



Non-closure of Quantum Correlation Matrices and Factorizable Channels that Require Infinite Dimensional Ancilla (With an Appendix by Narutaka Ozawa)

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Abstract: We show that there exist factorizable quantum channels in each dimension ≥ 11 which do not admit a factorization through any finite dimensional von Neumann algebra, and do require ancillas of type II_1 , thus witnessing new infinite-dimensional phenomena in quantum information theory. We show that the set of $n \times n$ matrices of correlations arising as second-order moments of projections in finite dimensional von Neumann algebras with a distinguished trace is non-closed, for all $n \geq 5$, and we use this to give a simplified proof of the recent result of Dykema, Paulsen and Prakash that the set of synchronous quantum correlations $C_q^s(5, 2)$ is non-closed. Using a trick originating in work of Regev, Slofstra and Vidick, we further show that the set of correlation matrices arising from second-order moments of unitaries in finite dimensional von Neumann algebras with a distinguished trace is non-closed in each dimension ≥ 11 , from which we derive the first result above.

1. Introduction

C. Anantharaman-Delaroche introduced in [1] the class of *factorizable* completely positive maps between von Neumann algebras equipped with a normal faithful state, while studying non-commutative analogues of classical ergodic theory results, including, e.g., G.-C. Rota’s “Alternierende Verfahren” theorem.

It was shown in [5] that not all unital completely positive, trace-preserving maps on $M_n(\mathbb{C})$ (also referred to as unital quantum channels in dimension n) are factorizable when $n \geq 3$, which also led to a negative answer to the so-called *asymptotic quantum Birkoff conjecture*, [15]. The tool was the characterization established in [5] that a unital, completely positive, trace-preserving map $T : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is factorizable if and

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only if there exist a finite von Neumann algebra N equipped with a (faithful, normal) tracial state τ and a unitary u in $M_n(\mathbb{C}) \otimes N$ such that $T(x) = (\text{id}_n \otimes \tau)(u(x \otimes 1_N)u^*)$, for all $x \in M_n(\mathbb{C})$. Following the terminology introduced in [6], we say in this case that T admits an *exact factorization through* $M_n(\mathbb{C}) \otimes N$, and N is called the *ancilla*. In all previously studied cases of factorizable maps (see [5, 6], and the recent paper [11] for the case $n = 2$), the ancilla could be taken to be finite dimensional, and even a full matrix algebra. It was, however, first remarked in [12] that one cannot always take the ancilla to be a full matrix algebra. In this paper we show that for each $n \geq 11$, there are factorizable maps on $M_n(\mathbb{C})$ that do not admit a finite dimensional ancilla, nor an ancilla of type I, and we give concrete examples of such maps, see Example 3.5 and Theorem 4.1. Therefore, one needs to employ ancillas of type II_1 to describe a general factorizable channel in dimension $n \geq 11$. Observe that all factorizable quantum channels do admit an exact factorization through a type II_1 von Neumann algebra (even a type II_1 factor), by the well-known fact that each finite von Neumann algebra equipped with a fixed faithful normal tracial state embeds in a trace-preserving way into a type II_1 factor.

The proof of our result uses very recent developments of analysis in quantum information theory concerning the non-closure of certain sets of correlation matrices. The first such result was due to Slofstra, who proved the failure of what is referred to as the *strong Tsirelson conjecture*. This was recently refined by Dykema, Paulsen and Prakash, [3], to show that the set of synchronous correlation matrices $C_q^s(n, k)$ is non-closed, when $n \geq 5$ and $k \geq 2$.

We consider the set $\mathcal{D}(n)$ of $n \times n$ matrices arising from second-order moments of n -tuples of projections in finite von Neumann algebras with a (normal, faithful) tracial state, and the subset $\mathcal{D}_{\text{fin}}(n)$ consisting of those matrices that arise likewise from n -tuples of projections in finite-dimensional von Neumann algebras (or C^* -algebras). We show that the set $\mathcal{D}_{\text{fin}}(n)$ is not closed, when $n \geq 5$. Our proof uses a theorem of Kruglyak, Rabanovich and Samoilenko, [10], also employed in [3], which describes which scalar multiples of the identity operator on a (finite dimensional) Hilbert space can arise as the sum of n projections. We use this to give a shorter and more direct proof of the main result from [3] that the set of synchronous quantum correlation matrices $C_q^s(n, 2)$ is non-closed, when $n \geq 5$.

Kirchberg, [9], reformulated the Connes Embedding Problem in terms of the set $\mathcal{G}(n)$ of $n \times n$ matrices of correlations arising from unitaries in finite von Neumann algebras with a (normal, faithful) tracial state. This result was further refined by Dykema and Juschenko, [2], and in their formulation, the Connes Embedding Problem is equivalent to the statement that $\mathcal{F}(n) = \mathcal{G}(n)$, for all $n \geq 3$, where $\mathcal{F}(n)$ is the closure of the set of $n \times n$ matrices of correlations arising from unitaries in full matrix algebras. A trick originating in (as of yet unpublished) work of Regev, Slofstra and Vidick (communicated to us by W. Slofstra in May 2018), which we carry out in the setting of finite von Neumann algebras in Sect. 3, allows us to conclude further that the set $\mathcal{F}_{\text{fin}}(2n + 1)$ of matrices of correlations arising from unitaries in finite dimensional von Neumann algebras is non-closed, whenever $\mathcal{D}_{\text{fin}}(n)$ is non-closed, i.e., for all $n \geq 5$.

A connection between the set $\mathcal{G}(n)$ and the set of factorizable Schur multipliers on $M_n(\mathbb{C})$ was established in [6]. This connection gives the final link between the established non-closure of the sets $\mathcal{F}_{\text{fin}}(2n + 1)$, for $n \geq 5$, and existence of factorizable Schur multipliers with no finite dimensional ancilla (or, even stronger, non type I), in each dimension ≥ 11 .

In the Appendix, written by N. Ozawa, it is shown that the construction by Kruglyak, Rabanovich and Samoilenko in [10] of an n -tuple of projections with sum equal to a

multiple α of the identity can be realized in the hyperfinite II_1 factor \mathcal{R} , for all admissible values of α , except, possibly, for two extremal ones. This, in turn, implies that the factorizable Schur multipliers with no finite dimensional ancilla found in this article *do* admit \mathcal{R} as an ancilla (except, possibly, for the cases corresponding to the above mentioned extremal values of α).

Different sets of matrices of correlations arising from the generators of L. Brown’s universal C^* -algebra were shown to be non-closed by Harris and Paulsen [7]. This was used by Gao, Harris and Junge [4] to obtain further non-closure results in a matrix-valued setting.

2. Non-closure of Sets of Matrices of Quantum Correlations

For $n \geq 2$, let $\mathcal{D}(n)$ and $\mathcal{D}_{\text{fin}}(n)$ be the set of $n \times n$ matrices $[\tau(p_j p_i)]_{i,j=1}^n$, where p_1, \dots, p_n are projections in some arbitrary finite von Neumann algebra, respectively, in some finite dimensional von Neumann algebra, equipped with a (normal) faithful tracial state τ . We show in this section that $\mathcal{D}_{\text{fin}}(n)$ is non-closed, when $n \geq 5$, and we use this to give a more direct proof, avoiding graph correlation functions, of the very recent result of Dykema, Paulsen and Prakash, [3], that the set of synchronous quantum correlations $C_q^s(5, 2)$ is non-closed.

Standard arguments involving ultralimits, as in the proof of Proposition 2.4 (v) below, respectively, direct sums of finite von Neumann algebras, show that the set $\mathcal{D}(n)$ is compact and convex. One can likewise show that the set $\mathcal{D}_{\text{fin}}(n)$ is convex. The subset $\mathcal{D}_{\text{matrix}}(n)$ of $\mathcal{D}_{\text{fin}}(n)$ consisting of $n \times n$ matrices $[\text{tr}_k(p_j p_i)]_{i,j=1}^n$, where p_1, \dots, p_n are projections in a matrix algebra $M_k(\mathbb{C})$, for $k \geq 1$, is not convex (and not closed), for any $n \geq 1$, since each diagonal entry of such a matrix is the (normalized) trace of a projection in a matrix algebra, which is a rational number. It is not hard to see that $\mathcal{D}_{\text{matrix}}(n)$ is a dense subset of $\mathcal{D}_{\text{fin}}(n)$.

Note that the closure of $\mathcal{D}_{\text{matrix}}(n)$ is equal to $\mathcal{D}(n)$, for all $n \geq 3$, if and only if the Connes Embedding Problem has an affirmative answer (see Sect. 3 for a sketch of a proof of this fact). Let us also observe that $\mathcal{D}_{\text{fin}}(2) = \mathcal{D}(2)$ is closed.

