Commun. Math. Phys. 376, 1053–1071 (2020)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-019-03530-4

Communications in
**Mathematical
Physics**

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Canonical Gauges in Higher Gauge Theory

Andreas Gastel

Fakultät für Mathematik, Universität Duisburg-Essen, 45117 Essen, Germany.
E-mail: andreas.gastel@uni-due.de

Received: 13 February 2019 / Accepted: 12 June 2019
Published online: 27 July 2019 –

Abstract: We study the problem of finding good gauges for connections in higher gauge theories. We find that, for 2-connections in strict 2-gauge theory and 3-connections in 3-gauge theory, there are local "Coulomb gauges" that are more canonical than in classical gauge theory. In particular, they are essentially unique, and no smallness of curvature is needed in the critical dimensions. We give natural definitions of 2-Yang–Mills and 3-Yang–Mills theory and find that the choice of good gauges makes them essentially linear. As an application, (anti-)selfdual 2-connections over B^6 are always 2-Yang–Mills, and (anti-)selfdual 3-connections over B^8 are always 3-Yang–Mills.

1. Introduction

An aspect of gauge theory that has proven to be very important in geometry and topology is the control of Sobolev norms by the Yang–Mills functional.

Assume that we are working on a trivial principal fibre bundle $B^m \times G$, where B^m is the unit ball in $\mathbb{R}^m$ and G is a compact Lie group. Assume $\mathfrak{g}$ is the Lie algebra of G, equipped with an ad_G-invariant scalar product. A connection of the bundle can be described as the differential operator d_A on $\mathfrak{g}$-valued functions acting as $d_A X := dX + [A, X]$. Here A is a $\mathfrak{g}$-valued 1-form. This d_A induces similar operators on $\mathfrak{g}$-valued k-forms also denoted by d_A. One of the important aspects of such connections is how they transform under pointwise coordinate transformations of the fibres. (This may look unnecessary for the trivial bundle, but it becomes essential when we glue trivial bundles to get nontrivial ones.) Assume we have a field of transformations $\mathrm{ad}_{g(x)^{-1}}$ acting on the fibres $\{x\} \times \mathfrak{g}$ of $B^m \times \mathfrak{g}$, which defines a mapping $g : B^m \to G$. Assume that g is C^1, say, and that we want to know what our d_A looks like after we have applied the coordinate change $\mathrm{ad}_{g(x)^{-1}}$ on all of our fibres. It turns out that the connection d_A is transformed to d_{g^*A}, where $g^*A := g^{-1}dg + g^{-1}Ag$. This is called a *gauge transformation* of A.

Observe that the (sufficiently regular) maps $g : B^m \to G$ form the *group of gauge transformations* acting from the right on the space of connections. That the action is from the right is not essential and a matter of convention. The important point of gauge theory is that there are quantities derived from A that transform more naturally under gauge transformations than A itself, i.e. like a tensor instead of a differential operator. The most important such quantity is the *curvature*

$$F_A := dA + A \wedge A = dA + \frac{1}{2}[A, A]$$

of A, a $\mathfrak{g}$-valued 2-form that simply transforms as $F_{g^*A} = g^{-1}F_A g$. This implies $|F_{g^*A}| = |F_A|$ since the norm on $\mathfrak{g}$ is ad_G-invariant. Hence the *Yang–Mills functional*

$$YM(A) := \frac{1}{2}\int_{B^m} |F_A|^2\,dx$$

does not change if we transform A by any gauge transformation. It is therefore a very natural functional to consider. As this is well-known, we do not go into details.

Given any connection A, natural norms like the $W^{1,2}$-norm of A are not gauge-invariant and hence depend on more than only "the geometric properties of A". Maybe $\|A\|_{W^{1,2}(B^m)}$ is rather large, but only because we look at A in an unfortunately chosen gauge. Can we find a gauge transformation such that $\|g^*A\|_{W^{1,2}(B^m)}$ is controlled by $\|F_{g^*A}\|_{L^2(B^m)} = \|F_A\|_{L^2(B^m)}$? The answer is yes if $m \le 4$ and the L^2-norm of the curvature is small enough. This is Uhlenbeck's [13] gauge theorem, which is one of the most important results in gauge theory. In dimensions $m \ge 5$, one can still control $\|g^*A\|_{W^{1,m/2}(B^m)}$ by $\|F_{g^*A}\|_{L^{m/2}(B^m)}$ after a suitable gauge transformation if $\|F_A\|_{L^{m/2}(B^m)}$ is sufficiently small. There are more global versions of this, but for us the following local version will be sufficient.

Theorem 1.1 (Uhlenbeck's gauge theorem [13]). *Assume we are given a compact Lie group G. Assume $A \in W^{1,p}\Lambda^1(B^m, \mathfrak{g})$ for some $p \ge 2$ if $2 \le m \le 4$, or $p \ge \frac{m}{2}$ if $m \ge 5$, represents some connection on the trivial bundle $B^m \times G$. There are constants $\kappa > 0$ and $c < \infty$ depending on m and p only such that, whenever $\|F_A\|_{L^p(B^m)} \le \kappa$, there is a gauge transformation $g \in W^{2,p}(B^m, G)$ such that the transformed connection $A' := g^*A$ fulfills $d^*A' = 0$, $A'_N = 0$ on ∂B^m, and*

$$\|A'\|_{W^{1,p}(B^m)} \le c\|F_{A'}\|_{L^p(B^m)}.$$

The gauge transformation can be estimated by

$$\|dg\|_{W^{1,p}(B^m)} \le c\|A\|_{W^{1,p}(B^m)}.$$

Hence the Yang–Mills functional on 4-dimensional manifolds (where it is also conformally invariant) locally controls Sobolev norms. Our paper will be concerned with the question whether there is something similar for *higher gauge theories.*

Higher gauge theories have evolved from attempts to deal with questions that involve parallel transport not only of vectors ("point locations"), which is what connections are made for, but also of higher-dimensional objects. In string theory, the notion of parallel transport of strings should be useful, and in M-theory, it could help to do likewise with 2-dimensional branes. In a rich interplay between ideas from physics and from higher category theory, several higher gauge theories have evolved, among them the (strict) 2-gauge theory and 3-gauge theory that we study in this paper. We cannot even attempt to summarize the rich history that has led to these ideas, and we refer to Baez' and Huerta's paper [3] for an excellent overview and an introduction of 2-gauge theory. An important step towards 2-gauge theory was a study of nonabelian gerbes by Breen and Messing [6]. For 2-gauge theory, the reader may also wish to consult foundational papers by Bartels [5] and by Baez and Schreiber [4] (as well as much more work by Baez and/or Schreiber). For 3-gauge theory, we refer to papers by Sämann and Wolf [11] where the theory has been developed, and by Wang [14].

Very roughly, 2-gauge theory is about 2-connections on 2-bundles. A trivial principal 2-bundle is described by several data that form a structure known as a *Lie crossed module.* We need two Lie groups G and H and homomorphisms $t : H \to G$ and $\alpha : G \to \mathrm{Aut}(H)$ satisfying certain relations. A 2-connection is described by a $\mathfrak{g}$-valued 1-form A and an $\mathfrak{h}$-valued 2-form B, related to each other by $\underline{t}(B) = F_A$, where here $\underline{t}$ is the differential of t at $e \in H$. There is a natural $\mathfrak{h}$-valued 3-form

$$Z_{A,B} := dB + \underline{\alpha}(A) \wedge B$$

which again transforms naturally under 2-gauge transformations. The latter are given by a pair (g, χ) of a function $g : B^m \to G$ and an $\mathfrak{h}$-valued 1-form χ. They also form a group acting from the right on the space of 2-connections. We will give precise formulae in Sect. 3. The L^2-norm of $Z_{A,B}$ turns out to be invariant under all 2-gauge transformations, just as $\|F_A\|_{L^2}$ is invariant under gauge transformations (but not under all 2-gauge transformations). Therefore we may reasonably hope that the L^2-norm of $Z_{A,B}$ plays a role in 2-gauge theory that is similar to the role of the Yang–Mills functional in gauge theory. We expect it to be particularly natural in 6 dimensions, where it is also conformally invariant. We therefore call

$$YM_2(A, B) := \int_{B^m} |Z_{A,B}|^2 \, dx$$

the 2-*Yang–Mills functional.* The attempt to provide a good notion of 2-Yang–Mills has already been undertaken in 2002 by Baez in the preprint [2]. Back then, the significance of the condition $\underline{t}(B) = F_A$ had not yet been fully established in the theory, hence Baez works without that condition and considers the functional $\int (|Z_{A,B}|^2 + |F_A - \underline{t}(B)|^2) \, dx$ instead. This is quite natural in the theory without $\underline{t}(B) = F_A$. In particular, it is also gauge invariant, but no longer conformally invariant in any dimension. Nevertheless, there are some interesting aspects in that paper, including some notion of self-duality in five (!) dimensions.

