



't Hooft Anomalies of Discrete Gauge Theories and Non-abelian Group Cohomology

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Abstract: We study discrete symmetries of Dijkgraaf–Witten theories and their gauging in the framework of (extended) functorial quantum field theory. Non-abelian group cohomology is used to describe discrete symmetries and we derive concrete conditions for such a symmetry to admit 't Hooft anomalies in terms of the Lyndon–Hochschild–Serre spectral sequence. We give an explicit realization of a discrete gauge theory with 't Hooft anomaly as a state on the boundary of a higher-dimensional Dijkgraaf–Witten theory. This allows us to calculate the 2-cocycle twisting the projective representation of physical symmetries via transgression. We present a general discussion of the bulk-boundary correspondence at the level of partition functions and state spaces, which we make explicit for discrete gauge theories.

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1. Introduction and Overview

A Dijkgraaf–Witten theory [DW90] is a topological gauge theory with a finite gauge group D . These theories are mathematically well-defined and hence provide an interesting toy model for the mathematical study of quantum gauge theory. Recently they have received a lot of attention in the physics literature due to their relevance to topological phases, see e.g. [WW15, WWH15, Yos17, HZvK17, CCW17a, CCW17b, Del17, TCSR18, WHT+18]. In this context properties of topologically protected boundary states can be described by quantum field theories with anomalies, which are compensated by an anomaly inflow from the bulk. Anomalies in discrete gauge theories show interesting new features that are absent in the continuous case [KT14, Tac17].

The purpose of the present paper is two-fold:

- We present a detailed study of ’t Hooft anomalies of Dijkgraaf–Witten theories in the framework of functorial quantum field theory. This is based on the framework originally discussed in [KT14]. Our formulation further enables the consideration of non-abelian gauge groups in terms of non-abelian group cohomology.
- We explicitly describe discrete gauge theories with anomalies as boundary phases in the language of extended functorial quantum field theory [Fre14a, Mon15]. This provides a mathematically rigorous toy model for this geometric approach to anomalies. These boundary theories can be regarded as functorial reformulations and extensions of the gapped boundary states originally proposed in [Wit16].

We begin with some physical background and motivation.

1.1. Anomalies and symmetry-protected topological phases. At the end of the last century it was realized that quantum phases of matter exist which cannot be described by Landau’s theory of symmetry breaking. Instead these phases can be distinguished by ‘topological order’ parameters which prevent them from being deformed to a trivial system whose ground state is a factorized state. Since their discovery over 30 years ago, immense progress in understanding and classifying these topological phases has been made. For instance, there exists a classification for non-interacting gapped systems in terms of twisted equivariant K-theory [FM13] (see also [BS17]).

A fruitful approach to the study of gapped interacting systems is to consider the effective low-energy (long-range) continuum theory of a lattice Hamiltonian model. Usually these field theories are topological. A famous example is the effective description of the integer quantum Hall effect in terms of Chern–Simons gauge theory. In this sense a gapped quantum phase may be thought of as a path-connected component of the moduli stack of topological quantum field theories. However, in the interacting case no complete classification exists. For this reason one usually restricts to tractable subclasses. In this paper we focus on ‘short-range entangled’ phases [CGW10]. A gapped phase Ψ is short-range entangled if there exists a phase Ψ^{-1} such that the ‘stacked’ phase $\Psi \otimes \Psi^{-1}$ can be deformed by an adiabatic transformation of the Hamiltonian to a trivial product state without closing the energy gap between the ground state and the first excited state. The stacking operation of topological phases corresponds to the tensor product of their low-energy effective topological field theories. A topological field theory is called invertible

if it admits an inverse with respect to the tensor product. This observation motivates a classification of short-range entangled phases in terms of invertible topological field theories [FH16].

Let G be a group. In the case of an additional global G -symmetry, a non-trivial short-range entangled phase may be trivial when the symmetry is ignored. Such a phase is called ‘ G -symmetry-protected’ [CGW10, CGLW13]. A G -symmetry-protected phase can be understood by studying its topological response to non-trivial background G -gauge fields, which is called ‘gauging’ the G -symmetry. For a finite symmetry group G , the low-energy effective field theories are G -equivariant topological field theories [KT17b]. Classical Dijkgraaf–Witten theories provide a particularly tractable class of invertible G -equivariant topological field theories. The corresponding lattice Hamiltonian models have been constructed in e.g. [WWH15, CCW17a, CCW17b, WLHW18]. They are classified by group cohomology. The corresponding classification of topological phases is called group cohomological classification [LG12]. However, this is not a complete classification of symmetry-protected phases and more refined classifications have been proposed, see e.g. [KTTW15, FH16, GJF19, WG18].

An essential feature of symmetry-protected topological phases is that they exhibit ‘topologically protected’ boundary states. These boundary states can be effectively described by anomalous quantum field theories. Under the bulk-boundary correspondence the anomaly is cancelled by an anomaly inflow from the bulk theory. The presence of non-trivial global anomalies forces the boundary theory to be non-trivial and topologically protected. The modern geometric point of view on field theories with anomaly is that they should be considered as a theory ‘relative to’ a higher-dimensional invertible field theory, as is naturally suggested by their appearance in condensed matter physics and quantum information theory. Reversing this logic, it follows that $n+1$ -dimensional invertible field theories should classify the possible anomalies in n dimensions [Wen13]. A class of gapped boundary states for the topological phases described by group cohomology are quantum Dijkgraaf–Witten theories based on a different gauge group D with an anomalous G -symmetry [Kap14]. Anomaly in this context means ’t Hooft anomaly, i.e. an obstruction to gauging the G -symmetry [tH80], and the bulk-boundary correspondence implements the ’t Hooft anomaly matching conditions.

The purpose of this paper is to study the appearance of ’t Hooft anomalies in the mathematically rigorous framework of functorial field theories and to realize anomalous field theories as gauge theories relative to higher-dimensional topological phases. Before outlining precisely what we do, let us first informally review some of the main mathematical background.

1.2. Anomalies in functorial quantum field theory. We formulate our results in the framework of functorial field theories. The idea is to give an axiomatic framework for the partition function of a quantum field theory. In physics, the partition function $Z(M)$ on an n -dimensional manifold M is calculated by the Feynman path integral of an exponentiated action functional over the space of dynamical field configurations on M ; so far there is no mathematically well-defined theory of such path integration in general. The axioms of functorial field theories are derived from the properties that such an integration would satisfy in the case that the action functional is an integral of a local Lagrangian density on M .¹ A quantum field theory should also assign a Hilbert space of states $Z(\Sigma)$

¹ In the case of discrete gauge theory there exists a well-defined integration theory (see for example [SW19, Appendix A]) which satisfies the axioms of a functorial field theory.

to every $n-1$ -dimensional manifold Σ . They satisfy $Z(\Sigma \sqcup \Sigma') \cong Z(\Sigma) \otimes_{\mathbb{C}} Z(\Sigma')$, i.e. the state space of non-interacting systems is given by the tensor product of the corresponding Hilbert spaces. In any quantum field theory there exists a time evolution operator (propagator)

$$Z([t_0, t_1] \times \Sigma): Z(\Sigma) \longrightarrow Z(\Sigma)$$

from time t_0 to t_1 . We think of this operator as associated to the cylinder $[t_0, t_1] \times \Sigma$. They satisfy $Z([t_1, t_2] \times \Sigma) \circ Z([t_0, t_1] \times \Sigma) = Z([t_0, t_2] \times \Sigma)$. The path integral should also allow for the construction of a more general operator $Z(M): Z(\Sigma_-) \longrightarrow Z(\Sigma_+)$ for every manifold M with boundary $\Sigma_- \sqcup \Sigma_+$, such that the gluing of manifolds corresponds to the composition of linear maps. Such a manifold is called a cobordism from Σ_- to Σ_+ .

These considerations motivate the definition of a functorial field theory, generalising Atiyah’s definition of topological field theories [Ati88] and Segal’s definition of conformal field theories [Seg88], as a symmetric monoidal functor

$$Z: \mathbf{Cob}_n^{\mathcal{F}} \longrightarrow \mathbf{Hilb} ,$$

where $\mathbf{Cob}_n^{\mathcal{F}}$ is a category modelling physical spacetimes with non-dynamical background fields \mathcal{F} and \mathbf{Hilb} is the category of complex Hilbert spaces. Roughly speaking, $\mathbf{Cob}_n^{\mathcal{F}}$ contains closed $n-1$ -dimensional manifolds with background fields as objects, and as morphisms the n -dimensional cobordisms as well as additional limit morphisms corresponding to diffeomorphisms which are compatible with the background fields. The additional morphisms encode symmetries. Evaluating Z on a closed n -dimensional manifold M gives rise to a linear map $\mathbb{C} \cong Z(\emptyset) \longrightarrow Z(\emptyset) \cong \mathbb{C}$ which can be identified with a complex number $Z(M)$, the partition function of Z on M . This definition can be thought of as a prescription for computing a manifold invariant $Z(M)$ by cutting manifolds into simpler pieces and studying the quantum field theory on these pieces.

We now turn our attention to the description of anomalies. The partition function of an $n-1$ -dimensional quantum field theory Z with anomaly described by an invertible field theory $L: \mathbf{Cob}_n^{\mathcal{F}} \longrightarrow \mathbf{Hilb}$ on an $n-1$ -dimensional manifold Σ takes values in the one-dimensional vector space $L(\Sigma)$, instead of \mathbb{C} . It is possible to pick a non-canonical isomorphism $L(\Sigma) \cong \mathbb{C}$ to identify the partition function with a complex number. The group of symmetries acts non-trivially on $L(\Sigma)$ encoding the breaking of the symmetry in the quantum field theory Z .

To also incorporate the description of the state space of Z on an $n-2$ -dimensional manifold S we need to promote L to an extended field theory E which assigns \mathbb{C} -linear categories to $n-2$ -dimensional manifolds such that $Z(S)$ can be considered as an object of $E(S)$. In other words, E should be an extended functorial field theory, i.e. a symmetric monoidal 2-functor

$$E: \mathbf{Cob}_{n,n-1,n-2}^{\mathcal{F}} \longrightarrow 2\mathbf{Vect} ,$$

where $\mathbf{Cob}_{n,n-1,n-2}^{\mathcal{F}}$ is a suitable extension of $\mathbf{Cob}_n^{\mathcal{F}}$; see Sect. 2 for details. There are different possible choices for the target bicategory. For simplicity we restrict ourselves to Kapranov–Voevodsky 2-vector spaces [KV94].

Requiring that E is an invertible field theory implies that there is a non-canonical equivalence of categories $E(S) \cong \mathbf{Vect}$ which allows one to identify the state space of the anomalous theory with a vector space. We can subsume the ideas sketched above

in the following concise definition: A quantum field theory with anomaly is a natural symmetric monoidal 2-transformation

$$Z: 1 \implies \text{tr } E$$

between a trivial field theory and a certain truncation of E ; see Sect. 4.1 for details. This formalism allows one to compute the 2-cocycle twisting the projective representation of the symmetry group on the state space completely in terms of the extended field theory E [Mon15]. Anomalous theories formulated in this way are a special case of relative field theories [FT14] and are closely related to twisted quantum field theories [ST11, JFS17]. The present paper describes 't Hooft anomalies of discrete gauge theories as relative field theories. We shall now give an overview of our constructions and findings.

1.3. Summary of results and outline. One of the main achievements of this paper is to give a mathematical description of symmetries of Dijkgraaf–Witten theories and their gauging in the framework of functorial field theory which is motivated by physical considerations. Let D be a finite group and n a natural number. The possible topological actions for n -dimensional Dijkgraaf–Witten theories with gauge group D are classified by the group cohomology of D or equivalently by the singular cohomology of the classifying space BD with coefficients in $U(1)$ [DW90, FQ93]. Let $\omega \in Z^n(BD; U(1))$ be an n -cocycle and M an n -dimensional manifold. Let P be a D -gauge field on M with classifying map $\psi_P: M \rightarrow BD$. The action of the Dijkgraaf–Witten theory L_ω evaluated at P is given by

$$\exp(2\pi i S_{\text{DW}}) := \int_M \psi_P^* \omega .$$

The quantum theory can be defined by appropriately summing over isomorphism classes of D -bundles. We review Dijkgraaf–Witten theories in detail in Sect. 2.

In general, a physical symmetry group G acts on gauge fields only up to gauge transformations. Since for finite gauge groups, gauge transformations can be naturally identified with homotopies of classifying maps, we define such an action as a homotopy coherent action of G on BD (Definition 3.2). We show that, up to equivalence, homotopy coherent actions on BD are described by non-abelian group 2-cocycles. If D is abelian, this description agrees with the description in [KT14]. Non-abelian 2-cocycles classify extensions of G by D :

$$1 \longrightarrow D \xrightarrow{\iota} \widehat{G} \xrightarrow{\lambda} G \longrightarrow 1 ,$$

as we review in Sect. 3.2. This extension has a natural physical interpretation: It describes how to combine D - and G -gauge fields into a single \widehat{G} -gauge field. When the extension is non-trivial, i.e. \widehat{G} is not a product group $D \times G$, one says that the G -symmetry is ‘fractionalized’ [WWW18].

A homotopy coherent action on BD induces a homotopy coherent action on the collection of classical D -gauge theories. Homotopy fixed points of this action are defined to be classical field theories with G -symmetry (Definition 3.5). An essential feature of homotopy fixed points is that they are a structure, not a property. In Proposition 3.9 we show that if the topological action is preserved by the action of G (Definition 3.7), then the corresponding Dijkgraaf–Witten theory can be equipped with a homotopy fixed point structure.

An internal symmetry of a quantum field theory acts on its Hilbert space of states. This motivates the definition of a functorial quantum field theory with internal G -symmetry as a functor

$$\text{Cob}_n^{\mathcal{F}} \longrightarrow G\text{-Rep}$$

to the category $G\text{-Rep}$ of representations of G . We show in Proposition 3.13 that classical symmetries of Dijkgraaf–Witten theories induce internal symmetries of the quantized theory. This shows that discrete gauge theories are anomaly-free in the sense that all symmetries extend to the quantum level. A discussion of symmetries of discrete classical and quantum gauge theories is the content of Sect. 3.1.

Anomalies appear as an obstruction to gauging the G -symmetry, i.e. to coupling it to non-trivial background gauge fields (Definition 3.19). Anomalies of this type are called 't Hooft anomalies. Gauging the G -symmetry can be achieved by finding a topological action $\widehat{\omega}$ for a \widehat{G} -gauge theory which restricts to ω and performing a path integral over D -gauge fields. Mathematically, this can be described by the partial orbifold (pushforward) construction of [SW19]. In Theorem 3.24 we prove that the partial orbifold construction of the classical gauge theory corresponding to $\widehat{\omega}$ gauges the G -symmetry. We discuss the gauging of discrete symmetries in Sect. 3.3.

However, in general it might be impossible to find a topological action which restricts correctly. In this case we say that the corresponding symmetry has a 't Hooft anomaly. The obstructions for $\widehat{\omega}$ to exist are encoded in the Lyndon–Hochschild–Serre spectral sequence. For an n -dimensional field theory there are n obstructions which need to vanish. In Proposition 3.31 we show that if all obstructions except the last one vanish then there exists an $n+1$ -dimensional topological action θ for a discrete G -gauge theory, together with an n -cochain ω' in $C^n(B\widehat{G}; U(1))$ satisfying $t^*\omega' = \omega$ and $\delta\omega' = \lambda^*\theta$; physically, this is interpreted as saying that the corresponding symmetry-protected topological state becomes trivial when its fields are regarded as n -dimensional \widehat{G} -gauge fields rather than as $n+1$ -dimensional G -gauge fields. These obstructions are studied in Sect. 3.4.

Based on this result we construct a boundary quantum field theory $Z_{\omega'}$ encoding the anomaly in Sect. 4. Let us give an informal description of $Z_{\omega'}$ here. The fact that ω' is not closed implies that $\int_M \psi_{\widehat{P}}^* \omega'$ is not gauge-invariant for a general \widehat{G} -bundle \widehat{P} on M . Under a gauge transformation $\widehat{h}: \widehat{P} \rightarrow \widehat{P}'$ the value of $\int_M \psi_{\widehat{P}}^* \omega'$ changes by multiplication with $\int_{[0,1] \times M} \widehat{h}^* \delta\omega'$, where we consider \widehat{h} as a homotopy $[0, 1] \times M \rightarrow B\widehat{G}$. We can rewrite this integral as $\int_{[0,1] \times M} (\lambda_* \widehat{h})^* \theta$. This is exactly the value of L_{θ} evaluated on $\lambda_* \widehat{h}$, which shows that the anomaly is controlled by the bulk classical gauge theory L_{θ} . The rough idea for the construction of $Z_{\omega'}$ is to modify the partial orbifold construction used in Sect. 3.3 in a way suited to the construction of boundary states. Let us fix a G -bundle P on M . To define the partition function we want to perform an integration over the preimage of P under λ_* . However, in the presence of gauge transformations, requiring two bundles to be the same is not natural. Hence we use the homotopy fibre $\lambda_*^{-1}[P]$ as a groupoid with objects the pairs (\widehat{P}, h) of a \widehat{G} -bundle \widehat{P} and a gauge transformation $h: \lambda_* \widehat{P} \rightarrow P$. Morphisms are gauge transformations $\widehat{h}: \widehat{P} \rightarrow \widehat{P}'$ which are compatible with h and h' . We show that

$$L_{\omega'}(M) := \int_M \psi_{\widehat{P}}^* \omega' \int_{[0,1] \times M} h^* \theta$$

² This integral is not actually well-defined as a complex number, see Sect. 2 for details. We ignore this subtlety in the present section.

is gauge-invariant with respect to morphisms in $\lambda_*^{-1}[P]$. Hence we define the partition function of $Z_{\omega'}$ on M as³

$$Z_{\omega'}(M) := \int_{\lambda_*^{-1}[P]} L_{\omega'}(M) .$$

Similar adaptations of the partial orbifold construction lead to the definition of state space in Sect. 4.2. The groupoid of symmetries acts only projectively on this state space. Using a result of [MW18] we show that the 2-cocycle twisting this projective representation is the transgression of θ to the groupoid of G -bundles. With this construction we provide an explicit demonstration of the anomaly inflow mechanism at the level of both partition functions and state spaces, which renders the composite bulk-boundary field theory free from anomalies.

Related work. Section 3 can be regarded as a formulation of the main ideas of [KT14] in the framework of functorial field theories and homotopy theory. The abstract formulation allows us to also treat non-abelian gauge groups D , which are not considered in [KT14]. A large part of our discussion should generalise and provide functorial descriptions of higher-form symmetries, as in e.g. [KT17a, KS14, GKSW15, TvK15, Tac17, DT18, BCH19], and of invertible topological sigma-models, as in [TE18, Tho17]. It should also be straightforward to include time-reversal symmetries using the techniques of [You18].

The relative field theory constructed in Sect. 4 can be regarded as a formulation of the gapped boundary states constructed in [Wit16], together with the explicit lattice gauge theory and Hamiltonian constructions of [WWW18], in the framework of relative field theories. We extend this construction to give the state space of the quantum field theory explicitly; our boundary field theory $Z_{\omega'}$ formally realises the new boundary degrees of freedom of the \widehat{G} -symmetry extended boundary states from [WWW18] in this language. Proposition 3.31 provides a clear relation between the works [KT14] and [Wit16]: We explicitly show that if all obstructions except the last one vanish in the spectral sequence, then one is in the set-up of [Wit16]. Similar considerations appear in [TvK15], see also [WWW18, Tac17] for different perspectives.

Recently the two-dimensional Ising model has been formulated as a field theory relative to a three-dimensional discrete gauge theory with trivial topological action [FT18]. It is conjectured that the low-energy effective field theory of the Ising model is topological, and topological boundary field theories are constructed using the Cobordism Hypothesis. Our formalism should provide an explicit construction of these field theories.

Conventions and notation. For the convenience of the reader, we summarise here our notation and conventions which are used throughout this paper. For 2-categories we use the definitions outlined in [MS18, Appendix B]. For integration over groupoids we use [SW19, Appendix A].

- Let G be a group. We denote by BG the classifying space of G , which for G finite is an Eilenberg-MacLane space $K(G, 1)$, i.e. its only non-trivial homotopy group is $\pi_1(BG) = G$. Let P be a principal G -bundle on a manifold M . We denote by $\psi_P : M \rightarrow BG$ the corresponding classifying map.
- Let T be a topological space, n a positive integer and A an abelian group. We denote the pairing of chains and cochains on T by $\langle \cdot, \cdot \rangle : C^n(T; A) \times C_n(T) \rightarrow A$.

³ We recall integration over essentially finite groupoids in Sect. 2.

