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The Low-Energy TQFT of the Generalized Double Semion Model

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Abstract: The generalized double semion (GDS) model, introduced by Freedman and Hastings, is a lattice system similar to the toric code, with a gapped Hamiltonian whose definition depends on a triangulation of the ambient manifold M, but whose space of ground states does not depend on the triangulation, but only on the underlying manifold. In this paper, we use topological quantum field theory (TQFT) to investigate the low-energy limit of the GDS model. We define and study a functorial TQFT Z_{GDS} in every dimension n such that for every closed $(n-1)$-manifold M, $Z_{\mathrm{GDS}}(M)$ is isomorphic to the space of ground states of the GDS model on M; the isomorphism can be chosen to intertwine the actions of the mapping class group of M that arise on both sides. Throughout this paper, we compare our constructions and results with their known analogues for the toric code.

1. Introduction

The classification of topological phases of matter is an active area of research in the theory of condensed-matter physics and in nearby mathematical fields. There are many different approaches to this classification problem (for an incomplete sample, see [12,38,41,49]), but from a mathematical point of view, a classification via low-energy limits is appealing: based on physical insights, it is believed that the low-energy effective theory of a gapped phase of matter is a topological quantum field theory (TQFT), possibly tensored with an invertible theory, and that passage to the low-energy effective theory should send physically distinct phases to distinct TQFTs [17,20,25,50]. As TQFTs have a purely mathematical description due to Atiyah [2] and Segal [54], this reframes the classification question within mathematics—though a systematic mathematical understanding of this physical ansatz relating lattice systems to effective field theories remains out of reach. Even at a physical level of rigor, it is not clear what the general definition of the low-energy effective theory of a lattice model should be, and without this it is impossible to rigorously verify the efficacy of the low-energy approach to classification in general.

Nonetheless, there are many examples of lattice models in the physical and mathematical literature, and it is instructive to study what can be said about their low-energy effective theories in order to gain insight into the general picture. Some examples include [1,5,8, 11,13,36].

In this paper, we investigate the low-energy effective theory of the generalized double semion (GDS) lattice model of Freedman–Hastings [22], which exists in every dimension. Freedman and Hastings define the GDS model and study its spaces of ground states on different manifolds, showing that in even (spacetime) dimensions n they are isomorphic to the state spaces of the $\mathbb{Z}/2$-Dijkgraaf–Witten theory with Lagrangian equal to 0, but that for odd $n > 3$, they are not isomorphic to the state spaces of any $\mathbb{Z}/2$-Dijkgraaf–Witten theory. For every dimension n, we define an n-dimensional TQFT $Z_{\mathrm{GDS}}\colon \mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}$ and show that for every closed $(n-1)$-manifold M, the state space $Z_{\mathrm{GDS}}(M)$ is isomorphic to the space of ground states of the GDS model on M, and that this isomorphism is equivariant with respect to the actions of $\mathrm{MCG}(M)$ coming from the GDS model and the TQFT. Along the way, we reformulate the GDS model as a lattice gauge theory with gauge group $\mathbb{Z}/2$: it is a theory formulated on manifolds with a triangulation, which plays the role that a Riemannian metric does in Wick-rotated quantum field theory. We find that, as for the toric code lattice model, the low-energy limit does not depend on the triangulation, and is described by the state spaces of a TQFT. For both the toric code and GDS models, this TQFT is a $\mathbb{Z}/2$-gauge theory, but unlike for the toric code, the GDS theory involves gravity, in that Stiefel–Whitney classes of the underlying manifold enter the effective action. This explains the above result of Freedman–Hastings that this TQFT cannot be any $\mathbb{Z}/2$-Dijkgraaf–Witten theory when n is odd and greater than 3 [22, Theorem 8.1].

The GDS model is closely analogous to the toric code; thus, throughout this paper, we will introduce ideas first for the toric code, which is simpler, and then turn to the GDS model. In Sect. 2, we define the toric code (Sect. 2.1) and GDS models (Sect. 2.2) in arbitrary dimension. These are both examples of lattice models, which are discretized analogues of quantum field theories studied in condensed-matter physics: one puts a combinatorial structure, such as a CW structure or a triangulation, on a manifold, and formulates all data of the theory, including the fields and the Hamiltonian, in terms of this combinatorial structure. The toric code and GDS models are typically written as spin liquids, meaning the fields are functions from the edges of a lattice to $\{\uparrow, \downarrow\}$. We reformulate them as lattice gauge theories, describing equivalent models whose fields are discretizations of principal $\mathbb{Z}/2$-bundles.

In Sect. 3, we construct a class of TQFTs called $\mathbb{Z}/2$-gauge–gravity theories. They generalize Dijkgraaf–Witten theories with gauge group $\mathbb{Z}/2$, but the Lagrangian includes Stiefel–Whitney classes of the underlying manifold in addition to characteristic classes of the principal $\mathbb{Z}/2$-bundle. First, in Sect. 3.1, we define "classical gauge–gravity theories," invertible TQFTs of manifolds with a principal $\mathbb{Z}/2$-bundle. Then, in Sect. 3.2, we quantize these theories, summing over the groupoid of principal $\mathbb{Z}/2$-bundles to produce TQFTs $Z_\beta\colon \mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}$ of unoriented manifolds given a cohomology class $\beta \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$.

In Sect. 4, we use these gauge–gravity TQFTs to study the low-energy behavior of the GDS model. The Hamiltonian in the GDS model has spectrum contained within $\mathbb{Z}_{\geq 0}$, and the space of ground states of the GDS model on an $(n-1)$-manifold M is defined to be the kernel of the Hamiltonian for M. In examples arising in physics from topological phases of matter, the space of ground states often depends only on M, and not on the triangulation. When this occurs, it is expected that this extends to a TQFT

$Z\colon \mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}$, in that for any closed $(n-1)$-manifold M, $Z(M)$ is isomorphic to the space of ground states on M. First, in Sect. 4.1, we discuss a way to strengthen this: given a closed $(n-1)$-manifold M, there is a natural action of $\mathrm{Diff}(M)$ of M on $Z(M)$. We provide a method for some lattice models of constructing a $\mathrm{Diff}(M)$-action on the space of ground states of M, and we will ask for the isomorphism of the space of ground states on M with $Z(M)$ to be $\mathrm{Diff}(M)$-equivariant. In Sect. 4.2, we implement this idea for the toric code, where we reprove the following known result.

Theorem 4.4. *If* $\mathrm{DW}_0\colon \mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}$ *denotes the* $\mathbb{Z}/2$*-Dijkgraaf–Witten theory with Lagrangian equal to* 0*, then for every closed* $(n-1)$*-manifold* M*, the space of ground states of the toric code on* M *is isomorphic to* $\mathrm{DW}_0(M)$ *as* $\mathrm{Diff}(M)$*-representations.*

In Sect. 4.3, we turn to the GDS model, where we prove the main theorem. Let $\alpha \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$ denote the generator and $w \in H^*(B\mathrm{O}_n;\mathbb{Z}/2)$ denote the total Stiefel–Whitney class. In the graded ring $H^*(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2) \cong H^*(B\mathrm{O}_n;\mathbb{Z}/2) \otimes_{\mathbb{Z}/2} H^*(B\mathbb{Z}/2;\mathbb{Z}/2)$, α is nilpotent, so $1+\alpha$ is invertible. Therefore we can form $w\alpha/(1+\alpha) \in H^*(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2)$, which is a sum of homogeneous elements of different degrees.

Theorem 4.18. *Let* $\beta \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2)$ *be the degree-n summand of* $w\alpha/(1+\alpha)$*. Then, for every closed* $(n-1)$*-manifold* M*, the space of ground states of the GDS model on* M *is isomorphic to* $Z_\beta(M)$ *as* $\mathrm{Diff}(M)$*-representations.*

Because of this, Z_β will also be denoted Z_{GDS}.

In Sect. 5, we provide some calculations with this low-energy TQFT, allowing us to prove a comparison theorem with $\mathbb{Z}/2$-Dijkgraaf–Witten theories.

Theorem. (1) *In dimension* 3*, there is an isomorphism between* Z_{GDS} *and the* $\mathbb{Z}/2$*-Dijkgraaf–Witten theory with Lagrangian equal to the nonzero element of* $H^3(B\mathbb{Z}/2;\mathbb{Z}/2)$.
(2) *In any even dimension, there is an isomorphism between* Z_{GDS} *and* DW_0.
(3) *For odd* $n \geq 5$*,* Z_{GDS} *is distinct from all* $\mathbb{Z}/2$*-Dijkgraaf–Witten theories.*

This theorem is a combination of Theorems 5.29, 5.31, and 5.32. Part (3) was first proven by [22], as was (2) for state spaces.

2. The Toric Code and GDS Models

Definition 2.1. Let X be a topological space with a CW structure Ξ. We let $\Delta^k(X)$ denote its set of k-cells and X^k denote its k-skeleton. When we need to make explicit that these are with respect to Ξ, we will write $\Delta^k(X;\Xi)$, resp. X^k_Ξ. If Π is a triangulation of X, we will also write $\Delta^k(X;\Pi)$ and X^k_Π for the k-simplices, resp. k-skeleton, of X with respect to Π.

When we need Ξ to be explicit, we will write $C^\Xi_k(X;A)$ (resp. $C^k_\Xi(X;A)$) for the group of cellular k-chains (resp. k-cochains) with coefficients in an abelian group A for the CW structure Ξ. We will employ analogous notation for cycles and cocycles, and for simplicial (co)chains and (co)cycles with respect to a given triangulation Π.

Definition 2.2. For a topological space X, let $\mathrm{Bun}_{\mathbb{Z}/2}(X)$ denote the groupoid of principal $\mathbb{Z}/2$-bundles on X, and if $Y \subset X$, let $\mathrm{Bun}_{\mathbb{Z}/2}(X,Y)$ denote the groupoid of principal $\mathbb{Z}/2$-bundles $P \to X$ equipped with a trivialization ξ over Y.

If X is a CW complex, then $(P,\xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(X^1,X^0)$ determines a function $\mathrm{spin}_{(P,\xi)}\colon \Delta^1(X) \to \mathbb{Z}/2$: if e is a 1-cell of X, $P|_e$ descends to a principal bundle $P' \to e/\partial e$, where we use the trivialization of P on ∂e to identify the fibers. Then $\mathrm{spin}_{(P,\xi)}(e)$ is 0 if P' is trivial, and 1 if it is nontrivial.

In other words, if $\partial e = \{v, w\}$, we can compare $\xi(v)$ and $\xi(w)$ by parallel-transporting along e; then $\mathrm{spin}_{(P,\xi)}(e)$ is their difference. The function $\mathrm{spin}_{(P,\xi)}$ determines (P, ξ) up to isomorphism.

2.1. The toric code. The toric code was originally studied by Kitaev [37]. He was interested in its properties as a quantum error-correcting code when put on a torus, hence the name "toric code;" a more descriptive name would be "lattice gauge theory for a finite group G." Subsequently, it has been generalized in many directions: defining it on nonorientable surfaces [23]; generalizing it to manifolds of any dimension [24]; placing the spins on k-cells, rather than edges [15]; considering a fermionic variant [28]; changing whether it is even a gauge theory at all [9]; and adding global symmetries [6,30,43]. In this paper, we will not consider most of these generalizations.

Fix a dimension n, which will always be the *spacetime dimension*; that is, lattice models are on $(n-1)$-manifolds, and TQFTs are formulated with n-dimensional cobordisms between $(n-1)$-dimensional manifolds. The toric code assigns to a closed $(n-1)$-manifold M together with a CW structure a finite-dimensional complex vector space $\mathcal{H}$, called the *state space*, and a self-adjoint operator $H \colon \mathcal{H} \to \mathcal{H}$, called the *Hamiltonian*. We proceed to define these.

The groupoid of fields for the toric code is $\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$, and the state space assigned to M is $\mathcal{H} := \mathbb{C}[\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)]$, the vector space of complex-valued functions on the groupoid of fields.[1] Given $(P, \xi) \in \pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$, let $\delta_{(P,\xi)} \in \mathcal{H}$ be the function sending $(P, \xi) \mapsto 1$ and all nonisomorphic (P', ξ') to 0. The set

$$\{\delta_{(P,\xi)} \mid (P, \xi) \in \pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)\} \tag{2.3}$$

is a basis for $\mathcal{H}$; endow $\mathcal{H}$ with the inner product for which it is an orthonormal basis.

Given a 0-cell v of M, let $A_v \colon \mathcal{H} \to \mathcal{H}$ denote the *shift operator* at v: if $\psi \in \mathcal{H}$ and $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$, let $\xi + \delta_v$ denote the section of P on M^0 which is identical to ξ except on v, where its value is $\xi(v) + 1$. Then,

$$A_v(\psi)(P, \xi) := \psi(P, \xi + \delta_v). \tag{2.4a}$$

Given a 2-cell f of M, let $B_f \colon \mathcal{H} \to \mathcal{H}$ be multiplication by the holonomy around ∂f:

$$B_f(\psi)(P, \xi) := (-1)^{\mathrm{Hol}_P(f)}\psi. \tag{2.4b}$$

There are operators associated to each 2-cell f and each 0-cell v, called *face operators*, resp. *vertex operators*:

$$H_f := \frac{1 - B_f}{2} \tag{2.5a}$$

$$H_v := \frac{1 - A_v}{2}, \tag{2.5b}$$

and the Hamiltonian assigned to M is

$$H_{\mathrm{TC}} := \sum_{v \in \Delta^0(M)} H_v + \sum_{f \in \Delta^2(M)} H_f. \tag{2.6}$$

[1] The space of functions on a groupoid G is defined to be the vector space of functions $\pi_0 G \to \mathbb{C}$.

Remark 2.7. The original definition of the toric code looked different, replacing (P, ξ) with the function $\mathrm{spin}_{(P,\xi)} : \Delta^1(M) \to \mathbb{Z}/2$ it defines. The state space is the free complex vector space on the finite set of these functions. The analogues of A_v and B_f for $v \in \Delta^0(M)$ and $f \in \Delta^2(M)$ are

$$A'_v := \prod_{e : v \in \partial e} \sigma^x_e \tag{2.8a}$$

$$B'_f := \prod_{e \in \partial f} \sigma^z_e. \tag{2.8b}$$

Here, σ^x and σ^z are the Pauli operators

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.9}$$

The state space $\mathcal{H}$ can be identified with the tensor product of local state spaces $\mathcal{H}_e := \mathbb{C} \cdot \{0, 1\}$ over each 1-cell e, and the notation σ^x_e and σ^z_e means these operators act on $\mathcal{H}_e$ by the matrices in (2.9), and by the identity on the remaining tensor factors.

We can identify A'_v with A_v by observing that switching the trivialization for (P, ξ) over v amounts to switching the value of $\mathrm{spin}_{(P,\xi)}$ on any 1-cell e adjacent to v, which is the action by σ^x_e. To identify B_f and B'_f, observe that the holonomy of (P, ξ) around ∂f is the product of the spins on the 1-cells in ∂f.

Proposition 2.10 (Kitaev [37] and Freedman–Meyer–Luo [24]).

(1) *The Hamiltonian H_{TC} is self-adjoint.*
(2) *The H_f and H_v operators are projectors, and pairwise commute.*
(3) $\mathrm{Spec}(H_{\mathrm{TC}}) \subset \mathbb{Z}_{\geq 0}$, *and* 0 *is always an eigenvalue.*

Proof sketch. Using the identifications of A_v with A'_v and B_f with B'_f, A_v and B_f are products of real symmetric matrices, hence are themselves real symmetric matrices; therefore H_v and H_f are too. Therefore H is a sum of real symmetric matrices, proving part (1).

Part (2) is directly analogous to Kitaev's original proof in dimension $n - 1 = 2$ [37]; see [24] for the generalization to higher dimensions.

Part (3) follows because the eigenvalues of A_f and B_v are in $\{\pm 1\}$, so the eigenvalues of H_f and H_v are in $\{0, 1\}$. The function dual to the trivial bundle with the identity trivialization is an eigenvector for 0. $\square$

2.2. Generalized double semion model. Our main focus is the generalized double semion (GDS) model.

The double semion model for $n = 3$ was first studied by Freedman–Nayak–Shtengel–Walker–Wang [21] and Levin–Wen [42, §VI.A], then generalized to all dimensions n by Freedman and Hastings [22].[2] The name comes from the description of this theory

[2] There are a few other generalizations of the double semion model in low dimensions [14,43,48,60], but we focus on Freedman–Hastings' construction.

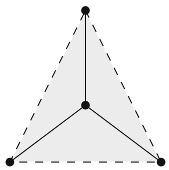

Fig. 1. The 0-clopen star of a vertex in a simplicial structure on a surface

in the case $n = 3$ as the lattice model associated to the Drinfeld double of the semion modular tensor category.[3]

Definition 2.11. Let M be a simplicial complex and c be a simplex of M.