Our focus is different here, since we have a 2-Yang–Mills functional that really resembles Yang–Mills. Therefore, it is tempting to ask whether there is a higher form of Uhlenbeck's gauge theorem, controlling norms like $\|A\|_{W^{2,2}} + \|B\|_{W^{1,2}}$ by $\|Z_{A,B}\|_{L^2}$ in dimensions $m \leq 6$ once a suitable 2-gauge is fixed, maybe under a smallness condition for the latter norm. One of our results will be that this really works. But, surprisingly enough for the author, it turns out that the good 2-gauge exists without a smallness condition, and moreover the transformed 2-connection has a canonical form that has the potential to simplify the theory very much. More precisely, we can 2-gauge transform (A, B) to get some (A', B') where $A' = 0$, $d^*B' = 0$, and B' takes its values in the abelian subalgebra $\mathrm{Ker}\, \underline{t}$ of $\mathfrak{h}$, plus of course the estimates mentioned above. We call this the *canonical* 2-*gauge* for (A, B), and we prove that it is even unique up to a constant gauge transformation. See Sect. 4 for details.

The proof of the existence of the canonical 2-gauge is considerably simpler than that of Uhlenbeck's theorem. Not surprisingly so, since we use the latter—but only for connections with $F_A = 0$, for which Uhlenbeck's theorem might look a bit trivial (but actually, under the weak regularity assumptions, it isn't). The other important ingredient in the proof is Hodge decomposition on manifolds with boundary.

The existence of a canonical gauge for which A' vanishes and B' maps to the abelian subalgebra Ker $\underline{t}$ makes 2-Yang–Mills theory an essentially *linear* theory, since $Z_{0,B'} = dB'$, and the 2-Yang–Mills equation becomes the Laplace equation for B'. Is this good news or bad news? On one hand, 2-gauge theory is a natural theory that has a geometric content in describing parallel transport of 1-dimensional objects, hence we should be happy to find that the theory turns out to be easier than classical gauge theory. On the other hand, a theory that is not genuinely nonlinear may be not the best candidate for for a Yang–Mills-like theory. That fact may actually diminish our hope for interesting topological implications. And the author suspects that also physics would be served better with a less linear theory.

Of course, the "essentially linear nature" of 2-gauge theory has been remarked before, e.g. in the introduction of [11], observing that $\underline{t}(Z_{A,B}) = 0$ follows from Bianchi type identities, which means the 2-curvature is always in the "abelian" part of the theory. Also, it has been remarked that there are no examples of solitons that are "non-abelian". Our result makes precise in which sense the theory is "linear". It says that after a canonical 2-gauge transformation, we can always work in abelian Lie subalgebras, where the Euler-Lagrange equations for curvature L^2-integrals are linear. The 2-gauge transformation itself, however, depends of course on the 2-connection and solves a nonlinear system of differential equations.

One way around having a 2-gauge theory that is "too abelian" may be to "embed" it into 3-gauge theory where the relations of 2-gauge theory do not hold strictly. (This can be given a precise meaning in the framework of categorification, see [11].) In 3-gauge theory, there is a third Lie group L involved, 3-connections additionally depend on an $\mathfrak{l}$-valued 3-form C, and 3-gauge transformations on an additional $\mathfrak{l}$-valued 2-form λ. There is a curvature 4-form $Y_{A,B,C}$, the L^2-norm of which is gauge-invariant, and conformally invariant in 8 dimensions. We can ask the same questions as for 2-gauge theory. For $m \leq 8$, we find canonical gauges where $A' = 0$, $B' = 0$, and this time C' is in some abelian Lie subalgebra of $\mathfrak{l}$.

One of the points [11] made in introducing 3-gauge theory is that in its framework the curvature 3-form $Z_{A,B}$ no longer is restricted to some abelian Lie algebra. The curvature 4-form $Y_{A,B,C}$, however, does have that restriction, and this is what makes our results on 3-gauges very similar to the ones for 2-gauges. Note also that $Z_{A,B}$ is not 3-gauge covariant in 3-gauge theory, just as F_A is not 2-gauge covariant in 2-gauge theory. The setting of 3-gauge theory will be described in Sect. 5, and our gauge theorem in Sect. 6.

Our gauge theorems can be applied to *flat* 2-connections satisfying $Z_{A,B} \equiv 0$, which then turn out to be 2-gauge equivalent to the trivial connection $(0, 0)$, and similarly to flat 3-connections ($Y_{A,B,C} \equiv 0$), which are seen to be 3-gauge equivalent to $(0, 0, 0)$. These two statements can be seen as higher generalizations of the Poincaré lemma and have been proven by Demessie and Sämann [7, Thm. 2.7 and Thm. 2.12]. The results are special cases of our gauge theorems, and they are proven here under rather weak regularity assumptions.

The curious reader may wonder about 4-gauge theory or even higher ones. The notion of a 3-crossed module, which should be the basis for 4-gauge theory, has been developed in [1]. The number of mappings and relations needed to describe a $(k-1)$-crossed module grows quickly with k, and that may limit our approach of writing down rather "explicit" gauge transformations to small values of k. There are more abstract ways to describe *differential* $(k-1)$-crossed modules using L_∞-algebras (see [9] for a thorough introduction), which may help for future extensions.

Starting from our gauge theorem, there is another aspect of gauge theory that turns out to be shared by 2-gauge and 3-gauge theory. In 4 dimensions, a connection A is called *selfdual or anti-selfdual* if $*F_A = \pm F_A$. One of the basic facts in gauge theory is that every (anti-)selfdual connection is Yang–Mills, i.e. it solves the Euler-Lagrange equation $d_A^* F_A = 0$ for YM. Using our canonical gauges, we prove that in 6 dimensions every 2-connection (A, B) with $*Z_{A,B} = \pm Z_{A,B}$ solves the Euler-Lagrange equation for YM_2. A similar result holds for 3-connections in 8 dimensions. We provide details in the Corollaries 4.2 and 6.2 below. (We always work on B^m with the Euclidean metric, which has turned out to be enough for all applications of Uhlenbeck's theorem. Our methods can be adapted without significant changes to domains with Riemannian metrics.)

It is a pleasure to thank the anonymous referee for many helpful suggestions.

2. Preliminaries on Differential Forms

We will need two nontrivial ingredients about differential forms in our study, the Hopf decomposition with boundaries and Gaffney's inequality. In preparation of these, we recall that there are two useful forms of boundary conditions for differential forms. If M is a Riemannian manifold with smooth boundary, then we can choose coordinates in the neighborhood V of any boundary point y such that dx_n is the dual of the outer normal on $\partial M \cap V$. Let $m := \dim M$ and $0 \le k \le m$. Any k-form ω on V can be decomposed as $\omega = \omega_T + \omega_N$, where

$$\omega_T := \sum_{1 \le i_1 < \ldots < i_k \le n-1} \omega_{i_1,\ldots,i_k}\, dx_{i_1} \wedge \ldots \wedge dx_{i_k},$$
$$\omega_N := \sum_{1 \le i_1 < \ldots < i_k = n} \omega_{i_1,\ldots,i_k}\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

On ∂M, this decomposition does not depend on the choice of coordinates, and therefore the boundary conditions $\omega_N = 0$ and $\omega_T = 0$ make sense for k-forms, and they do play a role in natural problems.

A version of the Hodge decomposition suitable for our needs has been given by Iwaniec, Scott, and Stroffolini—they, as well as Schwarz, proved several different decompositions with different sets of boundary conditions. The one we need is from [8, Remark 5.1 and Theorem 5.7], improved with arguments from [12, Lemma 2.4.11] for higher order Sobolev spaces. In what follows, $\mathcal{H}_N^\ell(M)$ is the space of harmonic forms on M with normal part vanishing on ∂M. Likewise, we use the index N to modify Sobolev spaces.

Proposition 2.1 (Hodge decomposition with boundary). *Let M be a compact m-dimensional manifold with smooth boundary. For $1 < p < \infty$, $k \in \mathbb{N}_0$ and every $\ell \in \{0, \ldots, m\}$, the space $W^{k,p}\Lambda^\ell(M)$ decomposes as a direct sum*

$$W^{k,p}\Lambda^\ell(M) = dW^{k+1,p}\Lambda^{\ell-1}(M) \oplus d^* W_N^{k+1,p}\Lambda^{\ell+1}(M) \oplus \mathcal{H}_N^\ell(M).$$

Correspondingly, any $\omega \in W^{k,p}\Lambda^\ell(M)$ can be decomposed as

$$\omega = d\alpha + d^*\beta + h \quad \text{with } \beta_N = 0 \text{ and } h_N = 0.$$

The forms α, β, h are uniquely determined under the further conditions

$$\alpha \in d^* W_N^{k+2,p}\Lambda^\ell(M), \quad \beta \in d W^{k+2,p}\Lambda^\ell(M).$$

There is a constant c depending on k, p, and M only such that

$$\|\alpha\|_{W^{k+1,p}(M)} + \|\beta\|_{W^{k+1,p}(M)} + \|h\|_{W^{k,p}(M)} \le c\|\omega\|_{W^{k,p}(M)}.$$

If $1 \le k \le m$, $m \ge 2$, *and* $M = B^m$, *we always have* $h \equiv 0$.