Proposition 2.1. *We have*

$$\mathcal{D}(2) = \left\{ \begin{pmatrix} s & u \\ u & t \end{pmatrix} : 0 \leq s, t \leq 1, \max\{0, s + t - 1\} \leq u \leq \min\{s, t\} \right\}. \tag{2.1}$$

Moreover, each matrix in $\mathcal{D}(2)$ arises from a pair of projections in the commutative finite-dimensional C^ -algebra $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ (with respect to a suitable trace).*

In particular, it follows that $\mathcal{D}_{\text{fin}}(2) = \mathcal{D}(2)$ and $\mathcal{D}_{\text{fin}}(2)$ is closed.

Proof. Let (M, τ) be a finite von Neumann algebra equipped with a tracial state, and let $p, q \in M$ be projections. The associated matrix in $\mathcal{D}(2)$ is

$$\begin{pmatrix} \tau(p) & \tau(pq) \\ \tau(qp) & \tau(q) \end{pmatrix} = \begin{pmatrix} s & u \\ u & t \end{pmatrix},$$

where $s = \tau(p), t = \tau(q)$ and $u = \tau(pq) = \tau(qp) = \tau(pqp) = \tau(qpq)$. It is clear that $0 \leq s, t \leq 1$. Since $0 \leq pqp \leq p$ and $0 \leq qpq \leq q$, we further conclude that $0 \leq u \leq \min\{s, t\}$.

By Kaplansky’s formula (see, e.g., [8]), we get $\tau(p \wedge q) = \tau(p) + \tau(q) - \tau(p \vee q) \geq \tau(p) + \tau(q) - 1$. As $p \wedge q = q(p \wedge q)q \leq qpq$, we infer that $u = \tau(qpq) \geq \tau(p \wedge q) \geq s + t - 1$.

To complete the proof, we show that each matrix in the right-hand side of (2.1) arises from projections p and q in $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, with respect to the trace with weight $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, satisfying $\alpha_j \geq 0$ and $\sum_{j=1}^4 \alpha_j = 1$. Set $p = (1, 1, 0, 0)$ and $q = (1, 0, 1, 0)$. Then $pq = (1, 0, 0, 0)$ and

$$\begin{pmatrix} \tau(p) & \tau(pq) \\ \tau(qp) & \tau(q) \end{pmatrix} = \begin{pmatrix} \alpha_1 + \alpha_2 & \alpha_1 \\ \alpha_1 & \alpha_1 + \alpha_3 \end{pmatrix}.$$

We must therefore choose $\alpha_1 = u \geq 0$, $\alpha_2 = s - u \geq 0$, $\alpha_3 = t - u \geq 0$, and $\alpha_4 = 1 - \alpha_1 - \alpha_2 - \alpha_3$. The inequality $u \geq s + t - 1$ ensures that $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$, whence $\alpha_4 \geq 0$. □

We do not know if the sets $\mathcal{D}_{\text{fin}}(n)$ are closed for $n = 3, 4$.

We proceed to prove that the sets $\mathcal{D}_{\text{fin}}(n)$ are non-closed, for $n \geq 5$, following the idea of Dykema, Paulsen and Prakash to use Theorem 2.2 below from [10]. As in [10], let Σ_n be the set of all $\alpha \geq 0$ for which there exist projections p_1, \dots, p_n on a Hilbert space H such that $\sum_{j=1}^n p_j = \alpha \cdot I_H$. The sets Σ_n are completely described in [10] and have the following properties: They are symmetric, i.e., if $\alpha \in \Sigma_n$, then $n - \alpha \in \Sigma_n$. Moreover, $\Sigma_2 = \{0, 1, 2\}$ and $\Sigma_3 = \{0, 1, \frac{3}{2}, 2, 3\}$. The set Σ_4 is a countably infinite subset of the rational numbers, \mathbb{Q} , with one accumulation point, namely 2. For $n \geq 5$, Σ_n is the union of the interval $[\frac{1}{2}(n - \sqrt{n^2 - 4n}), \frac{1}{2}(n + \sqrt{n^2 - 4n})]$ and a countably infinite discrete subset of rational numbers, containing $\{0, 1, \frac{n}{n-1}, n - \frac{n}{n-1}, n - 1, n\}$. Observe that $\frac{n}{n-1} < \frac{1}{2}(n - \sqrt{n^2 - 4n})$. Set

$$\Pi_n = [\frac{1}{2}(1 - \sqrt{1 - 4/n}), \frac{1}{2}(1 + \sqrt{1 - 4/n})] \subset \left(\frac{2n-1}{n(n-1)}, 1 - \frac{2n-1}{n(n-1)} \right). \tag{2.2}$$

Then Π_n is an interval with non-empty interior, and $n^{-1}\Sigma_n \setminus \Pi_n$ is contained in \mathbb{Q} .

Theorem 2.2 (Kruglyak, Rabanovich and Samoilenko, [10, Theorem 6]). *Let $n \geq 2$ be an integer. Then there exist projections p_1, \dots, p_n on some finite dimensional Hilbert space H such that $\sum_{j=1}^n p_j = \alpha \cdot I_H$ if and only if $\alpha \in \Sigma_n \cap \mathbb{Q}$.*

Note that the “only if” part of the theorem above is trivial: Apply the standard trace on $B(H)$ to both sides of the equation $\sum_{j=1}^n p_j = \alpha \cdot I_H$, to obtain $\sum_{j=1}^n \dim(p_j) = \alpha \dim(H)$. This argument also gives the following quantitative result for rational values of α : If $\alpha = a/b$ is irreducible, with a, b positive integers, and if there exist projections p_1, \dots, p_n on a Hilbert space H such that $\sum_{j=1}^n p_j = \alpha \cdot I_H$, then $\dim(H) \geq b$.

For each $n \geq 2$ and each $t \in [1/n, 1]$, define the $n \times n$ matrix $A_t^{(n)} = [A_t^{(n)}(i, j)]_{i,j=1}^n$ by

$$A_t^{(n)}(i, j) = \begin{cases} t, & i = j, \\ \frac{t(nt - 1)}{n - 1}, & i \neq j. \end{cases} \tag{2.3}$$

Proposition 2.3. *Let $n \geq 2$ be an integer, let $1/n \leq t \leq 1$, and let $\alpha = nt$.*

- (i) *Let \mathcal{A} be a unital C^* -algebra with a faithful tracial state τ and let p_1, \dots, p_n be projections in \mathcal{A} satisfying $\tau(p_j p_i) = A_t^{(n)}(i, j)$, for all $1 \leq i, j \leq n$. Then $\sum_{j=1}^n p_j = \alpha \cdot 1_{\mathcal{A}}$. Furthermore, if t is irrational, then \mathcal{A} is necessarily infinite dimensional. Respectively, if $\alpha = a/b$ is rational with a, b positive integers and a/b is irreducible, then \mathcal{A} has no representation on a Hilbert space of dimension less than b .*
- (ii) *Conversely, let \mathcal{A} be a unital C^* -algebra with a tracial state τ , and let p_1, \dots, p_n be projections in \mathcal{A} satisfying $\sum_{j=1}^n p_j = \alpha \cdot 1_{\mathcal{A}}$. Then there exist projections $\tilde{p}_1, \dots, \tilde{p}_n$ in some matrix algebra $M_m(\mathcal{A})$ over \mathcal{A} satisfying*

$$\sum_{j=1}^n \tilde{p}_j = \alpha \cdot 1_{M_m(\mathcal{A})}, \quad \tilde{\tau}(\tilde{p}_j \tilde{p}_i) = A_t^{(n)}(i, j), \quad 1 \leq i, j \leq n,$$

where $\tilde{\tau}$ is the normalized trace on $M_m(\mathcal{A})$ induced by τ .

Proof. (i). Set $q = (nt)^{-1} \sum_{j=1}^n p_j$. To show that $q = 1_{\mathcal{A}}$, it suffices to check that $1 = \tau(q) = \tau(q^2)$, since this will entail that $\tau((1 - q)^*(1 - q)) = \tau((1 - q)^2) = 0$. These are straightforward calculations: Indeed, $\tau(q) = (nt)^{-1} \sum_{j=1}^n \tau(p_j) = 1$, and

$$\tau(q^2) = (nt)^{-2} \left(\sum_{j=1}^n \tau(p_j) + \sum_{i \neq j} \tau(p_i p_j) \right) = (nt)^{-2} \left(nt + n(n - 1) \cdot \frac{t(nt - 1)}{n - 1} \right) = 1.$$

If t is irrational, then it follows from (the easy part of) Theorem 2.2 that \mathcal{A} does not have any finite dimensional representation on a Hilbert space, so \mathcal{A} must be infinite dimensional. Respectively, if $\alpha = a/b$ is rational with a, b positive integers and a/b irreducible, then \mathcal{A} has no representation on a Hilbert space of dimension less than b , by the (explicit) comment after the statement of Theorem 2.2.

