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Regularity of the Geodesic Flow of the Incompressible Euler Equations on a Manifold

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Abstract: In this paper we investigate the regularity in time of the volume-preserving geodesic flow, which is associated with the incompressible Euler equations on a compact d-dimensional Riemannian manifold with boundary. Our result, which completes the local-in-time well-posedness theory of Ebin and Marsden (Ann Math 92:102–163, 1970), states roughly that the time smoothness of geodesic curves is only limited by the smoothness of the manifold. Such regularity is measured in a broad class of ultradifferentiable functions, which includes the real analytic and Gevrey classes. A by-product of this simple and constructive proof is new ideas to design high-order semi-Lagrangian methods for integrating the incompressible Euler equations on a manifold.

1. Introduction and Main Result

As far as it concerns time regularity, there is a remarkable difference between the Eulerian and Lagrangian solutions to the incompressible Euler equations in the spatial non-too-smooth regime. For instance, in Eulerian coordinates if we consider solutions as a function of time with values in Hölder spaces, then this function is everywhere discontinuous in time for generic initial data [43,44]. On the contrary, in Lagrangian coordinates, where one is focusing on Lagrangian trajectories, an initial velocity field with limited smoothness (typically in Sobolev or Hölder classes with suitable indexes of regularity) launches geodesic curves, which are analytic in time. In Euclidean space such results have been known for about 20 years and are still studied until very recently [8,14,16,24,25,28,33,35,51–53,56]. For a brief history on how the issue of Lagrangian analyticity in ideal hydrodynamics was tackled so far, we refer the reader to [8].

To the best of our knowledge no time-regularity results for the Euler geodesic flow on a d-dimensional Riemaniann manifold seem available in the literature. Roughly speaking, our result states that an initial velocity with finite Sobolev regularity initiates a geodesic flow whose time regularity is given by the regularity of the manifold (see Theorem 2). This regularity is described by a broad class of ultradifferentiable functions, which encompasses the real analytic and Gevrey classes. For such study it is convenient to recall briefly the Arnold's geometric reinterpretation of the hydrodynamics of an ideal incompressible fluid [4]. Let M be a smooth compact Riemannian manifold of dimension d with boundary ∂M, usually called a Riemannian ∂-manifold (see, e.g., [50]). We will come back later on the precise regularity needed for M. In the Arnold geometric interpretation of the incompressible Euler equations [4,5], the solutions can be viewed as geodesics (i.e., extrema) of the *right-invariant* Riemannian metric given by the kinetic energy on the infinite-dimensional group of *volume-preserving* diffeomorphisms. Indeed, following [4], we define $\mathrm{SDiff}(M, \mu)$ as the infinite-dimensional group of diffeomorphisms $\eta : M \to M$ preserving the metric volume form μ, i.e., $\eta^*\mu = \mu$. Here η^* denotes the pullback operator associated with the diffeomorphism η. The standard notation and definitions of differential geometry that we use here are briefly recalled in "Appendix A". Since it is more convenient to work with Hilbert spaces (see [19]), we consider $\mathcal{D}^s(M, \mu)$ the infinite-dimensional group of diffeomorphisms of M, which preserve the Riemannian volume form μ and, which are of Sobolev class H^s. Its tangent space $T_\eta\mathcal{D}^s(M, \mu)$ at a point η consists of H^s sections V of the pullback bundle η^*TM, whose right-translations $V \circ \eta^{-1}$ to the identity element e are the divergence-free vector fields on M that are parallel to the boundary ∂M. In other words, we can associated with the group $\mathcal{D}^s(M, \mu)$ the Lie algebra $\mathfrak{g} := T_e\mathcal{D}^s(M, \mu)$, which consists of all vector fields $\mathfrak{u}$ satisfying $\nabla_i \mathfrak{u}^i = 0$, on M, and $(\mathfrak{u}, \nu)_g = 0$, on ∂M. Here ∇ denotes the covariant derivative, while $(\cdot, \cdot)_g$ stands for the Riemannian metric defined by the metric tensor g (see "Appendix A"). In fluid dynamics the space $T\mathcal{D}^s(M, \mu)$ represents the Lagrangian (material) description, while the space $\mathfrak{g}$ represents the Eulerian (spatial) description. The L^2 inner product for vector field,

$$\langle X, Y\rangle_{L^2(M)} = \int_M (X, Y)_g\, \mu, \quad X, Y \in \mathfrak{g}, \tag{1}$$

defines a right-invariant metric on $\mathcal{D}^s(M, \mu)$. Right-invariance comes from the property that $V \circ \eta^{-1}$ is independant of φ, when we replace η by the composition $\eta \circ \varphi$, for any fixed map $\varphi \in \mathcal{D}^s(M, \mu)$. In [4] Arnold showed that a perfect incompressible fluid is a minimal geodesic curve η_t with respect to the right-invariant L^2 metric on $\mathrm{SDiff}(M, \mu)$ (resp. $\mathcal{D}^s(M, \mu)$), starting from identity element e in the direction $\mathfrak{u}_0$ if, and only if, the time dependent vector field $\mathfrak{u} = \dot{\eta}_t \circ \eta_t^{-1}$ on M solves the Cauchy problem for the incompressible Euler equations,

$$\partial_t \mathfrak{u} + \nabla_{\mathfrak{u}}\, \mathfrak{u} + \mathrm{grad}\, p = 0, \quad \text{on } M, \quad t \in]-T, T[, \tag{2}$$

$$\mathrm{div}\, \mathfrak{u} = 0, \quad \text{on } M, \quad t \in]-T, T[, \tag{3}$$

$$\mathfrak{u}_{|t=0} = \mathfrak{u}_0, \quad \text{on } M, \tag{4}$$

$$(\mathfrak{u}, \nu)_g = 0, \quad \text{on } \partial M, \quad t \in]-T, T[, \tag{5}$$

with p the pressure function. In other words the Lagrangian flow η_t is an optimal (minimal) path in $\mathrm{SDiff}(M, \mu)$ (resp. $\mathcal{D}^s(M, \mu)$), i.e., a one-parameter family of smooth material deformation maps on M, which extremizes (minimizes) the time-integral of kinetic energy (i.e., the Maupertuis action). The geodesic curve $\eta_t = \eta(t, a)$, defined on $]-T, T[\times M$, is also called the Lagrangian map and satisfies obviously the following ODE,

$$\partial_t \eta(t, a) = \mathfrak{u}(t, \eta(t, a)), \quad \text{and} \quad \eta(0, a) = a \in M. \tag{6}$$

Soon after, Ebin and Marsden [19] observed that there is a technical advantage in rewriting the Euler equations in Lagrangian variables as a geodesic flow in $\mathcal{D}^s(M,\mu)$. They showed that the Cauchy problem for the corresponding geodesic equation on the group $\mathcal{D}^s(M,\mu)$ can be solved uniquely on short time intervals as an abstract ODE in Banach spaces by using Picard iteration and Banach fixed-point arguments, like in the proof of the Cauchy–Lipschitz–Picard theorem for ODE, but without using usual PDE-type estimates. From the key result of Ebin and Marsden [19], we only give the statement that is interesting for our purpose. We refer to Theorem 15.2 of [19] for complete statements.

Theorem 1 (Ebin and Marsden [19]). *Let M be a compact $\mathscr{C}^\infty$ d-dimensional Riemannian ∂-manifolds with $\mathscr{C}^\infty$ boundary ∂M. Let $s > d/2+1$. If $\mathfrak{u}_0$ is an H^s vector field on M, such that* $\operatorname{div}\mathfrak{u}_0 = 0$*, and $\mathfrak{u}_0$ parallel to ∂M (i.e., $(\mathfrak{u}_0,\nu)_g = 0$, with ν be the outward pointing unit normal to the boundary ∂M), then there exists a unique vector field $\mathfrak{u}$, defined for $-T < t < T$ for some $T > 0$, which is solution of the incompressible Euler equations* (2)–(5)*; $\mathfrak{u}$ is an H^s vector field and is $\mathscr{C}^1$ as a function of (t,x) for $-T < t < T$ and $x \in M$. The geodesic flow η_t is an H^s (in particular a $\mathscr{C}^1$) volume-preserving diffeomorphism.*

Remark 1. We recall that since the seminal work of Lichtenstein [39,40] and Gyunter [29,30] the Cauchy problem for the incompressible Euler equations in the whole space $\mathbb{R}^3$ is known to be well posed in time (at least for short time) when the initial velocity is in Hölder or Sobolev spaces with suitable indexes of regularity. In order to get rid of techniques of Riemannian geometry on infinite-dimensional manifolds and make results of Ebin and Mardsen [19] more accessible, the authors of [11] give a proof for bounded domains of $\mathbb{R}^d$ (flat Euclidean space) by reducing also the Euler equations to an ODE on a closed set of a Banach space.

In order to present the main result, we must first define some functional spaces that we use to describe time regularity and manifold smoothness. Let U be a domain in $\mathbb{R}^d$ and let $\mathfrak{B}$ be a Banach space endowed with the norm $\|\cdot\|_{\mathfrak{B}}$. Let $\mathcal{M} := \{\mathcal{M}_\sigma\}_{\sigma\geq 0}$ be a sequence of positive numbers. The ultradifferentiable class $\mathcal{C}\{\mathcal{M}\}(U;\mathfrak{B})$ is defined as the set of functions $f : U \longrightarrow \mathfrak{B}$ such that for any compact set $K \subset U$ there exist constants (depending on f) R_f, C_f such that for all $\sigma \in \mathbb{N}$,

$$\sup_{x\in K} |||D^\sigma f(x)||| \leq C_f R_f^{-\sigma} \mathcal{M}_\sigma. \tag{7}$$

The map $x \mapsto D^\sigma f(x)$ is a function defined on U with values in the set of symmetric σ-linear operators, which is endowed with the standard induced operator-norm $|||\cdot|||$. The class $\mathcal{C}\{\mathcal{M}\}$ is invariant under multiplication by a constant, i.e., $\mathcal{C}\{\lambda\mathcal{M}\}(U;\mathfrak{B}) = \mathcal{C}\{\mathcal{M}\}(U;\mathfrak{B})$ for $\lambda > 0$. As in [8], we choose the "log-superlinear Faà-di-Bruno" (LSL–FdB in short) class. For such class the sequence of weights $\{\mathcal{M}_\sigma/\sigma!\}_{\sigma\geq 0}$ (and $M_0 = \mathcal{M}_0$) satisfies

Definition 1. The log-superlinear Faà-di-Bruno class is the set of functions satisfying (7), where the weights $M_\sigma = \mathcal{M}_\sigma/\sigma!$ verify the following properties,

(i) differentiation stability:

$$\exists\, C_{\mathrm{D}} > 0: \quad M_{\sigma+1} \leq C_{\mathrm{D}}^\sigma M_\sigma, \quad \forall \sigma \in \mathbb{N}. \tag{8}$$

(ii) log-superlinearity:

$$M_\sigma M_\ell \leq M_0 M_{\sigma+\ell}, \quad \forall \sigma,\ \ell \in \mathbb{N}. \tag{9}$$

(iii) (FdB)-stability:

$$\forall \alpha_i \in \mathbb{N}^*, \quad \text{such that} \quad \alpha_1 + \cdots + \alpha_\ell = \sigma, \quad \text{we have} \quad M_\ell M_{\alpha_1} \ldots M_{\alpha_\ell} \leq M_\sigma. \tag{10}$$

Remark 2. Using the Leibniz differentiation rules, log-superlinearity implies that the class $\mathcal{C}\{\mathcal{M}\}(U;\mathfrak{B})$ is an algebra with respect to pointwise multiplication. Using the Faà di Bruno formula [17,20,31], (FdB)-stability implies stability under composition in the class $\mathcal{C}\{\mathcal{M}\}(U;\mathfrak{B})$ (see the proof of Proposition 3.1 in [48], or Proposition 1.4.2 in [37]). Finally, the differentiability stability property implies closure under differentiation in $\mathcal{C}\{\mathcal{M}\}(U;\mathfrak{B})$ [36,41,49].

Remark 3. Some well-known classes of functions belong to the LSL–FdB class. The first one is the real analytic functions class, which corresponds to $M_\sigma = 1$ (e.g., [12,42]). The second one is the log-convex class, which corresponds to $M_\sigma^2 \leq M_{\sigma-1}M_{\sigma+1}$, with $M_0 = M_1 = 1$ (see Lemma 2.9 of [38]). A particular case of the latter is the Gevrey class (see, e.g., [41,42]), which corresponds to $M_\sigma = (\sigma!)^r$, with $r \geq 0$.

From now on and throughout this work we suppose that the Riemannian ∂-manifold (M, g), where g denotes the Riemannian metric tensor, is ultradifferentiable in the LSL–FdB class. Its definition follows the standard definition of differentiable or real analytic manifolds (see, e.g., [10,50,54]) except that in our case charts and atlas belong to the LSL–FdB ultradifferentiable class. As a consequence, the components of the 2-covariant metric tensor g are LSL–FdB ultradifferentiable functions.

The main result of this paper is

Theorem 2. *Assume that the hypotheses of Theorem 1 hold, and in addition that the manifold M and its boundary ∂M belong to $\mathcal{C}\{\mathcal{M}\}$, where $\mathcal{M} := \{\sigma! M_\sigma\}_{\sigma\geq 0}$, with the sequence $\{M_\sigma\}_{\sigma\geq 0}$ satisfying Definition 1 (log-superlinear Faà-di-Bruno class). Then there exists a time $T = C(M, \|\mathrm{u}_0\|_{H^s(M)})$ such that the geodesic flow η satisfies*

$$\eta \in \mathcal{C}\{\mathcal{M}\}\big(]-T, T[;\, H^s(M)\big).$$

Theorem 2 is an extension, to d-dimensional manifolds, of the previous Theorem 2 of Besse and Frisch [8], which deals with bounded domain of $\mathbb{R}^3$. The proof makes use of a new Lagrangian formulation of Euler equations on d-dimensional manifolds [7], which is a generalization of the Cauchy invariants equation in $\mathbb{R}^3$ used in [8]. As in [8], the proof is based on this Lagrangian formulation, together with the Lagrangian incompressibility condition and the invariance of the boundary under the Lagrangian flow. Nevertheless the present proof differs from that of [8] in several points listed below and it is not a straightforward translate of [8] to the context of manifolds.

1. By constrast to [8], the fondamental object is not the Lagrangian map η_t, but a 1-form $\gamma_t^\flat$ constructed from it. We then obtained new recursion relations among time-Taylor coefficients of the time-series expansion of this 1-form.
2. Boundary conditions are handled in a different way. Indeed, in [8] the authors use the surface equation, $\mathcal{S}(a) = 0$, $\forall a \in \partial M$, as a functional representation of the two-dimensional boundary to derive boundary recursion relations, which involve successive derivatives of the function $a \mapsto \mathcal{S}(a)$. Here, we proceed differently by rewriting the boundary condition (5) in Lagrangian variables and by using the property that a particle being initially on the boundary remains on it forever. All the geometry of the boundary is then encoded in the metric tensor g and the normal vector ν.

3. There are some new technical estimates, which are crucially used throughout the proof. The authors of [8] obtained estimates in $\mathscr{C}^{1,\alpha}$-norm, which are then controlled by $\mathscr{C}^2$-norm. These estimates can not be extended to the general cases of Hölder spaces $\mathscr{C}^{s,\alpha}$ or Sobolev spaces H^s, for the following reason. In [8], by using a control in $\mathscr{C}^2$-norm, the authors obtained a second-order-in-time differential inequality. Because this inequality is only of second order, it can be easily integrated to obtain useful algebraic estimates. In the case of spaces $\mathscr{C}^{s,\alpha}$ or H^s, this strategy leads to a differential inequality of order $s+1$ in time, which becomes difficult to integrate and thus to exploit for obtaining useful estimates. Here, this difficulty is solved by new functional estimates, which are used throughout the proof and are given by Lemma 1. We then obtain new algebraic estimates (see Proposition 1) without requiring to integrate an entangled differential inequality of high order.
4. Another important difference is the Hodge decomposition method, which is more involved in the case of differential forms on manifolds than in the case of vectors in Euclidean spaces. We must deal with supplementary differential forms, called harmonic forms, which come from the non-emptyness of kernels of Laplace–De-Rham operators. By contrast with the case of bounded domains in $\mathbb{R}^3$, where the authors of [8] have considered the construction of a Neumann boundary value problem for the Hodge scalar potential and a Dirichlet boundary value problem for the Hodge vector potential, here we construct two Neumann boundary value problems for the two Hodge potentials. As a consequence, our Hodge decomposition is more general than the one used in [8] (see Remark 5). This construction also needs to consider refined decompositions in more involved functionnal spaces with some supplementary commutation relations between trace operators and (co-)exterior derivatives. Moreover for solving these non-homogeneous boundary value problems, we must show that right-hand sides satisfy suitable integrability conditions. This corresponds to the fact that right-hand sides of Neumann boundary value problems must be orthogonal to the kernel of the Neumann operator.
5. The paper [8] deals with the dimension three for bounded domains of the Euclidean space, while here we deal with ∂-manifolds of any dimension. This implies new recursion relations, especially those arising from the incompressibility constraint and the boundary conditions.

Finally it is worthwhile to stress the robustness of the present method to pass from the case of the Euclidean space to the case of manifolds. For other Lagrangian methods, which use a reformulation of the problem in the Euclidean space as an abstract ODE in some Banach spaces, it is not clear that an extension of such methods to manifolds is easily tractable. Indeed these methods require to work with intricate Green functions, which may be difficult to deal with.