The case $M = B^m$ mentioned in the last sentence is not explicitly discussed in the sources mentioned above. We can prove it as follows. For $1 \le k \le m$, by Poincaré's Lemma together with the Hodge isomorphism (see [12, Theorem 2.6.1] for details), we have $\mathcal{H}_N^k(B^m) = \{0\}$, which means $h \equiv 0$. □

Hodge theory on manifolds with boundary has been developed to quite some extent. Both [8,12] are excellent references. A closely related mathematical fact is Gaffney's inequality. We state it in a version that combines [8, Thms. 4.8 and 4.11] and the considerations we just made for the special case $M = B^m$.

Proposition 2.2 (Gaffney's inequality). *Let M be a compact m-dimensional Riemannian manifold with smooth boundary,* $1 < p < \infty$, $0 \le k \le m-1$. *For every k-form ω on M with ω, dω and d*ω in* L^p *and* $\omega_N = 0$ *or* $\omega_T = 0$ *on* ∂M, *we have* $\omega \in W^{1,p}$ *(in the sense that its full covariant derivative is also in* L^p*) with the estimate*

$$\|\omega\|_{W^{1,p}(M)} \le c(\|\omega\|_{L^p(M)} + \|d\omega\|_{L^p(M)} + \|d^*\omega\|_{L^p(M)}).$$

The constant c depends on M and p only.

Moreover, if $M = B^m$ *and* $1 \le k \le m-1$, *the simpler estimate*

$$\|\omega\|_{W^{1,p}(M)} \le c(\|d\omega\|_{L^p(M)} + \|d^*\omega\|_{L^p(M)})$$

holds.

3. The Setting of 2-Gauge Theory

A *crossed module* is (G, H, t, α), where G and H are groups, and $t : H \to G$ and $\alpha : G \to \mathrm{Aut}(H)$ are homomorphisms satisfying *G-equivariance* of t,

$$t(\alpha(g)(h)) = g t(h) g^{-1} \tag{1}$$

for all $g \in G$, $h \in H$, and the *Peiffer identity*,

$$\alpha(t(h_1))(h_2) = h_1 h_2 h_1^{-1} \tag{2}$$

for all $h_1, h_2 \in H$. If G, H are Lie groups and t, α are Lie group homomorphisms, then (G, H, t, α) is called a *Lie crossed module*.

Given a Lie crossed module, we can linearize everything and get Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ with Lie algebra homomorphisms $\underline{t} : \mathfrak{h} \to \mathfrak{g}$ and $\underline{\alpha} : \mathfrak{g} \to \mathfrak{aut}(\mathfrak{h})$. They satisfy the linearized versions of the identities above,

$$\underline{t}(\underline{\alpha}(x)(\xi)) = [x, \underline{t}(\xi)] \tag{3}$$

for all $x \in \mathfrak{g}$, $\xi \in \mathfrak{h}$, and

$$\underline{\alpha}(\underline{t}(\xi))(\nu) = [\xi, \nu] \tag{4}$$

for all $\xi, \nu \in \mathfrak{h}$. Such a structure $(\mathfrak{g}, \mathfrak{h}, \underline{t}, \underline{\alpha})$ is called a *differential crossed module.*

Besides the $\mathfrak{g}$-action on $\mathfrak{h}$ via $\underline{\alpha}$, we also have a G-action on $\mathfrak{h}$, induced by the action via α on H and also denoted by α. For later use, we assume that $\mathfrak{h}$ is equipped with a norm that is G-invariant under the action described by α. We will also assume that the Lie algebra $\mathfrak{g}$ carries a norm that is invariant under the adjoint action of G. In the theorems, we will formulate that shortly as $\mathfrak{g}$ and $\mathfrak{h}$ having G-invariant norms.

From (1) and (2), we infer the "mixed relations" (cf. [10, Section 2.1.1])

$$\underline{t}((\alpha(g)(\xi)) = g\underline{t}(\xi)g^{-1} \tag{5}$$

for all $g \in G, \xi \in \mathfrak{h}$, and

$$\alpha(t(h))(\xi) = h\xi h^{-1} \tag{6}$$

for all $h \in H$, $\xi \in \mathfrak{h}$. (We can always pretend working in matrix Lie algebras, and therefore write $h\xi h^{-1}$ instead of $\mathrm{ad}_h(\xi)$.)

Assume we are given a Lie crossed module $\mathcal{G} := (G, H, t, \alpha)$ and a manifold M. The trivial principal $\mathcal{G}$-2-bundle over M is just the product $M \times G \times H$ equipped with the homomorphisms t and α. A 2-*connection* on that 2-bundle is given by a pair (A, B) of a $\mathfrak{g}$-valued 1-form A and an $\mathfrak{h}$-valued 2-form B on M, where for the moment we assume them to be smooth. For (A, B) to represent a 2-connection, we further require the *vanishing fake curvature condition*

$$dA + A \wedge A - \underline{t}(B) = 0.$$

The notation is to be understood as follows. Let $\Lambda^k(U, \mathfrak{k})$ be the vector space of all $\mathfrak{k}$-valued k-forms on $U \subseteq \mathbb{R}^m$, where $\mathfrak{k}$ is any matrix Lie algebra. Every $V \in \Lambda^k(U, \mathfrak{k})$ can be written as $\sum_i V^i X_i$, where $X_i \in \mathfrak{k}$ (they need not form a basis), and the V^i are scalar k-forms. Similarly, $W \in \Lambda^\ell(U, \mathfrak{k})$ equals $\sum_j W^j X_j$. Then we write

$$V \wedge W := \sum_{i,j} V^i \wedge W^j X_i X_j,$$
$$[V \wedge W] := \sum_{i,j} V^i \wedge W^j [X_i, X_j].$$

For any 2-connection (A, B), we define F_A and the 2-*curvature* $Z_{A,B}$ by

$$F_A := dA + A \wedge A,$$
$$Z_{A,B} := dB + \underline{\alpha}(A) \wedge B.$$

Basic facts about the curvatures are the two Bianchi identities

$$dF_A + A \wedge F_A = 0,$$
$$dZ_{A,B} + \underline{\alpha}(A) \wedge Z_{A,B} = 0,$$

and their consequence

$$\underline{t}(Z_{A,B}) = 0.$$

A 2-*gauge transformation* is given by a function $g : U \to G$ and an $\mathfrak{h}$-valued 1-form χ on M. They transform a 2-connection (A, B) to another 2-connection (A', B') via

$$A' = g^{-1}Ag + g^{-1}dg - \underline{t}(\chi),$$
$$B' = \alpha(g^{-1})(B) - \underline{\alpha}(A') \wedge \chi - d\chi - \chi \wedge \chi.$$

We write $(A', B') =: (g, \chi)^*(A, B)$ and find

$$(g', \chi')^*(g, \chi)^*(A, B) = (gg', \alpha(g'^{-1})(\chi) + \chi')^*(A, B),$$

which means that the group of 2-gauge transformations is that of functions with values in the semi-direct product $G \ltimes_\alpha \mathfrak{h}$ acting from the right on the 2-connections.

The remarkable thing about the 2-curvature $Z_{A,B}$ is its covariance under 2-gauge transformations (g, χ). Those transform $Z_{A,B}$ according to

$$Z_{A',B'} = \alpha(g^{-1})(Z_{A,B}),$$

while F_A does not transform nicely. Since the norm on $\mathfrak{h}$ is assumed to be G-invariant via α, this implies that

$$YM_2(A, B) := \int_U |Z_{A,B}|^2 \, dx$$

is 2-gauge invariant (and conformally invariant on $\mathbb{R}^6$). Therefore it is a promising candidate for a "higher" variant of the Yang–Mills functional.

We have remarked that $\underline{t}(Z_{A,B}) = 0$. This gives the theory the abelian flavor we already mentioned, since $\operatorname{Ker} \underline{t}$ is an abelian subalgebra of $\mathfrak{h}$, which is seen immediately from (4) since $[\xi, \nu] = \underline{\alpha}(\underline{t}(\xi))(\nu) = \underline{\alpha}(0)(\nu) = 0$ for all $\xi \in \operatorname{Ker} \underline{t}$ and all $\nu \in \mathfrak{h}$.