- The *open star* of c, denoted $\mathrm{St}(c)$, is the subset of M consisting of all simplices whose closures contain c.
- The *closed star* of c, denoted $\overline{\mathrm{St}}(c)$, is the smallest subcomplex containing $\mathrm{St}(c)$.
- The *link* of c, denoted $S(c)$, is $\overline{\mathrm{St}}(c)\setminus \mathrm{St}(c)$.

For the GDS model, we need a neighborhood of v in between the open and closed stars of v.

Definition 2.12. Let M be a simplicial complex and e be a simplex of M. Define the 0-*clopen star* $\overline{\mathrm{St}}(0)(e)$ to be $\mathrm{St}(e) \cup \overline{\mathrm{St}}(e)^0$. That is, we include the 0-simplices of the closed star of e as well as all cells in the open star.

As before, fix a dimension n; we proceed to define the state space and Hamiltonian that the GDS model assigns. In order to avoid pathologies, one cannot define the GDS model for an arbitrary CW structure.

Definition 2.13. A *triangulation* of a smooth manifold M is a simplicial complex K together with a homeomorphism $f\colon |K| \to M$; if for every simplex e of K, the restriction of f to $|e|$ is smooth, we say (K, f) is a *smooth triangulation*.

When defining the GDS model, we choose a smooth triangulation Π such that the 0-clopen star of every vertex is contractible.[4] We discuss in Remark 2.34 why restricting to such triangulations, rather than more general combinatorial structures such as CW structures, is necessary.

The GDS model assigns to every closed $(n-1)$-manifold M with such a triangulation a state space and Hamiltonian, like the toric code does; the state space is $\mathbb{C}[\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)]$ as for the toric code, and we proceed to define the Hamiltonian, which is similar to that of the toric code, but with an extra sign.

Definition 2.14. Let M be a closed $(n-1)$-manifold with a smooth triangulation such that the 0-clopen star of every vertex is contractible. Then, given $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$ and a 0-simplex v, there is a unique maximal extension of ξ to a subset of $\overline{\mathrm{St}}(0)(v)$; we denote that subset $Y'_v(P, \xi)$.

[3] The semion modular tensor category is named such because the excitations in its corresponding lattice model are semions, anyonic quasiparticles with statistics intermediate between those of bosons and fermions. For $n > 3$, however, the name "generalized double semions" is somewhat of a misnomer, however: anyons cannot exist in dimension $n > 3$, because the braids that define their mutual statistics can be unlinked. See [50, §2.1]. It is also not clear that the theory is the double of another [22, §1]. At least it is generalized.

[4] The second constraint can always be satisfied after a refinement.

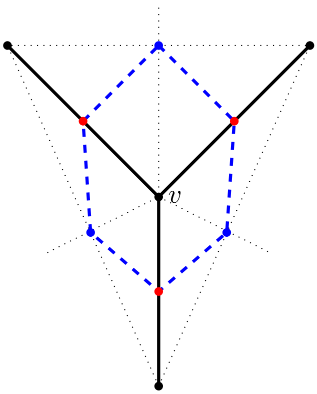

Fig. 2. The triangulation $\Pi_{S(v)}$ constructed in Definition 2.15. The black vertices and solid black edges are the original simplices in Π. The remaining edges are added in the barycentric subdivision Π_1 of Π. The blue (dashed) edges and the red and blue vertices are the link $S(v)$ of v in Π_1. To define $\Pi_{S(v)}$, we keep the red vertices as 0-simplices, but for 1-simplices, the blue vertices are merged with their adjacent edges. Thus $\Pi_{S(v)}$ has three 0-simplices and three 1-simplices, and each k-simplex is the intersection of a $(k+1)$-simplex of Π with $S(v)$

Definition 2.15. Let $v \in \Delta^0(M;\Pi)$ and $S(v)$ denote the link of v in the barycentric subdivision Π_1 of Π. Though $S(v)$ comes equipped with a triangulation $\Pi_1|_{S(v)}$, we define a new triangulation $\Pi_{S(v)}$ on $S(v)$. For $k \geq 0$, if e is a $(k+1)$-simplex of Π such that $v \in \partial e$, let

$$C(e) := \{c \in \Delta^*(S(v), \Pi_1|_{S(v)}) : |c| \subset |e|\}. \tag{2.16}$$

For each such e, we define a k-simplex of $\Pi_{S(v)}$, denoted $S(v) \cap e$, whose geometric realization is

$$|S(v) \cap e| := \bigcup_{c \in C(e)} c. \tag{2.17}$$

We say that $S(v) \cap e'$ is a face of $S(v) \cap e$ if every $c' \in C(e')$ is a face of some $c \in C(e)$, which may depend on c'. This data defines a triangulation on $S(v)$ such that if e is a simplex of Π with $v \in \partial e$,

$$|S(v) \cap e| = |S(v)| \cap |e|. \tag{2.18}$$

See Fig. 2 for a picture of this triangulation.

From now on, the triangulation on $S(v)$ is assumed to be $\Pi_{S(v)}$ unless stated otherwise.

Definition 2.19. Let $(P,\xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$. For any $v \in \Delta^0(M)$, let

$$Y_v(P,\xi) := \{S(v) \cap e \mid e \in Y'_v(P,\xi)\}, \tag{2.20}$$

which is a subcomplex of $S(v)$. The *GDS sign* [22, §4] is

$$\sigma(v,(P,\xi)) := (-1)^{1+\chi(|Y_v(P,\xi)|)}. \tag{2.21}$$

Here χ denotes the Euler characteristic.

Let U_v denote the operator on $\mathcal{H}$ defined by $U_v(\psi)(P,\xi) := \sigma(v,(P,\xi))A_v(\psi)$, where A_v is as in (2.4a). The Hamiltonian for the GDS model is

$$H_{\mathrm{GDS}} := \sum_{v\in\Delta^0(M)} \widetilde{H}_v + \sum_{f\in\Delta^2(M)} H_f, \tag{2.22}$$

where H_f is as in (2.5a) and

$$\widetilde{H}_v = \frac{1-U_v}{2}. \tag{2.23}$$

As for the toric code, we call $\widetilde{H}_v$ a *vertex operator* and H_f a *face operator*.

Remark 2.24. In our analysis of the GDS model, we will need to make use of the *dual cell complex* $\Pi^\vee$ to the specified triangulation Π, a CW complex on M with several nice properties.

- $\Pi^\vee$ comes with data of a bijection $(\cdot)^\vee\colon \Delta^k(M,\Pi)\to\Delta^{n-1-k}(M,\Pi^\vee)$, sending a simplex to its *dual cell*, and such that if $e\in\partial f$, then $f^\vee\in\partial e^\vee$, and conversely.
- The map $(\cdot)^\vee$ induces a chain map on the cellular chain complexes of Π and $\Pi^\vee$ which induces Poincaré duality for the cohomology of M with $\mathbb{Z}/2$ coefficients.
- Each cell in $\Pi^\vee$ is a union of cells of the barycentric subdivision Π_1 of Π. (One might think of Π_1 as a refinement of $\Pi^\vee$; though this is not strictly true, as $\Pi^\vee$ might not come from a triangulation, it is a useful piece of intuition.) In particular, $\Pi^\vee$ is a regular CW complex, meaning the closure of each cell is contractible.

This complex is unique up to equivalence of CW complexes. Proofs of these facts follow from the results in [31, §1.6].

We will also denote $((\cdot)^\vee)^{-1}$ by $(\cdot)^\vee$, but since we do not confuse Π and $\Pi^\vee$, the meaning will be clear from context. If S is a set of cells, we write $S^\vee := \{e^\vee \mid e\in S\}$.

Remark 2.25. Freedman–Hastings [22] study a dual version of the GDS model, in that our model for M and Π corresponds to their model for M and $\Pi^\vee$. Here we compare the two setups.

Let $(P,\xi)\in \mathrm{Bun}_{\mathbb{Z}/2}(M^1,M^0)$, which defines a function $\mathrm{spin}_{(P,\xi)}\colon \Delta^1(M,\Pi)\to\mathbb{Z}/2$ as in Definition 2.2; we also let $\mathrm{spin}_{(P,\xi)}$ denote the function $\Delta^{n-2}(M,\Pi^\vee)\to\mathbb{Z}/2$ defined by precomposing with $(\cdot)^\vee$.

For any $v\in\Delta^0(M,\Pi)$, let

$$T(v,(P,\xi)) := \mathrm{spin}^{-1}_{(P,\xi)}(0)\cap\partial v^\vee, \tag{2.26}$$

which is a closed union of cells of $\Pi^\vee$.

The GDS sign as defined by Freedman–Hastings [22, §4] is

$$\sigma'(v,(P,\xi)) := (-1)^{1+\chi(T(v,(P,\xi)))}. \tag{2.27}$$

Let $e\in\overline{\mathrm{St}}(0)(v)$. Unwinding the definitions, $e\cap S(v)\in Y_v(P,\xi)$ if and only if $e^\vee$ is a cell of $T(v,(P,\xi))$, so the number of simplices in $Y_v(P,\xi)$ equals the number of cells in $T(v,(P,\xi))$. Since both $T(v,(P,\xi))$ and $Y_v(P,\xi)$ are closed subsets of M that are unions of cells, their Euler characteristics are equal, so $\sigma=\sigma'$. This means there is an isomorphism between the state spaces of the model we define above and the model as defined by Freedman–Hastings, and this isomorphism intertwines their Hamiltonians, so on any closed $(n-1)$-manifold, the spaces of ground states of these two models are isomorphic.

Next, we prove analogues of Proposition 2.10 for the GDS model. In view of Remark 2.25, these also follow from results of Freedman–Hastings [22, Lemmas 4.1, 4.2], but are proven in a different way.

Lemma 2.28. *The Hamiltonian H_{GDS} is self-adjoint, and* $\mathrm{Spec}(H_{\mathrm{GDS}}) \subset \mathbb{Z}_{\geq 0}$.

Proof. The first part is true because the Hamiltonian is a sum of real symmetric matrices in a basis of δ-functions, just as in the proof of Proposition 2.10. For the second part, since the eigenvalues of A_v and B_f lie in $\{\pm 1\}$ and σ is valued in $\{\pm 1\}$, then the eigenvalues of H_f and $\widetilde{H}_v$ lie in $\{0, 1\}$. $\square$

Unlike for the toric code, it is not true that 0 is always an eigenvalue. Theorem 4.18 and Corollary 5.6 together imply this happens for $M = \mathbb{CP}^{2k}$.

Lemma 2.29. *All face operators commute, and all face operators commute with all vertex operators. After restricting to the intersection of the kernels of the face operators, $[U_{v_1}, U_{v_2}] = 0$ and hence all vertex operators commute when restricted to that intersection.*

Proof. The face operators are the same as in the toric code, hence commute by Proposition 2.10. Operators corresponding to simplices not in each others' closed stars commute. Therefore we have two things left to prove:

(1) Given a 2-simplex f and a 0-simplex $v \in \partial f$, $[H_f, \widetilde{H}_v] = 0$.
(2) Given a 1-simplex e and two 0-simplices $v_1, v_2 \in \partial e$, $[U_{v_1}, U_{v_2}] = 0$ when restricted to $\bigcap_{f \in \Delta^2(M)} H_f$.

For part (1): since the GDS sign factors out of $[B_f, U_v]$, then $[B_f, U_v] = \pm[B_f, A_v] = 0$ by Proposition 2.10, and therefore $[H_f, \widetilde{H}_v] = 0$.

For part (2), choose $\psi \in \mathcal{H}$ such that $H_f \psi = 0$ for all 2-simplices f, and choose $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$. Since B_f acts by multiplication by the holonomy of P around ∂f, then $\psi(P, \xi) = 0$ unless $\mathrm{Hol}_P(f) = 0$ for all f; equivalently, P must extend to all of M.[5] (This extension is necessarily unique up to isomorphism.) If this is the case,

$$\begin{aligned}[U_{v_1}, U_{v_2}]\psi(P, \xi) = \sigma(v_2, (P, \xi + \delta_{v_1}))\sigma(v_1, (P, \xi))\psi(P, \xi + \delta_{v_1} + \delta_{v_2}) \\ - \sigma(v_1, (P, \xi + \delta_{v_2}))\sigma(v_2, (P, \xi))\psi(P, \xi + \delta_{v_1} + \delta_{v_2}),\end{aligned} \tag{2.30}$$

so it suffices to show that if $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$,

$$\sigma(v_2, (P, \xi + \delta_{v_1}))\sigma(v_1, (P, \xi)) = \sigma(v_1, (P, \xi + \delta_{v_2}))\sigma(v_2, (P, \xi)). \tag{2.31}$$

Tracing through the definition of the GDS sign, this is equivalent to

$$\chi(|Y_{v_2}(P, \xi+\delta_{v_1})|)+\chi(|Y_{v_1}(P, \xi)|) \underset{\bmod 2}{\equiv} \chi(|Y_{v_1}(P, \xi+\delta_{v_2})|)+\chi(|Y_{v_2}(P, \xi)|). \tag{2.32}$$

Suppose $\mathrm{spin}_{(P,\xi)}(e) = 0$. For $i = 1, 2$, let $A(v_i)$ denote the set of simplices in $Y_{v_i}(P, \xi)$ contained in the closure of a simplex in $Y_{v_i}(P, \xi)$ whose closure also contains $S(v_i) \cap e$. Let $B(v_i) := Y_{v_i}(P, \xi) \backslash A(v_i)$. Then

$$\chi(|Y_{v_2}(P, \xi + \delta_{v_1})|) + \chi(|Y_{v_1}(P, \xi)|) \underset{\bmod 2}{\equiv} \#(A(v_1) \amalg B(v_1) \amalg B(v_2)) \tag{2.33a}$$

[5] We will return to this point in Sect. 4.2.

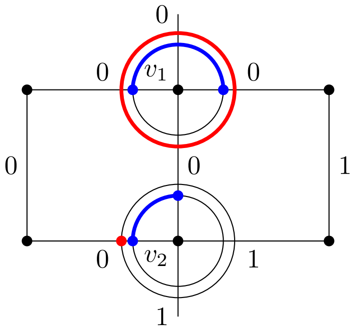

Fig. 3. Lemma 2.29 does not generalize from triangulations to CW structures. The straight lines in this figure depict a neighborhood on a smooth surface Σ with a CW structure. Choose $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(\Sigma^1, \Sigma^0)$ such that the number on each pictured 1-cell e is $\mathrm{spin}_{(P,\xi)}(e)$. The circles around the 0-cells v_1 and v_2 represent two copies of each of the links $S(v_1)$ and $S(v_2)$. The red region (shaded portions of the outer circles) is $|Y_{v_1}(P, \xi)| \amalg |Y_{v_2}(P, \xi + \delta_{v_1})|$, and the blue region (shaded portions of the inner circles) is $|Y_{v_2}(P, \xi)| \amalg |Y_{v_1}(P, \xi + \delta_{v_2})|$. By inspection, the Euler characteristics of these two regions are not equal mod 2, so (2.32) does not hold in this setting, and therefore Lemma 2.29 also does not apply to this CW structure: $\widetilde{H}_{v_1}$ and $\widetilde{H}_{v_2}$ do not commute even when restricted to $\bigcap_f H_f$

$$\chi(|Y_{v_1}(P, \xi + \delta_{v_2})|) + \chi(|Y_{v_2}(P, \xi)|) \underset{\text{mod } 2}{\equiv} \#(A(v_2) \amalg B(v_2) \amalg B(v_1)). \tag{2.33b}$$

It therefore suffices to prove that $\#A(v_1) = \#A(v_2)$. Let c_1 be a 1-simplex in $A(v_1)$. Since 2-simplices are triangles, there exists a unique 1-simplex c_2 whose closure contains v_2 and such that there is a 2-simplex f with $\partial f = c_1 + c_2 + e$. By assumption, $\mathrm{spin}_{(P,\xi)}(e) = \mathrm{spin}_{(P,\xi)}(c_1) = 0$, and since the holonomy of P around ∂f vanishes, $\mathrm{spin}_{(P,\xi)}(c_2) = 0$ too. Similarly, suppose c_1' and c_2' are 1-simplices such that v_1 is a face of c_1', v_2 is a face of c_2', $\mathrm{spin}_{(P,\xi)}(c_1') = 1$, and there is a 2-simplex f' with $\partial f' = c_1' + c_2' + e$; then $\mathrm{spin}_{(P,\xi)}(c_2') = 1$ too. This argument is obviously symmetric in v_1 and v_2.

The case $\mathrm{spin}_{(P,\xi)}(e) = 1$ is analogous. $\square$

Remark 2.34. The ideas that go into the GDS model still make sense when one generalizes to smooth manifolds with regular CW structures, rather than smooth triangulations, but Lemma 2.29 does not generalize. See Fig. 3 for a counterexample.