(ii). We follow the same strategy as in the proof of [3, Theorem 4.2]. Let $m = n!$ and let S_n denote the collection of all permutations of the set $\{1, 2, \dots, n\}$. For $1 \leq j \leq n$, set

$$\tilde{p}_j = \bigoplus_{\sigma \in S_n} p_{\sigma(j)} \in \bigoplus_{\sigma \in S_n} \mathcal{A} \subset M_m(\mathcal{A}),$$

where the inclusion above is given by identifying the diagonal of $M_m(\mathcal{A})$ with the direct sum $\bigoplus_{\sigma \in S_n} \mathcal{A}$ (for some enumeration of S_n). It is easy to check that $\sum_{j=1}^n \tilde{p}_j = \alpha \cdot 1_{M_m(\mathcal{A})}$. Moreover, for $1 \leq i \leq n$,

$$\tilde{\tau}(\tilde{p}_i) = \frac{1}{n!} \sum_{\sigma \in S_n} \tau(p_{\sigma(i)}) = \frac{1}{n} \sum_{k=1}^n \tau(p_k) =: t_0,$$

and for $1 \leq i \neq j \leq n$,

$$\tilde{\tau}(\tilde{p}_i \tilde{p}_j) = \frac{1}{n!} \sum_{\sigma \in S_n} \tau(p_{\sigma(i)} p_{\sigma(j)}) = \frac{1}{n(n - 1)} \sum_{k \neq \ell} \tau(p_k p_\ell) =: s_0.$$

Since $\sum_{j=1}^n p_j = \alpha \cdot 1_A$, we deduce that $t_0 = \alpha/n = t$, and that

$$n^2 t^2 = \alpha n t = \sum_{k=1}^n \tau(\alpha \cdot p_k) = \sum_{k=1}^n \tau(p_k \sum_{\ell=1}^n p_\ell) = \sum_{k,\ell=1}^n \tau(p_k p_\ell) = n t + n(n-1) s_0,$$

from which we conclude that $s_0 = (n-1)^{-1} t (n t - 1)$, as desired. □

Combining Theorem 2.2 and Proposition 2.3, we obtain the following:

Proposition 2.4. *For $n \geq 2$ and $1/n \leq t \leq 1$ we have:*

- (i) $A_t^{(n)} \in \mathcal{D}(n)$ if and only if $A_t^{(n)} \in \overline{\mathcal{D}_{\text{fin}}(n)}$ if and only if $t \in n^{-1} \Sigma_n$,
- (ii) $A_t^{(n)} \in \mathcal{D}_{\text{fin}}(n)$ if and only if $A_t^{(n)} \in \mathcal{D}_{\text{matrix}}(n)$ if and only if $t \in n^{-1} \Sigma_n \cap \mathbb{Q}$.

Moreover, for $n \geq 5$,

- (iii) $A_t^{(n)}$ belongs to $\overline{\mathcal{D}_{\text{fin}}(n)} \setminus \mathcal{D}_{\text{fin}}(n)$, for $t \in n^{-1} \Sigma_n \setminus \mathbb{Q} = \Pi_n \setminus \mathbb{Q}$.
- (iv) If $t \in n^{-1} \Sigma_n \setminus \mathbb{Q}$ and p_1, \dots, p_n are projections in a von Neumann algebra (N, τ) with faithful tracial state satisfying $A_t^{(n)}(i, j) = \tau(p_j p_i)$, for all $1 \leq i, j \leq n$, then N must be of type II_1 .
- (v) For each $A \in \overline{\mathcal{D}_{\text{fin}}(n)}$, there exist projections p_1, \dots, p_n in the ultrapower \mathcal{R}^ω of the hyperfinite type II_1 factor \mathcal{R} such that $A(i, j) = \tau_{\mathcal{R}^\omega}(p_j p_i)$, for all $1 \leq i, j \leq n$.

Proof. Suppose that $n \geq 2$ and $1/n \leq t \leq 1$. From the first part of Proposition 2.3 (i) we see that $A_t^{(n)} \in \mathcal{D}(n)$ implies $t \in n^{-1} \Sigma_n$, while its second part shows that $A_t^{(n)} \in \mathcal{D}_{\text{fin}}(n)$ implies $t \in n^{-1} \Sigma_n \cap \mathbb{Q}$. Proposition 2.3 (ii) gives that $t \in n^{-1} \Sigma_n$ implies $A_t^{(n)} \in \mathcal{D}(n)$.

Let $t \in n^{-1} \Sigma_n \cap \mathbb{Q}$. Then by Theorem 2.2, there exist projections p_1, \dots, p_n on a finite dimensional Hilbert space H satisfying $\sum_{j=1}^n p_j = \alpha \cdot I_H$, where $\alpha = n t$. Identifying the bounded operators on H with a full matrix algebra, we may assume that the projections p_j belong to $M_k(\mathbb{C})$, for some $k \geq 2$. By Proposition 2.3, we can find projections $\tilde{p}_1, \dots, \tilde{p}_n$ in some larger matrix algebra $M_m(\mathbb{C})$ with normalized trace tr , satisfying $A_t^{(n)}(i, j) = \text{tr}(\tilde{p}_i \tilde{p}_j)$, for all $1 \leq i, j \leq n$. This shows that $A_t^{(n)}$ belongs to $\mathcal{D}_{\text{matrix}}(n)$. As $\mathcal{D}_{\text{matrix}}(n) \subset \mathcal{D}_{\text{fin}}(n)$, this completes the proof of (ii).

To complete the proof of (i) we must show that $A_t^{(n)} \in \overline{\mathcal{D}_{\text{fin}}(n)}$ when $t \in n^{-1} \Sigma_n$. This follows directly from (ii) when t is rational. Suppose that t is irrational. As remarked below (2.2), $n^{-1} \Sigma_n \setminus \Pi_n \subset \mathbb{Q}$, for all $n \geq 2$. Hence $t \in \Pi_n \setminus \mathbb{Q}$. We can therefore find a sequence $\{t_k\}_{k=1}^\infty$ of rational numbers in the interval Π_n converging to t . Then $A_{t_k}^{(n)} \rightarrow A_t^{(n)}$, as $k \rightarrow \infty$, and $A_{t_k}^{(n)}$ belongs to $\mathcal{D}_{\text{fin}}(n)$, for each $k \geq 1$, by (i). This shows that $A_t^{(n)}$ belongs to the closure of $\mathcal{D}_{\text{fin}}(n)$.

(iii) follows from (i) and (ii).

(iv). The finite von Neumann algebra N can have no representation on a finite dimensional Hilbert space, by the second part of Proposition 2.3 (i), whence N must be of type II_1 .

(v). Suppose that A is the limit of a sequence $\{A_k\}_{k=1}^\infty$ of matrices in $\mathcal{D}_{\text{fin}}(n)$. Since each finite dimensional C^* -algebra with a distinguished trace can be embedded in a trace preserving way into the hyperfinite type II_1 factor \mathcal{R} , we can find projections $p_1^{(k)}, \dots, p_n^{(k)}$ in \mathcal{R} satisfying $\tau_{\mathcal{R}}(p_j^{(k)} p_i^{(k)}) = A_k(i, j)$, for all $1 \leq i, j \leq n$. Let p_i be the image in \mathcal{R}^ω of the sequence $\{p_i^{(k)}\}_{k=1}^\infty \in \ell^\infty(\mathcal{R})$. Then

$$\tau_{\mathcal{R}^\omega}(p_j p_i) = \lim_{\omega} \tau_{\mathcal{R}}(p_j^{(k)} p_i^{(k)}) = \lim_{\omega} A_k(i, j) = A(i, j),$$

for all $1 \leq i, j \leq n$, as wanted. □

By Proposition 2.4 (iii) and the fact that Π_n is an interval with non-empty interior, when $n \geq 5$, cf. (2.2), we obtain the following theorem:

Theorem 2.5. *The set $\mathcal{D}_{\text{fin}}(n)$ is non-compact, when $n \geq 5$.*

Remark 2.6. It is shown in Proposition 2.4 above that one can realize the $n \times n$ matrix $A_t^{(n)}$ using projections in \mathcal{R}^ω , for all $n \geq 2$ and all $t \in n^{-1}\Sigma_n$. If moreover t is rational, then $A_t^{(n)}$ can be realized using projections in some matrix algebra, by Proposition 2.3 (ii).

Using Proposition 2.3, this also shows that for each $n \geq 2$ and for each α in Σ_n , one can find an n -tuple of projections summing up to $\alpha \cdot 1_N$ in some type II_1 factor N , e.g., $N = \mathcal{R}^\omega$. One can also reach this conclusion directly from Theorem 2.2, using an ultraproduct argument as in the proof of Proposition 2.4 (v).