4. Canonical 2-Gauges in 2-Gauge Theory

A basic aim in gauge theory is to control natural quantities like norms of connections by gauge invariant quantities, after applying a suitable gauge transformation. A good example is Uhlenbeck's theorem discussed in the introduction. Its proof is quite non-trivial, and the smallness condition cannot be entirely removed. Compared to this, the corresponding 2-gauge theorem for 2-connections is much simpler.

Gauge theorems can be proven to hold on sufficiently smooth Riemannian manifolds. There is, however, hardly any need for that, since any of the norms involved defined with respect to one Riemannian metric can be estimated by a corresponding norm with respect to any other metric. Therefore, Uhlenbeck proved her theorem with respect to flat metrics only, and that turned out to be good enough. In her book [15], Wehrheim proved that all tools are sufficiently well developed to perform the arguments with a Riemannian metric. The same all but surely applies to our theorems, but we will always work on B^m with its flat metric as we want to keep the paper readable. We will also not bother about a formulation for $m = 2$, where the 2-curvature 3-form $Z_{A,B}$ always vanishes.

Theorem 4.1 (Canonical 2-gauges for 2-connections). *Assume we are given a Lie crossed module (G, H, t, α) where G is a compact Lie group, and that the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are equipped with G-invariant norms. Assume $3 \le m \le 6$ and that $(A, B) \in W^{2,2}\Lambda^1(B^m, \mathfrak{g}) \times W^{1,2}\Lambda^2(B^m, \mathfrak{h})$ represents a 2-connection of the trivial 2-bundle associated with (G, H, t, α) over B^m. Then there is a 2-gauge transformation $(g, \chi) \in W^{3,2}(B^m, G) \times W^{2,2}\Lambda^1(B^m, \mathfrak{h})$ such that $(A', B') := (g, \chi)^*(A, B)$ satisfies*

$$A' = 0, \quad \underline{t}(B') = 0, \quad d^* B' = 0, \quad (B'_N)_{|\partial B^m} = 0,$$

and its norm is controlled by the 2-curvature,

$$\|B'\|_{W^{1,2}(B^m)} \le c \|Z_{0,B'}\|_{L^2(B^m)} = c \|Z_{A,B}\|_{L^2(B^m)} \, .$$

The 2-gauge transformation obeys the estimates

$$\begin{aligned}\|dg\|_{W^{2,2}(B^m)} &\le c\|A\|_{W^{2,2}(B^m)},\\ \|\chi\|_{W^{2,2}(B^m)} &\le c(\|A\|_{W^{2,2}(B^m)} + \|A\|^3_{W^{2,2}(B^m)} + \|B\|_{W^{1,2}(B^m)} + \|B\|^{3/2}_{W^{1,2}(B^m)}).\end{aligned}$$

The 2*-connection* $(A', B') = (0, B')$ *is unique up to a constant gauge transformation, i.e. up to a* 2*-gauge transformation* $(g_0, 0)$ *with some* $g_0 \in G$*.*

Proof. Since G is compact, $\mathfrak{g}$ is a compact Lie algebra, i.e. the direct sum of a semisimple and an abelian Lie algebra. The image $\underline{t}(\mathfrak{h})$ of the Lie algebra homomorphism $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Even better, it is an ideal in $\mathfrak{g}$, $[\underline{t}(\mathfrak{h}), \mathfrak{g}] \subseteq \underline{t}(\mathfrak{h})$ because of (3). Now, for any ideal in a semisimple Lie algebra, the Lie algebra is the direct sum of the ideal and its orthogonal complement with respect to the Killing form. In our case, this also means that $\underline{t}(\mathfrak{h})^\perp$ is a Lie subalgebra of $\mathfrak{g}$.

Fix a right inverse $\underline{t}_{-1} : \underline{t}(\mathfrak{h}) \to \mathfrak{h}$ of $\underline{t}$ for which $\underline{t} \circ \underline{t}_{-1}$ is the identity of $\underline{t}(\mathfrak{h})$. Decompose $A = A^\top + A^\perp$ according to the direct sum $\underline{t}(\mathfrak{h}) \oplus \underline{t}(\mathfrak{h})^\perp$. Under the 2-gauge transformation $(e, \chi_1) := (e, \underline{t}_{-1}(A^\top))$, (A, B) transforms to

$$(A_1, B_1) = (A^\perp, B - \underline{\alpha}(A^\perp) \wedge \underline{t}_{-1}(A^\top) - d(\underline{t}_{-1}(A^\top)) - \underline{t}_{-1}(A^\top) \wedge \underline{t}_{-1}(A^\top)).$$

Since $A^\perp$ takes its values in $\underline{t}(\mathfrak{h})^\perp$, so do $dA^\perp$ and $A^\perp \wedge A^\perp$, the latter because $\underline{t}(\mathfrak{h})^\perp$ is a Lie subalgebra. Hence F_{A_1} is a $\underline{t}(\mathfrak{h})^\perp$-valued 2-form. But it is also $\underline{t}(\mathfrak{h})$-valued because of $\underline{t}(B_1) = F_{A_1}$. This means that $F_{A_1} = 0$.

Now that we know $F_{A_1} = 0$, we have $\|F_{A_1}\|_{L^3(B^m)} = 0$, and of course $A_1 \in W^{1,3}$ because of Sobolev's embedding $W^{2,2} \hookrightarrow W^{1,3}$. Hence Uhlenbeck's theorem gives us a gauge transformation $g_2 \in W^{2,3}(B^m, G)$ such that $g_2^* A_1 = 0$ on B^m because of $\|g_2^* A_1\|_{W^{1,3}(B^m)} \le c\|F_{A_1}\|_{L^3(B^m)} = 0$. This means we can apply the 2-gauge transformation $(g_2, 0)$ to (A_1, B_2) to find that (A, B) is 2-gauge equivalent to

$$(A_2, B_2) = (0, \alpha(g_2^{-1})(B_1)).$$

Apart from having $A_2 = 0$, we also know that B_2 is quite simple (like B_1, in fact) because we have $\underline{t}(B_2) = F_{A_2} = 0$, hence B_2 takes its values in $\operatorname{Ker} \underline{t}$, and we have seen above that on $\operatorname{Ker} \underline{t} \subseteq \mathfrak{h}$, the Lie bracket of $\mathfrak{h}$ vanishes. Remember that Z always takes its values in $\operatorname{Ker} \underline{t}$. And now that we have $A_2 = 0$, we have $Z_{A_2,B_2} = dB_2$.

The question of finding a good 2-gauge at this stage is reduced to a completely linear problem. We have $Z_{A_2,B_2} = dB_2$ and

$$(e, \chi_3)^*(0, B_2) = (0, B_2 - d\chi_3)$$

if we assume that also χ_3 takes its values in $\operatorname{Ker} \underline{t}$. But now that everything is linear and without Lie brackets, the gauge problem reduces simply to a question in Hodge theory. We use the Hodge decomposition from Proposition 2.1. Having $B_2 \in W^{1,2}\Lambda^2(B^m, \operatorname{Ker} \underline{t})$, we find unique forms $a \in d^* W^{3,2}\Lambda^2(B^m, \operatorname{Ker} \underline{t})$ and $b \in dW^{3,2}\Lambda^2(B^m, \operatorname{Ker} \underline{t})$ satisfying $b_N = 0$ on ∂B^m such that $B_2 = da + d^* b$. We then choose $\chi_3 := a$ and find

$$(A', B') := (e, \chi_3)^*(0, B_2) = (0, d^* b),$$

which proves the existence of a suitable gauge.

