2(G, A, \iota)$, note that the covariant Stone–von Neumann Theorem is not a statement about the unitary equivalence of Hilbert C^*-modules, but a statement about the unitary equivalence of Heisenberg module representations. Having said this, the next two results give a complete answer to the question above.

Proposition 9. *Let (G, A, α) and (G, A, β) be C^*-dynamical systems, with G abelian and $\alpha \neq \beta$. Then a direct sum of copies of the (G, A, α)-Schrödinger module representation cannot be unitarily equivalent to a direct sum of copies of the (G, A, β)-Schrödinger module representation.*

Proof. By way of contradiction, suppose that there are index sets I and J such that

$$\begin{aligned} &\bigoplus_{i \in I}\Big(\mathsf{L}^2(G, A, \alpha), \mathsf{M}^{(G,A,\alpha)}, \mathsf{U}^{(G,A,\alpha)}, \mathsf{V}^{(G,A,\alpha)}\Big) \\ &\qquad \sim_W \bigoplus_{j \in J}\Big(\mathsf{L}^2(G, A, \beta), \mathsf{M}^{(G,A,\beta)}, \mathsf{U}^{(G,A,\beta)}, \mathsf{V}^{(G,A,\beta)}\Big) \end{aligned}$$

for some $W \in \mathbb{U}\left(\bigoplus_{i \in I} \mathsf{L}^2(G, A, \alpha), \bigoplus_{j \in J} \mathsf{L}^2(G, A, \beta)\right)$. Then we have for all $x \in G$ and $a \in A$ that

$$\begin{aligned} \mathsf{U}^{(G,A,\alpha)}(x)\mathsf{M}^{(G,A,\alpha)}(a) &= \mathsf{M}^{(G,A,\alpha)}(\alpha_x(a))\mathsf{U}^{(G,A,\alpha)}(x), \\ \mathsf{U}^{(G,A,\beta)}(x)\mathsf{M}^{(G,A,\beta)}(a) &= \mathsf{M}^{(G,A,\beta)}(\beta_x(a))\mathsf{U}^{(G,A,\beta)}(x), \\ W\Big[\oplus_{i \in I}\mathsf{U}^{(G,A,\alpha)}(x)\Big]W^* &= \oplus_{j \in J}\mathsf{U}^{(G,A,\beta)}(x), \\ W\Big[\oplus_{i \in I}\mathsf{M}^{(G,A,\alpha)}(a)\Big]W^* &= \oplus_{j \in J}\mathsf{M}^{(G,A,\beta)}(a), \end{aligned}$$

so it follows that

$$\begin{aligned} \oplus_{j \in J}\mathsf{M}^{(G,A,\beta)}(\beta_x(a)) &= \oplus_{j \in J}\mathsf{U}^{(G,A,\beta)}(x)\mathsf{M}^{(G,A,\beta)}(a)\mathsf{U}^{(G,A,\beta)}(x)^{-1} \\ &= \Big[\oplus_{j \in J}\mathsf{U}^{(G,A,\beta)}(x)\Big]\Big[\oplus_{j \in J}\mathsf{M}^{(G,A,\beta)}(a)\Big]\Big[\oplus_{j \in J}\mathsf{U}^{(G,A,\beta)}(x)^{-1}\Big] \\ &= W\Big[\oplus_{i \in I}\mathsf{U}^{(G,A,\alpha)}(x)\Big]\Big[\oplus_{i \in I}\mathsf{M}^{(G,A,\alpha)}(a)\Big]\Big[\oplus_{i \in I}\mathsf{U}^{(G,A,\alpha)}(x)^{-1}\Big]W^* \end{aligned}$$

$$= W\Big[\oplus_{i \in I} \mathsf{U}^{(G,A,\alpha)}(x) \mathsf{M}^{(G,A,\alpha)}(a) \mathsf{U}^{(G,A,\alpha)}(x)^{-1}\Big] W^*$$
$$= W\Big[\oplus_{i \in I} \mathsf{M}^{(G,A,\alpha)}(\alpha_x(a))\Big] W^*$$
$$= \oplus_{j \in J} \mathsf{M}^{(G,A,\beta)}(\alpha_x(a)),$$

which yields $\mathsf{M}^{(G,A,\beta)}(\beta_x(a)) = \mathsf{M}^{(G,A,\beta)}(\alpha_x(a))$. Hence, for all $x \in G$, $a \in A$, and $\phi \in \mathrm{C_c}(G, A)$,