If one lets $n = 3$ and passes to the dual CW structure as in Remark 2.24, this recovers a fact known to condensed-matter theorists: the double semion model on a surface can be formulated on a hexagonal lattice (or more generally a trivalent lattice), but has an ambiguity when placed on a square lattice [22, §2]. This is because the dual CW structure to a trivalent lattice has triangular 2-cells, but the dual of a tetravalent lattice does not. For general n, this obstruction is encoded in the genericity assumption placed on the CW structure in Freedman–Hastings' construction [22, §4]; in our model this corresponds to the restriction to smooth triangulations.

Lemma 2.35. *The face operators are projectors. The operator U_v has order 2, and hence $\widetilde{H}_v$ is a projector.*

Proof. The face operators are the same as in the toric code, hence are projectors by Proposition 2.10. For U_v, choose a 0-simplex v, $\psi \in \mathcal{H}$, and $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$; then,

$$\begin{aligned} U_v^2 \psi(P, \xi) &= \sigma(v, (P, \xi + \delta_v) \sigma(v, (P, \xi)) \psi(P, \xi) \\ &= (-1)^{\chi(|Y_v(P, \xi + \delta_v)|) + \chi(|Y_v(P, \xi)|)} \psi(P, \xi). \end{aligned} \tag{2.36}$$

Unwinding the definition of Y_v, and using that $\chi(S(v)) \equiv 0 \bmod 2$, $\chi(|Y_v(P, \xi+\delta_v)|)+\chi(|Y_v(P, \xi)|)$ is equal mod 2 to the number of simplices e in $S(v)$ such that $\overline{e}$ contains a 0-simplex on which ξ extends and a 0-simplex on which $\xi+\delta_v$ extends (equivalently, on which ξ does not extend). Let Q be the set of such e.

Endow $S(v)$ with the Poincaré dual CW structure $\Pi^\vee_{S(v)}$ to the triangulation $\Pi_{S(v)}$, as in Remark 2.24. Let $R \subset \Pi_{S(v)}$ be the set of 1-simplices on which ξ extends; then, $|R^\vee|$ is a topological submanifold (with boundary) of $S(v)$, and $\partial|R^\vee| = |Q^\vee|$. Hence $\chi(|Q^\vee|) \equiv 0 \bmod 2$; since $Q^\vee$ is a subcomplex of $\Pi^\vee_{S(v)}$, this means $Q^\vee$ has an even number of cells, so Q has an even number of simplices. Thus $\chi(|Y_v(P, \xi+\delta_v)|)+\chi(|Y_v(P, \xi)|) \equiv 0 \bmod 2$, and this suffices by (2.36). □

There are a few other equivalent ways to define the GDS sign. We record one which we will use later.

Proposition 2.37. *Let $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$ and $v \in \Delta^0(M)$, and let N_v be the set of simplices c of M with $v \in \partial c$. If $Z_v(P, \xi) \subset N_v$ denotes the subset of simplices c such that either (1) c is a 1-simplex and $\mathrm{spin}_{(P,\xi)}(c) = 1$, or (2) there is a 1-simplex $e \in \partial c$ with $\mathrm{spin}_{(P,\xi)}(e) = 1$, then $(-1)^{1+\#Z_v(P,\xi)} = \sigma(v, (P, \xi))$.*

Proof. It suffices to show $\#Z_v(P, \xi) \equiv \#Y_v(P, \xi) \bmod 2$. If $W_v(P, \xi)$ denotes the subset of N_v consisting of simplices c such that either (1) c is a 1-simplex and $\mathrm{spin}_{(P,\xi)}(c) = 0$, or (2) $\mathrm{spin}_{(P,\xi)}(e) = 0$ for all $e \in \Delta^1(\partial c)$, then the map $c \mapsto c \cap S(v)$ for $c \in N_v$ restricts to a bijection from $W_v(P, \xi)$ to $Y_v(P, \xi)$.

By definition, $Z_v(P, \xi)$ is the complement of $W_v(P, \xi)$ inside N_v. Since $N_v^\vee = \partial v^\vee$ and $\chi(|\partial v^\vee|)$ is even, then $\#N_v$ is even and

$$\#Z_v(P, \xi) + \#Y_v(P, \xi) = \#Z_v(P, \xi) + \#W_v(P, \xi) = \#N_v \equiv 0 \bmod 2. \tag{2.38}$$

□

3. Gauge–Gravity TQFTs

As part of our goal of studying the low-energy behavior of the GDS model, we would like a description in terms of a TQFT whose state spaces we can compute relatively easily. The answer comes to us as one of a class of TQFTs, called $\mathbb{Z}/2$-gauge–gravity theories; these TQFTs are slight generalizations of $\mathbb{Z}/2$-Dijkgraaf–Witten theories [16,19], in which Stiefel–Whitney classes of the underlying manifold can enter the Lagrangian action. Theories of this sort have also been considered by Kapustin [33,34], Wen [62,63], and Lan–Kong–Wen [40], though not in this generality.

As in the construction of Dijkgraaf–Witten theories, we will construct the gauge–gravity theories in two steps. First, we will construct invertible theories for unoriented manifolds equipped with a principal $\mathbb{Z}/2$-bundle; these are the classical theories, and are examples of Turaev's homotopy quantum field theories with target $B\mathbb{Z}/2$ [59] (sometimes also called equivariant TQFTs [52]). Then, we will sum over principal $\mathbb{Z}/2$-bundles in a process called orbifoldization, producing what are called the quantum theories [18,19] or the orbifold theories [52].

3.1. Construction of the classical $\mathbb{Z}/2$-gauge–gravity theories. Let Bord_n denote the unoriented bordism category in dimension n, whose objects are closed $(n-1)$-manifolds and whose morphisms are diffeomorphism classes of bordisms between them, and, for a topological space X, let $\mathsf{Bord}_n(X)$ denote the bordism category of manifolds together with a map to X.

Definition 3.1. A TQFT $Z\colon \mathsf{Bord}_n(X) \to \mathsf{Vect}_{\mathbb{C}}$ is *invertible* if it factors through the subgroupoid $\mathsf{Line}_{\mathbb{C}} \hookrightarrow \mathsf{Vect}_{\mathbb{C}}$ of complex lines and nonzero homomorphisms.

This means, for example, that all partition functions are nonzero and all state spaces are one-dimensional.

Theorem 3.2. *Let $\beta \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$. Then there is an invertible TQFT $Z_\beta^{\mathrm{cl}}\colon \mathsf{Bord}_n(B\mathbb{Z}/2) \to \mathsf{Vect}_{\mathbb{C}}$ of n-manifolds equipped with a principal $\mathbb{Z}/2$-bundle, unique up to isomorphism, such that for any closed n-manifold M and principal $\mathbb{Z}/2$-bundle $P \to M$,*

$$Z_\beta^{\mathrm{cl}}(M, P) = (-1)^{\langle \beta(M,P), [M]\rangle}, \tag{3.3}$$

where $\beta(M, P)$ denotes the pullback of β under a map $M \to B\mathrm{O}_n \times B\mathbb{Z}/2$ classifying TM and P.[6]

Proof. The assignment (3.3) is a $\{\pm 1\}$-valued bordism invariant of manifolds equipped with a principal $\mathbb{Z}/2$-bundle. Composing with the unique nonzero map $\{\pm 1\} \hookrightarrow \mathrm{U}_1$, we obtain (3.3) as a U_1-valued bordism invariant. Assume for now that the bordism group $\Omega_{n-1}(B\mathbb{Z}/2)$ of $(n-1)$-dimensional manifolds with a principal $\mathbb{Z}/2$-bundle is finitely generated; using this assumption, Yonekura [64, Theorems 4.3 and 4.4] constructs an invertible TQFT valued in $\mathsf{Line}_{\mathbb{C}}$ whose partition function recovers the bordism invariant (3.3), and proves that it is unique up to isomorphism.

Now we show $\Omega_{n-1}(B\mathbb{Z}/2)$ is finite, hence finitely generated. Consider the E^2-page of the Atiyah-Hirzebruch spectral sequence for computing this bordism group, given by $E^2_{p,q} = H_p(B\mathbb{Z}/2; \Omega_q)$. The unoriented bordism group Ω_q is a direct sum of finitely many copies of $\mathbb{Z}/2$ [57], and hence $H_p(B\mathbb{Z}/2; \Omega_q)$ is also a direct sum of finitely many copies of $\mathbb{Z}/2$. In a spectral sequence, elements can be killed by differentials, but not created, so $|E^\infty_{p,q}| \le |E^2_{p,q}|$, and hence $E^\infty_{p,q}$ is also a finite abelian group. There is a filtration on $\Omega_{n-1}(B\mathbb{Z}/2)$ whose associated graded is

$$\bigoplus_{\substack{p,q\in\mathbb{Z}\\ p+q=n-1}} E^\infty_{p,q}; \tag{3.4}$$

since this is a first-quadrant spectral sequence, all but finitely many of these groups are zero. We saw that the rest are finite, so (3.4) is a finite abelian group, and therefore $\Omega_{n-1}(B\mathbb{Z}/2)$ is too. $\square$

We call Z_β^{cl} the *classical $\mathbb{Z}/2$-gauge–gravity theory* for β, and call β the *Lagrangian* for the theory.

Remark 3.5. The name "gauge–gravity" refers to the fact that the Lagrangian β can have terms depending both on the principal $\mathbb{Z}/2$-bundle (a gauge field) and characteristic classes of the underlying manifold (which, due to the relationship between characteristic classes and curvature, are sometimes called gravitational terms). This idea also appears for the anomaly TQFTs in [27,55], which are similar to the classical gauge–gravity theories considered in this paper.

Remark 3.6. It is also possible to describe Z_β^{cl} homotopically, following the Freed-Hopkins approach to invertible TQFTs [17]; we briefly sketch the construction. If a homomorphism of commutative monoids $A \to B$ factors through the subgroup of units

[6] The classifying map is unique up to homotopy, so $\beta(M, P)$ does not depend on this choice.

$B^\times \hookrightarrow B$, then it also factors through the group completion $A \to K(A)$; in a similar way, if a morphism of symmetric monoidal categories $\mathsf{C} \to \mathsf{D}$ factors through the Picard groupoid of units $\mathsf{D}^\times \hookrightarrow \mathsf{D}$, it also factors through the groupoid completion of C, which is also a Picard groupoid. The geometric realization of a Picard groupoid G is canonically an infinite loop space, and its associated spectrum, called the *classifying spectrum* of G and denoted $|\mathsf{G}|$, is a *stable 1-type*, i.e. its only nonzero homotopy groups are $\pi_0|\mathsf{G}|$ and $\pi_1|\mathsf{G}|$ [32]. The upshot is that an invertible TQFT $Z^{\mathrm{cl}}\colon \mathsf{Bord}_n(B\mathbb{Z}/2) \to \mathsf{Line}_{\mathbb{C}}$ determines and is determined up to isomorphism by the homotopy class of the map

$$|Z^{\mathrm{cl}}|\colon |\mathsf{Bord}_n(B\mathbb{Z}/2)| \to |\mathsf{Line}_{\mathbb{C}}| \tag{3.7}$$

it induces on classifying spectra.

If E is a spectrum, let $E\langle m, n\rangle$ denote the truncation of E to a spectrum with homotopy groups only in degrees between m and n, inclusive. Then there are weak equivalences

- $|\mathsf{Bord}_n(B\mathbb{Z}/2)| \simeq (\Sigma MTO_n \wedge (B\mathbb{Z}/2)_+)\langle 0, 1\rangle$ [26,47],[7] and
- $|\mathsf{Line}_{\mathbb{C}}| \simeq \Sigma H\mathbb{C}^\times$.

Here MTO_n is a *Madsen-Tillmann spectrum*: if $V_n \to B\mathrm{O}_n$ denotes the tautological bundle, MTO_n is the Thom spectrum of $-V_n \to B\mathrm{O}_n$.

Therefore an isomorphism class of invertible n-dimensional TQFTs for manifolds with a principal $\mathbb{Z}/2$-bundle is determined by an element of

$$[(\Sigma MTO_n \wedge (B\mathbb{Z}/2)_+)\langle 0, 1\rangle, \Sigma H\mathbb{C}^\times] \cong H^0(MTO_n \wedge (B\mathbb{Z}/2)_+; \mathbb{C}^\times), \tag{3.8}$$

and $\beta \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$ yields such an element through the mod 2 Thom isomorphism followed by the map induced on cohomology by $\mathbb{Z}/2 \cong \{\pm 1\} \hookrightarrow \mathbb{C}^\times$. Thus it defines an invertible TQFT $(Z^{\mathrm{cl}}_\beta)'$ up to isomorphism. Tracing through the Pontrjagin-Thom construction, one can prove that its partition functions agree with those in (3.3), and hence by Yonekura's uniqueness result [64, Theorem 4.4], $(Z^{\mathrm{cl}}_\beta)' \cong Z^{\mathrm{cl}}_\beta$.

This approach readily generalizes to extended invertible TQFTs, as in [51], and the classical gauge–gravity TQFTs can be realized as fully extended TQFTs valued in n-algebras, as in [18, §8], or n-vector spaces, using the calculation of the classifying spectrum of the n-category of n-vector spaces in [51, §7.4].

The partition functions of the classical gauge–gravity TQFT for β resemble those of classical Dijkgraaf–Witten theory [16,19] for the gauge group $\mathbb{Z}/2$, though the Lagrangians of the former can also contain Stiefel–Whitney classes. If β factors through the inclusion $H^n(B\mathbb{Z}/2; \mathbb{Z}/2) \hookrightarrow H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$, then Z^{cl}_β is isomorphic to a classical $\mathbb{Z}/2$-Dijkgraaf–Witten theory.

If $\gamma \in H^n(B\mathbb{Z}/2; \mathbb{R}/\mathbb{Z})$, we let $\mathrm{DW}^{\mathrm{cl}}_\gamma$ denote classical $\mathbb{Z}/2$-Dijkgraaf–Witten theory with Lagrangian γ.

Proposition 3.9. *Let $f\colon \mathbb{Z}/2 \hookrightarrow \mathbb{R}/\mathbb{Z}$ denote the map sending $1 \mapsto 1/2$, as well as the map $f\colon H^*(X; \mathbb{Z}/2) \to H^*(X; \mathbb{R}/\mathbb{Z})$ it induces on cohomology. Suppose β contains no Stiefel–Whitney terms, i.e. β factors through $H^n(B\mathbb{Z}/2; \mathbb{Z}/2) \hookrightarrow H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$. Then, as TQFTs of oriented manifolds equipped with principal $\mathbb{Z}/2$-bundles, $Z^{\mathrm{cl}}_\beta \cong \mathrm{DW}^{\mathrm{cl}}_{f(\beta)}$.*

[7] This fact has been proven or sketched in several additional ways: see also [3,4,10,44,51].

Proof. Let M be a closed, oriented n-manifold, $P \to M$ be a principal $\mathbb{Z}/2$-bundle, and β be as in the proposition statement. Let $\phi\colon M \to B\mathbb{Z}/2$ be a classifying map for P. Let $[M]_{\mathbb{Z}}$, resp. $[M]_{\mathbb{Z}/2}$, denote the fundamental class of M in integral, resp. $\mathbb{Z}/2$, homology.

The partition function of classical $\mathbb{Z}/2$-Dijkgraaf–Witten theory with Lagrangian $f(\beta)$ is $\mathrm{DW}^{\mathrm{cl}}_{f(\beta)}(M, P) = \mathrm{e}^{2\pi i \langle(\phi^*(f(\beta)), [M]_{\mathbb{Z}}\rangle}$ [19, Theorem 1.7]. Naturality of the cap product under change of coefficients implies $f(\langle x, [M]_{\mathbb{Z}/2}\rangle) = \langle f(x), [M]_{\mathbb{Z}}\rangle$ for any $x \in H^n(M; \mathbb{Z}/2)$, and naturality of the change-of-coefficients map on cohomology implies that $\phi^*(f(\beta)) = f(\phi^*(\beta))$, so $f(\langle \phi^*\beta, [M]_{\mathbb{Z}/2}\rangle) = \langle \phi^*(f(\beta)), [M]_{\mathbb{Z}}\rangle$. If $a \in \mathbb{Z}/2$, $(-1)^a = \mathrm{e}^{2\pi i f(a)}$, so

$$Z_\beta(M, P) = (-1)^{\langle \phi^*\beta, [M]_{\mathbb{Z}/2}\rangle} = \mathrm{e}^{2\pi i \langle \phi^*(f(\beta)), [M]_{\mathbb{Z}}\rangle} = \mathrm{DW}^{\mathrm{cl}}_{f(\beta)}(M, P). \tag{3.10}$$

Since the partition functions for these theories are identical, then by [64, Theorem 4.4], $Z_\beta \cong \mathrm{DW}^{\mathrm{cl}}_{f(\beta)}$. $\square$

Remark 3.11. One takeaway from Proposition 3.9 is that when β contains no Stiefel–Whitney terms, Z^{cl}_β is an extension of $\mathrm{DW}^{\mathrm{cl}}_{f(\beta)}$ to unoriented manifolds. Such extensions are studied in detail by Young [66] in both the classical and quantum settings, and are examples of Minkyu Kim's generalized Dijkgraaf–Witten theories [35].