Concerning the estimate of the norm of B' by Z, we note that $b_N = 0$ implies $d^*b_N = 0$ on ∂B^m. We therefore have $d^*B' = 0$, $dB' = Z_{0,B'}$, and $(B'_N)_{|\partial B^m} = 0$, which means we can use Gaffney's inequality Proposition 2.2 to estimate

$$\|B'\|_{W^{1,2}(B^m)} \le c(\|dB'\|_{W^{1,2}(B^m)} + \|d^*B'\|_{W^{1,2}(B^m)}) \le c\|Z_{0,B'}\|_{L^2(B^m)}.$$

What remains to be shown are estimates for g and χ, which are given by composition of (e, χ_1), $(g_2, 0)$, and (e, χ_3),

$$\begin{aligned} g &= g_2, \\ \chi &= \alpha(g_2)(\underline{t}_{-1}(A^\top)) + a. \end{aligned}$$

By an easy consequence of Uhlenbeck's theorem (that is, bootstrapping and using the equation $dg_2 = -A_1 g_2$ as in [15, Lemma A.7]), we have

$$\|dg_2\|_{W^{2,2}(B^m)} \le c\|A_1\|_{W^{2,2}(B^m)} \le c\|A\|_{W^{2,2}(B^m)}.$$

Before we proceed with estimating χ, a few words about the technique for readers not so familiar with that sort of analysis. We will need to estimate derivatives of $\alpha(\gamma)(\xi)$, where γ is a G-valued function and ξ an $\mathfrak{h}$-valued form. Note that that $\alpha(\gamma)(\xi)$ depends linearly on ξ, but nonlinearly on γ. We are going to write D for total derivatives, $\alpha(\gamma)(\xi) =: \alpha(\gamma, \xi)$, and $D_1\alpha$, $D_2\alpha$ for the total derivatives of α with respect to the γ-variables and the ξ-variables, respectively. By linear dependence on ξ, we have $D_2\alpha(\gamma, \xi)D\xi = \alpha(\gamma)(D\xi)$. The chain rule gives us

$$\begin{aligned} D[\alpha(\gamma)(\xi)] &= D_1\alpha(\gamma, \xi)D\gamma + \alpha(\gamma, D\xi), \\ D^2[\alpha(\gamma)(\xi)] &= D_1^2\alpha(\gamma, \xi)(D\gamma, D\gamma) + D_1\alpha(\gamma, \xi)D^2\gamma \\ &\quad + D_1\alpha(\gamma, D\xi)D\gamma + \alpha(\gamma, D^2\xi). \end{aligned}$$

By compactness of G and Linearity of α in ξ, we have

$$|D^k\alpha(\gamma, \xi)| \le c\,|\xi|,$$

with c depending on $k \in \mathbb{N}_0$ only. Therefore, the equations above imply the estimates

$$|D[\alpha(\gamma)(\xi)]| \le c(|\xi|\,|D\gamma| + |D\xi|), \tag{7}$$

$$|D^2[\alpha(\gamma)(\xi)]| \le c(|\xi|\,|D\gamma|^2 + |\xi|\,|D^2\gamma| + |D\xi|\,|D\gamma| + |D^2\xi|). \tag{8}$$

We will have to integrate such inequalities and proceed the right-hand side using Hölder's inequality and then Sobolev's, like in the simple example

$$\begin{aligned} \|\alpha(\gamma)(\xi)\|_{W^{1,2}} &= \|\alpha(\gamma)(\xi)\|_{L^2} + \|D[\alpha(\gamma)(\xi)]\|_{L^2} \\ &\le c\|\xi\|_{L^2} + c(\|D\xi\|_{L^2} + \||\xi||D\gamma|\|_{L^2}) \\ &\le c(\|\xi\|_{W^{1,2}} + \|\xi\|_{L^3}\|D\gamma\|_{L^6}) \\ &\le c(\|\xi\|_{W^{1,2}} + \|\xi\|_{W^{1,2}}\|D\gamma\|_{W^{2,2}}). \end{aligned}$$

Now we return to estimating $\chi = \alpha(g_2)(\underline{t}_{-1}(A^\top)) + a$, using (7), (8), and the techniques we just explained. Using also the trivial estimates $\|A^\top\|_{W^{2,2}(B^m)} \le \|A\|_{W^{2,2}(B^m)}$ and $\|g_2\|_{W^{3,2}(B^m)} \le c + \|dg_2\|_{W^{2,2}(B^m)}$, we find

$$\begin{aligned}
&\|\alpha(g_2)(\underline{t}_{-1}(A^\top))\|_{W^{2,2}}\\
&\quad\le c(\|A\|_{W^{2,2}} + \|A\|_{W^{1,3}}\|g_2\|_{W^{1,6}} + \|A\|_{L^6}(\|g_2\|^2_{W^{1,6}} + \|g_2\|_{W^{2,3}}))\\
&\quad\le c(\|g\|^2_{W^{3,2}}\|A\|_{W^{2,2}})\\
&\quad\le c(\|A\|_{W^{2,2}} + \|A\|^3_{W^{2,2}}),
\end{aligned}$$

with all norms to be taken on B^m. Observe that we are making frequent use of Sobolev's embedding in the form $W^{k,2} \hookrightarrow W^{k-1,3} \hookrightarrow W^{k-2,6}$ which holds when $m \le 6$. Combining that with the estimate from the Hodge decomposition,

$$\begin{aligned}
\|a\|_{W^{2,2}} \;&\le\; c\|B_2\|_{W^{1,2}}\\
&= c\|\alpha(g_2^{-1})(B - \underline{\alpha}(A^\perp) \wedge \underline{t}_{-1}(A^\top) - d(\underline{t}_{-1}(A^\top)) - \underline{t}_{-1}(A^\top) \wedge \underline{t}_{-1}(A^\top))\|_{W^{1,2}}\\
&\le c(\|B\|_{W^{1,2}} + \|A\|_{W^{1,3}}\|A\|_{L^6} + \|dA\|_{W^{1,2}})\\
&\quad + c\|dg_2\|_{L^6}(\|B\|_{L^3} + \|A\|^2_{L_6} + \|dA\|_{L^3})\\
&\le c(\|B\|_{W^{1,2}} + \|B\|^{3/2}_{W^{1,2}} + \|A\|_{W^{2,2}} + \|A\|^3_{W^{2,2}}),
\end{aligned}$$

we have proven the estimates for g and χ. (In case you are wondering where the exponent $\frac{3}{2}$ comes from: we have estimated $\|A\|\,\|B\|$ using Young's inequality $xy \le \frac{1}{3}|x|^3 + \frac{2}{3}|y|^{3/2}$.)

Now we turn to the uniqueness of B'. Assume that we have a "canonical gauge" $(0, B'')$ of (A, B) with the same properties as $(0, B')$. Then there exists a 2-gauge transformation, again denoted by (g, χ), such that $(0, B'') = (g, \chi)^*(0, B')$, which means

$$0 = g^{-1}dg - \underline{t}(\chi), \tag{9}$$

$$B'' = \alpha(g^{-1})(B') - d\chi - \chi \wedge \chi. \tag{10}$$

Observe that (9) implies $g^{-1}dg \in \underline{t}(\mathfrak{h})$ almost everywhere, which means that g is of the form $g_0 t(h)$ for some constant $g_0 \in G$ and some $h \in W^{3,2}(B^m, H)$. And $\alpha(g_0^{-1})(B') \in \operatorname{Ker}\underline{t}$ because $\underline{t}(B') = 0$ implies $\underline{t}(\alpha(g_0^{-1})(B')) = g_0^{-1}\underline{t}(B')g_0 = 0$ via (5). Since the adjoint action of H on the abelian subalgebra $\operatorname{Ker}\underline{t}$ of $\mathfrak{h}$ is trivial, using (6), we find

$$\alpha(g^{-1})(B') = \alpha(t(h^{-1}))(\alpha(g_0^{-1})(B')) = h^{-1}\alpha(g_0^{-1})(B')h = \alpha(g_0^{-1})(B'),$$

Now this means that (10) simplifies to

$$B'' = \alpha(g_0^{-1})(B') - d\chi - \chi \wedge \chi =: \alpha(g_0^{-1})(B') - \nu. \tag{11}$$

Applying d to (11), we have

$$dB'' = \alpha(g_0^{-1})(dB') - d\nu,$$

which we compare to an equation using the transformation behavior of Z and $A' = A'' = 0$,

$$dB'' = Z_{0,B''} = \alpha(g^{-1})(Z_{0,B'}) = \alpha(g_0^{-1})(Z_{0,B'}) = \alpha(g_0^{-1})(dB'),$$

where the third "=" is justified as above using $\underline{t}(Z_{0,B'}) = 0$. Comparing the last two equations, we find $d\nu = 0$.

We can also apply d^* to (11) to find $d^*\nu = 0$ because of $d^*B'' = 0$ and $d^*B' = 0$. And similarly, we observe $\nu_N = 0$ on ∂B^m since $B'_N = 0$ and $B''_N = 0$ on ∂B^m. Now we know $d\nu = 0$ and $d^*\nu = 0$ on B^m, and $\nu_N = 0$ on ∂B^m, which together imply $\nu = 0$, again by Gaffney's inequality. Then (11) reads $B'' = \alpha(g_0^{-1})(B')$, which is the asserted uniqueness of B' modulo constant gauge transformations. □

Remarks. (1) We have formulated our gauge theorem under the minimal regularity assumptions on A and B. If both A and B have more regularity, we will have more regularity of B', dg, and χ, by the same proof combined with some iterated estimates. This way, we can easily formulate $W^{k,p}$- and $C^{k,\alpha}$-versions of the gauge theorem. For example, if $A \in W^{k+1,2}$, $B \in W^{k,2}$ for some $k \geq 1$, we can choose the gauge transformation (g, χ) in $W^{k+2,2} \times W^{k+1,2}$ and control B' in $W^{k,2}$.

