Commun. Math. Phys. 372, 543–572 (2019)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-019-03385-9

Communications in **Mathematical Physics**

Critical Two-Point Function for Long-Range Models with Power-Law Couplings: The Marginal Case for $d \geq d_c$

Lung-Chi Chen[1,2], **Akira Sakai**[3]

[1] Department of Mathematical Sciences, National Chengchi University, Taipei, Taiwan
[2] Mathematics Division, National Center for Theoretical Sciences, Taipei, Taiwan
[3] Faculty of Science, Hokkaido University, Sapporo, Hokkaido, Japan. E-mail: sakai@math.sci.hokudai.ac.jp

Received: 20 August 2018 / Accepted: 20 January 2019
Published online: 28 February 2019 –

Abstract: Consider the long-range models on $\mathbb{Z}^d$ of random walk, self-avoiding walk, percolation and the Ising model, whose translation-invariant 1-step distribution/coupling coefficient decays as $|x|^{-d-\alpha}$ for some $\alpha > 0$. In the previous work (Chen and Sakai in Ann Probab 43:639–681, 2015), we have shown in a unified fashion for all $\alpha \neq 2$ that, assuming a bound on the "derivative" of the n-step distribution (the compound-zeta distribution satisfies this assumed bound), the critical two-point function $G_{p_c}(x)$ decays as $|x|^{\alpha\wedge 2-d}$ above the upper-critical dimension $d_c \equiv (\alpha \wedge 2)m$, where $m = 2$ for self-avoiding walk and the Ising model and $m = 3$ for percolation. In this paper, we show in a much simpler way, without assuming a bound on the derivative of the n-step distribution, that $G_{p_c}(x)$ for the marginal case $\alpha = 2$ decays as $|x|^{2-d}/\log|x|$ whenever $d \geq d_c$ (with a large spread-out parameter L). This solves the conjecture in Chen and Sakai (2015), extended all the way down to $d = d_c$, and confirms a part of predictions in physics (Brezin et al. in J Stat Phys 157:855–868, 2014). The proof is based on the lace expansion and new convolution bounds on power functions with log corrections.

Contents

1. Introduction and the Main Results

1.1. Introduction. The lace expansion has been successful in rigorously proving mean-field critical behavior for various models, such as self-avoiding walk [9], percolation [17], lattice trees and lattice animals [18], oriented percolation [24], the contact process [25], the classical Ising and φ^4 models [26,27]. It provides (a way to derive) a formal recursion equation for the two-point function $G_p(x)$, which is similar to the recursion equation for the random-walk Green function $S_p(x)$ generated by the non-degenerate (i.e., $D(o) < 1$) 1-step distribution $D(x)$ and the fugacity $p \in [0, 1]$:

$$S_p(x) = \delta_{o,x} + (pD * S_p)(x), \tag{1.1}$$

where, and in the rest of the paper, $(f * g)(x) \equiv \sum_y f(y)\, g(x - y)$ is the convolution of two functions f, g on $\mathbb{Z}^d$. The formal recursion equation for $G_p(x)$ is of the form

$$G_p(x) = \Pi_p(x) + (\Pi_p * pD * G_p)(x), \tag{1.2}$$

where $\Pi_p(x)$ is a series of the model-dependent lace-expansion coefficients. It is natural to expect that, once regularity of Π_p (e.g., absolute summability) is assured for all p up to the critical point p_c, the asymptotic behavior of $G_{p_c}(x)$ should be the same (modulo constant multiplication) as that for the random-walk Green function $S_1(x)$. If so, then sufficient conditions for the mean-field behavior, called the bubble condition for self-avoiding walk and the Ising model [1,22] and the triangle condition for percolation [6], hold for all dimensions above the model-dependent upper-critical dimension d_c, which is $2m$ for short-range models, where $m = 2$ for self-avoiding walk and the Ising model and $m = 3$ for percolation.

In recent years, long-range models defined by power-law couplings, $D(x) \approx |x|^{-d-\alpha}$ for some $\alpha > 0$, have attracted more attention, due to unconventional critical behavior and crossover phenomena (e.g., [7,8,13,21]). Under some mild assumptions, we have shown [13, Proposition 2.1] that, for $\alpha \neq 2$ and $d > \alpha \wedge 2$, the random-walk Green function $S_1(x)$ is asymptotically $\frac{\gamma_\alpha}{v_\alpha}|x|^{\alpha\wedge 2-d}$, where

$$\gamma_\alpha = \frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2}\pi^{d/2}\Gamma(\frac{\alpha\wedge 2}{2})}, \qquad v_\alpha = \lim_{|k|\to 0}\frac{1-\hat{D}(k)}{|k|^{\alpha\wedge 2}} \equiv \lim_{|k|\to 0}\sum_{x\in\mathbb{Z}^d}\frac{1-e^{ik\cdot x}}{|k|^{\alpha\wedge 2}}D(x). \tag{1.3}$$

For short-range models with variance $\sigma^2 = \sum_x |x|^2 D(x) < \infty$, the asymptotic behavior of $S_1(x)$ is well-known to be $\frac{d}{2}\Gamma(\frac{d-2}{2})\pi^{-d/2}\sigma^{-2}|x|^{2-d}$, which is consistent with (1.3) for large $\alpha > 2$. The crossover occurs at $\alpha = 2$, where the variance σ^2 diverges logarithmically and $S_1(x)$ was believed to have a log correction to the above standard Newtonian behavior.

An example of $D(x) \approx |x|^{-d-\alpha}$ is the compound-zeta distribution (see (1.15) for the precise definition). It has been shown [13] that this long-range distribution for $\alpha \neq 2$ also satisfies a certain bound on the "derivative" $|D^{*n}(x) - \frac{1}{2}(D^{*n}(x+y) + D^{*n}(x-y))|$ of the n-step distribution. Thanks to this extra bound, we have shown [13, Theorem 1.2] in a unified fashion for all $\alpha \neq 2$ that, whenever $d > d_c \equiv (\alpha \wedge 2)m$ (with a large spread-out parameter L), there is a model-dependent constant A close to 1 (in fact, $A = 1$ for $\alpha < 2$) such that $G_{p_c}(x) \sim \frac{A}{p_c} S_1(x)$. One of the key elements to showing this result is (slight improvement of) the convolution bounds on power functions [16, Proposition 1.7] that are used to prove regularity of Π_p in (1.2). However, since those convolution bounds are not good enough to properly control power functions with log corrections, we were unable to achieve an asymptotic result for $\alpha = 2$, until the current work.