Remark 3.12. These classical gauge–gravity theories are examples of homotopy quantum field theories (HQFTs) with target space $B\mathbb{Z}/2$, and in this setting they resemble primitive cohomological HQFTs [59, § I.2.1]; again the difference is whether the cohomology class can contain Stiefel–Whitney terms. The construction of primitive cohomological HQFTs is quite direct, and it seems likely that the classical gauge–gravity theories can be constructed in a similar way.

Lemma 3.13. *Let $\gamma \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$ be a cohomology class which vanishes when pulled back to all closed n-manifolds via a classifying map for the tangent bundle and any principal $\mathbb{Z}/2$-bundle. Then, $Z^{\mathrm{cl}}_\beta \cong Z^{\mathrm{cl}}_{\beta+\gamma}$.*

Proof. By (3.3), $Z_\beta(M) = Z_{\beta+\gamma}(M)$ for all closed n-manifolds M with a principal $\mathbb{Z}/2$-bundle. We have seen that invertible TQFTs of manifolds with a principal $\mathbb{Z}/2$-bundle are determined up to isomorphism by their partition functions, so $Z_\beta \cong Z_{\beta+\gamma}$. $\square$

For example, in dimension 3, $w_1^2\alpha = w_2\alpha$ on all 3-manifolds, so $Z^{\mathrm{cl}}_{w_1^2\alpha} \cong Z^{\mathrm{cl}}_{w_2\alpha}$.

Lemma 3.14. *If n is odd, the map $f\colon H^n(B\mathbb{Z}/2; \mathbb{Z}/2) \to H^n(B\mathbb{Z}/2; \mathbb{R}/\mathbb{Z})$ is surjective.*

Proof. Associated to the short exact sequence $0 \to \mathbb{Z}/2 \xrightarrow{f} \mathbb{R}/\mathbb{Z} \xrightarrow{2} \mathbb{R}/\mathbb{Z} \to 0$, there is a long exact sequence in cohomology:

$$\begin{array}{l} H^n(B\mathbb{Z}/2; \mathbb{Z}/2) \xrightarrow{\quad f \quad} H^n(B\mathbb{Z}/2; \mathbb{R}/\mathbb{Z}) \xrightarrow{\quad g \quad} H^n(B\mathbb{Z}/2; \mathbb{R}/\mathbb{Z}) \xrightarrow{\quad h \quad} \\ \\ H^{n+1}(B\mathbb{Z}/2; \mathbb{Z}/2) \longrightarrow H^{n+1}(B\mathbb{Z}/2; \mathbb{R}/\mathbb{Z}). \end{array} \tag{3.15}$$

Since $H^{n+1}(B\mathbb{Z}/2; \mathbb{R}/\mathbb{Z}) = 0$, h is surjective. Since n is odd, both $H^n(B\mathbb{Z}/2; \mathbb{R}/\mathbb{Z})$ and $H^{n+1}(B\mathbb{Z}/2; \mathbb{Z}/2)$ are isomorphic to $\mathbb{Z}/2$, so h is a surjective map $\mathbb{Z}/2 \to \mathbb{Z}/2$, hence an isomorphism. Thus $g = 0$, so f is surjective as desired. $\square$

Corollary 3.16. *If n is odd, every classical $\mathbb{Z}/2$-Dijkgraaf–Witten theory is isomorphic to Z^{cl}_β for some $\beta \in H^n(B\mathbb{Z}/2;\mathbb{Z}/2) \hookrightarrow H^n(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2)$.*

Proof. By Lemma 3.14, when n is odd, the map $f\colon H^n(B\mathbb{Z}/2;\mathbb{Z}/2) \to H^n(B\mathbb{Z}/2;\mathbb{R}/\mathbb{Z})$ is surjective; then the result follows from Proposition 3.9. □

3.2. Discussion of the quantum theories. We construct the quantum theory Z_β using the finite path integral approach of [18, §3]; see also [45,58] for a more detailed account and [52] for a related construction. This process is also known as *orbifolding*, and the quantum theory Z_β is sometimes called the *orbifold theory* for Z^{cl}_β.

Let Gpd denote the category of spans of essentially finite groupoids: the objects of Gpd are essentially finite groupoids, and a morphism from X_1 to X_2 is data of a essentially finite groupoid Y and functors $p_1\colon Y \to X_1$ and $p_2\colon Y \to X_2$, considered up to equivalence of (Y, p_1, p_2). Let $\mathsf{Gpd}(\mathsf{Vect}_\mathbb{C})$ denote the category whose objects are pairs (X, V), where X is an essentially finite groupoid and $V \to X$ is a complex vector bundle,[8] and whose morphisms are equivalence classes of spans

$$\begin{tikzcd} & Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 & & X_2 \end{tikzcd} \tag{3.17}$$

together with data of vector bundles $V_i \to X_i$ and $W \to Y$ and morphisms $\phi_i\colon p_i^* V_i \to W$ for $i = 1, 2$. For any $y \in Y$, this morphism determines a linear map $\varphi(y)\colon V_1(p_1(y)) \to V_2(p_2(y))$ by a push–pull construction. Disjoint union of groupoids defines a symmetric monoidal structure on $\mathsf{Gpd}(\mathsf{Vect}_\mathbb{C})$.

We next define the "quantization" functor $\Sigma\colon \mathsf{Gpd}(\mathsf{Vect}_\mathbb{C}) \to \mathsf{Vect}_\mathbb{C}$, which on to an object assigns

$$\Sigma\colon (X, V) \mapsto \Gamma(V) := \varinjlim_{x\in X} V(x), \tag{3.18}$$

i.e. regard V as a $\mathsf{Vect}_\mathbb{C}$-valued diagram indexed by the category X, and take the colimit of this diagram. Given a morphism (Y, W, ϕ_1, ϕ_2) as above, the maps $\varphi(y)$ for $y \in Y$ pass to the colimit to define a map

$$\widetilde{\varphi}\colon \pi_0 Y \to \mathrm{Hom}(\Gamma(X_1, V_1), \Gamma(X_2, V_2)). \tag{3.19}$$

Then, Σ assigns to this morphism the linear map

$$\Sigma(Y, W) := \sum_{[y]\in\pi_0 Y} \frac{\widetilde{\varphi}(y)}{|\mathrm{Aut}(y)|} \in \mathrm{Hom}(\Gamma(X_1, V_1), \Gamma(X_2, V_2)). \tag{3.20}$$

This functor is symmetric monoidal [58, Theorem 5.1].

Given a TQFT $Z^{\mathrm{cl}}\colon \mathsf{Bord}_n(B\mathbb{Z}/2) \to \mathsf{Vect}_\mathbb{C}$, the functor $F_{Z^{\mathrm{cl}}}\colon \mathsf{Bord}_n \to \mathsf{Gpd}(\mathsf{Vect}_\mathbb{C})$ sending

$$F_{Z^{\mathrm{cl}}}\colon M \mapsto \Big(\mathrm{Bun}_{\mathbb{Z}/2}(M), P \mapsto Z^{\mathrm{cl}}(M, P)\Big) \tag{3.21}$$

[8] A (complex) vector bundle over a groupoid $\mathcal{G}$, denoted $V \to \mathcal{G}$, is a functor $V\colon \mathcal{G} \to \mathsf{Vect}_\mathbb{C}$, and its space of sections is $\varinjlim V$. We will always assume these vector bundles are finite-dimensional, meaning they factor through the full subcategory of finite-dimensional vector spaces.

is also symmetric monoidal [53, Theorem 3.9], and therefore the composition

$$Z\colon \mathsf{Bord}_n \xrightarrow{F_{Z^{\mathrm{cl}}}} \mathsf{Gpd}(\mathsf{Vect}_{\mathbb{C}}) \xrightarrow{\Sigma} \mathsf{Vect}_{\mathbb{C}} \tag{3.22}$$

is symmetric monoidal, i.e. a (nonextended) TQFT of unoriented manifolds.

Definition 3.23. Given a TQFT $Z^{\mathrm{cl}}\colon \mathsf{Bord}_n(B\mathbb{Z}/2) \to \mathsf{Vect}_{\mathbb{C}}$, the TQFT Z in (3.22) above is called the *quantum theory* associated to Z^{cl}. In particular, we denote the quantum theory associated to Z_β^{cl} by Z_β, and call it the *(quantum) gauge–gravity theory* for β. In this case we call β the *Lagrangian* of the theory.

Proposition 3.24 ([53, Corollary 4.4]).

(1) *Let M be a closed n-manifold. Then, the partition function $Z_\beta(M)$ is*

$$Z_\beta(M) = \sum_{[P]\in\pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)} \frac{(-1)^{\langle \beta(P),[M]\rangle}}{|\mathrm{Aut}(P)|}. \tag{3.25}$$

(2) *Let N be a closed $(n-1)$-manifold. Then, define a line bundle $L_\beta \to \mathrm{Bun}_{\mathbb{Z}/2}(N)$ which*

- *assigns $\mathbb{C}$ to every object, and*
- *assigns to an automorphism $\phi \in \mathrm{Aut}(P)$ multiplication by $Z_\beta^{\mathrm{cl}}(S^1 \times N, P_\phi)$.*

Then the state space of N is $Z_\beta(N) \cong \Gamma(L_\beta)$.

Here $P_\phi \to S^1 \times N$ denotes the *mapping torus* of ϕ, i.e. the quotient of $[0,1] \times P$ by $(0,x) \sim (1,\phi(x))$. We sketch the proof; the details can be found in [53, §§3,4].

Proof. First, part (1). The partition function for M is $Z_\beta(M\colon \varnothing \to \varnothing)$. To this bordism, $F_{Z_\beta^{\mathrm{cl}}}$ assigns a span such that for any $P \in \mathrm{Bun}_{\mathbb{Z}/2}(M)$, the induced map $\varphi(P)\colon \mathbb{C} \to \mathbb{C}$ is multiplication by the classical partition function $Z_\beta^{\mathrm{cl}}(M, P)$. Applying Σ sums this over $[P] \in \pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)$, weighted by automorphisms, giving (3.25).

Now part (2). $F_{Z_\beta^{\mathrm{cl}}}$ sends N to a line bundle $L_N \to \mathrm{Bun}_{\mathbb{Z}/2}(N)$, which to a principal $\mathbb{Z}/2$-bundle $P \to N$ assigns the complex line $Z_\beta^{\mathrm{cl}}(N, P)$. Given a morphism, let $\mathrm{Cyl}^\phi(P) \to [0,1] \times N$ denote the *mapping cylinder* of ϕ, i.e. the space $P \times [0,1] \to N \times [0,1]$, interpreted as a bordism in which P is glued by the identity at 0 and by ϕ at 1. Then,

$$\begin{aligned} L_N(\phi) &= Z_\beta^{\mathrm{cl}}([0,1]\times N, \mathrm{Cyl}^\phi(P))\colon Z_\beta^{\mathrm{cl}}(N,P) \to Z_\beta^{\mathrm{cl}}(N,P) && \text{(3.26a)}\\ &= (\text{multiplication by } Z_\beta^{\mathrm{cl}}(S^1\times N, P_\phi))\colon Z_\beta^{\mathrm{cl}}(N,P) \to Z_\beta^{\mathrm{cl}}(N,P). && \text{(3.26b)} \end{aligned}$$

. Thus $L_N \to \mathrm{Bun}_{\mathbb{Z}/2}(N)$ is isomorphic to L_β from the proposition statement, so $Z_\beta(N) = \Gamma(L_\beta)$. $\square$

The finite path integral approach to defining the quantum gauge–gravity theories means a few of their basic properties are formal corollaries of their counterparts in the classical case, because an isomorphism of classical theories determines an isomorphism of quantum theories.

Corollary 3.27. *Let $\gamma \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$ be a cohomology class which vanishes when pulled back to all closed n-manifolds via a classifying map for the tangent bundle and any principal $\mathbb{Z}/2$-bundle. Then, $Z_\beta \cong Z_{\beta+\gamma}$.*

Corollary 3.28. *Suppose β contains no Stiefel–Whitney terms (in the sense of Proposition 3.9). Then, $Z_\beta \cong \mathrm{DW}_\beta$, the quantum $\mathbb{Z}/2$-Dijkgraaf–Witten theory with Lagrangian β.*

Corollary 3.29. *If n is odd, every quantum $\mathbb{Z}/2$-Dijkgraaf–Witten theory is isomorphic to Z_β for some $\beta \in H^n(B\mathbb{Z}/2;\mathbb{Z}/2) \hookrightarrow H^n(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2)$.*

There is a new phenomenon at this level, however: one can produce β and β' whose quantum theories are isomorphic, but whose classical theories are not.

Definition 3.30. Let $\beta \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2)$, so that there are coefficients $\gamma_1,\dots,\gamma_n \in H^*(B\mathrm{O}_n;\mathbb{Z}/2)$ such that

$$\beta = \gamma_n\alpha^n + \gamma_{n-1}\alpha^{n-1} + \cdots + \gamma_1\alpha + \gamma_0, \tag{3.31}$$

where $\alpha \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$ is the generator. If $w_1 \in H^1(B\mathrm{O}_n;\mathbb{Z}/2)$ denotes the first Stiefel–Whitney class, we call

$$\beta_{w_1} := \gamma_n(\alpha+w_1)^n + \gamma_{n-1}(\alpha+w_1)^{n-1} + \cdots + \gamma_1(\alpha+w_1) + \gamma_0 \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2) \tag{3.32}$$

the *orientation-twisting* of β.

Proposition 3.33. *Let β_{w_1} be the orientation-twisting of β. Then, $Z_\beta \cong Z_{\beta_{w_1}}$.*

The idea is that replacing β with β_{w_1} corresponds to tensoring with the orientation bundle, an involution on the space of fields. Since we are summing over the fields, this does not change the path integral.

Definition 3.34. We define a tensor product of principal $\mathbb{Z}/2$-bundles induced from the tensor product of real line bundles. Given two principal $\mathbb{Z}/2$-bundles $P_1, P_2 \to M$, define a real line bundle $L(P_i) \to M$ for $i = 1, 2$ by $L(P_i) := P_i \times_{\mathbb{Z}/2} \underline{\mathbb{R}}$, where $\mathbb{Z}/2$ acts on $\underline{\mathbb{R}}$ as $\{\pm 1\}$. The Euclidean metric on $\underline{\mathbb{R}}$ induces Euclidean metrics on $L(P_1)$ and $L(P_2)$, hence also on $L(P_1)\otimes L(P_2)$; we define the *tensor product* of P_1 and P_2, denoted $P_1 \otimes P_2 \to M$, to be the unit sphere bundle in $L(P_1)\otimes L(P_2)$, which is a principal $\mathbb{Z}/2$-bundle on M.

The characteristic class of $P \otimes Q$ is $\alpha(P\otimes Q) = \alpha(P) + \alpha(Q)$.

On any manifold M, there is a canonical principal $\mathbb{Z}/2$-bundle $\mathfrak{o}_M$, called the *orientation bundle*, whose fiber at $x \in M$ is the $\mathbb{Z}/2$-torsor of orientations at x. Its characteristic class is $\alpha(\mathfrak{o}_M) = w_1(M)$.

Proof of Proposition 3.33. Let PM_n denote the subcategory of $\mathsf{Gpd}(\mathsf{Vect}_\mathbb{C})$ whose objects are vector bundles over groupoids of the form $\mathrm{Bun}_{\mathbb{Z}/2}(N)$ for some closed $(n-1)$-manifold N and whose morphisms are induced from the spans

$$\begin{tikzcd} & \mathrm{Bun}_{\mathbb{Z}/2}(M) \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_{\mathbb{Z}/2}(N_1) & & \mathrm{Bun}_{\mathbb{Z}/2}(N_2), \end{tikzcd} \tag{3.35}$$

where M is a bordism between N_1 and N_2. For any β, $F_{Z^{\mathrm{cl}}_\beta}$ lands in PM_n. To simplify notation, we will let $F_\beta := F_{Z^{\mathrm{cl}}_\beta}$.

If M is a bordism between N_1 and N_2, $(\mathfrak{o}_M)|_{N_i} = \mathfrak{o}_{N_i}$. Thus the automorphism $-\otimes\mathfrak{o}_Y\colon \mathrm{Bun}_{\mathbb{Z}/2}(Y) \to \mathrm{Bun}_{\mathbb{Z}/2}(Y)$ induces an automorphism $\Phi\colon \mathsf{PM}_n \to \mathsf{PM}_n$ as follows.

