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Optimal Rate for Bose–Einstein Condensation in the Gross–Pitaevskii Regime

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Abstract: We consider systems of bosons trapped in a box, in the Gross–Pitaevskii regime. We show that low-energy states exhibit complete Bose–Einstein condensation with an optimal bound on the number of orthogonal excitations. This extends recent results obtained in Boccato et al. (Commun Math Phys 359(3):975–1026, 2018), removing the assumption of small interaction potential.

1. Introduction

We consider systems of N bosons trapped in the three-dimensional box $\Lambda = [0; 1]^3$, with periodic boundary conditions (the three dimensional torus with volume one), interacting through a repulsive potential with scattering length of the order N^{-1}, a scaling limit known as the Gross–Pitaevskii regime. The Hamilton operator is given by

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} N^2 V(N(x_i - x_j)) \tag{1.1}$$

and acts on a dense subspace of $L^2_s(\Lambda^N)$, the Hilbert space consisting of functions in $L^2(\Lambda^N)$ that are invariant with respect to permutations of the N particles. We assume here $V \in L^3(\mathbb{R}^3)$ to have compact support and to be pointwise non-negative (i.e. $V(x) \geq 0$ for almost all $x \in \mathbb{R}^3$).

Instead of trapping the Bose gas into the box $\Lambda = [0; 1]^3$ and imposing periodic boundary conditions, one could also confine particles through an external potential $V_{\text{ext}} : \mathbb{R}^3 \to \mathbb{R}$, with $V_{\text{ext}}(x) \to \infty$, as $|x| \to \infty$. In this case, the Hamilton operator would have the form

$$H_N^{\text{trap}} = \sum_{j=1}^{N} \left[-\Delta_{x_j} + V_{\text{ext}}(x_j)\right] + \sum_{i<j}^{N} N^2 V(N(x_i - x_j)) \tag{1.2}$$

and it would act on a dense subspace of $L^2_s(\mathbb{R}^{3N})$.

Lieb et al. proved in [12] that the ground state energy E_N^{trap} of (1.2) is such that, as $N \to \infty$,

$$\frac{E_N^{\text{trap}}}{N} \to \min_{\varphi \in L^2(\mathbb{R}^3): \|\varphi\|_2 = 1} \mathcal{E}_{\text{GP}}(\varphi)$$

with the Gross–Pitaevskii energy functional

$$\mathcal{E}_{\text{GP}}(\varphi) = \int \left[|\nabla \varphi|^2 + V_{\text{ext}} |\varphi|^2 + 4\pi \mathfrak{a}_0 |\varphi|^4 \right] dx \tag{1.3}$$

where $\mathfrak{a}_0$ denotes the scattering length of the unscaled interaction potential V.

In [10], Lieb–Seiringer also proved that the normalized ground state vector ψ_N^{trap} of (1.2) exhibits complete Bose–Einstein condensation in the minimizer φ_{GP} of (1.3), meaning that its reduced one-particle density matrix $\gamma_N = \text{tr}_{2,\dots,N} |\psi_N^{\text{trap}}\rangle\langle\psi_N^{\text{trap}}|$ (normalized so that $\text{tr}\, \gamma_N = 1$) satisfies

$$\gamma_N \to |\varphi_{\text{GP}}\rangle\langle\varphi_{\text{GP}}| \tag{1.4}$$

as $N \to \infty$ (convergence holds in the trace norm topology; since the limit is a rank-one projection, all reasonable notions of convergence are equivalent). Equation (1.4) asserts that, in the ground state of (1.2), all bosons, up to a fraction that vanishes in the limit $N \to \infty$, occupy the same one-particle state φ_{GP}. In [11], Lieb–Seiringer extended Eq. (1.4) to reduced density matrices associated with normalized sequences of approximate ground states, ie. states with expected energy per particle converging to the minimum of (1.3) (under the constraint $\|\varphi\| = 1$).

A new proof of the results described above has been later obtained by Nam et al. [14], making use of the quantum de Finetti theorem, first proposed in the mean-field setting by Lewin et al. [7,8].

The results of [10–12,14] can be translated to the Hamilton operator (1.1), defined on the torus, with no external potential. They imply, first of all, that the ground state energy E_N of (1.1) is such that

$$\lim_{N\to\infty} \frac{E_N}{N} = 4\pi \mathfrak{a}_0 \,. \tag{1.5}$$

Furthermore, they imply that for any sequence of approximate ground states, ie. for any sequence $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and

$$\lim_{N\to\infty} \frac{1}{N} \langle \psi_N, H_N \psi_N \rangle = 4\pi \mathfrak{a}_0 \,, \tag{1.6}$$

the reduced density matrices $\gamma_N = \text{tr}_{2,\dots,N} |\psi_N\rangle\langle\psi_N|$ are such that

$$\lim_{N\to\infty} \text{tr}\, |\gamma_N - |\varphi_0\rangle\langle\varphi_0|| = 0 \tag{1.7}$$

where $\varphi_0 \in L^2(\Lambda)$ is the zero momentum mode defined by $\varphi_0(x) = 1$ for all $x \in \Lambda$. Since we will make use of this result in our analysis and since, strictly speaking, the translation invariant Hamiltonian (1.1) is not treated in [11,14], in the version of this paper posted on the arXiv we added a sketch of the proof of (1.7), adapting the arguments of [14].

Under the additional assumption that the interaction potential V is sufficiently small, in [1] we recently improved (1.5) and (1.7), obtaining quantitative estimates showing, on the one hand, that $E_N - 4\pi \mathfrak{a}_0 N$ remains bounded, uniformly in N, and, on the other hand, that every sequence of approximate ground states ψ_N of (1.1) exhibit Bose–Einstein condensation, with number of excitations bounded by the excess energy $\langle \psi_N, H_N \psi_N \rangle - 4\pi \mathfrak{a}_0 N$. The goal of the present paper is to extend the results of [1], removing the assumption of small interaction.

Theorem 1.1. *Let $V \in L^3(\mathbb{R}^3)$ have compact support and be spherically symmetric and non-negative. Then there exists a constant $C > 0$ such that the ground state energy E_N of (1.1) satisfies*

$$|E_N - 4\pi \mathfrak{a}_0 N| \leq C. \tag{1.8}$$

Furthermore, consider a sequence $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and such that

$$\langle \psi_N, H_N \psi_N \rangle \leq 4\pi \mathfrak{a}_0 N + K$$

for a $K > 0$. Then the reduced density matrix $\gamma_N = \mathrm{tr}_{2,\dots,N} |\psi_N\rangle\langle\psi_N|$ associated with ψ_N is such that

$$1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle \leq \frac{C(K+1)}{N} \tag{1.9}$$

for all $N \in \mathbb{N}$ large enough.

Remark. Equation (1.9) gives a bound on the number of orthogonal excitations of the Bose–Einstein condensate, for low-energy states of the Hamilton operator (1.1). It implies that

$$\begin{aligned} \langle \psi_N, d\Gamma(1 - |\varphi_0\rangle\langle\varphi_0|)\psi_N \rangle &= N - \langle \psi_N, a^*(\varphi_0) a(\varphi_0) \psi_N \rangle \\ &= N \left[1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle \right] \leq C(K+1) \end{aligned} \tag{1.10}$$

and thus that, for low-energy states ψ_N with finite excess energy K, the number of excitations of the Bose–Einstein condensate remains bounded, uniformly in N. Notice that the bounds (1.9), (1.10) remain valid and non-trivial even if K grows, as $N \to \infty$, as long as $K \ll N$; in particular, they imply complete BEC for all sequences of approximate ground states ψ_N satisfying (1.6).

To prove Theorem 1.1, we are going to introduce, in Sect. 2, an excitation Hamiltonian $\mathcal{L}_N$, factoring out the Bose–Einstein condensate. In Sect. 3, we define generalized Bogoliubov transformations that are used in Sect. 4 to model correlations among particles and to define a renormalized excitation Hamiltonian $\mathcal{G}_{N,\ell}$; important properties of $\mathcal{G}_{N,\ell}$ are collected in Propositions 4.2 and 4.3. A second renormalization, this time through the exponential of an operator cubic in creation and annihilation operators, is performed in Sect. 5, leading to a new twice renormalized Hamiltonian $\mathcal{R}_{N,\ell}$; an important bound for $\mathcal{R}_{N,\ell}$ is stated in Proposition 5.2. In Sect. 6, we use the results of Propositions 4.2, 4.3 and 5.2 to show Theorem 1.1. Sections 7 and 8 are devoted to the proof of Proposition 4.2 and, respectively, of Proposition 5.2.

The main novelty, with respect to the analysis in [1] is the need for the second renormalization, through the exponential $S = e^A$ of a cubic operator A. Under the additional assumption of small potential, the analysis of $\mathcal{G}_{N,\ell}$ was enough in [1] to show Bose–Einstein condensation in the form (1.9). Here, this is not the case. The point is that

conjugation with a generalized Bogoliubov transformation renormalizes the quadratic terms in the excitation Hamiltonian, but it leaves the cubic term substantially unchanged. For small potentials, the cubic term can be controlled (by Cauchy–Schwarz) through the quartic interaction and through the gap in the kinetic energy. Without assumptions on the size of the potential, on the other hand, we need to conjugate with S, to renormalize the cubic term. After conjugation with S, we can apply techniques developed by Lewin et al. [9] (inspired by previous work of Lieb and Solovej [13]) based on localization of the number of excitations. On sectors with few excitations (the cutoff will be set at $M = cN$, for a sufficiently small constant $c > 0$), the renormalized cubic term is small and it can be controlled by the gap in the kinetic energy operator. On sectors with many excitations, on the other hand, we are going to bound the energy from below, using the estimate (1.7), due to [11,14] (since on these sectors we do not have condensation, the energy per particle must be strictly larger than $4\pi\mathfrak{a}_0$).

Theorem 1.1 is the first important step that we need in [3] to establish the validity of Bogoliubov theory, as proposed in [4], for the low-energy excitation spectrum of (1.1).

2. The Excitation Hamiltonian

The bosonic Fock space over $L^2(\Lambda)$ is defined as

$$\mathcal{F} = \bigoplus_{n\geq 0} L^2_s(\Lambda^n) = \bigoplus_{n\geq 0} L^2(\Lambda)^{\otimes_s n}$$

where $L^2_s(\Lambda^n)$ is the subspace of $L^2(\Lambda^n)$ consisting of wave functions that are symmetric w.r.t. permutations. The vacuum vector in $\mathcal{F}$ will be indicated with $\Omega = \{1, 0, \ldots\} \in \mathcal{F}$.

For $g \in L^2(\Lambda)$, the creation operator $a^*(g)$ and the annihilation operator $a(g)$ are defined by

$$(a^*(g)\Psi)^{(n)}(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} g(x_j)\Psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

$$(a(g)\Psi)^{(n)}(x_1, \ldots, x_n) = \sqrt{n+1} \int_\Lambda \bar{g}(x)\Psi^{(n+1)}(x, x_1, \ldots, x_n)\, dx.$$

Observe that $a^*(g)$ is the adjoint of $a(g)$ and that the canonical commutation relations

$$[a(g), a^*(h)] = \langle g, h\rangle, \quad [a(g), a(h)] = [a^*(g), a^*(h)] = 0$$

hold true for all $g, h \in L^2(\Lambda)$ ($\langle g, h\rangle$ is the inner product on $L^2(\Lambda)$).

It will be convenient for us to work in momentum space $\Lambda^* = 2\pi\mathbb{Z}^3$. For $p \in \Lambda^*$, we consider the plane wave $\varphi_p(x) = e^{-ip\cdot x}$ in $L^2(\Lambda)$. We define the operators

$$a^*_p = a^*(\varphi_p), \quad \text{and} \quad a_p = a(\varphi_p)$$

creating and, respectively, annihilating a particle with momentum p.

To exploit the non-negativity of the interaction potential V, it will sometimes be useful to switch to position space. To this end, we introduce operator valued distributions $\check{a}_x$, $\check{a}^*_x$ such that

$$a(f) = \int \bar{f}(x)\, \check{a}_x\, dx, \quad a^*(f) = \int f(x)\, \check{a}^*_x\, dx.$$

The number of particles operator, defined on a dense subspace of $\mathcal{F}$ by $(\mathcal{N}\Psi)^{(n)} = n\Psi^{(n)}$, can be expressed as

$$\mathcal{N} = \sum_{p\in\Lambda^*} a_p^* a_p = \int \check{a}_x^* \check{a}_x \, dx \, .$$

It is then easy to check that creation and annihilation operators are bounded with respect to the square root of $\mathcal{N}$, i.e.

$$\|a(f)\Psi\| \leq \|f\| \|\mathcal{N}^{1/2}\Psi\|, \quad \|a^*(f)\Psi\| \leq \|f\| \|(\mathcal{N}+1)^{1/2}\Psi\|$$

for all $f \in L^2(\Lambda)$.

Recall that $\varphi_0(x) = 1$ for all $x \in \Lambda$ is the zero-momentum mode in $L^2(\Lambda)$. We define $L^2_\perp(\Lambda)$ as the orthogonal complement in $L^2(\Lambda)$ of the one dimensional space spanned by φ_0. The Fock space over $L^2_\perp(\Lambda)$, generated by the creation operators a_p^* with $p \in \Lambda_+^* := 2\pi\mathbb{Z}^3 \backslash \{0\}$, will be denoted by

$$\mathcal{F}_+ = \bigoplus_{n\geq 0} L^2_\perp(\Lambda)^{\otimes_s n} \, .$$

On $\mathcal{F}_+$, the number of particles operator will be indicated by

$$\mathcal{N}_+ = \sum_{p\in\Lambda_+^*} a_p^* a_p .$$

For $N \in \mathbb{N}$, we also define the truncated Fock space

$$\mathcal{F}_+^{\leq N} = \bigoplus_{n=0}^{N} L^2_\perp(\Lambda)^{\otimes_s n} \, .$$

On this Hilbert space, we are going to describe the orthogonal excitations of the Bose–Einstein condensate. To this end, we are going to use a unitary map $U_N : L^2_s(\Lambda^N) \to \mathcal{F}_+^{\leq N}$, first introduced in [9], which removes the condensate. To define U_N, we notice that every $\psi_N \in L^2_s(\Lambda^N)$ can be uniquely decomposed as

$$\psi_N = \alpha_0 \varphi_0^{\otimes N} + \alpha_1 \otimes_s \varphi_0^{\otimes(N-1)} + \cdots + \alpha_N$$

with $\alpha_j \in L^2_\perp(\Lambda)^{\otimes_s j}$ (the symmetric tensor product of j copies of the orthogonal complement $L^2_\perp(\Lambda)$ of φ_0) for all $j = 0, \ldots, N$. Therefore, we can put $U_N\psi_N = \{\alpha_0, \alpha_1, \ldots, \alpha_N\} \in \mathcal{F}_+^{\leq N}$. We can also define U_N identifying ψ_N with the Fock space vector $\{0, 0, \ldots, \psi_N, 0, \ldots\}$ and using creation and annihilation operators; we find

$$U_N \, \psi_N = \bigoplus_{n=0}^{N} (1 - |\varphi_0\rangle\langle\varphi_0|)^{\otimes n} \frac{a(\varphi_0)^{N-n}}{\sqrt{(N-n)!}} \, \psi_N$$

for all $\psi_N \in L^2_s(\Lambda^N)$. It is then easy to check that $U_N^* : \mathcal{F}_+^{\leq N} \to L^2_s(\Lambda^N)$ is given by

$$U_N^* \, \{\alpha^{(0)}, \ldots, \alpha^{(N)}\} = \sum_{n=0}^{N} \frac{a^*(\varphi_0)^{N-n}}{\sqrt{(N-n)!}} \, \alpha^{(n)}$$

and that $U_N^* U_N = 1$, ie. U_N is unitary.

Using U_N, we can define the excitation Hamiltonian $\mathcal{L}_N := U_N H_N U_N^*$, acting on a dense subspace of $\mathcal{F}_+^{\leq N}$. To compute the operator $\mathcal{L}_N$, we first write the Hamiltonian (1.1) in momentum space, in terms of creation and annihilation operators. We find

$$H_N = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{1}{2N} \sum_{p,q,r \in \Lambda^*} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r} \tag{2.1}$$

where

$$\widehat{V}(k) = \int_{\mathbb{R}^3} V(x) e^{-ik\cdot x} dx$$

is the Fourier transform of V, defined for all $k \in \mathbb{R}^3$ (in fact, (1.1) is the restriction of (2.1) to the N-particle sector of the Fock space $\mathcal{F}$). We can now determine the excitation Hamiltonian $\mathcal{L}_N$ using the following rules, describing the action of the unitary operator U_N on products of a creation and an annihilation operator (products of the form $a_p^* a_q$ can be thought of as operators mapping $L^2_s(\Lambda^N)$ to itself). For any $p, q \in \Lambda_+^* = 2\pi\mathbb{Z}^3 \backslash \{0\}$, we find (see [9]):

$$\begin{aligned} U_N\, a_0^* a_0\, U_N^* &= N - \mathcal{N}_+ \\ U_N\, a_p^* a_0\, U_N^* &= a_p^* \sqrt{N - \mathcal{N}_+} \\ U_N\, a_0^* a_p\, U_N^* &= \sqrt{N - \mathcal{N}_+}\, a_p \\ U_N\, a_p^* a_q\, U_N^* &= a_p^* a_q . \end{aligned} \tag{2.2}$$

We conclude that

$$\mathcal{L}_N = \mathcal{L}_N^{(0)} + \mathcal{L}_N^{(2)} + \mathcal{L}_N^{(3)} + \mathcal{L}_N^{(4)} \tag{2.3}$$

with

$$\begin{aligned} \mathcal{L}_N^{(0)} &= \frac{N-1}{2N} \widehat{V}(0)(N - \mathcal{N}_+) + \frac{\widehat{V}(0)}{2N} \mathcal{N}_+ (N - \mathcal{N}_+) \\ \mathcal{L}_N^{(2)} &= \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p + \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \left[b_p^* b_p - \frac{1}{N} a_p^* a_p \right] \\ &\quad + \frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \left[b_p^* b_{-p}^* + b_p b_{-p} \right] \\ \mathcal{L}_N^{(3)} &= \frac{1}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N) \left[b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q} \right] \\ \mathcal{L}_N^{(4)} &= \frac{1}{2N} \sum_{\substack{p,q \in \Lambda_+^*, r \in \Lambda^* : \\ r \neq -p, -q}} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r} \end{aligned} \tag{2.4}$$

where we introduced generalized creation and annihilation operators

$$b_p^* = a_p^* \sqrt{\frac{N - \mathcal{N}_+}{N}}, \qquad \text{and} \qquad b_p = \sqrt{\frac{N - \mathcal{N}_+}{N}}\, a_p \tag{2.5}$$

for all $p \in \Lambda_+^*$. Observe that, by (2.2),

$$U_N^* b_p^* U_N = a_p^* \frac{a_0}{\sqrt{N}}, \qquad U_N^* b_p U_N = \frac{a_0^*}{\sqrt{N}} a_p.$$

In other words, b_p^* creates a particle with momentum $p \in \Lambda_+^*$ but, at the same time, it annihilates a particle from the condensate; it creates an excitation, preserving the total number of particles in the system. On states exhibiting complete Bose–Einstein condensation in the zero-momentum mode φ_0, we have $a_0, a_0^* \simeq \sqrt{N}$ and we can therefore expect that $b_p^* \simeq a_p^*$ and that $b_p \simeq a_p$. Modified creation and annihilation operators satisfy the commutation relations

$$\begin{aligned} [b_p, b_q^*] &= \left(1 - \frac{\mathcal{N}_+}{N}\right) \delta_{p,q} - \frac{1}{N} a_q^* a_p \\ [b_p, b_q] &= [b_p^*, b_q^*] = 0. \end{aligned} \tag{2.6}$$

Furthermore, we find

$$[b_p, a_q^* a_r] = \delta_{pq} b_r, \qquad [b_p^*, a_q^* a_r] = -\delta_{pr} b_q^* \tag{2.7}$$

for all $p, q, r \in \Lambda_+^*$; this implies in particular that $[b_p, \mathcal{N}_+] = b_p$, $[b_p^*, \mathcal{N}_+] = -b_p^*$. It is also useful to notice that the operators b_p^*, b_p, like the standard creation and annihilation operators a_p^*, a_p, can be bounded by the square root of the number of particles operators; we find

$$\begin{aligned} \|b_p \xi\| &\leq \left\| \mathcal{N}_+^{1/2} \left(\frac{N + 1 - \mathcal{N}_+}{N} \right)^{1/2} \xi \right\| \leq \|\mathcal{N}_+^{1/2} \xi\| \\ \|b_p^* \xi\| &\leq \left\| (\mathcal{N}_+ + 1)^{1/2} \left(\frac{N - \mathcal{N}_+}{N} \right)^{1/2} \xi \right\| \leq \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \end{aligned}$$

for all $\xi \in \mathcal{F}_+^{\leq N}$. Since $\mathcal{N}_+ \leq N$ on $\mathcal{F}_+^{\leq N}$, the operators b_p^*, b_p are bounded, with $\|b_p\|, \|b_p^*\| \leq (N+1)^{1/2}$.

We can also define modified operator valued distributions

$$\check{b}_x = \sqrt{\frac{N - \mathcal{N}_+}{N}}\, \check{a}_x, \qquad \text{and} \quad \check{b}_x^* = \check{a}_x^* \sqrt{\frac{N - \mathcal{N}_+}{N}}$$

in position space, for $x \in \Lambda$. The commutation relations (2.6) take the form

$$\begin{aligned} [\check{b}_x, \check{b}_y^*] &= \left(1 - \frac{\mathcal{N}_+}{N}\right) \delta(x - y) - \frac{1}{N} \check{a}_y^* \check{a}_x \\ [\check{b}_x, \check{b}_y] &= [\check{b}_x^*, \check{b}_y^*] = 0. \end{aligned}$$

Moreover, (2.7) translates to

$$[\check{b}_x, \check{a}_y^* \check{a}_z] = \delta(x - y) \check{b}_z, \qquad [\check{b}_x^*, \check{a}_y^* \check{a}_z] = -\delta(x - z) \check{b}_y^*$$

which also implies that $[\check{b}_x, \mathcal{N}_+] = \check{b}_x$, $[\check{b}_x^*, \mathcal{N}_+] = -\check{b}_x^*$.

3. Generalized Bogoliubov Transformations

Conjugation with U_N extracts, from the original quartic interaction in (2.1), some constant and some quadratic contributions, collected in $\mathcal{L}_N^{(0)}$ and $\mathcal{L}_N^{(2)}$ in (2.4). In the Gross–Pitevskii regime, however, this is not enough; there are still large contributions to the energy hidden among cubic and quartic terms in $\mathcal{L}_N^{(3)}$ and $\mathcal{L}_N^{(4)}$.

To extract the missing energy, we have to take into account the correlation structure. Since U_N only removes products of the zero-energy mode φ_0, correlations among particles, which play a crucial role in the Gross–Pitaevskii regime and carry an energy of order N, remain in the excitation vector $U_N\psi_N$. To factor out correlations, it is natural to conjugate $\mathcal{L}_N$ with a Bogoliubov transformation. In fact, to make sure that the truncated Fock space $\mathcal{F}_+^{\leq N}$ remains invariant, we will have to use generalized Bogoliubov transformations. Their definition and their main properties will be discussed in this section.

For $\eta \in \ell^2(\Lambda_+^*)$ with $\eta_{-p} = \eta_p$ for all $p \in \Lambda_+^*$, we define

$$B(\eta) = \frac{1}{2} \sum_{p \in \Lambda_+^*} \left(\eta_p b_p^* b_{-p}^* - \bar{\eta}_p b_p b_{-p} \right) \tag{3.1}$$

and we consider

$$e^{B(\eta)} = \exp \left[\frac{1}{2} \sum_{p \in \Lambda_+^*} \left(\eta_p b_p^* b_{-p}^* - \bar{\eta}_p b_p b_{-p} \right) \right]. \tag{3.2}$$

We refer to unitary operators of the form (3.2) as generalized Bogoliubov transformations, in analogy with the standard Bogoliubov transformations

$$e^{\widetilde{B}(\eta)} = \exp \left[\frac{1}{2} \sum_{p \in \Lambda_+^*} \left(\eta_p a_p^* a_{-p}^* - \bar{\eta}_p a_p a_{-p} \right) \right] \tag{3.3}$$

defined by means of the standard creation and annihilation operators. In this paper, we will work with (3.2), rather than (3.3), because the generalized Bogoliubov transformations, in contrast with the standard transformations, leave the truncated Fock space $\mathcal{F}_+^{\leq N}$ invariant. The price we will have to pay is the fact that, while the action of standard Bogoliubov transformation on creation and annihilation operators is explicitly given by

$$e^{-\widetilde{B}(\eta)} a_p e^{\widetilde{B}(\eta)} = \cosh(\eta_p) a_p + \sinh(\eta_p) a_{-p}^* \tag{3.4}$$

there is no such formula describing the action of generalized Bogoliubov transformations.

A first important tool to control the action of generalized Bogoliubov transformations is the following lemma, whose proof can be found in [5, Lemma 3.1] (a similar result has been previously established in [15]).

Lemma 3.1. *For every $n \in \mathbb{N}$ there exists a constant $C > 0$ such that, on $\mathcal{F}_+^{\leq N}$,*

$$e^{-B(\eta)} (\mathcal{N}_+ + 1)^n e^{B(\eta)} \leq C e^{C\|\eta\|} (\mathcal{N}_+ + 1)^n \tag{3.5}$$

for all $\eta \in \ell^2(\Lambda^*)$.

Bounds of the form (3.5) on the change of the number of particles operator are not enough for our purposes; we will need more precise information about the action of unitary operators of the form $e^{B(\eta)}$. To this end, we expand, for any $p \in \Lambda_+^*$,

$$\begin{aligned} e^{-B(\eta)}\, b_p\, e^{B(\eta)} &= b_p + \int_0^1 ds\, \frac{d}{ds} e^{-sB(\eta)} b_p e^{sB(\eta)} \\ &= b_p - \int_0^1 ds\, e^{-sB(\eta)} [B(\eta), b_p] e^{sB(\eta)} \\ &= b_p - [B(\eta), b_p] + \int_0^1 ds_1 \int_0^{s_1} ds_2\, e^{-s_2 B(\eta)} [B(\eta), [B(\eta), b_p]] e^{s_2 B(\eta)}. \end{aligned}$$

Iterating m times, we find

$$\begin{aligned} e^{-B(\eta)} b_p e^{B(\eta)} = &\sum_{n=1}^{m-1} (-1)^n \frac{\mathrm{ad}^{(n)}_{B(\eta)}(b_p)}{n!} \\ &+ \int_0^1 ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{m-1}} ds_m\, e^{-s_m B(\eta)} \mathrm{ad}^{(m)}_{B(\eta)}(b_p) e^{s_m B(\eta)} \end{aligned} \tag{3.6}$$

where we recursively defined

$$\mathrm{ad}^{(0)}_{B(\eta)}(A) = A \quad \text{and} \quad \mathrm{ad}^{(n)}_{B(\eta)}(A) = [B(\eta), \mathrm{ad}^{(n-1)}_{B(\eta)}(A)].$$

We are going to expand the nested commutators $\mathrm{ad}^{(n)}_{B(\eta)}(b_p)$ and $\mathrm{ad}^{(n)}_{B(\eta)}(b_p^*)$. To this end, we need to introduce some additional notation. We follow here [1,2,5]. For $f_1, \ldots, f_n \in \ell_2(\Lambda_+^*)$, $\sharp = (\sharp_1, \ldots, \sharp_n)$, $\flat = (\flat_0, \ldots, \flat_{n-1}) \in \{\cdot, *\}^n$, we set

$$\begin{aligned} &\Pi^{(2)}_{\sharp,\flat}\ (f_1, \ldots, f_n) \\ &\quad = \sum_{p_1, \ldots, p_n \in \Lambda^*} b^{\flat_0}_{\alpha_0 p_1} a^{\sharp_1}_{\beta_1 p_1} a^{\flat_1}_{\alpha_1 p_2} a^{\sharp_2}_{\beta_2 p_2} a^{\flat_2}_{\alpha_2 p_3} \ldots a^{\sharp_{n-1}}_{\beta_{n-1} p_{n-1}} a^{\flat_{n-1}}_{\alpha_{n-1} p_n} b^{\sharp_n}_{\beta_n p_n} \prod_{\ell=1}^n f_\ell(p_\ell) \end{aligned} \tag{3.7}$$

where, for $\ell = 0, 1, \ldots, n$, we define $\alpha_\ell = 1$ if $\flat_\ell = *$, $\alpha_\ell = -1$ if $\flat_\ell = \cdot$, $\beta_\ell = 1$ if $\sharp_\ell = \cdot$ and $\beta_\ell = -1$ if $\sharp_\ell = *$. In (3.7), we require that, for every $j = 1, \ldots, n-1$, we have either $\sharp_j = \cdot$ and $\flat_j = *$ or $\sharp_j = *$ and $\flat_j = \cdot$ (so that the product $a^{\sharp_\ell}_{\beta_\ell p_\ell} a^{\flat_\ell}_{\alpha_\ell p_{\ell+1}}$ always preserves the number of particles, for all $\ell = 1, \ldots, n-1$). With this assumption, we find that the operator $\Pi^{(2)}_{\sharp,\flat}(f_1, \ldots, f_n)$ maps $\mathcal{F}_+^{\leq N}$ into itself. If, for some $\ell = 1, \ldots, n$, $\flat_{\ell-1} = \cdot$ and $\sharp_\ell = *$ (i.e. if the product $a^{\flat_{\ell-1}}_{\alpha_{\ell-1} p_\ell} a^{\sharp_\ell}_{\beta_\ell p_\ell}$ for $\ell = 2, \ldots, n$, or the product $b^{\flat_0}_{\alpha_0 p_1} a^{\sharp_1}_{\beta_1 p_1}$ for $\ell = 1$, is not normally ordered) we require additionally that $f_\ell \in \ell^1(\Lambda_+^*)$. In position space, the same operator can be written as

$$\begin{aligned} \Pi^{(2)}_{\sharp,\flat}(f_1, \ldots, f_n) = \int &\check{b}^{\flat_0}_{x_1} \check{a}^{\sharp_1}_{y_1} \check{a}^{\flat_1}_{x_2} \check{a}^{\sharp_2}_{y_2} \check{a}^{\flat_2}_{x_3} \ldots \check{a}^{\sharp_{n-1}}_{y_{n-1}} \check{a}^{\flat_{n-1}}_{x_n} \check{b}^{\sharp_n}_{y_n} \\ &\times \prod_{\ell=1}^n \check{f}_\ell(x_\ell - y_\ell)\, dx_\ell dy_\ell. \end{aligned} \tag{3.8}$$

An operator of the form (3.7), (3.8) with all the properties listed above, will be called a $\Pi^{(2)}$-operator of order n.

For $g, f_1, \dots, f_n \in \ell_2(\Lambda_+^*), \sharp = (\sharp_1, \dots, \sharp_n) \in \{\cdot, *\}^n, \flat = (\flat_0, \dots, \flat_n) \in \{\cdot, *\}^{n+1}$, we also define the operator

$$\begin{aligned}\Pi^{(1)}_{\sharp,\flat}(f_1, \dots, f_n; g) &= \sum_{p_1,\dots,p_n \in \Lambda^*} b^{\flat_0}_{\alpha_0, p_1} a^{\sharp_1}_{\beta_1 p_1} a^{\flat_1}_{\alpha_1 p_2} a^{\sharp_2}_{\beta_2 p_2} a^{\flat_2}_{\alpha_2 p_3} \dots a^{\sharp_{n-1}}_{\beta_{n-1} p_{n-1}} a^{\flat_{n-1}}_{\alpha_{n-1} p_n} a^{\sharp_n}_{\beta_n p_n} a^{\flat_n}(g) \\ &\quad \times \prod_{\ell=1}^n f_\ell(p_\ell)\end{aligned} \tag{3.9}$$

where α_ℓ and β_ℓ are defined as above. Also here, we impose the condition that, for all $\ell = 1, \dots, n$, either $\sharp_\ell = \cdot$ and $\flat_\ell = *$ or $\sharp_\ell = *$ and $\flat_\ell = \cdot$. This implies that $\Pi^{(1)}_{\sharp,\flat}(f_1, \dots, f_n; g)$ maps $\mathcal{F}_+^{\leq N}$ back into $\mathcal{F}_+^{\leq N}$. Additionally, we assume that $f_\ell \in \ell^1(\Lambda_+^*)$ if $\flat_{\ell-1} = \cdot$ and $\sharp_\ell = *$ for some $\ell = 1, \dots, n$ (i.e. if the pair $a^{\flat_{\ell-1}}_{\alpha_{\ell-1} p_\ell} a^{\sharp_\ell}_{\beta_\ell p_\ell}$ is not normally ordered). In position space, the same operator can be written as

$$\begin{aligned}\Pi^{(1)}_{\sharp,\flat}(f_1, \dots, f_n; g) = \int \check{b}^{\flat_0}_{x_1} \check{a}^{\sharp_1}_{y_1} \check{a}^{\flat_1}_{x_2} \check{a}^{\sharp_2}_{y_2} \check{a}^{\flat_2}_{x_3} \dots \check{a}^{\sharp_{n-1}}_{y_{n-1}} \check{a}^{\flat_{n-1}}_{x_n} \check{a}^{\sharp_n}_{y_n} \check{a}^{\flat_n}(g) \\ \times \prod_{\ell=1}^n \check{f}_\ell(x_\ell - y_\ell)\, dx_\ell dy_\ell.\end{aligned} \tag{3.10}$$

An operator of the form (3.9), (3.10) will be called a $\Pi^{(1)}$-operator of order n. Operators of the form $b(f), b^*(f)$, for a $f \in \ell^2(\Lambda_+^*)$, will be called $\Pi^{(1)}$-operators of order zero.

The next lemma gives a detailed analysis of the nested commutators $\mathrm{ad}^{(n)}_{B(\eta)}(b_p)$ and $\mathrm{ad}^{(n)}_{B(\eta)}(b^*_p)$ for $n \in \mathbb{N}$; the proof can be found in [1, Lemma 2.5] (it is a translation to momentum space of [5, Lemma 3.2]).

Lemma 3.2. *Let $\eta \in \ell^2(\Lambda_+^*)$ be such that $\eta_p = \eta_{-p}$ for all $p \in \ell^2(\Lambda^*)$. To simplify the notation, assume also η to be real-valued (as it will be in applications). Let $B(\eta)$ be defined as in (3.1), $n \in \mathbb{N}$ and $p \in \Lambda^*$. Then the nested commutator $ad^{(n)}_{B(\eta)}(b_p)$ can be written as the sum of exactly $2^n n!$ terms, with the following properties.*

(i) *Possibly up to a sign, each term has the form*

$$\Lambda_1 \Lambda_2 \dots \Lambda_i\, N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta^s_p \varphi_{\alpha p}) \tag{3.11}$$

for some $i, k, s \in \mathbb{N}$, $j_1, \dots, j_k \in \mathbb{N}\backslash\{0\}$, $\sharp \in \{\cdot, *\}^k$, $\flat \in \{\cdot, *\}^{k+1}$ and $\alpha \in \{\pm 1\}$ chosen so that $\alpha = 1$ if $\flat_k = \cdot$ and $\alpha = -1$ if $\flat_k = *$ (recall here that $\varphi_p(x) = e^{-ip\cdot x}$). In (3.11), each operator $\Lambda_w : \mathcal{F}^{\leq N} \to \mathcal{F}^{\leq N}$, $w = 1, \dots, i$, is either a factor $(N - \mathcal{N}_+)/N$, a factor $(N - (\mathcal{N}_+ - 1))/N$ or an operator of the form

$$N^{-h} \Pi^{(2)}_{\sharp',\flat'}(\eta^{z_1}, \eta^{z_2}, \dots, \eta^{z_h}) \tag{3.12}$$

for some $h, z_1, \dots, z_h \in \mathbb{N}\backslash\{0\}$, $\sharp, \flat \in \{\cdot, *\}^h$.

(ii) *If a term of the form* (3.11) *contains* $m \in \mathbb{N}$ *factors* $(N-\mathcal{N}_+)/N$ *or* $(N-(\mathcal{N}_+-1))/N$ *and* $j \in \mathbb{N}$ *factors of the form* (3.12) *with* $\Pi^{(2)}$*-operators of order* $h_1, \dots, h_j \in \mathbb{N}\backslash\{0\}$*, then we have*

$$m + (h_1 + 1) + \cdots + (h_j + 1) + (k + 1) = n + 1.$$

(iii) *If a term of the form* (3.11) *contains (considering all* Λ*-operators and the* $\Pi^{(1)}$*-operator) the arguments* $\eta^{i_1}, \dots, \eta^{i_m}$ *and the factor* η_p^s *for some* $m, s \in \mathbb{N}$*, and* $i_1, \dots, i_m \in \mathbb{N}\backslash\{0\}$*, then*

$$i_1 + \cdots + i_m + s = n.$$

(iv) *There is exactly one term having of the form* (3.11) *with* $k = 0$ *and such that all* Λ*-operators are factors of* $(N - \mathcal{N}_+)/N$ *or of* $(N + 1 - \mathcal{N}_+)/N$*. It is given by*

$$\left(\frac{N-\mathcal{N}_+}{N}\right)^{n/2} \left(\frac{N+1-\mathcal{N}_+}{N}\right)^{n/2} \eta_p^n b_p$$

if n *is even, and by*

$$-\left(\frac{N-\mathcal{N}_+}{N}\right)^{(n+1)/2} \left(\frac{N+1-\mathcal{N}_+}{N}\right)^{(n-1)/2} \eta_p^n b_{-p}^*$$

if n *is odd.*

(v) *If the* $\Pi^{(1)}$*-operator in* (3.11) *is of order* $k \in \mathbb{N}\backslash\{0\}$*, it has either the form*

$$\sum_{p_1,\dots,p_k} b_{\alpha_0 p_1}^{\flat_0} \prod_{i=1}^{k-1} a_{\beta_i p_i}^{\sharp_i} a_{\alpha_i p_{i+1}}^{\flat_i} a_{-p_k}^* \eta_p^{2r} a_p \prod_{i=1}^{k} \eta_{p_i}^{j_i}$$

or the form

$$\sum_{p_1,\dots,p_k} b_{\alpha_0 p_1}^{\flat_0} \prod_{i=1}^{k-1} a_{\beta_i p_i}^{\sharp_i} a_{\alpha_i p_{i+1}}^{\flat_i} a_{p_k} \eta_p^{2r+1} a_p^* \prod_{i=1}^{k} \eta_{p_i}^{j_i}$$

for some $r \in \mathbb{N}$*,* $j_1, \dots, j_k \in \mathbb{N}\backslash\{0\}$*. If it is of order* $k = 0$*, then it is either given by* $\eta_p^{2r} b_p$ *or by* $\eta_p^{2r+1} b_{-p}^*$*, for some* $r \in \mathbb{N}$*.*

(vi) *For every non-normally ordered term of the form*

$$\sum_{q\in\Lambda^*} \eta_q^i a_q a_q^*, \quad \sum_{q\in\Lambda^*} \eta_q^i b_q a_q^*$$
$$\sum_{q\in\Lambda^*} \eta_q^i a_q b_q^*, \quad \text{or} \quad \sum_{q\in\Lambda^*} \eta_q^i b_q b_q^*$$

appearing either in the Λ*-operators or in the* $\Pi^{(1)}$*-operator in* (3.11)*, we have* $i \geq 2$*.*

With Lemma 3.2, it follows from (3.6) that, if $\|\eta\|$ is sufficiently small,

$$\begin{aligned} e^{-B(\eta)} b_p e^{B(\eta)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{ad}^{(n)}_{B(\eta)}(b_p) \\ e^{-B(\eta)} b^*_p e^{B(\eta)} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{ad}^{(n)}_{B(\eta)}(b^*_p) \end{aligned} \tag{3.13}$$

where the series converge absolutely (the proof is a translation to momentum space of [5, Lemma 3.3]).

While Lemma 3.2 gives a complete characterization of terms appearing in the expansions (3.13), to localize the number of particles as we do in Proposition 4.3, we will need to consider double commutators of $\mathrm{ad}^{(n)}_{-B(\eta)}(b_p)$ with a smooth function $f(\mathcal{N}_+/M)$ of the number of particles operator $\mathcal{N}_+$, varying on the scale $M \in \mathbb{N}\setminus\{0\}$. To this end, we will apply the following corollary, which is a simple consequence of Lemma 3.2.