The outline of the paper is as follows. In Sect. 2, we present the proof of Theorem 2 in three steps. First, in Sect. 2.1 we construct the time-Taylor coefficients of the dual 1-form $\gamma_t^\flat$ associated with the geodesic flow, by restricting our attention to the special case $d=3$. This construction consists in solving a sequence of non-homogeneous Neumann boundary value problems for potential differential forms stemming from the Hodge decomposition of the 1-form $\gamma_t^\flat$. Source terms and boundary conditions of these non-homogeneous boundary value problems are given by new recursion relations, which involve in a crucial way the regularity of the manifold (metric tensor). Then, in Sect. 2.2, we deal with convergence issues and we obtain a priori estimates in Sobolev spaces for time-Taylor coefficients of the 1-form $\gamma_t^\flat$. In Sect. 2.3, we extend the constructive scheme and its convergence analysis to the general case of a d-dimensional manifold with boundary. In conclusion, we discuss the applicability of such constructive proof to design new high-order semi-Lagrangian schemes for solving the incompressible Euler equations on a manifold. Finally there are three appendices. "Appendix A" recalls standard differential geometry notation that we use here. "Appendix B" states main results about the Hodge decomposition of differential forms on compact Riemannian manifolds with boundary [50]. "Appendix C" recalls and slightly corrects a derivation of the generalized Cauchy invariants equation [7].

2. Proof of Theorem 2

Here, we give a proof of Theorem 2, which is divided into three steps. In Sect. 2.1, the 1-form $\gamma_t^\flat$, associated with the geodesic flow, is constructed recursively as a formal time-Taylor expansion. In this first section we restrict our attention mainly to the special case $d = 3$. Sect. 2.2 is devoted to the convergence analysis of such formal time-expansions. The last step (Sect. 2.3) extends the previous constructive scheme and its convergence analysis to the general case of a d-dimensional manifold with boundary.

2.1. Construction of the 1*-form* $\gamma_t^\flat$. The starting point of the construction is a Lagrangian formulation, the origin of which traces back to Cauchy [13]. Indeed, in his pioneering paper [13], Cauchy used a Lagrangian formulation, called nowadays the Cauchy invariants equation, to integrate the Euler equations in the 3D Euclidean (flat) space by establishing the so-called Cauchy formula. In [7] the Cauchy invariants equation has been generalized to Riemaniann manifolds of any dimension and also to other hydrodynamic and magnetohydrodynamic models. In "Appendix C", we recall two derivations of the Cauchy invariants equation. The Lagrangian formulation of the incompressible Euler equation (2)–(5) that we consider is

$$dv_i \wedge dx^i = \omega_0 := dv_0^\flat, \quad \text{on } M, \tag{11}$$

$$\sqrt{\frac{|\mathrm{g}|(x)}{|\mathrm{g}|(a)}}\det\left(\frac{\partial x}{\partial a}\right) = 1, \qquad \text{on } M, \tag{12}$$

$$\mathrm{i}_{\nu(x)}\mathfrak{u}^\flat(x) = 0, \qquad \text{on } \partial M. \tag{13}$$

Here, the Lagrangian variable $x = x(t, a)$ (resp. $a = x(0, a)$) stands for the current (resp. initial) Lagrangian position of fluid particles. In a similar way, the Lagrangian variable $v = v(t, a)$ (resp. $v_0 = v(0, a)$) stands for the current (resp. initial) Lagrangian velocity of fluid particles. We then have $x := \eta_t$, and $v := \dot{\eta}_t = \mathfrak{u} \circ \eta_t$ ($v_0 = \mathfrak{u}_0$). In (11), x^i denotes the contravariant components of the Lagrangian position, while v_i denotes the covariant components of the Lagrangian velocity, i.e., $v_i = g_{ij}(\eta_t)\dot{\eta}_t^j$. Moreover, we use the standard convention that an index variable appearing twice in a single term, implies the summation of that term over all the values of the index. The 2-form $\omega_0 := dv_0^\flat = d\mathfrak{u}_0^\flat$ is the initial vorticity 2-form. The exterior derivative d corresponds to a differential operator in the Lagrangian variables, i.e., a differentiation with respect to the variable a. We also recall (see "Appendix A") the definition $|\mathrm{g}| := \det(g_{ij})$. Equation (13) means that the interior product of the 1-form $\mathfrak{u}^\flat$ with the vector field ν is zero on the boundary ∂M (for the definition of the interior product i, see, e.g., "Appendix A" and references therein). Equation (11) is the Cauchy invariants equation while the Lagrangian incompressibility condition is (12). Equation (13) reflects the flow-invariance of the impermeable boundary ∂M, i.e., the preservation by the Lagrangian flow of the prescribed rigid and impermeable boundary ∂M. We emphasize that Lagrangian formulation (11)–(13) is equivalent to Eulerian formulations (2)–(5) at least when everything is smooth (see [7]).

To construct a 1-form $\gamma_t^\flat$, associated with the geodesic flow η_t, we are going to derive from (11)–(13) recursion relations, which always involve a finite number of terms. To achieve this, we use formal time-Taylor series, whose convergence and time-regularity issues are postponed to Sect. 2.2. To keep the proof simple and to obtain explicit formula, we first consider the case $d = 3$, and we postpone to Sect. 2.3 the treatment of the general d-dimensional case. Anticipating a little bit, we will see that recursion relations coming from Cauchy invariants equation (11) and boundary condition (13) are the same for any dimension d, since the dimension only plays the role of a parameter. Recursion relations coming from incompressibility condition (12) will require to expand the determinant of the Jacobian matrix in terms of the trace of positive integer power of another matrix. For $d = 3$ the explicit formula is simple and well-known. For the general dimension d, we will use the Plemelj–Smithies formula (see, e.g., [27]), which leads to more intricate expressions.

From [19], we know that solutions of (2)–(5) depend continuously on the initial conditions, and the group $\mathcal{D}^s(M, \mu)$ has, at least for small time, a smooth exponential map in the usual sense of Lie group:

$$\begin{aligned} \exp_e : \mathfrak{g} &\longrightarrow \mathcal{D}^s(M, \mu) \\ \mathfrak{u}_0 &\longmapsto \eta_t = \exp_e(t\mathfrak{u}_0). \end{aligned} \tag{14}$$

Here, η_t is the unique geodesic of (1), starting from the identity e with initial velocity $\mathfrak{u}_0 \in \mathfrak{g}$. By the inverse function theorem, the exponential map is a local diffeomorphism from an open set around zero in $\mathfrak{g}$ onto a neighborhood of the identity in $\mathcal{D}^s(M, \mu)$. In the particular case $d = 2$, from the classical result of Wolibner [55] (see also [34] for a more recent result), the exponential map can be extended to the whole tangent space $\mathfrak{g}$. On one hand, time expansion of exponential map (14) gives

$$\begin{aligned} \exp(t\mathfrak{u}_0)a &= \Big(\mathrm{Id} + tv_0 + \frac{t^2}{2!}w_1 + \cdots + \frac{t^\sigma}{\sigma!}w_\sigma + \cdots\Big)a \\ &= a + tu_0 + \frac{t^2}{2!}u_1 + \cdots + \frac{t^\sigma}{\sigma!}u_\sigma + \cdots , \end{aligned} \tag{15}$$

where $v_0 = u_0^i \partial/\partial a^i \in TM_a$, and $w_\sigma = u_\sigma^i \partial/\partial a^i \in TM_a$, for $\sigma > 0$; the scalar coordinates $\{u_\sigma^i\}_{\{\sigma\in\mathbb{N},\, i=1,\ldots,d\}}$ are real numbers. On the other hand, time expansion of the geodesic flow η_t gives

$$\eta(t, a) = \eta_t a = \eta_t \circ a = \eta(0, a) + t\dot\eta(0, a) + \frac{t^2}{2!}\ddot\eta(0, a) + \cdots + \frac{t^\sigma}{\sigma!}(\partial_t^{(\sigma)}\eta)(0, a) + \cdots , \tag{16}$$

Comparing (15) and (16), we obtain

$$\begin{aligned} \eta(0, a) &= a, \\ \dot\eta(0, a) &= v_0 a = u_0^i \frac{\partial}{\partial a^i} a = u_0, \\ (\partial_t^{(\sigma)}\eta)(0, a) &= w_\sigma a = u_\sigma^i \frac{\partial}{\partial a^i} a = u_\sigma, \quad \sigma > 0. \end{aligned}$$

We then observe that (16) or (15) are simply relations between real scalar coordinates. Therefore, for $k \in \{1, \dots, d\}$, we can consider a sequence of scalar functions $\{\gamma^k_{(\sigma)}\}_{\sigma\in\mathbb{N}}$, with $\gamma^k_{(\sigma)} \in H^s\Omega^0(M)$, such that we can set the formal time series,

$$\eta^k(t,a) := \sum_{\sigma\geq 0} \gamma^k_{(\sigma)}(a)\frac{t^\sigma}{\sigma!} = a^k + \sum_{\sigma>0} \gamma^k_{(\sigma)}(a)\frac{t^\sigma}{\sigma!}. \tag{17}$$

In fact, the right object to consider is not the geodesic flow (17), but the 1-form $\gamma_t^\flat$ defined by

$$\gamma^\flat(t,a) := \sum_{\sigma\geq 0} \gamma_{i(\sigma)}(a)da^i\,\frac{t^\sigma}{\sigma!}, \tag{18}$$

with $\gamma_{i(\sigma)} = g_{ij}\gamma^j_{(\sigma)}$.

We recall that convergence issues of such time series is not the point here and is postponed to Sect. 2.2.

2.1.1. Recursion relations stemming from the Cauchy invariants equation. Here, we obtain recursion relations from the Cauchy invariants equation (11). Substituting formal time series (17)–(18) into the Cauchy invariants equation (11), and collecting terms of the same power $\sigma > 0$, we obtain, after some algebra,

$$\begin{aligned} d\gamma^\flat_{(\sigma)} = \delta_{1\sigma}dv^\flat_0 - &\sum_{0<m<\sigma}\binom{\sigma-1}{m-1} d\big(\gamma^j_{(m)}G_{ij(\sigma-m)}da^i + g_{ij}\gamma^j_{(m)}d\gamma^i_{(\sigma-m)}\big) \\ &- \sum_{\substack{m+n+l=\sigma\\ m,n,l>0}} \binom{\sigma-1}{m-1}\binom{\sigma-m}{n} d\big(G_{ij(n)}\gamma^j_{(m)}d\gamma^i_{(l)}\big), \end{aligned} \tag{19}$$

where δ_{ij} is the Kronecker symbol and $g_{ij} = g_{ij}(a)$. In (19), the 1-form $\gamma^\flat_{(\sigma)}$ is given by

$$\gamma^\flat_{(\sigma)} = \gamma_{i(\sigma)}da^i, \quad \text{with} \quad \gamma_{i(\sigma)} = g_{ij}\gamma^j_{(\sigma)}, \tag{20}$$

while terms $G_{ij(\sigma)}$ are defined by

$$G_{ij(\sigma)} = \sigma! \sum_{1\leq|\beta|\leq\sigma} \partial^\beta g_{ij}(a) \sum_{q=1}^{\sigma} \sum_{P_q(\sigma,\beta)} \prod_{r=1}^{q} \frac{(\gamma^1_{(\ell_r)})^{k^1_r}}{k^1_r!(\ell_r!)^{k^1_r}} \cdots \frac{(\gamma^d_{(\ell_r)})^{k^d_r}}{k^d_r!(\ell_r!)^{k^d_r}}, \tag{21}$$

for $\sigma > 0$, and $G_{ij(0)} = g_{ij}(a)$. In (21), the set $P_q(\sigma,\beta)$ is given by

$$\begin{aligned} P_q(\sigma,\beta) = \Big\{&(\ell_1,\dots,\ell_q),\ (k_1,\dots,k_q);\ \ 0<\ell_1<\cdots<\ell_q; \\ &|k_j|>0,\ j\in[1,q];\ \sum_{j=1}^q k_j = \beta,\ \sum_{j=1}^q |k_j|\ell_j = \sigma\Big\}. \end{aligned} \tag{22}$$

In (19), we have used the Faà di Bruno formula [17,20,31] to obtain the time series expansion of the composed function $g_{ij}(x) = g_{ij} \circ \eta(t)$, namely,

$$g_{ij}(\eta(t,a)) = \sum_{\sigma \geq 0} G_{ij(\sigma)}(a) \frac{t^\sigma}{\sigma!}. \tag{23}$$

We note that recursion relation (19) involves a finite number of terms and holds for any finite dimension d.

2.1.2. Recursion relations stemming from the incompressibility condition. Here, we derive recursion relations from incompressibility condition (12), by restricting our attention to the special case $d = 3$. Analysis of the general d-dimensional case, which uses Plemelj–Smithies formula (see, e.g., [27]), is postponed to Sect. 2.3. Since the 2-convariant metric tensor g can be identified to a symmetric definite positive matrix g_i^j, there exists a matrix J_i^j such that $g_i^j = J_i^k J_k^{Tj}$. Hence $|\mathrm{g}| = \det(g_i^j) = \det(J_i^j)^2 = \mathrm{J}^2$ or $\mathrm{J} = \sqrt{|\mathrm{g}|}$. Using the well-known formula $\det(I + A) = 1 + \mathrm{Tr}\,(A) + \frac{1}{2}[\mathrm{Tr}\,(A)^2 - \mathrm{Tr}\,(A^2)] + \det(A)$, the incompressibility condition can be rewritten as

$$\begin{aligned}
\mathrm{J}(a) &= \mathrm{J}(x) \det\left(\frac{\partial x^i}{\partial a^j}\right) \\
&= \mathrm{J}(\eta_t) \det\left(I + \frac{\partial \widehat{\gamma}^i}{\partial a^j}\right) \\
&= \mathrm{J}(\eta_t)\left(1 + \mathrm{Tr}\,\left(\frac{\partial \widehat{\gamma}^i}{\partial a^j}\right) + \frac{1}{2}\left[\mathrm{Tr}\,\left(\frac{\partial \widehat{\gamma}^i}{\partial a^j}\right)^2 - \mathrm{Tr}\left(\left(\frac{\partial \widehat{\gamma}^i}{\partial a^j}\right)^2\right)\right] + \det\left(\frac{\partial \widehat{\gamma}^i}{\partial a^j}\right)\right),
\end{aligned} \tag{24}$$

where we set

$$\widehat{\gamma}^i := \sum_{\sigma > 0} \gamma^i_{(\sigma)}(a) \frac{t^\sigma}{\sigma!}. \tag{25}$$

We also need a time series expansion of the Jacobian $\mathrm{J}(\eta_t)$. This is achevied by using the Faà di Bruno formula. We then obtain

$$\mathrm{J}(\eta_t) = \sum_{\sigma \geq 0} \mathrm{J}_{(\sigma)}(a) \frac{t^\sigma}{\sigma!} = \mathrm{J}(a) + \sum_{\sigma > 0} \mathrm{J}_{(\sigma)}(a) \frac{t^\sigma}{\sigma!}, \tag{26}$$

where

$$\begin{aligned}
\mathrm{J}_{(\sigma)} &= \sigma! \sum_{1 \leq |\beta| \leq \sigma} \partial^\beta \mathrm{J}(a) \sum_{q=1}^{\sigma} \sum_{P_q(\sigma,\beta)} \prod_{r=1}^{q} \frac{(\gamma^1_{(\ell_r)})^{k_r^1}}{k_r^1!(\ell_r!)^{k_r^1}} \cdots \frac{(\gamma^d_{(\ell_r)})^{k_r^d}}{k_r^d!(\ell_r!)^{k_r^d}} \\
&= D_a \mathrm{J} \cdot \gamma_{(\sigma)} + \sigma! \sum_{1 < |\beta| \leq \sigma} \partial^\beta \mathrm{J}(a) \sum_{q=1}^{\sigma} \sum_{P_q(\sigma,\beta)} \prod_{r=1}^{q} \frac{(\gamma^1_{(\ell_r)})^{k_r^1}}{k_r^1!(\ell_r!)^{k_r^1}} \cdots \frac{(\gamma^d_{(\ell_r)})^{k_r^d}}{k_r^d!(\ell_r!)^{k_r^d}} \\
&= D_a \mathrm{J} \cdot \gamma_{(\sigma)} \; + \; \mathcal{J}_{(\sigma)} = \gamma^i_{(\sigma)} \partial_i \mathrm{J} \; + \; \mathcal{J}_{(\sigma)},
\end{aligned} \tag{27}$$

for $\sigma > 0$ and $\mathrm{J}_{(0)} = \mathrm{J}(a)$. In (27) and below we use the following notation $\mathrm{J} = \mathrm{J}_{(0)} = \mathrm{J}(a)$. Moreover the operator $\cdot$ (resp. D_a) denotes the usual scalar-product (resp. derivative) in $\mathbb{R}^d$. Substituting formal time series (25) and (26) into (24), and using decomposition (27) and the following relations,

$$-\,d^*X^\flat = \nabla_i X^i = \frac{1}{\sqrt{|\mathrm{g}|}}\partial_i(\sqrt{|\mathrm{g}|}X^i) = \partial_i X^i + \frac{1}{\mathrm{J}}X^i\partial_i\mathrm{J}, \quad \forall X \in \mathfrak{X}(M), \tag{28}$$

we obtain

$$\begin{aligned}
&-\sum_{\sigma>0} d^*\gamma^\flat_{(\sigma)}\frac{t^\sigma}{\sigma!} + \mathrm{J}^{-1}\sum_{\sigma>0}\mathcal{J}_{(\sigma)}\frac{t^\sigma}{\sigma!}\\
&+\mathrm{J}^{-1}\left(\sum_{\sigma>0}\mathrm{J}_{(\sigma)}\frac{t^\sigma}{\sigma!}\right)\left(\sum_{\sigma>0}\partial_i\gamma^i_{(\sigma)}\frac{t^\sigma}{\sigma!}\right) + \mathcal{R} + \mathcal{R}\mathrm{J}^{-1}\sum_{\sigma>0}\mathrm{J}_{(\sigma)}\frac{t^\sigma}{\sigma!} = 0,
\end{aligned}\tag{29}$$

where

$$\begin{aligned}
\mathcal{R} &= \frac{1}{2}\left[\mathrm{Tr}\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right)^2 - \mathrm{Tr}\left(\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right)^2\right)\right] + \det\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right)\\
&= \frac{1}{2}\sum_{\sigma_1,\sigma_2>0}\left(\partial_i\gamma^i_{(\sigma_1)}\partial_j\gamma^j_{(\sigma_2)} - \partial_j\gamma^i_{(\sigma_1)}\partial_i\gamma^j_{(\sigma_2)}\right)\frac{t^{\sigma_1+\sigma_2}}{\sigma_1!\sigma_2!}\\
&\quad+\frac{1}{6}\varepsilon^{i_1i_2i_3}\varepsilon_{j_1j_2j_3}\sum_{\sigma_1,\sigma_2,\sigma_3>0}\partial_{i_1}\gamma^{j_1}_{(\sigma_1)}\partial_{i_2}\gamma^{j_2}_{(\sigma_2)}\partial_{i_3}\gamma^{j_3}_{(\sigma_3)}\frac{t^{\sigma_1+\sigma_2+\sigma_3}}{\sigma_1!\sigma_2!\sigma_3!}.
\end{aligned}$$