- Let G be a group. The groupoid of G -bundles on a manifold M is denoted by $\text{Bun}_G(M)$.
- For most constructions in this paper we fix a positive integer n . After fixing such an integer, we use M , Σ and S to denote manifolds of dimensions n , $n - 1$ and $n - 2$, respectively.
- Let M be an oriented manifold. We denote by $-M$ the same manifold equipped with the opposite orientation.
- If \mathcal{C} is a category we define $\underline{\mathcal{C}}$ to be the (strict) 2-category obtained by adding identity 2-morphisms to \mathcal{C} .
- Let \mathcal{C} be a monoidal category. We denote by $*//\mathcal{C}$ the bicategory with one object and \mathcal{C} as endomorphisms.
- Let $F: \mathcal{G} \rightarrow \mathcal{G}'$ be a functor between groupoids and $g' \in \mathcal{G}'$. We denote by $F^{-1}[g']$ the homotopy fibre of g' . Explicitly, the groupoid $F^{-1}[g']$ has as objects the pairs (g, h') , with $g \in \mathcal{G}$ and $h': F(g) \rightarrow g'$ an isomorphism, and morphisms $m: (g_1, h'_1) \rightarrow (g_2, h'_2)$ comprising of a morphism $m: g_1 \rightarrow g_2$ such that the diagram

$$\begin{array}{ccc}
 F(g_1) & \xrightarrow{F(m)} & F(g_2) \\
 & \searrow h'_1 & \swarrow h'_2 \\
 & & g'
 \end{array}$$

commutes.

- Let $\lambda: G \rightarrow G'$ be a homomorphism of groups. We denote the induced maps $BG \rightarrow BG'$ and $*//G \rightarrow *//G'$ again by λ .
- Let $F: \mathcal{C} \rightarrow \mathcal{C}'$ be a functor. We write the limit of F as an end $\int_{\mathcal{C}} F$.
- Let G be a finite group. We denote by $G\text{-Cob}_n$ the category and by $G\text{-Cob}_{n,n-1,n-2}$ the bicategory of cobordisms equipped with maps into BG .
- We denote by Grp the category of groups.
- We denote by Vect the category of finite-dimensional \mathbb{C} -vector spaces.
- We denote by 2Vect the bicategory of Kapranov–Voevodsky 2-vector spaces.
- We denote by Cat the bicategory of (small) categories.

2. Dijkgraaf–Witten Theory

Dijkgraaf–Witten theories are topological gauge theories with finite gauge group D . In this section we introduce the framework of (extended) functorial quantum field theory, and subsequently construct classical and quantum Dijkgraaf–Witten theories.

2.1. Equivariant functorial field theories. The idea of describing field theories as functors from a geometric category to a category of vector spaces goes back at least to Atiyah’s definition of topological quantum field theories [Ati88] and Segal’s definition of conformal field theories [Seg88]. The general framework allows for arbitrary background fields. We denote the collection of all background fields by⁴ \mathcal{F} . Typical examples of background fields in physics are metrics, spin or spin^c structures, framings and principal bundles with connection. If the collection of background fields does not contain a metric,

⁴ For concreteness one can model background fields as a (∞) -stack \mathcal{F} on the category of smooth n -dimensional manifolds (with corners). The language of stacks is not used in this paper.

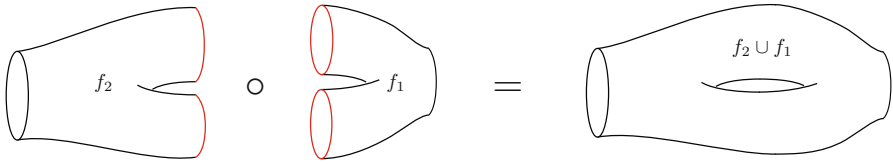


Fig. 1. Composition of two cobordisms equipped with background fields f_1 and f_2

the corresponding field theory is called topological. We only consider theories involving principal bundles with finite gauge group D as background fields in this paper.

An n -dimensional functorial field theory is a symmetric monoidal functor from the geometric category $\text{Cob}_n^{\mathcal{F}}$, modelling spacetimes with background fields, to the category of vector spaces. The category $\text{Cob}_n^{\mathcal{F}}$ is roughly defined as follows: Objects are $n-1$ -dimensional manifolds Σ equipped with \mathcal{F} -background fields. A morphism $\Sigma_1 \rightarrow \Sigma_2$ is an n -dimensional cobordism, i.e. a manifold M with boundary $\Sigma_1 \sqcup \Sigma_2$ equipped with compatible \mathcal{F} -background fields. Composition is defined by gluing along common boundaries (Fig. 1). The disjoint union of manifolds makes $\text{Cob}_n^{\mathcal{F}}$ into a symmetric monoidal category.

An n -dimensional functorial field theory with \mathcal{F} -background fields is a symmetric monoidal functor

$$Z: \text{Cob}_n^{\mathcal{F}} \rightarrow \text{Vect}_{\text{top}}$$

with target the category of topological vector spaces. All vector spaces appearing in this paper will be finite-dimensional and hence have a unique topology. We denote by Vect the category of finite-dimensional \mathbb{C} -vector spaces.

This definition ensures that $Z(\emptyset) \cong \mathbb{C}$ for every functorial field theory Z . For this reason, Z assigns to a closed n -dimensional manifold M with background fields f a linear map $Z(M, f): Z(\emptyset) \cong \mathbb{C} \rightarrow \mathbb{C} \cong Z(\emptyset)$, which can be identified with a complex number, the partition function of Z on M . The vector space associated to an $n-1$ -dimensional manifold Σ can be interpreted as the state space of the quantum field theory Z on Σ . Making the definition precise is quite involved, see e.g. [ST11]. A detailed discussion of this approach to quantum field theory can be found in [Seg11].

Remark 2.1. The functorial framework does not only describe quantum field theories, but also classical field theories. The invariant assigned to a closed manifold with background fields by a classical field theory is the exponentiated action. A classical field theory has no Hilbert space of states, however the exponentiated action on a manifold with boundary might not be well-defined without making additional choices on the boundary. In this case it is more natural to consider the exponentiated action not as a complex number, but as an element in a complex line associated to the additional choices. A famous example is Chern-Simons theory [Fre95], where the action functional on a manifold with boundary is not gauge-invariant and the gauge variation is controlled by the Wess-Zumino term. The n -dimensional functorial field theory corresponding to a classical field theory assigns these complex lines to $n-1$ -dimensional manifolds.

All theories appearing in this paper are homotopy quantum field theories [Tur10] with aspherical target. For these the background fields consist of principal bundles with finite gauge group D and an orientation. We describe principal D -bundles on a manifold M by their classifying map $M \rightarrow BD$. However, by a slight abuse of notation we

will freely switch between classifying maps and principal bundles without explicitly mentioning so. In particular, we will pull back cochains on BD along bundles, which should always be understood as a pullback along the classifying map. We denote the corresponding cobordism category by $D\text{-Cob}_n$, for which we can make the sketch above precise [Tur10].

Definition 2.2. For an oriented manifold M , we denote by $-M$ the same manifold equipped with the opposite orientation. Let D be a finite group. We denote by $D\text{-Cob}_n$ the symmetric monoidal category defined by:

- (a) An object is a closed oriented $n-1$ -dimensional manifold Σ equipped with a continuous map $\varphi: \Sigma \rightarrow BD$.
- (b) A morphism $(\Sigma_1, \varphi_1) \rightarrow (\Sigma_2, \varphi_2)$ is an equivalence class of triples (M, ψ, ϕ) , where M is a compact oriented manifold with boundary, $\phi: -\Sigma_1 \sqcup \Sigma_2 \rightarrow \partial M$ is an orientation-preserving diffeomorphism, and $\psi: M \rightarrow BD$ is a continuous map such that $\varphi_i = \psi \circ \phi|_{\Sigma_i}$ for $i = 1, 2$.

Two such triples are equivalent if the underlying manifolds differ by a diffeomorphism $M \rightarrow M'$ relative to the boundary such that $M \xrightarrow{\psi} BG$ and $M \rightarrow M' \xrightarrow{\psi'} BD$ differ by a homotopy relative to the boundary.

The composition is defined by gluing manifolds and continuous maps along boundaries. The monoidal structure is given by the disjoint union of manifolds.

Definition 2.3. An n -dimensional D -equivariant field theory is a symmetric monoidal functor

$$D\text{-Cob}_n \rightarrow \text{Vect}.$$

We denote by $D\text{-TFT}_n$ the category of n -dimensional D -equivariant field theories and natural symmetric monoidal transformations. Let Cob_n be the category $e\text{-Cob}_n$, where e is the unique group with one element. Objects of this category are oriented spacetimes without background gauge fields.

Later on we will also need extended field theories which assign Kapranov–Voevodsky 2-vector spaces, i.e. semi-simple \mathbb{C} -linear categories with finitely-many simple objects [KV94], to closed $n-2$ -dimensional manifolds S equipped with maps $\xi: S \rightarrow BD$, functors to $n-1$ -dimensional cobordisms equipped with compatible maps into BD , and natural transformations to n -dimensional cobordisms between cobordisms (manifolds with corners) equipped with compatible maps into BD . To define extended field theories properly one has to introduce a symmetric monoidal bicategory $D\text{-Cob}_{n,n-1,n-2}$ similar to the category of Definition 2.2. For a concrete definition we refer to [SW18a]. An n -dimensional extended D -equivariant field theory is a symmetric monoidal 2-functor

$$D\text{-Cob}_{n,n-1,n-2} \rightarrow 2\text{Vect}$$

where 2Vect is the symmetric monoidal 2-category of Kapranov–Voevodsky 2-vector spaces. The only extended field theory we need in this paper is the classical Dijkgraaf–Witten theory explicitly described in Sect. 2.2.

The symmetric monoidal structure of 2Vect , i.e. the Deligne tensor product \boxtimes , defines a tensor product of field theories. A field theory which has an inverse with respect to this tensor product is called invertible. Classical field theories regarded as functorial field theories are always invertible. Invertible field theories are physically interesting, since they describe the low-energy effective field theories of short-range entangled topological phases and field theories with anomalies can be described as boundary states of invertible field theories (see Sect. 4).

2.2. *Classical Dijkgraaf–Witten theories.* We provide a brief summary of the construction of an invertible extended field theory E_ω depending on an n -cocycle $\omega \in Z^n(BD; U(1))$ which is given in [MW18], to which we refer for more details and all proofs. We denote the corresponding unextended field theory by L_ω .

The possible actions of n -dimensional topological gauge theories with finite gauge group D are classified by the n -th cohomology group of the classifying space BD with coefficients in \mathbb{R}/\mathbb{Z} [DW90]. For a fixed representative $\omega' \in Z^n(BD; \mathbb{R}/\mathbb{Z})$ of a cohomology class, the action for a D -bundle $\psi : M \rightarrow BD$ on an oriented closed n -dimensional manifold M is given by

$$S_{\omega'}(M, \psi) = \int_M \psi^* \omega' .$$

As a real number, this action is only well-defined modulo \mathbb{Z} by definition. The quantity with physical relevance is the exponentiated action $\exp(2\pi i S_{\omega'}(M, \psi))$ which takes values in $U(1)$. For simplicity we work from now on with a cocycle $\omega \in Z^n(BD; U(1))$ and interpret its integration over the manifold as the exponentiated action. Formally, we can define integration as the evaluation of $\psi^* \omega$ on a representative σ_M of the fundamental class $[\sigma_M] \in H_n(M)$ of M . This defines the partition function

$$L_\omega(M, \psi) := \langle \psi^* \omega, \sigma_M \rangle .$$

Here the brackets denote the evaluation of cochains on chains.

On manifolds with boundary the exponentiated action is not well-defined as a complex number: To naively extend the exponentiated action to an n -dimensional manifold M with boundary $\partial M = \Sigma$ and D -bundle $\psi : M \rightarrow BD$ we would pick a representative $\sigma_M \in C_n(M)$ of the fundamental class $[\sigma_M] \in H_n(M, \Sigma)$ and evaluate $\psi^* \omega$ on this representative. The problem is that $[\sigma_M]$ is an element in relative homology, but $\psi^* \omega$ is an element of ordinary cohomology. As a representative of a relative homology class, σ_M has a boundary $\partial \sigma_M \in C_{n-1}(\Sigma)$ and $\langle \psi^* \omega, \sigma_M \rangle$ depends on $\partial \sigma_M$ in general. Hence the exponentiated action is not well-defined as a complex number, but rather it is an element of a complex line $L_\omega(\Sigma, \psi|_\Sigma)$. In [FQ93] it is shown that the vector spaces $L_\omega(\cdot, \cdot)$ are part of a functorial field theory $L_\omega : D\text{-Cob}_n \rightarrow \text{Vect}$. This field theory is the classical Dijkgraaf–Witten theory corresponding to the topological action ω .

Now we sketch how to promote L_ω to an extended field theory $E_\omega : D\text{-Cob}_{n,n-1,n-2} \rightarrow 2\text{Vect}$. It is convenient to introduce, for an arbitrary oriented manifold S of dimension k , the groupoid $\text{Fund}(S)$ whose objects are cycles $\sigma_S \in Z_k(S)$ representing the fundamental class of S and morphisms $\tau : \sigma_S \rightarrow \sigma'_S$ are given by chains $\tau \in C_{k+1}(S)$ satisfying $\partial \tau = \sigma'_S - \sigma_S$. To an object of $D\text{-Cob}_{n,n-1,n-2}$, i.e. an oriented $n-2$ -dimensional manifold S equipped with a continuous map $\xi : S \rightarrow BD$, we assign the 2-vector space $E_\omega(S, \xi)$ whose objects are finite formal sums $\sum_{i=1}^p V_i * \sigma_i$ of representatives σ_i of the fundamental class of S and finite-dimensional vector spaces V_i . We will abbreviate $\mathbb{C} * \sigma$ with σ . The space of morphisms between $\sigma, \sigma' \in E_\omega(S, \xi)$ is given by

$$\text{Hom}_{E_\omega(S, \xi)}(\sigma, \sigma') := \frac{\mathbb{C}[\text{Hom}_{\text{Fund}(S)}(\sigma, \sigma')]}{\sim_\omega} , \tag{2.4}$$

where $\mathbb{C}[\text{Hom}_{\text{Fund}(S)}(\sigma, \sigma')]$ is the complex vector space generated by the set $\text{Hom}_{\text{Fund}(S)}(\sigma, \sigma')$, and for two morphisms $\tau, \tilde{\tau} : \sigma \rightarrow \sigma'$ we make the identification

$$\tilde{\tau} \sim_\omega \langle \xi^* \omega, \Lambda \rangle \tau , \tag{2.5}$$

whenever there exists $\Lambda \in C_n(S)$ such that $\tilde{\tau} - \tau = \partial\Lambda$. In (2.5) the particular choice of Λ is immaterial. In order to obtain the morphism spaces between arbitrary objects in $E_\omega(S, \xi)$, we extend (2.4) bilinearly:

$$\begin{aligned} \text{Hom}_{E_\omega(S, \xi)} \left(\sum_{i=1}^p V_i * \sigma_i, \sum_{j=1}^m V_j * \sigma_j \right) \\ = \bigoplus_{i=1}^p \bigoplus_{j=1}^m \text{Hom}_{\mathbf{Vect}}(V_i, V_j) \otimes_{\mathbb{C}} \text{Hom}_{E_\omega(S, \xi)}(\sigma_i, \sigma_j). \end{aligned}$$

Composition is defined by matrix multiplication together with composition in \mathbf{Vect} and $\mathbf{Fund}(S)$. The category $E_\omega(S, \xi)$ carries a natural \mathbf{Vect} -module structure given by

$$\begin{aligned} \mathbf{Vect} \times E_\omega(S, \xi) &\longrightarrow E_\omega(S, \xi) \\ V \times (W * \sigma) &\longmapsto (V \otimes_{\mathbb{C}} W) * \sigma. \end{aligned}$$

Given a 1-morphism $(\Sigma, \varphi): (S_1, \xi_1) \longrightarrow (S_2, \xi_2)$ in $D\text{-Cob}_{n, n-1, n-2}$, together with representatives σ_1 and σ_2 for the fundamental classes of S_1 and S_2 , respectively, we define the vector space

$$\Sigma^\varphi(\sigma_2, \sigma_1) = \frac{\mathbb{C}[\mathbf{Fund}_{\sigma_1}^{\sigma_2}(\Sigma)]}{\sim_\varphi},$$

where $\mathbf{Fund}_{\sigma_1}^{\sigma_2}(\Sigma)$ is the set of cycles relative to $\partial\Sigma$ representing the fundamental class of Σ with boundary $\sigma_2 - \sigma_1$. The equivalence relation is defined by $\zeta' \sim_\varphi \langle \varphi^*\omega, \Lambda \rangle \zeta$ for arbitrary $\Lambda \in C_n(\Sigma)$ with boundary $\zeta - \zeta'$. This equivalence relation is different from (2.5). With this notation we define $E_\omega(\Sigma, \varphi)$ via the coend

$$E_\omega(\Sigma, \varphi)(\sigma_1) = \int^{\sigma_2 \in \mathbf{Fund}(S_2)} \Sigma^\varphi(\sigma_2, \sigma_1) * \sigma_2$$

and linear extensions. These functors are compatible with the composition of cobordisms only up to natural isomorphisms. For 1-morphisms $(\Sigma_a, \varphi_a): (S_1, \xi_1) \longrightarrow (S_2, \xi_2)$ and $(\Sigma_b, \varphi_b): (S_2, \xi_2) \longrightarrow (S_3, \xi_3)$ the natural isomorphism is induced by the linear map

$$\begin{aligned} \Sigma_b^{\varphi_b}(\sigma_3, \sigma_2) \otimes_{\mathbb{C}} \Sigma_a^{\varphi_a}(\sigma_2, \sigma_1) &\longrightarrow (\Sigma_b \circ \Sigma_a)^{\varphi_b \cup \varphi_a}(\sigma_3, \sigma_1) \\ \sigma_{\Sigma_b} \otimes_{\mathbb{C}} \sigma_{\Sigma_a} &\longmapsto \sigma_{\Sigma_b} + \sigma_{\Sigma_a}. \end{aligned}$$

In [MW18] it is shown that these coherence morphisms are closely related to the transgression of ω . They encode interesting physical properties as we will see in Sect. 4.1.

There are further coherence isomorphisms

$$\Phi_{(S, \xi)}: \text{id}_{E_\omega(S, \xi)} \implies E_\omega(\text{id}_{(S, \xi)}) \tag{2.6}$$

encoding the compatibility with identities for all objects (S, ξ) of $D\text{-Cob}_{n, n-1, n-2}$. They are defined as follows. Using the enriched co-Yoneda Lemma we can write the identity as the coend

$$\text{id}_{E_\omega(S, \xi)}(\cdot) \cong \int^{\sigma \in E_\omega(S, \xi)} \text{Hom}_{E_\omega(S, \xi)}(\sigma, \cdot) * \sigma.$$

Without loss of generality we can evaluate this at a representative σ_0 of the fundamental class of S :

$$\sigma_0 \cong \int^{\sigma \in E_\omega(S, \xi)} \text{Hom}_{E_\omega(S, \xi)}(\sigma, \sigma_0) * \sigma \cong \int^{\sigma \in \text{Fund}(S)} \text{Hom}_{E_\omega(S, \xi)}(\sigma, \sigma_0) * \sigma .$$

On the other hand we have

$$E_\omega(\text{id}_{(S, \xi)})(\sigma_0) = \int^{\sigma \in \text{Fund}(S)} (S \times [0, 1])^{\xi \times \text{id}_{[0, 1]}}(\sigma, \sigma_0) * \sigma .$$

There is a natural isomorphism

$$\begin{aligned} (S \times [0, 1])^{\xi \times \text{id}_{[0, 1]}}(\sigma, \sigma_0) &\longrightarrow \text{Hom}_{E_\omega(S, \xi)}(\sigma, \sigma_0) \\ \mu &\longmapsto -p_S * \mu \end{aligned}$$

defined by the projection $p_S : S \times [0, 1] \rightarrow S$, which induces the natural transformation (2.6).

To a 2-morphism $(M, \psi) : (\Sigma_a, \varphi_a) \rightrightarrows (\Sigma_b, \varphi_b)$ between 1-morphisms (Σ_a, φ_a) and (Σ_b, φ_b) from (S_1, ξ_1) to (S_2, ξ_2) in $D\text{-Cob}_{n, n-1, n-2}$, we assign the natural transformation

$$E_\omega(M, \psi) : E_\omega(\Sigma_a, \varphi_a) \rightrightarrows E_\omega(\Sigma_b, \varphi_b)$$

between the functors $E_\omega(\Sigma_a, \varphi_a), E_\omega(\Sigma_b, \varphi_b) : E_\omega(S_1, \xi_1) \rightarrow E_\omega(S_2, \xi_2)$ consisting of the natural maps

$$E_\omega(\Sigma_a, \varphi_a)(\sigma_1) \rightarrow E_\omega(\Sigma_b, \varphi_b)(\sigma_1)$$

for $\sigma_1 \in \text{Fund}(S_1)$, which are the maps between the respective coends induced by the linear maps

$$E_\omega(M, \psi)_{\sigma_2, \sigma_1} : \Sigma_a^{\varphi_a}(\sigma_2, \sigma_1) \rightarrow \Sigma_b^{\varphi_b}(\sigma_2, \sigma_1) \tag{2.7}$$

defined as follows: For $[\mu_a] \in \text{Fund}_{\sigma_1}^{\sigma_2}(\Sigma_a)$ and $[\mu_b] \in \text{Fund}_{\sigma_1}^{\sigma_2}(\Sigma_b)$ we can find a fundamental cycle ν of M which is compatible with these fundamental cycles on the boundary [MW18, eq. (3.11)]:

$$\partial \nu = \mu_b - \mu_a + (-1)^{n-2} (\sigma_1 \times [0, 1] - \sigma_2 \times [0, 1]) .$$

Mapping $[\mu_a]$ to $\langle \psi^* \omega, \nu \rangle [\mu_b]$ yields a well-defined linear map $\mathbb{C}[\text{Fund}_{\sigma_1}^{\sigma_2}(\Sigma_a)] \rightarrow \Sigma_b^{\varphi_b}(\sigma_2, \sigma_1)$, which descends to $\Sigma_a^{\varphi_a}(\sigma_2, \sigma_1)$ and induces the map (2.7).