Corollary 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real, smooth and bounded function. For $M \in \mathbb{N}\setminus\{0\}$, let $f_M = f(\mathcal{N}_+/M)$. Then, for any $n \in \mathbb{N}$, $p \in \Lambda^*_+$, the double commutator $[f_M, [f_M, ad^{(n)}_{-B(\eta)}(b_p)]]$ can be written as the sum of $2^n n!$ (possibly vanishing) terms, having the form*

$$F_{M,n}(\mathcal{N}_+)\, \Lambda_1 \Lambda_2 \dots \Lambda_i N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta^s_p \varphi_{\alpha p})$$

for some $i, k, s \in \mathbb{N}$, $j_1, \dots, j_k \in \mathbb{N}\setminus\{0\}$, $\sharp \in \{\cdot, *\}^k$, $\flat \in \{\cdot, *\}^{k+1}$ and $\alpha \in \{\pm 1\}$ chosen so that $\alpha = 1$ if $\flat_k = \cdot$ and $\alpha = -1$ if $\flat_k = *$, where the operators $\Lambda_1, \dots, \Lambda_i$ and $\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta^s_p \varphi_{\alpha p})$ satisfy all properties listed in the points (i)–(vi) *in Lemma 3.2 and where $F_{M,n}$ is a bounded function such that*

$$\|F_{M,n}(\mathcal{N}_+)\| \leq \frac{Cn^2}{M^2} \|f'\|^2_\infty \tag{3.14}$$

for a universal constant $C > 0$ (different terms will have different functions $F_{M,n}$, but they will all satisfy (3.14) with the same constant $C > 0$).

Proof. It follows from Lemma 3.2 that, for any $n \in \mathbb{N}$, $\mathrm{ad}^{(n)}_{-B(\eta)}(b_p)$ can be written as the sum of $2^n n!$ terms of the form (up to a sign)

$$\Lambda_1 \Lambda_2 \dots \Lambda_i\, N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta^s_p \varphi_{\alpha p}) \tag{3.15}$$

for some $i, k, s \in \mathbb{N}$, $j_1, \dots, j_k \in \mathbb{N}\setminus\{0\}$, $\sharp \in \{\cdot, *\}^k$, $\flat \in \{\cdot, *\}^{k+1}$ and $\alpha \in \{\pm 1\}$ chosen so that $\alpha = 1$ if $\flat_k = \cdot$ and $\alpha = -1$ if $\flat_k = *$. In (3.15), each operator $\Lambda_w : \mathcal{F}^{\leq N} \to \mathcal{F}^{\leq N}$, $w = 1, \dots, i$, is either a factor $(N - \mathcal{N}_+)/N$, a factor $(N - (\mathcal{N}_+ - 1))/N$ or an operator of the form

$$N^{-h} \Pi^{(2)}_{\sharp',\flat'}(\eta^{z_1}, \eta^{z_2}, \dots, \eta^{z_h}) \tag{3.16}$$

for some $h, z_1, \dots, z_h \in \mathbb{N}\backslash\{0\}$, $\sharp, \flat \in \{\cdot, *\}^h$. The commutator of (3.15) with f_M is therefore given by

$$\begin{aligned}
&[f_M, \Lambda_1\Lambda_2\dots\Lambda_i\, N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_k};\eta^s_p\varphi_{\alpha p})]\\
&\quad=\sum_{u=1}^{i}\Big(\prod_{t=1}^{u-1}\Lambda_t\Big)[f_M,\Lambda_u]\Big(\prod_{t=u+1}^{i}\Lambda_t\Big)N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_k};\eta^s_p\varphi_{\alpha p})\\
&\qquad+\Lambda_1\Lambda_2\dots\Lambda_i\, N^{-k}[f_M,\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_k};\eta^s_p\varphi_{\alpha p})].
\end{aligned}$$

Recalling (3.7) and (3.9) and using the identities $b_p\mathcal{N}_+ = (\mathcal{N}_+ + 1)b_p$, $b^*_p\mathcal{N}_+ = (\mathcal{N}_+ - 1)b^*_p$, we obtain that

$$[f_M,\Lambda_u]=\Big[f\Big(\frac{\mathcal{N}_+}{M}\Big)-f\Big(\frac{\mathcal{N}_+ + e_u}{M}\Big)\Big]\Lambda_u$$

with $e_u = 0$ if Λ_u is either $(N-\mathcal{N}_+)/N$ or $(N-(\mathcal{N}_+-1))/N$, while e_u takes values in the set $\{-2, 0, 2\}$ if Λ_u is of the form (3.16) ($\Pi^{(2)}_{\sharp,\flat}$-operators can either create or annihilate two excitations, or it can leave the number of excitations invariant). Moreover

$$\begin{aligned}
&\big[f_M,\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_k};\eta^s_p\varphi_{\alpha p})\big]\\
&\quad=\Big[f\Big(\frac{\mathcal{N}_+}{M}\Big)-f\Big(\frac{\mathcal{N}_+\pm 1}{M}\Big)\Big]\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_k};\eta^s_p\varphi_{\alpha p})
\end{aligned}$$

because $\Pi^{(1)}_{\sharp,\flat}$ can create or annihilate only one excitation. Therefore

$$\begin{aligned}
&[f_M,\Lambda_1\Lambda_2\dots\Lambda_i\, N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_k};\eta^s_p\varphi_{\alpha p})]\\
&=\sum_{u=1}^{i}\Big(\prod_{t=1}^{u-1}\Lambda_t\Big)\Big[f\Big(\frac{\mathcal{N}_+}{M}\Big)-f\Big(\frac{\mathcal{N}_+ + e_u}{M}\Big)\Big]\\
&\quad\times\Lambda_u\Big(\prod_{r=u+1}^{i}\Lambda_t\Big)N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_k};\eta^s_p\varphi_{\alpha p})\\
&\quad+\Lambda_1\Lambda_2\dots\Lambda_i\, N^{-k}\Big[f\Big(\frac{\mathcal{N}_+}{M}\Big)-f\Big(\frac{\mathcal{N}_+\pm 1}{M}\Big)\Big]\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_k};\eta^s_p\varphi_{\alpha p}).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&[f_M,\Lambda_1\Lambda_2\dots\Lambda_i\, N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_k};\eta^s_p\varphi_{\alpha p})]\\
&=\Big\{\sum_{u=1}^{i}\Big[f\Big(\frac{\mathcal{N}_+ + n_{u-1}}{M}\Big)-f\Big(\frac{\mathcal{N}_+ + e_u + n_{u-1}}{M}\Big)\Big]\\
&\quad+\Big[f\Big(\frac{\mathcal{N}_+ + n_i}{M}\Big)-f\Big(\frac{\mathcal{N}_+\pm 1 + n_i}{M}\Big)\Big]\Big\}\\
&\quad\times\Lambda_1\Lambda_2\dots\Lambda_i\, N^{-k}\Pi^{(1)}_{\sharp,\flat}(\eta^{j_1},\dots,\eta^{j_k};\eta^s_p\varphi_{\alpha p})
\end{aligned}$$

where $n_u = \sum_{t=1}^{u} e_t$. By the mean value theorem, we can find functions $\theta_1 : \mathbb{N} \to (0, \pm 1)$, $\theta_u : \mathbb{N} \to (0, e_u)$ such that

$$\begin{aligned}[f_M, \Lambda_1 \Lambda_2 \dots \Lambda_i \, N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta^s_p \varphi_{\alpha p})] \\ = \frac{1}{M} \left[\sum_{u=1}^{i} e_u f' \Big(\frac{\mathcal{N}_+ + \theta_u(\mathcal{N}_+)}{M} \Big) + f' \Big(\frac{\mathcal{N}_+ + \theta_1(\mathcal{N}_+)}{M} \Big) \right] \\ \times \Lambda_1 \Lambda_2 \dots \Lambda_i \, N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta^s_p \varphi_{\alpha p}).\end{aligned}$$

It follows that

$$\begin{aligned}[f_M, [f_M, \Lambda_1 \Lambda_2 \dots \Lambda_i \, N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta^s_p \varphi_{\alpha p})]] \\ = F_{M,n}(\mathcal{N}_+) \Lambda_1 \Lambda_2 \dots \Lambda_i \, N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta^s_p \varphi_{\alpha p})\end{aligned}$$

with

$$F_{M,n}(\mathcal{N}_+) = \frac{1}{M^2} \left[\sum_{u=1}^{i} e_u f' \Big(\frac{\mathcal{N}_+ + \theta_u(\mathcal{N}_+)}{M} \Big) + f' \Big(\frac{\mathcal{N}_+ + \theta_1(\mathcal{N}_+)}{M} \Big) \right]^2$$

depending on the precise form of the operator $\Lambda_1 \Lambda_2 \dots \Lambda_i \, N^{-k} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_k}; \eta^s_p \varphi_{\alpha p})$. Since $e_u \neq 0$ only if Λ_u is a $\Pi^{(2)}$ operator, since there are at most n $\Pi^{(2)}$ operators among $\Lambda_1, \dots, \Lambda_i$ and since $|e_u| \leq 2$ for all $u \in \{1, \dots, i\}$, we conclude that, for example,

$$\| F_{M,n} \| \leq \frac{3n^2}{M^2} \| f' \|^2_\infty .$$

□

As explained after their Definition (2.5), the generalized creation and annihilation operators b^*_p, b_p are close to the standard creation and annihilation operators on states with only few excitations, ie. with $\mathcal{N}_+ \ll N$. In particular, on these states we expect the action of the generalized Bogoliubov transformation (3.2) to be close to the action (3.4) of the standard Bogoliubov transformation (3.3). To make this statement more precise we define, under the assumption that $\|\eta\|$ is small enough, the remainder operators

$$\begin{aligned} d_q &= \sum_{m \geq 0} \frac{1}{m!} \Big[\mathrm{ad}^{(m)}_{-B(\eta)}(b_q) - \eta^m_q b^{\sharp_m}_{\alpha_m q} \Big], \\ d^*_q &= \sum_{m \geq 0} \frac{1}{m!} \Big[\mathrm{ad}^{(m)}_{-B(\eta)}(b^*_q) - \eta^m_q b^{\sharp_{m+1}}_{\alpha_m q} \Big] \end{aligned} \tag{3.17}$$

where $q \in \Lambda^*_+$, $(\sharp_m, \alpha_m) = (\cdot, +1)$ if m is even and $(\sharp_m, \alpha_m) = (*, -1)$ if m is odd. It follows then from (3.13) that

$$\begin{aligned} e^{-B(\eta)} b_q e^{B(\eta)} &= \gamma_q b_q + \sigma_q b^*_{-q} + d_q, \\ e^{-B(\eta)} b^*_q e^{B(\eta)} &= \gamma_q b^*_q + \sigma_q b_{-q} + d^*_q \end{aligned} \tag{3.18}$$

where we introduced the notation $\gamma_q = \cosh(\eta_q)$ and $\sigma_q = \sinh(\eta_q)$. It will also be useful to introduce remainder operators in position space. For $x \in \Lambda$, we define the operator valued distributions $\check{d}_x, \check{d}_x^*$ through

$$\begin{aligned} e^{-B(\eta)} \check{b}_x e^{B(\eta)} &= b(\check{\gamma}_x) + b^*(\check{\sigma}_x) + \check{d}_x, \\ e^{-B(\eta)} \check{b}_x^* e^{B(\eta)} &= b^*(\check{\gamma}_x) + b(\check{\sigma}_x) + \check{d}_x^* \end{aligned} \tag{3.19}$$

where $\check{\gamma}_x(y) = \sum_{q \in \Lambda^*} \cosh(\eta_q) e^{iq\cdot(x-y)}$ and $\check{\sigma}_x(y) = \sum_{q \in \Lambda^*} \sinh(\eta_q) e^{iq\cdot(x-y)}$.

The next lemma confirms the intuition that remainder operators are small, on states with $\mathcal{N}_+ \ll N$, and provides estimates that will be crucial for our analysis.

Lemma 3.4. *Let $\eta \in \ell^2(\Lambda_+^*)$, $n \in \mathbb{Z}$. For $p \in \Lambda_+^*$, let d_p be defined as in (3.17). If $\|\eta\|$ is small enough, there exists $C > 0$ such that*

$$\begin{aligned} \|(\mathcal{N}_+ + 1)^{n/2} d_p \xi\| &\leq \frac{C}{N} \Big[|\eta_p| \|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| + \|\eta\| \|b_p (\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| \Big], \\ \|(\mathcal{N}_+ + 1)^{n/2} d_p^* \xi\| &\leq \frac{C}{N} \|\eta\| \, \|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| \end{aligned} \tag{3.20}$$

for all $p \in \Lambda_+^*$, $\xi \in \mathcal{F}_+^{\leq N}$. With $\bar{\bar{d}}_p = d_p + N^{-1} \sum_{q \in \Lambda_+^*} \eta_q b_q^* a_{-q}^* a_p$, we also have, for $p \notin supp\, \eta$, the improved bound

$$\|(\mathcal{N}_+ + 1)^{n/2} \bar{\bar{d}}_p \xi\| \leq \frac{C}{N} \|\eta\|^2 \|a_p (\mathcal{N}_+ + 1)^{(n+2)/2} \xi\|. \tag{3.21}$$

In position space, with $\check{d}_x$ defined as in (3.19), we find

$$\|(\mathcal{N}_+ + 1)^{n/2} \check{d}_x \xi\| \leq \frac{C}{N} \|\eta\| \Big[\|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| + \|b_x (\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| \Big]. \tag{3.22}$$

Furthermore, letting $\check{\bar{d}}_x = \check{d}_x + (\mathcal{N}_+/N) b^*(\check{\eta}_x)$, we find

$$\begin{aligned} &\|(\mathcal{N}_+ + 1)^{n/2} \check{a}_y \check{\bar{d}}_x \xi\| \\ &\leq \frac{C}{N} \Big[\|\eta\|^2 \|(\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| + \|\eta\| |\check{\eta}(x-y)| \|(\mathcal{N} + 1)^{(n+2)/2} \xi\| \\ &\quad + \|\eta\| \|\check{a}_x (\mathcal{N}_+ + 1)^{(n+1)/2} \xi\| + \|\eta\|^2 \|\check{a}_y (\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| \\ &\quad + \|\eta\| \|\check{a}_x \check{a}_y (\mathcal{N} + 1)^{(n+2)/2} \xi\| \Big] \end{aligned} \tag{3.23}$$

and, finally,

$$\begin{aligned} &\|(\mathcal{N}_+ + 1)^{n/2} \check{d}_x \check{d}_y \xi\| \\ &\leq \frac{C}{N^2} \Big[\|\eta\|^2 \|(\mathcal{N}_+ + 1)^{(n+6)/2} \xi\| + \|\eta\| |\check{\eta}(x-y)| \|(\mathcal{N}_+ + 1)^{(n+4)/2} \xi\| \\ &\quad + \|\eta\|^2 \|a_x (\mathcal{N}_+ + 1)^{(n+5)/2} \xi\| + \|\eta\|^2 \|a_y (\mathcal{N}_+ + 1)^{(n+5)/2} \xi\| \\ &\quad + \|\eta\|^2 \|a_x a_y (\mathcal{N}_+ + 1)^{(n+4)/2} \xi\| \Big] \end{aligned} \tag{3.24}$$

for all $\xi \in \mathcal{F}_+^{\leq n}$.

Proof. To prove the first bound in (3.20), we notice that, from (3.17) and from the triangle inequality (for simplicity, we focus on $n = 0$, powers of $\mathcal{N}_+$ can be easily commuted through the operators d_p),

$$\|d_q \xi\| \leq \sum_{m\geq 0} \frac{1}{m!} \left\| \left[\mathrm{ad}^{(m)}_{-B(\eta)}(b_q) - \eta_q^m b^{\sharp_m}_{\alpha_m p} \right] \xi \right\| . \tag{3.25}$$

From Lemma 3.2, we can bound the norm $\|[\mathrm{ad}^{(m)}_{-B(\eta)}(b_q) - \eta_q^m b^{\sharp_m}_{\alpha_m p}]\xi\|$ by the sum of one term of the form

$$\left\| \left[\left(\frac{N-\mathcal{N}_+}{N} \right)^{\frac{m+(1-\alpha_m)/2}{2}} \left(\frac{N+1-\mathcal{N}_+}{N} \right)^{\frac{m-(1-\alpha_m)/2}{2}} - 1 \right] \eta_p^m b^{\sharp_m}_{\alpha_m p} \xi \right\| \tag{3.26}$$

and of exactly $2^m m! - 1$ terms of the form

$$\left\| \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1} \varphi_{\alpha_{\ell_1} p}) \xi \right\| \tag{3.27}$$

where $i_1, k_1, \ell_1 \in \mathbb{N}$, $j_1, \dots, j_{k_1} \in \mathbb{N}\backslash\{0\}$ and where each Λ_r-operator is either a factor $(N - \mathcal{N}_+)/N$, a factor $(N + 1 - \mathcal{N}_+)/N$ or a $\Pi^{(2)}$-operator of the form

$$N^{-h} \Pi^{(2)}_{\sharp,\flat}(\eta^{z_1}, \dots, \eta^{z_h}) \tag{3.28}$$

with $h, z_1, \dots, z_h \in \mathbb{N}\backslash\{0\}$. Furthermore, since we are considering the term (3.26) separately, each term of the form (3.27) must have either $k_1 > 0$ or it must contain at least one Λ-operator having the form (3.28). Since (3.26) vanishes for $m = 0$, it is easy to bound

$$\begin{aligned} &\left\| \left[\left(\frac{N-\mathcal{N}_+}{N} \right)^{\frac{m+(1-\alpha_m)/2}{2}} \left(\frac{N+1-\mathcal{N}_+}{N} \right)^{\frac{m-(1-\alpha_m)/2}{2}} - 1 \right] \eta_p^m b^{\sharp_m}_{\alpha_m p} \xi \right\| \\ &\quad \leq C^m |\eta_p|^m N^{-1} \|(\mathcal{N}_+ + 1)^{3/2} \xi\| . \end{aligned}$$

On the other hand, distinguishing the cases $\ell_1 > 0$ and $\ell_1 = 0$, we can bound

$$\begin{aligned} &\left\| \Lambda_1 \dots \Lambda_{i_1} N^{-k_1} \Pi^{(1)}_{\sharp,\flat}(\eta^{j_1}, \dots, \eta^{j_{k_1}}; \eta_p^{\ell_1} \varphi_{\alpha_{\ell_1} p}) \xi \right\| \\ &\leq C^m N^{-1} \left[\|\eta\|^{m-\ell_1} |\eta_p|^{\ell_1} \delta_{\ell_1 > 0} \|(\mathcal{N}_+ + 1)^{3/2} \xi\| + \|\eta\|^m \|b_p (\mathcal{N}_+ + 1) \xi\| \right] \\ &\leq C^m \|\eta\|^{m-1} N^{-1} \left[|\eta_p| \delta_{m>0} \|(\mathcal{N}_+ + 1)^{3/2} \xi\| + \|\eta\| \|b_p (\mathcal{N}_+ + 1) \xi\| \right] \end{aligned} \tag{3.29}$$

where in the last line we used $|\eta_p| \leq \|\eta\|$. Inserting the last two bounds in (3.25) and summing over m under the assumption that $\|\eta\|$ is small enough, we arrive at the first estimate (3.20). The second estimate in (3.20) can be proven similarly (notice that, when dealing with the second estimate in (3.20), contributions of the form (3.27) with $\ell_1 = 0$, can only be bounded by $\|b_p^*(\mathcal{N}_+ + 1)\xi\| \leq \|(\mathcal{N}_+ + 1)^{3/2}\xi\|$). To show (3.21), we notice that $\bar{\bar{d}}_p$ is exactly defined to cancel the only contribution with $m = 1$ that does not vanish for $p \notin \mathrm{supp}\, \eta$. Moreover, the assumption $\eta_p = 0$ implies that only terms with $\ell_1 = 0$ survive in (3.29). Also the bounds in (3.22) and (3.23) can be shown analogously, using [2, Lemma 7.2]. □

To localize the number of particles operator in Proposition 4.3, we will also need to control the double commutator of the remainder operators d_p, d_p^* with smooth functions $f(\mathcal{N}_+/M)$ of the number of particles operator, varying on the scale M. To this end, we use the next corollary, which is an immediate consequence of Corollary 3.3 and of Lemma 3.4 (and of its proof).

Corollary 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and bounded. For $M \in \mathbb{N}\backslash\{0\}$, let $f_M = f(\mathcal{N}_+/M)$. The bounds in (3.20), (3.21), (3.22), (3.23) and (3.24) remain true if we replace, on the left hand side, d_p by $[f_M, [f_M, d_p]]$, $\ddot{\overline{d}}_p$ by $[f_M, [f_M, \ddot{\overline{d}}_p]]$, $\check{d}_x$ by $[f_M, [f_M, \check{d}_x]]$, $\check{a}_y \ddot{\check{d}}_x$ by $[f_M, [f_M, \check{a}_y \ddot{\check{d}}_x]]$ and $\check{d}_x \check{d}_y$ by $[f_M, [f_M, \check{d}_x \check{d}_y]]$ and, on the right hand side, the constant C by $CM^{-2}\|f'\|_\infty^2$. For example, the first bound in (3.20) becomes*

$$\left\| (\mathcal{N}_+ + 1)^{n/2} [f_M, [f_M, d_p]] \xi \right\| \leq \frac{C\|f'\|_\infty^2}{NM^2} \left[|\eta_p| \|(\mathcal{N}_+ + 1)^{(n+3)/2} \xi\| + \|\eta\| \|b_p (\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| \right].$$

4. Quadratic Renormalization

We use now a generalized Bogoliubov transformation $\exp(B(\eta))$ of the form (3.2) to renormalize the excitation Hamiltonian. To make sure that $\exp(B(\eta))$ removes correlations that are present in low-energy states, we have to choose the coefficients $\eta \in \ell^2(\Lambda_+^*)$ appropriately. To this end, we consider the ground state solution of the Neumann problem

$$\left[-\Delta + \frac{1}{2} V \right] f_\ell = \lambda_\ell f_\ell \tag{4.1}$$

on the ball $|x| \leq N\ell$ (we omit here the N-dependence in the notation for f_ℓ and for λ_ℓ; notice that λ_ℓ scales as N^{-3}), with the normalization $f_\ell(x) = 1$ if $|x| = N\ell$. By scaling, we observe that $f_\ell(N.)$ satisfies the equation

$$\left[-\Delta + \frac{N^2}{2} V(Nx) \right] f_\ell(Nx) = N^2 \lambda_\ell f_\ell(Nx)$$

on the ball $|x| \leq \ell$. We choose $0 < \ell < 1/2$, so that the ball of radius ℓ is contained in the box $\Lambda = [-1/2; 1/2]^3$ (later, we will choose $\ell > 0$ small enough, but always of order one, independent of N). We extend then $f_\ell(N.)$ to Λ, by setting $f_{N,\ell}(x) = f_\ell(Nx)$, if $|x| \leq \ell$ and $f_{N,\ell}(x) = 1$ for $x \in \Lambda$, with $|x| > \ell$. Then

$$\left(-\Delta + \frac{N^2}{2} V(Nx) \right) f_{N,\ell} = N^2 \lambda_\ell f_{N,\ell} \chi_\ell \tag{4.2}$$

where χ_ℓ is the characteristic function of the ball of radius ℓ. The Fourier coefficients of the function $f_{N,\ell}$ are given by

$$\widehat{f}_{N,\ell}(p) := \int_\Lambda f_\ell(Nx) e^{-ip\cdot x} dx \tag{4.3}$$

for all $p \in \Lambda^*$. It is also useful to introduce the function $w_\ell(x) = 1 - f_\ell(x)$ for $|x| \leq N\ell$ and to extend it by setting $w_\ell(x) = 0$ for $|x| > N\ell$. Its rescaled version $w_{N,\ell} : \Lambda \to \mathbb{R}$

is then defined through $w_{N,\ell}(x) = w_\ell(Nx)$ if $|x| \leq \ell$ and $w_{N,\ell}(x) = 0$ if $x \in \Lambda$ with $|x| > \ell$. The Fourier coefficients of $w_{N,\ell}$ are then given, for $p \in \Lambda^*$, by

$$\widehat{w}_{N,\ell}(p) = \int_\Lambda w_\ell(Nx) e^{-ip\cdot x} dx = \frac{1}{N^3} \widehat{w}_\ell(p/N)$$

where

$$\widehat{w}_\ell(k) = \int_{\mathbb{R}^3} w_\ell(x) e^{-ik\cdot x} dx$$

denotes the Fourier transform of the (compactly supported) function w_ℓ. We find $\widehat{f}_{N,\ell}(p) = \delta_{p,0} - N^{-3}\widehat{w}_\ell(p/N)$. From (4.2), we obtain

$$\begin{aligned} &- p^2 \widehat{w}_\ell(p/N) + \frac{N^2}{2} \sum_{q\in\Lambda^*} \widehat{V}((p-q)/N) \widehat{f}_{N,\ell}(q) \\ &= N^5 \lambda_\ell \sum_{q\in\Lambda^*} \widehat{\chi}_\ell(p-q) \widehat{f}_{N,\ell}(q). \end{aligned} \tag{4.4}$$

In the next lemma we collect some important properties of w_ℓ, f_ℓ. The proof of the lemma is given in "Appendix A".

Lemma 4.1. *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, compactly supported and spherically symmetric. Fix $\ell > 0$ and let f_ℓ denote the solution of (4.1). For N large enough the following properties hold true.*

i) We have

$$\lambda_\ell = \frac{3\mathfrak{a}_0}{(\ell N)^3} \left(1 + \mathcal{O}(\mathfrak{a}_0/\ell N)\right). \tag{4.5}$$

ii) We have $0 \leq f_\ell, w_\ell \leq 1$. Moreover there exists a constant $C > 0$ such that

$$\left| \int V(x) f_\ell(x) dx - 8\pi \mathfrak{a}_0 \right| \leq \frac{C\mathfrak{a}_0^2}{\ell N} \tag{4.6}$$

for all $\ell \in (0; 1/2)$ and $N \in \mathbb{N}$.

iii) There exists a constant $C > 0$ such that

$$w_\ell(x) \leq \frac{C}{|x|+1} \quad \text{and} \quad |\nabla w_\ell(x)| \leq \frac{C}{x^2+1} \tag{4.7}$$

for all $x \in \mathbb{R}^3$, $\ell \in (0; 1/2)$ and all N large enough.

iv) There exists a constant $C > 0$ such that

$$|\widehat{w}_{N,\ell}(p)| \leq \frac{C}{Np^2}$$

for all $p \in \mathbb{R}^3$, all $\ell \in (0; 1/2)$ and all N large enough (such that $N \geq \ell^{-1}$).

We define $\eta : \Lambda^* \to \mathbb{R}$ through

$$\eta_p = -N\widehat{w}_{N,\ell}(p) = -\frac{1}{N^2}\widehat{w}_\ell(p/N).$$

With Lemma 4.1, we can bound

$$|\eta_p| \leq \frac{C}{|p|^2} \tag{4.8}$$

for all $p \in \Lambda^*_+ = 2\pi\mathbb{Z}^3\backslash\{0\}$, and for some constant $C > 0$ independent of N and $\ell \in (0; \frac{1}{2})$, if N is large enough. From (4.4), we also find the relation

$$p^2\eta_p + \frac{1}{2}(\widehat{V}(./N) * \widehat{f}_{N,\ell})(p) = N^3\lambda_\ell(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(p) \tag{4.9}$$

or equivalently, expressing the r.h.s. through the coefficients η_p,

$$\begin{aligned} p^2\eta_p + \frac{1}{2}\widehat{V}(p/N) + \frac{1}{2N}\sum_{q\in\Lambda^*}\widehat{V}((p-q)/N)\eta_q \\ = N^3\lambda_\ell\widehat{\chi}_\ell(p) + N^2\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q. \end{aligned} \tag{4.10}$$

Moreover, with (4.7), we find

$$\|\eta\|^2 = \|\check{\eta}\|^2 = \int_{|x|\leq\ell} N^2|w(Nx)|^2dx \leq C\int_{|x|\leq\ell}\frac{1}{|x|^2}dx \leq C\ell. \tag{4.11}$$

In particular, we can make $\|\eta\|$ arbitrarily small, choosing ℓ small enough.

For $\alpha > 0$, we now define the momentum set

$$P_H = \{p \in \Lambda^*_+ : |p| \geq \ell^{-\alpha}\}, \tag{4.12}$$

depending on the parameter $\ell > 0$ introduced in (4.1).[1] We set

$$\eta_H(p) = \eta_p\,\chi(p \in P_H) = \eta_p\chi(|p| \geq \ell^{-\alpha})\,. \tag{4.13}$$

Eq. (4.8) implies that

$$\|\eta_H\| \leq C\ell^{\alpha/2}. \tag{4.14}$$

For $\alpha > 1$, the last bound improves (4.11). As we will see later, this improvement, obtained through the introduction of a momentum cutoff, will play an important role in our analysis. Notice, on the other hand, that the H^1-norms of η and η_H diverge, as $N \to \infty$. From Lemma 4.1, part (iii), we find

$$\sum_{p\in P_H} p^2|\eta_p|^2 \leq \sum_{p\in\Lambda^*_+} p^2|\eta_p|^2 \leq CN \tag{4.15}$$

[1] At the end, we will need the high-momentum cutoff $\ell^{-\alpha}$ to be sufficiently large. To reach this goal, we will choose ℓ sufficiently small. Alternatively, we could decouple the cutoff from the radius ℓ introduced in (4.1), keeping $\ell \in (0; 1/2)$ fixed and choosing instead the exponent α sufficiently large.

for all $\ell \in (0; 1/2)$ and $N \in \mathbb{N}$ large enough. We will mostly use the coefficients η_p with $p \neq 0$. Sometimes, however, it will be useful to have an estimate on η_0 (because Eq. (4.10) involves η_0). From Lemma 4.1, part (iii) we find

$$|\eta_0| \leq N^{-2} \int_{\mathbb{R}^3} w_\ell(x) dx \leq C\ell^2. \tag{4.16}$$

It will also be useful to have bounds for the function $\check{\eta}_H : \Lambda \to \mathbb{R}$, having Fourier coefficients $\eta_H(p)$ as defined in (4.13). Writing $\eta_H(p) = \eta_p - \eta_p \chi(|p| \leq \ell^{-\alpha})$, we obtain

$$\check{\eta}_H(x) = \check{\eta}(x) - \sum_{\substack{p \in \Lambda^*: \\ |p| \leq \ell^{-\alpha}}} \eta_p e^{ip\cdot x} = -N w_\ell(Nx) - \sum_{\substack{p \in \Lambda^*: \\ |p| \leq \ell^{-\alpha}}} \eta_p e^{ip\cdot x}.$$

We obtain

$$|\check{\eta}_H(x)| \leq CN + \sum_{\substack{p \in \Lambda^*: \\ |p| \leq \ell^{-\alpha}}} |p|^{-2} \leq C(N + \ell^{-\alpha}) \leq CN \tag{4.17}$$

for all $x \in \Lambda$, if $N \in \mathbb{N}$ is large enough.

With the coefficients (4.13), we construct the generalized Bogoliubov transformation $e^{B(\eta_H)} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$, defined as in (3.2). Furthermore, we define a new, renormalized, excitation Hamiltonian $\mathcal{G}_{N,\ell} : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ by setting

$$\mathcal{G}_{N,\ell} = e^{-B(\eta_H)} \mathcal{L}_N e^{B(\eta_H)} = e^{-B(\eta_H)} U_N H_N U_N^* e^{B(\eta_H)}. \tag{4.18}$$

In the next proposition, we collect some important properties of the renormalized excitation Hamiltonian $\mathcal{G}_{N,\ell}$. In the following, we will use the notation

$$\mathcal{K} = \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p \quad \text{and} \quad \mathcal{V}_N = \frac{1}{2N} \sum_{\substack{p,q \in \Lambda_+^*, r \in \Lambda^*: \\ r \neq -p, -q}} \widehat{V}(r/N) a_{p+r}^* a_q^* a_{q+r} a_p \tag{4.19}$$

for the kinetic and potential energy operators, restricted on $\mathcal{F}_+^{\leq N}$. We will also write $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$.

Proposition 4.2. *Let $V \in L^3(\mathbb{R}^3)$ be compactly supported, pointwise non-negative and spherically symmetric. Then*

$$\mathcal{G}_{N,\ell} = 4\pi \mathfrak{a}_0 N + \mathcal{H}_N + \theta_{\mathcal{G}_{N,\ell}} \tag{4.20}$$

where for every $\delta > 0$ there exists a constant $C > 0$ such that

$$\pm\, \theta_{\mathcal{G}_{N,\ell}} \leq \delta \mathcal{H}_N + C\ell^{-\alpha}(\mathcal{N}_+ + 1) \tag{4.21}$$

and the improved lower bound

$$\theta_{\mathcal{G}_{N,\ell}} \geq -\delta \mathcal{H}_N - C\mathcal{N}_+ - C\ell^{-\alpha} \tag{4.22}$$

hold true for all $\alpha > 3$, $\ell \in (0; 1/2)$ small enough, $N \in \mathbb{N}$ large enough.

Furthermore, let

$$\begin{aligned}\mathcal{G}^{\mathit{eff}}_{N,\ell} :=\ & 4\pi\mathfrak{a}_0(N-\mathcal{N}_+) + \big[\widehat{V}(0) - 4\pi\mathfrak{a}_0\big]\mathcal{N}_+\frac{(N-\mathcal{N}_+)}{N} \\ & + \widehat{V}(0)\sum_{p\in P_H^c} a_p^* a_p (1-\mathcal{N}_+/N) + 4\pi\mathfrak{a}_0 \sum_{p\in P_H^c}\big[b_p^* b_{-p}^* + b_p b_{-p}\big] \\ & + \frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*: p+q\neq 0} \widehat{V}(p/N)\big[b_{p+q}^* a_{-p}^* a_q + \text{h.c.}\big] + \mathcal{H}_N .\end{aligned} \tag{4.23}$$

Then there exists a constant $C>0$ such that $\mathcal{E}_{\mathcal{G}_{N,\ell}} = \mathcal{G}_{N,\ell} - \mathcal{G}^{\mathit{eff}}_{N,\ell}$ is bounded by

$$\pm\,\mathcal{E}_{\mathcal{G}_{N,\ell}} \leq C\ell^{(\alpha-3)/2}\mathcal{H}_N + C\ell^{-\alpha} \tag{4.24}$$

for all $\alpha>3$, $\ell\in(0;1/2)$ small enough, and N large enough.

Finally, there exists a constant $C>0$ such that

$$\begin{aligned}&\pm\big[f(\mathcal{N}_+/M), \big[f(\mathcal{N}_+/M), \theta_{\mathcal{G}_{N,\ell}}\big]\big] \leq C\ell^{-\alpha/2}M^{-2}\|f'\|_\infty^2\big(\mathcal{H}_N+1\big) \\ &\pm\big[f(\mathcal{N}_+/M), \big[f(\mathcal{N}_+/M), \mathcal{E}_{\mathcal{G}_{N,\ell}}\big]\big] \leq C\ell^{(\alpha-3)/2}M^{-2}\|f'\|_\infty^2\big(\mathcal{H}_N+1\big)\end{aligned} \tag{4.25}$$

for all $\alpha>3$, $\ell\in(0;1/2)$ small enough, $f:\mathbb{R}\to\mathbb{R}$ smooth and bounded, $M\in\mathbb{N}$ and $N\in\mathbb{N}$ large enough.

The proof of Proposition 4.2 is technical and quite long; it is deferred to Sect. 7 below. Equation (4.25) allows us to prove a localization estimate for $\mathcal{G}_{N,\ell}$.

Proposition 4.3. *Let $f,g:\mathbb{R}\to[0;1]$ be smooth, with $f^2(x)+g^2(x)=1$ for all $x\in\mathbb{R}$. For $M\in\mathbb{N}$, let $f_M := f(\mathcal{N}_+/M)$ and $g_M := g(\mathcal{N}_+/M)$. There exists $C>0$ such that*

$$\mathcal{G}_{N,\ell} = f_M\,\mathcal{G}_{N,\ell}\,f_M + g_M\,\mathcal{G}_{N,\ell}\,g_M + \mathcal{E}_M$$

with

$$\pm\,\mathcal{E}_M \leq \frac{C\ell^{-\alpha/2}}{M^2}\big(\|f'\|_\infty^2 + \|g'\|_\infty^2\big)\big(\mathcal{H}_N+1\big)$$

for all $\alpha>3$, $\ell\in(0;1/2)$ small enough, $M\in\mathbb{N}$ and $N\in\mathbb{N}$ large enough.

Proof. An explicit computation shows that

$$\mathcal{G}_{N,\ell} = f_M\mathcal{G}_{N,\ell}f_M + g_M\mathcal{G}_{N,\ell}g_M + \frac{1}{2}\Big([f_M,[f_M,\mathcal{G}_{N,\ell}]] + [g_M,[g_M,\mathcal{G}_{N,\ell}]]\Big).$$

Writing as in (4.20), $\mathcal{G}_{N,\ell} = 4\pi\mathfrak{a}_0 N + \mathcal{H}_N + \theta_{\mathcal{G}_{N,\ell}}$, noticing that $4\pi\mathfrak{a}_0 N$ and $\mathcal{H}_N$ commute with f_M, g_M, and using the first bound in (4.25), we conclude that

$$\pm\Big([f_M,[f_M,\mathcal{G}_{N,\ell}]] + [g_M,[g_M,\mathcal{G}_{N,\ell}]]\Big) \leq \frac{C\ell^{-\alpha/2}}{M^2}\big(\|f_M'\|_\infty^2 + \|g_M'\|_\infty^2\big)\big(\mathcal{H}_N+1\big).$$

□

5. Cubic Renormalization

The quadratic renormalization leading to the excitation Hamiltonian $\mathcal{G}_{N,\ell}$ is not enough to show Theorem 1.1. In (4.22), the error term proportional to the number of particles operator cannot be controlled by the gap in the kinetic energy (in [1] this was possible, because the constant multiplying $\mathcal{N}_+$ is small, if the interaction potential is weak). To circumvent this problem, we have to conjugate the main part $\mathcal{G}^{\text{eff}}_{N,\ell}$ of $\mathcal{G}_{N,\ell}$, as defined in (4.23), with an additional unitary operator, given by the exponential of an expression cubic in creation and annihilation operators.

For a parameter $0 < \beta < \alpha$ we define the low-momentum set

$$P_L = \{p \in \Lambda^*_+ : |p| \leq \ell^{-\beta}\}$$

depending again on the parameter $\ell > 0$ introduced in (4.1).[2] Notice that the high-momentum set P_H defined in (4.12) and P_L are separated by a set of intermediate momenta $\ell^{-\beta} < |p| < \ell^{-\alpha}$. We introduce the operator $A : \mathcal{F}^{\leq N}_+ \to \mathcal{F}^{\leq N}_+$, by

$$A := \frac{1}{\sqrt{N}} \sum_{r \in P_H, v \in P_L} \eta_r \big[b^*_{r+v} a^*_{-r} a_v - \text{h.c.} \big]. \tag{5.1}$$

An important observation for our analysis is the fact that conjugation with e^A does not substantially change the number of excitations.

Proposition 5.1. *Suppose that A is defined as in (5.1). For any $k \in \mathbb{N}$ there exists a constant $C > 0$ such that the operator inequality*

$$e^{-A} (\mathcal{N}_+ + 1)^k e^A \leq C (\mathcal{N}_+ + 1)^k$$

holds true on $\mathcal{F}^{\leq N}_+$, for all $\alpha > \beta > 0$, $\ell \in (0; 1/2)$, and N large enough.

Proof. Let $\xi \in \mathcal{F}^{\leq N}_+$ and define $\varphi_\xi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_\xi(s) := \langle \xi, e^{-sA} (\mathcal{N}_+ + 1)^k e^{sA} \xi \rangle.$$

Then we have, using the notation $A_\gamma = N^{-1/2} \sum_{r \in P_H, v \in P_L} \eta_r b^*_{r+v} a^*_{-r} a_v$,

$$\partial_s \varphi_\xi(s) = 2\text{Re}\, \langle \xi, e^{-sA} \big[(\mathcal{N}_+ + 1)^k, A_\gamma \big] e^{sA} \xi \rangle.$$

We find

$$\begin{aligned} &\langle \xi, e^{-sA} \big[(\mathcal{N}_+ + 1)^k, A_\gamma \big] e^{sA} \xi \rangle \\ &\quad = \frac{1}{\sqrt{N}} \sum_{r \in P_H, v \in P_L} \eta_r \langle e^{sA} \xi, b^*_{r+v} a^*_{-r} a_{-v} \big[(\mathcal{N}_+ + 2)^k - (\mathcal{N}_+ + 1)^k \big] e^{sA} \xi \rangle. \end{aligned}$$

With the mean value theorem, we find a function $\theta : \mathbb{N} \to (0; 1)$ such that

$$(\mathcal{N}_+ + 2)^k - (\mathcal{N}_+ + 1)^k = k (\mathcal{N}_+ + \theta(\mathcal{N}_+) + 1)^{k-1}.$$

[2] At the end, we will need the low-momentum cutoff $\ell^{-\beta}$ to be sufficiently large (preserving however certain relations with the high-momentum cutoff). We will reach this goal by choosing ℓ small enough. Alternatively, as already remarked in the footnote after (4.12), also here we could decouple the low-momentum cutoff from the radius ℓ introduced in (4.1), by keeping $\ell \in (0; 1/2)$ fixed and varying instead the exponent β.