- An object of PM_n is a functor $F\colon \mathrm{Bun}_{\mathbb{Z}/2}(N) \to \mathsf{Vect}_{\mathbb{C}}$ for some $(n-1)$-manifold N. Let $\Phi(F)$ be $F \circ (-\otimes \mathfrak{o}_N)\colon \mathrm{Bun}_{\mathbb{Z}/2}(N) \to \mathrm{Bun}_{\mathbb{Z}/2}(N) \to \mathsf{Vect}_{\mathbb{C}}$.
- A morphism $F_1 \to F_2$ of PM_n is a push–pull map induced from a span as in (3.35). Since $(\mathfrak{o}_M)|_{N_i} = \mathfrak{o}_{N_i}$, the arrows in (3.35) intertwine the actions of $-\otimes \mathfrak{o}_M$ and $-\otimes \mathfrak{o}_{N_i}$, so this span induces a morphism $\Phi(F_1) \to \Phi(F_2)$ as desired.

Thus we may consider the diagram

$$\begin{tikzcd} \mathsf{Bord}_n \arrow[r, "F_\beta"] \arrow[dr, "F_{\beta_{w_1}}"'] & \mathsf{PM}_n \arrow[r, "\Sigma"] \arrow[d, "\Phi"] & \mathsf{Vect}_{\mathbb{C}} \\ & \mathsf{PM}_n, \arrow[ur, "\Sigma"'] & \end{tikzcd} \tag{3.36}$$

where the composition along the top is Z_β and the composition along the bottom is $Z_{\beta_{w_1}}$.

It suffices to prove this diagram commutes up to natural isomorphism, which means checking its two triangles.

- The left triangle commutes (up to natural isomorphism) by design, since $\alpha(P \otimes \mathfrak{o}_M) = \alpha(P) + w_1(M)$ and in β_{w_1}, we have replaced α with $\alpha + w_1$.
- The right triangle commutes because Σ takes a diagram and evaluates its colimit, and an automorphism of the indexing category does not change the value of the colimit. Hence $\Sigma(S)$ and $(\Sigma \circ \Phi)(S)$ are isomorphic for any object S, and since Φ is compatible with morphisms in PM_n, Σ and $\Sigma \circ \Phi$ also agree on morphisms. $\square$

Example 3.37. The orientation twisting of α^2 is $\alpha^2 + w_1^2$. The classical theories $Z^{\mathrm{cl}}_{\alpha^2}$ and $Z^{\mathrm{cl}}_{\alpha^2+w_1^2}$ are nonisomorphic; for example, they disagree on $\mathbb{RP}^2$ with the trivial principal $\mathbb{Z}/2$-bundle. But by Proposition 3.33, their quantum theories are isomorphic.

Remark 3.38. Lu–Vishwanath [43] observe a similar phenomenon in the physics of topological phases enriched by a global $\mathbb{Z}/2$-symmetry, in which distinct phases become equivalent after gauging the $\mathbb{Z}/2$ symmetry.

4. Low-Energy Limits

In this section, we return to the lattice, and investigate the spaces of ground states of the toric code and GDS models on closed $(n-1)$-manifolds. In both cases, we find a TQFT Z whose state space on M is isomorphic to the space of ground states of the lattice model on M.

4.1. Generalities.

Definition 4.1. Consider a lattice model which to all closed $(n-1)$-manifolds M together with some kind of lattice Π (e.g. a triangulation or a CW structure) associates a complex Hilbert space $\mathcal{H}_{M,\Pi}$ and a self-adjoint operator $H_{M,\Pi}\colon \mathcal{H}_{M,\Pi} \to \mathcal{H}_{M,\Pi}$ (respectively the state space and the Hamiltonian). In this setting, elements of $\ker(H_{M,\Pi})$ are called *ground states*. Assume that we can construct an action of $\mathrm{Diff}(M)$ on $\ker(H_{M,\Pi})$ from the data of the lattice model.

Let $Z\colon \mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}$ be a TQFT. We say that Z *captures the ground states* of the lattice model if for all closed $(n-1)$-manifolds M with a lattice Π, there is an isomorphism $Z(M) \cong \ker(H_{M,\Pi})$ intertwining the $\mathrm{Diff}(M)$-actions.

In the rest of this subsection, we discuss these $\mathrm{Diff}(M)$-actions. In Sect. 4.1.1, we recall the definition of the $\mathrm{Diff}(M)$-action on $Z(M)$, and in Sect. 4.1.2, we address the assumption of the $\mathrm{Diff}(M)$-action on $\ker(H_{M,\Pi})$, showing how to construct such an action given certain data present in the toric code and GDS models.

Let $\mathrm{Diff}_0(M) \subset \mathrm{Diff}(M)$ denote the connected component of the identity. The $\mathrm{Diff}(M)$-action on $Z(M)$ is trivial when restricted to $\mathrm{Diff}_0(M)$, hence induces an action of the *mapping class group* $\mathrm{MCG}(M) := \mathrm{Diff}(M)/\mathrm{Diff}_0(M)$. Thus, if Z captures the ground states of the lattice model, the $\mathrm{Diff}(M)$-action on $\ker(H_{M,\Pi})$ must also induce a mapping class group action in the same way.

Remark 4.2. When Z captures the ground states of a lattice model, it is believed to correspond to the physics notion of the low-energy effective theory of the model. The existence of such a low-energy TQFT for certain lattice models, called topological phases, is predicted by physics,[9] and the low-energy TQFT is expected to determine the lattice model up to some physically meaningful notion of equivalence; this correspondence is discussed in [17,20,25,50].

However, there is much left to understand, especially at a mathematical level of rigor. We do not intend for Definition 4.1 to be a mathematical definition of the physical notion of the low-energy effective theory of a lattice model. Providing such a mathematical definition is a major open question; as is, Definition 4.1 fails to address uniqueness (as shown in Remark 4.60) and existence (due to fracton phases; see, e.g., [7,29,65]).

4.1.1. The mapping class group action for TQFTs For any $\varphi \in \mathrm{Diff}(M)$, let C_φ denote the *mapping cylinder* of φ, i.e. the cobordism $[0, 1] \times M$ from M to itself, where M is attached via the identity at 0 and via φ at 1.

If $Z\colon \mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}$ is a TQFT, then the assignment $\varphi \mapsto Z(C_\varphi)\colon Z(M) \to Z(M)$ defines an action of $\mathrm{Diff}(M)$ on $Z(M)$. If $\varphi \in \mathrm{Diff}_0(M)$, then there is a smooth isotopy $\varphi_t\colon [0, 1] \times M \to M$ such that $\varphi_t(0, x) = x$ and $\varphi_t(1, x) = \varphi(x)$, and in particular there is a diffeomorphism of cobordisms $C_{\mathrm{id}} \cong C_\varphi$ defined by the map

$$\begin{aligned} [0,1] \times M &\to [0,1] \times M \\ (t, x) &\mapsto (t, \varphi_t(x)). \end{aligned} \tag{4.3}$$

Therefore $Z(C_\varphi) = Z(C_{\mathrm{id}}) = \mathrm{id}$, so this $\mathrm{Diff}(M)$-action is trivial on $\mathrm{Diff}_0(M)$, hence defines an $\mathrm{MCG}(M)$-action on $Z(M)$.

4.1.2. The $\mathrm{Diff}(M)$*-action for a lattice model* We will imitate the first half of the above argument for a lattice model with some assumptions, constructing a $\mathrm{Diff}(M)$-action on the space of ground states of the model on M; in Sects. 4.2.2 and 4.3.3, we will see that for the toric code and GDS models, these are trivial when restricted to $\mathrm{Diff}_0(M)$, defining actions of the mapping class group on the spaces of ground states of the toric code and GDS models.

We require the following of our lattice model.

(A1) The model is defined for closed $(n-1)$-manifolds equipped with a lattice, which here means a CW structure or a triangulation, or one of these structures subject to some condition that can be satisfied on all closed $(n-1)$-manifolds and for which any two such structures on a manifold admit a common refinement.

[9] One should allow TQFTs tensored with an invertible, non-topological theory, as in [17, §5.4]. The TQFTs we find in this paper are topological, so this distinction will not matter here.

(A2) Given a closed manifold M, a diffeomorphism $f\colon M \to M$ and a lattice Π on M, let $f(\Pi)$ denote the lattice obtained by postcomposing the attaching maps in Π with f. We ask for f to induce an isomorphism f_* from the state space of the model for Π to the state space of the model for $f(\Pi)$, for f_* to intertwine the Hamiltonians of these models, and for this to be functorial under composition of diffeomorphisms.

(A3) Data of, for every refinement $\Pi \to \Pi'$ of lattices, an isomorphism from the space of low-energy states of the model on Π to the space of low-energy states of the model on Π', which is functorial under composition of refinements, and which is compatible with the maps f_* in (A2).

Examples of conditions satisfying the constraint in (A1) include regular CW complexes and the class of smooth triangulations we considered when defining the GDS model.

With these assumptions in place, we define a category $\mathsf{Lat}(M)$ whose objects are the lattices on a closed manifold M and whose morphisms are generated by refinements and diffeomorphisms. Specifically, we add a morphism $r_{\Pi,\Pi'}\colon \Pi \to \Pi'$ for each refinement $\Pi \to \Pi'$, and for each diffeomorphism $f\colon M \to M$ we add a morphism $f_*\colon \Pi \to f(\Pi)$. These morphisms are subject to the relations establishing functoriality under composition of diffeomorphisms and under composition of refinements, and that $f_* \circ r_{\Pi,\Pi'} = r_{f(\Pi),f(\Pi')} \circ f_*$.

Then (A2) and (A3) define a functor $L\colon \mathsf{Lat}(M) \to \mathsf{Vect}_{\mathbb{C}}$ sending a lattice Π to the space of low-energy states of the model on Π; let $Z(M) := \varinjlim L$. Let $f \in \mathrm{Diff}(M)$. If $r_{\Pi,\Pi'}\colon \Pi \to \Pi'$ is a refinement, the fact that $f_* \circ r_{\Pi,\Pi'} = r_{f(\Pi),f(\Pi')} \circ f_*$ means that the action of f_* passes to the colimit, defining a map $f_*\colon Z(M) \to Z(M)$, and this is functorial with respect to diffeomorphisms, defining a $\mathrm{Diff}(M)$-action on $Z(M)$.

4.2. Review for the toric code. As a warmup, before tackling the GDS model, we determine a TQFT which captures the ground states of the toric code. Neither the answer nor this perspective on it are new.

Theorem 4.4. *Let* $\mathrm{DW}_0\colon \mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}$ *denote the* $\mathbb{Z}/2$*-Dijkgraaf–Witten theory with Lagrangian equal to* 0*. Then* DW_0 *captures the ground states of the toric code.*

Remark 4.5. This is not a new result. Because researchers consider different formulations of the toric code, there are some analogues of Theorem 4.4 in the literature for different classes of toric code models, e.g. in [5,11,37]. Though these results do not cover Theorem 4.4 in the case $n > 3$, it and its proof were certainly known before this paper.

Our proof of Theorem 4.4 will be slightly more complicated than necessary. This is so that it follows the same line of argument as the proof for the GDS model in Sect. 4.3. We hope that presenting the simpler example first makes the GDS example easier to understand.

Before we prove Theorem 4.4, we must define the $\mathrm{Diff}(M)$-action on the space of ground states of the toric code on M. First, though, in Sect. 4.2.1, we show the space of ground states on M is isomorphic as vector spaces to $\mathrm{DW}_0(M)$. Then, in Sect. 4.2.2, we use the argument of Sect. 4.1.2 to produce a $\mathrm{Diff}(M)$-action on the space of ground states on M, compare it with the $\mathrm{MCG}(M)$-action on $\mathrm{DW}_0(M)$, and conclude.

4.2.1. Identifying the vector spaces for the toric code Our goal is to prove the following proposition.

Proposition 4.6. *For a closed manifold M, the space of ground states of the toric code on M is isomorphic as vector spaces to* $\mathrm{DW}_0(M)$.

We can use the fact that the vertex and face operators commute to simplify our analysis of the Hamiltonian.

Lemma 4.7. *Let V be a vector space over a field k, and let $\Phi = \sum_{i=1}^m \phi_i$ be a finite sum of commuting projections $\phi_i \in \mathrm{End}_k(V)$. Then,* $\ker(\Phi) = \bigcap_{i=1}^m \ker(\phi_i)$.

Proof. By induction, it suffices to consider $m = 2$, so $\Phi = \phi_1 + \phi_2$. Clearly $\ker(\phi_1) \cap \ker(\phi_2) \subset \ker(\Phi)$, so assume $\Phi x = 0$ for some $x \in V$. Thus $\phi_1 x = -\phi_2 x$, so $\phi_1 x = \phi_1^2 x = -\phi_1\phi_2 x = -\phi_2(\phi_1 x)$, so $\phi_1 x$ is an eigenvector for ϕ_2 with eigenvalue -1. This means $\phi_2^2(\phi_1 x) = (-1)^2\phi_1 x = \phi_1 x$, and since ϕ_2 is a projection, $\phi_2^2(\phi_1 x) = \phi_2\phi_1 x = -\phi_1 x$, forcing $\phi_1 x = 0$. Since $\phi_2 = \Phi - \phi_1$, then $\phi_2 x = 0$ as well. □

Proof of Proposition 4.6. Let M be a closed manifold with a CW structure Ξ. As before, we will write (P, ξ) for an object of $\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$, meaning that $P \to M^1$ is a principal $\mathbb{Z}/2$-bundle and $\xi\colon M^0 \to P|_{M^0}$ is a trivialization of P over M^0.

By Lemma 4.7, the ground states of the toric code for M are those functions ψ on $\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$ such that $H_v\psi = 0$ for all 0-cells v and $H_f\psi = 0$ for all 2-cells f.

Let f be a 2-cell. Then, $H_f\psi = 0$ if and only if $B_f\psi = \psi$, or for all $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$, $(-1)^{\mathrm{Hol}_P(f)}\psi(P, \xi) = \psi(P, \xi)$. That is, either $\psi(P, \xi) = 0$ or $\mathrm{Hol}_P(f) = 0$, so ψ must vanish on all principal $\mathbb{Z}/2$-bundles with nontrivial holonomy around ∂f. Hence if $\psi \in \ker(H_f)$ for all 2-cells f, it can only be nonzero on the principal $\mathbb{Z}/2$-bundles with no holonomy around the boundary of any 2-cell, which are exactly the principal $\mathbb{Z}/2$-bundles which extend to M^2, hence to all of M, and such an extension is necessarily unique. That is, $\bigcap_f \ker(H_f)$ is the space of functions on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$.

Let $\mathcal{A} := C^0_\Xi(M; \mathbb{Z}/2)$ denote the group of cellular 0-cochains. We will describe the ground states of the toric code for M as invariant sections of an $\mathcal{A}$-equivariant line bundle on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$, then take the quotient by $\mathcal{A}$. For $v \in \Delta^0(M)$, let $\delta_v \in \mathcal{A}$ be the function equal to 1 on v and 0 elsewhere. Then, $\mathcal{A}$ has a presentation by the following generators and relations:

$$\mathcal{A} \cong \langle \delta_v \text{ for all } v \in \Delta^0(M) \mid \delta_v^2, [\delta_v, \delta_w]\rangle, \tag{4.8}$$

so an $\mathcal{A}$-action is the same data as commuting involutions associated to each δ_v. For example, $\mathcal{A}$ acts on the (discrete) groupoid $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ through the commuting involutions

$$\delta_v\colon (P, \xi) \mapsto (P, (w \mapsto \xi(w) + \delta_v(w))). \tag{4.9}$$

Consider the trivial line bundle $\underline{\mathbb{C}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ and give it the trivial $\mathcal{A}$-action. We can identify sections of $\underline{\mathbb{C}}$ with functions on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$, and the $\mathcal{A}$-actions match; in particular, if $\psi \in \Gamma(\underline{\mathbb{C}})$ and v is a 0-cell, then $\delta_v \cdot \psi = A_v\psi$. Therefore ψ is invariant under the $\mathcal{A}$-action if and only if $A_v\psi = \psi$ for all v, i.e. $H_v\psi = 0$ for all v. That is, the space of ground states is the space of $\mathcal{A}$-invariant sections of $\underline{\mathbb{C}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$.