Collecting terms of the same power $\sigma > 0$, after some algebra, we obtain from (29),

$$\begin{aligned}
d^*\gamma^\flat_{(\sigma)} &= \mathrm{J}^{-1}\mathcal{J}_{(\sigma)} + \mathrm{J}^{-1}\sum_{0<m<\sigma}\binom{\sigma}{m}\mathrm{J}_{(\sigma)}\partial_i\gamma^i_{(\sigma-m)}\\
&\quad+\frac{1}{2}\sum_{0<m<\sigma}\binom{\sigma}{m}\left(\partial_i\gamma^i_{(m)}\partial_j\gamma^j_{(\sigma-m)} - \partial_j\gamma^i_{(m)}\partial_i\gamma^j_{(\sigma-m)}\right)\\
&\quad+\frac{1}{6}\varepsilon^{i_1i_2i_3}\varepsilon_{j_1j_2j_3}\sum_{\substack{m+n+l=\sigma\\ m,n,l>0}}\binom{\sigma}{m}\binom{\sigma-m}{n}\partial_{i_1}\gamma^{j_1}_{(m)}\partial_{i_2}\gamma^{j_2}_{(n)}\partial_{i_3}\gamma^{j_3}_{(l)}\\
&\quad+\frac{1}{2}\mathrm{J}^{-1}\sum_{\substack{m+n+l=\sigma\\ m,n,l>0}}\binom{\sigma}{m}\binom{\sigma-m}{n}\left(\partial_i\gamma^i_{(m)}\partial_j\gamma^j_{(n)} - \partial_j\gamma^i_{(m)}\partial_i\gamma^j_{(n)}\right)\mathrm{J}_{(l)}\\
&\quad+\frac{1}{6}\mathrm{J}^{-1}\varepsilon^{i_1i_2i_3}\varepsilon_{j_1j_2j_3}\sum_{\substack{m+n+l+p=\sigma\\ m,n,l,p>0}}\binom{\sigma}{m}\binom{\sigma-m}{n}\binom{\sigma-m-n}{l}\\
&\quad\partial_{i_1}\gamma^{j_1}_{(m)}\partial_{i_2}\gamma^{j_2}_{(n)}\partial_{i_3}\gamma^{j_3}_{(l)}\mathrm{J}_{(p)}.
\end{aligned}\tag{30}$$

2.1.3. Recursion relations stemming from boundary conditions. Here, we derive recursion relations from the impermeability condition (12) on the boundary ∂M, which is smooth. To write the boundary condition

$$\mathrm{i}_{\nu(x)}\mathrm{u}^\flat(x) = g_{ij}(\eta_t)\dot{\eta}_t^i \nu^j(\eta_t) = 0, \tag{31}$$

as a time series, we need to expand in time the term $\nu^j(\eta_t)$, where ν is the outward pointing unit normal to the boundary ∂M. Using the Faà di Bruno formula, we obtain

$$\nu^j(\eta(t,a)) = \sum_{\sigma \geq 0} \nu^j_{(\sigma)}(a)\frac{t^\sigma}{\sigma!}, \tag{32}$$

where

$$\nu^j_{(\sigma)} = \sigma! \sum_{1\leq|\beta|\leq\sigma} \partial^\beta \nu^j(a) \sum_{q=1}^{\sigma} \sum_{P_q(\sigma,\beta)} \prod_{r=1}^{q} \frac{(\gamma^1_{(\ell_r)})^{k_r^1}}{k_r^1!(\ell_r!)^{k_r^1}} \cdots \frac{(\gamma^d_{(\ell_r)})^{k_r^d}}{k_r^d!(\ell_r!)^{k_r^d}}, \tag{33}$$

for $\sigma > 0$, and $\nu^j_{(0)} = \nu^j(a)$. Substituting formal time series (17), (23) and (32) into (31), and collecting terms of the same power $\sigma > 0$, we obtain, after some algebra,

$$\begin{aligned}\mathrm{i}_\nu \gamma^\flat_{(\sigma)} = &- \sum_{0<m<\sigma} \binom{\sigma-1}{m-1} \gamma^i_{(m)} \left(G_{ij(\sigma-m)}\nu^j + g_{ij}\nu^j_{(\sigma-m)}\right)\\ &- \sum_{\substack{m+n+l=\sigma\\ m,n,l>0}} \binom{\sigma-1}{m-1}\binom{\sigma-m}{n} \gamma^i_{(m)} G_{ij(n)}\nu^j_{(l)}.\end{aligned} \tag{34}$$

We observe that recursion relation (34) involves a finite number of terms, and holds for any finite dimension d.

2.1.4. Normalized recursion relations. To perform, in Sect. 2.2, the convergence analysis of the scheme described below (Sect. 2.1.5), it is more convenient to work with normalized recursion relations. To achieve this, we apply the following rescaling

$$\frac{\gamma^i_{(\sigma)}}{\sigma!} \longrightarrow \gamma^i_{(\sigma)}, \quad \frac{G_{ij(\sigma)}}{\sigma!} \longrightarrow G_{ij(\sigma)}, \quad \frac{\mathrm{J}_{(\sigma)}}{\sigma!} \longrightarrow \mathrm{J}_{(\sigma)}, \quad \frac{\nu^i_{(\sigma)}}{\sigma!} \longrightarrow \nu^i_{(\sigma)}. \tag{35}$$

Using rescaling (35), recursion relations of Sects. 2.1.1–2.1.3 become

$$\begin{aligned} d\gamma^\flat_{(\sigma)} = {}& \delta_{1\sigma} dv_0^\flat - \sum_{0<m<\sigma} \frac{m}{\sigma} d\big(\gamma^j_{(m)} G_{ij(\sigma-m)} da^i + g_{ij}\gamma^j_{(m)} d\gamma^i_{(\sigma-m)}\big)\\ &- \sum_{\substack{m+n+l=\sigma\\ m,n,l>0}} \frac{m}{\sigma} d\big(G_{ij(n)}\gamma^j_{(m)} d\gamma^i_{(l)}\big), \end{aligned} \tag{36}$$

$$\begin{aligned} d^*\gamma^\flat_{(\sigma)} = {}& \mathrm{J}^{-1}\mathcal{J}_{(\sigma)} + \mathrm{J}^{-1} \sum_{0<m<\sigma} \mathrm{J}_{(\sigma)}\partial_i \gamma^i_{(\sigma-m)}\\ &+ \frac{1}{2} \sum_{0<m<\sigma} \left(\partial_i\gamma^i_{(m)}\partial_j\gamma^j_{(\sigma-m)} - \partial_j\gamma^i_{(m)}\partial_i\gamma^j_{(\sigma-m)}\right)\\ &+ \frac{1}{6}\varepsilon^{i_1 i_2 i_3}\varepsilon_{j_1 j_2 j_3} \sum_{\substack{m+n+l=\sigma\\ m,n,l>0}} \partial_{i_1}\gamma^{j_1}_{(m)}\partial_{i_2}\gamma^{j_2}_{(n)}\partial_{i_3}\gamma^{j_3}_{(l)} \end{aligned}$$

$$+\frac{1}{2}\mathrm{J}^{-1}\sum_{\substack{m+n+l=\sigma\\ m,n,l>0}}\left(\partial_i\gamma^i_{(m)}\partial_j\gamma^j_{(n)}-\partial_j\gamma^i_{(m)}\partial_i\gamma^j_{(n)}\right)\mathrm{J}_{(l)}$$

$$+\frac{1}{6}\mathrm{J}^{-1}\varepsilon^{i_1i_2i_3}\varepsilon_{j_1j_2j_3}\sum_{\substack{m+n+l+p=\sigma\\ m,n,l>0}}\partial_{i_1}\gamma^{j_1}_{(m)}\partial_{i_2}\gamma^{j_2}_{(n)}\partial_{i_3}\gamma^{j_3}_{(l)}\mathrm{J}_{(p)}, \tag{37}$$

and

$$\mathrm{i}_\nu\gamma^\flat_{(\sigma)}=-\sum_{0<m<\sigma}\frac{m}{\sigma}\gamma^i_{(m)}\left(G_{ij(\sigma-m)}\nu^j+g_{ij}\nu^j_{(\sigma-m)}\right)-\sum_{\substack{m+n+l=\sigma\\ m,n,l>0}}\frac{m}{\sigma}\gamma^i_{(m)}G_{ij(n)}\nu^j_{(l)}, \tag{38}$$

where

$$G_{ij(\sigma)}=\sum_{1\le|\beta|\le\sigma}\partial^\beta g_{ij}(a)\sum_{q=1}^{\sigma}\sum_{P_q(\sigma,\beta)}\prod_{r=1}^{q}\frac{(\gamma^1_{(\ell_r)})^{k^1_r}}{k^1_r!}\cdots\frac{(\gamma^d_{(\ell_r)})^{k^d_r}}{k^d_r!}. \tag{39}$$

The term $\mathrm{J}_{(\sigma)}$ (resp. $\nu^j_{(\sigma)}$) is defined accordingly, i.e., by substituting $\mathrm{J}_{(\sigma)}$ (resp. $\nu^j_{(\sigma)}$) to $G_{ij(\sigma)}$ in (39) and by substituting J (resp. ν^j) to g_{ij} in (39).

2.1.5. Normal Hodge decomposition and Neumann boundary value problems. Here, from the normalized recursion relations (36)–(39), we design a scheme to compute the time-Taylor coefficients $\gamma^\flat_{(\sigma)}$. The heart of the method consists in using a normal Hodge decomposition (see "Appendix B") for the time-Taylor coefficients. Indeed, data (36)–(38), which give the exterior derivative, the coderivative and the normal trace of $\gamma^\flat_{(\sigma)}$, allow us to consider the normal Hodge decomposition problem for 1-forms $\gamma^\flat_{(\sigma)}$ [50]. The normal Hodge decomposition leads to the resolution of two Neumann boundary value problems for differential forms in which the boundary conditions (38) must be incorporated. Assuming $\gamma^\flat_{(\sigma)}\in H^s\Omega^1(M)$ and using Theorem 3 (See "Appendix B"), we have, for all $\sigma>0$, the decomposition

$$\gamma^\flat_{(\sigma)}=d\tilde{\phi}_{(\sigma)}+d\hat{\phi}_{(\sigma)}+d^*\Phi_{(\sigma)}+h_{(\sigma)}, \tag{40}$$

where $d\tilde{\phi}_{(\sigma)}\in H^s\mathcal{E}^1(M)$, $d\hat{\phi}_{(\sigma)}\in H^s\mathcal{H}^1_{\mathrm{ex}}(M)$, $d^*\Phi_{(\sigma)}\in H^s\mathcal{C}^1(M)$, and $h_{(\sigma)}\in\mathcal{H}^1_N(M)$. Setting $\phi_{(\sigma)}=\tilde{\phi}_{(\sigma)}+\hat{\phi}_{(\sigma)}$, decomposition (40) becomes

$$\gamma^\flat_{(\sigma)}=d\phi_{(\sigma)}+d^*\Phi_{(\sigma)}+h_{(\sigma)}. \tag{41}$$

The aim is now to determine the potential forms $\phi_{(\sigma)}$ (0-form) and $\Phi_{(\sigma)}$ (2-form), and the harmonic 1-form $h_{(\sigma)}$. Let us start with the 0-form $\phi_{(\sigma)}$. Applying the exterior coderivative operator to (41) and using the property $d^*\phi_{(\sigma)}=0$, we obtain $\Delta\phi_{(\sigma)}=d^*\gamma^\flat_{(\sigma)}$ on M. Applying the trace operator $\mathbf{n}$ to (41) and using the commutation property $\mathbf{n}d^*=d^*\mathbf{n}$ (see Proposition 1.2.6 of [50]) we obtain $\mathbf{n}d^*\Phi_{(\sigma)}=d^*\mathbf{n}\Phi_{(\sigma)}=0$ on ∂M, because $\Phi_{(\sigma)}\in H^{s+1}\Omega^2_N(M)$. Moreover $\mathbf{n}h_{(\sigma)}=0$ on ∂M (since $h_{(\sigma)}\in\mathcal{H}^1_N(M)$), hence $\mathbf{n}d\phi_{(\sigma)}=\mathbf{n}\gamma^\flat_{(\sigma)}$ on ∂M, where the right-hand side of this equality is given by the right-hand side of (38). Therefore, the potential 0-form $\phi_{(\sigma)}$ satisfies the following non-homogeneous Neumann boundary value problem,

$$\Delta\phi_{(\sigma)} = d^*\gamma^\flat_{(\sigma)} \text{ on } M, \qquad \mathbf{n}d\phi_{(\sigma)} = \mathbf{n}\gamma^\flat_{(\sigma)} = \mathrm{i}_\nu\gamma^\flat_{(\sigma)} \text{ and } \mathbf{n}\phi_{(\sigma)} = 0 \text{ on } \partial M, \tag{42}$$

where it is understood that the right-hand sides of (42) are taken from (37) and (38), and thus involve only coefficients of order less or equal to $\sigma - 1$. From Corollary 3.4.8 of [50], the problem (42) is solvable if, and only if, the data obey the following integrability condition,

$$\langle\langle d^*\gamma^\flat_{(\sigma)}, f\rangle\rangle_{L^2(M)} = -\int_{\partial M} \mathbf{t}f \wedge \star\mathbf{n}\gamma^\flat_{(\sigma)}, \quad \forall f \in \mathcal{H}^0_N(M).$$

This condition is satisfied since using the Green formula of Proposition 2.1.2 of [50], we obtain

$$\langle\langle d^*\gamma^\flat_{(\sigma)}, f\rangle\rangle_{L^2(M)} = \langle\langle\gamma^\flat_{(\sigma)}, df\rangle\rangle_{L^2(M)} - \int_{\partial M} \mathbf{t}f \wedge \star\mathbf{n}\gamma^\flat_{(\sigma)} = -\int_{\partial M} \mathbf{t}f \wedge \star\mathbf{n}\gamma^\flat_{(\sigma)},$$
$$\forall f \in \mathcal{H}^0_N(M).$$

Then, the resolution of (42) gives a unique solution $\phi_{(\sigma)}$ up to a Neumann field $\kappa_{(\sigma)} \in \mathcal{H}^0_N(M)$ (see Corollary 3.4.8 of [50]).

We now deal with the potential 2-form $\Phi_{(\sigma)}$. From Lemma 2.4.7 of [50], we observe that it is possible to modify $\Phi_{(\sigma)}$ by adding to it a gauge $\mathscr{G}_{(\sigma)} \in \mathcal{G}^2_{\mathcal{C}}(M) := \{\lambda \in H^{s+1}\Omega^2_N(M) \mid d^*\lambda = 0\}$. In particular we can choose $\mathscr{G}_{(\sigma)}$ such that $\Phi_{(\sigma)}$ is minimal in the sense that $\langle\langle\Phi_{(\sigma)}, \lambda\rangle\rangle_{L^2} = 0$, $\forall\lambda \in \mathcal{G}^2_{\mathcal{C}}(M)$. This implies that $d\Phi_{(\sigma)} = 0$ on M. Then, we naturally impose $\mathbf{n}d\Phi_{(\sigma)} = 0$ on ∂M. Applying the exterior derivative operator to (41) and using the property $d\Phi_{(\sigma)} = 0$, we obtain $\Delta\Phi_{(\sigma)} = d\gamma^\flat_{(\sigma)}$ on M. Since $d^*\Phi_{(\sigma)} \in H^s\mathcal{C}^1(M)$, we obtain $\mathbf{n}\Phi_{(\sigma)} = 0$ on ∂M, which implies $\mathbf{n}d^*\Phi_{(\sigma)} = 0$ on ∂M, by using the commutation property $\mathbf{n}d^* = d^*\mathbf{n}$. Therefore, the potential 2-form $\Phi_{(\sigma)}$ satisfies the following non-homogeneous Neumann boundary value problem,

$$\Delta\Phi_{(\sigma)} = d\gamma^\flat_{(\sigma)} \text{ on } M, \qquad \mathbf{n}d\Phi_{(\sigma)} = 0 \text{ and } \mathbf{n}\Phi_{(\sigma)} = 0 \text{ on } \partial M, \tag{43}$$

where the right-hand side of (43) is given by (36). From Corollary 3.4.8 of [50], the problem (43) is solvable if, and only if, the data obey the following integrability condition,

$$\langle\langle d\gamma^\flat_{(\sigma)}, \lambda\rangle\rangle_{L^2(M)} = 0, \quad \forall\lambda \in \mathcal{H}^2_N(M).$$

This condition is satisfied since using the Green formula of Proposition 2.1.2 of [50], we obtain

$$\langle\langle d\gamma^\flat_{(\sigma)}, \lambda\rangle\rangle_{L^2(M)} = \langle\langle\gamma^\flat_{(\sigma)}, d^*\lambda\rangle\rangle_{L^2(M)} + \int_{\partial M} \mathbf{t}\gamma^\flat_{(\sigma)} \wedge \star\mathbf{n}\lambda = 0, \quad \forall\lambda \in \mathcal{H}^2_N(M).$$

Therefore, the resolution of (43) gives a unique solution $\Phi_{(\sigma)}$ up to a Neumann field $\mathcal{K}_{(\sigma)} \in \mathcal{H}^2_N(M)$ (see Corollary 3.4.8 of [50]).