Since $b_p \mathcal{N}_+ = (\mathcal{N}_+ + 1) b_p$ and $b_p^* \mathcal{N}_+ = (\mathcal{N}_+ - 1) b_p^*$, we obtain, using Cauchy–Schwarz and the boundedness of θ,

$$\begin{aligned}
&\Big| \langle \xi, e^{-sA} \big[(\mathcal{N}_+ + 1)^k, A_\gamma \big] e^{sA} \xi \rangle \Big| \\
&\leq \frac{C}{\sqrt{N}} \sum_{r \in P_H, v \in P_L} |\eta_r| \big\| b_{r+v} a_{-r} (\mathcal{N}_+ + 1)^{-1/4 + (k-1)/2} e^{sA} \xi \big\| \\
&\quad \times \big\| a_{-v} (\mathcal{N}_+ + 1)^{1/4 + (k-1)/2} e^{sA} \xi \big\| \\
&\leq \frac{C}{\sqrt{N}} \Big[\sum_{r \in P_H, v \in P_L} \big\| b_{r+v} a_{-r} (\mathcal{N}_+ + 1)^{-1/4 + (k-1)/2} e^{sA} \xi \big\|^2 \Big]^{1/2} \\
&\quad \times \Big[\sum_{r \in P_H, v \in P_L} |\eta_r|^2 \big\| a_{-v} (\mathcal{N}_+ + 1)^{1/4 + (k-1)/2} e^{sA} \xi \big\|^2 \Big]^{1/2} \\
&\leq \frac{C}{\sqrt{N}} \|\eta_H\| \big\| (\mathcal{N}_+ + 1)^{3/4 + (k-1)/2} e^{sA} \xi \big\|^2 \\
&\leq \frac{C}{\sqrt{N}} \langle e^{sA} \xi, (\mathcal{N}_+ + 1)^{k+1/2} e^{sA} \xi \rangle \\
&\leq C \langle e^{sA} \xi, (\mathcal{N}_+ + 1)^k e^{sA} \rangle
\end{aligned}$$

for a constant $C > 0$ depending on k, but not on N or ℓ. This proves that

$$\partial_s \varphi_\xi(s) \leq C \varphi_\xi(s)$$

so that, by Gronwall's lemma, we find a constant C with

$$\langle \xi, e^{-A} (\mathcal{N}_+ + 1)^k e^A \xi \rangle = C \langle \xi, (\mathcal{N}_+ + 1)^k \xi \rangle .$$

□

We use now the cubic phase e^A to introduce a new excitation Hamiltonian, defining

$$\mathcal{R}_{N,\ell} := e^{-A} \, \mathcal{G}^{\text{eff}}_{N,\ell} \, e^A$$

on a dense subset of $\mathcal{F}_+^{\leq N}$. The operator $\mathcal{G}^{\text{eff}}_{N,\ell}$ is defined as in (4.23). As explained in the introduction, conjugation with e^A renormalizes the cubic term on the r.h.s. of (4.23), effectively replacing the singular potential $\widehat{V}(p/N)$ by a potential decaying already on momenta of order one. This allows us to show the following proposition.

Proposition 5.2. *Let $V \in L^3(\mathbb{R}^3)$ be compactly supported, pointwise non-negative and spherically symmetric. Then, for all $\alpha > 3$ and $\alpha/2 < \beta < 2\alpha/3$, there exists $\kappa > 0$ and a constant $C > 0$ such that*

$$\mathcal{R}_{N,\ell} \geq 4\pi \mathfrak{a}_0 N + \big(1 - C\ell^\kappa\big) \mathcal{H}_N - C\ell^{-3\alpha} \mathcal{N}_+^2 / N - C\ell^{-3\alpha}$$

for all $\ell \in (0; 1/2)$ small enough and N large enough.

The proof of Proposition 5.2 will be given in Sect. 8. In the next section, we show how Proposition 5.2, together with Proposition 4.2 and Proposition 4.3, implies Theorem 1.1.

6. Proof of Theorem 1.1

The next proposition combines the results of Propositions 4.2, 4.3 and 5.2.

Proposition 6.1. *Let $V \in L^3(\mathbb{R}^3)$ be compactly supported, pointwise non-negative and spherically symmetric. Let $\mathcal{G}_{N,\ell}$ be the renormalized excitation Hamiltonian defined as in (4.18). Then, for every $\alpha > 3$, $\ell \in (0; 1/2)$ small enough, there exist constants $C, c > 0$ such that*

$$\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N \geq c\mathcal{N}_+ - C \tag{6.1}$$

for all $N \in \mathbb{N}$ sufficiently large.

Proof. As in Proposition 4.3, let $f, g : \mathbb{R} \to [0; 1]$ be smooth, with $f^2(x) + g^2(x) = 1$ for all $x \in \mathbb{R}$. Moreover, assume that $f(x) = 0$ for $x > 1$ and $f(x) = 1$ for $x < 1/2$. We fix $M = \ell^{3\alpha+\kappa} N$ (with $\kappa > 0$ as in Proposition 5.2) and we set $f_M = f(\mathcal{N}_+/M)$, $g_M = g(\mathcal{N}_+/M)$. It follows from Proposition 4.3 that

$$\begin{aligned} \mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N \geq\ & f_M(\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N) f_M + g_M(\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N) g_M \\ & - C\ell^{-13\alpha/2-2\kappa} N^{-2}(\mathcal{H}_N + 1). \end{aligned} \tag{6.2}$$

Let us consider the first term on the r.h.s. of (6.2). From Proposition 4.2, there exists $C > 0$ such that

$$\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N \geq \mathcal{G}^{\text{eff}}_{N,\ell} - 4\pi \mathfrak{a}_0 N - C\ell^{(\alpha-3)/2}\mathcal{H}_N - C\ell^{-\alpha}$$

and also, from (4.20),

$$\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N \geq \frac{1}{2}\mathcal{H}_N - C\mathcal{N}_+ - C\ell^{-\alpha} \tag{6.3}$$

for all $\alpha > 3$, $\ell \in (0; 1/2)$ small enough and N large enough. Together, the last two bounds imply that

$$\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N \geq (1 - C\ell^{(\alpha-3)/2})(\mathcal{G}^{\text{eff}}_{N,\ell} - 4\pi \mathfrak{a}_0 N) - C\ell^{(\alpha-3)/2}\mathcal{N}_+ - C\ell^{-\alpha}.$$

Hence, for $\ell > 0$ small enough,

$$\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N \geq \frac{1}{2}(\mathcal{G}^{\text{eff}}_{N,\ell} - 4\pi \mathfrak{a}_0 N) - C\ell^{(\alpha-3)/2}\mathcal{N}_+ - C\ell^{-\alpha}.$$

With Proposition 5.2, choosing $\alpha > 3$ and $\alpha/2 < \beta < 2\alpha/3$, we find $\kappa > 0$ such that

$$\begin{aligned} & f_M(\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N) f_M \\ & \quad \geq \frac{1}{2} f_M(\mathcal{G}^{\text{eff}}_{N,\ell} - 4\pi \mathfrak{a}_0 N) f_M - C\ell^{(\alpha-3)/2} f_M^2 \mathcal{N}_+ - C\ell^{-\alpha} f_M^2 \\ & \quad \geq \frac{1}{2} f_M e^A \left[(1 - C\ell^\kappa)\mathcal{H}_N - C\ell^{-3\alpha}\frac{\mathcal{N}_+^2}{N} - C\ell^{-3\alpha} \right] e^{-A} f_M \\ & \qquad - C\ell^{(\alpha-3)/2} f_M^2 \mathcal{N}_+ - C\ell^{-\alpha} f_M^2 \\ & \quad \geq \frac{1}{2} f_M e^A \left[(1 - C\ell^\kappa)\mathcal{H}_N - C\ell^\kappa \mathcal{N}_+ \right] e^{-A} f_M - C\ell^{(\alpha-3)/2} f_M^2 \mathcal{N}_+ - C\ell^{-3\alpha} f_M^2. \end{aligned}$$

In the last inequality, we used Proposition 5.1 to estimate

$$\begin{aligned} f_M e^{-A} \mathcal{N}_+^2 e^A f_M &\leq C f_M (\mathcal{N}_+ + 1)^2 f_M \\ &\leq C N \ell^{3\alpha+\kappa} f_M (\mathcal{N}_+ + 1) f_M \leq C N \ell^{3\alpha+\kappa} f_M e^{-A} (\mathcal{N}_+ + 1) e^A f_M \end{aligned}$$

because we chose $M = \ell^{3\alpha+\kappa} N$. Since now $\mathcal{N}_+ \leq C\mathcal{K} \leq C\mathcal{H}_N$, we obtain that, for $\ell \in (0; 1/2)$ small enough,

$$f_M (\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N) f_M \geq C f_M e^A \mathcal{N}_+ e^{-A} f_M - C\ell^{(\alpha-3)/2} f_M^2 \mathcal{N}_+ - C\ell^{-3\alpha} f_M^2.$$

With Proposition 5.1, we conclude that, again for $\ell > 0$ small enough,

$$f_M (\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N) f_M \geq C f_M^2 \mathcal{N}_+ - C\ell^{-3\alpha} f_M^2. \tag{6.4}$$

Let us next consider the second term on the r.h.s. of (6.2). From now on, we keep $\ell > 0$ fixed (so that (6.4) holds true), and we will only worry about the dependence of N. We claim that there exists a constant $C > 0$ such that

$$g_M (\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N) g_M \geq C N g_M^2 \tag{6.5}$$

for all N sufficiently large. To prove (6.5) we observe that, since $g(x) = 0$ for all $x \leq 1/2$,

$$g_M (\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N) g_M \geq \left[\inf_{\xi \in \mathcal{F}_{\geq M/2}^{\leq N} : \|\xi\|=1} \frac{1}{N} \langle \xi, \mathcal{G}_{N,\ell} \xi \rangle - 4\pi \mathfrak{a}_0 \right] N g_M^2$$

where $\mathcal{F}_{\geq M/2}^{\leq N} = \{\xi \in \mathcal{F}_+^{\leq N} : \xi = \chi(\mathcal{N}_+ \geq M/2)\xi\}$ is the subspace of $\mathcal{F}_+^{\leq N}$ where states with at least $M/2$ excitations are described (recall that $M = \ell^{3\alpha+\kappa} N$). To prove (6.5) it is enough to show that there exists $C > 0$ with

$$\inf_{\xi \in \mathcal{F}_{\geq M/2}^{\leq N} : \|\xi\|=1} \frac{1}{N} \langle \xi, \mathcal{G}_{N,\ell} \xi \rangle - 4\pi \mathfrak{a}_0 \geq C \tag{6.6}$$

for all N large enough. From the result (1.7) of [10,11,14], we already know that

$$\begin{aligned} &\inf_{\xi \in \mathcal{F}_{\geq M/2}^{\leq N} : \|\xi\|=1} \frac{1}{N} \langle \xi, \mathcal{G}_{N,\ell} \xi \rangle - 4\pi \mathfrak{a}_0 \\ &\geq \inf_{\xi \in \mathcal{F}_+^{\leq N} : \|\xi\|=1} \frac{1}{N} \langle \xi, \mathcal{G}_{N,\ell} \xi \rangle - 4\pi \mathfrak{a}_0 = \frac{E_N}{N} - 4\pi \mathfrak{a}_0 \to 0 \end{aligned}$$

as $N \to \infty$. Hence, if we assume by contradiction that (6.6) does not hold true, then we can find a subsequence $N_j \to \infty$ with

$$\inf_{\xi \in \mathcal{F}_{\geq M_j/2}^{\leq N_j} : \|\xi\|=1} \frac{1}{N_j} \langle \xi, \mathcal{G}_{N_j,\ell} \xi \rangle - 4\pi \mathfrak{a}_0 \to 0$$

as $j \to \infty$ (here we used the notation $M_j = \ell^{3\alpha+\kappa} N_j$). This implies that there exists a sequence $\xi_{N_j} \in \mathcal{F}^{\leq N_j}_{\geq M_j/2}$ with $\|\xi_{N_j}\| = 1$ for all $j \in \mathbb{N}$ such that

$$\lim_{j\to\infty} \frac{1}{N_j} \langle \xi_{N_j}, \mathcal{G}_{N_j,\ell} \xi_{N_j} \rangle = 4\pi \mathfrak{a}_0.$$

Let now $S := \{N_j : j \in \mathbb{N}\} \subset \mathbb{N}$ and denote by ξ_N a normalized minimizer of $\mathcal{G}_{N,\ell}$ for all $N \in \mathbb{N} \setminus S$. Setting $\psi_N = U_N^* e^{B(\eta_H)} \xi_N$, for all $N \in \mathbb{N}$, we obtain that $\|\psi_N\| = 1$ and that

$$\lim_{N\to\infty} \frac{1}{N} \langle \psi_N, H_N \psi_N \rangle = \lim_{N\to\infty} \frac{1}{N} \langle \xi_N, \mathcal{G}_{N,\ell} \xi_N \rangle = 4\pi \mathfrak{a}_0.$$

In other words, the sequence ψ_N is an approximate ground state of H_N. From 1.7, we conclude that ψ_N exhibits complete Bose–Einstein condensation in the zero-momentum mode φ_0, meaning that

$$\lim_{N\to\infty} 1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle = 0.$$

Using Lemma 3.1 and the rules (2.2), we observe that

$$\begin{aligned} \frac{1}{N} \langle \xi_N, \mathcal{N}_+ \xi_N \rangle &= \frac{1}{N} \langle e^{-B(\eta_H)} U_N \psi_N, \mathcal{N}_+ e^{-B(\eta_H)} U_N \psi_N \rangle \\ &\leq \frac{C}{N} \langle \psi_N, U_N^* (\mathcal{N}_+ + 1) U_N \psi_N \rangle \\ &= \frac{C}{N} + C \left[1 - \frac{1}{N} \langle \psi_N, a^*(\varphi_0) a(\varphi_0) \psi_N \rangle \right] \\ &= \frac{C}{N} + C \left[1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle \right] \to 0 \end{aligned} \tag{6.7}$$

as $N \to \infty$. On the other hand, for $N \in S = \{N_j : j \in \mathbb{N}\}$, we have $\xi_N = \chi(\mathcal{N}_+ \geq M/2) \xi_N$ and therefore

$$\frac{1}{N} \langle \xi_N, \mathcal{N}_+ \xi_N \rangle \geq \frac{M}{2N} = \frac{\ell^{3\alpha+\kappa}}{2}$$

in contradiction with (6.7). This proves (6.6), (6.5) and therefore also

$$g_M (\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N) g_M \geq C \mathcal{N}_+ g_M^2. \tag{6.8}$$

Inserting (6.4) and (6.8) on the r.h.s. of (6.2), we obtain that

$$\mathcal{G}_{N,\ell} - 4\pi \mathfrak{a}_0 N \geq C \mathcal{N}_+ - C N^{-2} \mathcal{H}_N - C \tag{6.9}$$

for N large enough (the constants C are now allowed to depend on ℓ, since ℓ has been fixed once and for always after (6.4)). Interpolating (6.9) with (6.3), we obtain (6.1). □

We are now ready to show our main theorem.

Proof of Theorem 1.1. First of all, (4.20) and (4.21) in Proposition 4.2 imply that

$$\mathcal{G}_{N,\ell} - 4\pi\mathfrak{a}_0 N \leq 2\mathcal{H}_N + C\mathcal{N}_+ + C.$$

With the vacuum Ω as trial state, we obtain the upper bound $E_N \leq 4\pi\mathfrak{a}_0 N + C$ for the ground state energy E_N of $\mathcal{G}_{N,\ell}$ (which coincides with the ground state energy of H_N). With Eq. (6.1), we also find the lower bound $E_N \geq 4\pi\mathfrak{a}_0 N - C$. This proves (1.8).

Let now $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and

$$\langle \psi_N, H_N \psi_N \rangle \leq 4\pi\mathfrak{a}_0 N + K.$$

We define the excitation vector $\xi_N = e^{-B(\eta_H)} U_N \psi_N$. Then $\|\xi_N\| = 1$ and, recalling that $\mathcal{G}_{N,\ell} = e^{-B(\eta_H)} U_N H_N U_N^* e^{B(\eta_H)}$, we have

$$\langle \xi_N, \mathcal{N}_+ \xi_N \rangle \leq C \langle \xi_N, (\mathcal{G}_{N,\ell} - 4\pi\mathfrak{a}_0 N)\xi_N \rangle + C \leq C(K+1).$$

If γ_N denotes the one-particle reduced density matrix associated with ψ_N, we obtain

$$\begin{aligned} 1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle &= 1 - \frac{1}{N} \langle \psi_N, a^*(\varphi_0) a(\varphi_0) \psi_N \rangle \\ &= 1 - \frac{1}{N} \langle U_N^* e^{B(\eta_H)} \xi_N, a^*(\varphi_0) a(\varphi_0) U_N^* e^{B(\eta_H)} \xi_N \rangle \\ &= \frac{1}{N} \langle e^{B(\eta_H)} \xi_N, \mathcal{N}_+ e^{B(\eta_H)} \xi_N \rangle \leq \frac{C}{N} \langle \xi_N, \mathcal{N}_+ \xi_N \rangle \leq \frac{C(K+1)}{N} \end{aligned}$$

which concludes the proof of (1.9). □

7. Analysis of $\mathcal{G}_{N,\ell}$

From (2.3) and (4.18), we can decompose

$$\mathcal{G}_{N,\ell} = e^{-B(\eta_H)} \mathcal{L}_N e^{B(\eta_H)} = \mathcal{G}^{(0)}_{N,\ell} + \mathcal{G}^{(2)}_{N,\ell} + \mathcal{G}^{(3)}_{N,\ell} + \mathcal{G}^{(4)}_{N,\ell}$$

with

$$\mathcal{G}^{(j)}_{N,\ell} = e^{-B(\eta_H)} \mathcal{L}^{(j)}_N e^{B(\eta_H)}.$$

In the next subsections, we prove separate bounds for the operators $\mathcal{G}^{(j)}_{N,\ell}$, $j = 0, 2, 3, 4$. In Sect. 7.5, we combine these bounds to prove Propositions 4.2 and 4.3. Throughout this section, we will assume the potential $V \in L^3(\mathbb{R}^3)$ to be compactly supported, pointwise non-negative and spherically symmetric.

7.1. Analysis of $\mathcal{G}^{(0)}_{N,\ell} = e^{-B(\eta_H)} \mathcal{L}^{(0)}_N e^{B(\eta_H)}$. From (2.4), recall that

$$\mathcal{L}^{(0)}_N = \frac{(N-1)}{2N} \widehat{V}(0)(N - \mathcal{N}_+) + \frac{\widehat{V}(0)}{2N} \mathcal{N}_+(N - \mathcal{N}_+). \tag{7.1}$$

We define the error operator $\mathcal{E}^{(0)}_{N,\ell}$ through the identity

$$\begin{aligned} \mathcal{G}^{(0)}_{N,\ell} &= e^{-B(\eta_H)} \mathcal{L}^{(0)}_N e^{B(\eta_H)} \\ &= \frac{(N-1)}{2N} \widehat{V}(0)(N - \mathcal{N}_+) + \frac{\widehat{V}(0)}{2N} \mathcal{N}_+(N - \mathcal{N}_+) + \mathcal{E}^{(0)}_{N,\ell}. \end{aligned} \tag{7.2}$$

In the next proposition, we estimate $\mathcal{E}^{(0)}_{N,\ell}$ and its double commutator with a smooth and bounded function of $\mathcal{N}_+$.

Proposition 7.1. *There exists a constant $C > 0$ such that*

$$\pm \mathcal{E}^{(0)}_{N,\ell} \leq C \ell^{\alpha/2} (\mathcal{N}_+ + 1) \tag{7.3}$$

and

$$\pm [f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), \mathcal{E}^{(0)}_{N,\ell}]] \leq C \ell^{\alpha/2} M^{-2} \|f'\|^2_\infty (\mathcal{N}_+ + 1) \tag{7.4}$$

for all $\alpha > 0$, $\ell \in (0; 1/2)$, f smooth and bounded, $M \in \mathbb{N}$ and $N \in \mathbb{N}$ large enough.

Proof. From (7.1) we have

$$\mathcal{L}^{(0)}_N = \frac{(N-1)}{2} \widehat{V}(0) + \frac{1}{2N} \widehat{V}(0) \mathcal{N}_+ - \frac{1}{2N} \widehat{V}(0) \mathcal{N}_+^2. \tag{7.5}$$

In the last term, we rewrite

$$-\frac{\mathcal{N}_+^2}{N} = \mathcal{N}_+ \frac{N - \mathcal{N}_+}{N} - \mathcal{N}_+ = \sum_{q \in \Lambda_+^*} b^*_q b_q - \frac{\mathcal{N}_+}{N} - \mathcal{N}_+.$$

Inserting in (7.5), we obtain

$$\mathcal{L}^{(0)}_N = \frac{(N-1)}{2} \widehat{V}(0) + \frac{\widehat{V}(0)}{2} \left[\sum_{q \in \Lambda_+^*} b^*_q b_q - \mathcal{N}_+ \right].$$

From (7.2), it follows that

$$\begin{aligned} \mathcal{E}^{(0)}_{N,\ell} = \frac{\widehat{V}(0)}{2} \sum_{q \in \Lambda_+^*} \left[e^{-B(\eta_H)} b^*_q b_q e^{B(\eta_H)} - b^*_q b_q \right] \\ - \frac{\widehat{V}(0)}{2} \left[e^{-B(\eta_H)} \mathcal{N}_+ e^{B(\eta_H)} - \mathcal{N}_+ \right]. \end{aligned} \tag{7.6}$$

With (3.18), we can express

$$\sum_{q \in \Lambda_+^*} e^{-B(\eta_H)} b^*_q b_q e^{B(\eta_H)} = \sum_{q \in \Lambda_+^*} \left[\gamma_q b^*_q + \sigma_q b_{-q} + d^*_q \right] \left[\gamma_q b_q + \sigma_q b^*_{-q} + d_q \right]$$

where we set $\gamma_q = \cosh \eta_H(q)$, $\sigma_q = \sinh \eta_H(q)$ and where d_q, d_q^* are defined as in (3.17), with η replaced by $\eta_H(q) = \eta_q \chi(q \in P_H)$. Using $|\gamma_q^2 - 1| \leq C\eta_H(q)^2$, $|\sigma_q| \leq C|\eta_H(q)|$, the first bound in (3.20), Cauchy–Schwarz and the estimate $\|\eta_H\| \leq C\ell^{\alpha/2}$ from (4.14), we conclude that first term on the r.h.s. of (7.6) can be bounded by

$$\Big| \sum_{q \in \Lambda_+^*} \langle \xi, \left[e^{-B(\eta_H)} b_q^* b_q e^{B(\eta_H)} - b_q^* b_q \right] \xi \rangle \Big| \leq C\ell^{\alpha/2} \| (\mathcal{N}_+ + 1)^{1/2} \xi \|^2.$$

As for the second term on the r.h.s. of (7.6), we expand using again (3.18),

$$\begin{aligned} & e^{-B(\eta_H)} \mathcal{N}_+ e^{B(\eta_H)} - \mathcal{N}_+ \\ &= \int_0^1 e^{-sB(\eta_H)} [\mathcal{N}_+, B(\eta_H)] e^{sB(\eta_H)} ds \\ &= \int_0^1 \sum_{p \in P_H} \eta_p \, e^{-sB(\eta_H)} (b_p b_{-p} + b_p^* b_{-p}^*) e^{sB(\eta_H)} \, ds \\ &= \int_0^1 ds \sum_{p \in P_H} \eta_p \left[(\gamma_p^{(s)} b_p + \sigma_p^{(s)} b_{-p}^* + d_p^{(s)})(\gamma_p^{(s)} b_{-p} + \sigma_p^{(s)} b_{-p}^* + d_{-p}^{(s)}) + \text{h.c.} \right] \end{aligned}$$

with $\gamma_p^{(s)} = \cosh(s\eta_H(p))$, $\sigma_p^{(s)} = \sinh(s\eta_H(p))$ and where the operators $d_p^{(s)}$ are defined as in (3.17), with η replaced by $s\eta_H$. Using $|\gamma_p^{(s)}| \leq C$ and $|\sigma_p^{(s)}| \leq C|\eta_p|$, (3.20) in Lemma 3.4 and again (4.14), we arrive at

$$\begin{aligned} & \Big| \langle \xi, \left[e^{-B(\eta_H)} \mathcal{N}_+ e^{B(\eta_H)} - \mathcal{N}_+ \right] \xi \rangle \Big| \\ & \quad \leq C \| (\mathcal{N}_+ + 1)^{1/2} \xi \| \sum_{p \in P_H} |\eta_p| \left[|\eta_p| \| (\mathcal{N}_+ + 1)^{1/2} \xi \| + \| b_p \xi \| \right] \\ & \quad \leq C\ell^{\alpha/2} \| (\mathcal{N}_+ + 1)^{1/2} \xi \|^2. \end{aligned}$$

This concludes the proof of (7.3).

The bound (7.4) follows analogously, because, as observed in Corollary 3.5, the estimates (3.20) in Lemma 3.4 remain true if we replace d_p and d_p^* by $[f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), d_p]]$ and, respectively, $[f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), d_p^*]]$, provided we multiply the r.h.s. by an additional factor $M^{-2}\|f'\|_\infty^2$. The same observation holds true for bounds involving the operators b_p, b_p^*, since, for example,

$$[f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), b_p]] = (f(\mathcal{N}_+/M) - f((\mathcal{N}_+ + 1)/M))^2 b_p \tag{7.7}$$

and $\| f(\mathcal{N}_+/M) - f((\mathcal{N}_+ + 1)/M) \| \leq CM^{-1} \|f'\|_\infty$. □

7.2. Analysis of $\mathcal{G}_{N,\ell}^{(2)} = e^{-B(\eta_H)} \mathcal{L}_N^{(2)} e^{B(\eta_H)}$. With (2.4), we decompose $\mathcal{L}_N^{(2)} = \mathcal{K} + \mathcal{L}_N^{(2,V)}$, where $\mathcal{K} = \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p$ is the kinetic energy operator and

$$\mathcal{L}_N^{(2,V)} = \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) a_p^* a_p \frac{N - \mathcal{N}_+}{N} + \frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \left[b_p^* b_{-p}^* + b_p b_{-p} \right]. \tag{7.8}$$

Accordingly, we have

$$\mathcal{G}^{(2)}_{N,\ell} = e^{-B(\eta_H)} \mathcal{K} e^{B(\eta_H)} + e^{-B(\eta_H)} \mathcal{L}^{(2,V)}_N e^{B(\eta_H)}. \tag{7.9}$$

In the next two propositions, we analyse the two terms on the r.h.s. of the last equation.

Proposition 7.2. *There exists $C > 0$ such that*

$$\begin{aligned} e^{-B(\eta_H)} \mathcal{K} e^{B(\eta_H)} = \mathcal{K} &+ \sum_{p \in P_H} p^2 \eta_p (b_p b_{-p} + b^*_p b^*_{-p}) \\ &+ \sum_{p \in P_H} p^2 \eta_p^2 \Big(\frac{N - \mathcal{N}_+}{N} \Big) \Big(\frac{N - \mathcal{N}_+ - 1}{N} \Big) + \mathcal{E}^{(K)}_{N,\ell} \end{aligned} \tag{7.10}$$

where

$$\pm \mathcal{E}^{(K)}_{N,\ell} \leq C \ell^{(\alpha-3)/2} (\mathcal{H}_N + 1) \tag{7.11}$$

and

$$\pm \Big[f(\mathcal{N}_+/M), \Big[f(\mathcal{N}_+/M), \mathcal{E}^{(K)}_{N,\ell} \Big] \Big] \leq C M^{-2} \| f' \|^2_\infty \ell^{(\alpha-3)/2} \big(\mathcal{H}_N + 1 \big) \tag{7.12}$$

for all $\alpha > 3$, $\ell \in (0; 1/2)$ small enough, f smooth and bounded, $M \in \mathbb{N}$ and $N \in \mathbb{N}$ large enough.

Proof. To show (7.11), we write

$$\begin{aligned} e^{-B(\eta_H)} \mathcal{K} e^{B(\eta_H)} - \mathcal{K} &= \int_0^1 e^{-sB(\eta_H)} [\mathcal{K}, B(\eta_H)] e^{sB(\eta_H)} ds \\ &= \int_0^1 \sum_{p \in P_H} p^2 \eta_p \Big[e^{-sB(\eta_H)} b_p b_{-p} e^{sB(\eta_H)} \\ &\quad + e^{-sB(\eta_H)} b^*_p b^*_{-p} e^{sB(\eta_H)} \Big] ds. \end{aligned}$$

With relations (3.18), we can write

$$\begin{aligned} &e^{-B(\eta_H)} \mathcal{K} e^{B(\eta_H)} - \mathcal{K} \\ &= \int_0^1 ds \sum_{p \in P_H} p^2 \eta_p \Big[\big(\gamma^{(s)}_p b_p + \sigma^{(s)}_p b^*_{-p} \big) \big(\gamma^{(s)}_p b_{-p} + \sigma^{(s)}_p b^*_p \big) + \text{h.c.} \Big] \\ &\quad + \int_0^1 ds \sum_{p \in P_H} p^2 \eta_p \big[\big(\gamma^{(s)}_p b_p + \sigma^{(s)}_p b^*_{-p} \big) d^{(s)}_{-p} + d^{(s)}_p \big(\gamma^{(s)}_p b_{-p} + \sigma^{(s)}_p b^*_p \big) + \text{h.c.} \big] \\ &\quad + \int_0^1 ds \sum_{p \in P_H} p^2 \eta_p \big[d^{(s)}_p d^{(s)}_{-p} + \text{h.c.} \big] \\ &=: \mathrm{G}_1 + \mathrm{G}_2 + \mathrm{G}_3 \end{aligned} \tag{7.13}$$

with the notation $\gamma_p^{(s)} = \cosh(s\eta_H(p))$, $\sigma_p^{(s)} = \sinh(s\eta_H(p))$ and where $d_p^{(s)}$ is defined as in (3.17), with η_p replaced by $s\eta_H(p)$ (recall that $\eta_H(p) = \eta_p \chi(p \in P_H)$). We start by analysing G_1. Expanding the product, we obtain

$$\begin{aligned}\mathrm{G}_1 &= \int_0^1 ds \sum_{p\in P_H} p^2\eta_p \Big[\big((\gamma_p^{(s)})^2 + (\sigma_p^{(s)})^2\big)\big(b_p b_{-p} + b_{-p}^* b_p^*\big) \\ &\quad + \gamma_p^{(s)}\sigma_p^{(s)}(4b_p^* b_p - 2N^{-1}a_p^* a_p)\big)\Big] \\ &\quad + 2\int_0^1 ds \sum_{p\in P_H} p^2\eta_p \gamma_p^{(s)}\sigma_p^{(s)} \left(1 - \frac{\mathcal{N}_+}{N}\right) \\ &= \sum_{p\in P_H} p^2\eta_p\big(b_p b_{-p} + b_{-p}^* b_p^*\big) + \sum_{p\in P_H} p^2\eta_p^2 \left(1 - \frac{\mathcal{N}_+}{N}\right) + \mathcal{E}_1^K\end{aligned} \tag{7.14}$$

with

$$\begin{aligned}\mathcal{E}_1^K &= \int_0^1 ds \sum_{p\in P_H} p^2\eta_p\big[\big((\gamma_p^{(s)})^2 - 1\big) + (\sigma_p^{(s)})^2\big]\big(b_p b_{-p} + b_{-p}^* b_p^*\big) \\ &\quad + \int_0^1 ds \sum_{p\in P_H} p^2\eta_p\gamma_p^{(s)}\sigma_p^{(s)}(4b_p^* b_p - 2N^{-1}a_p^* a_p)\big) \\ &\quad + 2\int_0^1 ds \sum_{p\in P_H} p^2\eta_p \Big[(\gamma_p^{(s)} - 1)\sigma_p^{(s)} + (\sigma_p^{(s)} - s\eta_p)\Big]\left(1 - \frac{\mathcal{N}_+}{N}\right).\end{aligned}$$

For an arbitrary $\xi \in \mathcal{F}_+^{\leq N}$, we bound

$$\begin{aligned}&|\langle \xi, \mathcal{E}_1^K \xi\rangle| \\ &\leq C\sum_{p\in P_H} p^2|\eta_p|^3 \|b_p\xi\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + C\sum_{p\in P_H} p^2\eta_p^2\|a_p\xi\|^2 + C\sum_{p\in P_H} p^2\eta_p^4 \\ &\leq C\ell^{2\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2,\end{aligned} \tag{7.15}$$

since $|\big((\gamma_p^{(s)})^2 - 1\big)| \leq C\eta_p^2$, $(\sigma_p^{(s)})^2 \leq C\eta_p^2$ and $p^2\eta_p^2 \leq C\ell^{2\alpha}$, for all $p \in P_H$.

We consider now G_2 in (7.13). We split it as $\mathrm{G}_2 = \mathrm{G}_{21} + \mathrm{G}_{22} + \mathrm{G}_{23} + \mathrm{G}_{24}$, with

$$\begin{aligned}\mathrm{G}_{21} &= \int_0^1 ds \sum_{p\in P_H} p^2\eta_p \left(\gamma_p^{(s)} b_p d_{-p}^{(s)} + \text{h.c.}\right), \\ \mathrm{G}_{22} &= \int_0^1 ds \sum_{p\in P_H} p^2\eta_p \left(\sigma_p^{(s)} b_{-p}^* d_{-p}^{(s)} + \text{h.c.}\right), \\ \mathrm{G}_{23} &= \int_0^1 ds \sum_{p\in P_H} p^2\eta_p \left(\gamma_p^{(s)} d_p^{(s)} b_{-p} + \text{h.c.}\right), \\ \mathrm{G}_{24} &= \int_0^1 ds \sum_{p\in P_H} p^2\eta_p \left(\sigma_p^{(s)} d_p^{(s)} b_p^* + \text{h.c.}\right).\end{aligned} \tag{7.16}$$

We consider G_{21} first. We write

$$\begin{aligned}\mathrm{G}_{21} = &\int_0^1 ds \sum_{p\in P_H} p^2\eta_p(\gamma_p^{(s)}-1)b_p d_{-p}^{(s)} + \int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p b_p d_{-p}^{(s)} \\ &- \int_0^1 ds \sum_{p\in P_H^c} p^2\eta_p b_p \left[d_{-p}^{(s)} + \frac{1}{N}\sum_{q\in P_H} s\eta_q b_q^* a_{-q}^* a_{-p}\right] \\ &+ \int_0^1 ds\, \frac{s}{N} \sum_{p\in P_H^c, q\in P_H} p^2\eta_p\eta_q b_p b_q^* a_{-q}^* a_{-p} + \text{h.c.}\end{aligned}$$

Massaging a bit the second term (similarly as we do below, in (7.39), (7.40) in the proof of Proposition 7.3), we arrive at

$$\mathrm{G}_{21} = -\sum_{p\in P_H} p^2\eta_p \frac{\mathcal{N}_+ + 1}{N}\frac{N-\mathcal{N}_+}{N} + \left[\mathcal{E}_2^K + \text{h.c.}\right] \tag{7.17}$$

where $\mathcal{E}_2^K = \sum_{j=1}^5 \mathcal{E}_{2j}^K$, with

$$\begin{aligned}\mathcal{E}_{21}^K &= \frac{1}{2N}\sum_{p\in P_H} p^2\eta_p^2(\mathcal{N}_+ + 1)\big(b_p^* b_p - \frac{1}{N}a_p^* a_p\big), \\ \mathcal{E}_{22}^K &= \int_0^1 ds \sum_{p\in P_H} p^2\eta_p(\gamma_p^{(s)}-1)b_p d_{-p}^{(s)}, \\ \mathcal{E}_{23}^K &= \int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p b_p \bar{d}_{-p}^{(s)}, \\ \mathcal{E}_{24}^K &= -\int_0^1 ds \sum_{p\in P_H^c} p^2\eta_p b_p \bar{\bar{d}}_{-p}^{(s)}, \\ \mathcal{E}_{25}^K &= \frac{1}{2N}\sum_{p\in P_H^c, q\in P_H} p^2\eta_p\eta_q b_p b_q^* a_{-q}^* a_{-p}.\end{aligned} \tag{7.18}$$

Here we introduced the notation

$$\bar{d}_{-p}^{(s)} = d_{-p}^{(s)} + s\eta_H(p)\frac{\mathcal{N}_+}{N}b_p^*, \quad \text{and} \quad \bar{\bar{d}}_{-p}^{(s)} = d_{-p}^{(s)} + \frac{1}{N}\sum_{q\in P_H} s\eta_q b_q^* a_{-q}^* a_{-p}. \tag{7.19}$$

We can easily bound

$$|\langle\xi, \mathcal{E}_{21}^K\xi\rangle| \leq C\sum_{p\in P_H} p^2\eta_p^2 \|a_p\xi\|^2 \leq C\ell^{2\alpha}\|\mathcal{N}_+^{1/2}\xi\|^2 \tag{7.20}$$

and, using $|\gamma_p^{(s)} - 1| \leq C\eta_p^2$ and (3.20) in Lemma 3.4,

$$\begin{aligned} |\langle \xi, \mathcal{E}_{22}^K \xi\rangle| &\leq \sum_{p\in P_H} p^2 |\eta_p|^3 \|\mathcal{N}_+^{1/2}\xi\| \|d_{-p}^{(s)}\xi\| \\ &\leq \sum_{p\in P_H} p^2 |\eta_p|^3 \|\mathcal{N}_+^{1/2}\xi\| \Big[|\eta_p| \|\mathcal{N}_+^{1/2}\xi\| + \|\eta\| \|a_p\xi\| \Big] \\ &\leq C\ell^{3\alpha/2} \|\mathcal{N}_+ + 1)^{1/2}\xi\|^2. \end{aligned} \tag{7.21}$$

With (3.21) in Lemma 3.4, we can also estimate

$$\begin{aligned} |\langle \xi, \mathcal{E}_{24}^K \xi\rangle| &\leq \int_0^1 ds \sum_{p\in P_H^c} p^2 |\eta_p| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{-1/2} \bar{\bar{d}}_{-p}^{(s)} \xi\| \\ &\leq C\|\eta_H\|^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \sum_{p\in P_H^c} p^2 |\eta_p| \|a_p\xi\| \\ &\leq C\ell^\alpha \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\| \Big[\sum_{|p|\leq \ell^{-\alpha}} p^2 \eta_p^2 \Big]^{1/2} \\ &\leq C\ell^{\alpha/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\|. \end{aligned} \tag{7.22}$$

To bound the last term in (7.18), we commute b_p to the right (note that $p \neq q$). We find

$$\begin{aligned} |\langle \xi, \mathcal{E}_{25}^K \xi\rangle| &\leq CN^{-1} \sum_{p\in P_H^c, q\in P_H} p^2 |\eta_p| |\eta_q| \|a_q a_{-q}\xi\| \|a_p a_{-p}\xi\| \\ &\leq C \sum_{p\in P_H^c, q\in P_H} p^2 |\eta_p| |\eta_q| \|a_q\xi\| \|a_p\xi\| \\ &\leq C \Big[\sum_{p\in P_H^c, q\in P_H} p^2 \eta_p^2 q^2 \|a_q\xi\|^2 \Big]^{1/2} \\ &\quad \Big[\sum_{p\in P_H^c, q\in P_H} q^{-2} \eta_q^2 p^2 \|a_p\xi\|^2 \Big]^{1/2} \\ &\leq C\ell^\alpha \|\mathcal{K}^{1/2}\xi\|^2. \end{aligned} \tag{7.23}$$

To control the third term in (7.18), we first use (4.9) to write

$$\begin{aligned} \mathcal{E}_{23}^K = &\int_0^1 ds \sum_{p\in\Lambda_+^*} \Big(\widehat{V}(./N) * \widehat{f}_{N,\ell} \Big)(p) b_p \bar{d}_{-p}^{(s)} \\ &+ \int_0^1 ds\, N^3 \lambda_\ell \sum_{p\in\Lambda_+^*} \Big(\widehat{\chi}_\ell * \widehat{f}_{N,\ell} \Big)(p) b_p \bar{d}_{-p}^{(s)}. \end{aligned}$$