The construction sketched above fits into an extended field theory [MW18, Theorem 3.19], the classical Dijkgraaf–Witten theory with action ω . Restricting E_ω to the endomorphism category of the empty set reproduces the ordinary field theory

$$L_\omega : \text{End}_{D\text{-Cob}_{n, n-1, n-2}}(\emptyset) = D\text{-Cob}_n \rightarrow \text{Vect} = \text{End}_{2\text{Vect}}(\text{Vect})$$

mentioned above. The functor L_ω admits the following concrete description:

- To a closed $n-1$ -dimensional manifold Σ equipped with a map $\varphi : \Sigma \rightarrow BD$ it assigns the vector space $L_\omega(\Sigma, \varphi) = \Sigma^\varphi(\emptyset, \emptyset) = \mathbb{C}[\text{Fund}(\Sigma)] / \sim_\varphi$.
- To a morphism $(M, \psi) : (\Sigma_a, \varphi_a) \rightarrow (\Sigma_b, \varphi_b)$ it assigns the linear map

$$\begin{aligned} L_\omega(M, \psi) : L_\omega(\Sigma_a, \varphi_a) &\rightarrow L_\omega(\Sigma_b, \varphi_b) \\ [\sigma_a] &\longmapsto \langle \psi^* \omega, \sigma_M \rangle [\sigma_b] , \end{aligned}$$

with $\sigma_M \in \text{Fund}_{\sigma_a}^{\sigma_b}(M)$.

2.3. *Quantum Dijkgraaf–Witten theories.* Let ω be an n -cochain on BD with values in $U(1)$. We define the quantum Dijkgraaf–Witten theory Z_ω with topological action ω by performing a ‘path integral’ of the classical field theory L_ω over the space of field configurations $\mathbf{Bun}_D(M)$. One of the problems with finding a rigorous definition of general quantum field theories is the lack of a measure on the space of field configurations. In the case of Dijkgraaf–Witten theories the space of field configurations is the essentially finite groupoid⁵ $\mathbf{Bun}_D(M)$ of D -bundles on M and a well-defined integration theory exists. The groupoid cardinality [BHW10] induces a natural counting measure on $\mathbf{Bun}_D(M)$. The integral of a gauge-invariant function $f : \mathbf{Obj}(\mathbf{Bun}_D(M)) \rightarrow \mathbb{C}$ over $\mathbf{Bun}_D(M)$ is

$$\int_{\mathbf{Bun}_D(M)} f := \sum_{\varphi \in \pi_0(\mathbf{Bun}_D(M))} \frac{f(\varphi)}{|\mathbf{Aut}_{\mathbf{Bun}_D(M)}(\varphi)|},$$

where $\pi_0(\mathbf{Bun}_D(M))$ is the set of isomorphism classes of principal D -bundles on M , each of which is weighted by the inverse of the order of its automorphism group.

The existence of a well-defined path integral makes quantization in principal straightforward. A slight subtlety arises, since the action on a manifold with boundary is not a complex number, but rather an element of a complex line as we have seen in Sect. 2.2. For this reason, we have to assign to an $n-1$ -dimensional manifold Σ the vector space [FQ93]

$$Z_\omega(\Sigma) = \lim_{\mathbf{Bun}_D(\Sigma)} L_\omega = \int_{\mathbf{Bun}_D(\Sigma)} L_\omega, \tag{2.8}$$

writing the limit as an end in the second step. The functor L_ω can be evaluated on the groupoid $\mathbf{Bun}_D(\Sigma)$ by considering a gauge transformation h as a homotopy $h : [0, 1] \times \Sigma \rightarrow BD$ between classifying maps, i.e. as a morphism in $D\text{-Cob}_n$. The correspondence between homotopies of classifying maps and gauge transformations only holds for discrete gauge groups. A useful realisation of the limit is given by the vector space of parallel sections of $L_\omega(\Sigma, \cdot)$. A parallel section f of $L_\omega(\Sigma, \cdot)$ consists of an element $f(\varphi) \in L_\omega(\Sigma, \varphi)$ for all $\varphi \in \mathbf{Bun}_D(\Sigma)$ such that $L_\omega([0, 1] \times \Sigma, h)(f(\varphi)) = f(\varphi')$ for all gauge transformations $h : \varphi \rightarrow \varphi'$. The space of parallel sections can be regarded as the space of gauge-invariant functions on the set of classical gauge field configurations. For this reason the definition can be interpreted as an implementation of the Gauss Law in quantum gauge theory, which requires that physical states must be gauge-invariant.

Now consider a cobordism $M : \Sigma_1 \rightarrow \Sigma_2$. To define $Z_\omega(M)$ we introduce for $\varphi_2 \in \mathbf{Bun}_D(\Sigma_2)$ the homotopy fibre $\mathbf{Bun}_D(M)|_{\varphi_2}$ as the homotopy pullback

$$\begin{array}{ccc} \mathbf{Bun}_D(M)|_{\varphi_2} & \longrightarrow & \mathbf{Bun}_D(M) \\ \downarrow & & \downarrow \text{pr}_{\Sigma_2} \\ * & \xrightarrow{\varphi_2} & \mathbf{Bun}_D(\Sigma_2) \end{array}$$

where pr_{Σ_2} is the pullback functor induced by the inclusion $\Sigma_2 \hookrightarrow M$. The groupoid $\mathbf{Bun}_D(M)|_{\varphi_2} = \text{pr}_{\Sigma_2}^{-1}[\varphi_2]$ of gauge fields on M restricting, up to gauge transformations, to φ_2 on Σ_2 can be concretely described as follows: Objects are pairs (ψ, h) where

⁵ A groupoid is essentially finite if all morphism sets are finite and there exists an equivalence to a groupoid with a finite number of objects. The field configurations form a groupoid, since \mathbf{Bun}_D is a stack.

ψ is a bundle over M and $h: \psi|_{\Sigma_2} \rightarrow \varphi_2$ is a gauge transformation. Morphisms $(\psi, h) \rightarrow (\psi', h')$ are gauge transformations $\psi \rightarrow \psi'$ such that the diagram

$$\begin{array}{ccc} \psi|_{\Sigma_2} & \longrightarrow & \psi'|_{\Sigma_2} \\ & \searrow h & \swarrow h' \\ & & \varphi_2 \end{array}$$

commutes. We fix representatives σ_1 and σ_2 of the fundamental classes of Σ_1 and Σ_2 , respectively. This allows us to express the value of a parallel section $f \in Z_\omega(\Sigma_1)$ on a principal D -bundle $\varphi_1 \in \text{Bun}_D(\Sigma_1)$ as $f(\varphi_1) = \mathbf{f}(\varphi_1)[\sigma_1]$ with $\mathbf{f}(\varphi_1) \in \mathbb{C}$. Then we can define

$$Z_\omega(M)(f)(\varphi_2) = \left(\int_{(\psi, h) \in \text{Bun}_D(M)|_{\varphi_2}} \langle h^* \omega, [0, 1] \times \sigma_2 \rangle \langle \psi^* \omega, \sigma_M \rangle \mathbf{f}(\psi|_{\Sigma_1}) \right) [\sigma_2], \tag{2.9}$$

where σ_M is a representative for the fundamental class of M satisfying $\partial \sigma_M = \sigma_2 - \sigma_1$ and we consider the gauge transformation h as a homotopy $h: [0, 1] \times \Sigma_2 \rightarrow BD$. This definition is independent of all choices involved.

The assignments (2.8) and (2.9) define a functorial field theory [FQ93]

$$Z_\omega: \text{Cob}_n \rightarrow \text{Vect}.$$

Remark 2.10. A more systematic way of constructing Z_ω is given by applying the orbifold construction of [SW19] to the classical field theory L_ω . In [MW18] the extended orbifold construction of [SW18a] is used to construct Dijkgraaf–Witten theories as extended field theories. Three-dimensional extended Dijkgraaf–Witten theories are also constructed in [Mor15].

We conclude this section with a few examples, which we will consider throughout this paper.

Example 2.11. In two dimensions, for any finite group D and a 2-cocycle $\omega \in Z^2(BD; U(1))$, the vector space $Z_\omega(\mathbb{S}^1)$ associated to the circle \mathbb{S}^1 is the space of ω -twisted characters on D , because the groupoid of principal D -bundles on \mathbb{S}^1 is equivalent to the action groupoid $D//D$ for the adjoint action of D on itself, and hence a parallel section is just a map $f: D \rightarrow \mathbb{C}$ satisfying $f(d d' d^{-1}) = \omega(d, d') f(d')$ for all $d, d' \in D$. The invariant

$$Z_\omega(\mathbb{T}^2) = \frac{1}{|D|} \sum_{\substack{(d, d') \in D \times D \\ d d' = d' d}} \frac{\omega(d, d')}{\omega(d', d)}$$

associated to the two-dimensional torus \mathbb{T}^2 is the number of irreducible ω -twisted representations of D . We give a few concrete examples of 2-cocycles:

(a) The group cohomology $H^2(\mathbb{Z}_N \times \mathbb{Z}_N; U(1))$ is \mathbb{Z}_N . If we write the cyclic group \mathbb{Z}_N additively then the non-trivial 2-cocycle corresponding to $k \in \{0, 1, \dots, N - 1\}$ is

$$\omega_k((a_1, b_1), (a_2, b_2)) = \exp\left(\frac{2\pi i k}{N} a_1 b_2\right) \tag{2.12}$$

with $(a_1, b_1), (a_2, b_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$. For $N = 2$, the partition function $Z_{\omega_1}(\mathbb{T}^2)$ on \mathbb{T}^2 for the non-trivial $\mathbb{Z}_2 \times \mathbb{Z}_2$ cocycle is 1 corresponding to the fact that there exists only one ω_1 -twisted irreducible representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ [Wil08].

(b) The degree 2 group cohomology of the dihedral group $D_8 = \langle a, b \mid a^4 = b^2 = 1, b a b^{-1} = a^{-1} \rangle$ with values in $U(1)$ is \mathbb{Z}_2 . The non-trivial 2-cocycle is given by [Kar85, Section 3.7]

$$\omega(a^i b^j, a^{i'} b^{j'}) = \begin{cases} 1, & j = 0, \\ \exp\left(\frac{2\pi i}{4} i'\right), & j = 1. \end{cases}$$

Example 2.13. In three dimensions, for any finite group D and a 3-cocycle $\omega \in Z^3(BD; U(1))$, the invariant $Z_\omega(\mathbb{T}^3)$ associated to the three-dimensional torus \mathbb{T}^3 is the number of irreducible representations of the ω -twisted Drinfeld double of the group algebra $\mathbb{C}[D]$. The cohomology group $H^3(\mathbb{Z}_N; U(1))$ is \mathbb{Z}_N . The 3-cocycles have the concrete form [HLY14, Proposition 2.3]

$$\omega_k(a, b, c) = \exp\left(\frac{2\pi i k}{N} a \left\lfloor \frac{b+c}{N} \right\rfloor\right) \tag{2.14}$$

for $a, b, c, k \in \mathbb{Z}_N = \{0, 1, \dots, N - 1\}$, where $\lfloor r \rfloor$ denotes the integer part of the real number $r \in \mathbb{R}$, i.e. the largest integer less than or equal to r . These theories are studied in [KS14]. They have been extended to a product of an arbitrary number of cyclic groups \mathbb{Z}_{N_i} (i.e. a generic finite abelian group) in [CGLW13, WSW15].

3. Discrete Symmetries and 't Hooft Anomalies

In this section we study the action of a finite symmetry group G on a classical Dijkgraaf–Witten theory $L_\omega: D\text{-Cob}_n \rightarrow \text{Vect}$ with gauge group D and topological action $\omega \in Z^n(BD; U(1))$. General symmetries of abelian quantum Dijkgraaf–Witten theories are discussed in [FPSV15]. We only consider symmetries arising from an action of G on D -gauge fields which preserve $\omega \in Z^n(BD; U(1))$. Following [FPSV15] we call these kinematical symmetries. We show that they extend to the quantum theory and study their gauging. Gauging these symmetries is not always possible and the obstructions are encoded in a spectral sequence.

3.1. Discrete symmetries of Dijkgraaf–Witten theories. For a symmetry to be compatible with cutting and gluing of manifolds, we describe it as an endofunctor of $D\text{-Cob}_n$ acting by pullback along the inverse on a field theory. There is a natural way to construct endofunctors of $D\text{-Cob}_n$ from homeomorphisms of BD which is described by a 2-functor

$$\mathcal{R}: *//\Pi_1[BD, BD] \rightarrow *//\text{End}(D\text{-Cob}_n), \tag{3.1}$$

where $\Pi_1[BD, BD]$ is the category with continuous maps $BD \rightarrow BD$ as objects and equivalence classes of homotopies as morphisms. Concretely, \mathcal{R} sends a continuous map $\chi: BD \rightarrow BD$ to the endofunctor

$$\begin{aligned} \mathcal{R}(\chi): D\text{-Cob}_n &\rightarrow D\text{-Cob}_n \\ (\Sigma, \varphi: \Sigma \rightarrow BD) &\mapsto (\Sigma, \chi \circ \varphi: \Sigma \rightarrow BD) \\ ((M, \psi): (\Sigma_1, \varphi_1) \rightarrow (\Sigma_2, \varphi_2)) &\mapsto ((M, \chi \circ \psi): (\Sigma_1, \chi \circ \varphi_1) \rightarrow (\Sigma_2, \chi \circ \varphi_2)) \end{aligned}$$

and a homotopy $h: \chi_1 \rightarrow \chi_2$ to the natural transformation $\mathcal{R}(h): \mathcal{R}(\chi_1) \Rightarrow \mathcal{R}(\chi_2)$ with components

$$\mathcal{R}(h)_{(\Sigma, \varphi)} = ([0, 1] \times \Sigma, h \circ \text{id}_\varphi)$$

where $h \circ \text{id}_\varphi$ denotes the horizontal composition of homotopies. The naturality of $\mathcal{R}(h)$ follows from the proof of [SW19, Proposition 2.9]. By the bicategorical Yoneda Lemma, automorphisms of BD correspond to automorphisms of the stack of principal D -bundles (which is represented by BD). Hence a symmetry corresponding to a homeomorphism of BD acts on the space of field configurations.

To define a symmetry, the group G only has to act up to gauge transformations. For finite groups, gauge transformations and homotopies between classifying maps are in one-to-one correspondence. For this reason we expect a symmetry for every action of G on BD up to a ‘homotopy’ which preserves ω . Since BD is a homotopy 1-type we can work with the following concrete description.

Definition 3.2. An action of G on BD up to (coherent) homotopy is a 2-functor

$$\alpha: \ast//G \rightarrow \ast//\Pi_1[BD, BD],$$

where $\ast//G$ is the 2-category with one object, the group G as 1-morphisms and only identity 2-morphisms.

Remark 3.3. To unpack this compact definition note that the category $\Pi_1[BD, BD]$ is equivalent to the action groupoid

$$[\pi_1(BD), \pi_1(BD)]//D = \text{End}_{\text{Grp}}(D)//D,$$

where the action of D on a group homomorphism is by conjugation. Every action of G up to homotopy takes values in the full subgroupoid $\text{Aut}_{\text{Grp}}(D)//D$ of automorphisms of D . An arbitrary 2-functor $\ast//G \rightarrow \ast//(\text{Aut}_{\text{Grp}}(D)//D)$ is called a non-abelian group cocycle [BS06, BBF05]. Hence homotopy coherent actions on BD are classified by non-abelian group cocycles. Non-abelian cocycles also appear in the construction of equivariant Dijkgraaf–Witten theories [MNS12] under the name weak 2-cocycles. We discuss non-abelian group cohomology in more detail in Sect. 3.2.

If D is abelian there are no morphisms between different objects in $\text{Aut}_{\text{Grp}}(D)//D$. This implies that an action up to homotopy of G on BD is given by a proper action of G on D and a group 2-cocycle in $H^2(BG; D)$ describing the coherence isomorphisms of the corresponding 2-functor. This agrees with the physical description in [KT14].

For every action $\alpha: \ast//G \rightarrow \ast//\Pi_1[BD, BD]$ up to homotopy the 2-functor (3.1) induces via pullbacks a 2-functor

$$\begin{aligned} \rho: \ast//G &\rightarrow D\text{-TFT}_n//\text{End}_{\text{Cat}}(D\text{-TFT}_n) \hookrightarrow \text{Cat} \\ g &\mapsto \mathcal{R}(\alpha(g^{-1}))^*, \end{aligned} \tag{3.4}$$

where we denote by Cat the 2-category of categories.

The (exponentiated) action of a gauge theory can be considered as a gauge-invariant map from the space of field configurations on an n -dimensional manifold M , in our case $\text{Bun}_D(M)$, to $U(1)$. An action of G on the space of field configurations induces an action via pullbacks on the set of gauge-invariant functions from the space of field configurations to $U(1)$. A theory admits the symmetry G if its (exponentiated) action is invariant under this action, i.e. it is a fixed point. By categorification we arrive at the following description.

Definition 3.5. A D -equivariant field theory with kinematical symmetry described by

$$\rho : \underline{\ast//G} \longrightarrow D\text{-TFT}_n // \text{End}_{\text{Cat}}(D\text{-TFT}_n) ,$$

as in (3.4), is a homotopy fixed point of ρ , i.e. a natural 2-transformation $Z : 1 \implies \rho$, where 1 is the unique 2-functor sending \ast to the category with one object and only identity morphisms.

Remark 3.6. Unpacking the definition, a D -equivariant field theory with kinematical symmetry consists of

- (a) A functor $Z : 1 \longrightarrow D\text{-TFT}_n$; and
- (b) Natural transformations $\Upsilon_g : \rho(g)[Z] \implies Z$ for all $g \in G$;

satisfying natural coherence conditions. Since 1 represents the identity 2-functor on Cat this is the same as a field theory $Z \in D\text{-TFT}_n$, together with coherent natural symmetric monoidal transformations $\Upsilon_g : \mathcal{R}(\alpha(g^{-1}))^* Z \implies Z$ for $g \in G$.

An arbitrary Dijkgraaf–Witten theory with topological action $\omega \in Z^n(BD; U(1))$ does not admit a kinematical symmetry in general. On the other hand, there may be different ways to equip a given field theory with the structure of a homotopy fixed point. We give a sufficient condition for a kinematical symmetry to exist. For this we need to introduce the following notion.

Definition 3.7. A n -cocycle $\omega \in Z^n(BD; U(1))$ is preserved by the action α if it can be equipped with the structure of a homotopy fixed point for the induced action of G via the pullback along $\alpha(g^{-1})$ on the category $Z^n(BD; U(1))$ whose morphisms are $n-1$ -cochains up to coboundaries.

In general there are non-isomorphic choices for the fixed point structure. A necessary condition for such a fixed point to exist is $\alpha(g)^*[\omega] = [\omega]$ for all $g \in G$.

Remark 3.8. Concretely, the additional structure consists of an equivalence class of cochains $\Phi_g \in C^{n-1}(BD; U(1))$ up to coboundary satisfying⁶ $\delta\Phi_g = \omega - \alpha(g^{-1})^*\omega$. These cochains have to satisfy the coherence relations

$$\Phi_{g_1} + \alpha(g_1^{-1})^*\Phi_{g_2} = \Phi_{g_1 g_2} + \sigma_{g_1, g_2}[\omega] ,$$

where $\sigma_{g_1, g_2}[\omega]$ is the $n-1$ -cochain induced by the homotopy $\sigma_{g_1, g_2} : \alpha(g_2^{-1}) \circ \alpha(g_1^{-1}) \longrightarrow \alpha(g_2^{-1} g_1^{-1})$. The difference between two homotopy fixed point structures can be described by a group homomorphism $G \longrightarrow H^{n-1}(BD; U(1))$.

Proposition 3.9. Let $\omega \in Z^n(BD; U(1))$ be a topological action and $\alpha : \underline{\ast//G} \longrightarrow \underline{\ast//\Pi_1[BD, BD]}$ a homotopy coherent action of G on BD . If α preserves ω , then the classical Dijkgraaf–Witten theory $L_\omega : D\text{-Cob}_n \longrightarrow \text{Vect}$ admits a kinematical symmetry described by α .