The $\mathcal{A}$-equivariant line bundle $\underline{\mathbb{C}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ descends to a nonequivariant line bundle on the groupoid quotient $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)/\mathcal{A}$; since we began with the trivial $\mathcal{A}$-action, this will also be a trivial line bundle. Therefore it suffices to identify the quotient. □

Lemma 4.10. *The map* $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)/\mathcal{A} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$ *which forgets the trivialization is an equivalence of groupoids. Given* $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ *and* $\phi \in \mathrm{Aut}(P)$*, action by*

$$t_\phi := \sum_{\substack{v \in \Delta^0(M) \\ \phi|_v \mathit{nontrivial}}} \delta_v \in \mathcal{A} \tag{4.11}$$

on (P, ξ) *passes to* ϕ *in the quotient.*

Proof. $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ is a discrete groupoid, so we just have to determine the stabilizer subgroup for the $\mathcal{A}$-action. An automorphism ϕ of P switches the trivializations wherever ϕ is nontrivial, so defines an isomorphism $(P, \xi) \xrightarrow{\cong} (P, t_\phi \cdot \xi)$. To check these are the only isomorphisms that occur, suppose $(P, \xi) \cong (P, t \cdot \xi)$ for some $t \in \mathcal{A}$. Since the function $\mathrm{spin}_{(P,\xi)}$ is an isomorphism invariant of $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$, t must be the sum of δ_v as v ranges over a set S of 0-cells such that every 1-cell of M bounds an even number of 0-cells in S. Thus for any connected component M_0 of M, S includes either all 0-cells of M_0 or none, so t is realized by some t_ϕ. □

Therefore the space of ground states on M is the space of sections of $\underline{\mathbb{C}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$, i.e. the space of functions on $\mathrm{Bun}_{\mathbb{Z}/2}(M)$, which is what DW_0 assigns to M. □

4.2.2. The $\mathrm{MCG}(M)$*-action for the toric code* Recall the axioms (A1)–(A3) from Sect. 4.1.2 that allow us to produce a $\mathrm{Diff}(M)$-action on the space of ground states on M. It is clear how to satisfy (A1) and (A2); turning to (A3), a refinement $\varphi\colon \Xi \to \Xi'$ of CW structures on M induces a pullback map

$$\varphi^*\colon \mathrm{Bun}_{\mathbb{Z}/2}(M^1_{\Xi'}, M^0_{\Xi'}) \to \mathrm{Bun}_{\mathbb{Z}/2}(M^1_{\Xi}, M^0_{\Xi}). \tag{4.12}$$

hence a pushforward map on state spaces: $\varphi_*\colon \mathcal{H}(\Xi) \to \mathcal{H}(\Xi')$.

Remark 4.13. The pushforward φ_* does not restrict to an isomorphism on the spaces of ground states. Consider the refinement $\Xi \to \Xi'$ in Fig. 4 and (P, ξ) which induce the indicated spins on the 1-cells of Ξ'. If f is a ground state for Ξ', it must vanish on (P, ξ), because (P, ξ) has nontrivial holonomy around the boundaries of the pictured 2-cells, but pulled back to Ξ, this is no longer the case. Therefore $\mathrm{Im}(\varphi_*)$ contains states which do not vanish on (P, ξ), hence are not ground states.

The issue is that functions in the image of φ_* may not vanish on bundles with nontrivial holonomy around certain boundaries of 2-cells, so in order to satisfy (A3), we zero out their values on any such bundle. Let $\mathcal{P}\colon \mathcal{H}_{\Xi'} \to \mathcal{H}_{\Xi'}$ denote this projection: that is, if $f \in \mathcal{H}_{\Xi'}$ and $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1_{\Xi'}, M^0_{\Xi'})$, let

$$(\mathcal{P}f)(P, \xi) := \begin{cases} f(P, \xi), & \text{if } \mathrm{Hol}_P(e) = 0 \text{ for all } e \in \Delta^2(M; \Xi'), \\ 0, & \text{otherwise.} \end{cases} \tag{4.14}$$

Lemma 4.15. *The map* $\mathcal{P} \circ \varphi_*$ *sends ground states to ground states, hence restricts to an isomorphism* $L(\Xi) \xrightarrow{\cong} L(\Xi')$ *functorial in the sense of (A3), and this is compatible with the maps in (A2).*

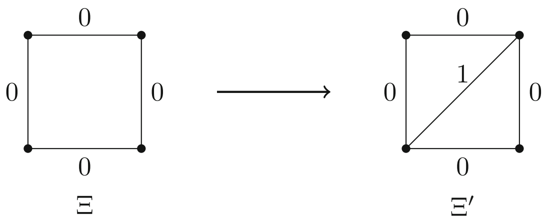

Fig. 4. Consider a refinement $\Xi \to \Xi'$ of CW structures as above, together with $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1_{\Xi'}, M^0_{\Xi'})$ such that the labels on the 1-simplices represent $\mathrm{spin}_{(P,\xi)}$, as in Remark 2.34. In Remark 4.13, we discuss how (P, ξ) illustrates a subtlety in defining the map from the ground states of the toric code for Ξ to those on Ξ'

Proof. Let $f \in L(\Xi)$. By construction $\mathcal{P}(\varphi_*(f))$ vanishes on principal $\mathbb{Z}/2$-bundles with nontrivial holonomy, so it suffices to check that it does not depend on the trivializations on the 0-cells. This is not changed by $\mathcal{P}$, so we can just think about $\varphi_*(f)$. Let $v \in \Delta^0(M, \Xi')$ and suppose v is also a 0-cell of Ξ. Then $\varphi_*(f)$ cannot depend on the trivialization at v, because f does not depend on the trivialization at v. If instead v is not a 0-cell of Ξ, so is created by the refinement, then $\varphi_*(f)$ also does not depend on the trivialization at v, because $\varphi_*(f)(P, \xi)$ is computed by pulling back to Ξ, where v is not a cell. $\square$

Therefore the argument of Sect. 4.1.2 applies to define for any closed $(n-1)$-manifold M an action of $\mathrm{Diff}(M)$ on the ground states of the toric code. Under the identification of this space with $\mathbb{C}[\mathrm{Bun}_{\mathbb{Z}/2}(M)]$, this representation is the one induced from the usual $\mathrm{Diff}(M)$-action on $\pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M) \cong H^1(M; \mathbb{Z}/2)$, which is trivial on the subgroup $\mathrm{Diff}_0(M)$ and therefore defines an action of the mapping class group.

Recall from Proposition 4.6 that for any closed $(n-1)$-manifold M, the state space of $\mathbb{Z}/2$-Dijkgraaf–Witten theory with Lagrangian equal to 0 on M, denoted $\mathrm{DW}_0(M)$, is isomorphic to the space of ground states of the toric code on M. Explicitly, $\mathrm{DW}_0(M) \cong \mathbb{C}[\mathrm{Bun}_{\mathbb{Z}/2}(M)]$, and DW_0 assigns to a cobordism a push–pull map, which implies that the $\mathrm{MCG}(M)$-action on $\mathrm{DW}_0(M)$ is also the action induced from the standard action on $\pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)$. Therefore the identification of the space of ground states of the toric code for M with $\mathrm{DW}_0(M)$ in Proposition 4.6 is equivariant with respect to the $\mathrm{MCG}(M)$-actions on both sides, proving Theorem 4.4.

Remark 4.16. The mapping class group action determines the partition functions of mapping tori: if $f \in \mathrm{MCG}(M)$, then $Z(M_f)$ is the trace of f acting on $Z(M)$. Though we can see these partition functions from the lattice, it is not clear in general how to extend this to arbitrary closed n-manifolds.

4.3. Derivation of the generalized double semion Lagrangian. We now answer the main question of this paper: identifying a TQFT whose state spaces are isomorphic to the spaces of ground states of the GDS model.

Definition 4.17. Fix a dimension n. Let $\alpha \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ denote the generator and $w \in H^*(B\mathrm{O}_n; \mathbb{Z}/2)$ denote the total Stiefel–Whitney class. In $H^*(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$, α is nilpotent, so $1+\alpha$ is invertible, and we can consider $w\alpha/(1+\alpha) \in H^*(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$, which is a sum of homogeneous elements of different degrees. Let β denote the degree-n summand of $w\alpha/(1+\alpha)$. We let $Z_{\mathrm{GDS}} \colon \mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}$ denote the quantum gauge–gravity theory Z_β from Definition 3.23; the dimension n will be clear from context when needed.

Our goal in this section is to prove the following.

Theorem 4.18. *The TQFT Z_{GDS} captures the ground states of the GDS model.*

As with the toric code, we first establish an isomorphism of vector spaces in Sects. 4.3.1 and 4.3.2. Then, in Sect. 4.3.3, we invoke the argument of Sect. 4.1.2 to define the $\mathrm{Diff}(M)$-action on the space of ground states of the GDS model on a closed manifold M and compare it with the action on Z_{GDS}, finishing the proof of Theorem 4.18.

4.3.1. Defining $L_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$ Our first goal is to prove the following theorem.

Theorem 4.19. *For a closed manifold M, the space of ground states of the GDS model on M is isomorphic as vector spaces to $Z_{\mathrm{GDS}}(M)$.*

Let M be a closed $(n-1)$-manifold with a smooth triangulation Π; as in Sect. 2.2, we assume the 0-clopen star of any vertex is contractible. We will prove Theorem 4.19 by identifying the ground states of the GDS model on M with the space of sections of a line bundle $L_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$ defined below. Proposition 3.24 identifies $Z_{\mathrm{GDS}}(M)$ with the sections of another line bundle $L_\beta \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$, and we will show that $L_{\mathrm{GDS}} \cong L_\beta$.

The commutativity relations for the operators in the GDS model are more complicated than those for the toric code, but we can still understand the spaces of ground states in terms of the vertex and face operators.

Lemma 4.20. *With V as in Lemma 4.7, let $\phi_i, \psi_j \in \mathrm{End}_k(V)$ and suppose*

$$H = \underbrace{\sum_{i=1}^{\ell} \phi_i}_{\Phi} + \underbrace{\sum_{j=1}^{m} \psi_j}_{\Psi}, \tag{4.21}$$

such that for all i and j,

(1) *ϕ_i and ψ_j are projections,*
(2) *$[\phi_i, \phi_j] = 0$,*
(3) *$[\phi_i, \psi_j] = 0$,*
(4) *for any $x \in \ker(\Phi)$, $[\psi_i, \psi_j]x = 0$.*

Then,

$$\ker(H) = \bigcap_{j=1}^{m} \ker(\psi_j \colon \ker(\Phi) \to \ker(\Phi)). \tag{4.22}$$

Proof. Lemma 4.7 tells us $\ker(H) = \ker(\Phi) \cap \ker(\Psi)$, so it suffices to restrict to $\ker(\Phi)$. Since ϕ_i and ψ_j commute, then $\psi_j(\ker \Phi) \subset \ker \Phi$ for each j, so we may consider ψ_j as an operator on $\ker(\Phi)$. Restricted to this subspace, $[\psi_i, \psi_j] = 0$, so we apply Lemma 4.7 again to conclude. □

The upshot is that for a Hamiltonian whose smallest eigenvalue is 0 and which is a sum of vertex and face operators satisfying the commutativity conditions in Lemma 4.20, the space of ground states can be computed by finding the $f \in \mathcal{H}$ with $\phi_i f = 0$ for all i, then taking the subspace of those such that $\psi_j f = 0$ for all j. By Lemmas 2.29 and Lemma 2.35, the vertex and face operators for the GDS model satisfy the commutation relations in Lemma 4.20, where the ϕ_i are the face operators and the ψ_j are the vertex operators, so we will use this method to find the space of ground states.

The first part of the derivation is to determine $\bigcap_f \ker(H_f)$. The H_f operators in the GDS model are the same as in the toric code, so the derivation proceeds as for the toric code (the first part of the proof of Theorem 4.4) to produce the space of functions on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$.

Next, we will use the vertex operators to define $L_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$ and characterize the ground states on M as its space of sections. Specifically, letting $\mathcal{A} := C^0_\Pi(M; \mathbb{Z}/2)$ as in the previous section, we will describe an $\mathcal{A}$-equivariant line bundle on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ whose invariant sections are the ground states, then let $L_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$ denote the induced bundle on the quotient.

Definition 4.23. First, we define the $\mathcal{A}$-equivariant line bundle $L'_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$. Begin with the trivial (nonequivariant) line bundle $\underline{\mathbb{C}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$, and give it an $\mathcal{A}$-action as follows: if $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ and $z \in \mathbb{C}$, let

$$\delta_v \colon ((P, \xi), z) \mapsto (\delta_v \cdot (P, \xi), \sigma(v, (P, \xi))z), \tag{4.24}$$

where $\sigma(v, (P, \xi))$ is the GDS sign from (2.21). By Lemmas 2.29 and 2.35, the actions of δ_{v_1} and δ_{v_2} on $\underline{\mathbb{C}}$ commute for 0-cells v_1 and v_2, so (4.24) defines an $\mathcal{A}$-action covering the $\mathcal{A}$-action on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$.

Identifying functions on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ with sections of the trivial line bundle, hence of $L'_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$, a section ψ is invariant under the $\mathcal{A}$-action if and only if $\psi \in \ker(\widetilde{H}_v)$ for all $v \in \Delta^0(M)$; hence, by Lemma 4.20, this identifies the ground states of the GDS model for M with the space $\Gamma(L'_{\mathrm{GDS}})^{\mathcal{A}}$ of invariant sections of L'_{GDS}. By Lemma 4.10, $L'_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ descends to a (nonequivariant) line bundle $L_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$, and there is an isomorphism $\Gamma(L'_{\mathrm{GDS}})^{\mathcal{A}} \cong \Gamma(L_{\mathrm{GDS}})$, so the space of ground states of the GDS model is isomorphic to $\Gamma(L_{\mathrm{GDS}})$.

4.3.2. Computing the isomorphism type of L_{GDS} Given a principal $\mathbb{Z}/2$-bundle $P \to M$, the action of $\mathrm{Aut}(P)$ on $(L_{\mathrm{GDS}})_P$ is a character of $\mathrm{Aut}(P)$, and the data of these characters for all $P \in \pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)$ determines L_{GDS} up to isomorphism. In this section, we compute these characters, describing the answer in Corollary 4.57.

Let $P \to M$ be a principal $\mathbb{Z}/2$-bundle and $\phi \in \mathrm{Aut}(P)$. Let $\mathcal{V}$ denote the set of vertices on which ϕ is nontrivial, and order this set as $\{v_1, \ldots, v_m\}$. Fix a trivialization ξ_0 of $P|_{M^0}$ and let

$$\xi_i := \delta_{v_i} \cdot (\delta_{v_{i-1}} \cdot (\cdots (\delta_{v_1} \cdot \xi_0) \cdots)). \tag{4.25}$$

In Lemma 4.10, we identified the action of ϕ on L_{GDS} with the action of t_ϕ on L'_{GDS}, which is multiplication by

$$\sigma_{\mathcal{V}} := \prod_{i=1}^m \sigma(v_i, (P, \xi_i)). \tag{4.26}$$

To compare L_{GDS} and L_β, we need to pass from this description of $\sigma_{\mathcal{V}}$ in terms of simplices to a description only depending on M and P. The following theorem makes this transition; afterwards we use characteristic classes to finish the calculation.

As in Proposition 3.24, let $P_\phi \to S^1 \times M$ denote the mapping torus of ϕ.

Theorem 4.27. *Let $N \subset S^1 \times M$ be an embedded submanifold representing the Poincaré dual to $\alpha(P_\phi) \in H^1(S^1 \times M; \mathbb{Z}/2)$. Then $\sigma_{\mathcal{V}} = (-1)^{\chi(N)}$.*

Our proof has two parts.

(1) First, the simplicial part: we construct an $(n-1)$-cycle C on $S^1 \times M$, cellular with respect to a certain CW structure, which represents the Poincaré dual of $\alpha(P_\phi)$ (Lemma 4.33) and such that if $|C|$ denotes the geometric realization of C, then $\sigma_{\mathcal{V}} = (-1)^{\chi(|C|)}$ (Proposition 4.36).
(2) Then, we show that replacing $|C|$ with a smoothly embedded representative of the homology class of C does not change the mod 2 Euler characteristic (Proposition 4.46).

The proof employs the dual CW structure $\Pi^\vee$ to the given triangulation Π; see Remark 2.24 for more information. Let $S^1(m)$ denote the simplicial structure on S^1 with m vertices, and choose an identification of the vertices with $\mathbb{Z}/m$ such that i and $i+1 \bmod m$ share an edge for each i. Then let $S^1(m) \times \Pi^\vee$ denote the product CW structure.

For any $i \in \mathbb{Z}/m$, the cellular 1-cochain $\mathrm{spin}_{(P,\xi_i)} \colon \Delta^1(M; \Pi) \to \mathbb{Z}/2$ is a cocycle representative for $\alpha(P) \in H^1(M; \mathbb{Z}/2)$, and therefore

$$Y_i := \{e^\vee \mid e \in \Delta^1(M; \Pi) \text{ and } \mathrm{spin}_{(P,\xi_i)}(e) = 1\} \subset \Delta^{n-2}(M; \Pi^\vee) \tag{4.28}$$

is a cellular $(n-2)$-cycle representative for the Poincaré dual of $\alpha(P)$ in $H_{n-2}(M; \mathbb{Z}/2)$. From the definitions of Y_i and of ξ_i (4.25) we see that

$$Y_i = Y_{i-1} + \partial v_i^\vee, \tag{4.29}$$

where $i-1$ is interpreted in $\mathbb{Z}/m$, and that

$$C := \sum_{i \in \mathbb{Z}/m} \left((i, i+1) \times Y_i + \{i\} \times v_i^\vee\right) \subset \Delta^n(S^1 \times M; S^1(m) \times \Pi^\vee) \tag{4.30}$$

is a cellular $(n-1)$-cycle on $S^1 \times M$.