Switching to position space, we obtain

$$\begin{aligned} \mathcal{E}_{23}^K = &\int_0^1 ds \int_{\Lambda^2} dx dy N^3 V(N(x-y)) f_{N,\ell}(x-y) \check{b}_x \check{\bar{d}}_y^{(s)} \\ &+ \int_0^1 ds N^3 \lambda_\ell \int_{\Lambda^2} dx dy \chi_\ell(x-y) f_{N,\ell}(x-y) \check{b}_x \check{\bar{d}}_y^{(s)}. \end{aligned}$$

With Lemma 4.1, we find

$$|\langle\xi, \mathcal{E}_{23}^K\xi\rangle| \leq \int_0^1 ds \int_{\Lambda^2} dxdy \left[N^3 V(N(x-y)) + \ell^{-3}\chi_\ell(x-y)\right] \times \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{-1/2}\check{a}_x \check{d}_y^{(s)}\xi\|.$$

Hence, with Eq. (3.23) in Lemma 3.4,

$$\begin{aligned}|\langle\xi, \mathcal{E}_{23}^K\xi\rangle| &\leq CN^{-1}\|\eta_H\| \int_0^1 ds \int_{\Lambda^2} dxdy \left[N^3 V(N(x-y)) + \ell^{-3}\chi_\ell(x-y)\right] \\ &\quad \times \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \Big[N\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{a}_x \mathcal{N}_+ \xi\| + \|\check{a}_y \mathcal{N}_+ \xi\| \\ &\quad + \|\check{a}_x \check{a}_y \mathcal{N}_+^{1/2}\xi\|\Big] \\ &\leq C\ell^{(\alpha-3)/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\ell^{\alpha/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\|.\end{aligned}$$

Combining the last bound with (7.20), (7.21), (7.22), (7.23), we conclude that

$$\pm \left[\mathcal{E}_2^K + \text{h.c.}\right] \leq C\ell^{(\alpha-3)/2}(\mathcal{H}_N + 1). \tag{7.24}$$

Next, we consider the term G_{22} in (7.16). With (3.20) in Lemma 3.4, we find

$$\begin{aligned}|\langle\xi, \mathrm{G}_{22}\xi\rangle| &\leq C \sum_{p\in P_H} p^2\eta_p^2 \|b_{-p}\xi\| \|d_{-p}\xi\| \\ &\leq C \sum_{p\in P_H} p^2\eta_p^2 \|b_{-p}\xi\| \left[|\eta_p| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta\| \|b_p\xi\|\right] \\ &\leq C\ell^{5\alpha/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.\end{aligned} \tag{7.25}$$

As for the term G_{23}, defined in (7.16), we split it as $\mathrm{G}_{23} = \sum_{j=1}^4 \mathcal{E}_{3j}^K + \text{h.c.}$, with

$$\begin{aligned}\mathcal{E}_{31}^K &= \int_0^1 ds \sum_{p\in P_H} p^2\eta_p\big(\gamma_p^{(s)} - 1\big) d_p^{(s)} b_{-p}\,, \\ \mathcal{E}_{32}^K &= \int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p d_p^{(s)} b_{-p}, \\ \mathcal{E}_{33}^K &= \frac{1}{2N} \sum_{p\in P_H^c, q\in P_H} p^2\eta_p\eta_q b_q^* a_{-q}^* a_p b_{-p}\,, \\ \mathcal{E}_{34}^K &= -\int_0^1 ds \sum_{p\in P_H^c} p^2\eta_p \bar{\bar{d}}_p^{(s)} b_{-p}\end{aligned}$$

with the notation for $\bar{\bar{d}}_p^{(s)}$ introduced in (7.19). With (3.20) in Lemma 3.4, we find

$$\begin{aligned}|\langle\xi, \mathcal{E}_{31}^K\xi\rangle| &\leq C\int_0^1 \sum_{p\in P_H} p^2|\eta_p|^3 \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|(\mathcal{N}_+ + 1)^{-1/2} d_p^{(s)} b_{-p}\xi\| \\ &\leq C\|\eta_H\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \sum_{p\in P_H} p^2|\eta_p|^3 \|b_p\xi\| \leq C\ell^{3\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\end{aligned}$$

and also, proceeding as in (7.22),

$$\begin{aligned}|\langle\xi,\mathcal{E}^K_{34}\xi\rangle| &\leq C\int_0^1 ds\sum_{p\in P_H^c} p^2|\eta_p|\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|(\mathcal{N}_+ +1)^{-1/2}\bar{\bar{d}}^{(s)}_p b_{-p}\xi\|\\ &\leq C\|\eta_H\|^2\|(\mathcal{N}_+ +1)^{1/2}\xi\|\sum_{p\in P_H^c} p^2|\eta_p|\|b_{-p}\xi\|\\ &\leq C\ell^{\alpha/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.\end{aligned} \tag{7.26}$$

The term $\mathcal{E}^K_{33}$ coincides with the contribution $\mathcal{E}^K_{25}$ in (7.18); from (7.23) we obtain $\pm\,\mathcal{E}^K_{33}\leq C\ell^\alpha\mathcal{K}$. As for $\mathcal{E}^K_{32}$, we use (4.9) and we switch to position space. Proceeding as we did above to control the term $\mathcal{E}^K_{23}$, we arrive at

$$\begin{aligned}|\langle\xi,\mathcal{E}^K_{32}\xi\rangle| &\leq \int_0^1 ds\int_{\Lambda^2} dxdy\left[N^3V(N(x-y))+\ell^{-3}\chi_\ell(x-y)\right]\\ &\quad\times\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|(\mathcal{N}_+ +1)^{-1/2}\check{d}^{(s)}_x\check{a}_y\xi\|.\end{aligned}$$

With (3.22) in Lemma 3.4, we find

$$\begin{aligned}|\langle\xi,\mathcal{E}^K_{32}\xi\rangle| &\leq CN^{-1}\|\eta_H\|\int_{\Lambda^2} dxdy\left[N^3V(N(x-y))+\ell^{-3}\chi_\ell(x-y)\right]\\ &\quad\times\|(\mathcal{N}_+ +1)^{1/2}\xi\|\left[\|\check{a}_y(\mathcal{N}_+ +1)\xi\|+\|\check{a}_x\check{a}_y(\mathcal{N}_+ +1)^{1/2}\xi\|\right]\\ &\leq C\ell^{(\alpha-3)/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2+C\ell^{\alpha/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|.\end{aligned}$$

Combining the last bounds, we conclude that

$$\pm\,\mathrm{G}_{23}\leq C\ell^{(\alpha-3)/2}(\mathcal{H}_N+1). \tag{7.27}$$

To estimate the term G_{24} in (7.16), we use (3.20) in Lemma 3.4; with (4.15), we find

$$\begin{aligned}&|\langle\xi,\mathrm{G}_{24}\xi\rangle|\\ &\leq C\int_0^1 ds\sum_{p\in P_H} p^2\eta_p^2\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|(\mathcal{N}_+ +1)^{-1/2}d^{(s)}_p b^*_p\xi\|\\ &\leq CN^{-1}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\sum_{p\in P_H} p^2\eta_p^2\left[|\eta_p|\|(\mathcal{N}_+ +1)^{3/2}\xi\|+\|\eta_H\|\|b_pb^*_p(\mathcal{N}_+ +1)^{1/2}\xi\|\right]\\ &\leq CN^{-1}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\\ &\quad\times\sum_{p\in P_H} p^2\eta_p^2\left[|\eta_p|\|(\mathcal{N}_+ +1)^{3/2}\xi\|+\|\eta_H\|\|(\mathcal{N}_+ +1)^{1/2}\xi\|+\|\eta_H\|\|a_p(\mathcal{N}_+ +1)\xi\|\right]\\ &\leq C\ell^{\alpha/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2.\end{aligned}$$

Together with (7.17), (7.24), (7.25), (7.27), this implies that

$$\mathrm{G}_2=-\sum_{p\in P_H} p^2\eta_p\frac{\mathcal{N}_+ +1}{N}\frac{N-\mathcal{N}_+}{N}+\mathcal{E}^K_4$$

where

$$\pm\, \mathcal{E}_4^K \leq C\ell^{(\alpha-3)/2}(\mathcal{H}_N + 1). \tag{7.28}$$

Finally, we consider G_3, defined in (7.13). We split it as $\mathrm{G}_3 = \mathcal{E}_{51}^K + \mathcal{E}_{52}^K + \text{h.c.}$, with

$$\mathcal{E}_{51}^K = \int_0^1 ds \sum_{p\in\Lambda_+^*} p^2\eta_p d_p^{(s)} d_{-p}^{(s)}, \qquad \mathcal{E}_{52}^K = -\int_0^1 ds \sum_{p\in P_H^c} p^2\eta_p d_p^{(s)} d_{-p}^{(s)}.$$

With (3.20) in Lemma 3.4 (using $\eta_H(p) = 0$ for $p \in P_H^c$) and proceeding as in (7.26), we obtain

$$\begin{aligned}
|\langle\xi, \mathcal{E}_{52}^K\xi\rangle| &\leq C\|\eta_H\| \sum_{p\in P_H^c} p^2|\eta_p| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|d_{-p}\xi\| \\
&\leq C\|\eta_H\|^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \sum_{p\in P_H^c} p^2|\eta_p| \|b_{-p}\xi\| \\
&\leq C\ell^{\alpha/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\|.
\end{aligned}$$

To estimate $\mathcal{E}_{51}^K$, we use (4.9) and we switch to position space. Similarly as in the analysis of the terms $\mathcal{E}_{23}^K$ and $\mathcal{E}_{32}^K$ above, we obtain

$$\begin{aligned}
|\langle\xi, \mathcal{E}_{51}^K\xi\rangle| \leq C\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \int_0^1 ds \int_{\Lambda^2} dxdy \Big[N^3 V(N(x-y)) + \ell^{-3}\chi_\ell(x-y)\Big] \\
\times \|(\mathcal{N}_+ + 1)^{-1/2} \check{d}_x^{(s)} \check{d}_y^{(s)} \xi\|.
\end{aligned}$$

With (3.24) in Lemma 3.4, we arrive at

$$\begin{aligned}
&|\langle\xi, \mathcal{E}_{51}^K\xi\rangle| \\
&\leq CN^{-2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \int_0^1 ds \int_{\Lambda^2} dxdy \\
&\quad \Big[N^3 V(N(x-y)) + \ell^{-3}\chi_\ell(x-y)\Big] \\
&\quad \times \Big[N\|\eta_H\| \|(\mathcal{N}_+ + 1)^{3/2}\xi\| + \|\eta_H\|^2 \|\check{a}_x \mathcal{N}_+^2 \xi\| + \|\eta_H\|^2 \|\check{a}_y \mathcal{N}_+^2 \xi\| \\
&\quad + \|\eta_H\|^2 \|\check{a}_x \check{a}_y \mathcal{N}_+^{3/2} \xi\|\Big] \\
&\leq C(\ell^{\alpha/2} + \ell^{\alpha-3/2}) \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \\
&\quad + C\ell^{3\alpha/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\|.
\end{aligned}$$

Hence, $\pm\mathrm{G}_3 \leq C(\ell^{\alpha/2} + \ell^{3(\alpha-1)/2})(\mathcal{H}_N + 1)$. With (7.14), (7.15), (7.28), we obtain (7.10) and (7.11), as desired.

As explained in Corollary 3.5, the bounds in Lemma 3.4 continue to hold, with an additional factor $M^{-2}\|f'\|_\infty^2$ on the r.h.s., if we replace the operators d_p, d_p^*, $\check{\check{d}}_p$, $\check{a}_y \check{\check{d}}_x$, $\check{d}_x \check{d}_y$ by their double commutators with $f(\mathcal{N}_+/M)$. From (7.7) we conclude that also bounds involving b_p and b_p^* or, analogously $\check{b}_x$ and $\check{b}_x^*$ remain true if we replace them by their double commutator with $f(\mathcal{N}_+/M)$. As a consequence, (7.12) follows through the same arguments that led us to (7.11). □

In the next proposition, we study the second term on the r.h.s. of (7.9).

Proposition 7.3. *There is a constant $C > 0$ such that*

$$\begin{aligned} e^{-B(\eta_H)}\mathcal{L}_N^{(2,V)}e^{B(\eta_H)} &= \sum_{p\in P_H}\widehat{V}(p/N)\eta_p\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big) \\ &\quad+\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)a_p^*a_p\frac{N-\mathcal{N}_+}{N} \\ &\quad+\frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\big(b_pb_{-p}+b_{-p}^*b_p^*\big)+\mathcal{E}_{N,\ell}^{(V)} \end{aligned} \tag{7.29}$$

where

$$\pm\,\mathcal{E}_{N,\ell}^{(V)} \le C\ell^{\alpha/2}(\mathcal{H}_N+1) \tag{7.30}$$

and

$$\pm\Big[f(\mathcal{N}_+/M),\Big[f(\mathcal{N}_+/M),\mathcal{E}_{N,\ell}^{(V)}\Big]\Big] \le C\ell^{\alpha/2}M^{-2}\|f'\|_\infty^2\big(\mathcal{H}_N+1\big) \tag{7.31}$$

for all $\alpha > 0$, $\ell \in (0;1/2)$ small enough, f smooth and bounded, $M \in \mathbb{N}$ and $N \in \mathbb{N}$ large enough.

Proof. To show (7.30), we start from (7.8) and we decompose

$$\begin{aligned} e^{-B(\eta_H)}\mathcal{L}_N^{(2,V)}e^{B(\eta_H)} &= \sum_{p\in\Lambda_+^*}\widehat{V}(p/N)e^{-B(\eta_H)}b_p^*b_pe^{B(\eta_H)} \\ &\quad-\frac{1}{N}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)e^{B(\eta_H)}a_p^*a_pe^{-B(\eta_H)} \\ &\quad+\frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)e^{-B(\eta_H)}\big[b_pb_{-p}+b_p^*b_{-p}^*\big]e^{B(\eta_H)} \\ &=: \mathrm{F}_1+\mathrm{F}_2+\mathrm{F}_3. \end{aligned} \tag{7.32}$$

With equations (3.18), we split F_1 as

$$\begin{aligned} \mathrm{F}_1 &= \sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\big[\gamma_pb_p^*+\sigma_pb_{-p}\big]\big[\gamma_pb_p+\sigma_pb_{-p}^*\big] \\ &\quad+\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\big[(\gamma_pb_p^*+\sigma_pb_{-p})d_p+d_p^*(\gamma_pb_p+\sigma_pb_{-p}^*)+d_p^*d_p\big] \\ &=: \mathrm{F}_{11}+\mathrm{F}_{12} \end{aligned}$$

with the notation $\gamma_p = \cosh\eta_H(p)$, $\sigma_p = \sinh\eta_H(p)$ and the operators d_p, as defined in (3.17), with η replaced by η_H. We decompose

$$\mathrm{F}_{11} = \sum_{p\in\Lambda_+^*}\widehat{V}(p/N)a_p^*a_p\frac{N-\mathcal{N}_+}{N}+\mathcal{E}_1^V$$

with

$$\mathcal{E}_1^V = \frac{1}{N}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)a_p^*a_p + \sum_{p\in P_H}\widehat{V}(p/N)\Big[(\gamma_p^2-1)b_p^*b_p + \gamma_p\sigma_p(b_{-p}b_p + b_p^*b_{-p}^*) + \sigma_p^2(b_p^*b_p - N^{-1}a_p^*a_p) + \sigma_p^2\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big]$$

where we used $\gamma_p = 1$ and $\sigma_p = 0$ for $p \in P_H^c$ to restrict the second sum. With $|\gamma_p^2 - 1| \leq C\eta_p^2$, $|\sigma_p| \leq C|\eta_p|$ for all $p \in P_H$ and since $\|\eta_H\| \leq \ell^{\alpha/2}$, we find

$$\pm\mathcal{E}_1^V \leq C(\ell^{\alpha/2} + N^{-1})(\mathcal{N}_+ + 1) \leq C\ell^{\alpha/2}(\mathcal{N}_+ + 1)$$

if N is large enough. With Lemma 3.4 (with η replaced by η_H), we can also bound $\pm \mathrm{F}_{12} \leq C\ell^{\alpha/2}(\mathcal{N}_+ + 1)$. We conclude that

$$\mathrm{F}_1 = \sum_{p\in\Lambda_+^*}\widehat{V}(p/N)a_p^*a_p\frac{N-\mathcal{N}_+}{N} + \mathcal{E}_2^V \tag{7.33}$$

with $\pm\mathcal{E}_2^V \leq C\ell^{\alpha/2}(\mathcal{N}_+ + 1)$. Let us now consider the second contribution on the r.h.s. of (7.32). We have $-\mathrm{F}_2 \geq 0$ and, by Lemma 3.1,

$$\begin{aligned} -\mathrm{F}_2 &= \frac{1}{N}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)e^{-B(\eta_H)}a_p^*a_pe^{B(\eta_H)} \\ &\leq \frac{\|\widehat{V}\|_\infty}{N}e^{-B(\eta_H)}\mathcal{N}_+e^{B(\eta_H)} \leq C\ell^{\alpha/2}(\mathcal{N}_+ + 1) \end{aligned} \tag{7.34}$$

if $N \in \mathbb{N}$ is large enough, Finally, we turn our attention to the last term on the r.h.s. of (7.32). With (3.18), we decompose F_3 as

$$\begin{aligned} \mathrm{F}_3 &= \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\Big[\gamma_pb_p + \sigma_pb_{-p}^*\Big]\Big[\gamma_pb_{-p} + \sigma_pb_p^*\Big] + \text{h.c.} \\ &\quad + \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\Big[(\gamma_pb_p + \sigma_pb_{-p}^*)\,d_{-p} + d_p\,(\gamma_pb_{-p} + \sigma_pb_p^*)\Big] + \text{h.c.} \\ &\quad + \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)d_pd_{-p} + \text{h.c.} \\ &=: \mathrm{F}_{31} + \mathrm{F}_{32} + \mathrm{F}_{33} + \text{h.c.} \end{aligned} \tag{7.35}$$

We decompose the first term as

$$\mathrm{F}_{31} = \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\big(b_pb_{-p} + b_{-p}^*b_p^*\big) + \sum_{p\in P_H}\widehat{V}(p/N)\eta_p\frac{N-\mathcal{N}_+}{N} + \mathcal{E}_3^V \tag{7.36}$$

with (recall that $\gamma_p = 1$ and $\sigma_p = 0$ for $p \in P_H^c$)

$$\mathcal{E}_3^V = \sum_{p\in P_H} \widehat{V}(p/N)\left[\frac{1}{2}(\gamma_p^2 - 1 + \sigma_p^2)\big(b_p b_{-p} + b_{-p}^* b_p^*\big) + 2\sigma_p\gamma_p b_p^* b_p \right.$$
$$\left. -N^{-1}\gamma_p\sigma_p a_p^* a_p + (\gamma_p\sigma_p - \eta_p)\frac{N-\mathcal{N}_+}{N}\right].$$

Using again the estimates $|\gamma_p^2 - 1| \leq C\eta_p^2$ and $|\sigma_p| \leq C|\eta_p|$ for all $p \in P_H$, we find

$$\pm\mathcal{E}_3^V \leq C\ell^{\alpha/2}(\mathcal{N}_+ + 1). \tag{7.37}$$

Let us now consider F_{32} in (7.35). We divide it into four parts

$$\begin{aligned}\mathrm{F}_{32} &= \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\left[(\gamma_p b_p + \sigma_p b_{-p}^*)\, d_{-p} + d_p\,(\gamma_p b_{-p} + \sigma_p b_p^*)\right] + \text{h.c.} \\ &=: \mathrm{F}_{321} + \mathrm{F}_{322} + \mathrm{F}_{323} + \mathrm{F}_{324}.\end{aligned} \tag{7.38}$$

We start with F_{321}, which we decompose as

$$\begin{aligned}\mathrm{F}_{321} &= \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N)(\gamma_p - 1)b_p d_{-p} \\ &\quad + \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N) b_p\left[d_{-p} + \eta_H(p)\frac{\mathcal{N}_+}{N} b_p^*\right] \\ &\quad - \frac{1}{2N}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\eta_H(p) b_p\,\mathcal{N}_+ b_p^* + \text{h.c.}\end{aligned} \tag{7.39}$$

Using (2.6), we commute

$$\begin{aligned}b_p\,\mathcal{N}_+ b_p^* &= (\mathcal{N}_+ + 1)b_p b_p^* = (\mathcal{N}_+ + 1)(1 - \mathcal{N}_+/N) \\ &\quad + (\mathcal{N}_+ + 1)(b_p^* b_p - N^{-1}a_p^* a_p).\end{aligned} \tag{7.40}$$

We arrive at

$$\mathrm{F}_{321} = -\sum_{p\in P_H} \widehat{V}(p/N)\eta_p\left(\frac{N-\mathcal{N}_+}{N}\right)\left(\frac{\mathcal{N}_+ + 1}{N}\right) + \mathcal{E}_4^V$$

where $\mathcal{E}_4^V = \mathcal{E}_{41}^V + \mathcal{E}_{42}^V + \mathcal{E}_{43}^V + \text{h.c.}$, with

$$\mathcal{E}_{41}^V = \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N)\,(\gamma_p - 1)b_p d_{-p}\,, \qquad \mathcal{E}_{42}^V = \frac{1}{2}\sum_{p\in\Lambda_+^*} \widehat{V}(p/N) b_p \bar{d}_{-p}$$
$$\mathcal{E}_{43}^V = -\frac{1}{2}\sum_{p\in P_H} \widehat{V}(p/N)\eta_p\frac{\mathcal{N}_+ + 1}{N}(b_p^* b_p - N^{-1}a_p^* a_p)$$

and with the notation $\bar{d}_{-p} = d_{-p} + N^{-1}\eta_H(p)\,\mathcal{N}_+ b_p^*$. Since $|\gamma_p - 1| \leq C\eta_p^2\chi(p \in P_H)$, we find easily with (3.20) in Lemma 3.4 that

$$\begin{aligned}|\langle\xi, \mathcal{E}_{41}^V\xi\rangle| &\leq C\sum_{p\in P_H}\eta_p^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\left[|\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta_H\|\|a_p\xi\|\right]\\ &\leq C\ell^{3\alpha}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.\end{aligned}$$

Furthermore

$$|\langle\xi, \mathcal{E}_{43}^V\xi\rangle| \leq C\sum_{p\in P_H}\eta_p\|a_p\xi\|^2 \leq C\ell^{2\alpha}\|\mathcal{N}_+^{1/2}\xi\|^2.$$

To control $\mathcal{E}_{42}^V$ we switch to position space. With (3.23) in Lemma 3.4, we find

$$\begin{aligned}|\langle\xi, \mathcal{E}_{42}^V\xi\rangle| &\leq C\int_{\Lambda^2} dxdy\, N^3V(N(x-y))\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{-1/2}\check{a}_x\check{\bar{d}}_y\xi\|\\ &\leq C\|\eta_H\|\int_{\Lambda^2} dxdy\, N^2V(N(x-y))\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\\ &\quad\times\left[N\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{a}_x\mathcal{N}_+\xi\| + \|\check{a}_y\mathcal{N}_+\xi\| + \|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|\right]\\ &\leq C\ell^{\alpha/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\ell^{\alpha/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|.\end{aligned}$$

We conclude that

$$\pm\mathcal{E}_4^V \leq C\ell^{\alpha/2}(\mathcal{H}_N + 1).$$

To estimate the term F_{322} in (7.38), we use (3.20) in Lemma 3.4 and $|\sigma_p| \leq C|\eta_H(p)|$; we obtain

$$\begin{aligned}|\langle\xi, \mathrm{F}_{322}\xi\rangle| &\leq C\sum_{p\in P_H}|\eta_p|\|b_{-p}\xi\|\|d_{-p}\xi\|\\ &\leq C\sum_{p\in P_H}|\eta_p|\|b_{-p}\xi\|\left[|\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta_H\|\|b_{-p}\xi\|\right]\\ &\leq C\ell^{5\alpha/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.\end{aligned}$$

Let us now consider the term F_{323} on the r.h.s. of (7.38). Here, we proceed as we did above to estimate F_{321}. We write $\mathrm{F}_{323} = \mathcal{E}_{51}^V + \mathcal{E}_{52}^V + \text{h.c.}$, with

$$\mathcal{E}_{51}^V = \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\,(\gamma_p - 1)\,d_pb_{-p}\,, \qquad \mathcal{E}_{52}^V = \frac{1}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/N)\,d_pb_{-p}.$$

With $|\gamma_p - 1| \leq C\eta_p^2\chi(p \in P_H)$, we obtain

$$|\langle\xi, \mathcal{E}_{51}^V\xi\rangle| \leq C\sum_{p\in P_H}\eta_p^2\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|a_p\xi\| \leq C\ell^{5\alpha/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.$$

Switching to position space, we find, by (3.22),

$$\begin{aligned}
|\langle\xi, \mathcal{E}^V_{52}\xi\rangle| &\leq C\int_{\Lambda^2} dxdy\, N^3V(N(x-y)) \\
&\quad\times \|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{-1/2}\check{d}_x\check{a}_y\xi\| \\
&\leq C\|\eta_H\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\int_{\Lambda^2} dxdy \\
&\quad\times N^2V(N(x-y))\Big[\|\check{a}_y\mathcal{N}_+\xi\| + \|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|\Big] \\
&\leq C\ell^{\alpha/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \\
&\quad + C\ell^{\alpha/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|.
\end{aligned}$$

Hence, $\pm \mathrm{F}_{323} \leq C\ell^{\alpha/2}(\mathcal{H}_N + 1)$.

To estimate the term F_{324} in (7.38), we use (3.20) in Lemma 3.4 and the estimate $\sum_{p\in\Lambda^*_+}\big|\widehat{V}(p/N)\big||\eta_p| \leq CN$; we find

$$\begin{aligned}
|\langle\xi, \mathrm{F}_{324}\xi\rangle| &\leq C\sum_{p\in P_H}\big|\widehat{V}(p/N)\big||\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|(\mathcal{N}_+ + 1)^{-1/2}d_p\, b^*_p\xi\| \\
&\leq \frac{C}{N}\sum_{p\in P_H}\big|\widehat{V}(p/N)\big||\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\
&\quad\times\Big[|\eta_p|\|(\mathcal{N}_+ + 1)^{3/2}\xi\| + \|\eta_H\|\|b_pb^*_p(\mathcal{N}_+ + 1)^{1/2}\xi\|\Big] \\
&\leq \frac{C}{N}\sum_{p\in P_H}\big|\widehat{V}(p/N)\big||\eta_p|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\
&\quad\times\Big[|\eta_p|\|(\mathcal{N}_+ + 1)^{3/2}\xi\| + \|\eta_H\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta_H\|\|a_p(\mathcal{N}_+ + 1)\xi\|\Big] \\
&\leq C\ell^{\alpha/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.
\end{aligned}$$

Combining the last bounds, we conclude that

$$\mathrm{F}_{32} = \sum_{p\in P_H}\widehat{V}(p/N)\eta_p\left(\frac{N-\mathcal{N}_+}{N}\right)\left(\frac{-\mathcal{N}_+ - 1}{N}\right) + \mathcal{E}^V_6$$

with

$$\pm\,\mathcal{E}^V_6 \leq C\ell^{\alpha/2}(\mathcal{H}_N + 1). \tag{7.41}$$

To bound the last term F_{33} in (7.35), we switch to position space. With Lemma 3.4, specifically (3.24), and (4.17), we obtain

$$\begin{aligned}
|\langle\xi, \mathrm{F}_{33}\xi\rangle| &\leq C\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\int_{\Lambda^2} dxdy\, N^3V(N(x-y))\|(\mathcal{N}_+ + 1)^{-1/2}\check{d}_x\check{d}_y\xi\| \\
&\leq C\|\eta_H\|\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\int_{\Lambda^2} dxdy\, NV(N(x-y)) \\
&\quad\times\Big[N\|(\mathcal{N}_+ + 1)^{3/2}\xi\| + \|\check{a}_x\mathcal{N}_+^2\xi\| + \|\check{a}_x\check{a}_y\mathcal{N}_+^{3/2}\xi\|\Big] \\
&\leq C\ell^{\alpha/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\ell^{\alpha/2}\|\mathcal{N}_+^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|.
\end{aligned}$$

The last equation, combined with (7.35), (7.36), (7.37) and (7.41), implies that

$$\begin{aligned} \mathrm{F}_3 = \frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N)(b_p b_{-p} + b^*_{-p} b^*_p) \\ + \sum_{p \in P_H} \widehat{V}(p/N) \eta_p \left(\frac{N - \mathcal{N}_+}{N} \right) \left(\frac{N - \mathcal{N}_+ - 1}{N} \right) + \mathcal{E}_7^V \end{aligned}$$

with

$$\pm \mathcal{E}_7^V \leq C \ell^{\alpha/2} (\mathcal{H}_N + 1).$$

Together with (7.33) and with (7.34), we obtain (7.29) with (7.30). Eq. (7.31) follows similarly, arguing as we did at the end of the proof of Proposition 7.2 to show (7.12). □

We conclude this section, summarizing the results of Propositions 7.2 and 7.3.

Proposition 7.4. *There exists a constant $C > 0$ such that*

$$\begin{aligned} \mathcal{G}^{(2)}_{N,\ell} = \mathcal{K} + \sum_{p \in P_H} \left[p^2 \eta_p^2 + \widehat{V}(p/N) \eta_p \right] \left(\frac{N - \mathcal{N}_+}{N} \right) \left(\frac{N - \mathcal{N}_+ - 1}{N} \right) \\ + \sum_{p \in P_H} p^2 \eta_p \left(b^*_p b^*_{-p} + b_p b_{-p} \right) + \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) a^*_p a_p \frac{N - \mathcal{N}_+}{N} \\ + \frac{1}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \left(b_p b_{-p} + b^*_{-p} b^*_p \right) + \mathcal{E}^{(2)}_{N,\ell} \end{aligned}$$

where

$$\pm \mathcal{E}^{(2)}_{N,\ell} \leq C \ell^{(\alpha-3)/2} (\mathcal{H}_N + 1)$$

and

$$\pm \left[f(\mathcal{N}_+/M), \left[f(\mathcal{N}_+/M), \mathcal{E}^{(2)}_{N,\ell} \right] \right] \leq C \ell^{(\alpha-3)/2} M^{-2} \| f' \|^2_\infty \left(\mathcal{H}_N + 1 \right)$$

for all $\alpha > 3$, $\ell \in (0; 1/2)$ small enough, f smooth and bounded, $M \in \mathbb{N}$, $N \in \mathbb{N}$ large enough.

7.3. Analysis of $\mathcal{G}^{(3)}_{N,\ell} = e^{-B(\eta_H)} \mathcal{L}^{(3)}_N e^{B(\eta_H)}$. From (2.4), we have

$$\mathcal{G}^{(3)}_{N,\ell} = \frac{1}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N) e^{-B(\eta_H)} b^*_{p+q} a^*_{-p} a_q e^{B(\eta_H)} + \text{h.c.} \tag{7.42}$$

Proposition 7.5. *There exists a constant $C > 0$ such that*

$$\mathcal{G}^{(3)}_{N,\ell} = \frac{1}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N) \left[b^*_{p+q} a^*_{-p} a_q + h.c. \right] + \mathcal{E}^{(3)}_{N,\ell} \tag{7.43}$$

where

$$\pm \mathcal{E}^{(3)}_{N,\ell} \leq C \ell^{\alpha/2} \left(\mathcal{H}_N + 1 \right) \tag{7.44}$$

and

$$\pm\,[f(\mathcal{N}_+/M),[f(\mathcal{N}_+/M),\mathcal{E}^{(3)}_{N,\ell}]]\leq CM^{-2}\|f'\|^2_\infty \ell^{\alpha/2}\big(\mathcal{H}_N+1\big) \tag{7.45}$$

for all $\alpha>0$, $\ell\in(0;1/2)$ *small enough,* f *smooth and bounded,* $M\in\mathbb{N}$, $N\in\mathbb{N}$ *large enough.*

Proof of Proposition 7.5. We start by writing

$$\begin{aligned} e^{-B(\eta_H)}a^*_{-p}a_q e^{B(\eta_H)} &= a^*_{-p}a_q+\int_0^1 ds\, e^{-sB(\eta_H)}[a^*_{-p}a_q,\,B(\eta_H)]e^{sB(\eta_H)}\\ &= a^*_{-p}a_q+\int_0^1 ds e^{-sB(\eta_H)}(\eta_H(p)b_qb_p+\eta_H(q)b^*_{-p}b^*_{-q})e^{sB(\eta_H)}. \end{aligned}$$

From (7.42), we find

$$\begin{aligned} \mathcal{G}^{(3)}_{N,\ell} &= \frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda^*_+:p+q\neq 0}\widehat{V}(p/N)e^{-B(\eta_H)}b^*_{p+q}e^{B(\eta_H)}\,a^*_{-p}a_q\\ &\quad+\frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda^*_+,p+q\neq 0}\widehat{V}(p/N)\eta_H(p)\,e^{-B(\eta_H)}b^*_{p+q}e^{B(\eta_H)}\\ &\quad\times\int_0^1 ds\,e^{-sB(\eta_H)}b_pb_qe^{sB(\eta_H)}\\ &\quad+\frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda^*_+,p+q\neq 0}\widehat{V}(p/N)\eta_H(q)\,e^{-B(\eta_H)}b^*_{p+q}e^{B(\eta_H)}\\ &\quad\times\int_0^1 ds\,e^{-sB(\eta_H)}b^*_{-p}b^*_{-q}e^{sB(\eta_H)}\\ &\quad+\text{h.c.} \end{aligned}$$

Using (3.18) we arrive at (7.43), with

$$\begin{aligned} \mathcal{E}^{(3)}_{N,\ell} &= \frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda^*_+:p+q\neq 0}\widehat{V}(p/N)\big((\gamma_{p+q}-1)b^*_{p+q}+\sigma_{p+q}b_{-p-q}+d^*_{p+q}\big)\,a^*_{-p}a_q\\ &\quad+\frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda^*_+,p+q\neq 0}\widehat{V}(p/N)\eta_H(p)\,e^{-B(\eta_H)}b^*_{p+q}e^{B(\eta_H)}\\ &\quad\times\int_0^1 ds\,e^{-sB(\eta_H)}b_pb_qe^{sB(\eta_H)}\\ &\quad+\frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda^*_+,p+q\neq 0}\widehat{V}(p/N)\eta_H(q)\,e^{-B(\eta_H)}b^*_{p+q}e^{B(\eta_H)}\\ &\quad\times\int_0^1 ds\,e^{-sB(\eta_H)}b^*_{-p}b^*_{-q}e^{sB(\eta_H)}\\ &\quad+\text{h.c.}\\ &=:\ \mathcal{E}^{(3)}_1+\mathcal{E}^{(3)}_2+\mathcal{E}^{(3)}_3+\text{h.c.} \end{aligned} \tag{7.46}$$

where we defined $\gamma_p = \cosh \eta_H(p)$, $\sigma_p = \sinh \eta_H(p)$ and where the operator d_p is defined as in (3.17), with η replaced by η_H. To complete the proof of the proposition, we have to show that the three error terms $\mathcal{E}_1^{(3)}, \mathcal{E}_2^{(3)}, \mathcal{E}_3^{(3)}$ all satisfy the bounds (7.44), (7.45). We start by considering $\mathcal{E}_1^{(3)}$. We decompose it as

$$\begin{aligned}\mathcal{E}_1^{(3)} &= \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*: p+q\neq 0} \widehat{V}(p/N)\big((\gamma_{p+q}-1)b^*_{p+q} + \sigma_{p+q} b_{-p-q} + d^*_{p+q}\big)\, a^*_{-p} a_q \\ &=: \mathcal{E}_{11}^{(3)} + \mathcal{E}_{12}^{(3)} + \mathcal{E}_{13}^{(3)}.\end{aligned}$$

Since $|\gamma_{p+q} - 1| \leq |\eta_H(p+q)|^2$ and $\|\eta_H\| \leq C\ell^{\alpha/2}$, we have

$$\begin{aligned}|\langle \xi, \mathcal{E}_{11}^{(3)}\xi\rangle| &\leq \frac{C}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*: p+q\neq 0} |\widehat{V}(p/N)||\eta_H(p+q)|^2 \, \|b_{p+q} a_{-p}\xi\| \|a_q\xi\| \\ &\leq \frac{C}{\sqrt{N}}\Big[\sum_{p,q\in\Lambda_+^*: p+q\neq 0} |\eta_H(p+q)|^2 \, \|a_{-p}(\mathcal{N}_+ + 1)^{1/2}\xi\|^2\Big]^{1/2} \\ &\quad \times \Big[\sum_{p,q\in\Lambda_+^*: p+q\neq 0} |\eta_H(p+q)|^2 \|a_q\xi\|^2\Big]^{1/2} \\ &\leq C\|\eta_H\|^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \leq C\ell^\alpha \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.\end{aligned} \tag{7.47}$$

To bound $\mathcal{E}_{12}^{(3)}$ we move a^*_{-p} to the left of b_{-p-q} (using $[a_{-p-q}, a^*_{-p}] = 0$, since $q \neq 0$). With $|\sigma_{p+q}| \leq C|\eta_H(p+q)|$, we obtain

$$\begin{aligned}|\langle \xi, \mathcal{E}_{12}^{(3)}\xi\rangle| &\leq \frac{C}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*: p+q\neq 0} |\widehat{V}(p/N)||\eta_H(p+q)| \, \|a_{-p}\xi\| \|a_q b_{-p-q}\xi\| \\ &\leq \frac{C}{\sqrt{N}}\Big[\sum_{p,q\in\Lambda_+^*: p+q\neq 0} |\eta_H(p+q)|^2 \, \|a_{-p}\xi\|^2\Big]^{1/2} \\ &\quad \times \Big[\sum_{p,q\in\Lambda_+^*: p+q\neq 0} \|a_q b_{-p-q}\xi\|^2\Big]^{1/2} \\ &\leq C\ell^{\alpha/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.\end{aligned} \tag{7.48}$$

In $\mathcal{E}_{13}^{(3)}$, on the other hand, we write $d^*_{p+q} = \bar{d}^*_{p+q} - \frac{(\mathcal{N}_+ + 1)}{N}\eta_H(p+q) b_{-p-q}$. We obtain $\mathcal{E}_{13}^{(3)} = \mathcal{E}_{131}^{(3)} + \mathcal{E}_{132}^{(3)}$, with

$$\begin{aligned}\mathcal{E}_{131}^{(3)} &= \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*: p+q\neq 0} \widehat{V}(p/N)\, \bar{d}^*_{p+q} a^*_{-p} a_q \\ \mathcal{E}_{132}^{(3)} &= -\frac{(\mathcal{N}_+ + 1)}{N}\frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*: p+q\neq 0} \widehat{V}(p/N)\eta_H(p+q)\, b_{-p-q} a^*_{-p} a_q.\end{aligned}$$

The term $\mathcal{E}_{132}^{(3)}$ can be bounded like $\mathcal{E}_{12}^{(3)}$, commuting a^*_{-p} to the left of b_{-p-q}; we find $\pm\mathcal{E}_{132}^{(3)} \leq C\ell^{\alpha/2}(\mathcal{N}_+ + 1)$. As for the term $\mathcal{E}_{131}^{(3)}$, we switch to position space:

$$\mathcal{E}_{131}^{(3)} = \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*: p+q\neq 0} \widehat{V}(p/N)\, \bar{d}^*_{p+q} a^*_{-p} a_q = \int_{\Lambda^2} dx dy N^{5/2} V(N(x-y)) \check{\bar{d}}^*_x \check{a}^*_y \check{a}_x.$$

With (3.23), we bound

$$\begin{aligned}
|\langle\xi, \mathcal{E}^{(3)}_{131}\xi\rangle| &\leq \int_{\Lambda^2} dxdyN^{5/2}V(N(x-y))\|\check{a}_x\xi\|\|\check{a}_y\check{\check{d}}_x\xi\| \\
&\leq C\|\eta_H\|\int_{\Lambda^2} dxdyN^{5/2}V(N(x-y))\|\check{a}_x\xi\| \\
&\quad\times\Big[\|(\mathcal{N}_+ + 1)\xi\| + \|\check{a}_x(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{a}_y(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{a}_x\check{a}_y\xi\|\Big] \\
&\leq C\|\eta_H\|\|\mathcal{N}_+^{1/2}\xi\|\Big[\int_{\Lambda^2} dxdyN^2V(N(x-y)) \\
&\quad\times\Big[\|(\mathcal{N}_+ + 1)\xi\|^2 + \|\check{a}_x(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + \|\check{a}_x\check{a}_y\xi\|^2\Big]\Big]^{1/2} \\
&\leq C\|\eta_H\|\|\mathcal{N}_+^{1/2}\xi\|\Big[\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\mathcal{V}_N^{1/2}\xi\|\Big] \\
&\leq C\ell^{\alpha/2}\Big[\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + \|\mathcal{V}_N^{1/2}\xi\|^2\Big].
\end{aligned}$$