(2) If $(0, B')$ is in the "canonical" gauge and 2-Yang–Mills, then it is stationary for the 2-YM functional among all connections in canonical gauge. And since $Z_{0,B'} = dB'$ for those connections, the Euler-Lagrange equation for that problem is $d^*dB' = 0$. Together with $d^*B' = 0$ (and the boundary condition for B'), we find that $\Delta B' = 0$ is equivalent to the 2-Yang–Mills equation for all connections in canonical gauge. This means that transforming to the canonical gauge, the (nonlinear) 2-Yang–Mills equation reduces to the (linear) Laplace equation.

In contrast to this, the Euler-Lagrange equation for the 2-Yang–Mills energy is more difficult to write down in general gauge, because the Euler-Lagrange equation for $\int |dB + \underline{\alpha}(A) \wedge B|^2\, dx$ has to be derived under the side condition $F_A - \underline{t}(B) = 0$. The full system looks like

$$
\begin{aligned}
d_A^* d_A B &= \underline{t}^*(\lambda), \\
d^*\lambda + [A \lrcorner \lambda] &= -\underline{\alpha}^*(B \lrcorner d_A B), \\
F_A &= \underline{t}(B),
\end{aligned}
$$

where $\lambda \in \Lambda^2(B^m, \mathfrak{g})$ is an unknown Lagrange multiplier, and "$\lrcorner$" denotes suitable contractions of forms.

(3) The previous remark has an interesting consequence, which generalizes the classical fact that (anti-)selfdual connections in 4 dimensions are Yang–Mils. For any 2-connection (A, B) over B^6, the 2-curvature $Z_{A,B}$ is (anti-)selfdual if $*Z_{A,B} = \pm Z_{A,B}$. We then call also (A, B) an (anti-)selfdual 2-connection. For a 2-connection $(0, B')$ in canonical gauge, (anti-)selfduality means $dB = \pm * dB$. We then have $d^*dB' = \pm * d * (dB') = \pm * d * *dB' = 0$. Again, together with $d^*B' = 0$, we find $\Delta B' = 0$. Hence any (anti-)selfdual 2-connection in canonical gauge is also 2-Yang–Mills. And since both the (anti-)selfduality and the 2-Yang–Mills functional (and hence its equations) are invariant under 2-gauge transformations, this proves:

Corollary 4.2. *Every (anti-)selfdual* 2*-connection* (A, B) *over* B^6 *is also* 2*-Yang–Mills.*

5. The Setting of 3-Gauge Theory

There is a notion of 3-gauge theory which is based on Lie 2-crossed modules. It has been developed systematically by Sämann and Wolf [11]. We refer to [14] for a concise presentation of the algebraic aspects of the local theory. It is described using a complex of Lie groups

$$
L \xrightarrow{\tau} H \xrightarrow{t} G
$$

(with $t \circ \tau \equiv e$). We also need homomorphisms $\alpha : G \to \mathrm{Aut}(H)$ and $\beta : G \to \mathrm{Aut}(L)$, with respect to which t and τ are again G-equivariant. That means (1) holds, and also $\tau(\beta(g)(\ell)) = \alpha(g)(\tau(\ell))$ for all $g \in G$, $\ell \in L$. The Peiffer identity is now replaced by a *Peiffer lifting* which is a smooth function $\{\cdot\,,\,\cdot\} : H \times H \to L$ that is G-equivariant in the sense that

$$\beta(g)(\{h,k\}) = \{\alpha(g)(h), \alpha(g)(k)\}$$

for all $g \in G$, $h, k \in H$. Moreover, it must satisfy the relations

$$\begin{aligned}
\tau(\{h,k\}) &= hkh^{-1}\alpha(t(h))(k^{-1}) =: \langle h,k\rangle,\\
lml^{-1}m^{-1} &= \{\tau(l),\tau(m)\},\\
\{hj,k\} &= \{h, jkj^{-1}\}\beta(t(h))(\{j,k\}),\\
\{h,jk\} &= \{h,j\}\{h,k\}\{\langle h,k\rangle^{-1}, \alpha(t(h))(j)\},\\
\{\tau(l),h\}\{h,\tau(l)\} &= l\beta(t(h))(l^{-1})
\end{aligned} \tag{12}$$

for all $h, j, k \in H$ and $l, m \in L$.

Correspondingly, a *differential* 2*-crossed module* is described by a complex of Lie algebras

$$\mathfrak{l} \xrightarrow{\underline{\tau}} \mathfrak{h} \xrightarrow{\underline{t}} \mathfrak{g}$$

with $\underline{t} \circ \underline{\tau} \equiv 0$ and a Peiffer lifting $\{\cdot\,,\,\cdot\} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{l}$ which is $\mathfrak{g}$-equivariant in the sense that

$$\underline{\beta}(a)(\{u,v\}) = \{\underline{\alpha}(a)(u), v\} + \{u, \underline{\alpha}(a)(v)\}$$

for all $a \in \mathfrak{g}$, $u, v \in \mathfrak{h}$, where here $\underline{\alpha} : \mathfrak{g} \to \mathfrak{aut}(\mathfrak{h})$ and $\underline{\beta} : \mathfrak{g} \to \mathfrak{aut}(\mathfrak{l})$ are Lie algebra homomorphisms, and $\underline{\tau}$ and $\underline{t}$ are $\mathfrak{g}$-invariant with respect to $\underline{\alpha}$ and $\underline{\beta}$. The relations for the Peiffer lifting in their linearized versions read

$$\underline{\tau}(\{u,v\}) = [u,v] - \underline{\alpha}(\underline{t}(u))(v), \tag{13}$$

$$[x,y] = \{\underline{\tau}(x), \underline{\tau}(y)\}, \tag{14}$$

$$\begin{aligned}
\{[u,v],w\} &= \underline{\beta}(\underline{t}(u))(\{v,w\}) + \{u,[v,w]\} - \underline{\beta}(\underline{t}(v))(\{u,w\}) - \{v,[u,w]\},\\
\{u,[v,w]\} &= \{\underline{\tau}(\{u,v\}),w\} - \{\underline{\tau}(\{u,w\}),v\},\\
\{\underline{\tau}(x),u\} + \{u,\underline{\tau}(x)\} &= -\underline{\beta}(\underline{t}(u))(x)
\end{aligned} \tag{15}$$

for $u, v, w \in \mathfrak{h}$ and $x, y \in \mathfrak{l}$.

Between (12) and (15), there is another "mixed relation" concerning the G-operation on $\mathfrak{l}$. We rewrite (12) as

$$\beta(t(h))(\ell) = \ell\{\tau(\ell^{-1}), h\}\{h, \tau(\ell^{-1})\}.$$

Using $\{e_H, h\} = \{h, e_H\} = e_L$ and letting $\ell = \exp(rx)$, we can differentiate at $r = 0$ to find

$$\begin{aligned}
\beta(t(h))(x) &= x\{\tau(e_L),h\}\{h,\tau(e_H)\} - e_L\{\underline{\tau}(x),h\}\{h,\tau(e_L)\} - e_L\{\tau(e_L),h\}\{h,\underline{\tau}(x)\}\\
&= x - \{\underline{\tau}(x),h\} - \{h,\underline{\tau}(x)\}
\end{aligned} \tag{16}$$

for all $h \in H$, $x \in \mathfrak{l}$, in a calculation that involves three different $\{\cdot,\ \cdot\}$ living on $H \times H$, $H \times \mathfrak{h}$, and $\mathfrak{h} \times H$. The mixed relation from the G-equivariance of τ is

$$\underline{\tau}(\beta(g)(x)) = \alpha(g)(\underline{\tau}(x))$$

for all $g \in G$, $x \in \mathfrak{l}$.