In Theorem A.1 in the Appendix by N. Ozawa it is shown that for each $n \geq 5$ and each $\alpha \in (\frac{1}{2}(n - \sqrt{n^2 - 4n}), \frac{1}{2}(n + \sqrt{n^2 - 4n}))$, one can find an n -tuple of projections in \mathcal{R} summing up to $\alpha \cdot 1_{\mathcal{R}}$. It follows that the matrix $A_t^{(n)}$ can be realized using projections in \mathcal{R} for all t in $n^{-1}\Sigma_n$, except, possibly, for the endpoints of the interval $[\frac{1}{2}(1 - \sqrt{1 - 4/n}), \frac{1}{2}(1 + \sqrt{1 - 4/n})]$.

Recall, e.g., from [3, Section 2], that for $n, k \geq 2$, the set $C_{qc}(n, k)$ consists of $nk \times nk$ quantum correlation matrices $[(p(i, j|v, w))]_{i,j,v,w}$ with entries given by

$$p(i, j|v, w) = \langle P_{v,i} Q_{w,j} \psi, \psi \rangle, \quad 1 \leq i, j \leq k, \quad 1 \leq v, w \leq n,$$

where, for each v and w , $\{P_{v,i}\}_{i=1}^k$ and $\{Q_{w,j}\}_{j=1}^k$ are projection-valued measures on some Hilbert space H , satisfying $P_{v,i} Q_{w,j} = Q_{w,j} P_{v,i}$, for all i, j, v, w , and where ψ is a unit vector in H . Let $C_q(n, k)$ be the same set of quantum correlation matrices, with the additional assumption that $H = H_A \otimes H_B$, for some finite dimensional Hilbert spaces H_A and H_B , and $P_{v,i}$ belongs to $B(H_A) \otimes I_{H_B}$, while $Q_{w,j}$ belongs to $I_{H_A} \otimes B(H_B)$. The closure of the set $C_q(n, k)$ is denoted by $C_{qa}(n, k)$. We have the following inclusions: $C_q(n, k) \subseteq C_{qa}(n, k) \subseteq C_{qc}(n, k)$. Furthermore, the sets of synchronous correlation matrices, denoted by $C_q^s(n, k)$, $C_{qa}^s(n, k)$, and $C_{qc}^s(n, k)$, respectively, consist of those quantum correlation matrices $[(p(i, j|v, w))]_{i,j,v,w}$ in $C_q(n, k)$, $C_{qs}(n, k)$, and $C_{qc}(n, k)$, respectively, where $p(i, j|v, v) = 0$, whenever $i \neq j$.

We use Theorem 2.5 and Proposition 2.7 below from [13] to give a shorter proof of [3, Theorem 4.2] of Dykema, Paulsen and Prakash, which, again, was refining Slofstra’s result (in the general, non-synchronous case) from [14]. For this, using the notation from [13], for $n, k \geq 2$ let $\mathcal{D}_q^s(n, k)$ be the set of matrices $[\tau(e_{v,i} e_{w,j})]_{i,j,v,w}$, where $e_{v,i}$ are projections in a finite dimensional von Neumann algebra with a (faithful) tracial state τ , satisfying $\sum_{i=1}^k e_{v,i} = 1$, for all $1 \leq v \leq n$. Note that $\mathcal{D}_q^s(n, k)$ is a subset of $\mathcal{D}(nk)$.

Proposition 2.7 (Paulsen, Severini, Stahlke, Todorov and Winter, [13]). *For all $n, k \geq 2$, one has $C_q^s(n, k) = \mathcal{D}_q^s(n, k)$.*

Theorem 2.8 (Dykema, Paulsen and Prakash, [3]). *The set $C_q^s(n, 2)$ of synchronous quantum correlation matrices is non-compact, for all $n \geq 5$.*

Proof. Use Theorem 2.5 to find a matrix A in $\mathcal{D}(n) \setminus \mathcal{D}_{\text{fin}}(n)$ and a sequence $\{A_k\}_{k=1}^\infty$ of matrices in $\mathcal{D}_{\text{fin}}(n)$ converging to A . Let p_1, \dots, p_n be projections in \mathcal{R}^ω such that $\tau_{\mathcal{R}^\omega}(p_j p_i) = A(i, j)$, for all i, j , cf. Proposition 2.4 (v), and let further $p_1^{(k)}, \dots, p_n^{(k)}$ be projections in some matrix algebra $M_{m_k}(\mathbb{C})$ with normalized trace tr_{m_k} such that $\text{tr}_{m_k}(p_j^{(k)} p_i^{(k)}) = A_k(i, j)$, $1 \leq i, j \leq n$.

Set $e_{0,v} = p_v$ and $e_{1,v} = 1 - p_v$ in \mathcal{R}^ω , and set $e_{0,v}^{(k)} = p_v^{(k)}$ and $e_{1,v}^{(k)} = 1 - p_v^{(k)}$ in $M_{m_k}(\mathbb{C})$, for all $k \geq 1$ and $1 \leq v \leq n$. It then follows that

$$\lim_{k \rightarrow \infty} [\text{tr}_{m_k}(e_{v,i}^{(k)} e_{w,j}^{(k)})]_{i,j,v,w} = [\tau_{\mathcal{R}^\omega}(e_{v,i} e_{w,j})]_{i,j,v,w},$$

and each of the matrices $[\text{tr}_{m_k}(e_{v,i}^{(k)} e_{w,j}^{(k)})]_{i,j,v,w}$ belongs to $\mathcal{D}_q^s(n, 2) = C_q^s(n, 2)$, cf. Proposition 2.7. However, the matrix $[\tau(e_{v,i} e_{w,j})]_{i,j,v,w}$ itself does not belong to $C_q^s(n, 2)$, since the matrix $A = [\tau(e_{v,0} e_{w,0})]_{v,w}$ does not belong to $\mathcal{D}_{\text{fin}}(n)$. \square

Note that the set $C_q(n, 2)$ of (non-synchronous) quantum correlation matrices contains $C_q^s(n, 2)$ as a relatively closed subset, which shows that $C_q(n, 2)$ also is non-closed, when $n \geq 5$.

We end this section with a remark on the matrices $A_t^{(n)}$ defined in (2.3).

Remark 2.9. For each integer $n \geq 2$ and for all $s, t \in [0, 1]$, consider the $n \times n$ matrix $A_{t,s}^{(n)}$, whose diagonal entries all are equal to t and whose off-diagonal entries are all equal to s . Note that $A_t^{(n)} = A_{t,s}^{(n)}$, with $s := t(nt - 1)/(n - 1)$, when $t \in [1/n, 1]$. The purpose of this remark is to describe the set of “admissible pairs” (t, s) , for which $A_{t,s}^{(n)}$ belongs to $\mathcal{D}(n)$, and to show that $s = t(nt - 1)/(n - 1)$ is the smallest number for which (t, s) is such an admissible pair, for each fixed $t \in n^{-1}\Sigma_n$,

Note first that for each fixed $t \in [0, 1]$, the set $I_t^{(n)}$ of those $s \in [0, 1]$, for which (t, s) is admissible, is a closed interval. This follows by convexity and compactness of the set $\mathcal{D}(n)$.

Fix $t \in [0, 1]$, take a projection p of trace t in some finite von Neumann algebra, and let $p_1 = \dots = p_n = p$. The matrix $A_{t,t}^{(n)}$ is then equal to $[\tau(p_j p_i)]_{i,j=1}^n$, and therefore belongs to $\mathcal{D}(n)$. Moreover, since $\tau(pq) \leq \tau(p)$, whenever p, q are projections in a von Neumann algebra with tracial state τ , we conclude that $t = \max I_t^{(n)}$.

For $0 \leq t \leq 1/n$ we can find pairwise orthogonal projections p_1, \dots, p_n in any type II_1 factor M with $\tau_M(p_j) = t$, for all j . The corresponding $n \times n$ matrix $[\tau(p_j p_i)]_{i,j=1}^n$ is equal to $A_{t,0}^{(n)}$. Hence 0 belongs to $I_t^{(n)}$. This shows that $I_t^{(n)} = [0, t]$, when $0 \leq t \leq 1/n$.

Let $s, t \in [0, 1]$, and suppose that (t, s) is an admissible pair. Let p_1, \dots, p_n be projections in a finite von Neumann algebra with tracial state τ such that $[\tau(p_j p_i)]_{i,j=1}^n = A_{t,s}^{(n)}$. Put $q_j = 1 - p_j$, $1 \leq j \leq n$. Then $[\tau(q_j q_i)]_{i,j=1}^n = A_{1-t, 1-2t+s}^{(n)}$, which shows that $(1-t, 1-2t+s)$ is an admissible pair. The map given by $(t, s) \mapsto (1-t, 1-2t+s)$ is involutive, and therefore it maps the set of admissible pairs in $[0, 1]^2$ onto itself. In particular, this involution maps $\{t\} \times I_t^{(n)}$ onto $\{1-t\} \times I_{1-t}^{(n)}$. Combining this fact with the result of the previous paragraph, we obtain that $I_t^{(n)} = [2t - 1, t]$, when $1 - 1/n \leq t \leq 1$.