With (7.47) and (7.48) we conclude that

$$\pm\,\mathcal{E}^{(3)}_1 \leq C\ell^{\alpha/2}(\mathcal{V}_N + \mathcal{N}_+ + 1) \leq C\ell^{\alpha/2}(\mathcal{H}_N + 1). \tag{7.49}$$

Next, we consider the term $\mathcal{E}^{(3)}_2$, defined in (7.46). Using Eq. (3.18) we rewrite

$$\begin{aligned}
\mathcal{E}^{(3)}_2 &= \frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*, p+q\neq 0}\widehat{V}(p/N)\eta_H(p)\,e^{-B(\eta_H)}b^*_{p+q}e^{B(\eta_H)} \\
&\quad\times\int_0^1 ds\,\big(\gamma^{(s)}_p\gamma^{(s)}_q b_p b_q + \sigma^{(s)}_p\sigma^{(s)}_q b^*_{-p}b^*_{-q} \\
&\quad+\gamma^{(s)}_p\sigma^{(s)}_q b^*_{-q}b_p + \sigma^{(s)}_p\gamma^{(s)}_q b^*_{-p}b_q\big) \\
&\quad+\frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*, p+q\neq 0}\widehat{V}(p/N)\eta_H(p)\,e^{-B(\eta_H)}b^*_{p+q}e^{B(\eta_H)} \\
&\quad\times\int_0^1 ds\,\gamma^{(s)}_p\sigma^{(s)}_q[b_p, b^*_{-q}] \\
&\quad+\frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*, p+q\neq 0}\widehat{V}(p/N)\eta_H(p)\,e^{-B(\eta_H)}b^*_{p+q}e^{B(\eta_H)} \\
&\quad\times\int_0^1 ds\,\Big[d^{(s)}_p\big(\gamma^{(s)}_q b_q + \sigma^{(s)}_q b^*_{-q}\big) \\
&\quad+\big(\gamma^{(s)}_p b_p + \sigma^{(s)}_p b^*_{-p}\big)d^{(s)}_q + d^{(s)}_p d^{(s)}_q\Big] \\
&=: \mathcal{E}^{(3)}_{21} + \mathcal{E}^{(3)}_{22} + \mathcal{E}^{(3)}_{23}
\end{aligned} \tag{7.50}$$

where, for any $s \in [0;1]$ and $p \in \Lambda_+^*$, $\gamma_p^{(s)} = \cosh(s\eta_H(p))$, $\sigma_p^{(s)} = \sinh(s\eta_H(p))$ and $d_p^{(s)}$ is the operator defined as in (3.17), with η replaced by $s\eta_H$. We have

$$\begin{aligned} |\langle \xi, \mathcal{E}_{21}^{(3)} \xi\rangle| &\leq \frac{C}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*: p\neq -q} |\eta_H(p)| \| b_{p+q} e^{B(\eta_H)}\xi\| \Big[\|b_p b_q \xi\| \\ &\quad + |\eta_H(p)| \|b_q (\mathcal{N}_+ + 1)^{1/2}\xi\| \\ &\quad + |\eta_H(q)| \|b_p (\mathcal{N}_+ + 1)^{1/2}\xi\| + |\eta_H(p)||\eta_H(q)| \|(\mathcal{N}_+ + 1)\xi\| \Big] \\ &\leq C\|\eta_H\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \leq C\ell^{\alpha/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2. \end{aligned} \tag{7.51}$$

Since $[b_p, b_{-q}^*] = -a_{-q}^* a_p/N$ for all $p \neq -q$, we find

$$\begin{aligned} |\langle \xi, \mathcal{E}_{22}^{(3)} \xi\rangle| &\leq \frac{C}{N^{3/2}} \sum_{p,q\in\Lambda_+^*, p+q\neq 0} |\eta_H(p)||\eta_H(q)| \|b_{p+q} e^{B(\eta_H)}\xi\| \|a_p (\mathcal{N}_+ + 1)^{1/2}\xi\| \\ &\leq \frac{C}{N} \|\eta_H\|^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \leq \frac{C\ell^\alpha}{N} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2. \end{aligned} \tag{7.52}$$

To bound the third term on the r.h.s. of (7.50), we switch to position space. We obtain

$$\begin{aligned} \mathcal{E}_{23}^{(3)} &= \int_{\Lambda^3} dxdydz N^{5/2} V(N(x-z)) \check{\eta}_H(z-y)\, e^{-B(\eta_H)} \check{b}_x^* e^{B(\eta_H)} \\ &\quad \times \int_0^1 ds \Big[\check{d}_y^{(s)} \big(b(\check{\gamma}_x^{(s)}) + b^*(\check{\sigma}_x^{(s)}) \big) + \big(b(\check{\gamma}_y^{(s)}) + b^*(\check{\sigma}_y^{(s)}) \big) \check{d}_x^{(s)} + \check{d}_y^{(s)} \check{d}_x^{(s)} \Big]. \end{aligned}$$

Using the bounds (3.22), (3.23), (3.24) and Lemma 3.1 we arrive at

$$\begin{aligned} |\langle \xi, \mathcal{E}_{23}^{(3)} \xi\rangle| &\leq C\|\eta_H\| \int_{\Lambda^3} dxdydz\, N^{5/2} V(N(x-z)) |\check{\eta}_H(y-z)|\, \|\check{b}_x e^{B(\eta_H)}\xi\| \\ &\quad \times \Big[\|\check{b}_x \check{b}_y \xi\| + \|(\mathcal{N}_+ + 1)\xi\| + \|\check{b}_x (\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{b}_y (\mathcal{N}_+ + 1)^{1/2}\xi\| \Big] \\ &\leq \frac{C\|\eta_H\|^2}{\sqrt{N}} \|\mathcal{N}_+^{1/2} e^{B(\eta_H)}\xi\| \|(\mathcal{N}_+ + 1)\xi\| \\ &\leq C\ell^\alpha \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2. \end{aligned}$$

Combined with (7.51) and (7.52), the last bound implies that

$$\pm\, \mathcal{E}_2^{(3)} \leq C\ell^{\alpha/2} (\mathcal{N}_+ + 1). \tag{7.53}$$

Finally, we consider the last term on the r.h.s. of (7.46). In fact, it is convenient to bound (in absolute value) the expectation of its adjoint, which we decompose as

$$\begin{aligned}
\mathcal{E}_3^{(3)*} &= \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p+q\neq 0} \widehat{V}(p/N)\eta_H(q) \int_0^1 ds\, e^{-sB(\eta_H)} b_{-q} e^{sB(\eta_H)} \\
&\quad \times \big(\gamma_p^{(s)} b_{-p} + \sigma_p^{(s)} b_p^* + d_{-p}^{(s)}\big)\big(\gamma_{p+q} b_{p+q} + \sigma_{p+q} b_{-p-q}^* + d_{p+q}\big) \\
&= \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p+q\neq 0} \widehat{V}(p/N)\eta_H(q) \int_0^1 ds\, e^{-sB(\eta_H)} b_{-q} e^{sB(\eta_H)} \\
&\quad \times \Big[\gamma_p^{(s)}\gamma_{p+q} b_{-p} b_{p+q} + \sigma_p^{(s)}\sigma_{p+q} b_p^* b_{-p-q}^* \\
&\quad + \gamma_p^{(s)}\sigma_{p+q} b_{-p-q}^* b_{-p} + \gamma_{p+q}\sigma_p^{(s)} b_p^* b_{p+q} \\
&\quad + d_{-p}^{(s)}\big(\gamma_{p+q} b_{p+q} + \sigma_{p+q} b_{-p-q}^*\big) \\
&\quad + \big(\gamma_p^{(s)} b_{-p} + \sigma_p^{(s)} b_p^*\big) d_{p+q} + d_{-p}^{(s)} d_{p+q} \Big] \\
&\quad + \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*, p+q\neq 0} \widehat{V}(p/N)\eta_H(q) \\
&\quad \times \int_0^1 ds\, e^{-sB(\eta_H)} b_{-q} e^{sB(\eta_H)} \gamma_p^{(s)}\sigma_{p+q} [b_{-p}, b_{-p-q}^*] \\
&=: \mathcal{E}_{31}^{(3)} + \mathcal{E}_{32}^{(3)}.
\end{aligned}$$

Using that $q \neq 0$ and thus that $[b_{-p}, b_{-p-q}^*] = -a_{-p-q}^* a_{-p}/N$, we can estimate the second term by

$$\begin{aligned}
&|\langle \xi, \mathcal{E}_{32}^{(3)}\xi\rangle| \\
&\quad \leq \frac{C}{N^{3/2}} \int_0^1 ds \sum_{p,q\in\Lambda_+^*, p+q\neq 0} |\eta_H(q)||\eta_H(p+q)|\, \|a_{-p-q}\, e^{-sB(\eta_H)} b_{-q}^* e^{sB(\eta_H)}\xi\|\, \|a_{-p}\xi\| \\
&\quad \leq \frac{C}{N^{3/2}} \int_0^1 ds \Big[\sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} |\eta_H(q)|^2\, \|a_{-p-q}\, e^{-sB(\eta_H)} b_{-q}^* e^{sB(\eta_H)}\xi\|^2 \Big]^{1/2} \\
&\qquad \times \Big[\sum_{\substack{p,q\in\Lambda_+^* \\ p+q\neq 0}} |\eta_H(p+q)|^2 \|a_{-p}\xi\|^2 \Big]^{1/2} \\
&\quad \leq \frac{C}{N} \|\eta_H\|^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 \leq \frac{C\ell^\alpha}{N} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2.
\end{aligned} \tag{7.54}$$

To bound the expectation of $\mathcal{E}_{31}^{(3)}$, it is convenient to switch to position space. We find

$$\begin{aligned}
\mathcal{E}_{31}^{(3)} &= \int_0^1 ds \int_{\Lambda^2} dx dy\, N^{5/2} V(N(x-y))\, e^{-sB(\eta_H)} b(\check{\eta}_{H,x}) e^{sB(\eta_H)} \\
&\quad \times \Big[b(\check{\gamma}_x^{(s)}) b(\check{\gamma}_y) + b^*(\check{\sigma}_x^{(s)}) b^*(\check{\sigma}_y) + b^*(\check{\sigma}_y) b(\check{\gamma}_x^{(s)}) + b^*(\check{\sigma}_x^{(s)}) b(\check{\gamma}_y) \\
&\quad + \check{d}_x^{(s)}\big(b(\check{\gamma}_y) + b^*(\check{\sigma}_y)\big) + \big(b(\check{\gamma}_x^{(s)}) + b^*(\check{\sigma}_x^{(s)})\big)\check{d}_y + \check{d}_x^{(s)}\check{d}_y \Big]
\end{aligned}$$

where we used the notation $\check{\eta}_H$, $\check{\gamma}^{(s)}$ and $\check{\sigma}^{(s)}$ to indicate the functions on Λ with Fourier coefficients $\eta_H(p)$, $\cosh(s\eta_H(p))$ and, respectively, $\sinh(s\eta_H(p))$, and where $\check{\eta}_{H,x}$, $\check{\gamma}_x$ and $\check{\sigma}_x$ denote the functions defined by $\check{\eta}_{H,x}(z) = \check{\eta}_H(z-x)$, $\check{\gamma}_x(z) = \check{\gamma}(z-x)$ and $\check{\sigma}_x(z) = \check{\sigma}^s(z-x)$. Using (3.22), (3.23), (3.24) and the bound (4.17), we find, for N large enough,

$$\begin{aligned}|\langle\xi, \mathcal{E}^{(3)}_{31}\xi\rangle| \leq \int_0^1 ds \int_{\Lambda^2} dxdy\, N^{5/2} V(N(x-y)) \|b^*(\check{\eta}_{H,x}) e^{sB(\eta_H)}\xi\| \\ \times \Big[\|\check{b}_x \check{b}_y \xi\| + \|\check{b}_x(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{b}_y(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|(\mathcal{N}_+ + 1)\xi\|\Big].\end{aligned}$$

With Lemma 3.1, we estimate

$$\|b^*(\check{\eta}_{H,x}) e^{sB(\eta_H)}\xi\| \leq C\|\eta_H\| \|(\mathcal{N}_+ + 1)^{1/2}\xi\|.$$

We conclude that

$$|\langle\xi, \mathcal{E}^{(3)}_{31}\xi\rangle| \leq C\ell^{\alpha/2}\Big[\|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + \|\mathcal{V}_N^{1/2}\xi\|^2\Big].$$

From (7.54), we find

$$\pm \mathcal{E}^{(3)}_3 \leq C\ell^{\alpha/2}(\mathcal{H}_N + 1)$$

and thus, combining this bound with (7.46), (7.49) and (7.53), we arrive at

$$\pm \mathcal{E}^{(3)}_{N,\ell} \leq C\ell^{\alpha/2}(\mathcal{H}_N + 1).$$

This proves (7.44). The bound (7.45) follows similarly, arguing as we did at the end of the proof of Proposition 7.2 to show (7.12). □

7.4. Analysis of $\mathcal{G}^{(4)}_{N,\ell} = e^{-B(\eta_H)}\mathcal{L}^{(4)}_N e^{B(\eta_H)}$. With $\mathcal{L}^{(4)}_N$ as defined in (2.4), we write

$$\begin{aligned}\mathcal{G}^{(4)}_{N,\ell} &= e^{-B(\eta_H)}\mathcal{L}^{(4)}_N e^{B(\eta_H)} \\ &= \mathcal{V}_N + \frac{1}{2N}\sum_{\substack{q\in\Lambda^*_+, r\in\Lambda^* \\ q,\, q+r\in P_H}} \widehat{V}(r/N)\eta_{q+r}\eta_q \left(1 - \frac{\mathcal{N}_+}{N}\right)\left(1 - \frac{\mathcal{N}_+ + 1}{N}\right) \\ &\quad + \frac{1}{2N}\sum_{\substack{q\in\Lambda^*_+, r\in\Lambda^*: \\ q+r\in P_H}} \widehat{V}(r/N)\,\eta_{q+r}\left(b_q b_{-q} + b^*_q b^*_{-q}\right) + \mathcal{E}^{(4)}_{N,\ell}.\end{aligned}$$

Proposition 7.6. *There exists a constant $C > 0$ such that*

$$\pm\, \mathcal{E}^{(4)}_{N,\ell} \leq C\ell^{\alpha/2}\big(\mathcal{H}_N + 1\big) \tag{7.55}$$

and

$$\pm\, [f(\mathcal{N}_+/M), [f(\mathcal{N}_+/M), \mathcal{E}^{(4)}_{N,\ell}]] \leq CM^{-2}\|f'\|^2_\infty \ell^{\alpha/2}\big(\mathcal{H}_N + 1\big) \tag{7.56}$$

for all $\alpha > 0$, $\ell \in (0; 1/2)$ small enough, f smooth and bounded, $M \in \mathbb{N}$, $N \in \mathbb{N}$ large enough.

The following lemma will be useful in the proof of Proposition 7.6.

Lemma 7.7. *Let $\eta_H \in \ell^2(\Lambda^*)$, as defined in (4.13). Then there exists a constant $C > 0$ such that*

$$\begin{aligned}\|(\mathcal{N}_+ + 1)^{n/2} e^{-B(\eta_H)} \check{b}_x \check{b}_y e^{B(\eta_H)} \xi\| \\ \leq C \Big[\|(\mathcal{N}_+ + 1)^{(n+2)/2} \xi\| + N \|(\mathcal{N}_+ + 1)^{n/2} \xi\| \\ + \|\check{a}_y (\mathcal{N}_+ + 1)^{(n+1)/2} \xi\| + \|\check{a}_x (\mathcal{N}_+ + 1)^{(n+1)/2} \xi\| + \|\check{a}_x \check{a}_y (\mathcal{N}_+ + 1)^{n/2} \xi\| \Big]\end{aligned} \tag{7.57}$$

for all $\xi \in \mathcal{F}_+^{\leq N}$, $n \in \mathbb{Z}$.

Proof. We consider $n = 0$, the general case follows similarly. With the notation $\gamma_p = \cosh \eta_H(p)$, $r_p = 1 - \gamma_p$, $\sigma_p = \sinh \eta_H(p)$ and denoting by $\check{\sigma}, \check{r}$ the functions in $L^2(\Lambda)$ with Fourier coefficients σ_p and r_p, we use (3.18) to write

$$\begin{aligned}\|e^{-B(\eta)} \check{b}_x \check{b}_y e^{B(\eta)} \xi\| &= \|\big(\check{b}_x + b(\check{r}_x) + b^*(\check{\sigma}_x) + \check{d}_x\big)\big(\check{b}_y + b(\check{r}_y) + b^*(\check{\sigma}_y) + \check{d}_y\big)\xi\| \\ &\leq \|\check{b}_x \check{b}_y \xi\| + C(\|\check{b}_x \mathcal{N}_+^{1/2} \xi\| + \|\check{b}_y \mathcal{N}_+^{1/2} \xi\|) + C|\check{\sigma}(x-y)|\|\xi\| \\ &\quad + \|\check{b}_x \check{d}_y \xi\| + \|\check{d}_x \big(\check{b}_y + b(\check{r}_y) + b^*(\check{\sigma}_y) + \check{d}_y\big)\xi\|\end{aligned}$$

because $\|r\|, \|\sigma\| \leq C\|\eta_H\| \leq C$. Using Eq. (3.24) and (after writing $\check{b}_x \check{d}_y = \check{b}_x \check{\check{d}}_y - \check{b}_x (\mathcal{N}_+/N) b^*(\check{\eta}_y)$) Eq. (3.23), and with the bound (4.17) (which also implies $|\check{\sigma}(x)| \leq CN$), we obtain (7.57). □

Proof of Proposition 7.6. We start by writing

$$\begin{aligned}&e^{-B(\eta_H)} \mathcal{L}_N^{(4)} e^{B(\eta_H)} \\ &= \frac{1}{2N} \sum_{p,q \in \Lambda_+^*, r \in \Lambda^* : r \neq -p,q} \widehat{V}(r/N) e^{-B(\eta_H)} a_p^* a_q^* a_{q-r} a_{p+r} e^{B(\eta_H)} \\ &= \mathcal{V}_N + \frac{1}{2N} \sum_{p,q \in \Lambda_+^*, r \in \Lambda^* : r \neq -p,q} \widehat{V}(r/N) \\ &\quad \times \int_0^1 ds\, e^{-sB(\eta_H)} \Big[a_p^* a_q^* a_{q-r} a_{p+r}, B(\eta_H) \Big] e^{sB(\eta_H)} \\ &= \mathcal{V}_N + \frac{1}{2N} \sum_{q \in \Lambda_+^*, r \in \Lambda^* : r \neq -q} \widehat{V}(r/N) \eta_H(q+r) \\ &\quad \times \int_0^1 ds \, \Big(e^{-sB(\eta_H)} b_q^* b_{-q}^* e^{sB(\eta_H)} + \text{h.c.} \Big) \\ &\quad + \frac{1}{N} \sum_{p,q \in \Lambda_+^*, r \in \Lambda^* : r \neq p, -q} \widehat{V}(r/N) \eta_H(q+r) \\ &\quad \times \int_0^1 ds \, \Big(e^{-sB(\eta_H)} b_{p+r}^* b_q^* a_{-q-r}^* a_p e^{sB(\eta_H)} + \text{h.c.} \Big) .\end{aligned} \tag{7.58}$$

Now we observe that

$$\begin{aligned} &e^{-sB(\eta_H)} a^*_{-q-r} a_p e^{sB(\eta_H)} \\ &\quad = a^*_{-q-r} a_p + \int_0^s d\tau\, e^{-\tau B(\eta_H)} \left[a^*_{-q-r} a_p, B(\eta_H) \right] e^{-\tau B(\eta_H)} \\ &\quad = a^*_{-q-r} a_p + \int_0^s d\tau\, e^{-\tau B(\eta_H)} \left(\eta_H(p) b^*_{-p} b^*_{-q-r} + \eta_H(q+r) b_p b_{q+r} \right) e^{-\tau B(\eta_H)}. \end{aligned}$$

Inserting in (7.58), we obtain

$$\mathcal{G}^{(4)}_{N,\ell} - \mathcal{V}_N = \mathrm{W}_1 + \mathrm{W}_2 + \mathrm{W}_3 + \mathrm{W}_4$$

where we defined

$$\begin{aligned} \mathrm{W}_1 &= \frac{1}{2N} \sum_{q \in \Lambda^*_+, r \in \Lambda^* : r \neq -q} \widehat{V}(r/N) \eta_H(q+r) \int_0^1 ds \big(e^{-sB(\eta_H)} b_q b_{-q}\, e^{sB(\eta_H)} + \text{h.c.} \big), \\ \mathrm{W}_2 &= \frac{1}{N} \sum_{p,q \in \Lambda^*_+, r \in \Lambda^* : r \neq p, -q} \widehat{V}(r/N)\, \eta_H(q+r) \\ &\quad \times \int_0^1 ds \big(e^{-sB(\eta_H)} b^*_q b^*_{-q} e^{sB(\eta_H)} a^*_{-q-r} a_p + \text{h.c.} \big), \\ \mathrm{W}_3 &= \frac{1}{N} \sum_{p,q \in \Lambda^*_+, r \in \Lambda^* : r \neq -p-q} \widehat{V}(r/N) \eta_H(q+r) \eta_H(p) \\ &\quad \times \int_0^1 ds \int_0^s d\tau \left(e^{-sB(\eta_H)} b^*_{p+r} b^*_q e^{sB(\eta_H)} e^{-\tau B(\eta_H)} b^*_{-p} b^*_{-q-r} e^{\tau B(\eta_H)} + \text{h.c.} \right), \\ \mathrm{W}_4 &= \frac{1}{N} \sum_{p,q \in \Lambda^*_+, r \in \Lambda^* : r \neq -p-q} \widehat{V}(r/N)\, \eta^2_H(q+r) \\ &\quad \times \int_0^1 ds \int_0^s d\tau \left(e^{-sB(\eta_H)} b^*_{p+r} b^*_q e^{sB(\eta_H)} e^{-\tau B(\eta_H)} b_p b_{q+r} e^{\tau B(\eta_H)} + \text{h.c.} \right). \end{aligned} \tag{7.59}$$

First, we consider the term W_1. With (3.18), we find

$$\begin{aligned} \mathrm{W}_1 = \frac{1}{2N} &\sum_{q \in \Lambda^*_+, r \in \Lambda^* : r \neq -q} \widehat{V}(r/N) \eta_H(q+r) \\ &\times \int_0^1 ds \big(\gamma^{(s)}_q b_q + \sigma^{(s)}_q b^*_{-q} + d^{(s)}_q \big) \big(\gamma^{(s)}_q b_{-q} + \sigma^{(s)}_q b^*_q + d^{(s)}_{-q} \big) + \text{h.c.} \end{aligned}$$

where we defined $\gamma_q^{(s)} = \cosh(s\eta_H(q))$, $\sigma_q^{(s)} = \sinh(s\eta_H(q))$ and where $d_q^{(s)}$ is defined as in (3.17), with η replaced by $s\eta_H$. We write

$$\begin{aligned}
\mathrm{W}_1 = &\frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r) \int_0^1 ds (\gamma_q^{(s)})^2 (b_q b_{-q} + \mathrm{h.c.}) \\
&+ \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r) \int_0^1 ds\, \gamma_q^{(s)}\sigma_q^{(s)} \big([b_q, b_q^*] + \mathrm{h.c.}\big) \\
&+ \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r) \int_0^1 ds\, \gamma_q^{(s)} \big(b_q d_{-q}^{(s)} + \mathrm{h.c.}\big) + \mathcal{E}_{10}^{(4)} \\
=: &\, \mathrm{W}_{11} + \mathrm{W}_{12} + \mathrm{W}_{13} + \mathcal{E}_{10}^{(4)}
\end{aligned} \tag{7.60}$$

where

$$\mathcal{E}_{10}^{(4)} = \mathcal{E}_{101}^{(4)} + \mathcal{E}_{102}^{(4)} + \mathcal{E}_{103}^{(4)} + \mathcal{E}_{104}^{(4)} + \mathcal{E}_{105}^{(4)} \tag{7.61}$$

with the errors

$$\begin{aligned}
\mathcal{E}_{101}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r) \\
&\quad \times \int_0^1 ds \Big[2\gamma_q^{(s)}\sigma_q^{(s)} b_q^* b_q + (\sigma_q^{(s)})^2 b_{-q}^* b_q^* + \mathrm{h.c.} \Big], \\
\mathcal{E}_{102}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r) \int_0^1 ds\, \sigma_q^{(s)} \big(b_{-q}^* d_{-q}^{(s)} + \mathrm{h.c.} \big), \\
\mathcal{E}_{103}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r) \int_0^1 ds\, \sigma_q^{(s)} \big(d_q^{(s)} b_q^* + \mathrm{h.c.} \big), \\
\mathcal{E}_{104}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r) \int_0^1 ds\, \gamma_q^{(s)} \big(d_q^{(s)} b_{-q} + \mathrm{h.c.} \big), \\
\mathcal{E}_{105}^{(4)} &= \frac{1}{2N} \sum_{q\in\Lambda_+^*, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r) \int_0^1 ds \big(d_q^{(s)} d_{-q}^{(s)} + \mathrm{h.c.} \big).
\end{aligned} \tag{7.62}$$

Since

$$\sup_{q\in\Lambda_+^*} \frac{1}{N} \sum_{r\in\Lambda_+^*} |\widehat{V}(r/N)||\eta_{q+r}| \leq C \, < \infty \tag{7.63}$$

uniformly in $N \in \mathbb{N}$ and $\ell \in (0; 1/2)$, we can bound the first term in (7.62) by

$$|\langle \xi, \mathcal{E}_{101}^{(4)} \xi \rangle| \leq C \sum_{q\in\Lambda_+^*} \Big[|\eta_q| \|b_q \xi\|^2 + \eta_q^2 \|b_q \xi\| \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \Big] \leq C\ell^{2\alpha} \|(\mathcal{N}_+ + 1)^{1/2} \xi\|^2.$$

To estimate the second term in (7.62), we use (7.63) and Lemma 3.4; we find

$$\begin{aligned}|\langle\xi,\mathcal{E}^{(4)}_{102}\xi\rangle| &\leq C\sum_{q\in\Lambda^*_+}|\eta_H(q)|\|b_{-q}\xi\|\left[|\eta_H(q)|\|(\mathcal{N}_++1)^{1/2}\xi\|+\|\eta_H\|\|b_{-q}\xi\|\right]\\ &\leq C\ell^{2\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2.\end{aligned}$$

For the third term in (7.62), we use (7.63), Lemma 3.4, and also

$$\frac{1}{N^2}\sum_{q\in\Lambda^*_+,r\in\Lambda^*,r\neq -q}|\widehat{V}(r/N)||\eta_H(q+r)||\eta_H(q)|\leq C<\infty$$

uniformly in N and $\ell\in(0;1/2)$. We obtain

$$\begin{aligned}|\langle\xi,\mathcal{E}^{(4)}_{103}\xi\rangle| &\leq \frac{C\|(\mathcal{N}_++1)^{1/2}\xi\|}{N}\sum_{q\in\Lambda^*_+,r\in\Lambda^*:r\neq -q}|\widehat{V}(r/N)||\eta_H(q+r)||\eta_H(q)|\\ &\quad\times\left[|\eta_q|\|b^*_q\xi\|+N^{-1}\|\eta_H\|\|b_qb^*_q\mathcal{N}_+^{1/2}\xi\|\right]\\ &\leq \frac{C\|(\mathcal{N}_++1)^{1/2}\xi\|}{N}\sum_{q\in\Lambda^*_+,r\in\Lambda^*:r\neq -q}|\widehat{V}(r/N)||\eta_H(q+r)||\eta_H(q)|\\ &\quad\times\left[(|\eta_q|+N^{-1}\|\eta_H\|)\|\mathcal{N}_+^{1/2}\xi\|+\|\eta_H\|\|b_q\xi\|\right]\\ &\leq C\ell^{\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2.\end{aligned}$$

Consider now the fourth term in (7.62). We write $\mathcal{E}^{(4)}_{104}=\mathcal{E}^{(4)}_{1041}+\mathcal{E}^{(4)}_{1042}$, with

$$\begin{aligned}\mathcal{E}^{(4)}_{1041}&=\frac{1}{2N}\sum_{q\in\Lambda^*_+,r\in\Lambda^*:r\neq -q}\widehat{V}(r/N)\eta_H(q+r)\int_0^1 ds\,(\gamma^{(s)}_q-1)d^{(s)}_qb_{-q},\\ \mathcal{E}^{(4)}_{1042}&=\frac{1}{2N}\sum_{q\in\Lambda^*_+,r\in\Lambda^*:r\neq -q}\widehat{V}(r/N)\eta_H(q+r)\int_0^1 ds\,d^{(s)}_qb_{-q}.\end{aligned}$$

With $|\gamma^{(s)}_q-1|\leq C|\eta_H(q)|^2$, (7.63) and Lemma 3.4, we easily find

$$|\langle\xi,\mathcal{E}^{(4)}_{1041}\xi\rangle|\leq C\ell^{3\alpha}\|(\mathcal{N}_++1)^{1/2}\xi\|^2.$$

As for the term $\mathcal{E}^{(4)}_{1042}$, we switch to position space. Using (4.17) and (3.22) in Lemma 3.4, we obtain

$$\begin{aligned}|\langle\xi,\mathcal{E}^{(4)}_{1042}\xi\rangle| &=\left|\frac{1}{2}\int_0^1 ds\int_{\Lambda^2}dxdyN^2V(N(x-y))\check{\eta}_H(x-y)\langle\xi,\check{d}^{(s)}_x\check{b}_y\xi\rangle\right|\\ &\leq C\int_0^1\int_{\Lambda^2}dxdyN^3V(N(x-y))\|(\mathcal{N}_++1)^{1/2}\xi\|\|(\mathcal{N}_++1)^{-1/2}\check{d}^{(s)}_x\check{b}_y\xi\|\\ &\leq C\|\eta_H\|\int_0^1\int_{\Lambda^2}dxdyN^2V(N(x-y))\|(\mathcal{N}_++1)^{1/2}\xi\|\\ &\quad\times\left[\|\check{a}_y\mathcal{N}_+\xi\|+\|\check{a}_x\check{a}_y\mathcal{N}_+^{1/2}\xi\|\right]\\ &\leq C\ell^{\alpha/2}\|(\mathcal{N}_++1)^{1/2}\xi\|^2+C\ell^{\alpha/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|.\end{aligned}$$

Let us now consider the last term in (7.62). Switching to position space and using (3.24) in Lemma 3.4 and again (4.17), we arrive at

$$\begin{aligned}|\langle\xi,\mathcal{E}^{(4)}_{105}\xi\rangle| &\leq C\int_{\Lambda^2} dxdy\, N^3V(N(x-y))\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|(\mathcal{N}_+ +1)^{-1/2}\check{d}_x\check{d}_y\xi\|\\ &\leq C\|\eta_H\|\|(\mathcal{N}_+ +1)^{1/2}\xi\|\int_{\Lambda^2} dxdy\, NV(N(x-y))\\ &\quad\times\Big[N\|(\mathcal{N}_+ +1)^{3/2}\xi\|+\|\check{a}_x\mathcal{N}_+^2\xi\|+\|\check{a}_y\mathcal{N}_+^2\xi\|+\|\check{a}_x\check{a}_y\mathcal{N}_+^{3/2}\xi\|\Big]\\ &\leq C\ell^{\alpha/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|^2+C\ell^{\alpha/2}\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|.\end{aligned}$$

We conclude that the error term (7.61) can be estimated by

$$\pm\mathcal{E}^{(4)}_{10}\leq C\ell^{\alpha/2}(\mathcal{H}_N+1).$$

Next, we come back to the terms W_{11}, W_{12}, W_{13} defined in (7.60). Using (7.63) and $|\gamma_q^{(s)}-1|\leq C\eta_H(q)^2$, we can write

$$\mathrm{W}_{11}=\frac{1}{2N}\sum_{q\in\Lambda_+^*,r\in\Lambda^*:r\neq -q}\widehat{V}(r/N)\eta_H(q+r)(b_qb_{-q}+\mathrm{h.c.})+\mathcal{E}^{(4)}_{11}\tag{7.64}$$

where $\mathcal{E}^{(4)}_{11}$ satisfies the estimate

$$\begin{aligned}|\langle\xi,\mathcal{E}^{(4)}_{11}\xi\rangle|&\leq\frac{C}{N}\sum_{q\in\Lambda_+^*,r\in\Lambda^*:r\neq -q}|\widehat{V}(r/N)||\eta_H(q+r)||\eta_H(q)|^2\|b_q\xi\|\|(\mathcal{N}_+ +1)^{1/2}\xi\|\\ &\leq C\ell^{5\alpha/2}\|(\mathcal{N}_+ +1)\xi\|^2.\end{aligned}$$

The second term in (7.60) can be decomposed as

$$\mathrm{W}_{12}=\frac{1}{2N}\sum_{q\in\Lambda_+^*,r\in\Lambda^*:r\neq -q}\widehat{V}(r/N)\eta_H(q+r)\eta_H(q)\left(1-\frac{\mathcal{N}_+}{N}\right)+\mathcal{E}^{(4)}_{12}\tag{7.65}$$

where the error

$$\begin{aligned}\mathcal{E}^{(4)}_{12}&=-\frac{1}{2N^2}\sum_{q\in\Lambda_+^*,r\in\Lambda^*:r\neq -q}\widehat{V}(r/N)\eta_H(q+r)\int_0^1 ds\gamma_q^{(s)}\sigma_q^{(s)}a_q^*a_q\\ &\quad+\frac{1}{2N}\sum_{q\in\Lambda_+^*,r\in\Lambda^*:r\neq -q}\widehat{V}(r/N)\eta_H(q+r)\int_0^1 ds(\gamma_q^{(s)}\sigma_q^{(s)}-s\eta_H(q))\left(1-\frac{\mathcal{N}_+}{N}\right)\end{aligned}$$

can be bounded, using (7.63) and $|\gamma_q^{(s)}\sigma_q^{(s)}-s\eta_H(q))|\leq C|\eta_H(q)|^3$, by

$$\pm\mathcal{E}^{(4)}_{12}\leq C\ell^{2\alpha}(\mathcal{N}_+ +1).$$

As for the third term on the r.h.s. of (7.60), we write

$$\begin{aligned}\mathrm{W}_{13}=&-\frac{1}{2N}\sum_{q\in\Lambda_+^*,r\in\Lambda^*:r\neq -q}\widehat{V}(r/N)\eta_H(q+r)\eta_H(q)\\ &\times\left(1-\frac{\mathcal{N}_+}{N}\right)\frac{\mathcal{N}_+ +1}{N}+\mathcal{E}^{(4)}_{13}\end{aligned}\tag{7.66}$$

where $\mathcal{E}^{(4)}_{13} = \mathcal{E}^{(4)}_{131} + \mathcal{E}^{(4)}_{132} + \mathcal{E}^{(4)}_{133} + \mathcal{E}^{(4)}_{134}$, with

$$\mathcal{E}^{(4)}_{131} = \frac{1}{2N} \sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r) \int_0^1 ds (\gamma_q^{(s)} - 1) b_q d^{(s)}_{-q} + \text{h.c.},$$

$$\mathcal{E}^{(4)}_{132} = \frac{1}{2N} \sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r) \int_0^1 ds\, b_q \left[d^{(s)}_{-q} + s\eta_H(q) \frac{\mathcal{N}_+}{N} b^*_q \right] + \text{h.c.},$$

$$\mathcal{E}^{(4)}_{133} = -\frac{1}{2N} \sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r)\eta_H(q) b^*_q b_q \frac{\mathcal{N}_+ + 1}{N},$$

$$\mathcal{E}^{(4)}_{134} = \frac{1}{2N^2} \sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r)\eta_H(q) a^*_q a_q \frac{\mathcal{N}_+ + 1}{N}.$$

It is easy to estimate the last two terms: with (7.63), we have

$$\pm \mathcal{E}^{(4)}_{133} \leq C\ell^{2\alpha}(\mathcal{N}_+ + 1), \qquad \pm \mathcal{E}^{(4)}_{134} \leq C\ell^{2\alpha}(\mathcal{N}_+ + 1).$$

With $|\gamma_q^{(s)} - 1| \leq C\eta_H(q)^2$, Lemma 3.4 and, again, (7.63), we also find

$$\begin{aligned} |\langle \xi, \mathcal{E}^{(4)}_{131}\xi\rangle| &\leq \frac{C}{N} \sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q} |\widehat{V}(r/N)||\eta_H(q+r)||\eta_H(q)|^2 \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ &\quad \times \Big[|\eta_H(q)| \|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\eta_H\| \|b_q\xi\| \Big] \\ &\leq C\ell^{3\alpha} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2. \end{aligned}$$

Let us now focus on $\mathcal{E}^{(4)}_{132}$. Switching to position space, making use of the notation $\check{d}^{(s)}_y = d^{(s)}_y + s(\mathcal{N}_+/N) b^*(\check{\eta}_{H,y})$ and using Lemma 3.4, specifically (3.23), we obtain

$$\begin{aligned} |\langle \xi, \mathcal{E}^{(4)}_{132}\xi\rangle| &= \Big| \int_0^1 ds \int_{\Lambda^2} dx dy N^2 V(N(x-y)) \check{\eta}_H(x-y) \langle \xi, \check{b}_x \check{\check{d}}_y \xi\rangle \Big| \\ &\leq C\|\eta_H\| \int_{\Lambda^2} dx dy N^2 V(N(x-y)) \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \\ &\quad \times \Big[N\|(\mathcal{N}_+ + 1)^{1/2}\xi\| + \|\check{a}_x \mathcal{N}_+ \xi\| + \|\check{a}_y \mathcal{N}_+ \xi\| + \|\check{a}_x \check{a}_y \mathcal{N}_+^{1/2}\xi\| \Big] \\ &\leq C\ell^{\alpha/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|^2 + C\ell^{\alpha/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{V}_N^{1/2}\xi\|. \end{aligned}$$

We conclude that $\pm \mathcal{E}^{(4)}_{13} \leq C\ell^{\alpha/2}(\mathcal{H}_N + 1)$. Combining this with (7.64), (7.65), (7.66), we obtain

$$\begin{aligned} \mathrm{W}_1 &= \frac{1}{2N} \sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r) \big(b_q b_{-q} + \text{h.c.} \big) \\ &\quad + \frac{1}{2N} \sum_{q\in\Lambda^*_+, r\in\Lambda^*: r\neq -q} \widehat{V}(r/N)\eta_H(q+r)\eta_H(q) \\ &\quad \times \left(1 - \frac{\mathcal{N}_+}{N}\right) \left(1 - \frac{\mathcal{N}_+ + 1}{N}\right) + \mathcal{E}^{(4)}_1 \end{aligned} \tag{7.67}$$

with

$$\pm \mathcal{E}_1^{(4)} \leq C\ell^{\alpha/2}(\mathcal{H}_N + 1).$$

Next, we consider the term W_2, in (7.59). To this end, it is convenient to switch to position space. We find

$$\mathrm{W}_2 = \int_{\Lambda^2} dxdy N^2 V(N(x-y)) \int_0^1 ds \big(e^{-sB(\eta_H)} \check{b}_x^* \check{b}_y^* e^{sB(\eta_H)} a^*(\check{\eta}_{H,x}) \check{a}_y + \text{h.c.} \big)$$

with the notation $\check{\eta}_{H,x}(z) = \check{\eta}_H(x-z)$. By Cauchy–Schwarz, we have

$$\begin{aligned} |\langle \xi, \mathrm{W}_2 \xi \rangle| \leq & \int_{\Lambda^2} dxdy\, N^2 V(N(x-y)) \int_0^1 ds \\ & \times \| (\mathcal{N}_+ + 1)^{1/2} e^{-sB(\eta_H)} \check{b}_x \check{b}_y e^{sB(\eta_H)} \xi \| \| (\mathcal{N}_+ + 1)^{-1/2} a^*(\check{\eta}_{H,x}) \check{a}_y \xi \|. \end{aligned}$$