Proof. We set $Z = L_\omega$ and define a family of natural transformations $\Upsilon_g : L_{\alpha(g^{-1})^*\omega} \implies L_\omega$ by

$$\begin{aligned} \Upsilon_{g(\Sigma, \varphi)} : L_{\alpha(g^{-1})^*\omega}(\Sigma, \varphi) &\longrightarrow L_\omega(\Sigma, \varphi) \\ [\sigma_\Sigma] &\longmapsto \langle \varphi^* \Phi_g, \sigma_\Sigma \rangle [\sigma_\Sigma] , \end{aligned}$$

⁶ Throughout we switch freely between the additive and multiplicative notation for $U(1)$ -valued cocycles.

where Φ_g is the $n-1$ -cochain of Remark 3.8 satisfying $\delta\Phi_g = \omega - \alpha(g^{-1})^*\omega$. For a morphism $(M, \psi): (\Sigma, \varphi) \rightarrow (\Sigma', \varphi')$ we have to show that the diagram

$$\begin{CD} L_{\alpha(g^{-1})^*\omega}(\Sigma, \varphi) @>\Upsilon_g(\Sigma, \varphi)>> L_\omega(\Sigma, \varphi) \\ @V L_{\alpha(g^{-1})^*\omega}(M, \psi) VV @VV L_\omega(M, \psi) V \\ L_{\alpha(g^{-1})^*\omega}(\Sigma', \varphi') @>\Upsilon_g(\Sigma', \varphi')>> L_\omega(\Sigma', \varphi') \end{CD}$$

commutes. We fix $\sigma_M \in \text{Fund}(M)_{\sigma_\Sigma}^{\sigma_{\Sigma'}}$ and compute

$$\begin{aligned} \langle \psi^*(\omega - \alpha(g^{-1})^*\omega), \sigma_M \rangle &= \langle \psi^*(\delta\Phi_g), \sigma_M \rangle = \langle \psi^*\Phi_g, \partial\sigma_M \rangle \\ &= \langle \psi^*\Phi_g, \sigma_{\Sigma'} \rangle - \langle \psi^*\Phi_g, \sigma_\Sigma \rangle. \end{aligned}$$

This shows that the diagram commutes. The coherence conditions follow from the fact that the collection Φ_g corresponds to a homotopy fixed point structure. \square

We describe the symmetries of quantum Dijkgraaf–Witten theory following [Fre14b, Section 2.4] by the following notion.

Definition 3.10. Let G be a finite group and denote by $G\text{-Rep}$ the category of finite-dimensional G -representations. Let $Z: \text{Cob}_n \rightarrow \text{Vect}$ be a topological field theory. An internal G -symmetry of Z is a lift

$$\begin{array}{ccc} & G\text{-Rep} & \\ Z_G \nearrow & & \searrow \\ \text{Cob}_n & \xrightarrow{Z} & \text{Vect} \end{array}$$

of Z , where $G\text{-Rep} \rightarrow \text{Vect}$ is the forgetful functor.

Remark 3.11. This definition is equivalent to fixing a group homomorphism $G \rightarrow \text{Aut}_\otimes(Z)$ to the group of symmetric monoidal natural automorphisms of Z .

Kinematical symmetries of classical Dijkgraaf–Witten theories extend to the quantum theory: For a fixed manifold Σ , Υ_g induces a natural isomorphism $L_\omega \circ \mathcal{R}(\alpha(g^{-1}))|_{\text{Bun}_D(\Sigma)} \rightarrow L_\omega|_{\text{Bun}_D(\Sigma)}$, which induces a linear map

$$\int_{\text{Bun}_D(\Sigma)} L_\omega \circ \mathcal{R}(\alpha(g^{-1})) \rightarrow \int_{\text{Bun}_D(\Sigma)} L_\omega = Z_\omega(\Sigma).$$

The equivalence $\mathcal{R}(\alpha(g^{-1}))$ induces a morphism

$$Z_\omega(\Sigma) = \int_{\text{Bun}_D(\Sigma)} L_\omega \rightarrow \int_{\text{Bun}_D(\Sigma)} L_\omega \circ \mathcal{R}(\alpha(g^{-1})).$$

The action of G consisting of $\varphi_g: Z_\omega(\Sigma) \rightarrow Z_\omega(\Sigma)$ is defined as the composition of these two maps. Fixing a fundamental class σ_Σ of Σ as in (2.9) and a parallel section $f(\cdot) = \mathbf{f}(\cdot)[\sigma_\Sigma] \in Z_\omega(\Sigma)$, the action of G on $Z_\omega(\Sigma)$ takes the concrete form

$$g \triangleright \mathbf{f}(\Sigma, \varphi) = \mathbf{f}(\mathcal{R}(\alpha(g^{-1}))[\Sigma, \varphi]) \langle \varphi^*\Phi_g, \sigma_\Sigma \rangle \tag{3.12}$$

with $\delta\Phi_g = \omega - \alpha(g^{-1})^*\omega$ as in the proof of Proposition 3.9.

Proposition 3.13. *The collection φ_g define a representation of G on $Z_\omega(\Sigma)$ such that Z_ω is a functor into the category $G\text{-Rep}$ of finite-dimensional G -representations.*

Proof. This is a direct consequence of the functoriality of the orbifold construction [SW19, Remark 3.43] and the coherence conditions for the homotopy fixed point. \square

Example 3.14. The trivial action of G on BD is always an internal G -symmetry. Any action of G is an internal G -symmetry for a theory with trivial topological Lagrangian. We will provide some more profound examples in Sects. 3.2 and 3.3 .

3.2. *Non-abelian group cohomology.* Following [BBF05] we review non-abelian group 2-cocycles and show how they classify extensions. For simplicity we only discuss groups, which is enough for the study of anomalies in Dijkgraaf–Witten theories. The generalisation to groupoids is straightforward. Let G and D be finite groups. Recall from Sect. 3.1 that a non-abelian 2-cocycle on G with coefficients in D is a 2-functor $\alpha : \ast//G \rightarrow \ast//(\text{Aut}_{\text{Grp}}(D)//D) \subset \text{Grpd}$, where Grpd is the 2-category of groupoids. The 2-category $\ast//(\text{Aut}_{\text{Grp}}(D)//D)$ can be considered as a sub-2-category of Grpd by sending the only object to $\ast//D$. We assume without loss of generality that α preserves identities strictly. Spelling out the definition, we see that α consists of maps of sets $\alpha : G \rightarrow \text{Aut}_{\text{Grp}}(D)$ and $\sigma_\alpha : G \times G \rightarrow D$ satisfying

$$\begin{aligned} \alpha(1) &= \text{id}_D , \\ \sigma_\alpha(1, 1) &= 1 , \\ \alpha(g_1 g_2)[d] &= \sigma_\alpha(g_1, g_2)^{-1} \alpha(g_1)[\alpha(g_2)[d]] \sigma_\alpha(g_1, g_2) , \\ \sigma_\alpha(g_1, g_2) \sigma_\alpha(g_1 g_2, g_3) &= \alpha(g_1)[\sigma_\alpha(g_2, g_3)] \sigma_\alpha(g_1, g_2 g_3) . \end{aligned} \tag{3.15}$$

Using the Grothendieck construction, the 2-functor $\alpha : \underline{BG} \rightarrow \text{Grpd}$ induces an op-fibration of groupoids

$$\int \alpha \rightarrow \ast//G$$

having the following concrete description: There is only one object which we denote by \ast , endomorphisms are given by pairs $(g, d) \in G \times D$ and composition is defined by

$$(g_2, d_2) (g_1, d_1) = (g_2 g_1, d_2 \alpha(g_2)[d_1] \sigma_\alpha(g_2, g_1)) .$$

The op-fibration corresponds to an extension of G by D , i.e. an exact sequence

$$1 \rightarrow D \rightarrow \widehat{G} \rightarrow G \rightarrow 1 ,$$

with $\widehat{G} = \text{End}_{f_\alpha}(\ast)$. Since the Grothendieck construction induces an equivalence of categories, it is natural to believe that extensions of G by D are classified by non-abelian 2-cocycles on G with coefficients in D . A proof of this expectation can be found in [BBF05].

Let us conclude this discussion with a few explicit examples.

Example 3.16. For every pair of groups (G, D) there is a trivial non-abelian 2-cocycle corresponding to the constant 2-functor $\ast//G \rightarrow \ast//(\text{Aut}_{\text{Grp}}(D)//D)$. The corresponding extension is

$$1 \rightarrow D \rightarrow D \times G \rightarrow G \rightarrow 1 .$$

Example 3.17. If D is abelian and $\alpha : G \rightarrow \text{Aut}_{\text{Grp}}(D)$ is trivial, then a non-abelian 2-cocycle reduces to an ordinary 2-cocycle $\sigma \in H^2(G; D)$ and the corresponding extensions are the usual central extensions classified by the abelian 2-cocycle. From a physical point of view such a 2-cocycle can appear if the G -action on matter fields only closes up to a D -gauge transformation. We give two concrete examples for later use. Let N and M be positive integers. Identifying the cyclic groups \mathbb{Z}_N and \mathbb{Z}_M with $\{0, 1, \dots, N - 1\}$ and $\{0, 1, \dots, M - 1\}$ we define the 2-cocycle

$$\begin{aligned} \sigma : \mathbb{Z}_M \times \mathbb{Z}_M &\longrightarrow \mathbb{Z}_N \\ (a, b) &\longmapsto \left\lfloor \frac{a + b}{M} \right\rfloor \bmod N . \end{aligned}$$

The corresponding central extension is

$$0 \longrightarrow \mathbb{Z}_N \xrightarrow{M \cdot} \mathbb{Z}_{NM} \longrightarrow \mathbb{Z}_M \longrightarrow 0 ,$$

where the first map is multiplication by M and the second map is reduction modulo N . This example can be adapted to an arbitrary number of copies of \mathbb{Z}_N and \mathbb{Z}_M . An example is the abelian 2-cocycle corresponding to

$$(0, 0) \longrightarrow \mathbb{Z}_N \times \mathbb{Z}_N \xrightarrow{(M, M) \cdot} \mathbb{Z}_{NM} \times \mathbb{Z}_{NM} \longrightarrow \mathbb{Z}_M \times \mathbb{Z}_M \longrightarrow (0, 0)$$

which is given by

$$\begin{aligned} (\mathbb{Z}_M \times \mathbb{Z}_M)^2 &\longrightarrow \mathbb{Z}_N \times \mathbb{Z}_N \\ ((a_1, b_1), (a_2, b_2)) &\longmapsto \left(\left\lfloor \frac{a_1 + a_2}{M} \right\rfloor \bmod N, \left\lfloor \frac{b_1 + b_2}{M} \right\rfloor \bmod N \right) . \end{aligned}$$

Example 3.18. Given a group homomorphism $\alpha : G \rightarrow \text{Aut}_{\text{Grp}}(D)$, we can consider α as a non-abelian 2-cocycle with trivial map σ_α . The corresponding extension is the semi-direct product

$$1 \longrightarrow D \longrightarrow G \ltimes_\alpha D \longrightarrow G \longrightarrow 1 .$$

3.3. Gauging discrete symmetries and 't Hooft anomalies. There is an inclusion of categories $i : \text{Cob}_n \hookrightarrow G\text{-Cob}_n$ for every group G by equipping every manifold with the trivial G -bundle. The pullback $i^*Z_G : \text{Cob}_n \rightarrow \text{Vect}$ of a G -equivariant field theory $Z_G : G\text{-Cob}_n \rightarrow \text{Vect}$ carries additional structure: By evaluating Z_G on gauge transformations of the trivial G -bundle on an $n-1$ -dimensional manifold Σ we get a representation of G on $i^*Z_G(\Sigma)$ which is compatible with the definition on cobordisms. Hence i^*Z_G is a quantum field theory with internal G -symmetry in the sense of Definition 3.10, i.e. a symmetric monoidal functor

$$i^*Z_G : \text{Cob}_n \longrightarrow G\text{-Rep} .$$

Recall that we considered a G -equivariant field theory as a field theory coupled to classical G -gauge fields. Given a field theory $Z : \text{Cob}_n \rightarrow G\text{-Rep}$ with internal G -symmetry we can ask if the symmetry can be gauged.

Definition 3.19. Let $Z : \text{Cob}_n \rightarrow G\text{-Rep}$ be a topological quantum field theory with internal G -symmetry. A G -equivariant field theory $Z_G : G\text{-Cob}_n \rightarrow \text{Vect}$ gauges the internal G -symmetry if $i^*Z_G = Z$ as functors $\text{Cob}_n \rightarrow G\text{-Rep}$.

In general it may be impossible to gauge a given symmetry due to cohomological obstructions. In this case we say that the symmetry has a 't Hooft anomaly. In the following we will study under which conditions the symmetries discussed in Sect. 3.1 have 't Hooft anomalies.

Remark 3.20. In three dimensions the question of whether a given field theory can be gauged is related to an interesting algebraic problem. A three-dimensional extended topological quantum field theory is described by a modular tensor category \mathbb{M} [BDSPV15]. An internal G -symmetry corresponds to a homotopy coherent action of G on \mathbb{M} via braided autoequivalences. The group of braided autoequivalences up to natural isomorphism is known as the Brauer-Picard group. The modular tensor category corresponding to the Dijkgraaf–Witten theory with gauge group D and topological action $\omega \in Z^3(BD; U(1))$ is the category of finite-dimensional modules over the ω -twisted Drinfeld double of the group algebra $\mathbb{C}[D]$ defined in [DPR90]. The corresponding Brauer-Picard group for $\omega = 0$ is studied in detail in [LP17a]. The more general case of the representation category of Hopf algebras which includes the case of non-trivial ω is studied in [LP17b]. The kinematical symmetries studied in this paper correspond to the subgroup of classical symmetries in [LP17b]. Three-dimensional G -equivariant extended field theories correspond to G -modular categories [TV14, SW18a]. The symmetry corresponding to a homotopy coherent action of G on a modular tensor category \mathbb{M} can be gauged if there exists a G -modular category $\mathbb{M}_G = \bigoplus_{g \in G} \mathbb{M}_g$ such that $\mathbb{M}_1 = \mathbb{M}$ in a compatible way. The question of under which conditions such an extension exists is answered in [ENOM10], whereby the case relevant for Dijkgraaf–Witten theories is discussed in their appendix.

In this algebraic framework the gauging of more complicated symmetries of arbitrary three-dimensional extended topological field theories can be addressed using the cobordism hypothesis and representation theoretic techniques. A detailed study of this would be interesting. However, we refrain from doing so in this paper and focus instead on a largely dimension-independent discussion.

The non-abelian group 2-cocycle describing the action of G on a Dijkgraaf–Witten theory with gauge group D and topological action $\omega \in Z^n(BD; U(1))$ determines an (not necessarily central) extension

$$1 \longrightarrow D \xrightarrow{\iota} \widehat{G} \xrightarrow{\lambda} G \longrightarrow 1. \tag{3.21}$$

The short exact sequence should be understood as a way to combine D - and G -gauge fields into a single \widehat{G} -gauge field. If there exists $\widehat{\omega} \in H^n(B\widehat{G}; U(1))$ such that $\iota^*\widehat{\omega} = \omega$ we say that the symmetry G is anomaly-free. In this case we can use the pushforward construction [SW19, Section 4] along λ to get a G -equivariant field theory gauging the symmetry. We review this construction in the concrete example we are interested in.

Let Σ be an $n-1$ -dimensional closed manifold. The group homomorphism λ induces an extension functor

$$\lambda_* : \text{Bun}_{\widehat{G}}(\Sigma) \longrightarrow \text{Bun}_G(\Sigma).$$

This functor acts on classifying maps by post-composition with the map $B\widehat{G} \rightarrow BG$ induced by $\lambda: \widehat{G} \rightarrow G$, which by a slight abuse of notation we denote again by λ . Equivalently this functor associates to a \widehat{G} -bundle $\widehat{\varphi}$ on Σ the induced G -bundle

$$\widehat{\varphi} \times_{\lambda} G = \widehat{\varphi} \times G / \sim,$$

where $(p, g) \sim (p \triangleleft \widehat{g}, \lambda(\widehat{g}^{-1})g)$ for all $p \in \widehat{\varphi}$, $g \in G$ and $\widehat{g} \in \widehat{G}$. For a bundle $\varphi \in \text{Bun}_G(\Sigma)$ we denote by $\lambda_*^{-1}[\varphi]$ the homotopy fibre

$$\begin{array}{ccc} \lambda_*^{-1}[\varphi] & \longrightarrow & \text{Bun}_{\widehat{G}}(\Sigma) \\ \downarrow & & \downarrow \lambda_* \\ * & \xrightarrow{\varphi} & \text{Bun}_G(\Sigma). \end{array}$$

Concretely, objects of $\lambda_*^{-1}[\varphi]$ are pairs $(\widehat{\varphi}, h)$ of a \widehat{G} -bundle $\widehat{\varphi}$ and a gauge transformation $h: \lambda_*\widehat{\varphi} \rightarrow \varphi$. Morphisms are gauge transformations $\widehat{h}: \widehat{\varphi} \rightarrow \widehat{\varphi}'$ such that the diagram

$$\begin{array}{ccc} \lambda_*\widehat{\varphi} & \xrightarrow{\lambda_*\widehat{h}} & \lambda_*\widehat{\varphi}' \\ & \searrow h & \swarrow h' \\ & \varphi & \end{array}$$

commutes. The pushforward of $L_{\widehat{\omega}}$ can now be defined on an object $(\Sigma, \varphi: \Sigma \rightarrow BG)$ as

$$\lambda_*L_{\widehat{\omega}}(\Sigma, \varphi) = \int_{\lambda_*^{-1}[\varphi]} \left(\lambda_*^{-1}[\varphi] \longrightarrow \text{Bun}_{\widehat{G}}(\Sigma) \xrightarrow{L_{\widehat{\omega}}} \text{Vect} \right). \tag{3.22}$$

This should be regarded as a quantization of the D -gauge fields while leaving the G -sector classical. We realise this limit again as the space of parallel sections. In this case a parallel section $f \in \lambda_*L_{\widehat{\omega}}(\Sigma, \varphi)$ consists of an element $f(\widehat{\varphi}, h) \in L_{\widehat{\omega}}(\Sigma, \widehat{\varphi})$ for all $(\widehat{\varphi}, h) \in \lambda_*^{-1}[\varphi]$.

Let $(M, \psi): (\Sigma_a, \varphi_a) \rightarrow (\Sigma_b, \varphi_b)$ be a morphism in $G\text{-Cob}_n$. To define the pushforward on a parallel section $f(\cdot) \in \lambda_*L_{\widehat{\omega}}(\Sigma_a, \varphi_a)$ we fix fundamental cycles σ_a and σ_b of Σ_a and Σ_b , respectively, and write f as $f(\cdot) = \mathbf{f}(\cdot)[\sigma_a]$ as in Sect. 2.3. We define

$$\begin{aligned} & \lambda_*L_{\widehat{\omega}}(M, \psi)[f](\widehat{\varphi}_b, h_b) \\ &= \left(\int_{(\widehat{\psi}, h, \widehat{h}) \in \lambda_*^{-1}[\psi]|_{(\widehat{\varphi}_b, h_b)}} \langle \widehat{h}^*\widehat{\omega}, [0, 1] \times \sigma_b \rangle \langle \widehat{\psi}^*\widehat{\omega}, \sigma_M \rangle \mathbf{f}(\widehat{\psi}|_{\Sigma_a}, h|_{\Sigma_a}) \right) [\sigma_b] \end{aligned} \tag{3.23}$$

with $\sigma_M \in \text{Fund}_{\sigma_a}^{\sigma_b}(M)$; here the homotopy pullback $\lambda_*^{-1}[\psi]|_{(\widehat{\varphi}_b, h_b)}$ is the groupoid with objects $(\widehat{\psi}, h, \widehat{h})$ where $\widehat{\psi}: M \rightarrow B\widehat{G}$ is a \widehat{G} -bundle, $h: \lambda_*\widehat{\psi} \rightarrow \psi$ is a gauge transformation, and $\widehat{h}: \widehat{\psi}|_{\Sigma_b} \rightarrow \widehat{\varphi}_b$ is a gauge transformation such that the diagram

$$\begin{array}{ccc} \lambda_*\widehat{\psi}|_{\Sigma_b} & \xrightarrow{\lambda_*\widehat{h}} & \lambda_*\widehat{\varphi}_b \\ & \searrow h|_{\Sigma_b} & \swarrow h_b \\ & \varphi_b & \end{array}$$

commutes. This defines a functorial field theory by [SW19, Proposition 4.2]. The field theories constructed here are called G -equivariant Dijkgraaf–Witten theories. In the case of trivial 2-cocycle they have been studied as extended field theories in [MNS12]. Extended G -equivariant Dijkgraaf–Witten theories with non-trivial 2-cocycles are constructed in [MW18].

Now we can formulate the central statement of this section.

Theorem 3.24. *Let Z_ω be a discrete gauge theory with topological action $\omega \in Z^n(BD; U(1))$ and kinematical G -symmetry described by an extension*

$$1 \longrightarrow D \xrightarrow{\iota} \widehat{G} \xrightarrow{\lambda} G \longrightarrow 1$$

such that there exists $\widehat{\omega} \in Z^n(B\widehat{G}; U(1))$ satisfying $\omega = \iota^*\widehat{\omega}$. Then the G -equivariant Dijkgraaf–Witten theory $\lambda_*L_{\widehat{\omega}}: G\text{-Cob}_n \longrightarrow \mathbf{Vect}$ gauges this symmetry.