Consider finally the case where $t \in [1/n, 1 - 1/n]$. Suppose that $s \in I_t$, and let p_1, \dots, p_n be projections in some finite von Neumann algebra with tracial state τ such that $[\tau(p_j p_i)]_{i,j=1}^n = A_{t,s}^{(n)}$. Then

$$nt + n(n - 1)s = \tau \left(\left(\sum_{i=1}^n p_i \right)^2 \right) \geq \left(\tau \left(\sum_{i=1}^n p_i \right) \right)^2 = n^2 t^2,$$

which implies that $s \geq t(nt - 1)/(n - 1)$. In other words, $I_t^{(n)} \subseteq [t(nt - 1)/(n - 1), t]$. It was noted in Remark 2.6 that $A_t^{(n)} = A_{t, t(nt-1)/(n-1)}^{(n)}$ belongs to $\mathcal{D}(n)$ if and only if t belongs to $n^{-1}\Sigma_n$. For those values of t , we therefore obtain that $s = t(nt - 1)/(n - 1)$ is the smallest number for which (t, s) is an admissible pair, which shows that $I_t^{(n)} = [t(nt - 1)/(n - 1), t]$.

It remains a curious open problem to determine the interval $I_t^{(n)}$, for t belonging to the (non-empty) set $[1/n, 1 - 1/n] \setminus n^{-1}\Sigma_n$. In this case, necessarily, $\min I_t^{(n)} > t(nt - 1)/(n - 1)$.

3. Correlation Matrices of Unitary Elements

Recall that a correlation matrix is a positive definite matrix whose diagonal entries are equal to 1. For each integer $n \geq 2$, let $\mathcal{G}(n)$, $\mathcal{F}_{\text{matrix}}(n)$, and $\mathcal{F}_{\text{fin}}(n)$ be the set of $n \times n$ correlation matrices $[\tau(u_j^* u_i)]_{i,j=1}^n$, where u_1, u_2, \dots, u_n are unitaries in some finite von Neumann algebra equipped with a faithful tracial state τ , respectively, in some full matrix algebra with its canonical tracial state, respectively, in some finite dimensional C^* -algebra with a faithful tracial state. Then the convex hull $\text{conv}(\mathcal{F}_{\text{matrix}}(n))$ of $\mathcal{F}_{\text{matrix}}(n)$ is equal to $\mathcal{F}_{\text{fin}}(n)$, and the two sets $\mathcal{F}_{\text{matrix}}(n)$ and $\mathcal{F}_{\text{fin}}(n)$ have the same closure, which is denoted by $\mathcal{F}(n)$. The sets $\mathcal{G}(n)$ and $\mathcal{F}(n)$ are compact and convex, see [2, Proposition 1.4].

As shown by Kirchberg, [9] (cf. Dykema–Juschenko, [2]), the Connes Embedding Problem has an affirmative answer if and only if $\mathcal{G}(n) = \mathcal{F}(n)$, for all $n \geq 3$. We can use this to show that $\mathcal{D}_{\text{matrix}}(n)$ is dense in $\mathcal{D}(n)$, for all $n \geq 3$, if and only if the Connes Embedding Problem has an affirmative answer. Indeed, the proof of the “if” part follows the same strategy as the proof of the “if” part of Kirchberg’s result, where a given finite subset of \mathcal{R}^ω is approximated in trace-norm with a finite subset of a matrix subalgebra of \mathcal{R}^ω . To see the “only if” part, assume that $\mathcal{D}_{\text{matrix}}(n)$ is dense in $\mathcal{D}(n)$, for all $n \geq 3$. We show that this implies $\mathcal{G}(n) = \mathcal{F}(n)$, for all $n \geq 3$. Take an n -tuple u_1, \dots, u_n of unitaries in some tracial von Neumann algebra (N, τ) . Fix an integer $m \geq 1$. Each u_j can be approximated in norm within $2\pi/m$ by unitaries v_1, \dots, v_n in N of the form $v_j = \sum_{k=1}^m \omega^k p_{k,j}$, where $\omega = \exp(2\pi i/m)$ and $p_{1,j}, \dots, p_{m,j}$ are pairwise orthogonal projections in N summing up to 1, for each j . Approximate the second-order moments of the collections of projections $\{p_{k,j}\}_{k,j}$ by second-order moments of projections $\{q_{k,j}\}_{k,j}$ in some matrix algebra $(M_r(\mathbb{C}), \text{tr}_r)$. Then $\text{tr}_r(q_{k,j} q_{\ell,j})$ are small when $k \neq \ell$, and $\text{tr}_r(\sum_{k=1}^m q_{k,j})$ is close to 1, for all j . A standard lifting argument allows us to replace the projections $\{q_{k,j}\}_{k,j}$ with new projections, close to the old ones in trace-norm, satisfying $\sum_{j=1}^m q_{k,j} = 1$, for all j . Set $w_j = \sum_{k=1}^m \omega^k q_{k,j}$. The second-order moments of the unitaries w_1, \dots, w_n are then close to those of u_1, \dots, u_n .

Remark 3.1. In the case where $n = 2$, the set $\mathcal{F}_{\text{matrix}}(2)$ is closed and convex, and $\mathcal{F}_{\text{matrix}}(2) = \mathcal{F}(2) = \mathcal{G}(2)$, which further is equal to the set of 2×2 matrices of the form

$$\begin{pmatrix} 1 & \bar{z} \\ z & 1 \end{pmatrix},$$

for $z \in \mathbb{C}$ with $|z| \leq 1$. To see that each of these 2×2 matrices belongs to $\mathcal{F}_{\text{matrix}}(2)$, take $z \in \mathbb{C}$ with $|z| \leq 1$ and find λ_1 and λ_2 on the complex unit circle \mathbb{T} , with $z = (\lambda_1 + \lambda_2)/2$. The correlation matrix arising from the unitary 2×2 matrices $u_1 = 1$ and $u_2 = \text{diag}(\lambda_1, \lambda_2)$ is then as desired.

We show in this section that the set $\mathcal{F}_{\text{fin}}(n)$ is not closed (hence, not compact), for all $n \geq 11$. This result originates in a remark made by T. Vidick during his talk at one of the workshops in the Quantitative Linear Algebra program at IPAM, Spring 2018, that led to subsequent discussions with W. Slofstra, who, in particular, communicated to us a version of the following result (to appear, in an approximate case, in a forthcoming paper by O. Regev, W. Slofstra and T. Vidick):

Proposition 3.2. *Let M be a finite von Neumann algebra with a faithful tracial state τ_M , and let p_1, \dots, p_n be projections in M . Further, let $u_0, u_1, \dots, u_n, u_{n+1}, \dots, u_{2n}$ be the unitaries in M given by*

$$u_0 = 1, \quad u_j = 2p_j - 1, \quad (1 \leq j \leq n), \quad u_j = (u_{j-n} + i \cdot 1)/\sqrt{2}, \quad (n + 1 \leq j \leq 2n).$$

Let N be another finite von Neumann algebra with a faithful tracial state τ_N . Then there exist $2n + 1$ unitaries v_0, v_1, \dots, v_{2n} in N satisfying

$$\tau_N(v_j^* v_i) = \tau_M(u_j^* u_i), \quad 0 \leq i, j \leq 2n, \tag{3.1}$$

if and only if there exist n projections q_1, \dots, q_n in N satisfying

$$\tau_N(q_j q_i) = \tau_M(p_j p_i), \quad 1 \leq i, j \leq n. \tag{3.2}$$

Proof. Assume that q_1, \dots, q_n are projections in N satisfying (3.2). Equip the vector spaces $\text{span}\{p_1, p_2, \dots, p_n\}$ and $\text{span}\{q_1, q_2, \dots, q_n\}$ with the Euclidean structure arising from the traces τ_M and τ_N , respectively. Using (3.2), we see that the map $p_j \mapsto q_j$, $1 \leq j \leq n$, extends to a well-defined linear isometry φ from $\text{span}\{p_1, \dots, p_n\}$ to $\text{span}\{q_1, \dots, q_n\}$. Set $v_0 = 1, v_j = 2q_j - 1$, for $1 \leq j \leq n$, and $v_j = (v_{j-n} + i \cdot 1)/\sqrt{2}$, for $n + 1 \leq j \leq 2n$, and use the isometric property of φ to check that (3.1) holds.