With

$$\| (\mathcal{N}_+ + 1)^{-1/2} a^*(\check{\eta}_{H,x}) \check{a}_y \xi \| \leq C \| \eta_H \| \| \check{a}_y \xi \| \leq C\ell^{\alpha/2} \| \check{a}_y \xi \|$$

and using Lemma 7.7, we obtain

$$\begin{aligned} |\langle \xi, \mathrm{W}_2 \xi \rangle| &\leq C\ell^{\alpha/2} \int_{\Lambda^2} dxdy\, N^2 V(N(x-y)) \| \check{a}_y \xi \| \\ &\quad \times \Big\{ N \| (\mathcal{N}_+ + 1)^{1/2} \xi \| + N \| \check{a}_x \xi \| + N \| \check{a}_y \xi \| + N^{1/2} \| \check{a}_x \check{a}_y \xi \| \Big\} \\ &\leq C\ell^{\alpha/2} \, \| (\mathcal{N}_+ + 1)^{1/2} \xi \| \| (\mathcal{V}_N + \mathcal{N}_+ + 1)^{1/2} \xi \|. \end{aligned} \tag{7.68}$$

Also for the term W_3 in (7.59), we switch to position space. We find

$$\begin{aligned} \mathrm{W}_3 = & \int_{\Lambda^2} dxdy\, N^2 V(N(x-y)) \\ & \times \int_0^1 ds \int_0^s d\tau \, \big(e^{-sB(\eta_H)} \check{b}_x^* \check{b}_y^* e^{sB(\eta_H)} \, e^{-\tau B(\eta_H)} b^*(\check{\eta}_{H,x}) b^*(\check{\eta}_{H,y}) e^{\tau B(\eta_H)} + \text{h.c.} \big) \end{aligned}$$

and thus

$$\begin{aligned} |\langle \xi, \mathrm{W}_3 \xi \rangle| \leq & \int_{\Lambda^2} dxdy\, N^2 V(N(x-y)) \int_0^1 ds \int_0^s d\tau \, \| (\mathcal{N}_+ + 1)^{1/2} e^{-sB(\eta_H)} \check{b}_x \check{b}_y e^{sB(\eta_H)} \xi \| \\ & \times \| (\mathcal{N}_+ + 1)^{-1/2} e^{-\tau B(\eta_H)} b^*(\check{\eta}_{H,x})) b^*(\check{\eta}_{H,y}) e^{\tau B(\eta_H)} \xi \|. \end{aligned}$$

With Lemma 3.1, we find

$$\| (\mathcal{N}_+ + 1)^{-1/2} e^{-\tau B(\eta_H)} b^*(\check{\eta}_{H,x})) b^*(\check{\eta}_{H,y}) e^{\tau B(\eta_H)} \xi \| \leq C \| \eta_H \|^2 \| (\mathcal{N}_+ + 1)^{1/2} \xi \|.$$

Using Lemma 7.7, we conclude that

$$\begin{aligned} |\langle \xi, \mathrm{W}_3 \xi \rangle| &\leq C\ell^{\alpha} \int_{\Lambda^2} dxdy\, N^2 V(N(x-y)) \| (\mathcal{N}_+ + 1)^{1/2} \xi \| \\ &\quad \times \Big\{ N \| (\mathcal{N}_+ + 1)^{1/2} \xi \| + N \| \check{a}_x \xi \| + N \| \check{a}_y \xi \| + N^{1/2} \| \check{a}_x \check{a}_y \xi \| \Big\} \\ &\leq C\ell^{\alpha} \, \| (\mathcal{N}_+ + 1)^{1/2} \xi \| \| (\mathcal{V}_N + \mathcal{N}_+ + 1)^{1/2} \xi \|. \end{aligned} \tag{7.69}$$

The term W_4 in (7.59) can be bounded similarly. Switching to position space, we find

$$\begin{aligned}\mathrm{W}_4 = &\int dxdy\, N^2 V(N(x-y)) \\ &\times \int_0^1 ds \int_0^s d\tau \left(e^{-sB(\eta_H)} \check{b}^*_x \check{b}^*_y\, e^{sB(\eta_H)}\, e^{-\tau B(\eta_H)} b(\check{\eta}^2_{H,x}) \check{b}_y e^{\tau B(\eta_H)} + \text{h.c.}\right)\end{aligned}$$

where $\check{\eta}^2_H$ denotes the function with Fourier coefficients $\eta^2_H(q)$, for $q \in \Lambda^*$, and where $\check{\eta}^2_{H,x}(y) := \check{\eta}^2_H(x-y)$. We conclude that $\|\check{\eta}^2_{H,x}\| = \|\eta^2_H\| \leq C\ell^{5\alpha/2}$. With Cauchy–Schwarz, we arrive at

$$\begin{aligned}|\langle \xi, \mathrm{W}_4 \xi\rangle| \leq\, & C\ell^{5\alpha/2} \int_0^1 ds \int_0^s d\tau \int dxdy N^2 V(N(x-y)) \\ &\times \|(\mathcal{N}_+ + 1)^{1/2} e^{-sB(\eta_H)} \check{b}_y \check{b}_x e^{sB(\eta_H)} \xi\| \|\check{b}_y e^{\tau B(\eta_H)} \xi\|.\end{aligned}$$

Applying Lemmas 3.1 and 7.7, we obtain

$$\begin{aligned}|\langle \xi, \mathrm{W}_4 \xi\rangle| &\leq C\ell^{5\alpha/2} \int_0^1 ds \int_0^s d\tau \int dxdy N^2 V(N(x-y)) \|\check{b}_y e^{\tau B(\eta_H)} \xi\| \\ &\quad \times \left\{ N\|(\mathcal{N}_+ + 1)^{1/2} \xi\| + N\|\check{a}_x \xi\| + N\|\check{a}_y \xi\| + N^{1/2} \|\check{a}_x \check{a}_y \xi\| \right\} \\ &\leq C\ell^{5\alpha/2} \int_0^1 ds \int_0^s d\tau\, \|(\mathcal{N}_+ + 1)^{1/2} e^{-\tau B(\eta_H)} \xi\| \|(\mathcal{V}_N + \mathcal{N}_+ + 1)^{1/2} \xi\| \\ &\leq C\ell^{5\alpha/2} \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \|(\mathcal{V}_N + \mathcal{N}_+ + 1)^{1/2} \xi\|.\end{aligned}$$

Combining (7.67), (7.68), (7.69) with the last bound, we find

$$\begin{aligned}\mathcal{G}^{(4)}_{N,\ell} = \mathcal{V}_N &+ \frac{1}{2N} \sum_{q \in \Lambda^*_+, r \in \Lambda^* : r \neq -q} \widehat{V}(r/N) \eta_H(q+r) \big(b_q b_{-q} + \text{h.c.} \big) \\ &+ \frac{1}{2N} \sum_{q \in \Lambda^*_+, r \in \Lambda^* : r \neq -q} \widehat{V}(r/N) \eta_H(q+r) \eta_H(q) \left(1 - \frac{\mathcal{N}_+}{N} \right) \left(1 - \frac{\mathcal{N}_+ + 1}{N} \right) + \mathcal{E}^{(4)}_{N,\ell}\end{aligned}$$

where $\mathcal{E}^{(4)}_{N,\ell}$ satisfies (7.55). As for the bound (7.56), it follows similarly, arguing as we did at the end of the proof of Proposition 7.2 to show (7.12). □

7.5. Proof of Propositions 4.2. We now combine the results of Sects. 7.1–7.4 to prove Proposition 4.2. From Propositions 7.1, 7.4, 7.5, 7.6, we conclude that the excitation Hamiltonian $\mathcal{G}_{N,\ell}$ defined in (4.18) is such that

$$
\begin{aligned}
\mathcal{G}_{N,\ell} = {} & \frac{\widehat{V}(0)}{2}\,(N+\mathcal{N}_+-1)\,\frac{N-\mathcal{N}_+}{N} \\
& + \sum_{p\in P_H} \eta_p\Big[p^2\eta_p + \widehat{V}(p/N) + \frac{1}{2N}\sum_{\substack{r\in\Lambda^*\\ p+r\in P_H}} \widehat{V}(r/N)\eta_{p+r}\Big]\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big) \\
& + \mathcal{K} + \sum_{p\in\Lambda_+^*} \widehat{V}(p/N) a_p^* a_p \,\frac{N-\mathcal{N}_+}{N} \\
& + \sum_{p\in P_H}\Big[\,p^2\eta_p + \frac{1}{2}\widehat{V}(p/N) + \frac{1}{2N}\sum_{r\in\Lambda^*:\ p+r\in P_H} \widehat{V}(r/N)\eta_{p+r}\,\Big]\big(b_p^* b_{-p}^* + b_p b_{-p}\big) \\
& + \frac{1}{2}\sum_{p\in P_H^c}\Big[\widehat{V}(p/N) + \frac{1}{2N}\sum_{r\in\Lambda^*:\ p+r\in P_H} \widehat{V}(r/N)\eta_{p+r}\Big]\big(b_p b_{-p} + b_{-p}^* b_p^*\big) \\
& + \frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*: p+q\neq 0} \widehat{V}(p/N)\Big[b_{p+q}^* a_{-p}^* a_q + \text{h.c.}\Big] + \mathcal{V}_N + \mathcal{E}_1
\end{aligned}
\tag{7.70}
$$

where

$$
\pm\mathcal{E}_1 \leq C\ell^{(\alpha-3)/2}\big(\mathcal{H}_N+1\big)
$$

and, with the notation $f_M = f(\mathcal{N}_+/M)$,

$$
\pm[f_M,[f_M,\mathcal{E}_1]] \leq C\ell^{(\alpha-3)/2}M^{-2}\|f'\|_\infty^2\big(\mathcal{H}_N+1\big)
$$

for every f bounded and smooth and $M\in\mathbb{N}$.

Our first goal is to show (4.24). With (4.10), we have

$$
\begin{aligned}
& \sum_{p\in P_H} \eta_p\Big[p^2\eta_p + \widehat{V}(p/N) + \frac{1}{2N}\sum_{r\in\Lambda^*:\ p+r\in P_H} \widehat{V}(r/N)\eta_{p+r}\Big] \\
& \quad = \sum_{p\in P_H} \eta_p\Big[\,\frac{1}{2}\widehat{V}(p/N) + \lambda_\ell N^3\widehat{\chi}_\ell(p) + \lambda_\ell N^2\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q\,\Big] \\
& \qquad - \frac{1}{2N}\sum_{\substack{p,q\in\Lambda^*:\\ p\in P_H,\,q\in P_H^c}} \widehat{V}((p-q)/N)\eta_p\eta_q .
\end{aligned}
$$

With Lemma 4.1 and estimating

$$
\begin{aligned}
&\|\widehat{\chi}_\ell\| = \|\chi_\ell\| \leq C\ell^{3/2}, \qquad \|\eta_H\| \leq \ell^{\alpha/2}, \\
&\|\widehat{\chi}_\ell * \eta_H\| = \|\chi_\ell\check{\eta}_H\| \leq \|\check{\eta}_H\| \leq \ell^{\alpha/2},
\end{aligned}
\tag{7.71}
$$

we conclude that

$$
\begin{aligned}
& \sum_{p\in P_H} \eta_p\Big[p^2\eta_p + \widehat{V}(p/N) + \frac{1}{2N}\sum_{\substack{r\in\Lambda^*\\ p+r\in P_H}} \widehat{V}(r/N)\eta_{p+r}\Big]\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big) \\
& \quad = \frac{1}{2}\sum_{p\in P_H} \widehat{V}(p/N)\eta_p\left(\frac{N-\mathcal{N}_+}{N}\right)\left(\frac{N-\mathcal{N}_+-1}{N}\right) + \mathcal{E}_2
\end{aligned}
$$

with $\pm\mathcal{E}_2 \leq C\ell^{-\alpha}$ (and with $[f_M,[f_M,\mathcal{E}_2]]=0$). Since $\sum_{p\in P_H^c}|V(p/N)||\eta_p| \leq C\ell^{-\alpha}$, and from (4.6), we further obtain

$$\begin{aligned}\sum_{p\in P_H}\eta_p\Big[p^2\eta_p+\widehat{V}(p/N)+\frac{1}{2N}\sum_{\substack{r\in\Lambda^*\\ p+r\in P_H}}\widehat{V}(r/N)\eta_{p+r}\Big]\Big(\frac{N-\mathcal{N}_+}{N}\Big)\Big(\frac{N-\mathcal{N}_+-1}{N}\Big)\\ =\left[4\pi\mathfrak{a}_0-\frac{\widehat{V}(0)}{2}\right](N-\mathcal{N}_+-1)\left(\frac{N-\mathcal{N}_+}{N}\right)+\mathcal{E}_3\end{aligned} \tag{7.72}$$

where $\pm\mathcal{E}_3 \leq C\ell^{-\alpha}$ (and $[f_M,[f_M,\mathcal{E}_3]]=0$). Using (4.10), we can also handle the fourth line of (7.70); we find

$$\begin{aligned}&\sum_{p\in P_H}\Big[p^2\eta_p+\frac{1}{2}\widehat{V}(p/N)+\frac{1}{2N}\sum_{r\in\Lambda^*:\ p+r\in P_H}\widehat{V}(r/N)\eta_{p+r}\Big]\big(b_p^*b_{-p}^*+b_pb_{-p}\big)\\ &\quad=\sum_{p\in P_H}\Big[N^3\lambda_\ell\widehat{\chi}_\ell(p)+N^2\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q\Big]\big(b_p^*b_{-p}^*+b_pb_{-p}\big)\\ &\qquad-\frac{1}{2N}\sum_{\substack{p,q\in\Lambda^*:\\ p\in P_H,\,q\in P_H^c}}\widehat{V}((p-q)/N)\eta_q\big(b_p^*b_{-p}^*+b_pb_{-p}\big).\end{aligned} \tag{7.73}$$

Observe that

$$\begin{aligned}\Big|\langle\xi,N^3\lambda_\ell\sum_{p\in P_H}\widehat{\chi}_\ell(p)b_pb_{-p}\xi\rangle\Big| &\leq C\ell^{-3}\|(\mathcal{N}_++1)^{1/2}\xi\|\sum_{p\in P_H}|p|^{-1}|\widehat{\chi}_\ell(p)||p|\|b_p\xi\|\\ &\leq C\ell^{-3+\alpha}\|\widehat{\chi}_\ell\|\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\\ &\leq C\ell^{\alpha-3/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.\end{aligned}$$

Using $\widehat{\chi}_\ell * \eta = \eta$ (because $\chi_\ell(x)w_\ell(x)=w_\ell(x)$ in position space), we also find

$$\begin{aligned}&\Big|\langle\xi,N^2\lambda_\ell\sum_{p\in P_H,q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\eta_q(b_p^*b_{-p}^*+b_pb_{-p})\xi\rangle\Big|\\ &\quad\leq CN^{-1}\ell^{-3+3\alpha/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\,.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\Big|\langle\xi,\frac{1}{2N}\sum_{\substack{p,q\in\Lambda^*:\\ p\in P_H,\,q\in P_H^c}}\widehat{V}((p-q)/N)\eta_qb_pb_{-p}\xi\rangle\Big|\\ &\quad\leq\frac{1}{2N}\Big[\sum_{\substack{p,q\in\Lambda^*:\\ p\in P_H,q\in P_H^c}}\frac{1}{|q|^2}\frac{|\widehat{V}((p-q)/N)|^2}{|p^2|}\Big]^{1/2}\\ &\qquad\times\Big[\sum_{\substack{p,q\in\Lambda^*:\\ p\in P_H,\,q\in P_H^c}}\frac{1}{|q|^2}|p|^2\|b_p\xi\|^2\Big]^{1/2}\|(\mathcal{N}_++1)^{1/2}\xi\|\\ &\quad\leq C\ell^{-\alpha}N^{-1/2}\|\mathcal{K}^{1/2}\xi\|\|(\mathcal{N}_++1)^{1/2}\xi\|.\end{aligned} \tag{7.74}$$

From (7.73), we conclude that

$$\pm \sum_{p\in P_H} \Big[p^2\eta_p + \frac{1}{2}\widehat{V}(p/N) + \frac{1}{2N} \sum_{\substack{r\in\Lambda^*:\\ p+r\in P_H}} \widehat{V}(r/N)\eta_{p+r} \Big] \big(b^*_p b^*_{-p} + b_p b_{-p} \big) \\ \leq C\ell^{\alpha-3/2}(\mathcal{K}+1) \tag{7.75}$$

for N large enough. As for the fifth line on the r.h.s. of (7.70), we can write it as

$$\frac{1}{2} \sum_{p\in P_H^c} \Big[\widehat{V}(p/N) + \frac{1}{N} \sum_{r\in\Lambda^*:\ p+r\in P_H} \widehat{V}(r/N)\eta_{p+r} \Big] \big(b_p b_{-p} + b^*_{-p} b^*_p \big) \\ = \frac{1}{2} \sum_{p\in P_H^c} (\widehat{V}(\cdot/N) * \widehat{f}_{N,\ell})_p \big(b_p b_{-p} + b^*_{-p} b^*_p \big) + \mathcal{E}_4 \tag{7.76}$$

where the error operator

$$\mathcal{E}_4 = \frac{1}{2N} \sum_{\substack{p,q\in\Lambda^*:\\ p,\,q\in P_H^c}} \widehat{V}((p-q)/N)\eta_q \big(b_p b_{-p} + b^*_{-p} b^*_p \big)$$

can be bounded by $\pm\mathcal{E}_4 \leq CN^{-1/2}\ell^{-\alpha}(\mathcal{K}+1)$, similarly as in (7.74).

Combining (7.70) with (7.72), (7.75) and (7.76), we conclude that

$$\begin{aligned} \mathcal{G}_{N,\ell} = {} & 4\pi\mathfrak{a}_0(N-1)\left(\frac{N-\mathcal{N}_+}{N}\right) + \big[\widehat{V}(0) - 4\pi\mathfrak{a}_0\big]\mathcal{N}_+\left(\frac{N-\mathcal{N}_+}{N}\right) \\ & + \mathcal{K} + \sum_{p\in\Lambda^*_+} \widehat{V}(p/N) a^*_p a_p \frac{N-\mathcal{N}_+}{N} + \frac{1}{2}\sum_{p\in P_H^c} (\widehat{V}(\cdot/N) * \widehat{f}_{N,\ell})_p \big(b_p b_{-p} + b^*_{-p} b^*_p \big) \\ & + \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda^*_+ : p+q\neq 0} \widehat{V}(p/N) \Big[b^*_{p+q} a^*_{-p} a_q + \text{h.c.} \Big] + \mathcal{V}_N + \mathcal{E}_5 \end{aligned}$$

with

$$\pm\mathcal{E}_5 \leq C\ell^{(\alpha-3)/2}\big(\mathcal{H}_N + 1\big) + C\ell^{-\alpha}.$$

Observing that

$$\pm \sum_{p\in P_H} \widehat{V}(p/N) a^*_p a_p \leq C\ell^{2\alpha}(\mathcal{K}+1)\,,$$

that $|\widehat{V}(p/N) - \widehat{V}(0)| \leq C|p|N^{-1}$, and that, by (4.6),

$$\begin{aligned} & |(\widehat{V}(\cdot/N) * \widehat{f}_{N,\ell})_p - 8\pi\mathfrak{a}_0| \\ & \quad \leq \int dx\, N^3 V(Nx) f_\ell(Nx) \big| e^{ip\cdot x} - 1 \big| + \left| \int N^3 V(Nx) f_\ell(Nx) - 8\pi\mathfrak{a}_0 \right| \\ & \quad \leq C(|p|+1)N^{-1} \end{aligned} \tag{7.77}$$

we arrive, with $\mathcal{G}^{\text{eff}}_{N,\ell}$ defined as in (4.23), at $\mathcal{G}_{N,\ell} = \mathcal{G}_{\text{eff}} + \mathcal{E}_{N,\ell}$, with an error $\mathcal{E}_{N,\ell}$ that satisfies

$$\pm\, \mathcal{E}_{N,\ell} \leq C\ell^{(\alpha-3)/2}\mathcal{H}_N + C\ell^{-\alpha} \tag{7.78}$$

for all N large enough. This completes the proof of (4.24). The second bound in (4.25) follows similarly, arguing as we did at the end of Proposition 7.2 (and noticing that the error term $\mathcal{E}_3$ in (7.72) which is responsible for the factor $\ell^{-\alpha}$ in (7.78) actually commutes with $f(\mathcal{N}_+/M)$).

Let us now prove (4.22) and the first bound in (4.25). We have to control the off-diagonal quadratic term and the cubic term appearing in $\mathcal{G}^{\text{eff}}_{N,\ell}$. We observe, first of all, that

$$\begin{aligned}\left|4\pi\mathfrak{a}_0 \sum_{p\in P_H^c} \langle \xi, (b_p b_{-p} + b^*_{-p} b^*_p)\xi\rangle\right| &\leq 4\pi\mathfrak{a}_0 \sum_{p\in P_H^c} \|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|b_p\xi\| \\ &\leq C\ell^{-\alpha/2}\|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.\end{aligned} \tag{7.79}$$

Using $[f_M, [f_M, b_p b_{-p}]] = (f(\mathcal{N}_+/M) - f((\mathcal{N}_+ + 2)/M))^2 b_p b_{-p}$, and a similar identity for $[f_M, [f_M, b^*_p b^*_{-p}]]$, we also obtain

$$\begin{aligned}&\left|4\pi\mathfrak{a}_0 \sum_{p\in P_H^c} \langle \xi, \left[f_M, \left[f_M, \left(b_p b_{-p} + b^*_p b^*_{-p}\right)\right]\right]\xi\rangle\right| \\ &\quad\leq CM^{-2}\ell^{-\alpha/2}\|f'\|^2_\infty \|(\mathcal{N}_+ + 1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.\end{aligned} \tag{7.80}$$

It is possible to show an improved lower bound for the operator on the l.h.s. of (7.79), by noticing that, for an arbitrary $\delta > 0$,

$$\begin{aligned}0 \leq &\sum_{p\in P_H^c} \left(\sqrt{\delta}|p| b^*_p + \frac{4\pi\mathfrak{a}_0}{\sqrt{\delta}|p|} b_{-p}\right)\left(\sqrt{\delta}|p| b_p + \frac{4\pi\mathfrak{a}_0}{\sqrt{\delta}|p|} b^*_{-p}\right) \\ &= \delta \sum_{p\in P_H^c} p^2 b^*_p b_p + \frac{(4\pi\mathfrak{a}_0)^2}{\delta} \sum_{p\in P_H^c} \frac{1}{p^2} b_{-p} b^*_{-p} + 4\pi\mathfrak{a}_0 \sum_{p\in P_H^c} (b_{-p} b_p + b^*_p b^*_{-p}).\end{aligned}$$

With (2.6), we commute

$$b_{-p} b^*_{-p} = b^*_{-p} b_{-p} + (1 - \mathcal{N}_+/N) - N^{-1} a^*_{-p} a_{-p}\,.$$

Observing that

$$b^*_p b_p = a^*_p \frac{N - \mathcal{N}_+}{N} a_p \leq a^*_p a_p$$

and that $\sum_{p\in P_H^c} |p|^{-2} \leq C\ell^{-\alpha}$, we conclude that there exists a constant $C > 0$, independent of $\ell \in (0; 1/2)$ and of N, such that

$$4\pi\mathfrak{a}_0 \sum_{p\in P_H^c} (b_{-p} b_p + b^*_p b^*_{-p}) \geq -\delta\mathcal{K} - C\delta^{-1}\mathcal{N}_+ - C\delta^{-1}\ell^{-\alpha} \tag{7.81}$$

for any $\delta > 0$. As for the cubic term on the r.h.s. of (4.23), we have, switching to position space,

$$\begin{aligned}\Big|\frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/N)\langle\xi,\big(b^*_{p+q}a^*_{-p}a_q+\text{h.c.}\big)\xi\rangle\Big| \\ \leq \int_{\Lambda^2} dxdy\, N^{5/2}V(N(x-y))\|\check{a}_x\xi\|\|\check{a}_x\check{a}_y\xi\| \leq C\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|\end{aligned} \tag{7.82}$$

and analogously

$$\begin{aligned}\Big|\frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/N)\langle\xi,\big[f_M,\big[f_M,\big(b^*_{p+q}a^*_{-p}a_q+\text{h.c.}\big)\big]\big]\xi\rangle\Big| \\ \leq CM^{-2}\|f'\|^2_\infty\|(\mathcal{N}_+ +1)^{1/2}\xi\|\|\mathcal{V}_N^{1/2}\xi\|.\end{aligned} \tag{7.83}$$

Combining (7.78) with (7.79) and (7.82), we obtain (4.21). From (7.78), (7.81) and (7.82), we infer (4.22). Combining instead the second bound in (4.25), with (7.80) and (7.83) we find the first bound in (4.25) (because all other contributions to $\mathcal{G}^{\text{eff}}_{N,\ell}$ commute with $\mathcal{N}_+$).

8. Analysis of the Excitation Hamiltonian $\mathcal{R}_{N,\ell}$

The goal of this section is to prove Proposition 5.2, which gives a lower bound on the excitation Hamiltonian $\mathcal{R}_{N,\ell} = e^{-A}\mathcal{G}^{\text{eff}}_{N,\ell}e^A$, with $\mathcal{G}^{\text{eff}}_{N,\ell}$ as in (4.23) and the cubic phase

$$A = \frac{1}{\sqrt{N}}\sum_{r\in P_H, v\in P_L}\eta_r\big[b^*_{r+v}a^*_{-r}a_v - \text{h.c.}\big] \tag{8.1}$$

introduced in (5.1), with the high momentum set $P_H = \{p\in\Lambda_+^* : |p|\geq \ell^{-\alpha}\}$ and the low momentum set $P_L = \{p\in\Lambda_+^* : |p|\leq \ell^{-\beta}\}$ for parameters $0<\beta<\alpha$ and $\ell\in(0;1/2)$ (in the proof of Proposition 5.2, we will assume $\alpha>3$ and $\alpha/2<\beta<2\alpha/3$). To study the properties of $\mathcal{R}_{N,\ell}$, it is convenient to decompose

$$\mathcal{G}^{\text{eff}}_{N,\ell} = \mathcal{D}_N + \mathcal{K} + \mathcal{Q}_{N,\ell} + \mathcal{C}_N + \mathcal{V}_N$$

with $\mathcal{K}$ and $\mathcal{V}_N$ being the kinetic and the potential energy operators, as in (4.19), and

$$\begin{aligned}\mathcal{D}_N &= 4\pi\mathfrak{a}_0(N-\mathcal{N}_+) + \big[\widehat{V}(0)-4\pi\mathfrak{a}_0\big]\mathcal{N}_+(1-\mathcal{N}_+/N),\\ \mathcal{Q}_{N,\ell} &= \widehat{V}(0)\sum_{p\in P_H^c}a^*_pa_p(1-\mathcal{N}_+/N) + 4\pi\mathfrak{a}_0\sum_{p\in P_H^c}\big[b^*_pb^*_{-p}+b_pb_{-p}\big],\\ \mathcal{C}_N &= \frac{1}{\sqrt{N}}\sum_{p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/N)\big[b^*_{p+q}a^*_{-p}a_q + h.c.\big]\end{aligned} \tag{8.2}$$

with $P_H^c = \Lambda_+^*\backslash P_H$. To study the contributions of these operators to $\mathcal{R}_{N,\ell}$ and to prove Proposition 5.2 we will need a-priori bounds controlling the growth of the expectation of the energy $\mathcal{H}_N = \mathcal{K}+\mathcal{V}_N$ through cubic conjugation; these estimates are obtained in the next subsection. As we did in Sect. 7, also in this Section we will always assume that $V\in L^3(\mathbb{R}^3)$ is compactly supported, pointwise non-negative and spherically symmetric.

8.1. A priori bounds on the energy. Our first proposition controls the commutator of the cubic phase (8.1) with the potential energy operator $\mathcal{V}_N$.

Proposition 8.1. *There exists a constant $C > 0$ such that*

$$[\mathcal{V}_N, A] = \frac{1}{N^{3/2}} \sum_{\substack{r\in\Lambda_+^*, v\in P_L \\ r\neq -v}} \big(\widehat{V}(\cdot/N) * \eta\big)(r)\big[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\big] + \delta_{\mathcal{V}_N} \tag{8.3}$$

where

$$|\langle\xi, \delta_{\mathcal{V}_N}\xi\rangle| \leq C\ell^{(\alpha-\beta)/2}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{K}_L^{1/2}\xi\| + C\ell^{3(\alpha-\beta)/2}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\| \tag{8.4}$$

for all $\alpha > \beta > 0$, $\ell \in (0; 1/2)$ and N large enough. Here $\mathcal{K}_L = \sum_{p\in P_L} p^2 a^*_p a_p$ denotes the kinetic energy associated to momenta $p \in P_L = \{p \in \Lambda_+^* : |p| \leq \ell^{-\beta}\}$.

Proof. With

$$\begin{aligned}
&[a^*_{p+u}a^*_q a_p a_{q+u}, b^*_{r+v}a^*_{-r}a_v] \\
&\quad = [a^*_{p+u}a^*_q a_p a_{q+u}, a^*_{r+v}]\sqrt{1-(\mathcal{N}_+/N)}a^*_{-r}a_v + b^*_{r+v}[a^*_{p+u}a^*_q a_p a_{q+u}, a^*_{-r}a_v] \\
&\quad = b^*_{p+u}a^*_q a_{q+u}a^*_{-r}a_v\delta_{p,r+v} + b^*_{p+u}a^*_q a_p a^*_{-r}a_v\delta_{q+u,r+v} \\
&\qquad + b^*_{r+v}a^*_{p+u}a^*_q a_p a_v\delta_{-r,q+u} + b^*_{r+v}a^*_{p+u}a^*_q a_{q+u}a_v\delta_{-r,p} \\
&\qquad - b^*_{r+v}a^*_{-r}a^*_{p+u}a_p a_{q+u}\delta_{q,v} - b^*_{r+v}a^*_{-r}a^*_q a_p a_{q+u}\delta_{v,p+u}
\end{aligned}$$

and normal ordering the first two terms, we obtain

$$[\mathcal{V}_N, A] = \frac{1}{N^{3/2}} \sum^*_{u\in\Lambda^*, r\in P_H, v\in P_L} \widehat{V}((u-r)/N)\eta_r b^*_{u+v}a^*_{-u}a_v + \Theta_2 + \Theta_3 + \Theta_4 + \text{h.c.}$$

with

$$\begin{aligned}
\Theta_2 &:= \frac{1}{N^{3/2}} \sum^*_{\substack{u\in\Lambda^*, p\in\Lambda_+^*, \\ r\in P_H, v\in P_L}} \widehat{V}(u/N)\eta_r b^*_{p+u}a^*_{r+v-u}a^*_{-r}a_p a_v, \\
\Theta_3 &:= \frac{1}{N^{3/2}} \sum^*_{\substack{u\in\Lambda^*, p\in\Lambda_+^*, \\ r\in P_H, v\in P_L}} \widehat{V}(u/N)\eta_r b^*_{r+v}a^*_{p+u}a^*_{-r-u}a_p a_v, \\
\Theta_4 &:= -\frac{1}{N^{3/2}} \sum^*_{\substack{u\in\Lambda^*, p\in\Lambda_+^*, \\ r\in P_H, v\in P_L}} \widehat{V}(u/N)\eta_r b^*_{r+v}a^*_{-r}a^*_{p+u}a_p a_{v+u}.
\end{aligned} \tag{8.5}$$

The notation $\sum^*$ indicates that we exclude choices of momenta for which the argument of a creation or annihilation operator vanishes. Writing

$$\begin{aligned}
&\frac{1}{N^{3/2}} \sum_{\substack{u\in\Lambda^*\\ r\in P_H, v\in P_L}}^{*} \widehat{V}((u-r)/N)\eta_r b^*_{u+v}a^*_{-u}a_v \\
&= \frac{1}{N^{3/2}} \sum_{\substack{u,r\in\Lambda^*,\\ v\in P_L}}^{*} \widehat{V}((u-r)/N)\eta_r b^*_{u+v}a^*_{-u}a_v \\
&\quad - \frac{1}{N^{3/2}} \sum_{\substack{u\in\Lambda^*, v\in P_L,\\ r\in P_H^c\cup\{0\}}}^{*} \widehat{V}((u-r)/N)\eta_r b^*_{u+v}a^*_{-u}a_v
\end{aligned}$$

and comparing with (8.3), we conclude that $\delta_{\mathcal{V}_N} = \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \text{h.c.}$, with

$$\Theta_1 = -\frac{1}{N^{3/2}} \sum_{\substack{u\in\Lambda^*, v\in P_L,\\ r\in P_H^c\cup\{0\}}}^{*} \widehat{V}((u-r)/N)\eta_r b^*_{u+v}a^*_{-u}a_v$$

and with $\Theta_2, \Theta_3, \Theta_4$ as defined in (8.5).

To conclude the proof of the lemma, we show next that each error term Θ_j, with $j = 1, 2, 3, 4$, satisfies (8.4). We start with Θ_1. For any $\xi \in \mathcal{F}_+^{\leq N}$, switching (partly) to position space and applying Cauchy–Schwarz, we find

$$\begin{aligned}
|\langle\xi, \Theta_1\xi\rangle| &\leq \frac{1}{\sqrt{N}}\bigg[\int_{\Lambda^2} dxdy\; N^2V(N(x-y)) \sum_{r\in\{0\}\cup P_H^c, v\in P_L} |\eta_r||v|^{-2}\|\check{b}_x\check{a}_y\xi\|^2\bigg]^{1/2} \\
&\quad\times\bigg[\int_{\Lambda^2} dxdy\; N^2V(N(x-y)) \sum_{r\in\{0\}\cup P_H^c, v\in P_L} |\eta_r||v|^2\|a_v\xi\|^2\bigg]^{1/2} \qquad (8.6)\\
&\leq \frac{C\ell^{-\alpha-\beta/2}}{N}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{K}_L^{1/2}\xi\|.
\end{aligned}$$

Denoting by $\check{\eta}_H \in L^2(\Lambda)$ the function with Fourier coefficients $\eta_H(p) = \eta_p\chi(p \in P_H)$ and using (4.14), we can bound the term Θ_2 on the r.h.s. of (8.5) by

$$\begin{aligned}
|\langle\xi, \Theta_2\xi\rangle| &= \bigg|\frac{1}{N^{1/2}}\int_{\Lambda^2} dxdy\; N^2V(N(x-y)) \sum_{v\in P_L} e^{ivy}\langle\xi, \check{b}^*_x\check{a}^*_y a^*(\check{\eta}_{H,y})\check{a}_x a_v\xi\rangle\bigg| \\
&\leq \frac{\|\check{\eta}_H\|}{N^{1/2}}\bigg[\int_{\Lambda^2} dxdy\; N^2V(N(x-y)) \sum_{v\in P_L} |v|^{-2}\|\mathcal{N}_+^{1/2}\check{b}_x\check{a}_y\xi\|^2\bigg]^{1/2} \\
&\quad\times\bigg[\int_{\Lambda^2} dxdy\; N^2V(N(x-y)) \sum_{v\in P_L} |v|^2\|\check{a}_x a_v\xi\|^2\bigg]^{1/2} \\
&\leq C\ell^{(\alpha-\beta)/2}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{K}_L^{1/2}\xi\|.
\end{aligned}$$

The remaining contributions Θ_3 and Θ_4 can be controlled similarly. We find

$$\begin{aligned}
|\langle\xi,\Theta_3\xi\rangle| &= \left|\frac{1}{\sqrt{N}}\int_{\Lambda^2} dxdy\; N^2V(N(x-y)) \sum_{r\in P_H, v\in P_L} e^{-iry}\eta_r \langle\xi, b^*_{r+v}\check{a}^*_x\check{a}^*_y\check{a}_x a_v\xi\rangle\right| \\
&\leq \frac{1}{\sqrt{N}}\left[\int_{\Lambda^2} dxdy\; N^2V(N(x-y)) \sum_{r\in P_H, v\in P_L} |v|^{-2}\|b_{r+v}\check{a}_x\check{a}_y\xi\|^2\right]^{1/2} \\
&\quad\times\left[\int_{\Lambda^2} dxdy\; N^2V(N(x-y)) \sum_{r\in P_H, v\in P_L} \eta_r^2|v|^2\|\check{a}_x a_v\xi\|^2\right]^{1/2} \\
&\leq \frac{C\ell^{-\beta/2}\|\eta_H\|}{N}\|\mathcal{N}_+^{1/2}\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\mathcal{K}_L^{1/2}\xi\| \leq C\ell^{(\alpha-\beta)/2}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{K}_L^{1/2}\xi\|
\end{aligned}$$

as well as

$$\begin{aligned}
|\langle\xi,\Theta_4\xi\rangle| &= \left|\frac{1}{\sqrt{N}}\int_{\Lambda^2} dxdy\; N^2V(N(x-y)) \sum_{r\in P_H, v\in P_L} \eta_r e^{-ivy}\langle\xi, b^*_{r+v}a^*_{-r}\check{a}^*_x\check{a}_x\check{a}_y\xi\rangle\right| \\
&\leq \frac{1}{\sqrt{N}}\left[\int_{\Lambda^2} dxdy\; N^2V(N(x-y)) \sum_{r\in P_H, v\in P_L} |r|^{-2}\eta_r^2\|\check{a}_x\check{a}_y\xi\|^2\right]^{1/2} \\
&\quad\times\left[\int_{\Lambda^2} dxdy\; N^2V(N(x-y)) \sum_{r\in P_H, v\in P_L} |r|^2\|b_{r+v}a_{-r}\check{a}_x\xi\|^2\right]^{1/2} \\
&\leq C\ell^{3(\alpha-\beta)/2}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.
\end{aligned}$$

Choosing $N > \ell^{-3\alpha/2}$ (to control the r.h.s. of (8.6)), we obtain (8.4). □

With the help of Proposition 8.1, we can now control the growth of the expectation of the energy $\mathcal{H}_N$ w.r.t. cubic conjugation.

Lemma 8.2. *There exists a constant $C > 0$ such that*

$$e^{-sA}\mathcal{H}_N e^{sA} \leq C\mathcal{H}_N + C\ell^{-\alpha}(\mathcal{N}_+ + 1) \tag{8.7}$$

for all $\alpha > \beta > 0$ with $\alpha > 4/3$, $s \in [0;1]$, $\ell \in (0;1/2)$ and $N \in \mathbb{N}$ large enough.