It has been proven in [14, Proposition 2.2(1)] that for 1-forms $\chi, \eta \in \Lambda^1(U, \mathfrak{h})$, (13) implies

$$\underline{\alpha}(\underline{t}(\chi)) \wedge \eta = [\chi \wedge \eta] - \underline{\tau}(\{\chi \wedge \eta\}), \tag{17}$$

and in particular

$$\chi \wedge \chi = \frac{1}{2}[\chi \wedge \chi] = \frac{1}{2}\underline{\alpha}(\underline{t}(\chi)) \wedge \chi + \frac{1}{2}\underline{\tau}(\{\chi \wedge \chi\}). \tag{18}$$

A 3-connection on the trivial 3-bundle over U is a triple

$$(A, B, C) \in \Lambda^1(U, \mathfrak{g}) \times \Lambda^2(U, \mathfrak{h}) \times \Lambda^3(U, \mathfrak{l})$$

satisfying two "fake curvature conditions"

$$dA + A \wedge A = \underline{t}(B), \tag{19}$$

$$dB + \underline{\alpha}(A) \wedge B = \underline{\tau}(C). \tag{20}$$

A 3-*gauge transformation* is a triple (g, χ, λ) of a function $g : U \to G$, an $\mathfrak{h}$-valued 1-form $\chi \in \Lambda^1(U, \mathfrak{h})$, and an $\mathfrak{l}$-valued 2-form $\lambda \in \Lambda^2(U, \mathfrak{l})$, acting on 3-connections via

$$\begin{aligned}
A' &= g^{-1}Ag + g^{-1}dg - \underline{t}(\chi),\\
B' &= \alpha(g^{-1})(B) - A' \wedge \chi - d\chi - \chi \wedge \chi - \underline{\tau}(\lambda),\\
C' &= \beta(g^{-1})(C) - d\lambda - \underline{\beta}(A') \wedge \lambda + \{B' \wedge \chi\} + \{\chi \wedge \alpha(g^{-1})(B)\} + \{\underline{\tau}(\lambda) \wedge \chi\}.
\end{aligned}$$

In particular, any $(g, 0, 0)$ acts via

$$\begin{aligned}
A' &= g^{-1}Ag + g^{-1}dg,\\
B' &= \alpha(g^{-1})(B),\\
C' &= \beta(g^{-1})(C),
\end{aligned}$$

while $(e, \chi, 0)$ acts via

$$\begin{aligned}
A' &= A - \underline{t}(\chi),\\
B' &= B - \underline{\alpha}(A') \wedge \chi - d\chi - \chi \wedge \chi,\\
C' &= C + \{B' \wedge \chi\} + \{\chi \wedge B\},
\end{aligned} \tag{21}$$

and $(e, 0, \lambda)$ via

$$\begin{aligned}
A' &= A,\\
B' &= B - \underline{\tau}(\lambda),\\
C' &= C - d\lambda - \underline{\beta}(A') \wedge \lambda.
\end{aligned}$$

The 3-curvature, transforming naturally under any of these, is an $\mathfrak{l}$-valued 4-form given by

$$Y_{A,B,C} := dC + \underline{\beta}(A) \wedge C + \{B \wedge B\}.$$

In particular, the 3-Yang–Mills functional

$$YM_3(A, B, C) := \int_U |Y_{A,B,C}|^2 \, dx$$

is invariant under all 3-gauge transformations, and conformally invariant if $m = 8$. Of course, this uses $Y_{A',B',C'} = \beta(g^{-1})(Y_{A,B,C})$ for $(A', B', C') := (g, \chi, \lambda)^*(A, B, C)$, and the asserted invariance of YM_3 can only hold if we have assumed G-invariance (via β) of the norm we have chosen on $\mathfrak{l}$.

Of course, the 3-gauge transformations form a group. The group law is interesting, and has (as far as I am aware) not been formulated before.

Proposition 5.1 (Composition of 3-gauge transformations). *In the setting described above, the composition of two* 3*-gauge transformations (acting from the right) is given by*

$$(g, \chi, \lambda)(g', \chi', \lambda') = \Big(gg', \alpha(g'^{-1})(\chi) + \chi', \beta(g'^{-1})(\lambda) + \lambda' - \{\chi' \wedge \alpha(g'^{-1})(\chi)\}\Big).$$

Proof. Since the dependence on g, g', λ, λ' is easy, we prove only

$$(e, \chi, 0)(e, \eta, 0) = (e, \chi + \eta, -\{\eta \wedge \chi\}),$$

with η instead of χ' for readability. Applying $(e, \eta, 0)$ to $(A', B', C') := (e, \chi, 0)^*(A, B, C)$ given in (21), we find that $(e, \eta, 0)^*(A', B', C') =: (A'', B'', C'')$ reads

$$\begin{aligned} A'' &= A - \underline{t}(\chi + \eta), \\ B'' &= B - \underline{\alpha}(A') \wedge \chi - d\chi - \chi \wedge \chi - \underline{\alpha}(A'') \wedge \eta - d\eta - \eta \wedge \eta \\ &= B - \underline{\alpha}(A'') \wedge \chi - \underline{\alpha}(\underline{t}(\eta)) \wedge \chi - \underline{\alpha}(A'') \wedge \eta - d\chi - d\eta \\ &\quad -(\chi + \eta) \wedge (\chi + \eta) + [\eta, \chi] \\ &= B - \underline{\alpha}(A'') \wedge (\chi + \eta) - d(\chi + \eta) - (\chi + \eta) \wedge (\chi + \eta) + \underline{\tau}(\{\eta, \chi\}), \end{aligned}$$

where we have used (17) in the last line. We read off that $(e, \eta, 0)^*(e, \chi, 0)^*$ transforms A and B just like $(e, \chi + \eta, -\{\eta \wedge \chi\})^*$. We still have to show the same for C. First we apply $(e, \eta, 0)$ to C' from (21) and find

$$\begin{aligned} C'' &= C + \{B' \wedge \chi\} + \{\chi \wedge B\} + \{B'' \wedge \eta\} + \{\eta \wedge B'\} \\ &= C + \{B'' \wedge (\chi + \eta)\} + \{(\chi + \eta) \wedge B\} \\ &\quad + \{(\underline{\alpha}(A'') \wedge \eta) \wedge \chi\} + \{d\eta \wedge \chi\} + \{(\eta \wedge \eta) \wedge \chi\} \\ &\quad - \{\eta \wedge (\underline{\alpha}(A'') \wedge \chi)\} - \{\eta \wedge (\underline{\alpha}(\underline{t}(\eta)) \wedge \chi)\} - \{\eta \wedge d\chi\} - \{\eta \wedge (\chi \wedge \chi)\}. \end{aligned}$$

This must coincide with $(e, \chi + \eta, -\{\eta \wedge \chi\})$ applied to C, which would give

$$\begin{aligned} C'' &= C + \{B'' \wedge (\chi + \eta)\} + \{(\chi + \eta) \wedge B\} \\ &\quad + d\{\eta \wedge \chi\} + \underline{\beta}(A'') \wedge \{\eta \wedge \chi\} - \{\underline{\tau}(\{\eta \wedge \chi\}) \wedge (\chi + \eta)\}. \end{aligned}$$

In order to show that the right-hand sides coincide, we need to prove four equations,

$$d\{\eta \wedge \chi\} = \{d\eta \wedge \chi\} - \{\eta \wedge d\chi\}, \tag{22}$$

$$\underline{\beta}(A'') \wedge \{\eta \wedge \chi\} = \{(\underline{\alpha}(A'') \wedge \eta) \wedge \chi\} - \{\eta \wedge (\underline{\alpha}(A'') \wedge \chi)\}, \tag{23}$$

$$-\{\underline{\tau}(\{\eta \wedge \chi\}) \wedge \chi\} = -\{\eta \wedge (\chi \wedge \chi)\}, \tag{24}$$

$$-\{\underline{\tau}(\{\eta \wedge \chi\}) \wedge \eta\} = \{(\eta \wedge \eta) \wedge \chi\} - \{\eta \wedge (\underline{\alpha}(\underline{t}(\eta)) \wedge \chi)\}. \tag{25}$$

Now (22) ist just the graded version of Leibnitz' rule, and (23) is the graded version of the $\mathfrak{g}$-equivariance of $\{\cdot, \cdot\}$, while (24) is [14, Proposition 2.2(4)]. Finally, to prove (25), we use [14, Proposition 2.2(5)], [14, Proposition 2.2(3)], and (17) in

$$\begin{aligned}\{(\eta \wedge \eta) \wedge \chi\} &= \{\eta \wedge [\eta \wedge \chi]\} + \underline{\beta}(\underline{t}(\eta)) \wedge \{\eta \wedge \chi\}\\ &= \{\eta \wedge [\eta \wedge \chi]\} - \{\eta \wedge \underline{\tau}(\{\eta \wedge \chi\})\} - \{\underline{\tau}(\{\eta \wedge \chi\}) \wedge \eta\}\\ &= \{\eta \wedge (\underline{\alpha}(\underline{t}(\eta)) \wedge \chi)\} - \{\underline{\tau}(\{\eta \wedge \chi\}) \wedge \eta\}.\end{aligned}$$

This completes the proof of the proposition. □

6. Canonical 3-Gauges in 3-Gauge Theory

Here is our analogue of Theorem 4.1 for 3-connections.