In this paper, we tackle the marginal case $\alpha = 2$. The headlines are the following:

- $S_1(x) \sim \frac{\gamma_2}{v_2} |x|^{2-d} / \log |x|$ whenever $d > 2$, where γ_2 is in (1.3), but v_2 is redefined as

$$v_2 = \lim_{|k| \to 0} \frac{1 - \hat{D}(k)}{|k|^2 \log(1/|k|)}. \tag{1.4}$$

- $G_{p_c}(x) \sim \frac{1}{p_c} S_1(x)$ whenever $d \geq d_c$ (with a large spread-out parameter L). This also implies that other critical exponents take on their mean-field values for $d \geq d_c$ (including equality).

The latter solves the conjecture [13, (1.29)], extended all the way down to $d = d_c$. It also confirms a part of predictions in physics [8, (3)]: the critical two-point function for percolation was proposed to decay as $|x|^{\alpha \wedge (2-\eta)-d}$ whenever $\alpha \neq 2 - \eta$, where $\eta = \eta(d)$ is the anomalous dimension for short-range percolation and is believed to be nonzero for $d < 6$, and as $|x|^{2-\eta-d} / \log |x|$ whenever $\alpha = 2 - \eta$.

We should emphasize that the proof of the asymptotic result in this paper is rather different from the one in [13] for $\alpha \neq 2$. In fact, we do not require the n-step distribution D^{*n} to satisfy the aforementioned derivative bound. Because of this, we can cover a wider class of models to which the same result applies, and can simplify the proof to some extent. Although the same proof works for $\alpha < 2$ (see Remark 3.7 below), we will focus on the marginal case $\alpha = 2$.

Before closing this subsection, we remark on recent progress in the renormalization group analysis for the $O(n)$ model, which is equivalent to self-avoiding walk when $n = 0$ and to the n-component $|\varphi|^4$ model when $n \geq 1$. Suppose that the above physics prediction is true for the $O(n)$ model as well, and that $\eta > 0$ for $d < 4$. Then, we can take a small $\varepsilon > 0$ to satisfy $\alpha = \frac{d+\varepsilon}{2} \in (\frac{d}{2}, 2 - \eta) \neq \varnothing$, hence $d = 2\alpha - \varepsilon < d_c$, and yet $G_{p_c}(x)$ is proven to decay as $|x|^{\alpha-d}$ [21]. This "sticking" at the mean-field behavior, even below the upper-critical dimension, has been proven by using a rigorous version of the ε-expansion.

In the next subsection, we give more precise definitions of the concerned models.

1.2. The models and the main results.

1.2.1. Random walk Let

$$|||x|||_r = \frac{\pi}{2}(|x| \vee r) \qquad [x \in \mathbb{R}^d,\ 1 \le r < \infty], \tag{1.5}$$

where $|\cdot|$ is the Euclidean norm. We require the 1-step distribution $D(x)$ to be bounded as

$$\begin{aligned} &D(x) \asymp \tfrac{1}{L^d} |||\tfrac{x}{L}|||_1^{-d-\alpha} \\ &\overset{\text{def}}{\Leftrightarrow} \quad \exists c > 0,\ \forall x \in \mathbb{Z}^d,\ \forall L \in [1, \infty) :\ c \le \frac{D(x)}{\frac{1}{L^d} |||\frac{x}{L}|||_1^{-d-\alpha}} \le \frac{1}{c}, \end{aligned} \tag{1.6}$$

where L is the spread-out parameter.

Let $\hat{D}$ and D^{*n} be the Fourier transform and the n-fold convolution of D, respectively:

$$\hat{D}(k) = \sum_{x \in \mathbb{Z}^d} e^{ik\cdot x} D(x) \qquad [k \in [-\pi, \pi)^d], \tag{1.7}$$

$$D^{*n}(x) = \begin{cases} \delta_{o,x} & [n = 0], \\ \sum_{y \in \mathbb{Z}^d} D^{*(n-1)}(y)\, D(x-y) & [n \ge 1]. \end{cases} \tag{1.8}$$

We also require D to satisfy the following properties.

Assumption 1.1 (Properties of $\hat{D}$). *There is a* $\Delta = \Delta(L) \in (0, 1)$ *such that*

$$1 - \hat{D}(k) \begin{cases} < 2 - \Delta & [\forall k \in [-\pi, \pi]^d], \\ > \Delta & [|k| > 1/L], \end{cases} \tag{1.9}$$

while, for $|k| \le 1/L$,

$$1 - \hat{D}(k) \asymp (L|k|)^{\alpha\wedge 2} \times \begin{cases} 1 & [\alpha \ne 2], \\ \log \frac{\pi}{2L|k|} & [\alpha = 2]. \end{cases} \tag{1.10}$$

Moreover, there is an $\epsilon > 0$ *such that, as* $|k| \to 0$,

$$1 - \hat{D}(k) = v_\alpha |k|^{\alpha\wedge 2} \times \begin{cases} \left(1 + O(L^\epsilon |k|^\epsilon)\right) & [\alpha \ne 2], \\ \left(\log \frac{1}{L|k|} + O(1)\right) & [\alpha = 2], \end{cases} \tag{1.11}$$

where the constant in the $O(1)$ *term is independent of* L.

Assumption 1.2 (Bounds on D^{*n}). *For* $n \in \mathbb{N}$ *and* $x \in \mathbb{Z}^d$,

$$\|D^{*n}\|_\infty \le O(L^{-d}) \times \begin{cases} n^{-d/(\alpha\wedge 2)} & [\alpha \ne 2], \\ (n \log \frac{\pi n}{2})^{-d/2} & [\alpha = 2], \end{cases} \tag{1.12}$$

$$D^{*n}(x) \le n \frac{O(L^{\alpha\wedge 2})}{|||x|||_L^{d+\alpha\wedge 2}} \times \begin{cases} 1 & [\alpha \ne 2], \\ \log |||\frac{x}{L}|||_1 & [\alpha = 2]. \end{cases} \tag{1.13}$$

It has been shown [10–13] that the following D is one of the examples that satisfy all the properties in the above assumptions:

$$D(x) = \begin{cases} \dfrac{|||x|||_L^{-d-\alpha}}{\sum_{y\in\mathbb{Z}^d\setminus\{o\}} |||y|||_L^{-d-\alpha}} & [x \neq o], \\ 0 & [x = o]. \end{cases} \tag{1.14}$$