Proof. We apply Gronwall's lemma. For a fixed $\xi \in \mathcal{F}_+^{\leq N}$ and $s \in [0;1]$, we define

$$f_\xi(s) := \langle\xi, e^{-sA}\mathcal{H}_N e^{sA}\xi\rangle.$$

Then

$$f'_\xi(s) = \langle\xi, e^{-sA}[\mathcal{K}, A]e^{sA}\xi\rangle + \langle\xi, e^{-sA}[\mathcal{V}_N, A]e^{sA}\xi\rangle. \tag{8.8}$$

Let us first consider the second term. From Proposition 8.1, we find

$$[\mathcal{V}_N, A] = \frac{1}{N^{3/2}}\sum_{r\in\Lambda^*_+, v\in P_L, r\neq -v} \big(\widehat{V}(\cdot/N) * \eta\big)(r)\left[b^*_{r+v}a^*_{-r}a_v + \text{h.c.}\right] + \delta_{\mathcal{V}_N}$$

where the operator $\delta_{\mathcal{V}_N}$ satisfies (8.4). Switching to position space and applying Cauchy–Schwarz, we find

$$\begin{aligned}
&\left| \frac{1}{N^{3/2}} \sum_{r\in\Lambda_+^*, v\in P_L, r\neq -v} \big(\widehat{V}(\cdot/N) * \eta\big)(r) \langle \xi, e^{-sA} b^*_{r+v} a^*_{-r} a_v e^{sA}\xi\rangle \right| \\
&= \left| \int_{\Lambda^2} dxdy\ N^{3/2} V(N(x-y)) \check{\eta}(x-y) \sum_{v\in P_L} e^{ivx} \langle \xi, e^{-sA} \check{a}^*_x \check{a}^*_y a_v e^{sA}\xi\rangle \right| \\
&\leq \frac{C\|\check{\eta}\|_\infty}{N} \|\mathcal{V}_N^{1/2} e^{sA}\xi\| \left[\int_{\Lambda^2} dxdy\ N^3 V(N(x-y)) \Big\| \sum_{v\in P_L} e^{ivx} a_v e^{sA}\xi \Big\|^2 \right]^{1/2} \\
&\leq C \|\mathcal{V}_N^{1/2} e^{sA}\xi\| \|\mathcal{N}_+^{1/2} e^{sA}\xi\|
\end{aligned} \tag{8.9}$$

because, by (4.17), $\|\check{\eta}\|_\infty \leq CN$ and

$$\int_\Lambda dx \Big\| \sum_{v\in P_L} e^{ivx} a_v e^{sA}\xi \Big\|^2 = \sum_{v\in P_L} \langle e^{sA}\xi, a^*_v a_v e^{sA}\xi\rangle \leq \langle e^{sA}\xi, \mathcal{N}_+ e^{sA}\xi\rangle.$$

Together with (8.4), using $\alpha > \beta$, we conclude that

$$\Big| \langle \xi, e^{-sA} [\mathcal{V}_N, A] e^{sA}\xi\rangle \Big| \leq C \langle \xi, e^{-sA} \mathcal{H}_N e^{sA}\xi\rangle$$

if N is large enough. Let us consider the first term on the r.h.s. of (8.8). We compute

$$\begin{aligned}
[\mathcal{K}, A] &= \frac{1}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} 2r^2 \eta_r \big[b^*_{r+v} a^*_{-r} a_v + \text{h.c.} \big] \\
&\quad + \frac{2}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} r\cdot v\ \eta_r \big[b^*_{r+v} a^*_{-r} a_v + \text{h.c.} \big] \\
&=: \mathrm{T}_1 + \mathrm{T}_2.
\end{aligned} \tag{8.10}$$

We use (4.10) to rewrite the first term on the r.h.s. of (8.10) as

$$\begin{aligned}
\mathrm{T}_1 = &- \frac{1}{\sqrt{N}} \sum_{\substack{r\in\Lambda_+^*, v\in P_L, \\ r\neq -v}} (\widehat{V}(\cdot/N) * \widehat{f}_{N,\ell})(r) \big[b^*_{r+v} a^*_{-r} a_v + \text{h.c.} \big] \\
&+ \frac{1}{\sqrt{N}} \sum_{\substack{r\in P_H^c, v\in P_L, \\ r\neq -v}} (\widehat{V}(\cdot/N) * \widehat{f}_{N,\ell})(r) \big[b^*_{r+v} a^*_{-r} a_v + \text{h.c.} \big] \\
&+ \frac{1}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} N^3 \lambda_\ell (\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(r) \big[b^*_{r+v} a^*_{-r} a_v + \text{h.c.} \big] \\
=: &\ \mathrm{T}_{11} + \mathrm{T}_{12} + \mathrm{T}_{13}.
\end{aligned} \tag{8.11}$$

Since $\|f_\ell\|_\infty \leq 1$, the contribution of T_{11} can be estimated as in (8.9); we obtain

$$\big| \langle \xi, e^{-sA} \mathrm{T}_{11}\, e^{sA}\xi\rangle \big| \leq C \|\mathcal{V}_N^{1/2} e^{sA}\xi\| \|\mathcal{N}_+^{1/2} e^{sA}\xi\|. \tag{8.12}$$

The second term in (8.11) can be controlled by

$$\begin{aligned}\big|\langle\xi, e^{-sA}\mathrm{T}_{12}\, e^{sA}\xi\rangle\big| &\leq \frac{C}{\sqrt{N}}\bigg[\sum_{r\in P_H^c, v\in P_L, r\neq -v} |r|^2 \|b_{r+v}a_{-r}e^{sA}\xi\|^2\bigg]^{1/2}\\ &\quad\times\bigg[\sum_{r\in P_H^c, v\in P_L, r\neq -v} |r|^{-2}\|a_v e^{sA}\xi\|^2\bigg]^{1/2}\\ &\leq C\ell^{-\alpha/2}\|\mathcal{K}^{1/2}e^{sA}\xi\|\,\|\mathcal{N}_+^{1/2}e^{sA}\xi\|.\end{aligned}$$

Finally, since $(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(r) = \widehat{\chi}_\ell(r) + N^{-1}\eta_r$, the explicit expression

$$\widehat{\chi}_\ell(r) = \frac{4\pi}{|r|^2}\bigg(\frac{\sin(\ell|r|)}{|r|} - \ell\cos(\ell|r|)\bigg)$$

and the bound (4.8) imply that $|(\widehat{\chi}_\ell * \widehat{f}_{N,\ell})(r)| \leq C\ell|r|^{-2}$, for N large enough. With Lemma 4.1, the third term on the r.h.s. of (8.11) can thus be estimated for $\alpha > 4/3$ by

$$\begin{aligned}&\big|\langle\xi, e^{-sA}\mathrm{T}_{13}e^{sA}\xi\rangle\big|\\ &\leq \frac{C\ell^{-2}}{\sqrt{N}}\bigg[\sum_{r\in P_H}|r|^2\|\mathcal{N}_+^{1/2}a_{-r}e^{sA}\xi\|^2\bigg]^{1/2}\bigg[\sum_{r\in P_H, v\in P_L}|r|^{-6}\|a_v e^{sA}\xi\|^2\bigg]^{1/2}\\ &\leq C\ell^{3\alpha/2-2}\|\mathcal{K}^{1/2}e^{sA}\xi\|\,\|\mathcal{N}_+^{1/2}e^{sA}\xi\| \leq C\|\mathcal{K}^{1/2}e^{sA}\xi\|\,\|\mathcal{N}_+^{1/2}e^{sA}\xi\|.\end{aligned} \tag{8.13}$$

So far, we proved that

$$|\langle\xi, \mathrm{T}_1\xi\rangle| \leq C\ell^{-\alpha/2}\|\mathcal{H}_N^{1/2}e^{sA}\xi\|\,\|\mathcal{N}_+^{1/2}e^{sA}\xi\| \tag{8.14}$$

for all $\xi \in \mathcal{F}_+^{\leq N}$. Let us now consider the second term on the r.h.s. of (8.10). We find

$$\begin{aligned}&\big|\langle\xi, e^{-sA}\mathrm{T}_2 e^{sA}\xi\rangle\big|\\ &\leq \frac{C}{\sqrt{N}}\bigg[\sum_{r\in P_H}|r|^2\|\mathcal{N}_+^{1/2}a_{-r}e^{sA}\xi\|^2\bigg]^{1/2}\bigg[\sum_{r\in P_H, v\in P_L}|v|^2\eta_r^2\|a_v e^{sA}\xi\|^2\bigg]^{1/2}\\ &\leq C\ell^{\alpha/2}\|\mathcal{K}^{1/2}e^{sA}\xi\|\,\|\mathcal{K}_L^{1/2}e^{sA}\xi\|.\end{aligned} \tag{8.15}$$

Together with (8.14), we conclude that

$$|\langle\xi, e^{-sA}[\mathcal{K}, A]e^{sA}\xi\rangle| \leq C\langle\xi, e^{-sA}\mathcal{H}_N e^{sA}\xi\rangle + C\ell^{-\alpha}\langle\xi, e^{-sA}\mathcal{N}_+e^{-sA}\xi\rangle.$$

With Proposition 5.1, we obtain the differential inequality

$$|f_\xi'(s)| \leq Cf_\xi(s) + C\ell^{-\alpha}\langle\xi, (\mathcal{N}_+ + 1)\xi\rangle.$$

By Gronwall's Lemma, we find (8.7). □

The bound (8.7) is not yet ideal, because of the large constant proportional to $\ell^{-\alpha}$ multiplying the number of particles operator $\mathcal{N}_+$. To improve it, it is useful to consider first the growth of the low-momentum part of the kinetic energy operator. For $\theta > 0$, we set

$$\mathcal{K}_\theta = \sum_{p \in \Lambda_+^* : |p| \leq \theta} p^2 a_p^* a_p .$$

Comparing with the definition given in Proposition 8.1, we have $\mathcal{K}_L \equiv \mathcal{K}_{\theta=\ell^{-\beta}}$.

Lemma 8.3. *There exists a constant $C > 0$ such that*

$$e^{-sA} \mathcal{K}_\theta e^{sA} \leq C\mathcal{K}_\theta + C\ell^{2(\alpha-\beta)}(\mathcal{H}_N + 1) \tag{8.16}$$

for all $\alpha > \beta > 0$ with $\alpha > 4/3$, $\ell \in (0; 1/2)$, $0 < \theta < \ell^{-\alpha} - \ell^{-\beta}$, $s \in [0; 1]$ and $N \in \mathbb{N}$ large enough.

Proof. For a fixed $\xi \in \mathcal{F}_+^{\leq N}$, we consider the function $g_\xi : [0; 1] \to \mathbb{R}$, defined by $g_\xi(s) := \langle \xi, e^{-sA} \mathcal{K}_\theta e^{sA} \xi \rangle$. For $r \in P_H$ and $v \in P_L$, we observe that $|r + v| \geq |r| - |v| \geq \ell^{-\alpha} - \ell^{-\beta} > \theta$. Hence, we obtain

$$\begin{aligned}
[\mathcal{K}_\theta, A] &= \frac{1}{\sqrt{N}} \sum_{r \in P_H, v \in P_L} \eta_r b_{r+v}^* a_{-r}^* [\mathcal{K}_\theta, a_v] + \text{h.c.} \\
&= -\frac{1}{\sqrt{N}} \sum_{r \in P_H, v \in P_L : |v| \leq \theta} |v|^2 \eta_r \, b_{r+v}^* a_{-r}^* a_v + \text{h.c.}
\end{aligned}$$

We estimate

$$\begin{aligned}
&\left| \frac{1}{\sqrt{N}} \sum_{r \in P_H, v \in P_L : |v| \leq \theta} |v|^2 \eta_r \langle \xi, e^{-sA} b_{r+v}^* a_{-r}^* a_v e^{sA} \xi \rangle \right| \\
&\leq \frac{1}{\sqrt{N}} \sum_{r \in P_H, v \in P_L : |v| \leq \theta} \frac{|v|}{|r+v|} |r+v| \| b_{r+v} a_{-r} e^{sA} \xi \| \, |\eta_r| |v| \| a_v e^{sA} \xi \| \\
&\leq \frac{C\ell^{\alpha-\beta}}{\sqrt{N}} \left[\sum_{r \in P_H, v \in P_L : |v| \leq \theta} |r+v|^2 \| b_{r+v} a_{-r} e^{sA} \xi \|^2 \right]^{1/2} \\
&\quad \times \left[\sum_{r \in P_H, v \in P_L : |v| \leq \theta} |\eta_r|^2 |v|^2 \| a_v e^{sA} \xi \|^2 \right]^{1/2} \\
&\leq C\ell^{3\alpha/2-\beta} \| \mathcal{K}^{1/2} e^{sA} \xi \| \| \mathcal{K}_\theta^{1/2} e^{sA} \xi \| .
\end{aligned}$$

Hence, using $\mathcal{K} \leq \mathcal{H}_N$ and Lemma 8.2,

$$\begin{aligned}
&|(\partial_s g_\xi)(s)| \\
&\quad \leq C\ell^{3\alpha-2\beta} \langle \xi, e^{-sA} \mathcal{H}_N e^{sA} \xi \rangle + C g_\xi(s) \leq C\ell^{2(\alpha-\beta)} \langle \xi, (\mathcal{H}_N + 1) \xi \rangle + C g_\xi(s).
\end{aligned}$$

Gronwall's Lemma implies (8.16). □

With Lemma 8.3 we can now improve the estimate of Lemma 8.2 for the growth of the expectation of the potential energy $\mathcal{V}_N$.

Corollary 8.4. *There exists a constant $C > 0$ such that*

$$e^{-sA}\mathcal{V}_N e^{sA} \leq C(\mathcal{H}_N + 1) \tag{8.17}$$

for all $\alpha > 4/3$ and $0 < \beta < 2\alpha/3$, $\ell \in (0; 1/2)$ small enough, $s \in [0; 1]$ and $N \in \mathbb{N}$ large enough.

Proof. For $\xi \in \mathcal{F}_+^{\leq N}$, consider the function $h_\xi : [0; 1] \to \mathbb{R}$ defined through $h_\xi(s) := \langle \xi, e^{-sA}\mathcal{V}_N e^{sA}\xi\rangle$. By Proposition 8.1, we have

$$\begin{aligned} h'_\xi(s) = \frac{1}{N^{3/2}} \sum_{r\in\Lambda_+^*, v\in P_L, r\neq -v} \big(\widehat{V}(\cdot/N) * \eta\big)(r)\langle \xi, e^{-sA}\big([b^*_{r+v}a^*_{-r}a_v + \text{h.c.}]\big)e^{sA}\xi\rangle \\ + \langle \xi, e^{-sA}\delta_{\mathcal{V}_N} e^{sA}\xi\rangle \end{aligned}$$

where

$$\big|\langle \xi, e^{-sA}\delta_{\mathcal{V}_N} e^{sA}\xi\rangle\big| \leq C\|\mathcal{V}_N^{1/2}e^{sA}\xi\|\|\mathcal{K}_L^{1/2}e^{sA}\xi\| + C\ell^{3(\alpha-\beta)/2}\|\mathcal{V}_N^{1/2}e^{sA}\xi\|\|\mathcal{K}^{1/2}e^{sA}\xi\|.$$

The estimate (8.9), in the proof of Lemma 8.2, shows moreover that

$$\begin{aligned} \bigg|\frac{1}{N^{3/2}} \sum_{r\in\Lambda_+^*, v\in P_L, r\neq -v} \big(\widehat{V}(\cdot/N) * \eta\big)(r)\langle \xi, e^{-sA}b^*_{r+v}a^*_{-r}a_v e^{sA}\xi\rangle\bigg| \\ \leq C\|\mathcal{V}_N^{1/2}e^{sA}\xi\|\|\mathcal{N}_+^{1/2}e^{sA}\xi\|. \end{aligned}$$

With Proposition 5.1 and Lemmas 8.2, 8.3 (with $\theta = \ell^{-\beta}$), we deduce that

$$|h'_\xi(s)| \leq C\|\mathcal{V}_N^{1/2}e^{sA}\xi\|^2 + C(1 + \ell^{2\alpha-3\beta})\langle \xi, (\mathcal{H}_N + 1)\xi\rangle \leq Ch_\xi(s) + C\langle \xi, (\mathcal{H}_N + 1)\xi\rangle$$

because $\beta < 2\alpha/3$. Notice that, for $\ell \in (0; 1/2)$ small enough, we have $2\ell^{-\beta} < \ell^{-\alpha}$; thus, we may choose indeed $\theta = \ell^{-\beta}$ in Lemma 8.2. Applying Gronwall's Lemma to the last bound concludes (8.17). □

Finally, we consider the growth of the kinetic energy operator; in this case, we do not get a bound uniform in ℓ; still, we can improve the result of Lemma 8.2 and the estimate we obtain is sufficient for our purposes.

Corollary 8.5. *There exists a constant $C > 0$ such that*

$$e^{-sA}\mathcal{K}e^{sA} \leq C\ell^{-(\alpha+\beta)/2}(\mathcal{H}_N + 1) \tag{8.18}$$

for all $\alpha > 4/3$ and $0 < \beta < 2\alpha/3$, $s \in [0; 1]$, $\ell \in (0; 1/2)$ small enough and $N \in \mathbb{N}$ large enough.

Proof. For a fixed $\xi \in \mathcal{F}_+^{\leq N}$ define $j_\xi : [0; 1] \to \mathbb{R}$ by $j_\xi(s) := \langle \xi, e^{-sA}\mathcal{K}e^{sA}\xi\rangle$. From (8.10) and (8.11), we infer that

$$[\mathcal{K}, A] = \mathrm{T}_{11} + \mathrm{T}_{12} + \mathrm{T}_{13} + \mathrm{T}_2$$

with $\mathrm{T}_{11}, \mathrm{T}_{12}, \mathrm{T}_{13}, \mathrm{T}_2$ as in (8.10) and (8.11). Combining (8.12) with Proposition 5.1 and Corollary 8.4, we find

$$|\langle \xi, e^{-sA}\mathrm{T}_{11}e^{sA}\xi\rangle| \leq C\|\mathcal{V}_N^{1/2}e^{sA}\xi\|\|\mathcal{N}_+^{1/2}e^{sA}\xi\| \leq C\langle \xi, (\mathcal{H}_N + 1)\xi\rangle. \tag{8.19}$$

From (8.13), Proposition 5.1 and Lemma 8.2, we obtain

$$\begin{aligned} |\langle \xi, e^{-sA} \mathrm{T}_{13} e^{sA} \xi \rangle| &\leq C\ell^{3\alpha/2-2} \|\mathcal{K}^{1/2} e^{sA}\xi\| \|\mathcal{N}_+^{1/2} e^{sA}\xi\| \\ &\leq C\ell^{\alpha-2} \langle \xi, (\mathcal{H}_N+1)\xi\rangle \leq C\langle \xi, (\mathcal{H}_N+1)\xi\rangle. \end{aligned} \tag{8.20}$$

Using (8.15), Lemmas 8.2 and 8.3, we arrive at

$$|\langle \xi, e^{-sA} \mathrm{T}_2 e^{sA} \xi \rangle| \leq C\ell^{\alpha/2} \|\mathcal{K}^{1/2} e^{sA}\xi\| \|\mathcal{K}_L^{1/2} e^{sA}\xi\| \leq C\langle \xi, (\mathcal{H}_N+1)\xi\rangle. \tag{8.21}$$

Hence, to show (8.18), we only need to improve the bound on T_{12}. To this end, we set $\theta = \ell^{-\alpha} - 5\ell^{-\beta}/4$ and we decompose

$$\begin{aligned} \mathrm{T}_{12} &= \frac{1}{\sqrt{N}} \sum_{\substack{0<|r|\leq\theta \\ v\in P_L, r\neq -v}} (\widehat{V}(\cdot/N) * \widehat{f}_{N,\ell})(r) b^*_{r+v} a^*_{-r} a_v \\ &\quad + \frac{1}{\sqrt{N}} \sum_{\substack{\theta<|r|\leq\ell^{-\alpha}, \\ v\in P_L, r\neq -v}} (\widehat{V}(\cdot/N) * \widehat{f}_{N,\ell})(r) b^*_{r+v} a^*_{-r} a_v \\ &=: \mathrm{T}_{121} + \mathrm{T}_{122}. \end{aligned}$$

With Proposition 5.1 and Lemma 8.3, we estimate

$$\begin{aligned} \left|\langle \xi, e^{-sA} \mathrm{T}_{121} e^{sA} \xi \rangle\right| &\leq \frac{C}{\sqrt{N}} \sum_{\substack{0<|r|\leq\theta, \\ v\in P_L, r\neq -v}} |r| \|a_{-r} b_{r+v} e^{sA}\xi\| \, |r|^{-1} \|a_v e^{sA}\xi\| \\ &\leq C\ell^{-\alpha/2} \|\mathcal{K}_\theta^{1/2} e^{sA}\xi\| \|\mathcal{N}_+^{1/2} e^{sA}\xi\| \leq C\ell^{-\alpha/2} \langle \xi, (\mathcal{H}_N+1)\xi\rangle. \end{aligned}$$

On the other hand, since $\sum_{\theta<|r|<\ell^{-\alpha}} |r|^{-2} \leq C\ell^{-\beta}$, we find, by Proposition 5.1 and Lemma 8.2,

$$\begin{aligned} \left|\langle \xi, e^{-sA} \mathrm{T}_{121} e^{sA} \xi \rangle\right| &\leq \frac{C}{\sqrt{N}} \sum_{\substack{\theta<|r|\leq\ell^{-\alpha}, \\ v\in P_L, r\neq -v}} |r| \|a_{-r} b_{r+v} e^{sA}\xi\| \, |r|^{-1} \|a_v e^{sA}\xi\| \\ &\leq C\ell^{-\beta/2} \|\mathcal{K}^{1/2} e^{sA}\xi\| \|\mathcal{N}_+^{1/2} e^{sA}\xi\| \leq C\ell^{-(\alpha+\beta)/2} \langle \xi, (\mathcal{H}_N+1)\xi\rangle. \end{aligned}$$

Combining the last two bounds with (8.19), (8.20), (8.21), we obtain

$$|j'_\xi(s)| \leq C\ell^{-(\alpha+\beta)/2} \langle \xi, (\mathcal{H}_N+1)\xi\rangle$$

for all $s \in [0; 1]$. Integrating over s, we arrive at (8.18). □

8.2. Analysis of $e^{-A}\mathcal{D}_N e^A$. In this section we study the contribution to $\mathcal{R}_{N,\ell}$ arising from the operator $\mathcal{D}_N$, defined in (8.2). To this end, it is convenient to use the following lemma.

Lemma 8.6. *There exists a constant $C > 0$ such that*

$$\begin{aligned}\Big| \sum_{p\in\Lambda_+^*} F_p \langle \xi_1, (e^{-A} a_p^* a_p e^A - a_p^* a_p)\xi_2\rangle \Big| \\ \leq C\ell^{\alpha/2} \|F\|_\infty \|(\mathcal{N}_+ + 1)^{1/2}\xi_1\| \|(\mathcal{N}_+ + 1)^{1/2}\xi_2\|\end{aligned} \tag{8.22}$$

for all $\alpha, \beta > 0$, $\xi_1, \xi_2 \in \mathcal{F}_+^{\leq N}$, $F \in \ell^\infty(\Lambda_+^*)$, $\ell \in (0; 1/2)$ and $N \in \mathbb{N}$ large enough.

Proof. The lemma is a simple consequence of Proposition 5.1. We write

$$\sum_{p\in\Lambda_+^*} F_p (e^{-A} a_p^* a_p e^A - a_p^* a_p) = \int_0^1 ds \sum_{p\in\Lambda_+^*} F_p e^{-sA} [a_p^* a_p, A] e^{sA}$$

and compute

$$\sum_{p\in\Lambda_+^*} F_p [a_p^* a_p, A] = \frac{1}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} (F_{r+v} + F_{-r} - F_v)\eta_r b_{r+v}^* a_{-r}^* a_v + \text{h.c.}.$$

By Cauchy–Schwarz, we find with the help of Proposition 5.1 that

$$\begin{aligned}&\Big| \frac{1}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} (F_{r+v} + F_{-r} - F_v)\eta_r \langle e^{sA}\xi_1, b_{r+v}^* a_{-r}^* a_v e^{sA}\xi_2\rangle \Big| \\ &\quad \leq \frac{C\|F\|_\infty}{\sqrt{N}} \sum_{r\in P_H, v\in P_L} |\eta_r| \|a_v e^{sA}\xi_2\| \|a_{-r} b_{r+v} e^{sA}\xi_1\| \\ &\quad \leq C\ell^{\alpha/2} \|F\|_\infty \|(\mathcal{N}_+ + 1)^{1/2}\xi_1\| \|(\mathcal{N}_+ + 1)^{1/2}\xi_2\|.\end{aligned}$$

Since the bound is uniform in the integration variable $s \in [0; 1]$, we obtain (8.22). □

Proposition 8.7. *There exists a constant $C > 0$ such that*

$$e^{-A}\mathcal{D}_N e^A = 4\pi\mathfrak{a}_0(N - \mathcal{N}_+) + \big[\widehat{V}(0) - 4\pi\mathfrak{a}_0\big]\mathcal{N}_+(1 - \mathcal{N}_+/N) + \delta_{\mathcal{D}_N}$$

where

$$|\langle \xi, \delta_{\mathcal{D}_N}\xi\rangle| \leq C\ell^{\alpha/2}\langle \xi, (\mathcal{N}_+ + 1)\xi\rangle$$

for all $\alpha, \beta > 0$, $\xi \in \mathcal{F}_+^{\leq N}$, $\ell \in (0; 1/2)$ and $N \in \mathbb{N}$ large enough.

Proof. Recall from (8.2) that

$$\mathcal{D}_N = 4\pi\mathfrak{a}_0(N - \mathcal{N}_+) + \big[\widehat{V}(0) - 4\pi\mathfrak{a}_0\big]\mathcal{N}_+(1 - \mathcal{N}_+/N).$$

Lemma 8.6 implies that

$$\begin{aligned}\pm \Big\{ &e^{-A} \big[4\pi\mathfrak{a}_0(N - \mathcal{N}_+) + \big[\widehat{V}(0) - 4\pi\mathfrak{a}_0\big]\mathcal{N}_+\big] e^A \\ &- \big[4\pi\mathfrak{a}_0(N - \mathcal{N}_+) + \big[\widehat{V}(0) - 4\pi\mathfrak{a}_0\big]\mathcal{N}_+\big] \Big\} \leq C\ell^{\alpha/2}(\mathcal{N}_+ + 1).\end{aligned}$$

As for the contribution quadratic in $\mathcal{N}_+$, we can write

$$\begin{aligned} N^{-1} &\left\langle \xi, \left[e^{-A} \mathcal{N}_+^2 e^A - \mathcal{N}_+^2 \right] \xi \right\rangle \\ &= N^{-1} \left\langle \xi_1, \left[e^{-A} \mathcal{N}_+ e^A - \mathcal{N}_+ \right] \xi \right\rangle + N^{-1} \left\langle \xi, \left[e^{-A} \mathcal{N}_+ e^A - \mathcal{N}_+ \right] \xi_2 \right\rangle \end{aligned}$$

with $\xi_1 = e^{-A} \mathcal{N}_+ e^A \xi$ and $\xi_2 = \mathcal{N}_+ \xi$. Applying again Lemma 8.6, we obtain

$$\begin{aligned} \Big| N^{-1} &\left\langle \xi, \left[e^{-A} \mathcal{N}_+^2 e^A - \mathcal{N}_+^2 \right] \xi \right\rangle \Big| \\ &\leq C N^{-1} \ell^{\alpha/2} \| (\mathcal{N}_+ + 1)^{1/2} \xi \| \Big[\| (\mathcal{N}_+ + 1)^{1/2} \xi_1 \| + \| (\mathcal{N}_+ + 1)^{1/2} \xi_2 \| \Big] . \end{aligned}$$

Using (twice) Proposition 5.1, we find

$$\| (\mathcal{N}_+ + 1)^{1/2} \xi_1 \| = \| (\mathcal{N}_+ + 1)^{1/2} e^{-A} \mathcal{N}_+ e^A \xi \| \leq C \| (\mathcal{N}_+ + 1)^{3/2} \xi \| .$$

Hence,we conclude that

$$\begin{aligned} \Big| N^{-1} &\left\langle \xi, \left[e^{-A} \mathcal{N}_+^2 e^A - \mathcal{N}_+^2 \right] \xi \right\rangle \Big| \\ &\leq C N^{-1} \ell^{\alpha/2} \| (\mathcal{N}_+ + 1)^{1/2} \xi \| \| (\mathcal{N}_+ + 1)^{3/2} \xi \| \leq C \ell^{\alpha/2} \| (\mathcal{N}_+ + 1)^{1/2} \xi \|^2 . \end{aligned}$$

□

8.3. Contributions from $e^{-A} \mathcal{K} e^A$. In this subsection, we consider contributions to $\mathcal{R}_{N,\ell}$ arising from conjugation of the kinetic energy operator $\mathcal{K} = \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p$. In particular, in the next proposition, we establish properties of the commutator $[\mathcal{K}, A]$.

Proposition 8.8. *There exists a constant $C > 0$ such that*

$$\begin{aligned} [\mathcal{K}, A] = &- \frac{1}{\sqrt{N}} \sum_{p \in \Lambda_+^*, q \in P_L, p \neq -q} (\widehat{V}(\cdot/N) * \widehat{f}_{N,\ell})(p) (b_{p+q}^* a_{-p}^* a_q + \text{h.c.}) \\ &+ \frac{8\pi \mathfrak{a}_0}{\sqrt{N}} \sum_{p \in P_H^c, q \in P_L, p \neq -q} \left[b_{p+q}^* a_{-p}^* a_q + \text{h.c.} \right] + \delta_{\mathcal{K}} \end{aligned}$$

where

$$|\langle \xi, \delta_{\mathcal{K}} \xi \rangle| \leq C (\ell^{3\alpha/2 - 2} + \ell^{\alpha/2}) \| \mathcal{K}^{1/2} \xi \| \| (\mathcal{N}_+ + \mathcal{K}_L)^{1/2} \xi \| \tag{8.23}$$

for all $\alpha, \beta > 0$, $\xi \in \mathcal{F}_+^{\leq N}$, $\ell \in (0; 1/2)$, $N \in \mathbb{N}$ large enough. Moreover, we have

$$\begin{aligned} \Big| \frac{8\pi \mathfrak{a}_0}{\sqrt{N}} &\sum_{p \in P_H^c, q \in P_L, p \neq -q} \langle \xi, \left[b_{p+q}^* a_{-p}^* a_q, A \right] \xi \rangle \Big| \\ &\leq C \ell^{3(\alpha-\beta)/2} \| (\mathcal{N}_+ + 1)^{1/2} \xi \| \| \mathcal{K}^{1/2} \xi \| + C \ell^{(\alpha-\beta)/2} \| \mathcal{K}_L^{1/2} \xi \| \| \mathcal{N}_+^{1/2} \xi \| \\ &\quad + C \ell^\alpha \| \mathcal{K}^{1/2} \xi \|^2 \end{aligned} \tag{8.24}$$

for all $\alpha, \beta > 0$, $\xi \in \mathcal{F}_+^{\leq N}$, $\ell \in (0; 1/2)$ and $N \in \mathbb{N}$ large enough.

Proof. The bound (8.23) is a consequence of Eqs. (8.10), (8.11), (8.13), (8.15) in the proof of Lemma 8.2, and of the observation that, from the estimate (7.77),

$$\begin{aligned}&\left|\frac{1}{\sqrt{N}}\sum_{p\in P_H^c, q\in P_L, p\neq -q}\left[(\widehat{V}(\cdot/N)*\widehat{f}_{N,\ell})(p)-8\pi\mathfrak{a}_0\right]\langle\xi, b^*_{p+q}a^*_{-p}a_q\xi\rangle\right|\\ &\leq CN^{-3/2}\sum_{p\in P_H^c, q\in P_L, p\neq -q}|p|\|b_{p+q}a_{-p}\xi\|\|a_q\xi\|\leq CN^{-1}\ell^{-3\alpha/2}\|\mathcal{K}^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|\end{aligned}$$

which is bounded by the r.h.s. of (8.23) if N is large enough. Let us now focus on (8.24). We have

$$\begin{aligned}&\frac{8\pi\mathfrak{a}_0}{\sqrt{N}}\sum_{p\in P_H^c, q\in P_L, p\neq -q}\left[b^*_{p+q}a^*_{-p}a_q, A\right]+\text{h.c.}\\ &=\frac{8\pi\mathfrak{a}_0}{N}\sum_{\substack{r\in P_H, p\in P_H^c,\\ q,v\in P_L, p\neq -q, r\neq -v}}\eta_r\left[b^*_{p+q}a^*_{-p}a_q, b^*_{r+v}a^*_{-r}a_v-a^*_va_{-r}b_{r+v}\right]+\text{h.c.}.\end{aligned}\tag{8.25}$$

We split the commutator into the four summands

$$\begin{aligned}[b^*_{p+q}a^*_{-p}a_q, b^*_{r+v}a^*_{-r}a_v-a^*_va_{-r}b_{r+v}]=&\left([b^*_{p+q}, b^*_{r+v}a^*_{-r}a_v]+[a^*_va_{-r}b_{r+v}, b^*_{p+q}]\right)a^*_{-p}a_q\\ &+b^*_{p+q}\left([a^*_{-p}a_q, b^*_{r+v}a^*_{-r}a_v]+[a^*_va_{-r}b_{r+v}, a^*_{-p}a_q]\right).\end{aligned}\tag{8.26}$$

We compute

$$a^*_{-p}a_q=-b^*_{r+v}b^*_{-r}a^*_{-p}a_q\delta_{p+q,v}=-b^*_{r+v}b^*_{-r}a^*_{q-v}a_q\delta_{p+q,v}\tag{8.27}$$

as well as

$$\begin{aligned}&[a^*_va_{-r}b_{r+v}, b^*_{p+q}]a^*_{-p}a_q\\ &=(1-\mathcal{N}_+/N)a^*_va^*_{r+q}a_qa_{r+v}\delta_{p+q,-r}+(1-\mathcal{N}_+/N)a^*_va_v\delta_{p+q,-r}\delta_{r+v,-p}\\ &\quad+(1-\mathcal{N}_+/N)a^*_va^*_{q-r-v}a_{-r}a_q\delta_{p+q,r+v}+(1-\mathcal{N}_+/N)a^*_va_v\delta_{p+q,r+v}\delta_{r,p}\\ &\quad-N^{-1}a^*_va^*_{p+q}a^*_{-p}a_{-r}a_{r+v}a_q-N^{-1}a^*_va^*_{q-r-v}a_{-r}a_q\delta_{r+v,-p}-N^{-1}a^*_va^*_{q+r}a_{r+v}a_q\delta_{p,r}.\end{aligned}\tag{8.28}$$

Similarly, we find

$$\begin{aligned}b^*_{p+q}[a^*_{-p}a_q, b^*_{r+v}a^*_{-r}a_v]=&\ b^*_{p+r+v}b^*_{-p}a^*_{-r}a_v\delta_{q,r+v}+b^*_{p-r}b^*_{r+v}a^*_{-p}a_v\delta_{q,-r}\\ &-b^*_{q-v}b^*_{r+v}a^*_{-r}a_q\delta_{-p,v}\end{aligned}\tag{8.29}$$

and

$$\begin{aligned}b^*_{p+q}[a^*_va_{-r}b_{r+v}, a^*_{-p}a_q]=&\ b^*_{q+r}a^*_va_qb_{r+v}\delta_{r,p}-b^*_{p+v}a^*_{-p}a_{-r}b_{r+v}\delta_{q,v}\\ &+b^*_{q-r-v}a^*_va_{-r}b_q\delta_{r+v,-p}.\end{aligned}\tag{8.30}$$

Taking into account that $\delta_{r,p}=\delta_{q,-r}=\delta_{r+v,q}=0$ for $r\in P_H$, $p\in P_H^c$, $q, v\in P_L$ we obtain, inserting these formulas into (8.25),

$$\frac{8\pi\mathfrak{a}_0}{\sqrt{N}}\sum_{p\in P_H^c, q\in P_L, p\neq -q}\left[b^*_{p+q}a^*_{-p}a_q, A\right]+\text{h.c.}=\sum_{j=1}^7\Upsilon_j+\text{h.c.}$$

where

$$\begin{aligned}
\Upsilon_1 &:= -\frac{16\pi\mathfrak{a}_0}{N} \sum_{\substack{r\in P_H;q,v\in P_L,\\ q\neq v,r\neq -v}} \eta_r b^*_{r+v} b^*_{-r} a^*_{q-v} a_q,\\
\Upsilon_2 &:= \frac{8\pi\mathfrak{a}_0}{N} \sum_{\substack{r\in P_H;q,v\in P_L,\\ q+r P_H^c, r\neq -q, r\neq -v}} \eta_r (1-\mathcal{N}_+/N) a^*_v a^*_{r+q} a_q a_{r+v},\\
\Upsilon_3 &:= \frac{8\pi\mathfrak{a}_0}{N} \sum_{\substack{r\in P_H, v\in P_L,\\ r+v\in P_H^c}} \eta_r (1-\mathcal{N}_+/N) a^*_v a_v,\\
\Upsilon_4 &:= \frac{8\pi\mathfrak{a}_0}{N} \sum_{\substack{r\in P_H;q,v\in P_L,\\ q-r-v\in P_H^c}} \eta_r (1-\mathcal{N}_+/N) a^*_v a^*_{q-r-v} a_{-r} a_q,\\
\Upsilon_5 &:= -\frac{8\pi\mathfrak{a}_0}{N^2} \sum_{\substack{r\in P_H, p\in P_H^c,\\ q,v\in P_L, p\neq -q, r\neq -v}} \eta_r a^*_v a^*_{p+q} a^*_{-p} a_{-r} a_{r+v} a_q,\\
\Upsilon_6 &:= -\frac{8\pi\mathfrak{a}_0}{N^2} \sum_{\substack{r\in P_H;q,v\in P_L,\\ r+v\in P_H^c, q\neq r+v}} \eta_r a^*_v a^*_{q-r-v} a_{-r} a_q,\\
\Upsilon_7 &:= -\frac{8\pi\mathfrak{a}_0}{N} \sum_{\substack{r\in P_H, p\in P_H^c,\\ v\in P_L; p, r\neq -v}} \eta_r b^*_{p+v} a^*_{-p} a_{-r} b_{r+v},\\
\Upsilon_8 &:= \frac{8\pi\mathfrak{a}_0}{N} \sum_{\substack{r\in P_H;q,v\in P_L,\\ r+v\in P_H^c, q\neq r+v}} \eta_r b^*_{q-r-v} a^*_v a_{-r} b_q.
\end{aligned} \tag{8.31}$$

In fact, Υ_1 collects the contribution from (8.27) and the non-vanishing contribution from (8.29), $\Upsilon_2 - \Upsilon_6$ corresponds to the five non-vanishing terms on the r.h.s. of (8.28), Υ_7 and Υ_8 reflect the two non-vanishing terms on the r.h.s. of (8.30).

To conclude the proof of Proposition 8.8, we show that all operators in (8.31) satisfy (8.24). By Cauchy–Schwarz, we observe that

$$\begin{aligned}
\big|\langle\xi, \Upsilon_1\xi\rangle\big| &\leq \frac{C\ell^\alpha}{N} \sum_{\substack{r\in P_H;q,v\in P_L,\\ q\neq v,r\neq -v}} |\eta_r| \|a_q (\mathcal{N}_+ + 1)^{1/2}\xi\| |r| \|a_{-r} a_{q-v} a_{r+v} (\mathcal{N}_+ + 1)^{-1/2}\xi\|\\
&\leq C\ell^{3(\alpha-\beta)/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\|.
\end{aligned}$$

The expectation of Υ_2 is bounded by

$$\begin{aligned}
\big|\langle\xi, \Upsilon_2\xi\rangle\big| &\leq \frac{C}{N} \sum_{\substack{r\in P_H;q,v\in P_L,\\ q+r\in P_H^c, r\neq -q, r\neq -v}} |\eta_r| |q| \|a_q a_{r+v}\xi\| |q|^{-1} \|a_v a_{r+q}\xi\|\\
&\leq C\ell^{(\alpha-\beta)/2} \|\mathcal{K}_L^{1/2}\xi\| \|\mathcal{N}_+^{1/2}\xi\|
\end{aligned}$$

where we recall the notation $\mathcal{K}_L = \mathcal{K}_{\ell^{-\beta}} = \sum_{|p|\leq \ell^{-\beta}} p^2 a_p^* a_p$ for the low-momenta kinetic energy. It is simple to see that $\pm \Upsilon_3 \leq CN^{-1}\ell^{-\alpha}\mathcal{N}_+$ and the expectations of the terms Υ_4, Υ_6 and Υ_8 can all be estimated by the expectation

$$\begin{aligned}\big|\langle \xi, (\Upsilon_4 + \Upsilon_6 + \Upsilon_8)\xi\rangle\big| &\leq \frac{C}{N} \sum_{\substack{r\in P_H; q,v\in P_L,\\ |r|\leq (\ell^{-\alpha}+2\ell^{-\beta}), q-r-v\neq 0}} |\eta_r||v|\|a_v a_{q-r-v}\xi\||v|^{-1}\|a_{-r}a_q\xi\| \\ &\leq C\ell^{(\alpha-\beta)/2}\|\mathcal{K}_L^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|.\end{aligned}$$

Finally, the expectations of Υ_5 and Υ_7 can be bounded by

$$\begin{aligned}&\big|\langle \xi, \Upsilon_5\xi\rangle\big| \\ &\quad\leq \frac{C\ell^\alpha}{N^2} \sum_{\substack{r\in P_H, p\in P_H^c,\\ q,v\in P_L, p\neq -q, r\neq -v}} |\eta_r||p|\|a_{-p}a_v a_{p+q}\xi\||p|^{-1}|r|\|a_{-r}a_{r+v}a_q\xi\| \leq C\ell^\alpha\|\mathcal{K}^{1/2}\xi\|^2\end{aligned}$$

and by

$$\big|\langle \xi, \Upsilon_7\xi\rangle\big| \leq \frac{C\ell^\alpha}{N} \sum_{\substack{r\in P_H, p\in P_H^c,\\ v\in P_L; p, r\neq -v}} |\eta_r||p|\|a_{-p}a_{p+v}\xi\||p|^{-1}|r|\|a_{-r}a_{r+v}\xi\| \leq C\ell^\alpha\|\mathcal{K}^{1/2}\xi\|^2.$$

□

8.4. Analysis of $e^{-A}\mathcal{Q}_{N,\ell}e^A$. In this subsection, we consider contributions to $\mathcal{R}_{N,\ell}$ arising from conjugation of $\mathcal{Q}_{N,\ell}$, as defined in (8.2).