Conversely, assume that we are given unitaries v_0, v_1, \dots, v_{2n} in N satisfying (3.1). Upon replacing v_j by $v_0^* v_j$, for all $0 \leq j \leq 2n$, we may assume that $v_0 = 1$. As above, equip the vector spaces $\text{span}\{u_0, u_1, \dots, u_{2n}\}$ and $\text{span}\{v_0, v_1, \dots, v_{2n}\}$ with the Euclidean structure arising from the traces τ_M and τ_N , respectively. Then, by (3.1), we have a well-defined linear isometry $\psi : \text{span}\{u_0, u_1, \dots, u_{2n}\} \rightarrow \text{span}\{v_0, v_1, \dots, v_{2n}\}$, mapping u_j to v_j , for $0 \leq j \leq 2n$. In particular, for $1 \leq j \leq n$,

$$\|v_{j+n} - (v_j + i \cdot v_0)/\sqrt{2}\|_2 = \|u_{j+n} - (u_j + i \cdot u_0)/\sqrt{2}\|_2 = 0,$$

so $v_{j+n} = (v_j + i \cdot 1)/\sqrt{2}$.

Note that if u and $(u + i \cdot 1)/\sqrt{2}$ are unitaries in some unital C^* -algebra, then u is necessarily a symmetry. Indeed, if λ is a complex number such that $|\lambda| = |(\lambda + i)/\sqrt{2}| = 1$, then $\lambda \in \mathbb{R}$. Hence, if u is as stated, then its spectrum is contained in \mathbb{R} , which entails that it is a symmetry.

We conclude that v_1, \dots, v_n are symmetries. For $1 \leq j \leq n$, set $q_j = (v_j + 1)/2$. Then q_j is a projection and $v_j = 2q_j - 1$. Use the isometric property of ψ to check (3.2). □

Corollary 3.3. *The set $\mathcal{F}_{\text{matrix}}(m)$ is not compact and not convex, whenever $m \geq 3$.*

Proof. Let $0 < \alpha < 1$ be irrational. Equip $M := \mathbb{C} \oplus \mathbb{C}$ with the trace τ given by $\tau(x, y) = \alpha x + (1 - \alpha)y$, for $x, y \in \mathbb{C}$.

Consider first the case where $m = 3$. Let $n = 1$ and let $p = p_1 = (1, 0) \in M$. Then $\tau(p) = \alpha$ is irrational. Let u_0, u_1, u_2 be the unitaries in M arising from this projection as in the proposition above (with $n = 1$). The matrix $[\tau(u_j^* u_i)]_{i,j=0}^2$ belongs to $\text{conv}(\mathcal{F}_{\text{matrix}}(3))$, and hence to $\mathcal{F}(3)$. However, by Proposition 3.2, it does not belong to $\mathcal{F}_{\text{matrix}}(3)$ itself, because no full matrix algebra contains a projection of irrational trace, and therefore contains no projection $q = q_1$ satisfying (3.2) (with $n = 1$).

Assume now that $m > 3$, let u_0, u_1, u_2 be as above, and let unitaries u_3, u_4, \dots, u_{m-1} in M be arbitrary. If v_0, v_1, \dots, v_{m-1} are unitaries in some tracial von Neumann algebra (N, τ_N) satisfying (3.1), then v_0, v_1, v_2 satisfy (3.1) with respect to the set $\{u_0, u_1, u_2\}$, so v_0, v_1, v_2 , and hence v_0, v_1, \dots, v_{m-1} cannot be found in a full matrix algebra. These arguments also yield the non-convexity of the set $\mathcal{F}_{\text{matrix}}(m)$ in all cases. \square

Example 3.4. For $m = 3$, the correlation matrix $B = [\tau(u_j^* u_i)]_{i,j=0}^2$ from the proof above, with unitaries given by

$$u_0 = (1, 1), \quad u_1 = (1, -1), \quad u_2 = \left(\frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}} \right),$$

has the following explicit form in terms of the parameter $\alpha \in (0, 1)$:

$$B = \begin{pmatrix} 1 & \gamma & \frac{\gamma-i}{\sqrt{2}} \\ \gamma & 1 & \frac{1-\gamma i}{\sqrt{2}} \\ \frac{\gamma+i}{\sqrt{2}} & \frac{1+\gamma i}{\sqrt{2}} & 1 \end{pmatrix},$$

where $\gamma = 2\alpha - 1 \in (-1, 1)$. Note that the matrix B belongs to $\mathcal{F}_{\text{fin}}(3)$, for all $\gamma \in (-1, 1)$, while it does not belong to $\mathcal{F}_{\text{matrix}}(3)$, whenever γ is irrational.

Example 3.5. For each $n \geq 2$ and each $t \in [1/n, 1]$, consider the self-adjoint $(1 + 2n) \times (1 + 2n)$ complex matrix

$$B_t^{(n)} = \begin{pmatrix} 1 & X^* & Y^* \\ X & D_1 & C^* \\ Y & C & D_2 \end{pmatrix},$$

where X, Y and C, D_1, D_2 are the $n \times 1$, respectively, $n \times n$ complex matrices given by

$$X = s \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad Y = \frac{s+i}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad C = \frac{1+is}{\sqrt{2}} I_n + \frac{4(r-t)+1+is}{\sqrt{2}} E,$$

$$D_1 = I_n + (4(r-t)+1) E, \quad D_2 = I_n + (2(r-t)+1) E,$$

where $s = 2t - 1$ and $r = (n - 1)^{-1}t(nt - 1)$, and

$$E = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{pmatrix} \in M_n(\mathbb{C}).$$

With this definition, Proposition 3.2 yields that the $(2n + 1) \times (2n + 1)$ matrix $B_t^{(n)}$ is the correlation matrix of an $(2n + 1)$ -tuple of unitaries in some finite von Neumann algebra M if and only if the $n \times n$ matrix $A_t^{(n)}$ is the correlation matrix of an n -tuple of projections in the same von Neumann algebra M . Indeed, if p_1, \dots, p_n are projections in M such that $[\tau_M(p_j p_i)]_{i,j=1}^n = A_t^{(n)}$, and if u_0, u_1, \dots, u_{2n} are the $2n + 1$ unitaries in M constructed from these projections as in Proposition 3.2, then $B_t^{(n)} = [\tau_M(u_j^* u_i)]_{i,j=0}^{2n}$. Conversely, if v_0, v_1, \dots, v_{2n} are unitaries in M such that $B_t^{(n)} = [\tau_M(v_j^* v_i)]_{i,j=0}^{2n}$, then there are projections $q_1, \dots, q_n \in M$ such that $A_t^{(n)} = [\tau_M(q_j q_i)]_{i,j=1}^n$.

In particular, $B_t^{(n)}$ belongs to $\mathcal{G}(2n + 1)$, respectively, to $\mathcal{F}_{\text{fin}}(2n + 1)$, if and only if $A_t^{(n)}$ belongs to $\mathcal{D}(n)$, respectively, to $\mathcal{D}_{\text{fin}}(n)$.

Theorem 3.6. *Let $n \geq 5$ and let $t \in \Pi_n$.*

- (i) *If t is rational, then $B_t^{(n)}$ belongs to $\mathcal{F}_{\text{fin}}(2n + 1)$, and if t is irrational, then $B_t^{(n)}$ belongs $\mathcal{F}(2n + 1) \setminus \mathcal{F}_{\text{fin}}(2n + 1)$.*
- (ii) *If t is irrational and if v_0, v_1, \dots, v_{2n} are unitaries in some finite von Neumann algebra N with a faithful tracial state τ_N such that $\tau_N(v_j^* v_i) = B_t^{(n)}(i, j)$, for $0 \leq i, j \leq 2n$, then N is necessarily of type II_1 .*
- (iii) *If $t \in (\frac{1}{2}(1 - \sqrt{1 - 4/n}), \frac{1}{2}(1 + \sqrt{1 - 4/n}))$, then there are unitaries v_0, v_1, \dots, v_{2n} in the hyperfinite II_1 factor \mathcal{R} such that $\tau_N(v_j^* v_i) = B_t^{(n)}(i, j)$, for $0 \leq i, j \leq 2n$.*
- (iv) *The convex sets $\mathcal{F}_{\text{fin}}(k)$ are non-compact, for all $k \geq 11$.*

Proof. (i). The map $t \mapsto B_t^{(n)}$, $t \in \Pi_n$, is clearly continuous. It follows from Proposition 2.3 and Example 3.5 that $B_t^{(n)} \in \mathcal{G}(2n + 1)$, whenever $A_t^{(n)} \in \mathcal{D}(n)$, and in particular whenever $t \in \Pi_n$. Moreover, if $t \in \Pi_n$, then $B_t^{(n)} \in \mathcal{F}_{\text{fin}}(2n + 1)$, when t is rational, and $B_t^{(n)} \notin \mathcal{F}_{\text{fin}}(2n + 1)$, when t is irrational. This implies that $B_t^{(n)} \in \mathcal{F}(2n + 1)$, for all $t \in \Pi_n$, and that $\mathcal{F}_{\text{fin}}(2n + 1)$ is non-compact, for each $n \geq 5$.