Proof. First we show that the trivial sector of $\lambda_*L_{\widehat{\omega}}$ is Z_ω . From the exact sequence of groups we get a fibration

$$BD \xrightarrow{\iota} B\widehat{G} \xrightarrow{\lambda} BG$$

of classifying spaces. Let N be a manifold of dimension n or $n - 1$. We have to evaluate the homotopy fibre $\lambda_*^{-1}[\star]$ of the trivial bundle $\star: N \longrightarrow BG$. Using the homotopy lifting property we can restrict ourselves to the full subgroupoid with objects $(\widehat{\varphi}: N \longrightarrow \lambda^{-1}(\star) = \iota(BD), \text{id})$, where $\lambda^{-1}(\star)$ is the preimage of the base point of BG . Morphisms in this subgroupoid are homotopies which are trivial after applying λ . Using again the homotopy lifting property of a fibration we see that they must be homotopic to a homotopy supported in BD . This shows that we can replace limits and integration over the homotopy fibre of the trivial bundle with limits and integration over $\mathbf{Bun}_D(N)$ for every manifold N . For this reason (3.22) and (3.23) reduce to (2.8) and (2.9) in this case, since $\widehat{\omega}$ pulls back to ω .

Next we show that this gauges the symmetry, see (3.12):

$$\lambda_*L_{\widehat{\omega}}(\Sigma, g)f(\iota_*\varphi_D, \text{id}) = f(\iota_*\mathcal{R}(\alpha(g^{-1}))\varphi_D, \text{id}) \langle \varphi_D^* \Phi_g, \sigma_\Sigma \rangle$$

for all closed $n-1$ -dimensional manifolds Σ , $f(\cdot) \in \lambda_*L_{\widehat{\omega}}(\Sigma, \star: \Sigma \longrightarrow BG)$ and $\varphi_D: \Sigma \longrightarrow BD$, where we interpret $g \in G$ as a homotopy from the constant map \star to itself. By [SW19, Proposition 4.2 (b)] we have

$$\lambda_*L_{\widehat{\omega}}(\Sigma, g)f(\iota_*\varphi_D, \text{id}) = f(\iota_*\varphi_D, g^{-1}).$$

We have to calculate a lift for the homotopy g^{-1} , i.e. a gauge transformation $\widehat{g}^{-1}: \iota_*\varphi_D \longrightarrow \iota_*\varphi'_D$ such that $\lambda(\widehat{g}^{-1}) = g^{-1}$. We use the concrete description of \widehat{G} -bundles as elements of the functor category $[\Pi_1(\Sigma), *//\widehat{G}]$, where $\Pi_1(\Sigma)$ is the fundamental groupoid of Σ . A lift of the gauge transformation is then given by conjugation with $(g^{-1}, 1) \in \widehat{G}$. We calculate its action on the image $d \in D \subset \widehat{G}$ of a path in Σ . The inverse is given by [BBF05]

$$(g, \sigma_\alpha(g, g^{-1})^{-1}).$$

Then

$$(g^{-1}, 1)(1, d)(g, \sigma_\alpha(g, g^{-1})^{-1}) = (g^{-1}, \alpha(g^{-1})[d])(g, \sigma_\alpha(g, g^{-1})^{-1})$$

$$\begin{aligned}
 &= (1, \alpha(g^{-1})[d]\alpha(g^{-1})[\sigma_\alpha(g, g^{-1})^{-1}]\sigma_\alpha(g^{-1}, g)) \\
 &= (1, \alpha(g^{-1})[d]\sigma_\alpha(g^{-1}, g)^{-1}\sigma_\alpha(g^{-1}, g)) \\
 &= (1, \alpha(g^{-1})[d]),
 \end{aligned}$$

where in the third equality we used $\sigma_\alpha(g^{-1}, g)^{-1} = \alpha(g^{-1})[\sigma_\alpha(g, g^{-1})^{-1}]$, which follows from (3.15) with $g_1 = g^{-1}, g_2 = g$ and $g_3 = g^{-1}$ using $\sigma_\alpha(1, g) = \sigma_\alpha(g, 1) = 1$ for all $g \in G$. This shows that $\varphi'_D = \mathcal{R}(\alpha(g^{-1}))\varphi_D$. That $f(\cdot)$ is a parallel section implies

$$f(\iota_*\varphi_D, g^{-1}) = L_{\widehat{\omega}}^{-1}(\Sigma, \widehat{g})f(\mathcal{R}(\alpha(g^{-1}))\varphi_D, \text{id}). \tag{3.25}$$

We define an $n-1$ -cochain Φ_g on D as follows: Let $\chi: \Delta^{n-1} \rightarrow BD$ be an $n-1$ -chain which we can include into $B\widehat{G}$ along ι . Putting \widehat{g} on the interval we get a map $[0, 1] \times \Delta^{n-1} \rightarrow B\widehat{G}$. Integration of the pullback of $\widehat{\omega}$ over $[0, 1] \times \Delta^{n-1}$ gives the inverse of the value of Φ_g evaluated on the $n-1$ -simplex. The value of $-\delta\Phi_g$ on an n -simplex $(d_1, \dots, d_n): \Delta^n \rightarrow BD$ is given by

$$\begin{aligned}
 &\langle [\widehat{g} \times (d_1, \dots, d_n)]^*\widehat{\omega}, [0, 1] \times \partial\Delta^n \rangle \\
 &= \langle [\widehat{g} \times (d_1, \dots, d_n)]^*\widehat{\omega}, \partial[0, 1] \times \Delta^n - \partial([0, 1] \times \Delta^n) \rangle \\
 &= \langle [\widehat{g} \times (d_1, \dots, d_n)]^*\widehat{\omega}, (\{1\} - \{0\}) \times \Delta^n \rangle \\
 &= \alpha(g^{-1})^*\omega(d_1, \dots, d_n) - \omega(d_1, \dots, d_n).
 \end{aligned}$$

By definition $L_{\widehat{\omega}}^{-1}(\Sigma, \widehat{g}) = \langle \varphi_D^*\Phi_g, \sigma_\Sigma \rangle$. Inserting this into (3.25) gives (3.12) where Φ_g provide the homotopy fixed point structure. \square

Remark 3.26. Theorem 3.24 provides a general mechanism for the gauging of symmetries. However, we cannot show that it is impossible to gauge the symmetry when the conditions of Theorem 3.24 are not satisfied, i.e. when no such $\widehat{\omega}$ exists.

Example 3.27. We describe a discrete two-dimensional gauge theory with gauge group $D = \mathbb{Z}_N \times \mathbb{Z}_N$ and topological action $\omega_k \in H^2(\mathbb{Z}_N \times \mathbb{Z}_N; U(1))$ as defined in (2.12). The action of the symmetry group G on D can be encoded in a short exact sequence

$$1 \rightarrow D \rightarrow \widehat{G} \rightarrow G \rightarrow 1.$$

Set $G = \mathbb{Z}_M \times \mathbb{Z}_M$ and consider the extension

$$(0, 0) \rightarrow \mathbb{Z}_N \times \mathbb{Z}_N \xrightarrow{(M, M)\cdot} \mathbb{Z}_{NM} \times \mathbb{Z}_{NM} \rightarrow \mathbb{Z}_M \times \mathbb{Z}_M \rightarrow (0, 0).$$

In this case we can gauge the symmetry in the manner of Theorem 3.24 for the topological action ω_k with $k \in \{0, 1, \dots, N-1\}$ if and only if k is divisible by M modulo N , i.e. there exists $k' \in \mathbb{Z}$ such that $k' M = k \pmod N$. Concretely, $\widehat{\omega} \in H^2(\mathbb{Z}_{NM} \times \mathbb{Z}_{NM}; U(1))$ is given by $\omega_{k'} \in Z^2(\mathbb{Z}_{NM} \times \mathbb{Z}_{NM}; U(1))$. This simple example already shows that we cannot gauge every symmetry using Theorem 3.24; it is discussed in [KS14, GKS15] in the context of 0-form and 1-form global symmetries. We will discuss obstructions to finding an appropriate lift $\widehat{\omega}$ in more detail and generality in Sect. 3.4.

Example 3.28. The cyclic group \mathbb{Z}_2 acts on the dihedral group D_8 by conjugation with the generator a . Since this is an action via inner automorphisms it preserves the non-trivial 2-cocycle $\omega \in H^2(BD_8; U(1))$ from Example 2.11.b. This action defines a non-abelian 2-cocycle with trivial map σ . The corresponding extension is

$$1 \longrightarrow D_8 \longrightarrow D_8 \rtimes \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 1 .$$

The Pauli group is the subgroup

$$P_1 = \{\pm 1_2, \pm i 1_2, \pm \sigma_x, \pm i \sigma_x, \pm \sigma_y, \pm i \sigma_y, \pm \sigma_z, \pm i \sigma_z\}$$

of the unitary group $U(2)$ with the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

There is an equivalence of extensions

$$\begin{array}{ccccccc}
 & & & D_8 \rtimes \mathbb{Z}_2 & & & \\
 & & \nearrow & \downarrow \vartheta & \searrow & & \\
 1 & \longrightarrow & D_8 & & & \mathbb{Z}_2 & \longrightarrow 1 \\
 & & \searrow & \downarrow & \nearrow & & \\
 & & & P_1 & & &
 \end{array}$$

given by $\vartheta(a^i b^j, k) = (i \sigma_x)^i \sigma_y^j \sigma_x^k$, showing that this extension is non-trivial even though it comes from an inner automorphism. The intuitive reason for this is that conjugation by a^2 is the identity even though a^2 itself is not. We will show in Example 3.34 that this symmetry cannot be gauged in the manner of Theorem 3.24.

Example 3.29. In three dimensions we can look at the extension

$$0 \longrightarrow \mathbb{Z}_N \xrightarrow{M} \mathbb{Z}_{NM} \longrightarrow \mathbb{Z}_M \longrightarrow 0 .$$

The 3-cocycle ω_k defined in (2.14) can always be gauged by the 3-cocycle $\widehat{\omega}_k \in H^3(B\mathbb{Z}_{NM}; U(1))$ corresponding to the same value of k .

3.4. Obstructions to gauging of symmetries. We shall now work with the group cohomology $H^n(G; U(1))$ which can be identified with the cohomology of BG with coefficients in $U(1)$. There are obstructions for $\widehat{\omega}$ to exist which follow from the Lyndon–Hochschild–Serre spectral sequence associated to the extension (3.21). We briefly review these obstructions. For a physical perspective on these obstructions, see [TvK15].

There is an action of G on $H^n(D; U(1))$ induced by conjugation in \widehat{G} . Every cocycle on \widehat{G} is invariant under conjugation and hence the first obstruction for $\widehat{\omega}$ to exist is

$$\omega \in H^n(D; U(1))^G .$$

By definition, the obstruction is always satisfied if the extension corresponds to a kinematical symmetry. The first quadrant Lyndon–Hochschild–Serre spectral sequence corresponding to the exact sequence (3.21) takes the form

$$E_2^{p,q} = H^p(G; H^q(D; U(1))) \implies H^{p+q}(\widehat{G}; U(1))$$

with edge maps $H^n(\widehat{G}; U(1)) \rightarrow E_\infty^{0,n} = E_{n+2}^{0,n} \hookrightarrow H^n(D; U(1))^G$ given by the restriction to D (see e.g. [Wei95, Section 6.8]). Hence we see that $\omega \in \text{im}(l^*) = E_{n+2}^{0,n}$ if and only if

$$d_i^{0,n} \omega = 0 \in E_i^{i,n+1-i} \tag{3.30}$$

for all $i \in \{2, \dots, n+1\}$. Note that $d_i^{0,n} \omega$ is only well-defined if $d_{i-1}^{0,n} \omega = 0$ and $E_i^{i,n+1-i}$ is a sub-quotient of $H^i(G; H^{n+1-i}(D; U(1)))$.

To understand these obstructions in more detail we introduce the algebraic model for the spectral sequence [HS53, Section 2]. The group cohomology of \widehat{G} can be computed from the normalised cochain complex $C^\bullet(\widehat{G}; U(1))$:

$$0 \rightarrow C^0(\widehat{G}; U(1)) \rightarrow C^1(\widehat{G}; U(1)) \rightarrow \dots$$

We introduce a filtration

$$C^\bullet(\widehat{G}; U(1)) = F^0 C^\bullet(\widehat{G}; U(1)) \supseteq F^1 C^\bullet(\widehat{G}; U(1)) \supseteq F^2 C^\bullet(\widehat{G}; U(1)) \supseteq \dots$$

where $F^i C^n(\widehat{G}; U(1))$ is 0 for $i > n$ and otherwise consists of all normalized n -cochains which are 0 as soon as $n-i+1$ entries are in the image of D . This filtration is compatible with the coboundary operator δ and hence induces a spectral sequence, which is the Lyndon–Hochschild–Serre spectral sequence. Concretely we set

$$\begin{aligned} Z_r^{p,q} &:= \ker\left(F^p C^{p+q}(\widehat{G}; U(1)) \xrightarrow{\delta} C^{p+q+1}(\widehat{G}; U(1)) / F^{p+r} C^{p+q+1}(\widehat{G}; U(1))\right), \\ B_r^{p,q} &:= \delta\left(F^{p-r+1} C^{p+q-1}(\widehat{G}; U(1))\right) \cap F^p C^{p+q}(\widehat{G}; U(1)), \\ E_r^{p,q} &:= Z_r^{p,q} / (B_r^{p,q} + Z_{r-1}^{p+1,q-1}). \end{aligned}$$

The differential $\delta: C^{p+q}(\widehat{G}; U(1)) \rightarrow C^{(p+r)+(q-r+1)}(\widehat{G}; U(1))$ induces the corresponding differentials

$$d_r^{p,q}: E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$$

in the spectral sequence.

We consider the two-dimensional case as a warm-up. We fix $\omega \in H^2(D; U(1))$. The corresponding element in $E_2^{0,2}$ is the 2-cochain

$$\begin{aligned} \tilde{\omega}: \widehat{G} \times \widehat{G} &\rightarrow U(1) \\ ((d, g), (d', g')) &\mapsto \omega(d, d'). \end{aligned}$$

This is not generally a cocycle, since the multiplication in \widehat{G} is twisted by the corresponding non-abelian 2-cocycle. This cochain obviously pulls back to ω . The ensuing calculation can be understood as trying to find a 2-cochain on \widehat{G} which is 0 when pulled back to D such that its sum with $\tilde{\omega}$ is closed.

The first obstruction $d_2^{0,2} \tilde{\omega} = 0$ is equivalent to $\delta \tilde{\omega} \in B_2^{2,1} + Z_1^{3,0}$. This implies that there exists $\gamma_1 \in F^1 C^2(\widehat{G}; U(1))$ such that

$$\delta \gamma_1 \in F^2 C^3(\widehat{G}; U(1)) \quad \text{and} \quad \delta(\tilde{\omega} - \gamma_1) \in E_1^{3,0}.$$

This means that we can consider $\tilde{\omega}$ as an element of $E_3^{0,2} \cong \ker d_2^{0,2}$. Note that the identification is not the identity, rather we have to map $\tilde{\omega}$ to $\tilde{\omega} - \gamma_1$. We have thus shown that if the first obstruction vanishes, then there exists $\theta \in Z_1^{3,0} = Z^3(G; U(1))$ and a cochain $\omega' = \tilde{\omega} - \gamma_1$ such that $\delta\omega' = \lambda^*\theta$ and $\iota^*\omega' = \omega$.

The next obstruction is $d_3^{0,2}\tilde{\omega} = 0$. This is equivalent to $\delta(\tilde{\omega} - \gamma_1) \in B_3^{3,0}$, hence there exists $\gamma_2 \in F^1C^2(\widehat{G}; U(1))$ such that $\delta\gamma_2 = \delta(\tilde{\omega} - \gamma_1) \in F^3C^2(\widehat{G}; U(1))$. This implies $\delta(\tilde{\omega} - \gamma_1 - \gamma_2) = 0$ and $\iota^*(\tilde{\omega} - \gamma_1 - \gamma_2) = \omega$, since γ_1 and γ_2 are elements in $F^1C^2(\widehat{G}; U(1))$. This gives the desired 2-cocycle $\widehat{\omega} = \tilde{\omega} - \gamma_1 - \gamma_2$.

The discussion above readily generalises to arbitrary dimension n . If the first obstruction vanishes then there exists $\gamma_1 \in F^1C^n(\widehat{G}; U(1))$ such that

$$\delta(\tilde{\omega} - \gamma_1) \in F^3C^{n+1}(\widehat{G}; U(1)) .$$

More generally if the first $m \leq n$ obstructions vanish, there are elements $\gamma_1, \dots, \gamma_m \in F^1C^n(\widehat{G}; U(1))$ such that

$$\begin{aligned} \delta\gamma_i &\in F^iC^{n+1}(\widehat{G}; U(1)) , \\ \delta\left(\tilde{\omega} - \sum_{i=1}^k \gamma_i\right) &\in F^{k+2}C^{n+1}(\widehat{G}; U(1)) , \end{aligned}$$

for all $i, k = 1, \dots, m$. In particular, if all obstructions vanish then

$$\delta\left(\tilde{\omega} - \sum_{i=1}^n \gamma_i\right) = 0$$

and

$$\iota^*\left(\tilde{\omega} - \sum_{i=1}^n \gamma_i\right) = \omega \in H^n(D; U(1)) .$$

We are mostly interested in the case when all obstructions except the last one vanish. In this case

$$\delta\left(\tilde{\omega} - \sum_{i=1}^{n-1} \gamma_i\right) = \lambda^*\theta$$

with $\theta \in Z^{n+1}(G; U(1))$, since closed elements of $F^{n+1}C^{n+1}(\widehat{G}; U(1))$ are in one-to-one correspondence with $Z^{n+1}(G; U(1))$. We summarize the present discussion in

Proposition 3.31. *Let*

$$1 \longrightarrow D \xrightarrow{\iota} \widehat{G} \xrightarrow{\lambda} G \longrightarrow 1$$

be a short exact sequence of groups, n a natural number and ω an n -cocycle on D with values in $U(1)$.

- (a) *When all obstructions in (3.30) vanish, then there exists $\widehat{\omega} \in Z^n(\widehat{G}; U(1))$ satisfying $\iota^*\widehat{\omega} = \omega$.*
- (b) *When the first $n - 1$ obstructions in (3.30) vanish, then there exist $\omega' \in C^n(\widehat{G}; U(1))$ and $\theta \in Z^{n+1}(G; U(1))$ satisfying $\iota^*\omega' = \omega$ and $\delta\omega' = \lambda^*\theta$.*

Remark 3.32. If the first $n - 1$ obstructions vanish we can realize the anomalous field theory as a boundary state of a classical $n+1$ -dimensional Dijkgraaf–Witten theory with topological action θ . In Sect. 4 we will explain this point in more detail.

Example 3.33. We have seen in Example 3.29 that for the extension

$$0 \longrightarrow \mathbb{Z}_N \longrightarrow \mathbb{Z}_{NM} \longrightarrow \mathbb{Z}_M \longrightarrow 0$$

all 3-cocycles on \mathbb{Z}_N arise as pullbacks of 3-cocycles on \mathbb{Z}_{NM} , hence all obstructions vanish in this case.

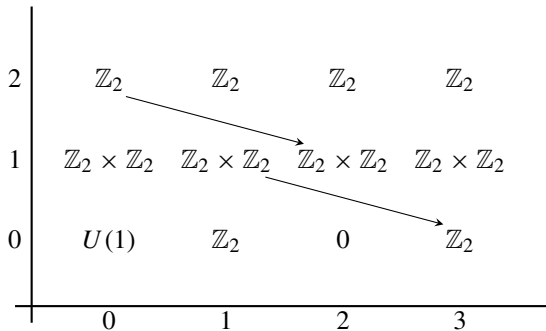
Example 3.34. Following up on Example 3.28 we show that for the symmetry described by

$$1 \longrightarrow D_8 \longrightarrow P_1 \longrightarrow \mathbb{Z}_2 \longrightarrow 1 \tag{3.35}$$

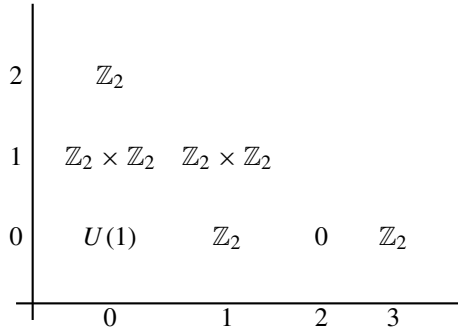
the non-trivial 2-cocycle $\omega \in H^2(D_8; U(1))$ cannot be gauged. The cohomology groups of the Pauli group P_1 can be computed using a computer algebra package such as GAP [Joy08] and the universal coefficient theorem to get

$$\begin{aligned} H^0(P_1; U(1)) &= U(1) , \\ H^1(P_1; U(1)) &= \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 , \\ H^2(P_1; U(1)) &= \mathbb{Z}_2 \times \mathbb{Z}_2 , \\ H^3(P_1; U(1)) &= \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8 . \end{aligned}$$

The E_2 page of the corresponding spectral sequence is



The two differentials drawn are 0 as can be checked by using the concrete description of the differentials in [Hue81] and the fact that (3.35) is the extension of \mathbb{Z}_2 by D_8 corresponding to the inner automorphism of D_8 given by conjugation with $a \in D_8$. Hence the E_3 page is given by



From $E_3^{1,1} = \mathbb{Z}_2 \times \mathbb{Z}_2 = E_\infty^{1,1}$ and $H^2(P_1; U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ we deduce that the differential $d_3^{0,2}: \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ is an isomorphism. This implies that the symmetry corresponding to (3.35) of the non-trivial topological action $\omega \in H^2(D_8; U(1))$ cannot be gauged using Theorem 3.24, since the second obstruction corresponding to $d_3^{0,2}$ does not vanish. However, since the first obstruction vanishes we can gauge the symmetry using the relative field theory constructed in Sect. 4.