Another such example is the following compound-zeta distribution [13]:

$$D(x) = \sum_{t\in\mathbb{N}} U_L^{*t}(x)\, T_\alpha(t) \qquad [x \in \mathbb{Z}^d], \tag{1.15}$$

where, with a probability distribution h on $[-1, 1]^d \subset \mathbb{R}^d$ and the Riemann-zeta function $\zeta(s) = \sum_{t\in\mathbb{N}} t^{-s}$,

$$U_L(x) = \frac{h(x/L)}{\sum_{y\in\mathbb{Z}^d\setminus\{o\}} h(y/L)} \qquad [x \in \mathbb{Z}^d], \tag{1.16}$$

$$T_\alpha(t) = \frac{t^{-1-\alpha/2}}{\zeta(1 + \alpha/2)} \qquad [t \in \mathbb{N}]. \tag{1.17}$$

We assume that the distribution h is bounded, non-degenerate, $\mathbb{Z}^d$-symmetric and piecewise continuous, such as $h(x) = 2^{-d}\mathbb{1}_{\{\|x\|_\infty \le 1\}}$.

Since the proof of (1.12) for $\alpha = 2$ is only briefly explained in [13, (1.19)], we will provide a full proof in Sect. 2.

Let S_p be the random-walk Green function generated by the 1-step distribution D:

$$S_p(x) = \sum_{\omega:o\to x} p^{|\omega|} \prod_{j=1}^{|\omega|} D(\omega_j - \omega_{j-1}) \qquad [x \in \mathbb{Z}^d], \tag{1.18}$$

where $o \in \mathbb{Z}^d$ is the origin, $p \ge 0$ is the fugacity and $|\omega|$ is the length of a path $\omega = (\omega_0, \omega_1, \ldots, \omega_{|\omega|})$. By convention, the contribution from the zero-step walk is the Kronecker delta $\delta_{o,x}$. It is convergent as long as $p < 1$ or $p = 1$ with $d > \alpha \wedge 2$. One of the main results of this paper is completion of the asymptotic picture of S_1 for all $\alpha > 0$, as follows.

Theorem 1.3. *Let $d > \alpha \wedge 2$ and suppose D satisfies Assumptions 1.1–1.2. Then, for any $p \in [0, 1]$,*

$$S_p(x) - \delta_{o,x} \le \frac{O(L^{-\alpha\wedge 2})}{|||x|||_L^{d-\alpha\wedge 2}} \times \begin{cases} 1 & [\alpha \neq 2], \\ \dfrac{1}{\log |||\frac{x}{L}|||_1} & [\alpha = 2]. \end{cases} \tag{1.19}$$

Moreover, there are $\epsilon, \eta > 0$ such that, for $L^{1+\eta} < |x| \to \infty$,

$$S_1(x) = \frac{\gamma_\alpha/v_\alpha}{|x|^{d-\alpha\wedge 2}} \times \begin{cases} \left(1 + \dfrac{O(L^\epsilon)}{|x|^\epsilon}\right) & [\alpha \neq 2], \\ \dfrac{1}{\log|x|}\left(1 + \dfrac{O(1)}{(\log|x|)^\epsilon}\right) & [\alpha = 2], \end{cases} \tag{1.20}$$

where the constant in the $O(1)$ term is independent of L.

1.2.2. Self-avoiding walk Self-avoiding walk (sometimes abbreviated as SAW) is a model for linear polymers. Taking into account the exclusion-volume effect among constituent monomers, we define the SAW two-point function as

$$G_p(x) = \sum_{\omega:o\to x} p^{|\omega|} \prod_{j=1}^{|\omega|} D(\omega_j - \omega_{j-1}) \prod_{s<t} (1 - \delta_{\omega_s,\omega_t}), \tag{1.21}$$

where the contribution from the zero-step walk is $\delta_{o,x}$, just as in (1.18). Notice that the difference between (1.18) and (1.21) is the last product, which is either 0 or 1 depending on whether ω intersects itself or does not. Because of this suppressing factor, the sum called the susceptibility

$$\chi_p = \sum_{x\in\mathbb{Z}^d} G_p(x) \tag{1.22}$$

is not bigger than $\sum_{x\in\mathbb{Z}^d} S_p(x)$, which is $(1-p)^{-1}$ when p is smaller than the radius of convergence 1, and therefore the critical point

$$p_c = \sup\{p : \chi_p < \infty\} \tag{1.23}$$

must be at least 1. It is known [22] that, if the bubble condition

$$G_{p_c}^{*2}(o) = \sum_{x\in\mathbb{Z}^d} G_{p_c}(x)^2 < \infty \tag{1.24}$$

holds, then

$$\chi_p \asymp (p_c - p)^{-1}, \tag{1.25}$$

meaning that the critical exponent for χ_p takes on its mean-field value 1.

1.2.3. Percolation Percolation is a model for random media. Each bond $\{u, v\} \subset \mathbb{Z}^d$ is assigned to be either occupied or vacant, independently of the other bonds. The probability of a bond $\{u, v\}$ being occupied is defined as $pD(v-u)$, where $p \geq 0$ is the percolation parameter. Since D is a probability distribution, the expected number of occupied bonds per vertex equals $p\sum_{x\neq o} D(x) = p(1-D(o))$. Let $G_p(x)$ denote the percolation two-point function, which is the probability that there is a self-avoiding path of occupied bonds from o to x. By convention, $G_p(o) = 1$.

For percolation, the susceptibility χ_p in (1.22) equals the expected number of vertices connected from o. It is known [6] that there is a critical point p_c defined as in (1.23) such that χ_p is finite if and only if $p < p_c$ and diverges as $p \uparrow p_c$. It is also known that, if the triangle condition

$$G_{p_c}^{*3}(o) = \sum_{x\in\mathbb{Z}^d} G_{p_c}(x)\, G_{p_c}^{*2}(x) < \infty \tag{1.26}$$

holds, then χ_p diverges in the same way as (1.25).

There is another order parameter θ_p called the percolation probability, which is the probability of the origin o being connected to infinity. It is known [2,14,23] that p_c in (1.23) can be characterized as $\inf\{p \geq 0 : \theta_p > 0\}$ and that, if the triangle condition (1.26) holds, then

$$\theta_p \asymp p - p_c, \tag{1.27}$$

meaning that the critical exponent for θ_p takes on its mean-field value 1, i.e., the value for the survival probability of the branching process.