Definition 4.31. If $P \to M$ is a principal $\mathbb{Z}/2$-bundle over a closed manifold M, there is an isomorphism $\mathrm{Aut}(P) \to H^0(M; \mathbb{Z}/2)$ sending $\phi \in \mathrm{Aut}(P)$ to the function on $\pi_0(M)$ which is 0 on a connected component if ϕ is trivial there and 1 if ϕ is nontrivial there. The image of $\phi \in \mathrm{Aut}(P)$ under this isomorphism is denoted $[\phi]$.

For example, if $x \in H^1(S^1; \mathbb{Z}/2)$ denotes the generator, then

$$\alpha(P_\phi) = \alpha(P) + x[\phi] \in H^1(S^1 \times M; \mathbb{Z}/2). \tag{4.32}$$

Lemma 4.33. *The homology class C represents is the Poincaré dual of $\alpha(P_\phi) \in H^1(S^1 \times M; \mathbb{Z}/2)$.*

Proof. Recall that $Y_0 \subset \Delta^{n-2}(M; \Pi^\vee)$ is a cellular $(n-2)$-cycle representing the Poincaré dual of $\alpha(P) \in H^1(M; \mathbb{Z}/2)$. The $(n-1)$-cycle in $S^1 \times M$ defined to be the set of $(n-1)$-cells of

$$(S^1 \times |Y_0|) \cup \bigcup_{\substack{M_i \in \pi_0(M) \\ [\phi](M_i)=1}} \{0\} \times M_i \tag{4.34}$$

represents the Poincaré dual to $\alpha(P) + x[\phi] = \alpha(P_\phi)$ (4.32), and is homologous to C in $Z_{n-1}^{S^1(m)\times\Pi^\vee}(S^1 \times M; \mathbb{Z}/2)$ by adding boundaries of the form $\partial((0, i) \times v_i^\vee)$. $\square$

Lemma 4.35. *For $1 \le i \le m$, let $Z_{v_i}(P, \xi_i)$ be as in Proposition 2.37. Then $\#(\overline{Y_i} \cap \partial v_i^\vee) = \#(Z_{v_i}(P, \xi_i))$ and therefore $(-1)^{1+\chi(|Y_i|\cap\partial v_i^\vee)} = \sigma(v_i, (P, \xi_i))$.*

Proof. This is a matter of unwinding the definitions: $c \in \overline{Y_i} \cap \partial v_i^\vee$ means that $v_i \in \partial c^\vee$ and either

(1) c is an $(n-2)$-cell and $\mathrm{spin}_{(P,\xi_i)}(c^\vee) = 1$, or
(2) there is an $(n-2)$-cell $e \in Y_i$ with $c \in \partial e$, i.e. $\mathrm{spin}_{(P,\xi_i)}(e^\vee) = 1$ and $e^\vee \in \partial c^\vee$.

These are exactly the conditions for $c^\vee$ to be in $Z_{v_i}(P, \xi_i)$, so $\#(\overline{Y_i} \cap \partial v_i) = \#(Z_{v_i}(P, \xi_i))$, and the rest of the conclusion then follows from Proposition 2.37. $\square$

Proposition 4.36. $(-1)^{\chi(|C|)} = \sigma_{\mathcal{V}}$.

Proof. The projection map $\pi\colon S^1 \times M \twoheadrightarrow S^1$ is cellular with respect to $S^1(m) \times \Pi^\vee$ and $S^1(m)$; if $D_i := |C| \cap \pi^{-1}([i, i+1))$, then each D_i is a union of cells and

$$|C| = \coprod_{i\in\mathbb{Z}/m} D_i. \tag{4.37}$$

Define A_i and B_i by $\pi^{-1}(\{i\}) = \{i\} \times A_i$ and $\pi^{-1}((i, i+1)) = (i, i+1) \times B_i$; A_i and B_i are also unions of cells. Then

$$A_i = |Y_i| \cup |Y_{i-1}| \cup |v_i^\vee| = |Y_i| \cup |v_i^\vee| \tag{4.38a}$$

because $Y_{i-1} = Y_i + \partial v_i^\vee$ (4.29), and

$$B_i = |Y_i|. \tag{4.38b}$$

Therefore

$$\begin{aligned}
\#(\text{cells of } D_i) &= \#(\text{cells of } A_i) + \#(\text{cells of } B_i) \\
&= \chi(|Y_i| \cup |v_i^\vee|) + \chi(|Y_i|) \\
&= \chi(|Y_i| \cup \mathrm{int}(|v_i^\vee|) \cup |\partial v_i^\vee|) + \chi(|Y_i|) \\
&= 1 + \chi(|Y_i|) + \chi(|\partial v_i^\vee|) - \chi(|Y_i| \cap |\partial v_i^\vee|) + \chi(|Y_i|) \\
&\equiv_2 1 + \chi(|Y_i| \cap |\partial v_i^\vee|),
\end{aligned} \tag{4.39}$$

since $\partial v_i^\vee \cong S^{n-1}$, which has even Euler characteristic. Looking at the definition of $\sigma_{\mathcal{V}}$ from (4.26), it suffices to equate $(-1)^{1+\chi(|Y_i|\cap\partial v_i^\vee)}$ with $\sigma(v_i, (P, \xi_i))$, which is taken care of by Lemma 4.35. $\square$

Now we show that we can replace $|C|$ with a smooth representative of the homology class of C.

Definition 4.40. Let M be a smooth manifold and $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. A C^r *triangulation* of M is a triangulation $(K, f\colon |K| \to M)$ of M such that for every simplex e of K, $f|_{|e|}$ is a C^r map.

Theorem 4.41 (Munkres [46, Theorem 10.6]). *Let W be a compact manifold and $r \in \mathbb{Z}_{>0} \cup \{\infty\}$. Then every C^r triangulation of ∂W extends to a C^r triangulation of W.*

Corollary 4.42. *Let X be a closed smooth manifold and $Y \subset X$ be a smooth codimension-one submanifold. Then there is a triangulation of X such that Y is a union of simplices.*

Proof. Let $\nu \to Y$ denote the normal bundle of $Y \hookrightarrow X$, $D(\nu) \to Y$ denote the unit disc bundle of ν, and $S(\nu) \to Y = \partial D(\nu)$ denote the unit sphere bundle of ν. Using the tubular neighborhood theorem, we choose an embedding $i\colon D(\nu) \hookrightarrow M$ such that the original embedding of Y in X is the zero section of $D(\nu) \to Y$ followed by i.

Let $r \geq 1$. Given a C^r triangulation $\Pi(N)$ of Y, we can triangulate $D(\nu)$: let $\Pi(I)$ denote the triangulation of $[-1, 1]$ which has vertices precisely at the integers, which is a smooth triangulation. For any simplex e of $\Pi(Y)$, $D(\nu)|_{|e|}$ is isomorphic to $|e| \times [-1, 1]$; choose an isomorphism ψ_e, and give $D(\nu)_{|e|}$ the product triangulation $|e| \times \Pi(I)$. These are compatible as e varies: if e' is another cell and $|e'|$ intersects $|e|$, $(\psi_{e'}^{-1} \circ \psi_e)|_{|e| \cap |e'|}$ is either the identity or multiplication by -1 on the fiber. Both of these send simplices to simplices, so we can glue the triangulations on $D(\nu)|_{|e|}$ and $D(\nu)|_{|e'|}$. Doing this for all simplices of Y defines a C^r triangulation $\Pi(D(\nu))$ of $D(\nu)$ in which $Y \subset D(\nu)$ is a union of simplices.

This induces a C^r triangulation of $S(\nu) = \partial(\overline{X \setminus D(\nu)})$, which by Theorem 4.41 extends to a triangulation of $\overline{X \setminus D(\nu)}$. We glue this triangulation to $\Pi(D(\nu))$, since both triangulations agree on $S(\nu)$, to obtain a triangulation of X in which Y is a union of simplices. $\square$

Lemma 4.43. *Let Π be a triangulation of an n-manifold X, $C \in Z^{\Pi}_{n-1}(X; \mathbb{Z}/2)$, and $f \in \Delta^n(X)$. Then*

$$\chi(|C|) \equiv \chi(|C + \partial f|) \bmod 2. \tag{4.44}$$

Proof. The sets of simplices in $|C|$ and $|C + \partial f|$ agree away from $|f|$, so if $R_0 := |C| \cap |\partial f|$ and $R_1 := |C + \partial f| \cap |\partial f|$, then it suffices to show $\chi(R_0) \equiv \chi(R_1) \bmod 2$.

Inclusion-exclusion implies

$$\chi(R_0) + \chi(R_1) \equiv \chi(|\partial f|) + \chi(R_0 \cap R_1) \bmod 2. \tag{4.45}$$

Since $|\partial f| \cong S^{d-1}$, its Euler characteristic is even. Next we show R_0 is a topological manifold with boundary: if R_0 is empty or all of $|\partial f|$, this is clear, and otherwise R_0 is an iterated boundary connect sum of its $(n-1)$-simplices. Since $R_0 \cap R_1 = \partial R_0$, $R_0 \cap R_1$ is null-bordant as a topological manifold, so its Euler characteristic is even, and (4.45) simplifies to $\chi(R_0) = \chi(R_1) \bmod 2$. $\square$

Proposition 4.46. *With C as in (4.30), if $N \hookrightarrow S^1 \times M$ is a smooth representative for the homology class of C (namely, the Poincaré dual of $\alpha(P_\phi)$), then $\chi(|C|) \equiv \chi(N) \bmod 2$.*

Proof. Let Π_1 be the barycentric subdivision of Π; as noted in Remark 2.24, this is also a "refinement" of $\Pi^\vee$, in that every cell of $\Pi^\vee$ is a union of simplices of Π_1. By Corollary 4.42, there is a triangulation Π_t of M such that N is a union of simplices; let Π' be a common refinement of Π_1 and Π_t, and $S^1(m) \times \Pi'$ be the product triangulation of $S^1 \times M$.

Let $C_{\mathrm{top}} \in Z_{n-1}^{S^1(m)\times\Pi'}(S^1 \times M; \mathbb{Z}/2)$ denote the cycle whose simplices are those contained in the cells of C; then $|C_{\mathrm{top}}| = |C|$. If $C_{\mathrm{sm}} \in Z_{n-1}^{S^1(m)\times\Pi'}(S^1 \times M; \mathbb{Z}/2)$ denotes the $(n-1)$-simplices in N, then $N = |C_{\mathrm{sm}}|$ and C_{top} and C_{sm} are homologous, so there are n-cells $f_1, \ldots, f_\ell$ such that

$$C_{\mathrm{sm}} = C_{\mathrm{top}} + \sum_{i=1}^{\ell} \partial f_i. \tag{4.47}$$

We apply Lemma 4.43 ℓ times and conclude. □

By combining this with Proposition 4.36, we have proven Theorem 4.27.

Next, we translate $(-1)^{\chi(N)}$ into an expression involving characteristic classes of M and P.

Proposition 4.48. *Let M be a closed manifold, $P \to M$ be a principal $\mathbb{Z}/2$-bundle, and $N \subset M$ be a smoothly embedded, codimension-1 submanifold representing the Poincaré dual to $\alpha(P)$. Then,*

$$\chi(N) \bmod 2 = \left\langle \frac{w(M)\alpha(P)}{1+\alpha(P)}, [M] \right\rangle. \tag{4.49}$$

But before we prove this:

Lemma 4.50. *Let $L \to X$ be a line bundle over a closed manifold X and $Y \hookrightarrow X$ be a smoothly embedded closed submanifold representing the Poincaré dual to $w_1(L)$, with normal bundle $\nu \to Y$. Then, as line bundles over Y, $\nu \cong L|_Y$.*

Proof. If $i_!\colon H^*(Y; \mathbb{Z}/2) \hookrightarrow H^{*+1}(X; \mathbb{Z}/2)$ denotes the Gysin map (which is Poincaré dual to restriction $H^*(X; \mathbb{Z}/2) \to H^*(Y; \mathbb{Z}/2)$), then $i_!(1)$ is Poincaré dual to $[Y] \in H_{d-1}(X; \mathbb{Z}/2)$ and $i^* i_!(1) = w_1(\nu)$. By construction, $[Y]$ is Poincaré dual to $w_1(L)$, so $i^* w_1(L) = w_1(L|_Y) = w_1(\nu)$. As line bundles are classified by their Stiefel–Whitney classes, $\nu \cong L|_Y$. □

Proof of Proposition 4.48. Let $j\colon N \hookrightarrow M$ be inclusion. Since N represents the Poincaré dual of $\alpha(P)$, then for any $x \in H^{n-1}(M; \mathbb{Z}/2)$,

$$\langle j^* x, [N] \rangle = \langle \alpha(P) x, [M] \rangle. \tag{4.51}$$

We will use this to carry the mod 2 Euler characteristic of N, which is equal to $\langle w(N), [N] \rangle$, to the cohomology of M; in order to do so, we must show $w(N) \in \mathrm{Im}(j^*)$.

If $\nu \to N$ denotes the normal bundle of N, there is a short exact sequence of vector bundles on N

$$0 \longrightarrow TN \longrightarrow j^* TM \longrightarrow \nu \longrightarrow 0, \tag{4.52}$$

so $w(j^* TM) = j^* w(M) = w(N) w(\nu)$. Since ν is a line bundle,

$$w(\nu) = 1 + w_1(\nu) = 1 + j^* \alpha(P) = j^*(1 + \alpha(P)) \tag{4.53}$$

by Lemma 4.50. Hence

$$j^* w(M) = w(N) j^*(1 + \alpha(P)). \tag{4.54}$$

Since $\alpha(P) \in H^*(X; \mathbb{Z}/2)$ is nilpotent, $j^*(1 + \alpha(P))$ is invertible, and therefore

$$w(N) = \frac{j^* w(M)}{j^*(1 + \alpha(P))} = j^* \left(\frac{w(M)}{1 + \alpha(P)} \right). \tag{4.55}$$

Thus we can invoke Poincaré duality:

$$\chi(N) \bmod 2 = \langle w(N), [N]\rangle = \left\langle \alpha(P) \cdot \frac{w(M)}{1+\alpha(P)}, [M] \right\rangle. \tag{4.56}$$

□

Combining this with Theorem 4.27, we get:

Corollary 4.57. *If $P \in \mathrm{Bun}_{\mathbb{Z}/2}(M)$, the character of* $\mathrm{Aut}(P)$ *acting on* $(L_{\mathrm{GDS}})_P$ *has* ϕ *act by multiplication by*

$$(-1)^{\langle \alpha(P_\phi) w(S^1\times M)/(1+\alpha(P_\phi)), [S^1\times M]\rangle} \in \{\pm 1\} \subset \mathbb{C}^\times. \tag{4.58}$$

Next, we compare this with the character of $\mathrm{Aut}(P)$ acting on $(L_\beta)_P$ and conclude.

Proof of Theorem 4.19. Proposition 3.24 tells us that in the character of $\mathrm{Aut}(P)$ acting on $(L_\beta)_P$, ϕ acts by $Z_\beta^{\mathrm{cl}}(S^1 \times M, P_\phi)$; by Theorem 3.2, this is exactly (4.58). Hence $L_{\mathrm{GDS}} \cong L_\beta$. □

4.3.3. The $\mathrm{MCG}(M)$*-action for the GDS model* Let $\mathsf{Cell}(M)$ denote the poset category whose objects are smooth triangulations on M such that the 0-clopen star of every vertex is contractible, and whose morphisms are generated by diffeomorphisms and refinements similarly to the construction of $\mathsf{Lat}(M)$ in Sect. 4.1.2. Just as for the toric code, given a diffeomorphism $f\colon M \to M$ and $\Pi \in \mathsf{Cell}(M)$, we obtain a map f_* from the state space for Π to the state space for $f(\Pi)$, and this assignment satisfies (A2).

Let $\varphi\colon \Pi \to \Pi'$ be a refinement. Define φ_* and $\mathcal{P}$ as in the previous section, and let $\mathcal{P}'\colon \mathcal{H}_{\Pi'} \to \mathcal{H}_{\Pi'}$ be the projection onto $\bigcap_v \widetilde{H}_v$ which is orthogonal with respect to the inner product in which the δ-functions on elements of $\pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$ are an orthonormal basis.

Lemma 4.59. *The map $\mathcal{P} \circ \mathcal{P}' \circ \varphi_*$ sends ground states to ground states, hence restricts to an isomorphism $L(\Pi) \xrightarrow{\cong} L(\Pi')$ functorial as in (A3), and this is compatible with the maps in (A2).*

Proof. Suppose φ adds no 0-simplices and 1-simplices to Π, so $\mathcal{H}_{\Pi'} \cong \mathcal{H}_{\Pi'}$ and φ_* is the identity. Then φ adds no cells at all, because it is not possible to add cells to a manifold that is a simplicial complex without adding 0- or 1-simplices, so φ is the identity refinement and the lemma follows because $\mathcal{P}$ and $\mathcal{P}'$ are projections.