Before dealing with the harmonic 1-form $h_{(\sigma)}$, we make some remarks about boundary value problems (42) and (43).

Remark 4 (Non-uniqueness). We have seen that solutions to (42) and (43) are unique up to some Neumann fields. As far as it concerns the proof of the regularity property of Theorem 2, this non-uniqueness default has no consequence since a priori estimates of Sect. 2.2 only involve $d\phi_{(\sigma)}$ and $d^*\Phi_{(\sigma)}$. Moreover, from Theorem 2.6.1 of [50], we have the Hodge isomorphism

$$\mathcal{H}_N^k(M) = \mathrm{H}^k(M, d) := \mathrm{Ker}\, d_{|_{\Omega^k(M)}} / \mathrm{Im}\, d_{|_{\Omega^{k-1}(M)}},$$

where $\mathrm{H}^k(M, d)$ is the k-th De Rham cohomology. Then elements of $\mathcal{H}_N^k(M)$ are purely of geometrical nature and depends only on the topological properties of M. Moreover, from Theorem 22 of [18], the space $\mathcal{H}_N^k(M)$ is finite dimensional, with a dimension given by the Betti number $b_k(M) := \dim \mathrm{H}^k(M, d)$. From Theorem 26 of [18], elements of $\mathcal{H}_N^k(M)$ are analytic (resp. ultradifferentiable) in space if M is analytic (resp. ultradifferentiable). Therefore, the Neumann fields $\kappa_{(\sigma)} \in \mathcal{H}_N^0(M)$ and $\mathcal{K}_{(\sigma)} \in \mathcal{H}_N^2(M)$ belong to the LSL–FdB ultradifferentiable class. Hereafter, we give some basic examples of determination of $\mathcal{H}_N^0(M)$ and $\mathcal{H}_N^2(M)$; more examples can be found in [1,22,50]. If M is compact and (path-)connected (i.e., any two points of M can be connected by a piecewise smooth curve), then $\mathrm{H}^0(M, d) = \mathbb{R}$ and $b_0(M) = 1$. If M is compact but not connected, i.e., it consists of k connected pieces, then $\mathrm{H}^0(M, d) = \mathbb{R}^k$ and $b_0(M) = k$. From the Poincaré lemma (see, e.g., Lemma 6.4.18 of [1]), if M is a compact contractible[1] manifold, then all the Betti numbers $b_k(M)$ with $k \geq 1$ vanish (in particular $b_2(M) = 0$) and $b_0(M) = 1$.

Remark 5 (Other boundary conditions). In the setting of boundary value problem (43), we observe the following. Using the commutation property $\mathbf{t}d = d\mathbf{t}$ (see Proposition 1.2.6 of [50]) and $d\Phi_{(\sigma)} = 0$, it is also consistent to impose $\mathbf{t}\Phi_{(\sigma)} = 0$ on ∂M. This condition is a particular solution of the more general solution to $d(\mathbf{t}\Phi_{(\sigma)}) = 0$ on ∂M, which is given by $\mathbf{t}\Phi_{(\sigma)} = \mathbf{t}\mathscr{G}^2_{(\sigma)}$ on ∂M, with $\mathscr{G}^2_{(\sigma)} \in \mathcal{G}^2(M) := \{\lambda \in H^{s+1}\Omega^2(M) \mid d\lambda = 0\}$. In particular, we can take $\mathscr{G}^2_{(\sigma)} = d\mathscr{G}^1_{(\sigma)}$, with $\mathscr{G}^1_{(\sigma)}$ an arbitrary 1-form in $H^{s+2}\Omega^1(M)$. The condition $\mathbf{t}\Phi_{(\sigma)} = 0$ and $d^*\Phi_{(\sigma)} \in H^s\mathcal{C}^1(M)$, lead to $\Phi_{(\sigma)} = 0$ on ∂M, which is the boundary condition used in [8] to solve the Dirichlet boundary value problem satisfied by the Helmholtz–Hodge vector potential. Therefore, this Dirichlet problem and its associated Helmholtz–Hodge decomposition are less general than the normal Hodge decomposition we present here. Since the Helmholtz–Hodge decomposition of [8] is more restrictive, it can not cover all physical cases.

Remark 6 (Case of a bounded and simply-connected domain D of $\mathbb{R}^3$). Here, we rewrite boundary value problems (42) and (43) for a bounded and simply-connected domain D of $\mathbb{R}^3$. The problem (42) reduces exactly to the same Neumann boundary value problem than the one established in [8] (see equation (30) of [8]). The corresponding Neumann field $\kappa_{(\sigma)}$ is an arbitrary constant.

We next deal with the problem (43). For this, we use standard notation for vectors of $\mathbb{R}^3$, i.e., e.g., $\vec{\Phi}_{(\sigma)}$. The vector $\vec{\Phi}_{(\sigma)}$ is associated to the 2-form $\Phi_{(\sigma)}$, while the vector $\vec{\gamma}_{(\sigma)}$ is associated to the 1-form $\gamma^\flat_{(\sigma)}$ and corresponds to the Taylor coefficients of time

[1] A manifold M is contractible if there exists a vector field X on M which generates a flow $\eta_t : M \to M$, with $t \in [0, 1]$, that gradually and smoothly shrinks the whole manifold M to the point a, i.e., $\eta_0 = \mathrm{Id}_M$ and $\eta_1(x) = a$, $\forall x \in M$, where the point a is fixed and independent of x. For more details see, e.g., Section 1.6 of [1].

series expansion (17). Equation $\Delta\Phi_{(\sigma)} = d\gamma^{\flat}_{(\sigma)}$ on D, rewrites as

$$\Delta\vec{\Phi}_{(\sigma)} = -\nabla \times \vec{\gamma}_{(\sigma)}, \quad \text{on} \;\; D. \tag{44}$$

This equation was established in [8] (see equation (23) of [8]). The gauge (resp. boundary) condition $d\Phi_{(\sigma)} = 0$ on D (resp. $\mathbf{n}d\Phi_{(\sigma)} = 0$ on ∂D), rewrites as $\nabla \cdot \vec{\Phi}_{(\sigma)} = 0$ on D (resp. $\nabla \cdot \vec{\Phi}_{(\sigma)} = 0$ on ∂D), which implies

$$\nabla \cdot \vec{\Phi}_{(\sigma)} = 0, \quad \text{on} \;\; \bar{D}. \tag{45}$$

The boundary condition $\mathbf{n}\Phi_{(\sigma)} = 0$ on ∂D, rewrites as

$$\vec{\Phi}_{(\sigma)} \times \vec{\nu} = 0. \tag{46}$$

Indeed, using the definition of the normal trace operator $\mathbf{n}$ (see "Appendix A"), we obtain for all $X, Y \in \Gamma(TD_{|\partial D}) = \mathbb{R}^3$,

$$\begin{aligned} 0 &= \mathbf{n}\Phi_{(\sigma)}(X, Y) \\ &= \Phi_{(\sigma)}(X, Y) - \mathbf{t}\Phi_{(\sigma)}(X, Y) \\ &= \Phi_{(\sigma)}(X, Y) - \Phi_{(\sigma)}(X^{\|}, Y^{\|}) \\ &= \Phi_{(\sigma)}(X^{\perp}, Y^{\|}) + \Phi_{(\sigma)}(X^{\|}, Y^{\perp}) + \Phi_{(\sigma)}(X^{\perp}, Y^{\perp}). \end{aligned} \tag{47}$$

In terms of vectors of $\mathbb{R}^3$, the term $\Phi_{(\sigma)}(X, Y)$ rewrites as the triple-product $\vec{X} \cdot \vec{\Phi}_{(\sigma)} \times \vec{Y}$. Without loss of generality we can choose $X^{\perp} = Y^{\perp} = \nu$, and then (47) rewrites as

$$\vec{\nu} \times \vec{\Phi}_{(\sigma)} \, (Y^{\|} - X^{\|}) = 0, \quad \forall X^{\|}, Y^{\|} \in \mathbb{R},$$

which is equivalent to $\vec{\Phi}_{(\sigma)} \times \vec{\nu} = 0$. Moreover the following implications,

$$\mathbf{n}\Phi_{(\sigma)} = 0 \implies d^*\mathbf{n}\Phi_{(\sigma)} = 0 \implies \mathbf{n}d^*\Phi_{(\sigma)} = 0,$$

rewrite as

$$\vec{\Phi}_{(\sigma)} \times \vec{\nu} = 0 \implies \nabla \cdot (\vec{\Phi}_{(\sigma)} \times \vec{\nu}) = 0 \implies \vec{\nu} \cdot \nabla \times \vec{\Phi}_{(\sigma)} = 0,$$

since $\nabla \times \vec{\nu} = 0$. Using the Stokes' theorem,

$$\int_D d\lambda = \int_{\partial D} \jmath^*\lambda, \quad \forall \lambda \in H^1\Omega^2(D),$$

the gauge condition, $d\Phi_{(\sigma)} = 0$ on D, implies

$$\int_{\partial D} \jmath^*\Phi_{(\sigma)} = 0,$$

which rewrites as

$$\int_{\partial D} \vec{\Phi}_{(\sigma)} \cdot \vec{\nu} ds = 0.$$

Here, ds is the elementary measure of a two-dimensional surface. Finally, since the simply-connected domain D is contractible, the Poincaré lemma shows that the 2-form $\mathcal{K}_{(\sigma)}$ vanishes. Therefore, the boundary value problem (44)–(46) has a unique solution (see, e.g., [3,26]). We note that this boundary value problem differs from the one used in [8].

Finally we deal with the harmonic 1-form $h_{(\sigma)}$. On one hand, setting

$$\gamma^\flat(t,a) := \sum_{\sigma\geq 0}\big(\gamma_{i(\sigma)}(a)da^i\big)t^\sigma, \tag{48}$$

and using Theorem 3, we obtain the Hodge decomposition

$$\gamma^\flat(t,a) = d\phi_\gamma(t,a) + d^*\Phi_\gamma(t,a) + h_\gamma(t,a), \tag{49}$$

with $\phi_\gamma = \tilde{\phi}_\gamma + \hat{\phi}_\gamma$, and where $d\tilde{\phi}_\gamma \in H^s\mathcal{E}^1(M)$, $d\hat{\phi}_\gamma \in H^s\mathcal{H}^1_{\text{ex}}(M)$, $d^*\Phi_\gamma \in H^s\mathcal{C}^1(M)$, and $h_\gamma \in \mathcal{H}^1_N(M)$. From Theorem 2.6.1 of [50], we have the Hodge isomorphism $\mathcal{H}^1_N(M) = \mathrm{H}^1(M,d) := \mathrm{Ker}\, d_{|_{\Omega^1(M)}} / \mathrm{Im}\, d_{|_{\Omega^0(M)}}$, where $\mathrm{H}^1(M,d)$ is the 1-th De Rham cohomology. Then elements of $\mathcal{H}^1_N(M)$ are independent of time, but depend only on the topology of M. Moreover, using Theorem 22 and Theorem 26 of [18], the space $\mathcal{H}^1_N(M)$ is finite dimensional and its elements are analytic (resp. ultradifferentiable) in space if M is analytic (resp. ultradifferentiable). Therefore, $h_\gamma = h_\gamma(a)$ is independent of time and belongs to the LSL–FdB ultradifferentiable class. On the other hand from uniqueness and linearity of Hodge decompositions (41) and (49), and using (48), we obtain

$$\phi_\gamma = \sum_{\sigma\geq 0}\phi_{(\sigma)}t^\sigma, \quad \Phi_\gamma = \sum_{\sigma\geq 0}\Phi_{(\sigma)}t^\sigma, \quad h_\gamma = h_{(0)}, \quad \text{and} \quad h_{(\sigma)} = 0,\ \forall\sigma > 0.$$

Then, Hodge decomposition (41) becomes

$$\gamma^\flat_{(\sigma)} = d\phi_{(\sigma)} + d^*\Phi_{(\sigma)} + h_0\delta_{\sigma 0}, \tag{50}$$

where we set $h_0 := h_{(0)}$. It is straightforward to solve the Hodge decomposition (50) at the order $\sigma = 0$. Since $\phi_{(0)}$ satisfies $\Delta\phi_{(0)} = 0$ on M, with the boundary conditions $\mathbf{n}d\phi_{(0)} = \mathbf{n}\phi_{(0)} = 0$ on ∂M, we obtain $\phi_{(0)} = 0$ (up to smooth Neumann fields $\kappa_{(0)} \in \mathcal{H}^0_N(M)$). Since $\Phi_{(0)}$ satisfies $\Delta\Phi_{(0)} = 0$ on M, with the boundary conditions $\mathbf{n}\Phi_{(0)} = \mathbf{n}d\Phi_{(0)} = 0$ on ∂M, we obtain $\Phi_{(0)} = 0$ (up to smooth Neumann fields $\mathcal{K}_{(0)} \in \mathcal{H}^2_N(M)$). Therefore, we obtain

$$\gamma^\flat_{(0)} = h_0. \tag{51}$$

Since $h_0 \in \mathcal{H}^1_N(M)$, the harmonic field h_0 belongs to the (finite dimensional) kernel of the Laplace–De-Rham operator with homogeneous normal boundary conditions, i.e., $\Delta h_0 = 0$ on M, and $\mathbf{n}h_0 = 0$ on ∂M. It is also straightforward to solve the Hodge decomposition (50) at the order $\sigma = 1$. Indeed, from (36) we obtain $d(\gamma^\flat_{(1)} - v^\flat_0) = 0$, which together with decomposition (50), imply that there exists a 0-form f such that $df = \gamma^\flat_{(1)} - v^\flat_0$. Then, from (37) we obtain $d^*\gamma^\flat_{(1)} = 0$, which together with the initial incompressibility condition $d^*v^\flat_0 = 0$, imply $\Delta f = d^*(\gamma^\flat_{(1)} - v^\flat_0) = 0$. Adding the consistent homogeneous Neumann boundary conditions $\mathbf{n}df = \mathrm{i}_\nu\gamma^\flat_{(1)} - \mathrm{i}_\nu v^\flat_0 = 0$, and $\mathbf{n}f = 0$ on ∂M, to the homogeneous Laplace–De-Rham (also called Laplace–Beltrami for 0-form) equation $\Delta f = 0$ on M, we obtain $f = 0$ in M (up to smooth Neumann fields $\kappa \in \mathcal{H}^0_N(M)$) i.e.,

$$\gamma^\flat_{(1)} = v^\flat_0. \tag{52}$$

Differential forms (51)–(52) initiate the recursive algorithm for computing the time-Taylor coefficients $\{\gamma^\flat_{(\sigma)}\}_{\sigma>0}$ of the 1-form $\gamma^\flat_t$.

Remark 7. In the case of a compact Riemannian ∂-manifold, particular topologies of M lead to $\mathcal{H}_N^1 = \{\emptyset\}$. Some examples are given in Theorem 2.6.4 of [50]. In particular, if M is compact and simply-connected (i.e., path-connected and every path between two points can be continuously transformed, staying on M, into any other such path while preserving the two endpoints in question; in other words M is connected and every loop in M is contractible to a point), then $\mathrm{H}^1(M, d) = \{\emptyset\}$ (hence $\mathcal{H}_N^1 = \{\emptyset\}$) and $b_1(M) = 0$. In the case of a compact Riemannian without boundary, the celebrated result of Bochner [9] says that the first cohomology $\mathrm{H}^1(M, d)$ (and thus $\mathcal{H}_N^1$) is empty if the Ricci tensor of (M, g) is positive definite. More details can be found in Section 2.6 of [50] and references therein.