1.2.4. The Ising model The Ising model is a model for magnets. Let $\Lambda \subset \mathbb{Z}^d$ and define the Hamiltonian (under the free-boundary condition) for a spin configuration $\varphi = \{\varphi_v\}_{v\in\Lambda} \in \{\pm 1\}^\Lambda$ as

$$H_\Lambda(\varphi) = -\sum_{\{u,v\}\subset\Lambda} J_{u,v}\varphi_u\varphi_v, \tag{1.28}$$

where $J_{u,v} = J_{o,v-u} \ge 0$ is the ferromagnetic coupling and is to satisfy the relation

$$D(x) = \frac{\tanh(\beta J_{o,x})}{\sum_{y\in\mathbb{Z}^d}\tanh(\beta J_{o,y})}, \tag{1.29}$$

where $\beta \ge 0$ is the inverse temperature. Let

$$\langle\varphi_o\varphi_x\rangle_{\beta,\Lambda} = \sum_{\varphi\in\{\pm1\}^\Lambda} \varphi_o\varphi_x\, e^{-\beta H_\Lambda(\varphi)} \Big/ \sum_{\varphi\in\{\pm1\}^\Lambda} e^{-\beta H_\Lambda(\varphi)}. \tag{1.30}$$

Using $p = \sum_{x\in\mathbb{Z}^d}\tanh(\beta J_{o,x})$, we define the Ising two-point function $G_p(x)$ as a unique infinite-volume limit of $\langle\varphi_o\varphi_x\rangle_{\beta,\Lambda}$:

$$G_p(x) = \lim_{\Lambda\uparrow\mathbb{Z}^d}\langle\varphi_o\varphi_x\rangle_{\beta,\Lambda}. \tag{1.31}$$

It is known [20] that the susceptibility χ_p defined as in (1.22) is finite if and only if $p < p_c$ and diverges as $p \uparrow p_c$. It is also known [3,14] that p_c is unique in the sense that the spontaneous magnetization

$$\theta_p = \sqrt{\lim_{|x|\to\infty} G_p(x)} \tag{1.32}$$

also exhibits a phase transition at p_c. (Unlike the case for percolation, the continuity of θ_p in p has been proven for all dimensions, as long as $J_{o,x}$ satisfies a strong symmetry condition called the reflection positivity [4].) Furthermore, it is known [1,5] that, if the bubble condition (1.24) holds for the critical Ising model, then

$$\chi_p \asymp (p_c - p)^{-1}, \qquad \theta_p \asymp \sqrt{p - p_c}, \tag{1.33}$$

meaning that the critical exponents for χ_p and θ_p take on their mean-field values 1 and 1/2, respectively.

1.2.5. The main results Let

$$d_c = (\alpha \wedge 2) \times m, \qquad m = \begin{cases} 2 & \text{[SAW and Ising]}, \\ 3 & \text{[percolation]}, \end{cases} \tag{1.34}$$

where m is the number of G_{p_c} involved in the bubble/triangle conditions (1.24) and (1.26).

In the previous paper [13], we investigated asymptotic behavior of $G_{p_c}(x)$ for $\alpha \neq 2$, $d > d_c$ and $L \gg 1$ (see Theorem 1.7). In the current paper, we investigate the marginal case $\alpha = 2$, for which the variance of D diverges logarithmically, and prove the following:

Theorem 1.4. *Let $\alpha = 2$ and $d \ge d_c$ (including equality) and suppose that D satisfies Assumptions 1.1–1.2. Then there is a model-dependent $L_0 < \infty$ such that, for any $L \ge L_0$,*

$$G_{p_c}(x) \le \delta_{o,x} + \frac{O(L^{-2})}{|||x|||_L^{d-2} \log |||\frac{x}{L}|||_1}. \tag{1.35}$$

Moreover, there is an $\epsilon > 0$ such that, as $|x| \to \infty$,

$$G_{p_c}(x) = \frac{1}{p_c} \frac{\gamma_2/v_2}{|x|^{d-2} \log |x|} \bigg(1 + \frac{O(1)}{(\log |x|)^\epsilon} \bigg), \tag{1.36}$$

where the $O(1)$ term is independent of L.

Due to the log correction to the standard Newtonian behavior in (1.35)–(1.36), we can show that the bubble/triangle conditions hold, even at the critical dimension $d = d_c$. For example, the tail of the sum in the bubble condition (1.24) can be estimated, for any $R > 1$, as

$$\sum_{x:|x|>R} G_{p_c}(x)^2 \approx \int_R^\infty \frac{\mathrm{d}r}{r} \frac{r^{4-d}}{(\log r)^2}, \tag{1.37}$$

which is finite even when $d = 4$, due to the log-squared term in the denominator. Also, by the convolution bounds in Lemma 3.5 below, which is one of the novelties of this paper, we can show that $G_{p_c}^{*2}(x)$ for $d \ge 4$ is bounded above by a multiple of $|x|^{4-d}/\log |x|$. Therefore, the tail of the sum in the triangle condition (1.26) can be estimated as

$$\sum_{x:|x|>R} G_{p_c}(x)\, G_{p_c}^{*2}(x) \approx \int_R^\infty \frac{\mathrm{d}r}{r} \frac{r^{6-d}}{(\log r)^2}, \tag{1.38}$$

which is finite even when $d = 6$, again due to the log-squared term in the denominator. Therefore:

Corollary 1.5. *The mean-field results (1.25), (1.27) and (1.33) hold for all three models with $\alpha = 2$ and sufficiently large L, in dimensions $d \ge d_c$ (including equality).*

Remark 1.6. 1. In the previous paper [13], we investigated the other case $\alpha \ne 2$ and proved the following:

Theorem 1.7 (Theorems 1.2 and 3.3 of [13]). *Let $\alpha \ne 2$ and $d > d_c$ and suppose that D satisfies Assumptions 1.1–1.2 and the following bound on the "derivative" of D^{*n}: for $n \in \mathbb{N}$ and $x, y \in \mathbb{Z}^d$ with $|y| \le \frac{1}{3}|x|$,*

$$\bigg| D^{*n}(x) - \frac{D^{*n}(x+y) + D^{*n}(x-y)}{2} \bigg| \le n \frac{O(L^{\alpha\wedge 2})\, |||y|||_L^2}{|||x|||_L^{d+\alpha\wedge 2+2}}. \tag{1.39}$$

Then, there is a model-dependent $L_0 < \infty$ such that, for any $L \ge L_0$,

$$G_{p_c}(x) \le \delta_{o,x} + \frac{O(L^{-\alpha\wedge 2})}{|||x|||_L^{d-\alpha\wedge 2}}. \tag{1.40}$$