If otherwise, we show that φ_* of a nonzero ground state is not a ground state, so that the orthogonal projection thereafter sends it to a nonzero ground state. If φ adds any 1-simplices to Π that do not arise from splitting preexisting 1-simplices into smaller ones, the construction in Remark 4.13 shows that φ_* of a nonzero ground state is not a ground state; if the only 1-simplices it adds are split from preexisting ones, then it must add a 0-simplex. If φ adds any 0-simplices to Π, it must add a 1-simplex that is not split from a preexisting 1-simplex, because all 0-simplices must be trivalent. □

Therefore the argument of Sect. 4.1.2 applies to define for any closed $(n-1)$-manifold M an action of $\mathrm{Diff}(M)$ on the ground states of the GDS model. Under the identification of the space of ground states with the space of functions on the set of $P \in \pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)$ such that $\langle \alpha(P)w(M)/(1+\alpha(P)), [M]\rangle = 0$, this representation is the one induced from the usual $\mathrm{Diff}(M)$-action on this space, which is an invariant subspace of $\mathbb{C}[\mathrm{Bun}_{\mathbb{Z}/2}(M)]$, and once again this is trivial on $\mathrm{Diff}_0(M)$, so it defines an $\mathrm{MCG}(M)$-action.

Recall from Theorem 4.19 that $Z_{\mathrm{GDS}}(M)$ is isomorphic to the space of ground states of the GDS model on M; using the push–pull map Z_{GDS} assigns to a cobordism, its $\mathrm{MCG}(M)$-action is the same, again induced from the standard action on $\pi_0 \operatorname{Bun}_{\mathbb{Z}/2}(M)$, finishing the proof of Theorem 4.18.

Remark 4.60. Suppose n is even, and let $Z_2\colon \mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}$ denote the quantum gauge–gravity TQFT with Lagrangian β_2 equal to the degree-n summand of $w\alpha/(1+\alpha^2) \in H^*(B\mathrm{O}_n \times B\mathbb{Z}/2)$. Then $Z_{\mathrm{GDS}}(\mathbb{RP}^n) = 1$ and $Z_2(\mathbb{RP}^n) = 0$, so $Z_{\mathrm{GDS}} \neq Z_2$. However, a characteristic-class computation shows that for any closed $(n-1)$-manifold M, there is an isomorphism $Z_{\mathrm{GDS}}(M) \cong Z_2(M)$ equivariant with respect to the $\mathrm{MCG}(M)$-action on the state spaces. This means that in the sense of Definition 4.1, both Z_{GDS} and Z_2 capture the ground states of the GDS model, and that it is not clear how to distinguish them using data from the lattice. In physics, however, the low-energy effective theory of a lattice model is expected to be unique.

Freed–Hopkins [17, §7.3], following Kong–Wen [39], suggest that the low-energy effective theory may only be defined on manifolds which locally have a direction of time, i.e. manifolds M together with a reduction of the structure group of TM from O_n to O_{n-1}. That is, it should be possible to calculate the partition function on such manifolds using locality of the lattice model, and it might not be possible to calculate further in general. Alternatively, Shapourian–Shiozaki–Ryu [56] describe a method to compute partition functions on $\mathbb{RP}^2$ for 2D symmetry-protected topological phases defined by a Hamiltonian, and it is possible their method would generalize, though we have not pursued this.

5. Calculations

In this section, we perform some calculations with the GDS Lagrangian in order to understand when Z_{GDS} is isomorphic to a $\mathbb{Z}/2$-Dijkgraaf–Witten theory. First, we fix some notation.

- Recall that α denotes the generator of $H^1(B\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2$; in particular, it defines a characteristic class for principal $\mathbb{Z}/2$-bundles by pullback, and if $P \in \operatorname{Bun}_{\mathbb{Z}/2}(X)$, this characteristic class evaluated on P is denoted $\alpha(P) \in H^1(X;\mathbb{Z}/2)$.
- $\mathrm{DW}_0\colon \mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}$ denotes $\mathbb{Z}/2$-Dijkgraaf–Witten theory with the zero Lagrangian and $Z_{\alpha^n}\colon \mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}$ denotes $\mathbb{Z}/2$-Dijkgraaf–Witten theory with Lagrangian $\alpha^n \in H^n(B\mathbb{Z}/2;\mathbb{Z}/2)$.
- Recall from Definition 4.31 that if $P \to M$ is a principal $\mathbb{Z}/2$-bundle, the image of $\phi \in \operatorname{Aut}(P)$ under the isomorphism $\operatorname{Aut}(P) \to H^0(M;\mathbb{Z}/2)$ is denoted $[\phi]$. Letting $x \in H^1(S^1;\mathbb{Z}/2)$ denote the generator, $\alpha(P_\phi) = \alpha(P) + x[\phi]$ in $H^*(S^1 \times M;\mathbb{Z}/2)$.

We begin with a few example calculations. We will call a principal $\mathbb{Z}/2$-bundle $P \to M$ *permitted* if the GDS action $\langle w(M)\alpha(P_\phi)/(1+\alpha(P_\phi)), [M]\rangle$ vanishes for all $\phi \in \operatorname{Aut}(P)$; thus $Z_{\mathrm{GDS}}(M)$ is the space of functions on the set of isomorphism classes of permitted bundles.

Proposition 5.1. *If M is a closed $(n-1)$-manifold, then the trivial bundle $P_{\mathrm{triv}} \to M$ is permitted if and only if $\chi(M)$ is even.*

Proof. The action for P_{triv} and $\phi \in \operatorname{Aut}(P_{\mathrm{triv}})$ is

$$\left\langle \frac{w(M)\alpha((P_{\mathrm{triv}})_\phi)}{1+\alpha((P_{\mathrm{triv}})_\phi)}, [S^1 \times M]\right\rangle = \left\langle \frac{w(M)(x[\phi]+\alpha(P_{\mathrm{triv}}))}{1+(x[\phi]+\alpha(P_{\mathrm{triv}}))}, [S^1 \times M]\right\rangle \tag{5.2}$$

by (4.32). Since P_{triv} is trivial, $\alpha(P_{\mathrm{triv}}) = 0$, so

$$= \left\langle \frac{w(M)x[\phi]}{1 + x[\phi]}, [S^1 \times M] \right\rangle. \tag{5.3}$$

Since $(x[\phi])^2 \in H^2(S^1; \mathbb{Z}/2) = 0$,

$$= \langle w(M)x[\phi], [S^1 \times M] \rangle, \tag{5.4}$$

which by a Fubini theorem is

$$= \langle x[\phi], [S^1] \rangle \langle w(M), [M] \rangle. \tag{5.5}$$

If ϕ is nontrivial, $\langle x[\phi], [S^1] \rangle = 1$. Hence the action is zero for all ϕ if and only if $\langle w(M), [M] \rangle$, which is $\chi(M) \bmod 2$, vanishes. □

Corollary 5.6. *Let M be simply connected. Then,*

$$Z_{\mathrm{GDS}}(M) \cong \begin{cases} 0, & \chi(M) odd \\ \mathbb{C}, & \chi(M) even. \end{cases} \tag{5.7}$$

Proof. All principal $\mathbb{Z}/2$-bundles over such a manifold are trivial, so we just have to check whether the trivial bundle is permitted. □

It is worth comparing this to the α^n Dijkgraaf–Witten theory.

Lemma 5.8. *If $n > 1$ and M is a closed $(n-1)$-manifold, $Z^{\mathrm{cl}}_{\alpha^n}(S^1 \times M, (P_{\mathrm{triv}})_\phi) = 0$ for any automorphism ϕ. In particular, if M is simply connected, $Z_{\alpha^n}(M) \cong \mathbb{C}$.*

Proof. Let $\phi \in \mathrm{Aut}(P_{\mathrm{triv}})$, so

$$\alpha((P_{\mathrm{triv}})_\phi) = \alpha(P_{\mathrm{triv}}) + x[\phi] = x[\phi]. \tag{5.9}$$

The action is

$$\langle \alpha(P_\phi)^n, [S^1 \times M] \rangle = \langle (x[\phi])^n, [S^1 \times M] \rangle = 0. \tag{5.10}$$

□

Proposition 5.11.

$$Z_{\mathrm{GDS}}(\mathbb{CP}^n \times \mathbb{RP}^2) \cong \begin{cases} \mathbb{C}, & n \text{ even} \\ \mathbb{C}^2, & n \text{ odd}. \end{cases} \tag{5.12}$$

Proof. Let $X := \mathbb{CP}^n \times \mathbb{RP}^2$, and let z be the generator of $H^1(X; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Since

$$\chi(X) = \chi(\mathbb{CP}^n)\chi(\mathbb{RP}^2) = \begin{cases} 0 \bmod 2, & n \text{ odd} \\ 1 \bmod 2, & n \text{ even}, \end{cases} \tag{5.13}$$

then by Proposition 5.1, the trivial bundle is permitted if and only if n is odd.

The other isomorphism class of principal $\mathbb{Z}/2$-bundles on X is the one whose total space is the universal cover of X, which we denote P. Then $\alpha(P) = z$, and for $\phi \in \mathrm{Aut}(P)$, the Lagrangian for $S^1 \times X$ and P_ϕ is

$$\frac{\alpha(P_\phi)w(S^1 \times X)}{1 + \alpha(P_\phi)} = \frac{(z + x[\phi])w(\mathbb{RP}^2)w(\mathbb{CP}^n)}{1 + z + x[\phi]}. \tag{5.14}$$

Since $z + x[\phi]$ is nilpotent, $1 + z + x[\phi]$ is invertible, so

$$= \frac{(z + x[\phi])w(\mathbb{RP}^2)w(\mathbb{CP}^n)(1 + z + x[\phi])}{(1 + z + x[\phi])^2}. \tag{5.15}$$

Since $(x[\phi])^2 = 0$,

$$= \frac{(1 + z)^3(z + z^2 + x[\phi])w(\mathbb{CP}^n)}{1 + z^2} \tag{5.16}$$

$$= (1 + z)(z + z^2 + x[\phi])w(\mathbb{CP}^n) \tag{5.17}$$

$$= (z + x[\phi] + zx[\phi])w(\mathbb{CP}^n). \tag{5.18}$$

We want to pair this with $[S^1 \times X]$, but (5.18) has no terms of degree $\dim(S^1 \times X) = 2n+3$. Thus

$$\langle (z + x[\phi] + zx[\phi])w(\mathbb{CP}^n), [S^1 \times X]\rangle = 0, \tag{5.19}$$

so this bundle is always permitted. $\square$

Proposition 5.20. *For $n \geq 2$,*

$$Z_{\mathrm{GDS}}(\mathbb{RP}^n) \cong \begin{cases} \mathbb{C}, & n \text{ even} \\ \mathbb{C}^2, & n \text{ odd.} \end{cases} \tag{5.21}$$

Proof. Let $z \in H^1(\mathbb{RP}^n; \mathbb{Z}/2)$ denote the generator. By Proposition 5.1, the trivial principal $\mathbb{Z}/2$-bundle is permitted if and only if n is odd. The other isomorphism class of principal $\mathbb{Z}/2$-bundles is the universal cover $S^n \twoheadrightarrow \mathbb{RP}^n$, with $\alpha(S^n) = z$, so it suffices to prove this bundle is always permitted. Let ϕ be an automorphism of this principal bundle. The action is

$$\frac{\alpha(S^n_\phi)w(\mathbb{RP}^n)}{1 + \alpha(S^n_\phi)} = \frac{(z + x[\phi])(1 + z)^{n+1}}{1 + z + x[\phi]}. \tag{5.22}$$

Again, $z + x[\phi]$ is nilpotent, so $1 + z + x[\phi]$ is invertible, so

$$= \frac{(z + x[\phi])(1 + z)^{n+1}(1 + z + x[\phi])}{(1 + z + x[\phi])^2} \tag{5.23}$$

$$= \frac{(1 + z)^{n+1}(z + z^2 + x[\phi])}{(1 + z)^2} \tag{5.24}$$

$$= (1 + z)^{n-1}(z + z^2 + x[\phi]). \tag{5.25}$$

But in (5.25), only the $(1+z)^{n-1}z^2$ term contributes anything of degree $\dim(S^1 \times \mathbb{RP}^n) = n + 1$, and this lives in $H^{n+1}(\mathbb{RP}^n; \mathbb{Z}/2) \otimes H^0(S^1; \mathbb{Z}/2)$, hence must be 0. Thus (5.25) has no terms of top degree, so

$$\langle (1 + z)^{n+1}(z + z^2 + x[\phi]), [S^1 \times \mathbb{RP}^n]\rangle = 0, \tag{5.26}$$

and this bundle is always permitted. $\square$

We now compare Z_{GDS} with $\mathbb{Z}/2$-Dijkgraaf–Witten theories.

Lemma 5.27. *Let M be a closed $(2k+1)$-manifold and $y \in H^1(M;\mathbb{Z}/2)$. Then $w_1(M)y^{2k} = 0$.*

Proof. Let v_1 denote the first Wu class. Then,

$$w_1 y^{2k} = v_1 y^{2k} = \mathrm{Sq}^1((y^k)^2) = 0. \tag{5.28}$$

□

Theorem 5.29. *In dimension* 3*, Z_{GDS} is isomorphic to Z_{α^3}.*

Proof. This follows from Proposition 3.33 after observing

$$(\alpha + w_1)^3 = \alpha^3 + w_1\alpha^2 + w_1^2\alpha + w_1^3. \tag{5.30}$$

On any closed 3-manifold, $w_1^3 = 0$ because all closed 3-manifolds bound, and $w_1\alpha^2 = 0$ by Lemma 5.27. Thus (5.30) agrees with the Lagrangian for Z_{GDS}. □

The relationship in dimension 3 between the double semion model and the $\mathbb{Z}/2$-Dijkgraaf–Witten theory with Lagrangian α^3 is known to physicists (see, e.g., [61, §II]), though not previously proven in this form.

Theorem 5.31. *For even n, Z_{GDS} is isomorphic to* DW_0.

Proof. By Corollary 3.27, it suffices to prove that $w(M)\alpha/(1+\alpha) = 0$ for any even-dimensional manifold M and $\alpha \in H^1(M;\mathbb{Z}/2)$. In Proposition 4.48, we saw $\langle w(M)\alpha/(1+\alpha), [M]\rangle$ is the mod 2 Euler characteristic of a submanifold N representing the Poincaré dual of α. Since N is a closed, odd-dimensional manifold, its mod 2 Euler characteristic vanishes, so $w(M)\alpha/(1+\alpha) = 0$. □

[22, Thm. 5.3] proved this for state spaces, and the proof idea is the same.

Theorem 5.32. *For odd $n \geq 4$, Z_{GDS} is not isomorphic to any $\mathbb{Z}/2$-Dijkgraaf–Witten theory.*

Proof. By Corollary 3.29, it suffices to prove that Z_{GDS} is not isomorphic to DW_0 and Z_{α^n}.

If $n = 4k+1$ for some $k \geq 1$, then $Z_{\mathrm{GDS}}(\mathbb{CP}^{2k}) = 0$ by Corollary 5.6, but $\mathrm{DW}_0(\mathbb{CP}^{2k}) \cong \mathbb{C}$, and $Z_{\alpha^n}(\mathbb{CP}^{2k}) \cong \mathbb{C}$ by Lemma 5.8.

If $n = 4k+3$ for some $k \geq 1$, then $Z_{\mathrm{GDS}}(\mathbb{CP}^{2k} \times \mathbb{RP}^2) \cong \mathbb{C}$ by Proposition 5.11 and $\mathrm{DW}_0(\mathbb{CP}^{2k} \times \mathbb{RP}^2) \cong \mathbb{C}^2$. For the theory with Lagrangian α^n, Lemma 5.8 gives us one copy of $\mathbb{C}$ from the trivial bundle. If $P \to \mathbb{CP}^{2k} \times \mathbb{RP}^2$ denotes the nontrivial bundle and $z \in H^1(\mathbb{RP}^2;\mathbb{Z}/2)$ denotes the generator, then $\alpha(P) = z$. For any $\phi \in \mathrm{Aut}(P)$,

$$\langle \alpha(P_\phi)^n, [S^1 \times \mathbb{CP}^{2k} \times \mathbb{RP}^2]\rangle = \langle (z + x[\phi])^n, [S^1 \times \mathbb{CP}^{2k} \times \mathbb{RP}^2]\rangle. \tag{5.33}$$

Since $(x[\phi])^2 = 0$, this is

$$= \langle z^n + nz^{n-1}x[\phi], [S^1 \times \mathbb{CP}^{2k} \times \mathbb{RP}^2]\rangle, \tag{5.34}$$

and since $z^3 = 0$, this is 0. Thus the state space picks up another factor of $\mathbb{C}$, and $Z_{\alpha^n}(\mathbb{CP}^{2k} \times \mathbb{RP}^2) \cong \mathbb{C}^2$. □

This was also proven in [22, Thm. 8.1], with the same manifolds as counterexamples.

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