2.2. Convergence analysis. Here, we prove that the time-series expansion of the 1-form $\gamma_t^\flat$, which is defined by

$$\gamma^\flat(t, a) := \sum_{\sigma \geq 0} \left(\gamma_{i(\sigma)}(a) da^i\right) t^\sigma, \tag{53}$$

are time-ultradifferentiable in the log-superlinear Faà di Bruno class $\mathcal{C}(\mathcal{M})(]-T, T[;\, H^s(M))$ (see Definition 1). As a by-product, we obtain the convergence of time-series expansions (17) and (53), which were explicitly constructed in Sect. 2.1. To simplify the notation, the norm $\|\cdot\|_{H^s\Omega^k(M)}$ will be denoted by $\|\cdot\|_{H^s(M)}$ and sometimes even shorter by $\|\cdot\|_{H^s}$. The 1-form $\gamma_t^\flat$ belongs to the space $\mathcal{C}(\mathcal{M})(]-T, T[;\, H^s(M))$ if, and only if, there exists a real positive number ρ such that the set

$$\left\{\frac{\|\partial_t^\sigma \gamma_t^\flat\|_{H^s}}{\rho^\sigma \sigma! M_\sigma}, \quad \sigma \in \mathbb{N}, \quad t \in]-T, T[\right\}, \tag{54}$$

is bounded. This will be the case if the generatrice function $t \mapsto \zeta(t)$, defined by

$$\zeta(t) = \sum_{\sigma > 0} \|\gamma_{(\sigma)}^\flat\|_{H^s} \rho^{-\sigma} M_\sigma^{-1} t^\sigma, \tag{55}$$

is uniformly bounded on $]-T, T[$. In the sum (55) we have excluded the term of order $\sigma = 0$, i.e., $\gamma_{(0)}^\flat = h_0$, because the harmonic field h_0 belongs the LSL–FdB ultradifferentiable class (see Sect. 2.1.5); hence bounded in any Sobolev norm. To derive a priori estimates we will often use

Lemma 1. *Let f be a LSL–FdB ultradifferentiable 0-form on M. Then, there exist positive constants C and R, which depend on f, (M, g), s, M_0, and C_{D}, such that*

$$\|\partial^\alpha f\|_{H^s} \leq C R^{-|\alpha|} |\alpha|! M_{|\alpha|}, \quad |\alpha| \geq 0. \tag{56}$$

Proof. Since f is a LSL–FdB ultradifferentiable function on M, for all compact set K in M, there exist constants $C_{f,K,g}$ and $R_{f,K,g}$ such that for all $a \in K$, we have

$$|\partial^\alpha f(a)| \leq C_{f,K,g} R_{f,K,g}^{-|\alpha|} M_{|\alpha|}, \quad |\alpha| \geq 0. \tag{57}$$

Using Stirling formula, we can show that there exists a constant C_s ($\sim \sqrt{1+s}\exp(2s(1+s))$) such that $(|\alpha| + |\beta|)! \leq C_s |\alpha|! 2^{|\alpha|}$, for $|\beta| \leq s$. Moreover using the property

i) of Definition 1, we obtain the bound $M_{|\beta|} \le M_s \le C_{C_\mathrm{D},s,M_0} := C_\mathrm{D}^{s(s-1)/2} M_0$. Therefore, using the log-superlinearity property ii) of Definition 1 and (57), we obtain

$$\begin{aligned}
\|\partial^\alpha f\|_{H^s} &\le \sum_{0\le|\beta|\le s} \|\partial^{\alpha+\beta} f\|_{L^2} \le \sum_{0\le|\beta|\le s} C_{f,M,g} R_{f,M,g}^{-|\alpha+\beta|} |\alpha+\beta|! M_{|\alpha+\beta|} \\
&\le \sum_{0\le|\beta|\le s} C_{f,M,g} R_{f,M,g}^{-|\alpha|-|\beta|} (|\alpha|+|\beta|)! M_0 M_{|\alpha|} M_{|\beta|} \\
&\le \sum_{0\le|\beta|\le s} C_s C_{f,M,g} C_{C_\mathrm{D},s,M_0} M_0 \max(R_{f,M,g}^{-s}, 1) \left(\frac{R_{f,M,g}}{2}\right)^{-|\alpha|} |\alpha|! M_{|\alpha|} \\
&\le \mathcal{C}_{f,M,g,s,C_\mathrm{D},M_0} \mathcal{R}_{f,M,g}^{-|\alpha|} |\alpha|! M_{|\alpha|}.
\end{aligned}$$

□

We will also use frequently some basic estimates summarized in the following

Lemma 2. *Let $s > d/2 + 1$. The mappings $(\cdot)^\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ and $(\cdot)^\flat : \mathfrak{X}(M) \to \Omega^1(M)$ are continuous mappings in $H^{s-1}(M)$ and we have*

$$\|\gamma_{(\sigma)}^\flat\|_{H^{s-1}} \le C_\flat \|\gamma_{(\sigma)}\|_{H^{s-1}}, \quad \textit{and} \quad \|\gamma_{(\sigma)}\|_{H^{s-1}} \le C_\sharp \|\gamma_{(\sigma)}^\flat\|_{H^{s-1}}, \tag{58}$$

where $C_\flat = \|g\|_{H^{s-1}}$ and $C_\sharp = \|g^{-1}\|_{H^{s-1}}$.

Proof. The proof is obvious since $H^{s-1}(M)$ is an algebra for $s > d/2 + 1$. □

We now derive the a priori estimates. For this, we will use elliptic regularity estimates in Sobolev spaces for the non-homogeneous Neumann boundary value problem. Indeed, following the proof of Lemma 3.4.7 of [50] and using Lemma 3.3.2 of [50] (see also Corollary 3.4.8 of [50]), we have the elliptic estimates

$$\|\Phi_{(\sigma)}\|_{H^{s+2}(M)} \le C_\mathrm{N}^1 \|d\gamma_{(\sigma)}^\flat\|_{H^s(M)}, \tag{59}$$

for the non-homogeneous Neumann problem (43), and,

$$\|\phi_{(\sigma)}\|_{H^{s+2}(M)} \le C_\mathrm{N}^2 \left(\|d^*\gamma_{(\sigma)}^\flat\|_{H^s(M)} + \|\mathbf{n}\gamma_{(\sigma)}^\flat\|_{H^{s+1/2}(\partial M)}\right), \tag{60}$$

for the non-homogeneous Neumann problem (42). Using Hodge decomposition (50), continuous mappings (91) of "Appendix A", and elliptic estimates (59)–(60), we obtain, for $\sigma > 0$,

$$\begin{aligned}
\|\gamma_{(\sigma)}^\flat\|_{H^s(M)} &\le \|d\phi_{(\sigma)}\|_{H^s(M)} + \|d^*\Phi_{(\sigma)}\|_{H^s(M)} \\
&\le C_\mathrm{cm} \left(\|\phi_{(\sigma)}\|_{H^{s+1}(M)} + \|\Phi_{(\sigma)}\|_{H^{s+1}(M)}\right) \\
&\le C_\mathrm{N} \left(\|d\gamma_{(\sigma)}^\flat\|_{H^{s-1}(M)} + \|d^*\gamma_{(\sigma)}^\flat\|_{H^{s-1}(M)} + \|\mathbf{n}\gamma_{(\sigma)}^\flat\|_{H^{s-1/2}(\partial M)}\right),
\end{aligned} \tag{61}$$

where $C_\mathrm{N} := C_\mathrm{cm} \max\{C_\mathrm{N}^1, C_\mathrm{N}^2\}$ and C_cm is the constant of the continuous mappings (91). Multiplying (61) by $\rho^{-\sigma} M_\sigma^{-1} t^\sigma$ and summing over index σ, we obtain

$$\zeta(t) \le C_\mathrm{N}\Bigg(\sum_{\sigma>0} \|d\gamma_{(\sigma)}^\flat\|_{H^{s-1}(M)} \rho^{-\sigma} M_\sigma^{-1} t^\sigma$$

$$+\sum_{\sigma>0}\|d^*\gamma^\flat_{(\sigma)}\|_{H^{s-1}(M)}\rho^{-\sigma}M_\sigma^{-1}t^\sigma+\sum_{\sigma>0}\|\mathbf{n}\gamma^\flat_{(\sigma)}\|_{H^{s-1/2}(\partial M)}\rho^{-\sigma}M_\sigma^{-1}t^\sigma\Bigg). \tag{62}$$

We then must estimate the right-hand side of (62). Before this, we must write precisely the regularity assumptions for the manifold M and its boundary ∂M. Since (M, g) and its boundary ∂M are LSL–FdB ultradifferentiable manifolds then, using Lemma 1, there exist positive real constants C_g, C_{J}, C_ν, R_g, R_{J}, and R_ν such that, for $0 \le s < \infty$, and $|\alpha| \ge 0$,

$$\begin{aligned}&\|\partial^\alpha g\|_{H^s(M)} \le C_g R_g^{-|\alpha|}|\alpha|!M_{|\alpha|},\ \ \|\partial^\alpha \mathrm{J}\|_{H^s(M)} \le C_{\mathrm{J}} R_{\mathrm{J}}^{-|\alpha|}|\alpha|!M_{|\alpha|},\\ &\|\partial^\alpha \nu\|_{H^s(\partial M)} \le C_\nu R_\nu^{-|\alpha|}|\alpha|!M_{|\alpha|},\end{aligned} \tag{63}$$

where the sequence $\{M_\sigma\}_{\sigma\ge 0}$ satisfies Definition 1. An estimate of the right-hand side of (62) is given by

Proposition 1. *Let $s > d/2+1$. Then there exist positive real constants $C_d = C_d(C_a, C_g, C_\sharp, M_0, d)$, $C_{d^*} = C_{d^*}(C_a, C_{\mathrm{J}}, C_\sharp, M_0, d, K_{\mathrm{J}^{-1}})$ and $C_{\mathbf{n}} = C_{\mathbf{n}}(C_a, C_\nu, C_\sharp, C_g, C_\partial, M_0, d, K_{\mathrm{J}^{-1}})$ such that*

$$\begin{aligned}&\sum_{\sigma>0}\|d\gamma^\flat_{(\sigma)}\|_{H^{s-1}(M)}\rho^{-\sigma}M_\sigma^{-1}t^\sigma\\ &\quad\le \|v_0^\flat\|_{H^s}\rho^{-1}M_1^{-1}t + C_d\left\{\zeta^2(t)+\zeta(t)(1+\zeta(t))\big(1-K_g^{-1}\zeta(t)\big)^{-1}\right\},\end{aligned} \tag{64}$$

$$\begin{aligned}&\sum_{\sigma>0}\|d^*\gamma^\flat_{(\sigma)}\|_{H^{s-1}(M)}\rho^{-\sigma}M_\sigma^{-1}t^\sigma\\ &\quad\le C_{d^*}\left\{\big(1-K_{\mathrm{J}}^{-1}\zeta(t)\big)^{-1}\left[1+\zeta(t)+\zeta^2(t)+\zeta^3(t)\right]+\zeta^2(t)+\zeta^3(t)\right\},\end{aligned} \tag{65}$$

and

$$\begin{aligned}&\sum_{\sigma>0}\|\mathbf{n}\gamma^\flat_{(\sigma)}\|_{H^{s-1/2}(\partial M)}\rho^{-\sigma}M_\sigma^{-1}t^\sigma\\ &\quad\le C_{\mathbf{n}}\zeta(t)\big(1-K_g^{-1}\zeta(t)\big)^{-1}\big(1-K_\nu^{-1}\zeta(t)\big)^{-1}\left\{1+\big(1-K_g^{-1}\zeta(t)\big)+\big(1-K_\nu^{-1}\zeta(t)\big)\right\},\end{aligned} \tag{66}$$

where

$$K_g^{-1} := C_a C_\sharp/R_g,\quad K_{\mathrm{J}}^{-1} := C_a C_\sharp/R_{\mathrm{J}},\quad K_\nu^{-1} := C_a C_\sharp C_\partial/R_\nu,\quad \textit{and}\quad K_{\mathrm{J}^{-1}} := \|\mathrm{J}^{-1}\|_{H^s}. \tag{67}$$

Proof. We start with the proof of (64). Using estimate (90) of "Appendix A", Lemma 2, and the superlinearity property (9), we obtain from (36),

$$\begin{aligned}
\|d\gamma^{\flat}_{(\sigma)}\|_{H^{s-1}}\rho^{-\sigma}M_\sigma^{-1} &\le \|v_0^{\flat}\|_{H^s}\rho^{-1}M_1^{-1}\\
&+C_aC_\sharp\sum_{0<m<\sigma}\left(\frac{\|\gamma^{\flat}_{(m)}\|_{H^s}}{\rho^m M_m}\sum_{i,j}\frac{\|G^{\flat}_{ij(\sigma-m)}\|_{H^s}}{\rho^{\sigma-m}M_{\sigma-m}}+\frac{\|\gamma^{\flat}_{(m)}\|_{H^s}}{\rho^m M_m}\frac{\|\gamma^{\flat}_{(\sigma-m)}\|_{H^s}}{\rho^{\sigma-m}M_{\sigma-m}}\right)\\
&+C_a^2C_\sharp^2\sum_{\substack{m+n+l=\sigma\\ m,n,l>0}}\sum_{i,j}\frac{\|\gamma^{\flat}_{(m)}\|_{H^s}}{\rho^m M_m}\frac{\|\gamma^{\flat}_{(l)}\|_{H^s}}{\rho^l M_l}\frac{\|G^{\flat}_{ij(n)}\|_{H^s}}{\rho^n M_n}. \qquad (68)
\end{aligned}$$

We have to control $\|G^{\flat}_{ij(\sigma)}\|_{H^s}$. Using (90), Lemma 1 and 2, and (63), we obtain from (39),

$$\begin{aligned}
\|G_{ij(\sigma)}\|_{H^s} &\le C_a\sum_{1\le|\beta|\le\sigma}\|\partial^\beta g_{ij}\|_{H^s}C_a^{|\beta|}\sum_{q=1}^{\sigma}\sum_{P_q(\sigma,\beta)}\prod_{j=1}^{q}\frac{\|\gamma^1_{(\ell_r)}\|_{H^s}^{k_r^1}}{k_r^1!}\cdots\frac{\|\gamma^d_{(\ell_r)}\|_{H^s}^{k_r^d}}{k_r^d!}\\
&\le C_aC_g\rho^\sigma\sum_{1\le|\beta|\le\sigma}\left(\frac{C_aC_\sharp}{R_g}\right)^{|\beta|}|\beta|!\sum_{q=1}^{\sigma}\sum_{P_q(\sigma,\beta)}M_{|\beta|}M_{\ell_1}^{|k_1|}\cdots\\
&\qquad M_{\ell_q}^{|k_q|}\prod_{r=1}^{q}\left(\frac{\|\gamma^{\flat}_{(\ell_r)}\|_{H^s}}{\rho^{l_r}M_{\ell_r}}\right)^{|k_r|}\frac{1}{k_r!}\\
&\le C_aC_g\frac{\rho^\sigma}{\sigma!}M_\sigma\,\sigma!\sum_{1\le|\beta|\le\sigma}\left(\frac{C_aC_\sharp}{R_g}\right)^{|\beta|}|\beta|!\sum_{q=1}^{\sigma}\sum_{P_q(\sigma,\beta)}\\
&\qquad\frac{M_{|\beta|}M_{\ell_1}^{|k_1|}\cdots M_{\ell_q}^{|k_q|}}{M_\sigma}\prod_{r=1}^{q}\frac{\left(\partial^{\ell_r}\zeta(0)\right)^{|k_r|}}{k_r!(\ell_r!)^{|k_r|}}. \qquad (69)
\end{aligned}$$

It is now convenient to introduce the following notation,

$$\alpha_1:=\ell_1,\ \ldots,\ \alpha_{|k_1|}:=\ell_1,\ \ \alpha_{|k_1|+1}:=\ell_2,\ \ldots,\ \alpha_{|k_1|+|k_2|}:=\ell_2,\ \ldots,\\ \alpha_{|k_1|+\cdots+|k_q|}:=\ell_q,$$

in terms of which we have

$$\begin{aligned}
M_{\ell_1}^{|k_1|}\ldots M_{\ell_q}^{|k_q|} &= M_{\alpha_1}\ldots M_{\alpha_{|k_1|}}M_{\alpha_{|k_1|+1}}\ldots M_{\alpha_{|k_1|+|k_2|}}\ldots M_{\alpha_{|k_1|+\cdots+|k_q|}}\\
&= M_{\alpha_1}\ldots M_{\alpha_{|\beta|}}.
\end{aligned}$$

Using the FdB-stability property (9), we obtain

$$\frac{M_{|\beta|}M_{\alpha_1}\ldots M_{\alpha_{|\beta|}}}{M_\sigma}\le\frac{M_{\alpha_1+\cdots+\alpha_{|\beta|}}}{M_\sigma}\le 1,$$

and (69) becomes

$$\begin{aligned}\|G_{ij(\sigma)}\|_{H^s} \;\le\; & C_a C_g \frac{\rho^\sigma}{\sigma!} M_\sigma\, \sigma! \sum_{1\le|\beta|\le\sigma} \left(\partial^{|\beta|}\mathcal{K}\right)(0,\dots,0) \sum_{q=1}^{\sigma} \sum_{P_q(\sigma,\beta)} \\ & \frac{M_{|\beta|} M_{\ell_1}^{|k_1|}\dots M_{\ell_q}^{|k_q|}}{M_\sigma} \prod_{r=1}^{q} \frac{\left(\partial^{\ell_r}\zeta(0)\right)^{|k_r|}}{k_r!(\ell_r!)^{|k_r|}}, \\ =: & \; C_a C_g \frac{\rho^\sigma}{\sigma!} M_\sigma f^{(\sigma)}(0), \end{aligned} \tag{70}$$

where the map $\mathcal{K}:\mathbb{R}^d \to \mathbb{R}$ is defined by

$$\mathcal{K}(x_1,\dots,x_d) = \prod_{i=1}^{d} \left(1 - K_g^{-1} x_i\right)^{-1/d}, \quad K_g^{-1} := C_a C_\sharp / R_g, \text{ with}$$
$$(\partial^\beta \mathcal{K})(0,\dots,0) = |\beta|! K_g^{-|\beta|}.$$

Using the Faà di Bruno formula, and (70), we obtain

$$\begin{aligned}&\sum_{\sigma>0} \|G_{ij(\sigma)}\|_{H^s} \rho^{-\sigma} M_\sigma^{-1} t^\sigma \\ &\quad\le C_a C_g \sum_{\sigma>0} f^{(\sigma)}(0) \frac{t^\sigma}{\sigma!} \le C_a C_g \mathcal{K}\left(\zeta(t),\dots,\zeta(t)\right) \le C_a C_g \left(1 - K_g^{-1}\zeta(t)\right)^{-1}.\end{aligned} \tag{71}$$