As a result, the bubble/triangle conditions (1.24) and (1.26) hold, and therefore the critical exponents for χ_p and θ_p take on their respective mean-field values. Moreover, there are $A = 1 + O(L^{-2})\mathbb{1}_{\{\alpha>2\}}$ and $\epsilon > 0$ such that, as $|x| \to \infty$,

$$G_{p_c}(x) = \frac{A}{p_c} \frac{\gamma_\alpha/v_\alpha}{|x|^{d-\alpha\wedge 2}} \left(1 + \frac{O(L^\epsilon)}{|x|^\epsilon}\right). \tag{1.41}$$

The extra assumption (1.39) is hard to verify in a general setup. However, we have shown [13] that the compound-zeta distribution (1.15) for $\alpha \neq 2$ satisfies (1.39). In fact, as explained in Sect. 3.2 (see also Remark 3.7), the proof of Theorem 1.4 for $\alpha = 2$ also works for the case $\alpha < 2$, so that we do not have to require (1.39) for $\alpha \leq 2$, but not for $\alpha > 2$. This is somewhat related to the fact that the multiplicative constant A in (1.41) becomes 1 for $\alpha \leq 2$.

2. The possibility to extend the mean-field results down to $d = d_c$ was already hinted in [19, Theorem 1.1], where we have shown that, for $d > d_c$ and $L \gg 1$, the Fourier transform $\hat{G}_p(k)$ obeys the following infrared bound, uniformly in $k \in [-\pi, \pi]^d$ and $p < p_c$:

$$\hat{G}_p(k) = \frac{1 + O(\delta_m)}{\chi_p^{-1} + p(1 - \hat{D}(k))}, \tag{1.42}$$

where

$$\delta_m = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\hat{D}(k)^2}{(1 - \hat{D}(k))^m}. \tag{1.43}$$

In fact, we can follow the same line of proof of [19, Theorem 1.1] to obtain (1.42), as long as δ_m is sufficiently small. However, for $\alpha = 2$ and $d \geq d_c$ (including equality), we have

$$\delta_m \leq \underbrace{\int_{|k|>1/L} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\hat{D}(k)^2}{\Delta^m}}_{\because (1.9)} + \underbrace{O(L^{-2m}) \int_{|k|\leq 1/L} \frac{\mathrm{d}^d k}{(|k|^2 \log \frac{\pi}{2L|k|})^m}}_{\because (1.10)} = O(L^{-d}). \tag{1.44}$$

Therefore, by taking L sufficiently large and using monotonicity in p, we obtain

$$G_{p_c}^{*m}(o) = \lim_{p\uparrow p_c} G_p^{*m}(o) = \lim_{p\uparrow p_c} \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \hat{G}_p(k)^m < \infty, \tag{1.45}$$

as long as $d \geq d_c$, hence the mean-field results for all $d \geq d_c$.

2. Analysis for the Underlying Random Walk

In Sect. 2.1, we prove Theorem 1.3 for $\alpha = 2$ (the results for $\alpha \neq 2$ have been proven in [13]). In Sect. 2.2, we complete the proof of (1.12) for $\alpha = 2$.

2.1. Proof of Theorem 1.3. The results for $\alpha \neq 2$ are already proven in [13, Proposition 2.1]. The proof of (1.19) for $\alpha = 2$ is easy, as we split the sum at $N \equiv |||\frac{x}{L}|||_1^2 / \log |||\frac{x}{L}|||_1$ and use (1.13) for $n \leq N$ and (1.12) for $n \geq N$, as follows:

$$\begin{aligned} S_q(x) - \delta_{o,x} &\leq \sum_{n=1}^{N} D^{*n}(x) + \sum_{n=N}^{\infty} \|D^{*n}\|_\infty \\ &\leq O(L^{-d}) \bigg(\frac{\log |||\frac{x}{L}|||_1}{|||\frac{x}{L}|||_1^{d+2}} \sum_{n=1}^{N} n + \sum_{n=N}^{\infty} (n \log n)^{-d/2} \bigg) \\ &\leq O(L^{-d}) \bigg(\frac{\log |||\frac{x}{L}|||_1}{|||\frac{x}{L}|||_1^{d+2}} N^2 + \frac{N^{1-d/2}}{(\log N)^{d/2}} \bigg) = \frac{O(L^{-d}) |||\frac{x}{L}|||_1^{2-d}}{\log |||\frac{x}{L}|||_1}. \end{aligned} \tag{2.1}$$

It remains to show (1.20) for $\alpha = 2$. First, we rewrite $S_1(x)$ for $d > 2$ as

$$S_1(x) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{-ik\cdot x}}{1 - \hat{D}(k)} = \int_0^\infty \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x - t(1-\hat{D}(k))}. \tag{2.2}$$

Let

$$\mu \in (0, \tfrac{2}{d+2}), \qquad T = \frac{(\frac{|x|}{L})^2}{(\log \frac{|x|}{L})^{1+\mu}}. \tag{2.3}$$

Then, for $|x| > L^{1+\eta}$ (so that $\delta_{o,x} = 0$),

$$\begin{aligned} I_1 &\equiv \int_0^T \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x - t(1-\hat{D}(k))} \mathrm{d}t \\ &= \int_0^T \mathrm{d}t\, e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x} \hat{D}(k)^n \\ &= \int_0^T \mathrm{d}t\, e^{-t} \bigg(\delta_{o,x} + \sum_{n=1}^{\infty} \frac{t^n}{n!} D^{*n}(x) \bigg) \\ &\overset{(1.13)}{\leq} \frac{O(L^2) \log \frac{|x|}{L}}{|x|^{d+2}} T^2 = \frac{O(L^{-2}) |x|^{2-d}}{(\log \frac{|x|}{L})^{1+2\mu}}, \end{aligned} \tag{2.4}$$

which is an error term.