Example 3.36. We have seen in Example 3.27 that for the extension

$$(0, 0) \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \times \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow (0, 0)$$

the 2-cocycle $\omega_1 \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2; U(1))$ cannot be obtained as the pullback of a 2-cocycle on $\mathbb{Z}_4 \times \mathbb{Z}_4$. The corresponding 2-cochain is given by

$$\begin{aligned} \tilde{\omega}_1: (\mathbb{Z}_4 \times \mathbb{Z}_4)^2 &\rightarrow U(1) \\ ((a_1, b_1), (a_2, b_2)) &\mapsto \exp\left(\pi i \left\lfloor \frac{a_1}{2} \right\rfloor \left\lfloor \frac{b_2}{2} \right\rfloor\right). \end{aligned}$$

To find the corresponding obstructions we calculate using $\lfloor \frac{a+b}{2} \rfloor = ab + \lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor \pmod 2$ to get

$$\delta \tilde{\omega}_1((a_1, b_1), (a_2, b_2), (a_3, b_3)) = \exp\left(\pi i \left(a_1 a_2 \left\lfloor \frac{b_3}{2} \right\rfloor + \left\lfloor \frac{a_1}{2} \right\rfloor b_2 b_3\right)\right).$$

Using the computer algebra program Maple [Map] we verified by checking all possibilities that there are no solutions to the equation

$$\delta(\tilde{\omega}_1 - \gamma_1) = \lambda^* \theta$$

with $\gamma_1 \in F^1 C^2(\mathbb{Z}_4 \times \mathbb{Z}_4; U(1))$ and $\theta \in Z^3(\mathbb{Z}_2 \times \mathbb{Z}_2; U(1))$. Hence the first obstruction $d_2^{0,2} \omega_1$ does not vanish.

4. Bulk-Boundary Correspondence

In this section we realise anomalous gauged Dijkgraaf–Witten theories on the boundaries of higher-dimensional Dijkgraaf–Witten theories. We start by reviewing the general description of boundary field theories and anomaly inflow through the bulk-boundary correspondence in the functorial framework. Afterwards we make the general theory explicit in the case of Dijkgraaf–Witten theories.

4.1. *Anomaly inflow in functorial field theories.* We work with the geometric bicategory $G\text{-Cob}_{n,n-1,n-2}$ for definiteness, but the description generalises to arbitrary background fields. The description of boundary field theories and anomalies in the framework of extended field theories is worked out in [Fre14a, FT14, Mon15, MS18] (see also [JFS17, FV15] for an (∞, n) -categorical discussion in the case of topological field theories). This description is closely related to the twisted field theories discussed in [ST11]. Here we follow [MS18, Section 3.3]. Let $E: G\text{-Cob}_{n,n-1,n-2} \rightarrow 2\mathbf{Vect}$ be an extended field theory. We define its truncation $\text{tr } E$ to be the restriction of E to the sub-bicategory of $G\text{-Cob}_{n,n-1,n-2}$ containing only invertible 2-morphisms.

Definition 4.1. An $n-1$ -dimensional G -equivariant anomalous field theory with anomaly described by an invertible n -dimensional extended field theory $E: G\text{-Cob}_{n,n-1,n-2} \rightarrow 2\mathbf{Vect}$ is a natural symmetric monoidal 2-transformation

$$Z: 1 \implies \text{tr } E,$$

where 1 is the trivial theory assigning \mathbf{Vect} to every object, the identity functor to 1-morphisms and the identity natural transformation to 2-morphisms.

Remark 4.2. There are different definitions for natural symmetric monoidal 2-transformations corresponding to different levels of strictness. Here we use [MS18, Definition B.13], which seems to be best suited for physical applications.

Concretely $Z: 1 \implies \text{tr } E$ consists of a linear functor $Z(S, \xi): \mathbf{Vect} \rightarrow E(S, \xi)$ for all objects $(S, \xi) \in G\text{-Cob}_{n,n-1,n-2}$, and a natural transformation $Z(\Sigma, \varphi): E(\Sigma, \varphi) \circ Z(S_1, \xi_1) \implies Z(S_2, \xi_2)$ for all 1-morphisms $((\Sigma, \varphi): (S_1, \xi_1) \rightarrow (S_2, \xi_2)) \in G\text{-Cob}_{n,n-1,n-2}$. The functor $Z(S, \xi): \mathbf{Vect} \rightarrow E(S, \xi)$ can be described by an object $Z(S, \xi)[\mathbb{C}] \in E(S, \xi)$ which by a slight abuse of notation we denote again by $Z(S, \xi)$. The natural transformation $Z(\Sigma, \varphi): E(\Sigma, \varphi) \circ Z(S_1, \xi_1) \implies Z(S_2, \xi_2)$ can be described by a morphism $Z(\Sigma, \varphi): E(\Sigma, \varphi)[Z(S_1, \xi_1)] \rightarrow Z(S_2, \xi_2)$ in $E(S_2, \xi_2)$. Requiring Z to be a natural 2-transformation reduces explicitly to the following: Let (S, ξ) , (S_1, ξ_1) , (S_2, ξ_2) and (S_3, ξ_3) be objects of $G\text{-Cob}_{n,n-1,n-2}$, and $(\Sigma_a, \varphi_a): (S_1, \xi_1) \rightarrow (S_2, \xi_2)$ and $(\Sigma_b, \varphi_b): (S_2, \xi_2) \rightarrow (S_3, \xi_3)$ 1-morphisms in $G\text{-Cob}_{n,n-1,n-2}$. Then the diagrams

$$\begin{array}{ccc}
 E(\Sigma_b, \varphi_b) \circ E(\Sigma_a, \varphi_a)[Z(S_1, \xi_1)] & \longrightarrow & E(\Sigma_b \circ \Sigma_a, \varphi_b \cup \varphi_a)[Z(S_1, \xi_1)] \\
 \downarrow E(\Sigma_b, \varphi_b)[Z(\Sigma_a, \varphi_a)] & & \downarrow Z(\Sigma_b \circ \Sigma_a, \varphi_b \cup \varphi_a) \\
 E(\Sigma_b, \varphi_b)[Z(S_2, \xi_2)] & \xrightarrow{Z(\Sigma_b, \varphi_b)} & Z(S_3, \xi_3)
 \end{array} \tag{4.3}$$

and

$$\begin{array}{ccc}
 Z(S, \xi) & \longrightarrow & E(\text{id}_{(S, \xi)})[Z(S, \xi)] \\
 \searrow \text{id} & & \swarrow Z(\text{id}_{(S, \xi)}) \\
 & Z(S, \xi) &
 \end{array} \tag{4.4}$$

commute, where the unlabelled morphisms are part of the structure of the extended field theory E . The symmetric monoidal structure on Z is described explicitly by specifying natural morphisms

$$M^{-1}: Z(\emptyset) \rightarrow \iota_E(\mathbb{C})$$

in $E(\emptyset)$ and

$$\Pi_Z((S_1, \xi_1), (S_2, \xi_2)) : \chi_E[Z(S_1, \xi_1) \boxtimes Z(S_2, \xi_2)] \longrightarrow Z(S_1 \sqcup S_2, \xi_1 \sqcup \xi_2)$$

in $E(S_1 \sqcup S_2, \xi_1 \sqcup \xi_2)$, where $\iota_E : \mathbf{Vect} \longrightarrow E(\emptyset)$ and $\chi_E : E(\cdot) \boxtimes E(\cdot) \implies E(\cdot \sqcup \cdot)$ are part of the structure corresponding to a symmetric monoidal 2-functor [MS18, Definition B.12]. There is a long list of coherence and compatibility conditions that these morphisms have to satisfy, see [MS18, Proposition 3.14] for details.

Remark 4.5. If the extended field theory E is trivial, it follows from this discussion that a field theory relative to E is the same as an ordinary $n-1$ -dimensional field theory.

It is instructive to decategorify Definition 4.1 [MS18, Section 2]. We denote by $\text{tr } G\text{-Cob}_n$ the maximal subgroupoid of $G\text{-Cob}_n$. For a functor $L : G\text{-Cob}_n \longrightarrow \mathbf{Vect}$ we denote by $\text{tr } L : \text{tr } G\text{-Cob}_n \longrightarrow \mathbf{Vect}$ its restriction to $\text{tr } G\text{-Cob}_n$.

Definition 4.6. A partition function Z with anomaly described by an invertible n -dimensional field theory $L : G\text{-Cob}_n \longrightarrow \mathbf{Vect}$ is a natural symmetric monoidal transformation

$$Z : 1 \implies \text{tr } L .$$

Remark 4.7. Restricting Definition 4.1 to the endomorphisms of \emptyset induces a natural transformation $\text{tr } L \implies 1$. Since in most physically relevant examples all vector spaces are Hilbert spaces, this discrepancy can be resolved by taking the adjoint. Here we stick to Definition 4.6, because it has a natural geometric interpretation in terms of line bundles.

Unpacking Definition 4.6, we get for every object $(\Sigma, \varphi) \in G\text{-Cob}_n$ a linear map $Z(\Sigma, \varphi) : \mathbb{C} \longrightarrow L(\Sigma, \varphi)$, such that for all invertible morphisms $\phi : (\Sigma_a, \varphi_a) \longrightarrow (\Sigma_b, \varphi_b)$ in $G\text{-Cob}_n$ the diagram

$$\begin{CD} \mathbb{C} @>Z(\Sigma_a, \varphi_a)>> L(\Sigma_a, \varphi_a) \\ @V\text{id}VV @VV L(\phi) V \\ \mathbb{C} @>Z(\Sigma_b, \varphi_b)>> L(\Sigma_b, \varphi_b) \end{CD} \tag{4.8}$$

commutes. We think of ϕ as a symmetry of the classical background fields. Since L is an invertible field theory, $L(\Sigma, \varphi)$ is a one-dimensional vector space isomorphic to \mathbb{C} , though not necessarily in a canonical way. Picking such an isomorphism for all (Σ, φ) induces a \mathbb{C}^\times -valued 1-cocycle of the groupoid of symmetries $\text{tr } G\text{-Cob}_n$, since we can then identify the linear map $L(\phi)$ with a non-zero complex number.

Definition 4.6 naturally encodes properties of field theories with anomalies. For example, in physically relevant theories one should also require that the vector spaces $L(\Sigma, \cdot)$ form a line bundle over the space of field configurations as in the case of smooth field theories [ST11]. In the case of $G\text{-Cob}_n$ the space of field configurations is a discrete groupoid and every invertible functorial field theory gives a flat line bundle over the groupoid of field configurations in the sense of [Wil08]. The partition function is now a parallel section of this line bundle. This reproduces the more geometric description of anomalies as the non-triviality of a line bundle over the space of field configurations [Nas91].

In the extended framework of Definition 4.1, evaluating E on manifolds of dimension $n - 2$ can be regarded as a 2-line bundle over the groupoid of field configurations as

defined in e.g. [SW18b]. In particular, an anomalous field theory defines a section of this 2-line bundle. This reproduces the description of Hamiltonian anomalies in terms of line bundle gerbes [CMM97]. The extended functorial framework thus naturally combines the Hamiltonian and Lagrangian descriptions of anomalies. It further formulates the field theory on all possible spacetimes simultaneously, and also requires compatibility with gluing and cutting of manifolds. Hence it is a mathematically concrete formulation of the requirements from [Wit15] of local quantum field theory.

Of particular interest in the Hamiltonian description of anomalies is the projective representation of the symmetry group on the Hilbert space of the field theory. To describe these in the functorial framework we briefly recall the 2-categorical description of projective representations. An element of $H^2(BG; \mathbb{C}^\times)$ corresponds to a homotopy class of maps from BG to the Eilenberg-MacLane space $K(\mathbb{C}^\times, 2)$. The classifying space BG is the nerve of a category, namely the action groupoid $*//G$. The Eilenberg-MacLane space $K(\mathbb{C}^\times, 2)$ is the nerve of a 2-category $B^2\mathbb{C}^\times$ with one object, one 1-morphism and the group \mathbb{C}^\times as the set of 2-morphisms. Hence we can describe 2-cocycles alternatively as 2-functors up to natural isomorphisms.

Definition 4.9. Let \mathcal{G} be a groupoid and A an abelian group. A 2-cocycle on \mathcal{G} with values in A is a 2-functor

$$\alpha: \underline{\mathcal{G}} \longrightarrow B^2A .$$

Remark 4.10. Spelling out the coherence isomorphisms corresponding to a 2-functor reproduces the usual definition of groupoid cohomology [MS18, Section 3.4].

There is a natural 2-functor $B^2\mathbb{C}^\times \hookrightarrow 2\mathbf{Vect}$ sending the unique object to the category of vector spaces, the unique 1-morphism to the identity functor and a non-zero complex number λ to the natural transformation of the identity functor induced by scalar multiplication with λ . Using this embedding a 2-cocycle α on \mathcal{G} with values in \mathbb{C}^\times induces a 2-functor $\alpha: \underline{\mathcal{G}} \longrightarrow 2\mathbf{Vect}$. Using this 2-functor we can give a categorical definition of projective representations (see [FV15] for the higher categorical framework).

Definition 4.11. A projective representation of a groupoid \mathcal{G} twisted by a 2-cocycle α with values in \mathbb{C}^\times is a natural 2-transformation

$$\begin{array}{ccc}
 & \alpha & \\
 \curvearrowright & & \curvearrowleft \\
 \underline{\mathcal{G}} & \uparrow \rho & 2\mathbf{Vect} \\
 \curvearrowleft & & \curvearrowright \\
 & 1 &
 \end{array}$$

Remark 4.12. Spelling out the definition reproduces the usual definition of a projective representation [MS18, Section 3.4].

The assumption that the anomaly field theory E is invertible implies that for every $n-2$ -dimensional manifold S there exists a 2-cocycle $\alpha: \underline{\mathbf{Bun}}_G(S) \longrightarrow B^2\mathbb{C}^\times$ such that the diagram

$$\begin{array}{ccc}
 \underline{\mathbf{Bun}}_G(S) & \xrightarrow{E} & 2\mathbf{Vect} \\
 \searrow \alpha & & \nearrow \\
 & B^2\mathbb{C}^\times &
 \end{array}$$

commutes up to a non-canonical isomorphism. A quantum field theory with anomaly Z induces a projective representation of $\mathbf{Bun}_G(S)$ with respect to the 2-cocycle α . Morphisms in $\mathbf{Bun}_G(S)$ are gauge transformations, which are symmetries of the field theory. Hence we shall sometimes refer to $\mathbf{Bun}_G(S)$ as the symmetry groupoid. Projective representations corresponding to different choices of $\alpha : \mathbf{Bun}_G(S) \rightarrow B^2\mathbb{C}^\times$ are equivalent. The projective representation can be concretely constructed by picking trivialisations of the categories that E assigns to field configurations. The 2-cocycle twisting the projective representation is completely determined by E and independent of the explicit form of Z .

We shall now discuss in more detail how to couple the bulk field theory (E, L) and boundary field theory Z to construct an anomaly-free theory. We start with the unextended framework corresponding to Definition 4.6. This involves the full quantum field theory $L : G\text{-Cob}_n \rightarrow \mathbf{Vect}$ and not just its truncation. Let M be an n -dimensional manifold with boundary $\partial M = -\Sigma$, and $\psi : M \rightarrow BG$ a G -bundle. The field theory with anomaly defines an element $Z(\Sigma, \psi|_\Sigma) \in L(\Sigma, \psi|_\Sigma)$. We can interpret (M, ψ) as a morphism $(M, \psi) : (\Sigma, \psi|_\Sigma) \rightarrow \emptyset$ in $G\text{-Cob}_n$. The partition function of the composite system can now be defined as

$$Z_{\text{bb}}(M, \psi, \Sigma) = L(M, \psi)[Z(\Sigma, \psi|_\Sigma)] \in L(\emptyset) \cong \mathbb{C}. \tag{4.13}$$

This definition does not depend on any additional choices. Let $\psi' : M \rightarrow BG$ be a principal G -bundle and $\nu : \psi \rightarrow \psi'$ a gauge transformation. We then calculate

$$\begin{aligned} Z_{\text{bb}}(M, \psi', \Sigma) &= L(M, \psi')[Z(\Sigma, \psi'|_\Sigma)] \\ &= L(M, \psi') \circ L([0, 1] \times \Sigma, \nu|_\Sigma)[Z(\Sigma, \psi|_\Sigma)] \\ &= L(M, \psi)[Z(\Sigma, \psi|_\Sigma)] \\ &] = Z_{\text{bb}}(M, \psi, \Sigma), \end{aligned}$$

where we used (4.8) in the second equality and in the third equality the fact that L is invariant under gauge transformations relative to the boundary. This shows that the composite partition function is gauge-invariant.

Definition 4.1 also allows us to formulate the composite system at the level of state spaces. Let $E : G\text{-Cob}_{n,n-1,n-2} \rightarrow 2\mathbf{Vect}$ be an invertible extended field theory. Consider an $n-1$ -dimensional manifold Σ with boundary $\partial\Sigma = -S$ and a principal G -bundle $\varphi : \Sigma \rightarrow BG$. The anomalous field theory Z defines an element $Z(S, \varphi|_S) \in E(S, \varphi|_S)$. The composite state space is given by

$$Z_{\text{bb}}(\Sigma, \varphi, S) = E(\Sigma, \varphi)[Z(S, \varphi|_S)] \in E(\emptyset) \cong \mathbf{Vect}. \tag{4.14}$$

This vector space does not depend on any additional choices. Let $\nu : \varphi \rightarrow \varphi'$ be a gauge transformation. Then there is an induced linear map

$$\begin{aligned} Z_{\text{bb}}(\Sigma, \varphi, S) = E(\Sigma, \varphi)[Z(S, \varphi|_S)] &\xrightarrow{E(\nu)} E((\Sigma, \varphi') \circ ([0, 1] \times S, \nu|_S))[Z(S, \varphi|_S)] \\ &\rightarrow E(\Sigma, \varphi') \circ E([0, 1] \times S, \nu|_S)[Z(S, \varphi|_S)] \\ &\xrightarrow{Z([0, 1] \times S, \nu|_S)} E(\Sigma, \varphi')[Z(S, \varphi'|_S)] = Z_{\text{bb}}(\Sigma, \varphi', S). \end{aligned} \tag{4.15}$$

It follows from the coherence conditions that this defines an honest representation of the symmetry groupoid. Hence we have described a way of coupling bulk and boundary degrees of freedom to an anomaly-free state space. In condensed matter physics applications the invertible field theory E arises as the low-energy effective theory of the bulk system.

4.2. *Anomalous Dijkgraaf–Witten theories as boundary states.* We will now illuminate this construction in the case of anomalies of Dijkgraaf–Witten theories. Let $Z_\omega : \mathbf{Cob}_{n-1} \rightarrow \mathbf{Vect}$ be a Dijkgraaf–Witten theory with gauge group D , topological action $\omega \in Z^{n-1}(BD; U(1))$ and kinematical G -symmetry described by an extension

$$1 \rightarrow D \xrightarrow{\iota} \widehat{G} \xrightarrow{\lambda} G \rightarrow 1.$$

Recall from Proposition 3.31 that when the first $n - 2$ obstructions in the spectral sequence vanish, a cochain $\omega' \in C^{n-1}(B\widehat{G}; U(1))$ and a cocycle $\theta \in Z^n(BG; U(1))$ exist such that $\omega = \iota^*\omega'$ and $\delta\omega' = \lambda^*\theta$. We realise the anomalous gauged theory as a field theory $Z_{\omega'}$ living on the boundary of a classical n -dimensional Dijkgraaf–Witten theory $E_\theta : G\text{-Cob}_{n,n-1,n-2} \rightarrow 2\mathbf{Vect}$ with topological action θ .

Partition function We shall first construct the state as an unextended relative field theory, i.e. a natural transformation $Z_{\omega'} : 1 \Rightarrow \text{tr } L_\theta$ (see Definition 4.6). The following construction is similar to the one in [Wit16, Section 3.3]. However, we use the language of functorial field theories and homotopy fibres to describe the construction. The approach to boundary field theories in the present paper is to some extent the reverse of the approach in [Wit16], where anomalous boundary field theories are constructed starting from a bulk Dijkgraaf–Witten theory. Instead we start from a field theory with anomaly and show how to realize this theory as a boundary field theory. The construction of the state space below is not given in [Wit16].