$$\begin{aligned} q_{(G,A,\beta)}\left(\begin{Bmatrix} G \to & A \\ y \mapsto & \alpha_x(a)\phi(y) \end{Bmatrix}\right) &= \Big[\mathsf{M}^{(G,A,\beta)}(\alpha_x(a))\Big]\big(q_{(G,A,\beta)}(\phi)\big) \\ &= \Big[\mathsf{M}^{(G,A,\beta)}(\beta_x(a))\Big]\big(q_{(G,A,\beta)}(\phi)\big) \\ &= q_{(G,A,\beta)}\left(\begin{Bmatrix} G \to & A \\ y \mapsto & \beta_x(a)\phi(y) \end{Bmatrix}\right), \end{aligned}$$

from which we get $(\alpha_x(a) - \beta_x(a))\phi(y) = 0_A$ for all $y \in G$. As we can choose ϕ to assume any value at any point, we obtain $\alpha_x(a) = \beta_x(a)$ for all $x \in G$ and $a \in A$, which contradicts $\alpha \neq \beta$. □

Corollary 2. *Let $(G, \mathbb{K}(\mathcal{H}), \alpha)$ and $(G, \mathbb{K}(\mathcal{H}), \beta)$ be C^*-dynamical systems, with G abelian, $\mathcal{H}$ a non-trivial Hilbert space, and $\alpha \neq \beta$. Then any $(G, \mathbb{K}(\mathcal{H}), \alpha)$-Heisenberg module representation cannot be unitarily equivalent to any $(G, \mathbb{K}(\mathcal{H}), \beta)$-Heisenberg module representation.*

Proof. This follows immediately from Propositions 7 and 9. □

Corollary 2 should remind physicists of Haag's Theorem in quantum field theory (QFT), which posits the failure of the uniqueness of the canonical commutation relations within QFT in general [6].

We finally arrive at a discussion of Takai–Takesaki Duality.

Theorem 7 (Takai–Takesaki Duality [14,22]). *Let (G, A, α) be a C^*-dynamical system. Then*

$$C^*\big(\widehat{G}, C^*(G, A, \alpha), \widehat{\alpha}\big) \cong \mathbb{K}\Big(L^2(G)\Big) \otimes A,$$

where $\widehat{\alpha}$ denotes the dual action of $\widehat{G}$ on $C^*(G, A, \alpha)$.

In his proof of Takai–Takesaki Duality in [14], Iain Raeburn first showed that

$$C^*\big(\widehat{G}, C^*(G, A, \alpha), \widehat{\alpha}\big) \cong C^*(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha).$$

He then formed a C^*-isomorphism $C^*\big(\widehat{G}, C^*(G, A, \alpha), \widehat{\alpha}\big) \cong \mathbb{K}\big(L^2(G)\big) \otimes A$ as a composition of a series of C^*-isomorphisms shown below, each requiring a lengthy justification except for the last one:

$$\begin{aligned} C^*(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha) \cong C^*(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \iota) \cong C^*(G, \mathrm{C}_0(G), \mathsf{lt}) \otimes A \\ \cong \mathbb{K}\Big(L^2(G)\Big) \otimes A. \end{aligned}$$

His "untwisting" of α is thus performed at the level of C^*-crossed products, with the last C^*-isomorphism being given by the classical Stone–von Neumann Theorem, which relies on Green's Imprimitivity Theorem. However, by taking full advantage of Green's Imprimitivity Theorem, we can derive a shorter proof of this C^*-isomorphism, which "untwists" α at the level of Hilbert C^*-modules:

Proposition 10. *Let (G, A, α) be a C^*-dynamical system. Then* $C^*(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha) \cong \mathbb{K}\big(L^2(G)\big) \otimes A$.

Proof. By Remark 1, Proposition 8, and basic results about Hilbert C^*-modules, we have a one-line proof:

$$C^*(G, \mathrm{C}_0(G, A), \mathsf{lt} \otimes \alpha) \cong \mathbb{K}\Big(\mathsf{L}^2(G, A, \alpha)\Big) \cong \mathbb{K}\Big(\mathsf{L}^2(G, A, \iota)\Big) \cong \mathbb{K}\Big(L^2(G) \otimes A_A\Big) \cong \mathbb{K}\Big(L^2(G)\Big) \otimes A. \quad \square$$

7. Conclusions

We would like to present here some questions and thoughts that naturally arose while writing this paper:

(1) Is there a C^*-algebra A not C^*-isomorphic to $\mathbb{K}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ such that any C^*-dynamical system of the form (G, A, α) has the von Neumann Uniqueness Property? As C^*-subalgebras of $\mathbb{K}(\mathcal{H})$ are C^*-isomorphic to a direct sum $\bigoplus_{i \in I} \mathbb{K}(\mathcal{H}_i)$, where the $\mathcal{H}_i$'s are Hilbert spaces, we think that a series of technical extensions can be made to accommodate the Covariant Stone–von Neumann Theorem for such C^*-algebras.
(2) The results of this paper suggest that quantum mechanics could be developed using Hilbert C^*-modules as state spaces, in which case the expectations of observables would assume values in a C^*-algebra. Can this idea be developed further?
(3) As mentioned in the introduction, we suspect that the covariant Stone–von Neumann Theorem could be generalized to actions of non-abelian groups using techniques of non-abelian duality.

While interesting in a purely-mathematical context, the Covariant Stone–von Neumann Theorem has a rich interpretation from the perspective of quantum mechanics. By including representations of C^*-dynamical systems, it allows for the consideration of time-dependence of observables in addition to time-dependence of states. To contrast, recall that a time-independent quantum system is modeled by a Hilbert space $\mathcal{H}$ and a Hamiltonian $\widehat{H}$ whose corresponding one-parameter unitary family, $\Big(e^{-(it/\hbar)\cdot \widehat{H}}\Big)_{t \in \mathbb{R}}$, determines the time evolution of the state space via

$$\forall \psi \in \mathcal{H},\ \forall t \in \mathbb{R}: \qquad \psi(t) = e^{-(it/\hbar)\cdot \widehat{H}} \cdot \psi(0).$$

The time evolution of the state space determined by $\widehat{H}$ can also be viewed as time evolution of the algebra $\mathbb{B}(\mathcal{H})$ of bounded observables via $\left\{ \begin{matrix} \mathbb{R} \to & \mathbb{B}(\mathcal{H}) \\ t \ \mapsto & e^{(it/\hbar)\cdot \widehat{H}} T e^{-(it/\hbar)\cdot \widehat{H}} \end{matrix} \right\}$, for all $T \in \mathbb{B}(\mathcal{H})$. From this perspective, one may state the time-independent version of Ehrenfest's Theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \psi | T(\psi) \rangle_{\mathcal{H}} = \big\langle \psi \big| \big[i \cdot \widehat{H}, T \big](\psi) \big\rangle_{\mathcal{H}}.$$

As the Covariant Stone–von Neumann Theorem applies to C^*-dynamical systems of the form $(G, \mathbb{K}(\mathcal{H}), \alpha)$, and as all $*$-automorphisms of $\mathbb{K}(\mathcal{H})$ are implemented via conjugation by unitaries, we make a convenient but natural restriction in the case when $G = \mathbb{R}$ to the action α^C, where $C \in \mathbb{B}(\mathcal{H})$ is self-adjoint, and

$$\forall T \in \mathbb{B}(\mathcal{H}),\ \forall t \in \mathbb{R}: \qquad \alpha_t^C(T) \stackrel{\text{df}}{=} e^{(it/\hbar)\cdot C} T e^{-(it/\hbar)\cdot C}.$$

The covariance conditions present in the definition of an $\left(\mathbb{R}, \mathbb{K}(\mathcal{H}), \alpha^C\right)$-Heisenberg module representation (X, ρ, R, S) then reduce to commutation relations between C and the infinitesimal generators of R and S. It is in this context that we are able to get an infinitesimal version of the Covariant Stone–von Neumann Theorem, which will appear in a sequel to this article.

As mentioned in the introduction, a catalyst for the Stone–von Neumann Theorem was to investigate the uniqueness of pairs (A, B) of self-adjoint Hilbert-space operators satisfying the Heisenberg Commutation Relation. Nelson's counterexample [12] shows that uniqueness fails in general, and decades of research have been devoted to identifying sufficient conditions for (A, B) that imply that $\left(e^{is\cdot A}\right)_{s\in\mathbb{R}}$ and $\left(e^{it\cdot B}\right)_{t\in\mathbb{R}}$ satisfy the Weyl Commutation Relation.

In the sequel, we follow the strategy in [8]—which takes place in the Hilbert-space setting—to provide necessary and sufficient conditions for when a pair (A, B) of unbounded self-adjoint operators on a Hilbert $\mathbb{K}(\mathcal{H})$-module yield one-parameter unitary groups that satisfy the Weyl Commutation Relation.