Proposition 8.9. *There exists a constant $C > 0$ such that*

$$e^{-A}\mathcal{Q}_{N,\ell}e^A = \widehat{V}(0)\sum_{p\in P_H^c} a_p^* a_p(1-\mathcal{N}_+/N) + 4\pi\mathfrak{a}_0 \sum_{p\in P_H^c}\big[b_p^* b_{-p}^* + b_p b_{-p}\big] + \delta_{\mathcal{Q}_{N,\ell}}$$

where

$$\pm\,\delta_{\mathcal{Q}_{N,\ell}} \leq C\ell^{(\alpha-\beta)/2}(\mathcal{H}_N + 1) \tag{8.32}$$

for all $\alpha > 4/3$, $0 < \beta < 2\alpha/3$, $\ell \in (0; 1/2)$ small enough and $N \in \mathbb{N}$ large enough.

Proof. Proceeding as in the proof of Proposition 8.7, it follows from Lemma 8.6 that

$$\begin{aligned}&\pm\left[\widehat{V}(0)\sum_{p\in P_H^c} e^{-A}a_p^* a_p(1-N/\mathcal{N}_+)e^A - \widehat{V}(0)\sum_{p\in P_H^c} a_p^* a_p(1-N/\mathcal{N}_+)\right] \\ &\quad\leq C\ell^{\alpha/2}(\mathcal{N}_+ + 1).\end{aligned} \tag{8.33}$$

Let us thus focus on the remaining part of $\mathcal{R}_{N,\ell}^{(2,V)}$. We expand

$$\begin{aligned}&4\pi\mathfrak{a}_0\sum_{p\in P_H^c}\Big(e^{-A}\big[b_p^* b_{-p}^* + b_p b_{-p}\big]e^A - \big[b_p^* b_{-p}^* + b_p b_{-p}\big]\Big) \\ &\quad= 4\pi\mathfrak{a}_0\int_0^1 ds\sum_{p\in P_H^c} e^{-sA}\big[b_p^* b_{-p}^*, A\big]e^{sA} + \text{h.c.}\end{aligned} \tag{8.34}$$

We compute

$$\left[b_p^* b_{-p}^*, b_{r+v}^* a_{-r}^* a_v - a_v^* a_{-r} b_{r+v}\right] = b_{r+v}^* \left[b_p^* b_{-p}^*, a_{-r}^* a_v\right] + \left[a_v^* a_{-r} b_{r+v}, b_p^* b_{-p}^*\right]$$

where

$$b_{r+v}^* \left[b_p^* b_{-p}^*, a_{-r}^* a_v\right] = -b_{r+v}^* b_{-v}^* b_{-r}^* (\delta_{-p,v} + \delta_{p,v})$$

and

$$\begin{aligned}\left[a_v^* a_{-r} b_{r+v}, b_p^* b_{-p}^*\right] &= b_v^* b_r^* b_{r+v} (\delta_{-r,p} + \delta_{r,p}) + (1 - \mathcal{N}_+/N) b_{-r-v}^* a_v^* a_{-r} (\delta_{r+v,p} + \delta_{r+v,-p}) \\ &\quad - 2N^{-1} b_v^* a_r^* a_{r+v} (\delta_{p,-r} + \delta_{r,p}) - 2N^{-1} b_p^* a_{-p}^* a_v^* a_{-r} a_{r+v}.\end{aligned}$$

Using the fact that $\delta_{p,-r} = \delta_{p,r} = 0$ for $r \in P_H$ and $p \in P_H^c$, we find that $\sum_{p \in P_H^c} \left[b_p^* b_{-p}^*, A\right] + \text{h.c.} = \sum_{i=1}^3 (\Phi_i + \text{h.c.})$, where

$$\begin{aligned}\Phi_1 &:= -\frac{2}{\sqrt{N}} \sum_{r \in P_H, v \in P_L} \eta_r b_{r+v}^* b_{-r}^* b_{-v}^*, \\ \Phi_2 &:= \frac{2}{\sqrt{N}} \sum_{r \in P_H, v \in P_L : r+v \in P_H^c} \eta_r (1 - \mathcal{N}_+/N) b_{-r-v}^* a_v^* a_{-r}, \\ \Phi_3 &:= -\frac{2}{N^{3/2}} \sum_{r \in P_H, v \in P_L, p \in P_H^c} \eta_r b_p^* a_{-p}^* a_v^* a_{-r} a_{r+v}.\end{aligned}$$

Let us now bound the expectation of the operators Φ_i, $i = 1, 2, 3,$. By Cauchy–Schwarz, we find that

$$\begin{aligned}|\langle \xi, \Phi_1 \xi \rangle| &\leq \left| \frac{2}{\sqrt{N}} \sum_{r \in P_H, v \in P_L} \eta_r \langle \xi, b_{r+v}^* b_{-r}^* b_{-v}^* \xi \rangle \right| \\ &\leq \frac{C}{\sqrt{N}} \sum_{r \in P_H, v \in P_L} |\eta_r| |v|^{-1} \| (\mathcal{N}_+ + 1)^{1/2} \xi \| \, |v| \| b_{-v} b_{r+v} b_{-r} (\mathcal{N}_+ + 1)^{-1/2} \xi \| \\ &\leq C \ell^{(\alpha-\beta)/2} \| (\mathcal{N}_+ + 1)^{1/2} \xi \| \| \mathcal{K}_L^{1/2} \xi \|\end{aligned}$$

as well as

$$\begin{aligned}|\langle \xi, \Phi_2 \xi \rangle| &\leq \left| \frac{2}{\sqrt{N}} \sum_{r \in P_H, v \in P_L : r+v \in P_H^c} \eta_r \langle \xi, (1 - \mathcal{N}_+/N) b_{-r-v}^* a_v^* a_{-r} \xi \rangle \right| \\ &\leq \frac{C}{\sqrt{N}} \sum_{r \in P_H, v \in P_L} |\eta_r| |v|^{-1} \| (\mathcal{N}_+ + 1)^{1/2} \xi \| |v| \| a_{-v} b_{r+v} \xi \| \\ &\leq C \ell^{(\alpha-\beta)/2} \| (\mathcal{N}_+ + 1)^{1/2} \xi \| \| \mathcal{K}_L^{1/2} \xi \|.\end{aligned}$$

To bound Φ_3 we notice that

$$\begin{aligned}\left| \langle \xi, \Phi_3 \xi \rangle \right| &\leq \frac{C \ell^\alpha}{N^{3/2}} \sum_{r \in P_H, v \in P_L, p \in P_H^c} |\eta_r| |p| \| a_p a_v (\mathcal{N}_+ + 1)^{1/2} \xi \| |p|^{-1} |r| \| a_{-r} a_{r+v} \xi \| \\ &\leq C \ell^\alpha \| \mathcal{K}^{1/2} \xi \|^2.\end{aligned}$$

With (8.34), we conclude that

$$\begin{aligned}\pm\Big[4\pi\mathfrak{a}_0\sum_{p\in P_H^c}\Big(e^{-A}\big[b_p^*b_{-p}^*+b_pb_{-p}\big]e^A-\big[b_p^*b_{-p}^*+b_pb_{-p}\big]\Big)\Big]\\ \leq C\int_0^1 ds\; e^{-sA}\big[\ell^{(\alpha-\beta)/2}(\mathcal{N}_++\mathcal{K}_L+1)+\ell^\alpha\mathcal{K}\big]e^{sA}.\end{aligned}$$

Finally, we apply Proposition 5.1, Lemma 8.3 and Corollary 8.5 to conclude that

$$\pm\Big[4\pi\mathfrak{a}_0\sum_{p\in P_H^c}\Big(e^{-A}\big[b_p^*b_{-p}^*+b_pb_{-p}\big]e^A-\big[b_p^*b_{-p}^*+b_pb_{-p}\big]\Big)\Big]\leq C\ell^{(\alpha-\beta)/2}(\mathcal{H}_N+1).$$

Together with the estimate (8.33), we arrive at (8.32). □

8.5. Contributions from $e^{-A}\mathcal{C}_Ne^A$. In this subsection, we consider contributions to $\mathcal{R}_{N,\ell}$ arising from conjugation of the cubic operator $\mathcal{C}_N$ defined in (8.2). In particular, in the next proposition, we establish properties of the commutator $[\mathcal{C}_N, A]$.

Proposition 8.10. *There exists a constant $C>0$ such that*

$$\big[\mathcal{C}_N,A\big]=\frac{2}{N}\sum_{r\in P_H,v\in P_L}\big[\widehat{V}(r/N)\eta_r+\widehat{V}((r+v)/N)\eta_r\big]a_v^*a_v\frac{(N-\mathcal{N}_+)}{N}+\delta_{\mathcal{C}_N}$$

where

$$\begin{aligned}|\langle\xi,\delta_{\mathcal{C}_N}\xi\rangle|\leq{}&C\ell^{3(\alpha-\beta)/2}\|(\mathcal{V}_N+\mathcal{N}_++1)^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|\\&+C\ell^{(\alpha-\beta)/2}\|(\mathcal{K}_L+\mathcal{V}_N+\mathcal{N}_+)^{1/2}\xi\|^2\end{aligned}\tag{8.35}$$

for all $\alpha,\beta>0$, $\ell\in(0;1/2)$ and $N\in\mathbb{N}$ large enough.

Proof. We have

$$\big[\mathcal{C}_N,A\big]=\frac{1}{N}\sum_{\substack{p,q\in\Lambda_+^*:p+q\neq0\\r\in P_H,v\in P_L}}\widehat{V}(p/N)\eta_r\big[b_{p+q}^*a_{-p}^*a_q,b_{r+v}^*a_{-r}^*a_v-a_v^*a_{-r}b_{r+v}\big]+\text{h.c.}$$

From (8.26), (8.27), (8.28), (8.29) and (8.30) we arrive at

$$\begin{aligned}\big[\mathcal{C}_N,A\big]={}&\frac{2}{N}\sum_{r\in P_H,v\in P_L}\big[\widehat{V}(r/N)\eta_r+\widehat{V}((r+v)/N)\eta_r\big]a_v^*a_v\frac{N-\mathcal{N}_+}{N}\\&+\sum_{j=1}^{12}(\Xi_j+\text{h.c.})\end{aligned}\tag{8.36}$$

where

$$\Xi_1 := -\frac{1}{N} \sum_{\substack{r\in P_H, v\in P_L,\\ p\in\Lambda_+^*: p\neq v}} \widehat{V}(p/N)\eta_r b^*_{r+v} b^*_{-r} a^*_{-p} a_{v-p},$$

$$\Xi_2 := \frac{1}{N} \sum_{\substack{r\in P_H, v\in P_L,\\ p\in\Lambda_+^*: r\neq -p}} \widehat{V}(p/N)\eta_r (1-\mathcal{N}_+/N) a^*_v a^*_{-p} a_{-r-p} a_{r+v},$$

$$\Xi_3 := \frac{1}{N} \sum_{\substack{r\in P_H, v\in P_L,\\ p\in\Lambda_+^*: r+v\neq p}} \widehat{V}(p/N)\eta_r (1-\mathcal{N}_+/N) a^*_v a^*_{-p} a_{-r} a_{r+v-p},$$

$$\Xi_4 := -\frac{1}{N^2} \sum_{\substack{r\in P_H, v\in P_L,\\ p,q\in\Lambda_+^*: p+q\neq 0}} \widehat{V}(p/N)\eta_r a^*_v a^*_{p+q} a^*_{-p} a_{-r} a_{r+v} a_q,$$

$$\Xi_5 := -\frac{1}{N^2} \sum_{\substack{r\in P_H, v\in P_L,\\ q\in\Lambda_+^*: r+v\neq q}} \widehat{V}((r+v)/N)\eta_r a^*_v a^*_{q-r-v} a_{-r} a_q,$$

$$\Xi_6 := -\frac{1}{N^2} \sum_{\substack{r\in P_H, v\in P_L,\\ q\in\Lambda_+^*: r\neq -q}} \widehat{V}(r/N)\eta_r a^*_v a^*_{q+r} a_{r+v} a_q,$$

$$\Xi_7 := \frac{1}{N} \sum_{\substack{r\in P_H, v\in P_L,\\ p\in\Lambda_+^*: r+v\neq -p}} \widehat{V}(p/N)\eta_r b^*_{p+r+v} b^*_{-p} a^*_{-r} a_v,$$

$$\Xi_8 := \frac{1}{N} \sum_{\substack{r\in P_H, v\in P_L,\\ p\in\Lambda_+^*: r\neq -p}} \widehat{V}(p/N)\eta_r b^*_{p-r} b^*_{r+v} a^*_{-p} a_v,$$

$$\Xi_9 := -\frac{1}{N} \sum_{\substack{r\in P_H, v\in P_L,\\ q\in\Lambda_+^*: q\neq v}} \widehat{V}(v/N)\eta_r b^*_{q-v} b^*_{r+v} a^*_{-r} a_q,$$

$$\Xi_{10} := \frac{1}{N} \sum_{\substack{r\in P_H, v\in P_L,\\ q\in\Lambda_+^*: r\neq -q}} \widehat{V}(r/N)\eta_r b^*_{q+r} a^*_v a_q b_{r+v},$$

as well as

$$\Xi_{11} := -\frac{1}{N} \sum_{\substack{r\in P_H, v\in P_L,\\ p\in\Lambda_+^*: p\neq -v}} \widehat{V}(p/N)\eta_r b^*_{p+v} a^*_{-p} a_{-r} b_{r+v},$$

$$\Xi_{12} := \frac{1}{N} \sum_{\substack{r\in P_H, v\in P_L,\\ q\in\Lambda_+^*: q\neq r+v}} \widehat{V}((r+v)/N)\eta_r b^*_{q-r-v} a^*_v a_{-r} b_q.$$

In fact, the first term on the r.h.s. of (8.36) arises from the second and fourth terms on the r.h.s. of (8.28), together with their Hermitean conjugates. The commutator (8.27) yields

Ξ_1, the remaining terms from (8.28) produce the contributions Ξ_2 to Ξ_6, from (8.29) we find the operators Ξ_7 to Ξ_9 and from (8.30) we obtain Ξ_{10}, Ξ_{11}, Ξ_{12}.

To conclude the proof of the proposition, we have to show that all terms Ξ_j, $j = 1, \dots, 12$, satisfy the bound (8.35). The expectation of Ξ_1 can be controlled with Cauchy–Schwarz by

$$\begin{aligned}\big|\langle \xi, \Xi_1 \xi\rangle\big| &\leq \frac{C\ell^\alpha}{N} \sum_{\substack{r\in P_H, v\in P_L,\\ p\in\Lambda_+^*: p\neq v}} |\eta_r| \|(\mathcal{N}_+ + 1)^{1/2} a_{v-p}\xi\| |r| \|a_{-r} a_{r+v} a_{-p} (\mathcal{N}_+ + 1)^{-1/2}\xi\| \\ &\leq C\ell^{3(\alpha-\beta)/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\|.\end{aligned}$$

The same bound applies (after relabeling) to Ξ_9; we find

$$\big|\langle \xi, \Xi_9 \xi\rangle\big| \leq C\ell^{3(\alpha-\beta)/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\|.$$

Also the expectations of the terms Ξ_2, Ξ_3 and (again after relabeling) of the terms Ξ_5, Ξ_6, Ξ_{10}, Ξ_{12} can be bounded similarly. We find

$$\begin{aligned}&|\langle \xi, \Xi_2 \xi\rangle| + |\langle \xi, \Xi_3 \xi\rangle| + |\langle \xi, \Xi_5 \xi\rangle| + |\langle \xi, \Xi_6 \xi\rangle| + |\langle \xi, \Xi_{10} \xi\rangle| + |\langle \xi, \Xi_{12} \xi\rangle| \\ &\quad\leq \frac{C\ell^\alpha}{N} \sum_{r\in P_H, v\in P_L, p\in\Lambda_+^*} \Big(|\eta_r| \|a_v a_{-p}\xi\| |r+v| \|a_{r+v} a_{-r-p}\xi\| + |\eta_r| \|a_{-p} a_v \xi\| |r| \|a_{-r} a_{r+v-p}\xi\| \\ &\qquad + |\eta_r| \|a_v a_{p-r-v}\xi\| |r| \|a_{-r} a_p \xi\| + |\eta_r| \|a_v a_{p+r}\xi\| |r+v| \|a_{r+v} a_p \xi\| \\ &\qquad + |\eta_r| \|a_{p+r} a_v \xi\| |r+v| \|a_{r+v} a_p \xi\| + |\eta_r| \|a_{p-r-v} a_v \xi\| |r| \|a_{-r} a_p \xi\| \Big) \\ &\quad\leq C\ell^{3(\alpha-\beta)/2} \|(\mathcal{N}_+ + 1)^{1/2}\xi\| \|\mathcal{K}^{1/2}\xi\|.\end{aligned}$$

To control the remaining terms, we switch to position space and use the potential energy operator $\mathcal{V}_N$. We start with Ξ_4. Applying Cauchy–Schwarz, we find

$$\begin{aligned}|\langle \xi, \Xi_4 \xi\rangle| &= \left| \frac{1}{N} \int_{\Lambda^2} dxdy\, N^2 V(N(x-y)) \sum_{r\in P_H, v\in P_L} \eta_r \langle \xi, \check{a}_x^* \check{a}_y^* a_v^* a_{-r} a_{r+v} \check{a}_x \xi\rangle \right| \\ &\leq \frac{1}{N} \int_{\Lambda^2} dxdy\, N^2 V(N(x-y)) \sum_{r\in P_H, v\in P_L} |\eta_r| \|a_v \check{a}_x \check{a}_y \xi\| \|a_{-r} a_{r+v} \check{a}_x \xi\| \\ &\leq C\ell^{\alpha/2} \|\mathcal{V}_N^{1/2}\xi\| \|\mathcal{N}_+^{1/2}\xi\|.\end{aligned}$$

Next, we rewrite Ξ_7, Ξ_8 and Ξ_{11} as

$$\begin{aligned}\Xi_7 &= \int_{\Lambda^2} dxdy\, N^2 V(N(x-y)) \sum_{r\in P_H, v\in P_L} e^{i(r+v)x} \eta_r \check{b}_x^* \check{b}_y^* a_{-r}^* a_v, \\ \Xi_8 &= \int_{\Lambda^2} dxdy\, N^2 V(N(x-y)) \sum_{r\in P_H, v\in P_L} e^{-irx} \eta_r \check{b}_x^* \check{b}_y^* a_{r+v}^* a_v, \\ \Xi_{11} &= -\int_{\Lambda^2} dxdy\, N^2 V(N(x-y)) \sum_{r\in P_H, v\in P_L} e^{ivx} \eta_r \check{b}_x^* \check{b}_y^* a_{-r} b_{r+v}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}|\langle\xi, \Xi_7\xi\rangle| &\leq \int_{\Lambda^2} dxdy\ N^2V(N(x-y))\sum_{r\in P_H}\|\check{a}_x\check{a}_y a_{-r}\xi\|\,|\eta_r|\Big\|\sum_{v\in P_L} e^{ivx}a_v\xi\Big\| \\ &\leq C\ell^{\alpha/2}\|\mathcal{V}_N^{1/2}\xi\|\bigg[\int_\Lambda dx\sum_{v,v'\in P_L} e^{i(v-v')x}\langle\xi, a_{v'}^*a_v\xi\rangle\bigg]^{1/2} \\ &\leq C\ell^{\alpha/2}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}_+^{1/2}\xi\|\end{aligned}$$

as well as

$$\begin{aligned}&|\langle\xi, \Xi_8\xi\rangle| + |\langle\xi, \Xi_{11}\xi\rangle| \\ &\leq C\int_{\Lambda^2} dxdy\ N^2V(N(x-y)) \\ &\quad\times\sum_{r\in P_H, v\in P_L}\Big(|v|^{-1}\|\check{a}_x\check{a}_y a_{r+v}\xi\|\,|\eta_r|\,|v|\|a_v\xi\| + C\ell^\alpha|\eta_r|\|\check{a}_x\check{a}_y\xi\|\,|r|\|a_{-r}b_{r+v}\xi\|\Big) \\ &\leq C\ell^{(\alpha-\beta)/2}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{K}_L^{1/2}\xi\| + C\ell^{3(\alpha-\beta)/2}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{K}^{1/2}\xi\|.\end{aligned}$$

Collecting all the bounds above, we arrive at (8.35). □

8.6. Proof of Proposition 5.2. Let us now combine the results of Sects. 8.1–8.5 to prove Proposition 5.2. Here, we assume $\alpha > 3$ and $\alpha/2 < \beta < 2\alpha/3$.

From Propositions 8.7 and 8.9 we obtain that

$$\begin{aligned}\mathcal{R}_{N,\ell} &\geq 4\pi\mathfrak{a}_0(N-\mathcal{N}_+) + \big[\widehat{V}(0) - 4\pi\mathfrak{a}_0\big]\mathcal{N}_+(1-\mathcal{N}_+/N) \\ &\quad + \widehat{V}(0)\sum_{p\in P_H^c} a_p^*a_p(1-\mathcal{N}_+/N) + 4\pi\mathfrak{a}_0\sum_{p\in P_H^c}\big[b_p^*b_{-p}^* + b_pb_{-p}\big] \\ &\quad + \mathcal{K} + \mathcal{C}_N + \mathcal{V}_N + \int_0^1 ds\ e^{-sA}\big[\mathcal{K}+\mathcal{C}_N+\mathcal{V}_N, A\big]e^{sA} \\ &\quad - C\ell^{(\alpha-\beta)/2}(\mathcal{H}_N+1)\end{aligned}$$

with $\mathcal{C}_N$ defined as in (8.2). From Propositions 8.1, 8.8 and 8.10, we can write, for N large enough,

$$\begin{aligned}&\big[\mathcal{K}+\mathcal{C}_N+\mathcal{V}_N, A\big] \\ &\geq -\frac{1}{\sqrt{N}}\sum_{\substack{p\in\Lambda_+^*, q\in P_L,\\ p\neq -q}}\widehat{V}(p/N)\big[b_{p+q}^*a_{-p}^*a_q + \text{h.c.}\big] + \frac{8\pi\mathfrak{a}_0}{\sqrt{N}}\sum_{\substack{p\in P_H^c, q\in P_L,\\ p\neq -q}}\big[b_{p+q}^*a_{-p}^*a_q + \text{h.c.}\big] \\ &\quad + \frac{2}{N}\sum_{r\in P_H, v\in P_L}\big[\widehat{V}(r/N)\eta_r + \widehat{V}((r+v)/N)\eta_r\big]a_v^*a_v(1-\mathcal{N}_+/N) \\ &\quad - C(\ell^{\alpha-2}+\ell^{(\alpha-\beta)/4})(\mathcal{N}_+ + \mathcal{V}_N + \mathcal{K}_L) - C(\ell^{5(\alpha-\beta)/2} + \ell^{(3\alpha+\beta)/4} + \ell^{2\alpha-2})\mathcal{K}.\end{aligned}$$

From Proposition 5.1, Lemma 8.3, Corollaries 8.4 and 8.5 and recalling the Definition (8.2) of the operator $\mathcal{C}_N$, we deduce that

$$\begin{aligned}
&\int_0^1 ds\, e^{-sA}[\mathcal{K}+\mathcal{C}_N+\mathcal{V}_N, A]e^{sA} \\
&\geq \int_0^1 ds\, e^{-sA}\Big[-\mathcal{C}_N+\frac{8\pi\mathfrak{a}_0}{\sqrt{N}}\sum_{\substack{p\in P_H^c, q\in P_L,\\ p\neq -q}}\big[b^*_{p+q}a^*_{-p}a_q+\text{h.c.}\big] \\
&\quad+\frac{2}{N}\sum_{r\in P_H, v\in P_L}\big[\widehat{V}(r/N)\eta_r+\widehat{V}((r+v)/N)\eta_r\big]a^*_v a_v\frac{(N-\mathcal{N}_+)}{N}\Big]e^{sA} \qquad (8.37)\\
&\quad+\frac{1}{\sqrt{N}}\int_0^1 ds\sum_{\substack{p\in\Lambda^*_+, q\in P_L^c,\\ p\neq -q}}\widehat{V}(p/N)e^{-sA}\big[b^*_{p+q}a^*_{-p}a_q+\text{h.c.}\big]e^{sA} \\
&\quad-C(\ell^{(\alpha-\beta)/4}+\ell^{\alpha-2}+\ell^{2\alpha-3\beta})(\mathcal{H}_N+1).
\end{aligned}$$

The expectation of the operator on the fourth line can be estimated after switching to position space with Corollaries 8.4 and 8.5. We find

$$\begin{aligned}
&\Big|\frac{1}{\sqrt{N}}\int_0^1 ds\sum_{\substack{p\in\Lambda^*_+, q\in P_L^c,\\ p\neq -q}}\widehat{V}(p/N)\langle\xi, e^{-sA}b^*_{p+q}a^*_{-p}a_q e^{sA}\xi\rangle\Big| \\
&\quad\leq\int_0^1 ds\int_{\Lambda^2}dxdy\, N^{5/2}V(N(x-y))\|\check{a}_x\check{a}_y e^{sA}\xi\|\Big\|\sum_{q\in P_L^c}e^{iqx}a_q e^{sA}\xi\Big\| \\
&\quad\leq C\int_0^1 ds\,\|\mathcal{V}_N^{1/2}e^{sA}\xi\|\bigg[\int_\Lambda dx\sum_{q,q'\in P_L^c}e^{i(q-q')x}\langle e^{sA}\xi, a^*_{q'}a_q e^{sA}\xi\rangle\bigg]^{1/2} \\
&\quad\leq C\ell^\beta\int_0^1 ds\,\|\mathcal{V}_N^{1/2}e^{sA}\xi\|\|\mathcal{K}^{1/2}e^{sA}\xi\|\leq C\ell^{(3\beta-\alpha)/4}\|(\mathcal{H}_N+1)^{1/2}\xi\|^2.
\end{aligned} \qquad (8.38)$$

Next, we consider the term on the third line of (8.37). With Lemma 4.1, part (ii), and since $\alpha>1$, we have

$$\Big|\frac{1}{N}\sum_{r\in P_H}\big[\widehat{V}(r/N)\eta_r+\widehat{V}((r+v)/N)\eta_r\big]-\big[16\pi\mathfrak{a}_0-2\widehat{V}(0)\big]\Big|\leq\frac{C\ell^{-\alpha}|v|}{N}$$

for every $v\in P_L$. With Lemmas 8.3, 8.6 and Proposition 5.1 we obtain, for $N\geq\ell^{-3\alpha}$,

$$\begin{aligned}
&\pm\bigg[\frac{1}{N}\sum_{r\in P_H, v\in P_L}\big[\widehat{V}(r/N)\eta_r+\widehat{V}((r+v)/N)\eta_r\big]e^{-sA}a^*_v a_v\frac{(N-\mathcal{N}_+)}{N}e^{sA} \\
&\quad-\big[16\pi\mathfrak{a}_0-2\widehat{V}(0)\big]\sum_{v\in P_L}a^*_v a_v\frac{(N-\mathcal{N}_+)}{N}\bigg] \qquad (8.39)\\
&\quad\leq C(N^{-1}\ell^{-\beta}+\ell^{\alpha/2})(\mathcal{H}_N+1)\leq C\ell^{\alpha/2}(\mathcal{H}_N+1).
\end{aligned}$$

To handle the second term on the second line of (8.37), we apply Propositions 8.8 and 5.1, Lemma 8.3 and Corollary 8.5 to conclude, again for $N \geq \ell^{-3\alpha}$,

$$\begin{aligned}
&\pm \left(\frac{8\pi \mathfrak{a}_0}{\sqrt{N}} \int_0^1 ds \sum_{\substack{p\in P_H^c, q\in P_L, \\ p\neq -q}} \Big[e^{-sA} b^*_{p+q} a^*_{-p} a_q e^{sA} - b^*_{p+q} a^*_{-p} a_q \Big] + \text{h.c.} \right) \\
&= \pm \left(\frac{8\pi \mathfrak{a}_0}{\sqrt{N}} \int_0^1 ds \int_0^s dt \sum_{\substack{p\in P_H^c, q\in P_L, \\ p\neq -q}} e^{-tA} \Big[b^*_{p+q} a^*_{-p} a_q, A \Big] e^{tA} \right) \\
&\leq C \big(\ell^{(2\alpha-3\beta)} + \ell^{(\alpha-\beta)/2} \big) (\mathcal{H}_N + 1).
\end{aligned}$$

As for the first term on the second line of (8.37), we use again Proposition 8.10. Proceeding then as in (8.39), we have

$$\begin{aligned}
\int_0^1 ds\, e^{-sA} \mathcal{C}_N e^{sA} &= \mathcal{C}_N + \int_0^1 ds \int_0^s dt\, e^{-tA} [\mathcal{C}_N, A] e^{tA} \\
&\leq \mathcal{C}_N + \big[16\pi \mathfrak{a}_0 - 2\widehat{V}(0) \big] \sum_{p\in P_L} a^*_p a_p \frac{(N-\mathcal{N}_+)}{N} \\
&\quad + C \big(\ell^{(\alpha-\beta)/2} + \ell^{2\alpha-3\beta} \big) (\mathcal{H}_N + 1).
\end{aligned} \tag{8.40}$$

Inserting the bounds (8.38)–(8.40) into (8.37) and using additionally the simple bounds

$$0 \leq \sum_{p\in P_L^c \cap P_H} a^*_p a_p \leq \sum_{p\in P_L^c} a^*_p a_p \leq \ell^{2\beta} \mathcal{K}$$

and

$$\begin{aligned}
\Big| \frac{8\pi \mathfrak{a}_0}{\sqrt{N}} \sum_{\substack{p\in P_H^c, q\in P_L^c, \\ p\neq -q}} \langle \xi, b^*_{p+q} a^*_{-p} a_q \xi \rangle \Big| &\leq \frac{C\ell^\beta}{\sqrt{N}} \sum_{\substack{p\in P_H^c, q\in P_L^c, \\ p\neq -q}} |p| \|a_{-p} a_{p+q} \xi\| |p|^{-1} |q| \|a_q \xi\| \\
&\leq \frac{C\ell^{\beta-\alpha/2}}{\sqrt{N}} \|\mathcal{K}^{1/2} \mathcal{N}_+^{1/2} \xi\| \Big[\sum_{q\in P_L^c} |q|^2 \|a_q \xi\|^2 \Big]^{1/2} \\
&\leq C\ell^{\beta-\alpha/2} \|\mathcal{K}^{1/2} \xi\|^2
\end{aligned}$$

we arrive at

$$\begin{aligned}
\mathcal{R}_{N,\ell} &\geq 4\pi \mathfrak{a}_0 (N - \mathcal{N}_+) + 4\pi \mathfrak{a}_0 \mathcal{N}_+ \frac{(N-\mathcal{N}_+)}{N} \\
&\quad + 8\pi \mathfrak{a}_0 \sum_{p\in P_H^c} a^*_p a_p \frac{(N-\mathcal{N}_+)}{N} + 4\pi \mathfrak{a}_0 \sum_{p\in P_H^c} \big[b^*_p b^*_{-p} + b_p b_{-p} \big] \\
&\quad + \frac{8\pi \mathfrak{a}_0}{\sqrt{N}} \sum_{p\in P_H^c, q\in \Lambda_+^*: p\neq -q} \big[b^*_{p+q} a^*_{-p} a_q + \text{h.c.} \big] + \big(1 - C\ell^\kappa \big) (\mathcal{H}_N + 1)
\end{aligned} \tag{8.41}$$

with $\kappa = \min[(\alpha-\beta)/4; \alpha - 3; \beta - \alpha/2; 2\alpha - 3\beta] > 0$ under the assumptions $\alpha > 3$ and $\alpha/2 < \beta < 2\alpha/3$.

We define now the function $v_\ell \in L^\infty(\Lambda)$ by setting

$$v_\ell(x) := 8\pi\mathfrak{a}_0 \sum_{p\in\{0\}\cup P_H^c} e^{ip\cdot x} = 8\pi\mathfrak{a}_0 \sum_{p\in\Lambda^*:|p|\leq \ell^{-\alpha}} e^{ip\cdot x}.$$

In other words, v_ℓ is defined so that $\widehat{v}_\ell(p) = 8\pi\mathfrak{a}_0$ for all $p \in \Lambda^*$ with $|p| \leq \ell^{-\alpha}$ and $\widehat{v}_\ell(p) = 0$ otherwise. Observe, in particular, that $\widehat{v}_\ell(p) \geq 0$ for all $p \in \Lambda^*$. Proceeding as in (2.4), but now with $\widehat{V}(p/N)$ replaced by $\widehat{v}_\ell(p)$, we find that

$$\begin{aligned}
&U_N \left[\frac{1}{N}\sum_{i<j}^N v_\ell(x_i - x_j)\right] U_N^* \\
&= \frac{(N-1)}{N} 4\pi\mathfrak{a}_0 (N-\mathcal{N}_+) + 4\pi\mathfrak{a}_0 \mathcal{N}_+ \frac{(N-\mathcal{N}_+)}{N} \\
&\quad + 8\pi\mathfrak{a}_0 \sum_{p\in P_H^c} a_p^* a_p \frac{(N-\mathcal{N}_+)}{N} + 4\pi\mathfrak{a}_0 \sum_{p\in P_H^c} (b_p^* b_{-p}^* + b_p b_{-p}) \\
&\quad + \frac{8\pi\mathfrak{a}_0}{\sqrt{N}} \sum_{p\in P_H^c, q\in\Lambda_+^*, p\neq -q} [b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q}] \\
&\quad + \frac{4\pi\mathfrak{a}_0}{N} \sum_{p,q\in\Lambda_+^*, r\in P_H^c : r\neq -p,-q} a_{p+r}^* a_q^* a_p a_{q+r}.
\end{aligned}$$

Comparing with (8.41) and noticing that

$$\begin{aligned}
\frac{4\pi\mathfrak{a}_0}{N} \sum_{\substack{p,q\in\Lambda_+^*, r\in P_H^c: \\ r\neq -p,-q}} \langle \xi, a_{p+r}^* a_q^* a_p a_{q+r} \xi\rangle &\leq \frac{C}{N} \sum_{\substack{p,q\in\Lambda_+^*, r\in P_H^c: \\ r\neq -p,-q}} \|a_{p+r} a_q \xi\| \|a_p a_{q+r}\xi\| \\
&\leq \frac{C\ell^{-3\alpha}}{N} \|\mathcal{N}_+ \xi\|^2
\end{aligned}$$

we conclude that

$$\begin{aligned}
\mathcal{R}_{N,\ell} \geq\ & U_N \left[\frac{1}{N}\sum_{i<j}^N v_\ell(x_i - x_j)\right] U_N^* \\
&+ (1 - C\ell^\kappa)\mathcal{H}_N - C\ell^{-3\alpha}\mathcal{N}_+^2/N - C\ell^\kappa.
\end{aligned} \tag{8.42}$$

Following standard arguments, for example from [15, Lemma 1], we observe now that, since $\widehat{v}_\ell(p) \geq 0$ for all $p \in \Lambda^*$,

$$\begin{aligned}
0 &\leq \int_{\Lambda^2} dxdy\, v_\ell(x-y) \left[\sum_{j=1}^N \delta(x - x_j) - N\right] \left[\sum_{i=1}^N \delta(y - x_i) - N\right] \\
&= \sum_{i,j=1}^N v_\ell(x_i - x_j) - N^2 \widehat{v}_\ell(0) = 2\sum_{i<j}^N v_\ell(x_i - x_j) + N v_\ell(0) - N^2\widehat{v}_\ell(0).
\end{aligned}$$

This implies that

$$\frac{1}{N}\sum_{i<j}^{N} v_\ell(x_i - x_j) \geq \frac{N}{2}\widehat{v}_\ell(0) - v_\ell(0) \geq 4\pi\mathfrak{a}_0 N - C\ell^{-3\alpha}.$$

From (8.42), we finally obtain

$$\mathcal{R}_{N,\ell} \geq 4\pi\mathfrak{a}_0 N + (1 - C\ell^\kappa)\mathcal{H}_N - C\ell^{-3\alpha}\mathcal{N}_+^2/N - C\ell^{-3\alpha}.$$

This completes the proof of Proposition 5.2.

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A. Properties of the Scattering Function

In this appendix we give a proof of Lemma 4.1 containing the basic properties of the solution of the Neumann problem (4.1).

Proof of Lemma 4.1. Part (i) and the bounds $0 \leq f_\ell, w_\ell \leq 1$ in part (ii) follow from [6, Lemma A.1]. We prove (4.6). We set $r = |x|$ and $m_\ell(r) = r f_\ell(r)$. We rewrite (4.1) as

$$-m_\ell''(r) + \frac{1}{2}V(r)m_\ell(r) = \lambda_\ell m_\ell(r). \tag{A.1}$$

Let $R > 0$ be the radius of the support of V, so that $V(x) = 0$ for all $x \in \mathbb{R}^3$ with $|x| > R$. For $r \in (R, N\ell]$ we can solve (A.1) explicitly; since the boundary conditions $f_\ell(N\ell) = 1$ and $(\partial_r f_\ell)(N\ell) = 0$ translate into $m_\ell(N\ell) = N\ell$ and $m_\ell'(N\ell) = 1$, we find

$$m_\ell(r) = \lambda_\ell^{-1/2}\sin(\lambda_\ell^{1/2}(r - N\ell)) + N\ell\cos(\lambda_\ell^{1/2}(r - N\ell)). \tag{A.2}$$

With the result of part (i), we obtain

$$m_\ell(r) = r - \mathfrak{a}_0 + \frac{3}{2}\frac{\mathfrak{a}_0}{N\ell}r - \frac{1}{2}\frac{\mathfrak{a}_0}{(N\ell)^3}r^3 + \mathcal{O}(\mathfrak{a}_0^2(N\ell)^{-1}) \tag{A.3}$$

for all $r \in (R, N\ell]$ (the error is uniform in r). Using the scattering equation we can write

$$\int V(x)f_\ell(x)dx = 4\pi\int_0^{N\ell} dr\, rV(r)m_\ell(r) = 8\pi\int_0^{N\ell} dr\,(rm_\ell''(r) + \lambda_\ell r m_\ell(r)).$$

Integrating by parts, we observe that the first contribution on the r.h.s. vanishes (because $m_\ell(N\ell) = N\ell$, $m'_\ell(N\ell) = 1$ and $m_\ell(0) = 0$). With the result of part (i) and with (A.3), we get

$$8\pi\lambda_\ell \int_0^{N\ell} dr\, r m_\ell(r) = 8\pi\lambda_\ell \left(\frac{(N\ell)^3}{3} + \mathcal{O}\big(\mathfrak{a}_0 (N\ell)^2\big) \right) = 8\pi \mathfrak{a}_0 + \mathcal{O}\big(\mathfrak{a}_0^2/\ell N\big)$$

which proves (4.6).

We consider now part (iii). Combining (A.3) for $r \in (R, N\ell]$ with $w_\ell(r) \leq 1$ for $r \leq R$, we obtain the first bound in (4.7). To show the second bound in (4.7), we observe that, for $r \in (R, N\ell]$, (A.2) and the estimate in part (i) imply that $|f'_\ell(r)| \leq Cr^{-2}$, for a constant $C > 0$ independent of N and ℓ, provided $N\ell \geq 1$. For $r < R$ we write, integrating by parts,

$$f'_\ell(r) = \frac{m'_\ell(r) r - m_\ell(r)}{r^2} = \frac{1}{r^2} \int_0^r ds\, s\, m''_\ell(s).$$

With (A.1) and since $0 \leq f_\ell \leq 1$, we obtain

$$\begin{aligned} |f'_\ell(r)| &= \Big| \frac{1}{r^2} \int_0^r ds\, s \Big[\frac{1}{2} V(s) m_\ell(s) - \lambda_\ell m_\ell(s) \Big] \Big| \\ &= \frac{1}{r^2} \Big[\frac{1}{8\pi} \int_{|x|<r} dx\ V(x) f_\ell(x) + \lambda_\ell \int_{|x|<r} dx\ f_\ell(x) \Big] \leq C(\|V\|_3 + 1) \end{aligned}$$

for a constant $C > 0$ independent of N and ℓ, if $N\ell \geq 1$ and for all $0 < r < R$. This concludes the proof of the second bound in (4.7).

To show part (iv), we use (4.4) and we observe that, by (4.5), (4.6) and $f_\ell \leq 1$, there exists a constant $C > 0$ such that

$$\begin{aligned} |\widehat{w}_\ell(p/N)| &\leq \frac{N^2}{p^2} \Big[\big(\widehat{V}(./N) * \widehat{f}_{N,\ell}\big)(0) + C\ell^{-3} \big(\widehat{\chi}_\ell * \widehat{f}_{N,\ell}\big)(0) \Big] \\ &\leq \frac{N^2}{p^2} \Big[\int V(x) f_\ell(x) dx + C\ell^{-3} \int \chi_\ell(x) f_\ell(Nx) dx \Big] \leq \frac{CN^2}{p^2} \end{aligned}$$

for all $N \in \mathbb{N}$ and $\ell > 0$, if $N\ell \geq 1$. □

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