(ii). Let v_0, v_1, \dots, v_{2n} be unitaries in some finite von Neumann algebra N with faithful tracial state τ_N , satisfying $\tau_N(v_j^* v_i) = B_t^{(n)}(i, j)$, for $0 \leq i, j \leq 2n$. Then, by Example 3.5, there exist projections q_1, \dots, q_n in N such that $A_t^{(n)}(i, j) = \tau_N(q_j q_i)$. This entails that N has no finite dimensional representations, by Proposition 2.3, so N must be of type II_1 .

(iii). Let $t \in (\frac{1}{2}(1 - \sqrt{1 - 4/n}), \frac{1}{2}(1 + \sqrt{1 - 4/n}))$. It follows from Theorem A.1 in the Appendix by Ozawa that $A_t^{(n)}$ can be realized using projections in \mathcal{R} , cf. Remark 2.6. The claim now follows from Example 3.5.

(iv). We show that if $\mathcal{F}_{\text{fin}}(k)$ is non-compact, for some positive integer k , then so is $\mathcal{F}_{\text{fin}}(k + 1)$. To this end, define a map $\rho: \mathcal{F}(k) \rightarrow \mathcal{F}(k + 1)$ by

$$\rho\left([\tau_M(u_j^* u_i)]_{i,j=1}^k\right) = [\tau_M(u_j^* u_i)]_{i,j=1}^{k+1},$$

whenever u_1, \dots, u_k are unitaries in some von Neumann algebra M with a faithful tracial state τ , and where u_{k+1} is chosen to be equal to u_1 . Then the last row and the last column of $[\tau_M(u_j^* u_i)]_{i,j=1}^{k+1}$ is equal to the first row and the first column of the matrix $[\tau_M(u_j^* u_i)]_{i,j=1}^k$, which shows that ρ is well-defined and continuous. Moreover, $\rho^{-1}(\mathcal{F}_{\text{fin}}(k + 1)) = \mathcal{F}_{\text{fin}}(k)$. Hence, if $\mathcal{F}_{\text{fin}}(k + 1)$ is compact, and thus closed, then $\mathcal{F}_{\text{fin}}(k)$ is closed, and thus compact. \square

4. Factorizable Maps that Require Infinite Dimensional Ancilla

We prove here our claimed result about existence of factorizable quantum channels in all dimensions ≥ 11 , requiring infinite dimensional ancilla. We first recall necessary prerequisites.

Let $\text{UCPT}(n)$ denote the convex and compact set of all unital completely positive trace preserving linear maps $T : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$, $n \geq 2$. Maps in $\text{UCPT}(n)$ are also called unital quantum channels in dimension n . Anantharaman-Delaroche defined in [1] a channel to be *factorizable* if it admits a factorization (in a suitable way) through a finite von Neumann algebra with a faithful tracial state. This notion was studied extensively in [5], and the following characterization (which we will take to be our definition of factorizable maps) was established therein (cf., [5, Theorem 2.2]): A unital quantum channel T in dimension n is factorizable if and only if there exist a finite von Neumann algebra N , equipped with a normal faithful tracial state τ_N , and a unitary $u \in M_n(\mathbb{C}) \otimes N$ such that

$$T(x) = (\text{id}_n \otimes \tau_N)(u(x \otimes 1_N)u^*), \quad x \in M_n(\mathbb{C}). \tag{4.1}$$

The von Neumann algebra N above is also called the *ancilla*, and, following Definition 3.1 in [6], we say that T has an *exact factorization* through $M_n(\mathbb{C}) \otimes N$. The set of all factorizable unital channels in dimension n is denoted by $\mathcal{FM}(n)$. This set is convex and compact, as shown by standard arguments.

Further, let $\mathcal{FM}_{\text{matrix}}(n)$ and $\mathcal{FM}_{\text{fin}}(n)$ denote the set of factorizable maps in $\text{UCPT}(n)$ that exactly factor through a full matrix algebra, respectively, through a finite dimensional C^* -algebra, equipped with a faithful tracial state. It was shown in [6, Theorem 3.7] that a positive answer to the Connes Embedding Problem is equivalent to $\mathcal{FM}_{\text{matrix}}(n)$ being dense in $\mathcal{FM}(n)$, for all $n \geq 3$. Moreover, if a unital quantum channel $T : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ belongs to the closure of $\mathcal{FM}_{\text{matrix}}(n)$, then T admits an exact factorization through an ultrapower \mathcal{R}^ω of the hyperfinite type II_1 factor \mathcal{R} .

It is shown in the upcoming manuscript [12] that $\mathcal{FM}_{\text{fin}}(n) = \text{conv}(\mathcal{FM}_{\text{matrix}}(n))$, and that $\mathcal{FM}_{\text{matrix}}(n)$ is non-compact and non-convex, when $n \geq 3$.

Let $\mathcal{S}(n)$ be the set of all *Schur multipliers* $T_B : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$, where $B \in M_n(\mathbb{C})$, i.e., $T_B(x)$ is the Schur product of B and x , for $x \in M_n(\mathbb{C})$. Let $\mathcal{FMS}(n) = \mathcal{FM}(n) \cap \mathcal{S}(n)$ be the set of all factorizable Schur multipliers, and write $\mathcal{FMS}_{\text{fin}}(n) = \mathcal{FM}_{\text{fin}}(n) \cap \mathcal{S}(n)$.

For the theorem below, recall the definition (2.2) of the set Π_n , and the $(2n + 1) \times (2n + 1)$ matrix $B_t^{(n)}$ constructed in Example 3.5.

Theorem 4.1. *The set $\mathcal{FM}_{\text{fin}}(k)$ is not compact, for all $k \geq 11$. Moreover, for each $n \geq 5$ and each irrational number $t \in \Pi_n$, the Schur multiplier T_B , where $B = B_t^{(n)}$, is a factorizable map belonging to the closure of $\mathcal{FM}_{\text{fin}}(2n + 1)$, but not to $\mathcal{FM}_{\text{fin}}(2n + 1)$, and it requires an ancilla of type II_1 . This ancilla can be taken to be the hyperfinite II_1 factor \mathcal{R} , when $\frac{1}{2}(1 - \sqrt{1 - 4/n}) < t < \frac{1}{2}(1 + \sqrt{1 - 4/n})$.*

Proof. It was shown in [5, Proposition 2.8] that if $B \in M_k(\mathbb{C})$ is a correlation matrix, then the associated Schur multiplier T_B admits an exact factorization through $M_k(\mathbb{C}) \otimes N$, where N is a finite von Neumann algebra with normal faithful tracial state τ_N , if and only if $B = [\tau_N(u_j^* u_i)]_{i,j=1}^k$, for some unitaries $u_1, \dots, u_k \in N$. In particular, T_B belongs to $\mathcal{FMS}(k)$, $\mathcal{FMS}_{\text{fin}}(k)$, and the closure of $\mathcal{FMS}_{\text{fin}}(k)$, respectively, if and only if B belongs to $\mathcal{G}(k)$, $\mathcal{F}_{\text{fin}}(k)$, and $\mathcal{F}(k)$, respectively.

Since the map from $\mathcal{G}(k)$ to $\mathcal{FM}(k)$ given by $B \mapsto T_B$ is continuous, it follows from Theorem 3.6 that $\mathcal{FMS}_{\text{fin}}(k)$ is non-closed in $\mathcal{FM}(k)$. As the set $\mathcal{S}(k)$ is closed, we conclude that $\mathcal{FMS}_{\text{fin}}(k)$ is non-compact.

To prove the second part of the theorem, let $n \geq 5$, let $t \in \Pi_n$ be irrational, and let $B = B_t^{(n)}$. Then B belongs to $\mathcal{F}(2n+1) \setminus \mathcal{F}_{\text{fin}}(2n+1)$ by Theorem 3.6. By the argument in the first paragraph, we conclude that T_B belongs to the closure of $\mathcal{FMS}_{\text{fin}}(2n+1)$, but not to $\mathcal{FMS}_{\text{fin}}(2n+1)$. Moreover, if T_B admits an exact factorization through $M_n(\mathbb{C}) \otimes N$, with ancilla (N, τ_N) as in the first paragraph, then B is the matrix of correlations of unitaries u_0, u_1, \dots, u_{2n} in N , which by Theorem 3.6 implies that N must be of type II_1 .