Using (68) and (71) we obtain (64) with $C_d := d^2 C_a^2 C_g C_\sharp M_0 \max\{1,\; C_a C_\sharp M_0\}$. We now deal with (65). Using (90), Lemma 2, and the superlinearity property (9), we obtain from (37),

$$\begin{aligned}&\|d^* \gamma^\flat_{(\sigma)}\|_{H^{s-1}} \rho^{-\sigma} M_\sigma^{-1} \\ &\quad\le C_a \|\mathrm{J}^{-1}\|_{H^{s-1}} \frac{\|\mathcal{J}_{(\sigma)}\|_{H^{s-1}}}{\rho^\sigma M_\sigma} \\ &\qquad + C_a^2 C_\sharp \|\mathrm{J}^{-1}\|_{H^{s-1}} \sum_{0<m<\sigma} \frac{\|\mathrm{J}_{(m)}\|_{H^{s-1}}}{\rho^m M_m} \frac{\|\gamma^\flat_{(\sigma-m)}\|_{H^s}}{\rho^{\sigma-m} M_{\sigma-m}} \\ &\qquad + C_a C_\sharp^2 \sum_{0<m<\sigma} \frac{\|\gamma^\flat_{(m)}\|_{H^s}}{\rho^m M_m} \frac{\|\gamma^\flat_{(\sigma-m)}\|_{H^s}}{\rho^{\sigma-m} M_{\sigma-m}} \\ &\qquad + C_a^2 C_\sharp^3 \sum_{\substack{m+n+l=\sigma\\ m,n,l>0}} \frac{\|\gamma^\flat_{(m)}\|_{H^s}}{\rho^m M_m} \frac{\|\gamma^\flat_{(n)}\|_{H^s}}{\rho^n M_n} \frac{\|\gamma^\flat_{(l)}\|_{H^s}}{\rho^l M_l} \\ &\qquad + C_a^3 C_\sharp^2 \|\mathrm{J}^{-1}\|_{H^{s-1}} \sum_{\substack{m+n+l=\sigma\\ m,n,l>0}} \frac{\|\gamma^\flat_{(m)}\|_{H^s}}{\rho^m M_m} \frac{\|\gamma^\flat_{(n)}\|_{H^s}}{\rho^n M_n} \frac{\|\mathrm{J}_{(l)}\|_{H^{s-1}}}{\rho^l M_l}\end{aligned}$$

$$+ C_a^4 C_\sharp^3 \|\mathrm{J}^{-1}\|_{H^{s-1}} \sum_{\substack{m+n+l+p=\sigma \\ m,n,l,p>0}} \frac{\|\gamma^\flat_{(m)}\|_{H^s}}{\rho^m M_m} \frac{\|\gamma^\flat_{(n)}\|_{H^s}}{\rho^n M_n} \frac{\|\gamma^\flat_{(l)}\|_{H^s}}{\rho^l M_l} \frac{\|\mathrm{J}_{(p)}\|_{H^{s-1}}}{\rho^p M_p}. \tag{72}$$

In the similar way we have obtained bound (71) for $\{G_{ij(\sigma)}\}_{\sigma>0}$, we also obtain for $\{\mathrm{J}_{(\sigma)}\}_{\sigma>0}$ and $\{\mathcal{J}_{(\sigma)}\}_{\sigma>0}$ the following bounds,

$$\begin{aligned} &\sum_{\sigma>0} \|\mathrm{J}_{(\sigma)}\|_{H^{s-1}} \rho^{-\sigma} M_\sigma^{-1} t^\sigma, \quad \sum_{\sigma>0} \|\mathcal{J}_{(\sigma)}\|_{H^{s-1}} \rho^{-\sigma} M_\sigma^{-1} t^\sigma \\ &\quad \le C_a C_{\mathrm{J}} \left(1 - K_{\mathrm{J}}^{-1} \zeta(t)\right)^{-1}, \end{aligned} \tag{73}$$

with $K_{\mathrm{J}}^{-1} := C_a C_\sharp / R_{\mathrm{J}}$. Using (72) and (73), we obtain (65) with

$$\begin{aligned} C_{d^*} := \max\{&C_{\mathrm{J}} K_{\mathrm{J}^{-1}} C_a^2,\ C_{\mathrm{J}} K_{\mathrm{J}^{-1}} C_a^3 C_\sharp M_0,\ C_a C_\sharp^2 M_0, C_a^2 C_\sharp^3 M_0^2,\ C_{\mathrm{J}} K_{\mathrm{J}^{-1}} C_a^4 C_\sharp^2 M_0^2, \\ &C_{\mathrm{J}} K_{\mathrm{J}^{-1}} C_a^5 C_\sharp^3 M_0^3\}. \end{aligned}$$

It remains to show bound (66). Using (90), Lemma 2, the superlinearity property (9), and the continuous surjection of the normal trace operator $\mathbf{n}$ from $H^s\Omega^k(M)$ to $H^{s-1/2}\Omega^k(\partial M)$ with continuity constant C_∂, (see, e.g., Theorem 1.3.7 of [50]), we obtain from (38),

$$\begin{aligned} &\|\mathbf{n}\gamma^\flat_{(\sigma)}\|_{H^{s-1/2}(\partial M)} \rho^{-\sigma} M_\sigma^{-1} \\ &\quad \le C_\partial^2 C_a^2 C_\nu C_\sharp \sum_{0<m<\sigma} \sum_{i,j} \frac{\|\gamma^\flat_{(m)}\|_{H^s}}{\rho^m M_m} \frac{\|G_{ij(\sigma-m)}\|_{H^s}}{\rho^{\sigma-m} M_{\sigma-m}} \\ &\quad + C_\partial^2 C_a^2 C_g C_\sharp \sum_{0<m<\sigma} \sum_{k} \frac{\|\gamma^\flat_{(m)}\|_{H^s}}{\rho^m M_m} \frac{\|\nu^k_{(\sigma-m)}\|_{H^{s-1/2}(\partial M)}}{\rho^{\sigma-m} M_{\sigma-m}} \\ &\quad + C_\partial^2 C_a^2 C_\sharp \sum_{\substack{m+n+l=\sigma \\ m,n,l>0}} \sum_{i,j,k} \frac{\|\gamma^\flat_{(m)}\|_{H^s}}{\rho^m M_m} \frac{\|G_{ij(n)}\|_{H^s}}{\rho^n M_n} \frac{\|\nu^k_{(l)}\|_{H^{s-1/2}(\partial M)}}{\rho^l M_l}. \end{aligned} \tag{74}$$

Following the proof of bound (71), and using the continuous surjection of the normal trace operator $\mathbf{n}$ from $H^s\Omega^k(M)$ to $H^{s-1/2}\Omega^k(\partial M)$, we obtain

$$\sum_{\sigma>0} \|\nu_{(\sigma)}\|_{H^{s-1/2}(\partial M)} \rho^{-\sigma} M_\sigma^{-1} t^\sigma \le C_a C_\nu \left(1 - K_\nu^{-1} \zeta(t)\right)^{-1}, \tag{75}$$

with $K_\nu^{-1} := C_a C_\sharp C_\partial / R_\nu$. Finally from (74) and (75), we obtain (66) with

$$C_{\mathbf{n}} := d C_g C_\nu C_\sharp C_\partial^2 C_a^3 M_0 \max\{d,\ 1,\ C_a M_0 d^2\},$$

which completes the proof. □

Combining (62) and estimates of Proposition 1, we obtain the following algebraic inequality,

$$\left(1 - K_*^{-1}\,\zeta(t)\right)^2 \left(C_0 t - C_{\mathrm{N}}^{-1}\zeta(t) + C_*\left(\zeta^2(t) + \zeta^3(t)\right)\right)$$
$$+ C_*\left(1 - K_*^{-1}\,\zeta(t)\right)\left(1 + \zeta(t) + \zeta^2(t) + \zeta^3(t)\right) + C_*\zeta(t) \geq 0, \tag{76}$$

where

$$C_0 := \|v_0^\flat\|_{H^s(M)}\rho^{-1}M_1^{-1}, \quad C_* := 4\max\{C_d,\ C_{d^*},\ C_{\mathbf{n}}\}, \quad \text{and}$$
$$K_* := \min\left\{K_g,\ K_\nu,\ K_{\mathrm{J}}\right\}.$$

A sufficient condition for inequality (76) to hold is to have both

$$K_* \geq \zeta(t), \quad \text{and} \quad Q(\zeta) := \lambda(t) - P(\zeta)$$
$$:= \lambda(t) - C_{\mathrm{N}}^{-1}\zeta(t) + C_*\zeta^2(t) + C_*\zeta^3(t) \geq 0, \tag{77}$$

where we have set $\lambda(t) := (\|v_0^\flat\|_{H^s}/(\rho M_1))\,t$. Let λ_c be the value of λ for which the discriminant of the cubic polynomial $Q(\zeta)$ vanishes. Following [8], there exist two times $t_c := \lambda_c \rho M_1/\|v_0^\flat\|_{H^s}$ and $t_* := P(K_*)\rho M_1/\|v_0^\flat\|_{H^s}$, such that an upper bound for the radius of convergence T of the generating function ζ is given by the smaller of t_* and t_c. Then, there exists a bounded positive constant C_{K_*,λ_c}, depending on K_* and λ_c, such that, $\zeta(t) \leq C_{K_*,\lambda_c} \leq K_*$, for $t \in]-T, T[$, with $T = \min\{t_*, t_c\}$. The sufficient condition (77) is then satisfied, which ends the proof.

2.3. Extension to arbitrary dimension. Here, we extend the analysis of Sects. 2.1 and 2.2 to the general d-dimensional case. We observe that recursion relations (36) and (38) hold for any finite dimension d. Then, estimates (64) and (66) of Proposition 1 are still valid in the d-dimensional case. The only recursion relation to generalize is (37), which will lead to a new estimate for (65). For this, we use the Plemelj-Smithies recursion formula (see, e.g., § 7 of Chapter 1 in [27]) to expand the determinant

$$\det\left(I + \frac{\partial\widehat{\gamma}^i}{\partial a^j}\right),$$

as follows,

$$\det\left(I + \frac{\partial\widehat{\gamma}^i}{\partial a^j}\right) = \sum_{k=0}^{d}\frac{1}{k!}\Gamma_k\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right). \tag{78}$$

Coefficients Γ_k are polynomials of $\{\partial\widehat{\gamma}^i/\partial a^j\}_{i,j\in\{1,\ldots,d\}}$ and are given by the following recursion formula

$$\Gamma_k\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right) = \sum_{l=1}^{k}(-1)^{l+1}\frac{(k-1)!}{(k-l)!}\,\mathrm{Tr}\left[\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right)^l\right]\Gamma_{k-l}\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right), \tag{79}$$

with the starting coefficient $\Gamma_0 = 1$. In particular we have

$$\Gamma_1\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right) = \mathrm{Tr}\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right), \quad \text{and} \quad \Gamma_d\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right) = \det\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right).$$

We observe that the coefficient Γ_k is a polynomial of degree k with respect to matrix coefficients $\{\partial\widehat{\gamma}^i/\partial a^j\}_{i,j\in\{1,\ldots,d\}}$. Then, expansion (78) can be rewritten as

$$\det\left(I+\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right)=1+\mathrm{Tr}\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right)+\mathscr{P}_d\left(\left\{\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right\}_{i,\,j}\right), \tag{80}$$

where $\mathscr{P}_d$ denotes a polynomial of degree less than or equal to d, and of degree greater than or equal to 2. Coefficients of polynomial $\mathscr{P}_d$, which are purely numerical, can be found explicitly because $\mathscr{P}_d$ is obviously given by

$$\mathscr{P}_d\left(\left\{\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right\}_{i,\,j}\right)=\sum_{k=2}^{d}\frac{1}{k!}\Gamma_k\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right).$$

Using rescaling (35) and relation (28), we obtain from (24) and (80),

$$\begin{aligned}\sum_{\sigma>0}d^*\gamma^{\flat}_{(\sigma)}t^{\sigma} &= \mathrm{J}^{-1}\sum_{\sigma>0}\mathcal{J}_{(\sigma)}t^{\sigma}\;+\;\mathrm{J}^{-1}\sum_{\sigma_1,\,\sigma_2>0}\mathrm{J}_{(\sigma_1)}\partial_i\gamma^i_{(\sigma_2)}t^{\sigma_1+\sigma_2}\\ &\quad+\;\mathscr{P}_d\Big(\big\{\partial_j\gamma^i_{(\sigma)}t^{\sigma}\big\}_{i,\,j,\,\sigma>0}\Big)+\mathrm{J}^{-1}\mathscr{P}_d\Big(\big\{\partial_j\gamma^i_{(\sigma)}t^{\sigma}\big\}_{i,\,j,\,\sigma>0}\Big)\sum_{\sigma>0}\mathrm{J}_{(\sigma)}t^{\sigma}.\end{aligned} \tag{81}$$

Equation (81) leads to the normalized recursion relations

$$\begin{aligned}d^*\gamma^{\flat}_{(\sigma)} &= \mathrm{J}^{-1}\mathcal{J}_{(\sigma)}+\mathrm{J}^{-1}\sum_{0<m<\sigma}\mathrm{J}_{(\sigma)}\partial_i\gamma^i_{(\sigma-m)}\\ &\quad+\sum_{k=2}^{d}\sum_{\substack{\sigma_1+\ldots+\sigma_k=\sigma\\ \sigma_1,\ldots,\sigma_k>0}}\mathcal{P}_k\Big(\big\{\partial_j\gamma^i_{(\sigma_1)},\;\ldots,\;\partial_j\gamma^i_{(\sigma_k)}\big\}_{i,j}\Big)\\ &\quad+\mathrm{J}^{-1}\sum_{k=2}^{d}\sum_{\substack{\sigma_1+\cdots+\sigma_{k+1}=\sigma\\ \sigma_1,\ldots,\sigma_{k+1}>0}}\mathrm{J}_{(\sigma_{k+1})}\mathcal{P}_k\Big(\big\{\partial_j\gamma^i_{(\sigma_1)},\;\cdots,\;\partial_j\gamma^i_{(\sigma_k)}\big\}_{i,j}\Big),\end{aligned} \tag{82}$$

where polynomials $\mathcal{P}_k$, homogeneous of degree k, have purely numerical coefficients, which can be determined from the Plemelj-Smithies formulas (79) or (86), and also from formula (89). In the same way as the one yielding estimate (65), we obtain from (82),

$$\sum_{\sigma>0}\|d^*\gamma^{\flat}_{(\sigma)}\|_{H^{s-1}(M)}\rho^{-\sigma}M_{\sigma}^{-1}t^{\sigma}\leq C_{d^*}\left\{\big(1-K_{\mathrm{J}}^{-1}\zeta(t)\big)^{-1}\sum_{k=0}^{d}\zeta^k(t)+\sum_{k=2}^{d}\zeta^k(t)\right\}. \tag{83}$$

Putting together estimates (62), (64), (66) and (83), we obtain the following inequality,

$$\begin{aligned}&\Big(1-K_*^{-1}\,\zeta(t)\Big)^2\left(\lambda(t)-C_{\mathrm{N}}^{-1}\zeta(t)+C_*\sum_{k=2}^{d}\zeta^k(t)\right)+C_*\Big(1-K_*^{-1}\,\zeta(t)\Big)\\ &\sum_{k=0}^{d}\zeta^k(t)+C_*\zeta(t)\geq 0.\end{aligned} \tag{84}$$

Once again, a sufficient condition for inequality (84) to hold is to have both

$$K_* \geq \zeta(t), \quad \text{and} \quad Q(\zeta) := \lambda(t) - C_{\mathrm{N}}^{-1}\zeta(t) + C_* \sum_{k=2}^{d} \zeta^k(t) \geq 0. \tag{85}$$

Finally, there still exists a time $T > 0$ such that constraints (85) are satisfied simultaneously. This completes the proof.

Remark 8. Coefficients Γ_k can be written with two (at least) differents explicit formulas. The first one is the Plemelj-Smithies formula (see, e.g., Theorem 7.2 of Chapter I in [27]) which is given by

$$\Gamma_k\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right) = \det\begin{pmatrix} \tau_1 & k-1 & 0 & 0 & \dots & 0 & 0 \\ \tau_2 & \tau_1 & k-2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tau_{k-1} & \tau_{k-2} & \tau_{k-3} & \tau_{k-4} & \dots & \tau_1 & 1 \\ \tau_k & \tau_{k-1} & \tau_{k-2} & \tau_{k-3} & \dots & \tau_2 & \tau_1 \end{pmatrix},$$

$$\text{where} \quad \tau_\ell = \mathrm{Tr}\left[\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right)^\ell\right]. \tag{86}$$

The second explicit formula uses the Faà di Bruno formula and a well-known expression for the determinant. This expression involves only the trace of positive integer power of the Jacobian matrix $(\partial_j\widehat{\gamma}^i)_{i,j}$. Indeed, using Lidskii's theorem and the product expansion for the determinant, we can show, for ϵ small enough, the following formula (see, e.g., Theorem 3.3 of Chapter I in [27]),

$$\det\left(I + \epsilon\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right) = \exp\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\mathrm{Tr}\left[\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right)^n\right]\epsilon^n\right) =: \exp(f(\epsilon)). \tag{87}$$

We next use the Faà di Bruno formula to obtain a series expansion in power of ϵ for the composition of functions $(\exp\circ f)(\epsilon)$ in (87). We then obtain

$$\det\left(I + \epsilon\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right) = \sum_{k\geq 0}\Gamma_k\frac{\epsilon^k}{k!}, \tag{88}$$

where,

$$\begin{aligned}\Gamma_k &= k!\sum_{n=1}^{k}\left(\frac{d^n}{dx^n}\exp\right)(0)\sum_{p(k,n)}\prod_{\ell=1}^{k}\frac{1}{\lambda_\ell!(\ell!)^{\lambda_\ell}}\left(\frac{d^\ell}{d\epsilon^\ell}f\right)^{\lambda_\ell}(0) \\ &= k!\sum_{n=1}^{k}(-1)^{n+k}\sum_{p(k,n)}\prod_{\ell=1}^{k}\frac{1}{\ell^{\lambda_\ell}\lambda_\ell!}\mathrm{Tr}\left[\left(\frac{\partial\widehat{\gamma}^i}{\partial a^j}\right)^\ell\right]^{\lambda_\ell}.\end{aligned} \tag{89}$$

In (89), the set $p(k,n)$ is defined by

$$p(k,n) = \left\{(\lambda_1,\dots,\lambda_k) \mid \lambda_\ell \in \mathbb{N}^*, \ \sum_{\ell=1}^{k}\lambda_\ell = n, \ \sum_{\ell=1}^{k}\ell\lambda_\ell = k\right\}.$$

Since $\det(I + \epsilon\partial_j \widehat{\gamma}^i)$ is a polynomial of degree at most d with respect to the variable ϵ, formulas (87) and (88) hold for all $\epsilon \in \mathbb{C}$. Truncating expansion (88) at the order d, and then taking $\epsilon = 1$, we obtain the desired result. Of course formulas (89) and (86) give the same value for Γ_k.