Next, we investigate $S_1(x) - I_1$. Let

$$\omega = \frac{1}{\eta \log L} \in (0, 1), \qquad LR = \bigg(\frac{|x|}{L} \bigg)^{-\omega}. \tag{2.5}$$

Then, we can rewrite $S_1(x) - I_1$ as

$$\begin{aligned} S_1(x) - I_1 &= \int_T^\infty \mathrm{d}t \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x - t(1-\hat{D}(k))} \mathrm{d}t \\ &= \int_T^\infty \mathrm{d}t \int_{|k|\leq R} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x - v_2 t |k|^2 \log \frac{1}{L|k|}} + \sum_{j=2}^{4} I_j, \end{aligned} \tag{2.6}$$

where

$$I_2 = \int_T^\infty \mathrm{d}t \int_{|k| \le R} \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-ik\cdot x}\Big(e^{-t(1-\hat{D}(k))} - e^{-v_2 t|k|^2 \log\frac{1}{L|k|}}\Big), \tag{2.7}$$

$$I_3 = \int_{R<|k|\le 1/L} \frac{\mathrm{d}^d k}{(2\pi)^d}\, \frac{e^{-ik\cdot x - T(1-\hat{D}(k))}}{1-\hat{D}(k)}, \tag{2.8}$$

$$I_4 = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d}\, \frac{e^{-ik\cdot x - T(1-\hat{D}(k))}}{1-\hat{D}(k)}\, \mathbb{1}_{\{|k|>1/L\}}. \tag{2.9}$$

For I_2, we first note that, by (1.11),

$$\Big|e^{-t(1-\hat{D}(k))} - e^{-v_2 t|k|^2 \log\frac{1}{L|k|}}\Big| \le O(L^2)t|k|^2 e^{-v_2 t|k|^2 \log\frac{1}{L|k|}}. \tag{2.10}$$

Let $s = v_2 t|k|^2 \log\frac{1}{L|k|}$ and $r = |k| \le R$. Since $|x| > L^{1+\eta}$, we have

$$\frac{\mathrm{d}s}{s} = \Big(2 - \frac{1}{\log\frac{1}{Lr}}\Big)\frac{\mathrm{d}r}{r} \ge \Big(2 - \frac{1}{\log\frac{1}{LR}}\Big)\frac{\mathrm{d}r}{r} > \Big(2 - \frac{1}{\omega\eta\log L}\Big)\frac{\mathrm{d}r}{r} = \frac{\mathrm{d}r}{r}. \tag{2.11}$$

Therefore, for $d > 2$,

$$\begin{aligned}
|I_2| &\le O(L^2)\int_T^\infty \mathrm{d}t\; t\int_0^R \frac{\mathrm{d}r}{r}\, r^{d+2} e^{-v_2 t r^2 \log\frac{1}{Lr}} \\
&\le O(L^2)\int_T^\infty \mathrm{d}t\; t\int_0^{v_2 t R^2 \log\frac{1}{LR}} \frac{\mathrm{d}s}{s}\Big(\frac{s}{v_2 t\log\frac{1}{LR}}\Big)^{(d+2)/2} e^{-s} \\
&\overset{(2.5)}{\le} O(L^{-d})\Big(\log\frac{|x|}{L}\Big)^{-(d+2)/2} T^{1-d/2} \\
&\overset{(2.3)}{=} \frac{O(L^{-2})|x|^{2-d}}{(\log\frac{|x|}{L})^{2-(d-2)\mu/2}},
\end{aligned} \tag{2.12}$$

which is an error term because

$$2 - \frac{(d-2)\mu}{2} \overset{(2.3)}{>} 2 - \frac{d-2}{d+2} = 1 + \frac{4}{d+2} > 1. \tag{2.13}$$

For I_3, since (1.10) holds and $\log\frac{\pi}{2L|k|} \ge \log\frac{\pi}{2} > 0$ for $|k| \le 1/L$, there is a $c > 0$ such that

$$|I_3| \le O(L^{-2})\int_R^{1/L} \frac{\mathrm{d}r}{r}\, r^{d-2} e^{-cL^2 T r^2} = O(L^{-d})T^{1-d/2}\int_{cL^2TR^2}^{cT} \frac{\mathrm{d}s}{s}\, s^{(d-2)/2} e^{-s}. \tag{2.14}$$

Since $TR^2 \to \infty$ as $|x| \to \infty$ (cf., (2.3) and (2.5)), the integral is bounded by a multiple of $(L^2TR^2)^{(d-4)/2}e^{-cL^2TR^2}$, which is a bound on the incomplete gamma function. Therefore, for $N \in \mathbb{N}$ large enough to ensure $2N+4 > d$,

$$\begin{aligned} |I_3| \le O(L^{-d})\frac{(LR)^{d-4}}{T}e^{-cL^2TR^2} &\le \frac{(LR)^{d-4}}{T}\frac{O(L^{-d})}{(L^2TR^2)^N} \\ &= \frac{O(L^{-2+(2N+4-d)(1-\omega)})(\log\frac{|x|}{L})^{(1+\mu)(N+1)}}{|x|^{d-2+(2N+4-d)(1-\omega)}} \\ &\overset{|x|>L^{1+\eta}}{\le} \frac{O(L^{-2})(\log\frac{|x|}{L})^{(1+\mu)(N+1)}}{|x|^{d-2+(2N+4-d)(1-\omega)\eta/(1+\eta)}}, \end{aligned} \tag{2.15}$$

which is an error term.

For I_4, we use (1.9) and a similar argument to (2.15) to obtain that, for $N \in \mathbb{N}$ large enough to ensure $2(N\eta-1)/(1+\eta) > d-2$,

$$|I_4| \le O(1)e^{-T\Delta} \le \frac{O(1)}{T^N} \le \frac{O(L^{2N})(\log\frac{|x|}{L})^N}{|x|^{2N}} \overset{|x|>L^{1+\eta}}{\le} \frac{O(L^{-2})(\log\frac{|x|}{L})^N}{|x|^{2(N\eta-1)/(1+\eta)}}, \tag{2.16}$$

which is an error term.