The remaining part of the theorem follows from Theorem 3.6 (iii) and the argument in the first paragraph. □

We also obtain the following quantitative version of the theorem above, in the case where $t \in \Pi_n$ is rational and $n \geq 5$: Let $B = B_t^{(n)}$. If $nt = a/b$, with a, b positive integers and a/b irreducible, then the Schur channel T_B admits an exact factorization through a finite dimensional von Neumann algebra $M = M_n(\mathbb{C}) \otimes N$, where the von Neumann algebra N can have no representation on a Hilbert space of dimension smaller than b . This follows as in the proof of the theorem above and by appealing to Proposition 2.3 (i).

We conclude that for every fixed integer $n \geq 11$, there is a sequence of factorizable unital quantum channels in dimension n , each admitting finite-dimensional ancillas, but where the size of any such ancillas must tend to infinity.

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Appendix (by Narutaka Ozawa¹)

Realizing the Kruglyak–Rabonovich–Samoilenko Projections in the Hyperfinite II_1 Factor

Kruglyak, Rabanovich, and Samoilenko proved in [10, Theorem 6], cf. Theorem 2.2, that for any $n \geq 5$ and any $\alpha \in [\frac{1}{2}(n - \sqrt{n^2 - 4n}), \frac{1}{2}(n + \sqrt{n^2 - 4n})]$, there are orthogonal projections p_1, \dots, p_n such that $\sum_i p_i = \alpha$. In this appendix, we observe that the Kruglyak–Rabonovich–Samoilenko construction shows that these projections are realized in the hyperfinite II_1 factor \mathcal{R} , possibly except for the extremities.

Theorem A.1. *For any $n \geq 5$ and any $\alpha \in (\frac{1}{2}(n - \sqrt{n^2 - 4n}), \frac{1}{2}(n + \sqrt{n^2 - 4n}))$, there are projections $p_1, \dots, p_n \in \mathcal{R}$ which satisfy $\sum_i p_i = \alpha$.*

Let (M, τ) be a finite von Neumann algebra. By a *matricial approximation* (or *matricial microstates*) of a d -tuple (a_1, \dots, a_d) in M^{sa} , we mean a sequence $(x_1(n), \dots, x_d(n))$ in $(M_{k(n)}(\mathbb{C})^{\text{sa}})^d$ such that $\lim_n \text{tr}(p(x_1(n), \dots, x_d(n))) = \tau(p(a_1, \dots, a_d))$ for every polynomial p in d non-commuting variables. A matricial approximation of a generating d -tuple (a_1, \dots, a_d) of M gives rise to an embedding of (M, τ) into the tracial ultraproduct of $(M_{k(n)}(\mathbb{C}), \text{tr}_{k(n)})$. Recall that M satisfies the Connes Embedding Conjecture, i.e.,

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$(M, \tau) \leftrightarrow (\mathcal{R}^\omega, \tau^\omega)$, if and only if every (or some) generating d -tuple $(a_i)_{i=1}^d$ in M^{sa} admits a matricial approximation. We will give a sufficient condition for hyperfiniteness of M in terms of a matricial approximation. For $x = (x_{ij}) \in M_k(\mathbb{C})$, we define its *propagation* to be $\max\{|i - j| : x_{ij} \neq 0\}$.

Lemma A.2. *Let (M, τ) be a finite von Neumann algebra generated by $a_1, \dots, a_d \in M^{\text{sa}}$. Assume that (a_1, \dots, a_d) admits a matricial approximation $(x_1(n), \dots, x_d(n))$ with uniformly bounded propagations. Then M is hyperfinite.*

Proof. Let k be a positive integer and consider the shift unitary matrix $z_k \in M_k(\mathbb{C})$ given by $(z_k)_{i,j} = \delta_{i+1,j}$ (modulo k). It normalizes the diagonal maximal abelian subalgebra $D_k \subset M_k(\mathbb{C})$. Observe that any $y \in M_k(\mathbb{C})$ that has propagation at most l can be written as $y = \sum_{m=-l}^l f_m z_k^m$ for some $f_m \in D_k$ with $\|f_m\| \leq \|y\|$.

Now, let a matricial approximation $(x_1(n), \dots, x_d(n))$ be given as in the statement. We denote by M_ω the tracial ultraproduct of $(M_{k(n)}(\mathbb{C}), \text{tr}_{k(n)})$ and by D_ω the subalgebra arising from the diagonal maximal abelian subalgebras $D_{k(n)}$. From the above discussion, one sees that the element $x_i \in M_\omega$ that corresponds to $(x_i(n))_n$ belongs to the von Neumann subalgebra generated by D_ω and z , where z is the unitary element corresponding to $(z_{k(n)})_n$. The von Neumann subalgebra generated by x_1, \dots, x_d is isomorphic to M and the von Neumann subalgebra generated by D_ω and z is hyperfinite (as it is isomorphic to $D_\omega \rtimes \mathbb{Z}$, assuming $k(n) \rightarrow \infty$). \square

Proof of Theorem A.1. Firstly, note that every separable finite von Neumann algebra (N, τ) with a faithful normal tracial state is embeddable in a trace-preserving way into a separable II_1 factor M , which can be taken to be the hyperfinite II_1 factor \mathcal{R} if M is hyperfinite. Indeed, as observed by U. Haagerup, we may take M to be $(\bigotimes_{n=1}^\infty N) \rtimes S_\infty$, where the infinite tensor product is with respect to the standard representation of N on $L^2(N, \tau)$, and where S_∞ is the (locally finite) group of permutations on the natural numbers with finite support. It therefore suffices to find the projections p_1, \dots, p_n in any hyperfinite finite von Neumann algebra N .

For each $\alpha \in \mathbb{Q} \cap [3/2, 2]$, the projections $P_1(\alpha), \dots, P_5(\alpha)$ in $M_{k(\alpha)}(\mathbb{C})$ that satisfy $\sum_i P_i(\alpha) = \alpha$ are constructed in [10, Theorem 6] as R_i . The proof of Theorem 6 (and Lemma 7) in [10] reveals that the projections R_i are obtained by *sewing* (see [10, Definition 1]) the projections $P_i^{(k)} \in M_{k_i+2}(\mathbb{C})$, $k_i \in \{1, 2, 3\}$. Since $P_i^{(k)}$'s have propagation at most 4, the projections R_i have propagation at most 8, regardless of α .

Let $\alpha \in (3/2, 2)$ be given and take a rational sequence $(\alpha_n)_n$ which converges to α . Then after passing to a convergent subsequence, $(P_1(\alpha_n), \dots, P_5(\alpha_n))$ is a matricial approximation of (P_1, \dots, P_5) in the tracial ultraproduct M_ω of $(M_{k(n)}(\mathbb{C}), \text{tr}_{k(n)})$ and (P_1, \dots, P_5) satisfies $\sum_i P_i = \alpha$. By Lemma A.2, the projections P_1, \dots, P_5 generate a hyperfinite von Neumann subalgebra. This proves Theorem A.1 for $n = 5$ and $\alpha \in [3/2, 2]$. By [10, Lemma 5], this implies Theorem A.1 for every $n \geq 5$ and $\alpha \in [2, n - 2]$.

Finally, note that all values in $(\frac{1}{2}(n - \sqrt{n^2 - 4n}), \frac{1}{2}(n + \sqrt{n^2 - 4n}))$ are obtained by iterating the numerical mappings Φ^+ and Φ^- (see [10, Section 1.2]) starting at $\alpha \in [2, n - 2]$ (see [10, Lemma 6]). Thus it suffices to show the functors S and T constructed in Section 1.2 in [10] preserve hyperfiniteness. This is clear for the linear reflection T . For the reader's convenience, we replicate here the construction of the hyperbolic reflection S , adapted to our setting. Let $P_1, \dots, P_n \in N$ be projections such that $\sum_{i=1}^n P_i = \alpha$. We will construct projections Q_1, \dots, Q_n such that $\sum_{i=1}^n Q_i = \frac{\alpha}{\alpha-1}$. Put

$$V_i := (\alpha^2 - \alpha)^{-1/2} P_i [-P_1 \cdots \alpha - P_i \cdots -P_n] \in M_{1,n}(N).$$

Then, $V_i V_i^* = (\alpha^2 - \alpha)^{-1} P_i (\alpha^2 - 2\alpha P_i + \sum_{k=1}^n P_k) P_i = P_i$ and V_i is a partial isometry. Hence $Q_i := V_i^* V_i \in M_n(N)$ is a projection. A calculation shows

$$\sum_k Q_k = (\alpha^2 - \alpha)^{-1} (\alpha^2 \operatorname{diag}(P_1, \dots, P_n) - \alpha [P_i P_j]_{i,j}) \in M_n(N).$$

Note that $Q := \operatorname{diag}(P_1, \dots, P_n) - \alpha^{-1} [P_i P_j]_{i,j}$ is a projection and one has $\sum_k Q_k = \frac{\alpha}{\alpha-1} Q$. Thus, viewing Q_k as projections in $QM_n(N)Q$, we are done. When N is hyperfinite, so is the amplification $QM_n(N)Q$. \square

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