3. Conclusion

In this paper we have obtained a relatively simple and constructive proof for time ultra-differentiability of the incompressible Euler geodesic flow on a compact d-dimensional Riemaniann manifold with boundary. This proof makes use of a new Lagrangian formulation of the incompressible Euler equations on d-dimensional manifold [7], which is a generalization of the Cauchy invariants equation in $\mathbb{R}^3$ [8,13]. This Lagrangian formulation, together with the incompressibility condition for the Jacobian of the Lagrangian map, and the invariance of the boundary under the Lagrangian flow allow us to derive new recursion relations among time-Taylor coefficients of the time-series expansion of a dual 1-form $\gamma_t^\flat$ associated with the geodesic flow.

Such proof could be also usefull to construct high-order semi-Lagrangian methods for integrating numerically the incompressible Euler equations on a manifold with very high accuracy, in the spirit of [47]. Indeed, a crucial point in semi-Lagrangian schemes is the construction of a Lagrangian-map with high accuracy. Recursion relations obtained here offer this possiblity, and not only give some estimates for successive time derivatives of the Lagrangian flow. Of course nonlocality of the problem (due to the pressure function) reappears in the resolution of some elliptic problems, which here come from the Hodge decomposition of time-Taylor coefficients of the 1-form $\gamma_t^\flat$. Nevertheless there exist several numerical methods (finite element methods, discontinuous Galerkin methods, pseudo-spectral methods, domain decomposition methods,...) to solve efficiently these elliptic problems formulated in terms of PDEs rather than in terms of singular integral operators.

For instance, such ideas could be used to solve numerically a challenging problem in 2D turbulence, which is the following: what is the features at large scales of the energy inverse cascade in the infinite 2D Euclidean space or in presence of a very large scale friction? Let us remind that in a bounded 2D domain the energy is transferred towards large scales. If there is no friction mechanism removing energy at large scales, then, upon reaching the largest but finite scale of the bounded domain, the inverse cascade will develop large-scale coherent structures and, eventually, a stationary state with no cascades. To mimic an infinite 2D Euclidean space, which could be tractable from a numerical point of view, Falkovich and Gawedzki [21] use an hyperbolic plane which is an infinite surface of constant negative curvature. Such numerical studies could complete the analytical investigations performed in [21].

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A. Differential Geometry Notation

In this appendix, we recall differential geometry notation, which follow mainly the standard notation of classical monographs. To keep this presentation as short as possible we will not always give the definition of these differential geometry tools. For a precise definition of tools left undefined, we refer the reader to the following classical monographs of differential geometry on manifolds [1,15,18,22,50,54]. We also refer the reader to Appendix "Differential geometry in a nutshell" of [7]. Indeed this very short overview of differential geometry tools, used here, can serve as a reminder for the reader.

Consider a LSL–FdB ultradifferentiable compact Riemannian ∂-manifold of dimension d with boundary ∂M. The set of tangent vectors to M at $a \in M$ forms a vector space TM_a, which is called the tangent space to M at a. The union of the tangent spaces to M at the various points of M, i.e., $TM := \cup_{a\in M} TM_a$, is called the tangent bundle of M. A vector field on M is a (cross-)section of TM. Let us recall that a (cross-)section of a vector bundle assigns to each base point $a \in M$ a vector in the fiber $\pi^{-1}(a)$ over a and the addition and scalar multiplication of sections take place within each fiber. Here, the mapping $\pi : TM \to M$, which takes a tangent vector X to the point $a \in M$ at which the vector is tangent to M (i.e., $X \in TM_a$), is called the natural projection. The inverse image of a point $a \in M$ under the natural projection, i.e., $\pi^{-1}(a)$, is the tangent space TM_a. This space is called the fiber of the tangent bundle over the point a. The space $\Gamma(TM)$ of all smooth sections of TM is noted $\mathfrak{X}(M) := \Gamma(TM)$ and describes all smooth vector fields on M. The dual of the tangent bundle, noted T^*M, can be constructed through linear forms, called 1-forms or cotangent vectors, acting on vectors of the tangent bundle TM. The cotangent space to M at a is noted T^*M_a, and the cotangent bundle is the union of the cotangent spaces to the manifold M at all its points, that is $T^*M := \cup_{a\in M} T^*M_a$. The space of all smooth sections $\Omega^1(M) := \Gamma(T^*M)$ is called the space of differential 1-forms on M. The space of 0-forms is the space of smooth functions on M and is noted $\Omega^0(M)$.

The Riemannian metric is given by the infinitesimal line element ds^2, which is defined by the metric tensor g: $ds^2 = g = g_{ij}da^i da^j = g_{ij}(a)da^i \otimes da^j$. The tensor g endows each tangent vector space TM_a with an inner or scalar product, $(\cdot,\cdot)_g$ called also Riemannian metric and defined as: $\forall a \in M, \forall X, Y \in TM_a, (X, Y)_g = g_{ij}(a)X^i(a)Y^j(a)$. The components of g are LSL–FdB ultradifferentiable functions. Therefore, using the inner product $(\cdot,\cdot)_g$, we get an isomorphism between the tangent bundle TM and the cotangent bundle T^*M. In particular, it induces an isomorphism of spaces of sections, which is called the raising operator $(\cdot)^\sharp : \Omega^1(M) \to \mathfrak{X}(M)$. It is defined by: $\forall \omega \in \Omega^1(M)$, $\omega^\sharp = (\omega_i da^i)^\sharp = (\omega^\sharp)^i \partial/\partial a^i = \omega^i \partial/\partial a^i$, where contravariant components are given by $(\omega^\sharp)^i = \omega^i = g^{ij}\omega_j$. The inverse of the raising operator, named the lowering operator $(\cdot)^\flat : \mathfrak{X}(M) \to \Omega^1(M)$, is defined by: $\forall X \in \mathfrak{X}(M)$, $X^\flat = (X^i\partial/\partial a^i)^\flat = (X^\flat)_i da^i = X_i da^i$, where covariant components are given by $(X^\flat)_i = X_i = g_{ij}X^j$. Of course we have $g_{ik}g^{kj} = \delta_i^j$ where δ_i^j is the constant diagonal metric with unity on the diagonal.

The space of all anti-symmetric k-linear maps $\omega_{|_a} : TM_a \times \cdots \times TM_a \to \mathbb{R}$ at the point $a \in M$ is noted $\Lambda^k(T_aM)$. Then the exterior k-form bundle is defined as $\Lambda^k(TM) = \cup_{a\in M}\Lambda^k(T_aM)$. The space all smooth sections $\Omega^k(M) := \Gamma(\Lambda^k(TM))$ is called the space of differential k-forms on M. The Riemannian ∂-manifold is endowed with a metric volume d-form $\mu = \sqrt{|\mathrm{g}|}da^1 \wedge \ldots \wedge da^d \equiv \sqrt{|\mathrm{g}|}da$, where $\sqrt{|\mathrm{g}|} = \sqrt{\det(g_{ij})}$. We continue by fixing the notation of standard operators acting on differential forms, the properties of which can be found in [1,15,18,50,54]. The operator $\mathrm{i}_X : \Omega^k(M) \to \Omega^{k-1}(M)$ denotes the interior product of a k-form with the vector field $X \in$

$\mathfrak{X}(M)$. The Riemann-Levi-Civita connection or the covariant derivative on M is noted ∇ and is an operator from $\Omega^k(M)$ to $\Omega^{k+1}(M)$. The exterior derivative (resp. coderivative), denoted d (resp. d^*), is an operator from $\Omega^k(M)$ to $\Omega^{k+1}(M)$ (resp. $\Omega^{k-1}(M)$). In some textbooks the coderivative d^* is denoted by δ, but we prefer to avoid this notation for not be confused with Kronecker symbol or diagonal metric. The Laplace–De-Rham operator Δ is defined by $\Delta := dd^* + d^*d$. The Hodge dual operator is defined as the unique isomorphism $\star : \Omega^k(M) \to \Omega^{d-k}(M)$, which satisfies $\omega \wedge \star\gamma = (\!(\omega, \gamma)\!)_g\, \mu$, $\forall \omega, \gamma \in \Omega^k(M)$, with $(\!(\omega, \gamma)\!)_g = \omega_{i_1 \ldots i_k} \gamma^{i_1 \ldots i_k}/k! = \omega_{i_1 \ldots i_k} \gamma_{j_1 \ldots j_k} g^{i_1 j_1} \ldots g^{i_k j_k}/k!$, and where the wedge symbol $\wedge$ denotes the exterior product. Using the metric $(\!(\cdot, \cdot)\!)_g$ for k-form on M, we can equipped the space of k-form field $\Omega^k(M)$ with the L^2 scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{L^2(M)} = \int_M (\!(\cdot, \cdot)\!)_g\, \mu$ and the induced norm $\| \cdot \|_{L^2\Omega^k(M)} := \sqrt{\langle\!\langle \cdot, \cdot \rangle\!\rangle_{L^2(M)}}$. Let $\varphi : M \to M$ be a diffeomorphism between compact ∂-manifolds. Then, the operator $\varphi^* : \Omega^k(M) \to \Omega^k(M)$ denotes the pullback transformation associated with φ.

To study differential forms on the boundary of M, we consider the inclusion $\jmath : \partial M \to M$ and its pullback $\jmath^* : \Omega^k(M) \to \Omega^k(\partial M)$. The boundary manifold ∂M carries a metric $\jmath^* g$, which is canonically induced from the metric g on M. The corresponding Riemannian volume form μ_∂ is computed as $\mu_\partial = \mathrm{i}_\nu \mu_{|\partial M}$, where the vector field ν is the outward pointing unit normal to the boundary ∂M. The restriction $\omega_{|\partial M} \in \Omega^k(M)_{|\partial M} := \Gamma(\Lambda^k(TM)_{|\partial M})$ is called the boundary value of $\omega \in \Omega^k(M)$, and in particular one has $(\omega \wedge \gamma)_{|\partial M} = \omega_{|\partial M} \wedge \gamma_{|\partial M}$ and $\star(\omega_{|\partial M}) = (\star\omega)_{|\partial M}$. Every vector field $X \in \Gamma(TM_{|\partial M})$ can be decomposed into its tangential and normal parts as $X = X^\parallel + X^\perp$, where $X^\perp = (X, \nu)_g\, \nu$ and $(X^\parallel, \nu)_g = 0$. The tangent bundle $T\partial M$ should not be confused with the bundle $TM_{|\partial M}$. The latter is the restriction of the full tangent bundle TM to ∂M, and contains $T\partial M$ as a sub-bundle of co-dimension 1. Nevertheless there exists a smooth map, the tangent map $T\jmath : T\partial M \to TM_{|\partial M}$, which induces a natural inclusion from $\Gamma(T\partial M)$ of the vector field on the boundary manifold into the space $\Gamma(TM_{|\partial M})$ of vectors field on M sitting over the boundary. By means of the decomposition of a vector field $X \in \Gamma(TM_{|\partial M})$ into its tangential and normal parts $X = X^\parallel + X^\perp$, we denote the tangential and normal trace operators respectively by $\mathbf{t}$ and $\mathbf{n}$, which are defined as follows: for $k \geq 1$,

$$\begin{aligned} &\forall X_1, \ldots, X_k \in \Gamma(TM_{|\partial M}), \quad \omega \in \Omega^k(M), \\ &\quad \mathbf{t}\omega(X_1, \ldots, X_k) = \omega(X_1^\parallel, \ldots, X_k^\parallel) \ \text{ and } \ \mathbf{n}\omega = \omega_{|\partial M} - \mathbf{t}\omega. \end{aligned}$$

For $k = 0$, $\mathbf{t}\omega = \omega$ and $\mathbf{n}\omega = 0$. The component $\mathbf{t}\omega$ (resp. $\mathbf{n}\omega$) is called the tangential (resp. normal) component of $\omega \in \Omega^k(M)$. The tangential component $\mathbf{t}\omega$ is uniquely determinded by $\jmath^*\omega$, i.e., one has $\mathbf{t}\omega = \jmath^*\mathbf{t}\omega = \jmath^*\omega$. We have the commutation relations $\star(\mathbf{n}\omega) = \mathbf{t}(\star\omega)$, $\star(\mathbf{t}\omega) = \mathbf{n}(\star\omega)$, $\mathbf{t}(d\omega) = d(\mathbf{t}\omega)$ and $\mathbf{n}(d^*\omega) = d^*(\mathbf{n}\omega)$. For more details on how to handle differential forms on the boundary of M, we refer the reader to Section 1 of Chapter 1 of [50].

The regularity in space of initial data, i.e., the regularity with respect to the Lagrangian spatial variables, is for convenience measured in Sobolev spaces. Of course the proof can be extended, for instance, in Hölder spaces or ultradifferentiable spaces (e.g., analytic functions). We refer the reader to [6,32,50] for an introduction to Sobolev spaces $W^{s,p}\Gamma(\mathbb{F})$ of sections of a general Riemannian vector bundle $\mathbb{F}$ over a ∂-manifold M, and especially for the particular case $W^{s,p}\Omega^k(M)$ of differential forms. We use the notation $H^s\Omega^k(M) := W^{s,2}\Omega^k(M)$. The classical Sobolev spaces theory in domains of $\mathbb{R}^d$ (see, e.g., [2]) is covered by the general definition of spaces $W^{s,p}\Gamma(\mathbb{F})$. To make the bridge between the theory on section spaces and the classical one in $\mathbb{R}^d$ the main ingredient is the invariance of $W^{s,p}$-Sobolev topology under diffeomorphism between compact manifolds. Therefore, a number of central results (continuous and compact Sobolev embeddings, Sobolev inequalities,...) from theory in $\mathbb{R}^d$ can be generalized to section spaces $\Gamma(\mathbb{F})$. In particular $\omega \in W^{s,p}\Omega^k(M)$ implies that coefficients of the k-form ω are in $W^{s,p}\Omega^0(M)$. Since $H^s\Omega^0(M)$ is an algebra with respect to the pointwise multiplication provided that $s > d/2$, there exists a constant $C_a := C_a(s)$, which depends on s such that, for $s > d/2$, and $k,\ l \geq 0$,

$$\|\omega \wedge \gamma\|_{H^s\Omega^{k+l}(M)} \leq C_a \|\omega\|_{H^s\Omega^k(M)} \|\gamma\|_{H^s\Omega^l(M)}, \quad \omega \in H^s\Omega^k(M), \ \ \gamma \in H^s\Omega^l(M). \tag{90}$$

Let $s,\ p,\ k$ be integers such that $s \in \mathbb{N}$, $1 < p < \infty$ and $k \geq 1$. Since the differential operators considered below can be described in terms of the induced covariant derivative ∇ on $\Omega^k(M)$, we then have the following continuous mappings,

$$\begin{aligned}
d &: W^{s+1,p}\Omega^k(M) \longrightarrow W^{s,p}\Omega^{k+1}(M),\\
d^* &: W^{s+1,p}\Omega^k(M) \longrightarrow W^{s,p}\Omega^{k-1}(M),\\
\Delta &: W^{s+2,p}\Omega^k(M) \longrightarrow W^{s,p}\Omega^k(M),\\
\star &: W^{s,p}\Omega^k(M) \longrightarrow W^{s,p}\Omega^{d-k}(M),\\
\varphi^* &: W^{s,p}\Omega^k(M) \longrightarrow W^{s,p}\Omega^k(M),
\end{aligned} \tag{91}$$

where $\varphi : M \to M$ is a diffeomorphism between compact ∂-manifolds and X is a smooth vector field on M (e.g., $X \in \mathfrak{X}(M)$ and is bounded or X is an element of $W^{s+1,p}$ sections of TM with $s > d/p$). More details can be found in the monograph [50].