Following the general theory outlined in Sect. 4.1 we have to specify an element $Z_{\omega'}(\Sigma, \varphi : M \rightarrow BG)$ of $L_\theta(\Sigma, \varphi)$ for all objects $(\Sigma, \varphi) \in G\text{-Cob}_n$. Let σ_Σ be a representative for the fundamental class of Σ . We set

$$Z_{\omega'}(\Sigma, \varphi) = \left(\int_{(\widehat{\varphi}, h) \in \lambda_*^{-1}[\varphi]} \langle \widehat{\varphi}^* \omega', \sigma_\Sigma \rangle \langle h^* \theta, [0, 1] \times \sigma_\Sigma \rangle \right) [\sigma_\Sigma] \in L_\theta(\Sigma, \varphi). \tag{4.16}$$

Proposition 4.17. *$Z_{\omega'}$ is a partition function with anomaly $L_\theta : G\text{-Cob}_n \rightarrow \mathbf{Vect}$.*

Proof. We have to show that $Z_{\omega'}$ is a well-defined natural transformation. This is an immediate consequence of Theorem 4.22 below. To become acquainted with the constructions involved, we present here part of the proof. We start by showing that $\langle \widehat{\varphi}^* \omega', \sigma_\Sigma \rangle \langle h^* \theta, [0, 1] \times \sigma_\Sigma \rangle$ is well-defined on isomorphism classes of $\lambda_*^{-1}[\varphi]$. Let $\widehat{h} : (\widehat{\varphi}_1, h_1) \rightarrow (\widehat{\varphi}_2, h_2)$ be a morphism in $\lambda_*^{-1}[\varphi]$, i.e. a homotopy $\widehat{h} : \widehat{\varphi}_1 \rightarrow \widehat{\varphi}_2$ such that the diagram

$$\begin{array}{ccc} \lambda_* \widehat{\varphi}_1 & \xrightarrow{\lambda_* \widehat{h}} & \lambda_* \widehat{\varphi}_2 \\ & \searrow h_1 & \swarrow h_2 \\ & \varphi & \end{array}$$

commutes. The homotopy induces a chain homotopy $H : \widehat{\varphi}_{1*} \rightarrow \widehat{\varphi}_{2*}$ between the induced maps on singular chains given by $H(c) = \widehat{h}_*([0, 1] \times c)$ for all chains $c \in C_\bullet(\Sigma)$. Hence, writing $U(1) = \mathbb{R}/\mathbb{Z}$ additively for the calculation, we find

$$\begin{aligned}
 \langle \widehat{\varphi}_2^* \omega', \sigma_\Sigma \rangle - \langle \widehat{\varphi}_1^* \omega', \sigma_\Sigma \rangle &= \langle \omega', \partial H(\sigma_\Sigma) - H(\partial \sigma_\Sigma) \rangle \\
 &= \langle \omega', \partial H(\sigma_\Sigma) \rangle \\
 &= \langle \widehat{h}^* \lambda^* \theta, [0, 1] \times \sigma_\Sigma \rangle \\
 &= \langle h_1^* \theta - h_2^* \theta, [0, 1] \times \sigma_\Sigma \rangle.
 \end{aligned}$$

This shows that the integration in (4.16) is well-defined.

Let σ'_Σ be a different representative for the fundamental class of Σ and χ an n -chain satisfying $\partial \chi = \sigma'_\Sigma - \sigma_\Sigma$. To show that (4.16) is an element of $L_\theta(\Sigma, \varphi)$ we calculate

$$\begin{aligned}
 \langle \widehat{\varphi}^* \omega', \sigma'_\Sigma - \sigma_\Sigma \rangle \langle h^* \theta, [0, 1] \times (\sigma'_\Sigma - \sigma_\Sigma) \rangle &= \langle \widehat{\varphi}^* \omega', \partial \chi \rangle \langle h^* \theta, [0, 1] \times \partial \chi \rangle \\
 &= \langle \widehat{\varphi}^* \lambda^* \theta, \chi \rangle \langle h^* \theta, -\{0\} \times \chi + \{1\} \times \chi \rangle \\
 &= \langle \varphi^* \theta, \chi \rangle.
 \end{aligned}$$

This is exactly the required transformation behaviour. We leave the verification of naturality to the reader. □

Remark 4.18. Before extending the field theory we give the precise form of the composite partition function (4.13). We fix an n -dimensional manifold M with boundary $\partial M = -\Sigma$ and a principal G -bundle $\psi : M \rightarrow BG$. Evaluating L_θ on (M, ψ) gives a linear map $L_\theta(M, \psi) : L_\theta(\Sigma, \psi|_\Sigma) \rightarrow \mathbb{C}$. The composite partition function is then

$$\begin{aligned}
 Z_{\omega' \text{ bb}}(M, \psi, \Sigma) &= L_\theta(M, \psi)[Z_{\omega'}(\Sigma, \psi|_\Sigma)] \\
 &= \left(\int_{(\widehat{\varphi}, h) \in \lambda_*^{-1}[\psi|_\Sigma]} \langle \widehat{\varphi}^* \omega', \partial \sigma_M \rangle^{-1} \langle h^* \theta, [0, 1] \times \partial \sigma_M \rangle^{-1} \right) \\
 &\quad \cdot \langle \psi^* \theta, \sigma_M \rangle,
 \end{aligned}$$

which is gauge-invariant according to the general theory outlined in Sect. 4.1.

State space Let (S, ξ) be an object of $G\text{-Cob}_{n, n-1, n-2}$ and denote by $\text{Fund}_\theta(S, \xi)$ the full subcategory of simple objects of $E_\theta(S, \xi)$. Recall from Sect. 2.2 that objects of $\text{Fund}_\theta(S, \xi)$ are representatives σ_S of the fundamental class of S and morphism spaces are given by

$$\text{Hom}_{E_\theta(S, \xi)}(\sigma_S, \sigma'_S) = \mathbb{C}[\{\Lambda \in C_{n-1}(S) \mid \partial \Lambda = \sigma'_S - \sigma_S\}] / \sim_\theta.$$

We construct a functor

$$\widetilde{Z}_{\omega'}^{(S, \xi)} : \text{Fund}_\theta(S, \xi)^{\text{op}} \rightarrow \text{Vect}.$$

To this end we first construct a functor $L_{\xi, \omega'}(\sigma_S) : \lambda_*^{-1}[\xi] \rightarrow \text{Vect}$ as follows: To every object $(\widehat{\xi}, h)$ we assign \mathbb{C} and to a homotopy $\widehat{h} : (\widehat{\xi}, h) \rightarrow (\widehat{\xi}', h')$ we assign the complex number $\langle \widehat{h}^* \omega', [0, 1] \times \sigma_S \rangle$. Let σ_S and σ'_S be representatives for the fundamental class of S and $\Lambda \in C_{n-1}(S)$ an $n-1$ -chain satisfying $\partial \Lambda = \sigma'_S - \sigma_S$, i.e. a morphism in $\text{Fund}_\theta(S, \xi)$. We construct a natural transformation

$$\begin{aligned}
 L_{\xi, \omega'}(\Lambda) : L_{\xi, \omega'}(\sigma'_S) &\implies L_{\xi, \omega'}(\sigma_S) \\
 L_{\xi, \omega'}(\Lambda)_{(\widehat{\xi}, h)} : \mathbb{C} &\rightarrow \mathbb{C}, \quad 1 \mapsto \langle \widehat{\xi}^* \omega', \Lambda \rangle^{-1} \langle h^* \theta, [0, 1] \times \Lambda \rangle^{-1}. \quad (4.19)
 \end{aligned}$$

We show in "Appendix A" that $L_{\xi, \omega'}: \text{Fund}_\theta(S, \xi)^{\text{op}} \longrightarrow [\lambda_*^{-1}[\xi], \text{Vect}]$ is a well-defined functor to the category $[\lambda_*^{-1}[\xi], \text{Vect}]$ of functors from $\lambda_*^{-1}[\xi]$ to Vect (Lemma A.1). Composition with the limit functor $\int_{\lambda_*^{-1}[\xi]}: [\lambda_*^{-1}[\xi], \text{Vect}] \longrightarrow \text{Vect}$ then constructs the desired functor

$$\tilde{Z}_{\omega'}^{(S, \xi)} = \int_{\lambda_*^{-1}[\xi]} L_{\xi, \omega'} .$$

The limit can again be realised by parallel sections. This allows us to define

$$\begin{aligned} Z_{\omega'}(S, \xi) &= \int^{\sigma \in \text{Fund}_\theta(S, \xi)} \tilde{Z}_{\omega'}^{(S, \xi)}(\sigma) * \sigma \\ &= \int^{\sigma \in \text{Fund}_\theta(S, \xi)} \int_{\lambda_*^{-1}[\xi]} L_{\xi, \omega'}(\sigma) * \sigma \in E_\theta(S, \xi) . \end{aligned} \tag{4.20}$$

Let $(\Sigma, \varphi): (S_1, \xi_1) \longrightarrow (S_2, \xi_2)$ be a 1-morphism in $G\text{-Cob}_{n, n-1, n-2}$. We construct a morphism

$$\begin{aligned} Z_{\omega'}(\Sigma, \varphi) &: E_\theta(\Sigma, \varphi)[Z_{\omega'}(S_1, \xi_1)] \\ &= \int^{\sigma_2 \in \text{Fund}_\theta(S_2, \xi_2)} \left(\int^{\sigma_1 \in \text{Fund}_\theta(S_1, \xi_1)} \Sigma^\varphi(\sigma_2, \sigma_1) \otimes_{\mathbb{C}} \tilde{Z}_{\omega'}^{(S_1, \xi_1)}(\sigma_1) \right) * \sigma_2 \\ &\longrightarrow \int^{\sigma_2 \in \text{Fund}_\theta(S_2, \xi_2)} \tilde{Z}_{\omega'}^{(S_2, \xi_2)}(\sigma_2) * \sigma_2 = Z_{\omega'}(S_2, \xi_2) \end{aligned}$$

in $E_\theta(S_2, \xi_2)$ from the universal property of the coend by realising $\tilde{Z}_{\omega'}^{(S_2, \xi_2)}(\sigma_2)$ as a cowedge. We define the required linear maps for the concrete description of the limit as parallel sections by

$$\begin{aligned} Z_{\omega'}(\Sigma, \varphi)_{\sigma_1} &: \Sigma^\varphi(\sigma_2, \sigma_1) \otimes_{\mathbb{C}} \tilde{Z}_{\omega'}^{(S_1, \xi_1)}(\sigma_1) \longrightarrow \tilde{Z}_{\omega'}^{(S_2, \xi_2)}(\sigma_2) \\ &\Lambda \otimes_{\mathbb{C}} f(\cdot) \longmapsto Z_{\omega'}(\Sigma, \varphi)_{\sigma_1}(f, \Lambda)(\cdot) , \end{aligned}$$

with

$$\begin{aligned} Z_{\omega'}(\Sigma, \varphi)_{\sigma_1}(f, \Lambda)(\widehat{\xi}_2, h_2) &= \int_{(\widehat{\varphi}, g, \widehat{h}) \in \lambda_*^{-1}[\varphi]_{(\widehat{\xi}_2, h_2)}} \langle \widehat{\varphi}^* \omega', \Lambda \rangle \langle g^* \theta, [0, 1] \times \Lambda \rangle \langle \widehat{h}^* \omega', [0, 1] \times \sigma_2 \rangle \\ &\cdot f(\widehat{\varphi}|_{S_1}, g|_{S_1}) \in \mathbb{C} . \end{aligned} \tag{4.21}$$

The domain of integration here is the groupoid with objects consisting of triples of a map

$$\widehat{\varphi}: \Sigma \longrightarrow B\widehat{G} ,$$

a gauge transformation

$$g: \lambda_* \widehat{\varphi} \longrightarrow \varphi ,$$

and a gauge transformation

$$\widehat{h}: \widehat{\varphi}|_{S_2} \longrightarrow \widehat{\xi}_2$$

such that the diagram

$$\begin{array}{ccc}
 \lambda_* \widehat{\varphi}|_{S_2} & \xrightarrow{\lambda_* \widehat{h}} & \lambda_* \widehat{\xi}_2 \\
 \searrow g|_{S_2} & & \swarrow h_2 \\
 & \xi_2 &
 \end{array}$$

commutes. We show in "Appendix A" that this induces the desired morphism (Lemma A.2).

We can now formulate and prove the main result of this section.

Theorem 4.22. *Let $Z_\omega: \mathbf{Cob}_{n-1} \rightarrow \mathbf{Vect}$ be an $n-1$ -dimensional discrete gauge theory with gauge group D , topological action $\omega \in Z^{n-1}(BD; U(1))$ and kinematical G -symmetry described by an extension*

$$1 \rightarrow D \xrightarrow{\iota} \widehat{G} \xrightarrow{\lambda} G \rightarrow 1.$$

Let ω' be an $n-1$ -chain on $B\widehat{G}$ and $\theta \in Z^n(BG; U(1))$ an n -cocycle on BG satisfying $\iota^* \omega' = \omega$ and $\delta \omega' = \lambda^* \theta$. Then $Z_{\omega'}$ defined in (4.20) and (4.21) is an anomalous field theory with anomaly $E_\theta: G\text{-Cob}_{n,n-1,n-2} \rightarrow 2\mathbf{Vect}$.

Proof. Let (S_1, ξ_1) and (S_2, ξ_2) be objects of $G\text{-Cob}_{n,n-1,n-2}$. We first construct the missing structure corresponding to the compatibility with the monoidal structures. This involves a morphism

$$M^{-1}: Z_{\omega'}(\emptyset) = \sigma_\emptyset \rightarrow \sigma_\emptyset$$

in $E_\theta(\emptyset)$ which we choose to be the identity, and a natural isomorphism

$$\Pi_{Z_{\omega'}}((S_1, \xi_1), (S_2, \xi_2)): \chi_{E_\theta}[Z_{\omega'}(S_1, \xi_1) \boxtimes Z_{\omega'}(S_2, \xi_2)] \rightarrow Z_{\omega'}(S_1 \sqcup S_2, \xi_1 \sqcup \xi_2). \tag{4.23}$$

Spelling out (4.20) we find that $\chi_{E_\theta}[Z_{\omega'}(S_1, \xi_1) \boxtimes Z_{\omega'}(S_2, \xi_2)]$ is given explicitly by

$$\begin{aligned}
 & \int^{\sigma_2 \in \text{Fund}_\theta(S_2, \xi_2)} \int^{\sigma_1 \in \text{Fund}_\theta(S_1, \xi_1)} \left(\tilde{Z}_{\omega'}^{(S_2, \xi_2)}(\sigma_2) \otimes_{\mathbb{C}} \tilde{Z}_{\omega'}^{(S_1, \xi_1)}(\sigma_1) \right) * [\sigma_1 \sqcup \sigma_2] \\
 & \cong \int^{\sigma_1 \sqcup \sigma_2 \in \text{Fund}_\theta(S_1 \sqcup S_2, \xi_1 \sqcup \xi_2)} \left(\tilde{Z}_{\omega'}^{(S_2, \xi_2)}(\sigma_2) \otimes_{\mathbb{C}} \tilde{Z}_{\omega'}^{(S_1, \xi_1)}(\sigma_1) \right) * [\sigma_1 \sqcup \sigma_2],
 \end{aligned}$$

where we used Fubini's Theorem for coends together with the fact that we can naturally identify $\text{Fund}_\theta(S_1, \xi_1) \times \text{Fund}_\theta(S_2, \xi_2)$ with $\text{Fund}_\theta(S_1 \sqcup S_2, \xi_1 \sqcup \xi_2)$. Concretely, $Z_{\omega'}(S_1 \sqcup S_2, \xi_1 \sqcup \xi_2)$ is given by

$$\int^{\sigma_1 \sqcup \sigma_2 \in \text{Fund}_\theta(S_1 \sqcup S_2, \xi_1 \sqcup \xi_2)} \tilde{Z}_{\omega'}^{(S_2 \sqcup S_1, \xi_2 \sqcup \xi_1)}(\sigma_2 \sqcup \sigma_1) * [\sigma_1 \sqcup \sigma_2].$$

The isomorphism (4.23) is induced by the collection of natural linear isomorphisms

$$\tilde{Z}_{\omega'}^{(S_2, \xi_2)}(\sigma_2) \otimes_{\mathbb{C}} \tilde{Z}_{\omega'}^{(S_1, \xi_1)}(\sigma_1) \rightarrow \tilde{Z}_{\omega'}^{(S_2 \sqcup S_1, \xi_2 \sqcup \xi_1)}(\sigma_2 \sqcup \sigma_1)$$

given on parallel sections by

$$f_2(\cdot) \otimes_{\mathbb{C}} f_1(\cdot) \mapsto f_2(\cdot|_{S_2}) f_1(\cdot|_{S_1}).$$

A straightforward but tedious calculation shows that these definitions satisfy [MS18, eqs. (3.18)–(3.22)].

Next we show that $Z_{\omega'}$ is compatible with identities, i.e. that the diagram (4.4) commutes. Since all constructions are natural it is enough to check this on the terms appearing in the coends of (4.20) and (2.6). Explicitly, we have to show that the diagram

$$\begin{array}{ccc} \tilde{Z}_{\omega'}^{(S,\xi)}(\sigma) & \xrightarrow{(-1)^n \sigma \times [0,1] \otimes \mathbb{C} \cdot} & (S \times [0,1])^{\xi \times \text{id}_{[0,1]}}(\sigma, \sigma) \otimes_{\mathbb{C}} \tilde{Z}_{\omega'}^{(S,\xi)}(\sigma) \\ & \searrow \text{id} & \swarrow \\ & \tilde{Z}_{\omega'}^{(S,\xi)}(\sigma) & \end{array}$$

commutes for all objects $(S, \xi) \in G\text{-Cob}_{n,n-1,n-2}$ and representatives σ of the fundamental class of S . Let $(\widehat{\xi}, h)$ be an object of $\lambda_*^{-1}[\xi]$. The groupoid $\lambda_*^{-1}[\xi \times \text{id}_{[0,1]}]_{(\widehat{\xi}, h)}$ is contractible. For this reason we may fix the object $(\widehat{\xi} \times \text{id}_{[0,1]}, h \times \text{id}_{[0,1]}, \text{id}) \in \lambda_*^{-1}[\xi \times \text{id}_{[0,1]}]_{(\widehat{\xi}, h)}$ without loss of generality. The upper composition evaluated on a parallel section f at $(\widehat{\xi}, h)$ is

$$\begin{aligned} & \langle (\widehat{\xi} \times \text{id}_{[0,1]})^* \omega', (-1)^n \sigma \times [0,1] \rangle \langle (\text{id}_{[0,1]} \times \xi \times \text{id}_{[0,1]})^* \theta, [0,1] \times (-1)^n (\sigma \times [0,1]) \rangle \\ & \cdot \langle (\text{id}_{[0,1]} \times \widehat{\xi})^* \omega', [0,1] \times \sigma \rangle f(\widehat{\xi}, h) \\ & = \langle (\text{id}_{[0,1]} \times \xi \times \text{id}_{[0,1]})^* \theta, [0,1] \times (-1)^n (\sigma \times [0,1]) \rangle f(\widehat{\xi}, h) \\ & = \langle \xi^* \theta, \text{pr}_2([0,1] \times (-1)^n (\sigma \times [0,1])) \rangle f(\widehat{\xi}, h) \\ & = f(\widehat{\xi}, \text{id}_{[0,1]} \times \xi), \end{aligned}$$

where in the first equality we used the fact that f is a parallel section and in the last equality that the projection to the second factor $\text{pr}_2([0,1] \times (-1)^n (\sigma \times [0,1]))$ is a boundary, which follows from

$$\partial \text{pr}_2([0,1] \times (-1)^n (\sigma \times [0,1])) = 0$$

and by dimensional reasons.