Theorem 6.1 (Canonical 3-gauges for 3-connections). *Assume we are given a Lie 2-crossed module $(G, H, L, t, \tau, \alpha, \beta)$ where G is a compact Lie group, and that the Lie algebras $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$ are equipped with G-invariant norms. Assume $4 \leq m \leq 8$ and that $(A, B, C) \in W^{3,2}\Lambda^1(B^m, \mathfrak{g}) \times W^{2,2}\Lambda^2(B^m, \mathfrak{h}) \times W^{1,2}\Lambda^3(B^m, \mathfrak{l})$ represents a 3-connection of the trivial 3-bundle associated with $(G, H, L, t, \tau, \alpha, \beta)$ over B^m. Then there is a 3-gauge transformation $(g, \chi, \lambda) \in W^{4,2}(B^m, G) \times W^{3,2}\Lambda^1(B^m, \mathfrak{h}) \times W^{2,2}\Lambda^2(B^m, \mathfrak{l})$ such that $(A', B', C') := (g, \chi, \lambda)^*(A, B)$ satisfies*

$$A' = 0, \quad B' = 0, \quad \underline{\tau}(C') = 0, \quad d^*C' = 0, \quad (C'_N)_{|\partial B^m} = 0,$$

and its norm is controlled by the 3-curvature,

$$\|C'\|_{W^{1,2}(B^m)} \leq c\|Y_{0,0,C'}\|_{L^2(B^m)} = c\|Y_{A,B,C}\|_{L^2(B^m)}.$$

The 3-gauge transformation obeys the estimates

$$\begin{aligned}\|dg\|_{W^{3,2}(B^m)} &\leq c\|A\|_{W^{3,2}(B^m)},\\ \|\chi\|_{W^{3,2}(B^m)} &\leq c(\|A\|_{W^{3,2}(B^m)} + \|A\|^4_{W^{3,2}(B^m)} + \|B\|_{W^{2,2}(B^m)} + \|B\|^2_{W^{2,2}(B^m)}),\\ \|\lambda\|_{W^{2,2}(B^m)} &\leq c(\|A\|_{W^{3,2}(B^m)} + \|A\|^6_{W^{3,2}(B^m)} + \|B\|_{W^{2,2}(B^m)} + \|B\|^{7/2}_{W^{2,2}(B^m)}\\ &\quad +\|C\|_{W^{1,2}(B^m)} + \|C\|^{4/3}_{W^{1,2}(B^m)}).\end{aligned}$$

Proof. Exactly as in the proof of Theorem 4.1, we make gauge transformations that transform (A, B, C) to $(0, B_2, C_2)$ with B_2 a $\operatorname{Ker} \underline{t}$-valued 1-form. Now decompose $B_2 = B_2^\top + B_2^\perp$ according to the direct sum $\underline{\tau}(\mathfrak{l}) \oplus \underline{\tau}(\mathfrak{l})^\perp$. Assume again we have chosen some fixed right inverse $\underline{\tau}_{-1} : \underline{\tau}(\mathfrak{l}) \to \mathfrak{l}$ of $\underline{\tau}$. Apply the gauge transformation

$$(A_3, B_3, C_3) := (e, 0, \underline{\tau}_{-1}(B_2^\top))^*(0, B_2, C_2)$$

and find

$$\begin{aligned} A_3 &= 0, \\ B_3 &= B_2^{\perp}, \\ C_3 &= C_2 - \underline{\tau}_{-1}(dB_2^{\top}). \end{aligned}$$

Now that we know B_3 is a $\underline{\tau}(\mathfrak{l})^{\perp}$-valued 1-form, we can Hodge-decompose B_3 in the space of such forms, which means we find unique forms $a \in d^* W^{4,2}\Lambda^2(B^m, \underline{\tau}(\mathfrak{l})^{\perp})$ and $b \in dW^{4,2}\Lambda^2(B^m, \underline{\tau}(\mathfrak{l})^{\perp})$ satisfying $b_N = 0$ on ∂B^m such that $B_3 = da + d^*b$. Having that, we perform the gauge transformation

$$(A_4, B_4, C_4) := (e, a, 0)^*(0, B_3, C_3)$$

with the result

$$\begin{aligned} A_4 &= 0, \\ B_4 &= d^*b - a \wedge a, \\ C_4 &= C_3 + \{B_4 \wedge a\} + \{a \wedge B_3\}. \end{aligned}$$

From $\underline{t}(a) = 0$ and (18), we find

$$a \wedge a = \frac{1}{2}\underline{\tau}(\{a \wedge a\}),$$

which means our next step should be the gauge transformation

$$(A_5, B_5, C_5) := (e, 0, -\frac{1}{2}\{a \wedge a\})^*(0, B_4, C_4),$$

where here

$$\begin{aligned} A_5 &= 0, \\ B_5 &= d^*b, \\ C_5 &= C_4 + \frac{1}{2}d\{a, a\}. \end{aligned}$$

Now we have that B_5 is a $\underline{\tau}(\mathfrak{l})^{\perp}$-valued 2-form, which implies that also dB is $\underline{\tau}(\mathfrak{l})^{\perp}$-valued. On the other hand, (20) (together with $A_5 = 0$) implies that $dB_5 = Z_{0,B_5} = \underline{\tau}(C_5)$ takes its values in $\underline{\tau}(\mathfrak{l})$, which means $dB_5 = 0$. Since also $d^*B_5 = d^*d^*b = 0$, and $(B_5)_N = 0$, we now know that actually $B_5 = 0$ by Gaffney's inequality.

We have reached a situation that parallels that for $(0, B_2)$ in the previous section. We know that $(A_5, B_5, C_5) = (0, 0, C_5)$, and because of $\underline{\tau}(C_5) = Z_{0,0} = 0$, we know that C_5 takes its values in $\operatorname{Ker}\underline{\tau} \subset \mathfrak{l}$. Because of (14), the Lie subalgebra $\operatorname{Ker}\underline{\tau}$ of $\mathfrak{l}$ is abelian. We can again perform Hodge decomposition, this time $C_5 = dp + d^*q$, $p \in d^*W^{3,2}\Lambda^3(B^m, \operatorname{Ker}\underline{\tau})$ and $q \in dW^{3,2}\Lambda^3(B^m, \operatorname{Ker}\underline{\tau})$ with $q_N = 0$ on ∂B^m. We let

$$(A', B', C') := (e, 0, p)^*(0, 0, C_5) = (0, 0, d^*q),$$

which is the transformed 3-connection as stated in the theorem. The estimates for the gauge transformation put together from those in the proof are a long routine calculation along the lines of the proof of Theorem 4.1. In that calculation, it is crucial to know the composition law Proposition 5.1 in order to compute (g, χ, λ) from the six 3-gauge transformations performed above.

The uniqueness proof for C' is also similar, but needs a few modifications. Assume again that we have two canonical gauges $(0, 0, C')$ and $(0, 0, C'')$ of (A, B, C). Then there exists a 3-gauge transformation, again denoted by (g, χ, λ), such that $(0, 0, C'') = (g, \chi, \lambda)^*(0, 0, C')$, meaning

$$\underline{t}(\chi) = g^{-1}dg, \tag{26}$$

$$\underline{\tau}(\lambda) = -d\chi - \chi \wedge \chi, \tag{27}$$

$$C'' = \beta(g^{-1})(C') - d\lambda + \{\underline{\tau}(\lambda) \wedge \chi\}. \tag{28}$$

As before, (26) implies $g = g_0 t(h)$ with $g_0 \in G$ constant. From (16), we have

$$\beta(t(h^{-1}))(X) = X - \{\underline{\tau}(X), h^{-1}\} - \{h^{-1}, \underline{\tau}(X)\} = X$$

for any k-form X with values in Ker $\underline{\tau}$. We can apply this to $X = \beta(g_0^{-1})(C')$ to simplify (28), finding

$$C'' = \beta(g_0^{-1})(C') - d\lambda + \{\underline{\tau}(\lambda) \wedge \chi\} =: \beta(g_0^{-1})(C') - \xi, \tag{29}$$

and to $X = dC'$ in

$$dC'' = Y_{0,0,C''} = \beta(g^{-1})(Y_{0,0,C'}) = \beta(g_0^{-1})(Y_{0,0,C'}) = \beta(g_0^{-1})(dC').$$

Comparing the latter to d(29), we have $d\xi = 0$, and from d^*(29) and $(29)_N$, we again have $d^*\xi = 0$ on B^m and $\xi_N = 0$ on ∂B^m. Hence $\xi = 0$, which completes the uniqueness proof. □

All remarks made about our 2-gauge theorem apply similarly here. In particular, by the same reasoning as for Corollary 4.2, we have a selfduality theorem. In the case of an 8-dimensional base of the 3-bundle, we call a 3-connection (A, B, C) (anti-)selfdual if $*Y_{A,B,C} = \pm Y_{A,B,C}$.

Corollary 6.2. *Every (anti-)selfdual 3-connection (A, B, C) over B^8 is also 3-Yang–Mills.*

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Communicated by C. Schweigert