So far, we have obtained

$$S_1(x) = \int_T^\infty \mathrm{d}t \int_{|k|\le R} \frac{\mathrm{d}^dk}{(2\pi)^d} e^{-ik\cdot x - v_2t|k|^2\log\frac{1}{L|k|}} + \sum_{j=1}^4 I_j. \tag{2.17}$$

To investigate the above integral, we introduce $\xi \equiv x/|x|$ and change variables as $\kappa = |x|k$. Then, by changing time variables as $\tau = \frac{v_2t}{|x|^2}\log\frac{|x|}{L}$, the integral in (2.6) can be written as

$$\begin{aligned} &|x|^{-d}\int_T^\infty \mathrm{d}t \int_{|\kappa|\le|x|R} \frac{\mathrm{d}^d\kappa}{(2\pi)^d}\, \exp\bigg(-i\kappa\cdot\xi - \frac{v_2t|\kappa|^2}{|x|^2}\log\frac{|x|}{L|\kappa|}\bigg) \\ &= \frac{|x|^{2-d}}{v_2\log\frac{|x|}{L}} \int_{\frac{v_2T}{|x|^2}\log\frac{|x|}{L}}^\infty \mathrm{d}\tau \int_{|\kappa|\le|x|R} \frac{\mathrm{d}^d\kappa}{(2\pi)^d}\, \exp\bigg(-i\kappa\cdot\xi - \tau|\kappa|^2\frac{\log\frac{|x|}{L|\kappa|}}{\log\frac{|x|}{L}}\bigg) \\ &= \frac{|x|^{2-d}}{v_2\log\frac{|x|}{L}}\bigg(\int_0^\infty \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{\mathrm{d}^d\kappa}{(2\pi)^d}\, e^{-i\kappa\cdot\xi-\tau|\kappa|^2} - \sum_{j=1}^3 M_j\bigg), \end{aligned} \tag{2.18}$$

where

$$M_1 = \int_0^{\frac{v_2T}{|x|^2}\log\frac{|x|}{L}} \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{\mathrm{d}^d\kappa}{(2\pi)^d}\, e^{-i\kappa\cdot\xi-\tau|\kappa|^2}, \tag{2.19}$$

$$M_2 = \int_{\frac{v_2T}{|x|^2}\log\frac{|x|}{L}}^\infty \mathrm{d}\tau \int_{|\kappa|>|x|R} \frac{\mathrm{d}^d\kappa}{(2\pi)^d}\, e^{-i\kappa\cdot\xi-\tau|\kappa|^2}, \tag{2.20}$$

$$M_3 = \int_{\frac{v_2 T}{|x|^2} \log \frac{|x|}{L}}^{\infty} \mathrm{d}\tau \int_{|\kappa| \le |x| R} \frac{\mathrm{d}^d \kappa}{(2\pi)^d} \, e^{-i\kappa\cdot\xi - \tau|\kappa|^2} \left(1 - \exp\left(\tau|\kappa|^2 \frac{\log|\kappa|}{\log\frac{|x|}{L}}\right)\right). \tag{2.21}$$

Notice that the first term in the parentheses in (2.18) gives the leading term:

$$\int_0^{\infty} \mathrm{d}\tau \int_{\mathbb{R}^d} \frac{\mathrm{d}^d \kappa}{(2\pi)^d} \, e^{-i\kappa\cdot\xi - \tau|\kappa|^2} = \int_0^{\infty} \mathrm{d}\tau \, \frac{e^{-1/(4\tau)}}{(4\pi\tau)^{d/2}} = \frac{\Gamma(\frac{d-2}{2})}{4\pi^{d/2}} = \gamma. \tag{2.22}$$

For M_1, since $|x|^2/(v_2 T \log\frac{|x|}{L}) = \frac{L^2}{v_2}(\log\frac{|x|}{L})^{\mu} \to \infty$, we obtain that, for $N \in \mathbb{N}$ large enough to ensure $2N + 4 > d$,

$$\begin{aligned} M_1 &= \int_0^{\frac{v_2 T}{|x|^2}\log\frac{|x|}{L}} \mathrm{d}\tau \, \frac{e^{-1/(4\tau)}}{(4\pi\tau)^{d/2}} = \frac{1}{4\pi^{d/2}} \int_{|x|^2/(4v_2 T \log\frac{|x|}{L})}^{\infty} \frac{\mathrm{d}s}{s} \, s^{(d-2)/2} e^{-s} \\ &\le O(1) \left(\frac{|x|^2}{v_2 T \log\frac{|x|}{L}}\right)^{(d-4)/2} e^{-|x|^2/(4v_2 T \log\frac{|x|}{L})}. \end{aligned} \tag{2.23}$$

Using the exponentially decaying term yields

$$M_1 \overset{\forall N}{\le} \frac{O(1)}{(\log\frac{|x|}{L})^{(2N+4-d)\mu/2}}, \tag{2.24}$$

which gives an error term as long as $2N + 4 > d$.

For M_2, changing the order of integrations and changing variables as $r = |\kappa|^2 \frac{v_2 T}{|x|^2} \log\frac{|x|}{L}$ yields

$$\begin{aligned} |M_2| &\le \int_{|\kappa| > |x| R} \frac{\mathrm{d}^d \kappa}{(2\pi)^d} \int_{\frac{v_2 T}{|x|^2}\log\frac{|x|}{L}}^{\infty} \mathrm{d}\tau \, e^{-\tau|\kappa|^2} \\ &= \int_{|\kappa| > |x| R} \frac{\mathrm{d}^d \kappa}{(2\pi)^d} \, \frac{1}{|\kappa|^2} \exp\left(-|\kappa|^2 \frac{v_2 T}{|x|^2} \log\frac{|x|}{L}\right) \\ &= O(1) \left(\frac{|x|^2}{v_2 T \log\frac{|x|}{L}}\right)^{(d-2)/2} \int_{v_2 T R^2 \log\frac{|x|}{L}}^{\infty} \frac{\mathrm{d}r}{r} \, r^{(d-2)/2} e^{-r} \\ &= O(1) \, (|x| R)^{d-4} \frac{|x|^2}{v_2 T \log\frac{|x|}{L}} \, e^{-v_2 T R^2 \log\frac{|x|}{L}}. \end{aligned} \tag{2.25}$$

Using the exponentially decaying term and $|x| > L^{1+\eta}$ as in (2.15)–(2.16), we obtain that, for $N \in \mathbb{N}$ large enough to ensure $2N + 4 > d$,

$$|M_2| \le \frac{O(1)(\log\frac{|x|}{L})^{(N+1)\mu}}{(\frac{|x|}{L})^{(2N+4-d)(1-\omega)}} \overset{|x| > L^{1+\eta}}{\le} \frac{O(1)(\log\frac{|x|}{L})^{(N+1)\mu}}{|x|^{(2N+4-d)(1-\omega)\eta/(1+\eta)}}, \tag{2.26}$$

which gives another error term.