B. Hodge Decomposition of Differential Forms on a Manifold

In this appendix, we present the Hodge decomposition theorem for k-forms on compact ∂-manifolds. We refer the reader to the monograph [50] for a proof of the following Theorem 3, and especially to Sect. 2.4 of [50]. For a compact ∂-manifolds, Hodge decomposition of differential forms in the spaces L^2 and H^1, was first given by Friedrichs [23] and Morrey [45,46]. Its generalisation to differential forms of Sobolev class $W^{s,p}$, with $s \in \mathbb{N}$ and $1 < p < \infty$ was proven by Schwarz [50]. Before stating the Hodge decomposition theorem, we need to introduce some spaces described by

Definition 2. Let M be a compact ∂-manifold. Let $s,\ p,\ k$ be integers such that $s \in \mathbb{N}$, $1 < p < \infty$, and $k \geq 1$. We set the following spaces,

$$\begin{aligned}
H^1\Omega^k_D(M) &= \left\{\omega \in H^1\Omega^k(M) \mid \mathbf{t}\omega = 0\right\},\\
H^1\Omega^k_N(M) &= \left\{\omega \in H^1\Omega^k(M) \mid \mathbf{n}\omega = 0\right\},\\
\mathcal{H}^k(M) &= \left\{\omega \in H^1\Omega^k(M) \mid d\omega = 0 \text{ and } d^*\omega = 0\right\}, \quad \text{(space of harmonic fields)},\\
\mathcal{H}^k_N(M) &= H^1\Omega^k_N(M) \cap \mathcal{H}^k(M), \quad \text{(space of Neumann fields)},\\
\mathcal{H}^k_D(M) &= H^1\Omega^k_D(M) \cap \mathcal{H}^k(M), \quad \text{(space of Dirichlet fields)},\\
\mathcal{H}^k_{\text{ex}}(M) &= \left\{\omega \in \mathcal{H}^k(M) \mid \omega = d\gamma\right\},\\
\mathcal{E}^k(M) &= \left\{d\omega \mid \omega \in H^1\Omega^{k-1}_D(M)\right\} \subset L^2\Omega^k(M),
\end{aligned}$$

$$\mathcal{C}^k(M) = \left\{ d^*\gamma \mid \gamma \in H^1\Omega_N^{k+1}(M) \right\} \subset L^2\Omega^k(M),$$
$$W^{s,p}\mathcal{E}^k(M) = W^{s,p}\Omega^k(M) \cap \mathcal{E}^k(M),$$
$$W^{s,p}\mathcal{C}^k(M) = W^{s,p}\Omega^k(M) \cap \mathcal{C}^k(M).$$

Using Definition 2 we have

Theorem 3 (Hodge decomposition, Schwarz [50]). *On a compact ∂-manifold M, the space $W^{s,p}\Omega^k(M)$, with $k \geq 0$, $s \in \mathbb{N}$, and $1 < p < \infty$, can be orthogonally decomposed into*

$$W^{s,p}\Omega^k(M) = W^{s,p}\mathcal{E}^k(M) \oplus W^{s,p}\mathcal{C}^k(M) \oplus \mathcal{H}^k_{\mathrm{ex}}(M) \oplus \mathcal{H}^k_N(M). \tag{92}$$

Proof. See Corollary 2.4.9 of [50]. □

Remark 9. The space $W^{s,p}\Omega^k(M)$ admits other Hodge decompositions, which depend on boundary conditions, among other criteria (see Section 2.4 of [50]). The orthogonal decomposition (92) is the so-called normal Hodge decomposition. There exist other orthogonal decompositions such as the tangential or the mixed Hodge decompositions (see, e.g., [3,26,50]). In Theorem 3, we choose the normal Hodge decomposition, because it is well suited for the boundary data available in our problem.

C. Derivation of the Cauchy Invariants Equation from the Relabelling Symmetry and a Variational Principle

In this appendix, from the relabelling symmetry, i.e., the invariance of the action under relabelling transformations, we obtain the Cauchy invariants equation on a d-dimensional Riemannian manifold. This derivation was originally performed in [7], but with an improper treatment of boundary terms. We give here the corrected version of this derivation, which does not directly make use of Noether's theorem, but is reminiscent of its proof. Before stating the result, we give the formal definition of a relabelling transformation and we establish a Green formula. Another proof of existence of Cauchy invariants equation, rooted in differential geometry, is given by the particular application of Theorem 1 of [7] to the Lie-advected vorticity 2-form $\omega := d\mathrm{u}^\flat$.

Definition 3 (*Relabelling transformation*) .

Let (M, g, ∇, μ) be a Riemannian ∂-manifold. A relabelling transformation is a map $M \ni a \to \gamma(a) \in M$ such that

$$\gamma(a) = a + \delta a(a), \quad \delta a \in \mathfrak{g},$$

i.e., with,

$$\nabla_i \delta a^i = 0 \quad \text{and} \quad (\delta a, \nu)_g = 0.$$

In other words the vector field δa is the infinitesimal generator of a group of volume-preserving diffeomorphisms of M that leave the boundary ∂M invariant.

Lemma 3 (Green formula). *Let μ_∂ be the metric volume form on the boundary manifold ∂M. Let ν be the outward pointing unit normal to the boundary ∂M. Then we have, for all $f \in \Omega^0(M)$ and $X \in \mathfrak{X}(M)$,*

$$\int_M \mu f \nabla_i X^i = -\int_M \mu\, \mathrm{i}_X df + \int_{\partial M} \mu_\partial f\, \mathrm{i}_\nu X^\flat. \tag{93}$$

Proof. Equation (93) is the same as the following Green formula,

$$\langle\langle df, X^\flat\rangle\rangle_{L^2(M)} = \langle\langle f, d^* X^\flat\rangle\rangle_{L^2(M)} + \int_{\partial M} \mathbf{t} f \wedge \star\mathbf{n} X^\flat. \tag{94}$$

The proof of (94), which is based on Stokes' theorem and the commutation relation $\mathbf{t}(\star\omega) = \star(\mathbf{n}\omega)$, is given in Proposition 2.1.2 of [50]. Let us now retrieve (93) from (94). Using the definition of the scalar product, $\langle\langle\omega, \gamma\rangle\rangle_{L^2(M)} := \int_M \omega\wedge\star\gamma,\ \forall\omega,\ \gamma \in \Omega^k(M)$, and using the relations $\star X^\flat = \mathrm{i}_X\mu$ and $df \wedge \mathrm{i}_X\mu = \mathrm{i}_X df \wedge \mu$, we obtain

$$\langle\langle df, X^\flat\rangle\rangle_{L^2(M)} = \int_M df \wedge \mathrm{i}_X\mu = \int_M \mu\, \mathrm{i}_X df.$$

Using the relations $\star d^* X^\flat = (d^* X^\flat) \star 1 = \mu d^* X^\flat = -\mu\nabla_i X^i$, we obtain

$$\langle\langle f, d^* X^\flat\rangle\rangle_{L^2(M)} = -\int_M \mu f \nabla_i X^i.$$

The proof of (93) ends by using the relation $\mathbf{t} f \wedge \star\mathbf{n} X^\flat = \mu_\partial f \mathrm{i}_\nu X^\flat$, which is proved in Proposition 1.2.6 of [50]. □

Theorem 4. (Cauchy invariants equation from the relabelling symmetry and variational principle) *Let $\eta_t \in \mathrm{SDiff}(M, \mu)$ be the geodesic flow solving the incompressible Euler equations. We set $x := \eta_t$, and $v := \dot\eta_t := \partial_t\eta_t = \mathfrak{u} \circ \eta_t$, with $v_0 = \dot\eta_0 = \mathfrak{u}_0$. Then the invariance of the action*

$$\mathcal{A} := \frac{1}{2}\int_0^T dt\, \langle\mathfrak{u}(t), \mathfrak{u}(t)\rangle_{L^2(M)} = \frac{1}{2}\int_0^T dt \int_M \mu\, (\mathfrak{u}(t), \mathfrak{u}(t))_g, \tag{95}$$

under relabelling transformations of Definition 3 implies the following Cauchy invariants conservation law,

$$dv_k \wedge dx^k = \omega_0 := dv_0^\flat. \tag{96}$$

Proof. The idea is first to compute the first-order variation of the action integral

$$\mathcal{A}(\eta, M) = \frac{1}{2}\int_0^T dt \int_M \mu(a) g_{ij}(\eta_t(a))\partial_t\eta_t^i(a)\partial_t\eta_t^j(a),$$

induced by the relabelling transformations of Definition 3. The variation of $\mathcal{A}(\eta, M)$ is given by

$$\begin{aligned}\delta\mathcal{A}(\eta, M)[\delta\eta] &= \frac{1}{2}\delta\int_0^T dt \int_M \mu(a)\, g_{ij}(\eta_t(a))\partial_t\eta_t^i(a)\partial_t\eta_t^j(a)\\ &= \frac{1}{2}\int_0^T dt \int_M \mu(a)\, \partial_l g_{ij}(\eta_t(a))\delta\eta_t^l(a)\partial_t\eta_t^i(a)\partial_t\eta_t^j(a)\\ &\quad + \int_0^T dt \int_M \mu(a)\, g_{ij}(\eta_t(a))\partial_t\delta\eta_t^i(a)\partial_t\eta_t^j(a).\end{aligned} \tag{97}$$

The relabelling transformation of Definition 3 induces a change in the Lagrangian flow η_t at time t, given by

$$\delta\eta_t = \frac{\partial\eta_t}{\partial a^i}\delta\gamma^i = \frac{\partial\eta_t}{\partial a^i}\delta a^i. \tag{98}$$

Substituting (98) in (97), and using the product rule, we obtain

$$
\begin{aligned}
&\delta\mathcal{A}(\eta, M)[\delta a] \\
&= \int_0^T \int_M \mu(a) \left\{ \frac{1}{2}\partial_l g_{ij}(\eta_t(a)) \frac{\partial \eta_t^l(a)}{\partial a^m} \partial_t \eta_t^i(a) \partial_t \eta_t^j(a) \delta a^m \right. \\
&\quad \left. + g_{ij}(\eta_t(a)) \partial_t \left(\frac{\partial \eta_t^i(a)}{\partial a^n} \right) \partial_t \eta_t^j(a) \delta a^n \right\} \\
&= \int_0^T \int_M \mu(a) \left\{ \frac{1}{2}\partial_l g_{ij}(\eta_t(a)) \frac{\partial \eta_t^l(a)}{\partial a^m} \partial_t \eta_t^i(a) \partial_t \eta_t^j(a) \delta a^m \right. \\
&\quad \left. + \partial_t \left(g_{ij}(\eta_t(a)) \frac{\partial \eta_t^i(a)}{\partial a^n} \partial_t \eta_t^j(a) \right) \delta a^n - \partial_t \left(g_{ij}(\eta_t(a)) \partial_t \eta_t^j(a) \right) \frac{\partial \eta_t^i(a)}{\partial a^n} \delta a^n \right\} \\
&= \int_0^T \int_M \mu(a) \left\{ \frac{1}{2}\partial_l g_{ij}(\eta_t(a)) \frac{\partial \eta_t^l(a)}{\partial a^m} \partial_t \eta_t^i(a) \partial_t \eta_t^j(a) \delta a^m \right. \\
&\quad \left. - \partial_k g_{ij}(\eta_t(a)) \partial_t \eta_t^k(a) \partial_t \eta_t^j(a) \frac{\partial \eta_t^i(a)}{\partial a^m} \delta a^m - g_{ij}(\eta_t(a)) \partial_t^2 \eta_t^j(a) \frac{\partial \eta_t^i(a)}{\partial a^m} \delta a^m \right\} \\
&\quad + \int_0^T \int_M \mu(a)\, \partial_t \left(g_{ij}(\eta_t(a)) \frac{\partial \eta_t^i(a)}{\partial a^n} \partial_t \eta_t^j(a) \right) \delta a^n \\
&= I_1 + I_2.
\end{aligned} \tag{99}
$$

First, we show that $I_1 = 0$. From (99) and using the definition of the covariant derivative, we obtain

$$
\begin{aligned}
I_1 &= \int_0^T dt \int_M \mu(a) \frac{\partial \eta_t^j}{\partial a^m} \delta a^m \left\{ -g_{ij}(\eta_t) \left[\partial_t \mathrm{u}^i(t, \eta_t) + \mathrm{u}^k(t, \eta_t) \partial_k \mathrm{u}^i(t, \eta_t) \right] \right. \\
&\quad \left. + \frac{1}{2} \partial_j g_{ik}(\eta_t) \mathrm{u}^i(t, \eta_t) \mathrm{u}^k(t, \eta_t) - \partial_k g_{ij}(\eta_t) \mathrm{u}^i(t, \eta_t) \mathrm{u}^k(t, \eta_t) \right\} \\
&= - \int_0^T dt \int_M \mu(a) \frac{\partial \eta_t^j}{\partial a^m} \delta a^m g_{ij}(\eta_t) \left\{ \partial_t \mathrm{u}^i(t, \eta_t) + \mathrm{u}^k(t, \eta_t) \partial_k \mathrm{u}^i(t, \eta_t) \right. \\
&\quad \left. + \frac{1}{2} g^{im}(t, \eta_t) (\partial_k g_{lm}(t, \eta_t) + \partial_l g_{km}(t, \eta_t) - \partial_m g_{lk}(t, \eta_t)) \mathrm{u}^k(t, \eta_t) \mathrm{u}^l(t, \eta_t) \right\} \\
&= - \int_0^T dt \int_M \mu(a) \frac{\partial \eta_t^j}{\partial a^m} \delta a^m g_{ij}(\eta_t) \left\{ \partial_t \mathrm{u}^i(t, \eta_t) + \mathrm{u}^k(t, \eta_t) \nabla_k \mathrm{u}^i(t, \eta_t) \right\}.
\end{aligned}
$$

Using the Euler equations, $\partial_t \mathrm{u}^i + \mathrm{u}^k \nabla_k \mathrm{u}^i = -g^{ik} \partial_k p$, the term I_1 becomes

$$
\begin{aligned}
I_1 &= \int_0^T dt \int_M \mu(a)\, \delta a^m \frac{\partial \eta_t^j}{\partial a^m} g_{ij}(\eta_t) g^{ik}(\eta_t) \partial_k p(t, \eta_t) \\
&= \int_0^T dt \int_M \mu(a)\, \delta a^m \frac{\partial \eta_t^j}{\partial a^m} \delta_j^k \partial_k p(t, \eta_t) \\
&= \int_0^T dt \int_M \mu(a)\, \delta a^m \frac{\partial \eta_t^k}{\partial a^m} \partial_k p(t, \eta_t) = \int_0^T dt \int_M \mu(a)\, \delta a^m \frac{\partial p}{\partial a^m}.
\end{aligned}
$$

Now, we recall that $\nabla_i \delta a^i = |g|^{-1/2}\partial_i(\sqrt{|g|}\delta a^i) = 0$, and $(\delta a, \nu)_g = 0$. Therefore, using Green formula (93), the term I_1 becomes

$$I_1 = \int_0^T dt \int_M \mu(a)\,\delta a^i \frac{\partial p}{\partial a^i} = -\int_0^T dt \int_M \mu(a)\,\nabla_i \delta a^i\, p + \int_0^T dt \int_{\partial M} \mu_\partial\, p\,(\delta a, \nu)_g = 0.$$

Finally, we deal with the term I_2 defined in (99). For this, we use the property that $\delta a \in \mathfrak{g}$, i.e., $\nabla_i \delta a^i = 0$, and $(\delta a, \nu)_g = 0$. Such a vector δa can be constructed from a skew-symmetric 2-contravariant tensor π satisfying the following constraints,

$$\pi^{ij} + \pi^{ji} = 0 \text{ on } M, \quad g_{ij}\pi^{ik}\nu^j = 0 \;\forall k \text{ on } \partial M, \quad \text{and} \quad g_{ij}\pi^{ik}\nabla_k \nu^j = 0 \text{ on } \partial M. \tag{100}$$

Indeed, if we set

$$\delta a^i := \nabla_j \pi^{ij} = \frac{1}{\sqrt{|g|}}\partial_j(\sqrt{|g|}\pi^{ij}), \tag{101}$$

then, using (100), we find that $\nabla_i \delta a^i = 0$, and $(\delta a, \nu)_g = 0$ on ∂M. We observe that a smooth skew-symmetric 2-contravariant tensor π with compact support in M and vanishing smoothly at the boundary ∂M satisfies (100). Using (100)–(101) and Green formula (93), the term I_2 becomes

$$\begin{aligned} I_2 &= \int_0^T \int_M \mu(a)\,\partial_t\left(g_{ij}(\eta_t(a))\frac{\partial \eta_t^i(a)}{\partial a^n}\partial_t \eta_t^j(a)\right)\delta a^n \\ &= -\int_0^T \int_M \mu(a)\,\partial_k \partial_t\left(g_{ij}(\eta_t(a))\frac{\partial \eta_t^i(a)}{\partial a^n}\partial_t \eta_t^j(a)\right)\pi^{nk} \\ &\quad + \int_0^T dt \int_{\partial M} \mu_\partial(a)\,\partial_t\left(g_{ij}(\eta_t(a))\frac{\partial \eta_t^i(a)}{\partial a^n}\partial_t \eta_t^j(a)\right)\pi^{nk}\nu^m g_{km} \\ &= -\int_0^T \int_M \mu(a)\,\partial_t \partial_k\left(g_{ij}(\eta_t(a))\frac{\partial \eta_t^i(a)}{\partial a^n}\partial_t \eta_t^j(a)\right)\pi^{nk}. \end{aligned}$$

The action $\mathcal{A}(\eta, M)$ should be invariant under relabelling transformations. Thus the variation of the action integral, i.e., $\delta\mathcal{A}$, must vanish. Therefore we have $I_2 = 0$, i.e.,

$$-\int_0^T \int_M \mu(a)\,\partial_t \partial_k\left(g_{ij}(\eta_t(a))\frac{\partial \eta_t^i(a)}{\partial a^n}\partial_t \eta_t^j(a)\right)\pi^{nk} = 0.$$

Since the π^{nk}'s are arbitrary, we obtain

$$\frac{d}{dt}\partial_k\left(g_{ij}(\eta_t(a))\frac{\partial \eta_t^i(a)}{\partial a^n}\partial_t \eta_t^j(a)\right) = 0, \quad \forall k,\ n = 1, \ldots, d.$$

Integration in time of these equations leads to

$$\partial_k\left(\mathfrak{u}_i(t, \eta_t(a))\frac{\partial \eta_t^i(a)}{\partial a^n}\right) = \partial_k v_{0n}, \quad \forall k,\ n = 1, \ldots, d. \tag{102}$$

Using the identity $v_i(t,a) = \mathrm{u}_i(t,\eta_t(a))$ and multiplying (102) by $da^k \wedge da^n$, followed by a summation over the indices k and n, we obtain

$$d(v_i dx^i) = dv_0^\flat \quad \text{i.e.,} \quad dv_i \wedge dx^i = \omega_0 := dv_0^\flat,$$

which ends the proof. □

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