We now show that the diagram (4.3) commutes. Again it is enough to check the commutativity on elements of the coends. Let $(\Sigma_a, \varphi_a): (S_1, \xi_1) \rightarrow (S_2, \xi_2)$ and $(\Sigma_b, \varphi_b): (S_2, \xi_2) \rightarrow (S_3, \xi_3)$ be 1-morphisms in $G\text{-Cob}_{n,n-1,n-2}$. We fix representatives σ_1, σ_2 and σ_3 for the fundamental classes of S_1, S_2 and S_3 , respectively. The upper composition in (4.3) corresponds to the linear map

$$\begin{aligned} \Sigma_b^{\varphi_b}(\sigma_2, \sigma_3) \otimes_{\mathbb{C}} \Sigma_a^{\varphi_a}(\sigma_1, \sigma_2) \otimes_{\mathbb{C}} \tilde{Z}_{\omega'}^{(S_1, \xi_1)}(\sigma_1) & \longrightarrow \tilde{Z}_{\omega'}^{(S_3, \xi_3)}(\sigma_3) \\ \Lambda_b \otimes_{\mathbb{C}} \Lambda_a \otimes_{\mathbb{C}} f(\cdot) & \longmapsto \tilde{f}(\cdot) \end{aligned}$$

with

$$\begin{aligned} \tilde{f}(\widehat{\xi}_3, h_3) & = \int_{(\widehat{\varphi}, g, \widehat{h}) \in \lambda_*^{-1}[\varphi_b \cup \varphi_a]_{(\widehat{\xi}_3, h_3)}} \langle \widehat{\varphi}^* \omega', \Lambda_a + \Lambda_b \rangle \\ & \cdot \langle g^* \theta, [0,1] \times (\Lambda_a + \Lambda_b) \rangle \langle \widehat{h}^* \omega', [0,1] \times \sigma_3 \rangle \\ & \cdot f(\widehat{\varphi}|_{S_1}, \widehat{h}|_{S_1}). \end{aligned} \tag{4.24}$$

The lower composition gives

$$\Sigma_b^{\varphi_b}(\sigma_2, \sigma_3) \otimes_{\mathbb{C}} \Sigma_a^{\varphi_a}(\sigma_1, \sigma_2) \otimes_{\mathbb{C}} \tilde{Z}_{\omega'}^{(S_1, \xi_1)}(\sigma_1) \longrightarrow \tilde{Z}_{\omega'}^{(S_3, \xi_3)}(\sigma_3)$$

$$\Lambda_b \otimes_{\mathbb{C}} \Lambda_a \otimes_{\mathbb{C}} f(\cdot) \longmapsto \underline{f}(\cdot)$$

with

$$\begin{aligned} \underline{f}(\widehat{\xi}_3, h_3) &= \int_{(\widehat{\varphi}_b, g_b, \widehat{h}_b) \in \lambda_*^{-1}[\varphi_b]|_{(\widehat{\xi}_3, h_3)}} \\ &\int_{(\widehat{\varphi}_a, g_a, \widehat{h}_a) \in \lambda_*^{-1}[\varphi_a]|_{(\widehat{\varphi}_b, \widehat{h}_b)|_{S_2}} \langle \widehat{\varphi}_b^* \omega', \Lambda_b \rangle \langle g_b^* \theta, [0, 1] \times \Lambda_b \rangle \\ &\cdot \langle \widehat{h}^* \omega', [0, 1] \times \sigma_3 \rangle \langle \widehat{\varphi}_a^* \omega', \Lambda_a \rangle \langle g_a^* \theta, [0, 1] \times \Lambda_a \rangle \\ &\cdot \langle \widehat{h}^* \omega', [0, 1] \times \sigma_2 \rangle f(\widehat{\varphi}_a|_{S_1}, \widehat{h}_a|_{S_1}). \end{aligned} \tag{4.25}$$

Using the descent property for the stack Bun_G of G -bundles we can write the domain of the first integral as a homotopy pullback⁷

$$\begin{array}{ccc} \lambda_*^{-1}[\varphi_b \cup \varphi_a]|_{(\widehat{\xi}_3, h_3)} & \xrightarrow{\text{pr}_{\Sigma_b}} & \lambda_*^{-1}[\varphi_b]|_{(\widehat{\xi}_3, h_3)} \\ \text{pr}_{\Sigma_a} \downarrow & & \downarrow \text{pr}_{S_2} \\ \lambda_*^{-1}[\varphi_a] & \xrightarrow{\text{pr}_{S_2}} & \lambda_*^{-1}[\varphi_a|_{S_2}] \end{array}$$

where pr. denotes the pullback functor over the indicated submanifold. From the fibre-wise characterization of homotopy pullbacks [CPS06] it follows that

$$\begin{array}{ccccc} \text{pr}_{\Sigma_b}^{-1}[(\widehat{\varphi}_b, \widehat{h}_b)] & \longrightarrow & \lambda_*^{-1}[\varphi_b \cup \varphi_a]|_{(\widehat{\xi}_3, h_3)} & \xrightarrow{\text{pr}_{\Sigma_b}} & \lambda_*^{-1}[\varphi_b]|_{(\widehat{\xi}_3, h_3)} \\ \Xi \downarrow & & \text{pr}_{\Sigma_a} \downarrow & & \downarrow \text{pr}_{S_2} \\ \text{pr}_{S_2}^{-1}[(\widehat{\varphi}_b, \widehat{h}_b)|_{S_2}] & \longrightarrow & \lambda_*^{-1}[\varphi_a] & \xrightarrow{\text{pr}_{S_2}} & \lambda_*^{-1}[\varphi_a|_{S_2}] \end{array}$$

is a homotopy commuting diagram containing an equivalence Ξ . The equality of the integrals (4.24) and (4.25) now follows from the generalized Cavalieri Principle [SW19, Proposition A.15] applied to the functor

$$\text{pr}_{\Sigma_b} : \lambda_*^{-1}[\varphi_b \cup \varphi_a]|_{(\widehat{\xi}_3, h_3)} \longrightarrow \lambda_*^{-1}[\varphi_b]|_{(\widehat{\xi}_3, h_3)}$$

and the ordinary Cavalieri Principle [SW19, Proposition A.14] for Ξ .

Finally, the invariance with respect to gauge transformations [MS18, eq. (3.17)] follows directly from the invariances of the integrands which is part of the statement of Lemma A.1. \square

Similarly to the proof of Theorem 3.24, it should be possible to show that the relative field theory gauges the G -symmetry. Let us explain in more detail what this means: The pullback $i^* E_\theta$ along the inclusion $i : \text{Cob}_{n, n-1, n-2} \longrightarrow G\text{-Cob}_{n, n-1, n-2}$ is naturally isomorphic to the trivial theory $1 : \text{Cob}_{n, n-1, n-2} \longrightarrow 2\text{Vect}$. The pullback

⁷ Here we use the homotopy invariance of the stack Bun_G to identify the evaluation on an open neighbourhood of S_2 with the evaluation on S_2 .

$i^*Z_{\omega'} : 1 \implies \text{tri}^*E_\theta \cong 1$ is a field theory relative to the trivial theory. From [MS18, Proposition 3.14] it follows that $i^*Z_{\omega'}$ is an $n - 1$ dimensional topological quantum field theory. This field theory comes with an internal G symmetry from the evaluation of $Z_{\omega'}$ on gauge transformations of the trivial bundle. Gauging the symmetry means that the field theory $i^*Z_{\omega'}$ recovers the Dijkgraaf–Witten theory Z_ω together with its internal symmetry induced by the group extensions

$$1 \longrightarrow D \xrightarrow{\iota} \widehat{G} \xrightarrow{\lambda} G \longrightarrow 1 .$$

Remark 4.26. Let S be a closed oriented $n - 2$ -dimensional manifold and σ_S a representative of its fundamental class. The general theory outlined in Sect. 4.1 implies that the vector spaces $\widetilde{Z}_\omega^{(S, \cdot)}(\sigma_S)$ form a projective representation of $\text{Bun}_G(S)$. The 2-cocycle α twisting the projective representation is completely described by the coherence isomorphisms for E_θ . In [MW18, Theorem 4.5] it is shown that the class of this 2-cocycle is given by the transgression of $\theta \in Z^n(BG; U(1))$ to the groupoid $\text{Bun}_G(S)$, i.e. α is induced by the 2-cocycle $\tau_S\theta$ on the underlying mapping space $|\text{Bun}_G(S)|$ (with the compact-open topology) given by

$$(\tau_S\theta)(\chi) := (\text{ev}^*\theta)(\chi \times \sigma_S)$$

for any 2-simplex $\chi : \Delta^2 \longrightarrow |\text{Bun}_G(S)|$, where $\text{ev} : |\text{Bun}_G(S)| \times S \longrightarrow BG$ is the evaluation map. This generalizes the low-dimensional descriptions of anomalies and projective representations on state spaces discussed in [TY17, Section 2.1]: In the simplest $n = 1$ case, with $S = \{*\}$ the 2-cocycles α and θ may be identified, and describe the same 2-cocycle specifying both the two-dimensional bulk G -symmetry protected phase and the class of the projective G -representation on the one-dimensional boundary state, whereas for $n = 3$ with $S = \mathbb{S}^1$ transgression induces a homomorphism $H^3(BG; U(1)) \longrightarrow H^2(BG; U(1))$ specifying the two-dimensional G -symmetry protected phase on the boundary of the three-dimensional G -symmetry protected phase.

In a more geometric language this means that the state spaces of the gauged theory form a section of the transgression 2-line bundle of the flat $n - 1$ -gerbe on the classifying space BG described by θ , as the classical gauge theory corresponding to θ describes the parallel transport for the $n - 1$ -gerbe. This 2-line bundle is trivial if and only if the corresponding 2-cocycle is a boundary. Hence the obstruction for the projective representation to form an honest representation is the non-triviality of the transgression 2-line bundle.

Remark 4.27. Let $(\Sigma, \varphi) : (S, \xi) \longrightarrow \emptyset$ be a 1-morphism in $G\text{-Cob}_{n,n-1,n-2}$. According to (4.14) the state space of the composite system is given by

$$Z_{\omega' \text{ bb}}(\Sigma, \varphi, S) = E_\theta(\Sigma, \varphi)[Z_{\omega'}(S, \varphi|_S)] \cong \Sigma^\varphi(\emptyset, \sigma_S) \otimes_{\mathbb{C}} \widetilde{Z}_{\omega'}^{(S, \varphi|_S)}(\sigma_S) .$$

It is independent of the choice of σ_S up to unique isomorphism corresponding to the choice of a representative of the coend. The composite state space carries an honest representation of the gauge group G described in (4.15).

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A. Lemmata

Lemma A.1. $L_{\xi, \omega'} : \text{Fund}_\theta(S, \xi)^{\text{op}} \longrightarrow [\lambda_*^{-1}[\xi], \text{Vect}]$ is a well-defined functor.

Proof. Let (S, ξ) be an object of $G\text{-Cob}_{n, n-1, n-2}$, σ_S and σ'_S representatives of the fundamental class of S , $\Lambda \in C_{n-1}(S)$ an $n-1$ -chain satisfying $\partial\Lambda = \sigma'_S - \sigma_S$ and $\widehat{h} : (\widehat{\xi}, h) \longrightarrow (\widehat{\xi}', h')$ a morphism in $\lambda_*^{-1}[\xi]$. The only subtle part is the naturality of $L_{\xi, \omega'}(\Lambda)$, i.e. commutativity of the diagram

$$\begin{array}{ccc}
 L_{\xi, \omega'}(\sigma'_S)(\widehat{\xi}, h) = \mathbb{C} & \xrightarrow{\langle \widehat{h}^* \omega', [0, 1] \times \sigma'_S \rangle} & L_{\xi, \omega'}(\sigma'_S)(\widehat{\xi}', h') = \mathbb{C} \\
 \downarrow \langle \widehat{\xi}^* \omega', \Lambda \rangle^{-1} \langle h^* \theta, [0, 1] \times \Lambda \rangle^{-1} & & \downarrow \langle \widehat{\xi}'^* \omega', \Lambda \rangle^{-1} \langle h'^* \theta, [0, 1] \times \Lambda \rangle^{-1} \\
 L_{\xi, \omega'}(\sigma_S)(\widehat{\xi}, h) = \mathbb{C} & \xrightarrow{\langle \widehat{h}^* \omega', [0, 1] \times \sigma_S \rangle} & L_{\xi, \omega'}(\sigma_S)(\widehat{\xi}', h') = \mathbb{C}
 \end{array}$$

We check this by calculating the lower path to get

$$\begin{aligned}
 & \langle \widehat{\xi}^* \omega', \Lambda \rangle^{-1} \langle h^* \theta, [0, 1] \times \Lambda \rangle^{-1} \langle \widehat{h}^* \omega', [0, 1] \times \sigma_S \rangle \\
 &= \langle \widehat{\xi}^* \omega', \Lambda \rangle^{-1} \langle h^* \theta, [0, 1] \times \Lambda \rangle^{-1} \langle \widehat{h}^* \omega', [0, 1] \times \sigma'_S \rangle \langle \widehat{h}^* \omega', [0, 1] \times \partial\Lambda \rangle^{-1} \\
 &= \langle \widehat{\xi}^* \omega', \Lambda \rangle^{-1} \langle h^* \theta, [0, 1] \times \Lambda \rangle^{-1} \langle \widehat{h}^* \omega', [0, 1] \times \sigma'_S \rangle \\
 & \quad \cdot \langle \widehat{h}^* \omega', -\partial([0, 1] \times \Lambda) - \{0\} \times \Lambda + \{1\} \times \Lambda \rangle^{-1} \\
 &= \langle h^* \theta, [0, 1] \times \Lambda \rangle^{-1} \langle \widehat{h}^* \omega', [0, 1] \times \sigma'_S \rangle \langle \widehat{h}^* \omega', -\partial([0, 1] \times \Lambda) \rangle^{-1} \langle \widehat{\xi}'^* \omega', \Lambda \rangle^{-1} \\
 &= \langle \widehat{\xi}'^* \omega', \Lambda \rangle^{-1} \langle h'^* \theta, [0, 1] \times \Lambda \rangle^{-1} \langle \widehat{h}^* \omega', [0, 1] \times \sigma'_S \rangle,
 \end{aligned}$$

where in the first step we used $\sigma_S = \sigma'_S - \partial\Lambda$, in the second step the graded product rule for \times , in the third step that \widehat{h} is a homotopy from $\widehat{\xi}$ to $\widehat{\xi}'$, and in the last step $\delta\omega' = \lambda^*\theta$ and $\lambda_*\widehat{h} = h'^{-1}h$. □

Lemma A.2. $Z_{\omega'}(\Sigma, \varphi)_{\sigma_1}(f, \Lambda)(\cdot)$ is a well-defined parallel section and the collection $Z_{\omega'}(\Sigma, \varphi)_{\sigma_1}$ defines a cowedge inducing the desired maps, which again form a cowedge defining $Z_{\omega'}(\Sigma, \varphi)$.

Proof. We first check that the integrand of (4.21) is gauge-invariant, i.e. the integral is well-defined. Let $\widehat{H} : (\widehat{\varphi}, g, \widehat{h}) \longrightarrow (\widehat{\varphi}', g', \widehat{h}')$ be an isomorphism in $\lambda_*^{-1}[\varphi]_{(\widehat{\xi}_2, h_2)}$, i.e. a homotopy $\widehat{H} : \widehat{\varphi} \longrightarrow \widehat{\varphi}'$ such that the diagrams

$$\begin{array}{ccc}
 \lambda_*\widehat{\varphi} & \xrightarrow{\lambda_*\widehat{H}} & \lambda_*\widehat{\varphi}' \\
 \searrow g & & \swarrow g' \\
 & \varphi &
 \end{array} \tag{A.3}$$

and

$$\begin{array}{ccc}
 \widehat{\varphi}|_{S_2} & \xrightarrow{\widehat{H}|_{S_2}} & \widehat{\varphi}'|_{S_2} \\
 \searrow \widehat{h} & & \swarrow \widehat{h}' \\
 & \widehat{\xi}_2 &
 \end{array}
 \tag{A.4}$$

commute. We study the transformation of the first term in (4.21). The homotopy $\widehat{H}: [0, 1] \times \Sigma \rightarrow BG$ induces a chain homotopy between the chain maps $\widehat{\varphi}_*$ and $\widehat{\varphi}'_*$ given by

$$\widehat{\varphi}'_{*p} - \widehat{\varphi}_{*p} = \partial \circ H_p + H_{p-1} \circ \partial$$

for all $p \in \mathbb{Z}$, where $H_p := \widehat{H}_{*p+1} \circ D_p$ and

$$D_p : C_p(\Sigma) \rightarrow C_{p+1}([0, 1] \times \Sigma), \quad c \mapsto [0, 1] \times c.$$

Hence

$$\begin{aligned}
 \langle \widehat{\varphi}'^* \omega', \Lambda \rangle \langle \widehat{\varphi}^* \omega', \Lambda \rangle^{-1} &= \langle \omega', \partial H_{n-1}(\Lambda) + H_{n-2}(\partial \Lambda) \rangle \\
 &= \langle (\lambda_* \widehat{H})^* \theta, [0, 1] \times \Lambda \rangle \langle \widehat{H}^* \omega', [0, 1] \times (\sigma_2 - \sigma_1) \rangle \\
 &= \langle g^* \theta, [0, 1] \times \Lambda \rangle \langle g'^* \theta, [0, 1] \times \Lambda \rangle^{-1} \langle \widehat{h}^* \omega', [0, 1] \times \sigma_2 \rangle \langle \widehat{h}'^* \omega', [0, 1] \times \sigma_2 \rangle^{-1} \\
 &\quad \cdot \langle \widehat{H}|_{S_1}^* \omega', [0, 1] \times \sigma_1 \rangle^{-1} \\
 &= \langle g^* \theta, [0, 1] \times \Lambda \rangle \langle g'^* \theta, [0, 1] \times \Lambda \rangle^{-1} \langle \widehat{h}^* \omega', [0, 1] \times \sigma_2 \rangle \langle \widehat{h}'^* \omega', [0, 1] \times \sigma_2 \rangle^{-1} \\
 &\quad \cdot f(\widehat{\varphi}'|_{S_1}, g'|_{S_1}) f(\widehat{\varphi}|_{S_1}, g|_{S_1})^{-1},
 \end{aligned}
 \tag{A.5}$$

where we used (A.3) and (A.4) in the third step, and that $f(\cdot)$ is a parallel section in the last step. Comparing (A.5) with the transformation behaviours of the other terms in (4.21) shows that the integrand is gauge-invariant.

We now check that (4.21) is a parallel section of $L_{\widehat{\xi}_2, \omega'}(\sigma_2): \lambda_*^{-1}[\xi_2] \rightarrow \mathbf{Vect}$. Let $\widehat{h}': (\widehat{\xi}_2, h_2) \rightarrow (\widehat{\xi}'_2, h'_2)$ be a morphism in $\lambda_*^{-1}[\xi_2]$. The map \widehat{h}' induces an equivalence of groupoids

$$\begin{aligned}
 \widehat{h}'_* : \lambda_*^{-1}[\varphi]|_{(\widehat{\xi}_2, h_2)} &\rightarrow \lambda_*^{-1}[\varphi]|_{(\widehat{\xi}'_2, h'_2)} \\
 (\widehat{\varphi}, g, \widehat{h}) &\mapsto (\widehat{\varphi}, g, \widehat{h}' \circ \widehat{h}).
 \end{aligned}$$

Pulling back the integrand of (4.21) along this equivalence, reparametrization invariance of the integral over groupoids and $\langle (\widehat{h}' \circ \widehat{h})^* \omega', [0, 1] \times \sigma_2 \rangle = \langle \widehat{h}^* \omega', [0, 1] \times \sigma_2 \rangle + \langle \widehat{h}'^* \omega', [0, 1] \times \sigma_2 \rangle$ then shows that (4.21) is a parallel section.

Next we show that this defines a cowedge with respect to σ_1 , i.e. the diagram

$$\begin{array}{ccc}
 \Sigma^\varphi(\sigma_2, \sigma_1) \otimes_{\mathbb{C}} \widetilde{Z}_{\omega'}^{(S_1, \xi_1)}(\sigma'_1) & \xrightarrow{\Sigma^\varphi(\zeta) \otimes_{\mathbb{C}} \text{id}} & \Sigma^\varphi(\sigma_2, \sigma'_1) \otimes_{\mathbb{C}} \widetilde{Z}_{\omega'}^{(S_1, \xi_1)}(\sigma'_1) \\
 \text{id} \otimes_{\mathbb{C}} \widetilde{Z}_{\omega'}^{(S_1, \xi_1)}(\zeta) \downarrow & & \downarrow \\
 \Sigma^\varphi(\sigma_2, \sigma_1) \otimes_{\mathbb{C}} \widetilde{Z}_{\omega'}^{(S_1, \xi_1)}(\sigma_1) & \longrightarrow & \widetilde{Z}_{\omega'}^{(S_2, \xi_2)}(\sigma_2)
 \end{array}$$

commutes for all morphisms $(\zeta : \sigma_1 \rightarrow \sigma'_1) \in \text{Fund}_\theta(S_1, \xi_1)$. The upper composition evaluated on an element $\Lambda \otimes_{\mathbb{C}} f(\cdot) \in \Sigma^\varphi(\sigma_2, \sigma_1) \otimes_{\mathbb{C}} \tilde{Z}_{\omega'}^{(S_1, \xi_1)}(\sigma'_1)$ gives

$$\begin{aligned} & \int_{(\widehat{\varphi}, g, \widehat{h}) \in \lambda_*^{-1}[\varphi]_{(\widehat{\xi}_2, h_2)}} \langle \widehat{\varphi}^* \omega', \Lambda - \zeta \rangle \langle g^* \theta, [0, 1] \times (\Lambda - \zeta) \rangle \\ & \cdot \langle \widehat{h}^* \omega', [0, 1] \times \sigma_2 \rangle f(\widehat{\varphi}|_{S_1}, g|_{S_1}) \\ & = \int_{(\widehat{\varphi}, g, \widehat{h}) \in \lambda_*^{-1}[\varphi]_{(\widehat{\xi}_2, h_2)}} \langle g^* \theta, [0, 1] \times \zeta \rangle^{-1} \langle \widehat{\varphi}^* \omega', \zeta \rangle^{-1} \langle \widehat{\varphi}^* \omega', \Lambda \rangle \\ & \cdot \langle g^* \theta, [0, 1] \times \Lambda \rangle \langle \widehat{h}^* \omega', [0, 1] \times \sigma_2 \rangle \\ & \cdot f(\widehat{\varphi}|_{S_1}, g|_{S_1}) \\ & = \int_{(\widehat{\varphi}, g, \widehat{h}) \in \lambda_*^{-1}[\varphi]_{(\widehat{\xi}_2, h_2)}} \langle g|_{S_1}^* \theta, [0, 1] \times \zeta \rangle^{-1} \langle \widehat{\varphi}|_{S_1}^* \omega', \zeta \rangle^{-1} \langle \widehat{\varphi}^* \omega', \Lambda \rangle \\ & \cdot \langle g^* \theta, [0, 1] \times \Lambda \rangle \langle \widehat{h}^* \omega', [0, 1] \times \sigma_2 \rangle \\ & \cdot f(\widehat{\varphi}|_{S_1}, g|_{S_1}) . \end{aligned}$$

Comparing the term $\langle g|_{S_1}^* \theta, [0, 1] \times \zeta \rangle^{-1} \langle \widehat{\varphi}|_{S_1}^* \omega', \zeta \rangle^{-1}$ here with (4.19) makes it clear that this is the same as the lower composition.

Finally, we have to show naturality with respect to σ_1 . This follows from the same calculation as above if we replace ζ with $-\zeta$ everywhere. □

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