Acknowledgements. The first author wishes to thank the second author for her warm hospitality and collaborative energy when he visited the University of Nebraska-Lincoln. He would also like to thank his former PhD advisor, Professor Albert Sheu of the University of Kansas, for cultivating his deep-seated interest in the Stone–von Neumann Theorem in graduate school. The second author would like to thank the first author for his invitation to collaborate on this project. His passion and breadth of knowledge have been a privilege to work alongside. The second author is also grateful to her advisors, Allan Donsig and David Pitts, as well as Ruy Exel, for their thoughtful comments and questions. Finally, the authors would like to express their heartfelt thanks to the referees who gave the paper such a meticulous reading and offered invaluable advice to streamline and polish the exposition herein.

Appendix

Proof of Lemma 1. Fix $f \in \mathrm{C_c}(X, V)$ and $\epsilon > 0$. Let $K \stackrel{\text{df}}{=} \mathrm{Supp}(f)$, and let $\left(U_p\right)_{p\in K}$ be a K-indexed sequence of open subsets of X with the following properties:

- U_p is an open neighborhood of p in X for each $p \in K$.
- $\|f(x) - f(p)\|_V < \dfrac{\epsilon}{3}$ for all $x \in U_p$.

Clearly, $\left\{U_p\right\}_{p\in K}$ covers K, and as K is compact, there is a finite subset F of K such that $\left\{U_p\right\}_{p\in F}$ also covers K and $U_p \neq U_{p'}$ for distinct $p, p' \in F$. As X is locally compact and Hausdorff, there is a partition of unity $\left(\gamma_p\right)_{p\in F}$ for K that is subordinate to $\left\{U_p\right\}_{p\in F}$, i.e.,

- $\gamma_p \in \mathrm{C_c}(X, [0, 1])$ and $\mathrm{Supp}\left(\gamma_p\right) \subseteq U_p$ for each $p \in F$, and
- $\displaystyle\sum_{p\in F} \gamma_p(x) \leq 1$ for all $x \in X$, with equality holding for all $x \in K$.

Define $P \in C_c(X, V)$ by

$$\forall x \in X: \qquad P(x) \overset{\text{df}}{=} \sum_{p \in F} \gamma_p(x) \cdot f(p).$$

For all $x \in X$, we have $f(x) = \sum_{p \in F} \gamma_p(x) \cdot f(x)$ (if $x \in K$, then $\sum_{p \in F} \gamma_p(x) = 1$; otherwise, $f(x) = 0_V$), so

$$\begin{aligned}\|f(x) - P(x)\|_V &= \left\| \sum_{p \in F} \gamma_p(x) \cdot [f(x) - f(p)] \right\|_V \leq \sum_{p \in F} \gamma_p(x) \|f(x) - f(p)\|_V \\ &\leq \sum_{p \in F} \gamma_p(x) \cdot \frac{\epsilon}{3} \leq \frac{\epsilon}{3}.\end{aligned}$$

As D is a dense subset of V, we can find an F-indexed sequence $\left(v_p\right)_{p \in F}$ in D such that $\left\| f(p) - v_p \right\|_V < \dfrac{\epsilon}{3}$ for each $p \in F$. Define $Q \in C_c(X, V)$ by

$$\forall x \in X: \qquad Q(x) \overset{\text{df}}{=} \sum_{p \in F} \gamma_p(x) \cdot v_p.$$

Then for all $x \in X$,

$$\begin{aligned}\|P(x) - Q(x)\|_V &= \left\| \sum_{p \in F} \gamma_p(x) \cdot \left[f(p) - v_p\right] \right\|_V \leq \sum_{p \in F} \gamma_p(x) \left\| f(p) - v_p \right\|_V \\ &\leq \sum_{p \in F} \gamma_p(x) \cdot \frac{\epsilon}{3} \leq \frac{\epsilon}{3}.\end{aligned}$$

Therefore, by the Triangle Inequality, we have for all $x \in X$ that

$$\|f(x) - Q(x)\|_V \leq \|f(x) - P(x)\|_V + \|P(x) - Q(x)\|_V \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3} < \epsilon.$$

As Q has the desired form, the first part of the theorem is therefore established.

Let U be an open neighborhood of K whose closure is compact (such a neighborhood exists because X is locally compact and Hausdorff). Then U is a locally compact Hausdorff space and $f|_U \in C_c(U, V)$, so we may apply the first part to find $\gamma_1, \ldots, \gamma_n \in C_c(U)$ and $v_1, \ldots, v_n \in D$ such that for all $x \in U$,[1]

$$\left\| f|_U(x) - \sum_{i=1}^{n} \gamma_i(x) \cdot v_i \right\|_V < \frac{\epsilon}{1 + \lambda(U)}.$$

Let $\widetilde{\gamma_1}, \ldots, \widetilde{\gamma_n}$ denote the respective extensions of $\gamma_1, \ldots, \gamma_n$ to X by 0_V. Then $\widetilde{\gamma_1}, \ldots, \widetilde{\gamma_n} \in C_c(X)$, and

$$\int_X \left\| f(x) - \sum_{i=1}^{n} \widetilde{\gamma_i}(x) \cdot v_i \right\|_V \mathrm{d}\lambda(x) = \int_U \left\| f|_U(x) - \sum_{i=1}^{n} \gamma_i(x) \cdot v_i \right\|_V \mathrm{d}\lambda(x)$$

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[1] The measure of a pre-compact subset, with respect to a regular Borel measure, is finite.

$$\leq \int_U \frac{\epsilon}{1 + \lambda(U)} \, \mathrm{d}\lambda(x)$$
$$= \lambda(U) \cdot \frac{\epsilon}{1 + \lambda(U)}$$
$$< \epsilon.$$

The proof of the second part of the theorem is therefore established. □

References

1. Arveson, W.: An Invitation to C^*-Algebras. Graduate Texts in Mathematics, vol. 39. Springer, New York (1976)
2. Bakić, D., Guljaš, B.: Operators on Hilbert H^*-modules. J. Oper. Theory **46**(1), 123–137 (2001)
3. Bakić, D., Guljaš, B.: Hilbert C^*-modules over C^*-algebras of compact operators. Acta Sci. Math. (Szeged) **68**(1), 249–269 (2002)
4. Cavallaro, S., Morchio, G., Strocchi, F.: A generalization of the Stone–von Neumann theorem to nonregular representations of the CCR-algebra. Lett. Math. Phys. **47**(4), 307–320 (1999)
5. Green, P.: The structure of imprimitivity algebras. J. Funct. Anal. **36**(1), 88–104 (1980)
6. Haag, R.: On quantum field theories. Danske Vid. Selsk. Mat. Fys. Medd. **29**(12), 1–37 (1955)
7. Heisenberg, W.: Die physikalischen Prinzipien der Quantentheorie. Hirzel, Leipzig (1930)
8. Huang, L.: An infinitesimal version of the Stone–von Neumann theorem. arXiv:1704.03859, pp. 1–6 (2017)
9. Lance, E.: Unitary operators on Hilbert C^*-modules. Bull. Lond. Math. Soc. **26**(4), 363–366 (1994)
10. Mackey, G.: A theorem of Stone and von Neumann. Duke Math. J. **16**(2), 313–326 (1949)
11. Magajna, B.: Hilbert modules in which all closed submodules are complemented. Proc. Am. Math. Soc. **125**(3), 849–852 (1997)
12. Nelson, E.: Analytic vectors. Ann. Math. **70**(3), 572–615 (1959)
13. Palma, R.: Crossed Products by Hecke Pairs, vol. 252(1204). American Mathematical Society, Providence (2018)
14. Raeburn, I.: On crossed products and Takai duality. Proc. Edinb. Math. Soc. (2) **31**(2), 321–330 (1988)
15. Raeburn, I., Williams, D.: Morita Equivalence and Continuous-Trace C^*-Algebras. Mathematical Surveys and Monographs, vol. 60. American Mathematical Society, Providence (1998)
16. Rieffel, M.: On the uniqueness of the Heisenberg commutation relations. Duke Math. J. **39**(4), 745–752 (1972)
17. Rieffel, M.: Induced representations of C^*-algebras. Adv. Math. **13**, 176–257 (1974)
18. Rosenberg, J.: A Selective History of the Stone-von Neumann Theorem, Operator Algebras, Quantization, and Noncommutative Geometry. Contemporary Mathematics, vol. 365, pp. 331–353. American Mathematical Society, Providence (2004)
19. Stone, M.H.: Linear Transformations in Hilbert Space, vol. 15 (1930). American Mathematical Society, Providence (1990)
20. von Neumann, J.: Die Eindeutigkeit der Schrödingerschen operatoren. Math. Ann. **104**(1), 570–578 (1931)
21. Wegge-Olsen, N.E.: K-Theory and C^*-Algebras. Oxford University Press, New York (1993)
22. Williams, D.: Crossed Products of C^*-Algebras. Mathematical Surveys and Monographs, vol. 134. American Mathematical Society, Providence (2007)

Communicated by Y. Kawahigashi