For M_3, we first note that, since $|\kappa| \le |x|R = (|x|/L)^{1-\omega}$,

$$\left|1 - \exp\left(\tau|\kappa|^2 \frac{\log|\kappa|}{\log\frac{|x|}{L}}\right)\right| \le \tau|\kappa|^2 \frac{|\log|\kappa||}{\log\frac{|x|}{L}} \exp\left(\tau|\kappa|^2 \frac{\log|x|R}{\log\frac{|x|}{L}}\right)$$
$$= \tau|\kappa|^2 \frac{|\log|\kappa||}{\log\frac{|x|}{L}} e^{(1-\omega)\tau|\kappa|^2}. \tag{2.27}$$

Then, by changing the order of integrations and changing variables as $s = \omega|\kappa|^2 \frac{v_2 T}{|x|^2} \log\frac{|x|}{L}$, we obtain

$$\begin{aligned}|M_3| &\le \frac{1}{\log\frac{|x|}{L}} \int_{|\kappa|\le|x|R} \frac{\mathrm{d}^d\kappa}{(2\pi)^d} |\kappa|^2 |\log|\kappa|| \int_{\frac{v_2T}{|x|^2}\log\frac{|x|}{L}}^{\infty} \mathrm{d}\tau\, \tau\, e^{-\omega\tau|\kappa|^2} \\ &= \frac{1}{\log\frac{|x|}{L}} \int_{|\kappa|\le|x|R} \frac{\mathrm{d}^d\kappa}{(2\pi)^d} \frac{|\log|\kappa||}{\omega^2|\kappa|^2} \left(1 + \omega|\kappa|^2 \frac{v_2T}{|x|^2} \log\frac{|x|}{L}\right) e^{-\omega|\kappa|^2 \frac{v_2T}{|x|^2}\log\frac{|x|}{L}} \\ &= \underbrace{\frac{O(1)}{\log\frac{|x|}{L}} \left(\frac{|x|^2}{v_2 T \log\frac{|x|}{L}}\right)^{(d-2)/2}}_{(\log\frac{|x|}{L})^{-1+(d-2)\mu/2}} \int_0^{\omega v_2 T R^2 \log\frac{|x|}{L}} \frac{\mathrm{d}s}{s} \left|\log\frac{s|x|^2}{\omega v_2 T \log\frac{|x|}{L}}\right| s^{(d-2)/2}(1+s)e^{-s}.\end{aligned} \tag{2.28}$$

Using the triangle inequality

$$\left|\log\frac{s|x|^2}{\omega v_2 T \log\frac{|x|}{L}}\right| \overset{(2.3)}{\le} \left|\log\frac{L^2 s}{\omega v_2}\right| + \mu \log\log\frac{|x|}{L}, \tag{2.29}$$

we obtain

$$\begin{aligned}&\int_0^{\omega v_2 T R^2 \log\frac{|x|}{L}} \frac{\mathrm{d}s}{s} \left|\log\frac{s|x|^2}{\omega v_2 T \log\frac{|x|}{L}}\right| s^{(d-2)/2}(1+s)e^{-s} \\ &\le \underbrace{\int_0^{\infty} \frac{\mathrm{d}s}{s} \left|\log\frac{L^2 s}{\omega v_2}\right| s^{(d-2)/2}(1+s)e^{-s}}_{\text{convergent as long as } d>2} + \Big(\Gamma(\tfrac{d-2}{2}) + \Gamma(\tfrac{d}{2})\Big)\mu \log\log\frac{|x|}{L} \\ &= O(1)\left(1 + \log\log\frac{|x|}{L}\right).\end{aligned} \tag{2.30}$$

As a result,

$$|M_3| \le \frac{O(1)\log\log\frac{|x|}{L}}{(\log\frac{|x|}{L})^{1-(d-2)\mu/2}} \overset{\mu<\frac{2}{d+2}}{\le} \frac{O(1)\log\log\frac{|x|}{L}}{(\log\frac{|x|}{L})^{4/(d+2)}}, \tag{2.31}$$

which gives another error term.

Summarizing (2.17)–(2.18) and (2.22), we arrive at

$$S_1(x) = \frac{|x|^{2-d}}{v_2 \log\frac{|x|}{L}} \left(\gamma - \sum_{j=1}^{3} M_j\right) + \sum_{j=1}^{4} I_j, \tag{2.32}$$

with the error estimates (2.4), (2.12), (2.15)–(2.16), (2.24), (2.26) and (2.31). This completes the proof of Theorem 1.3 assuming the properties in Assumptions 1.1–1.2. □

2.2. Proof of the bound (1.12) *on* $\|D^{*n}\|_\infty$ *for* $\alpha = 2$. For $n = 1$, $\|D\|_\infty = O(L^{-d})$ is obvious. For $n \geq 2$, we recall that $\|D^{*n}\|_\infty$ is bounded as (cf., [10, (A.2) and (A.4)])

$$\|D^{*n}\|_\infty \leq O(L^{-d}) \int_0^1 \frac{\mathrm{d}r}{r}\, r^d e^{-nr^2 \log \frac{\pi}{2r}} + \|D\|_\infty (1-\Delta)^{n-2}. \tag{2.33}$$

Since the second term decays exponentially in n, it suffices to show that

$$\int_0^1 \frac{\mathrm{d}r}{r}\, r^d e^{-nr^2 \log \frac{\pi}{2r}} \leq O\big((n \log \tfrac{\pi n}{2})^{-d/2}\big). \tag{2.34}$$

Let $t = n^{-1/4}$ (so that $nt^2 = \sqrt{n}$). Notice that $\log \frac{\pi}{2r} \geq \log \frac{\pi}{2} > 0$ for $r \leq 1$. By changing variables as $s = nr^2 \log \frac{\pi}{2}$, we have

$$\int_t^1 \frac{\mathrm{d}r}{r}\, r^d e^{-nr^2 \log \frac{\pi}{2r}} \leq O(n^{-d/2}) \int_{\sqrt{n}\log \frac{\pi}{2}}^\infty \frac{\mathrm{d}s}{s}\, s^{d/2} e^{-s} \leq O(n^{-\frac{d+2}{4}})\, e^{-\sqrt{n}\log \frac{\pi}{2}}, \tag{2.35}$$

which decays much faster than $O((n \log \frac{\pi n}{2})^{-d/2})$. For the remaining integral over $r \in (0, t)$, we change variables as $s = nr^2 \log \frac{\pi}{2r}$. Then, there is a $c > 0$ such that

$$\frac{\mathrm{d}s}{s} = \bigg(2 - \frac{1}{\log \frac{\pi}{2r}}\bigg)\frac{\mathrm{d}r}{r} \geq \bigg(2 - \frac{1}{\log \frac{\pi}{2t}}\bigg)\frac{\mathrm{d}r}{r} \geq c\frac{\mathrm{d}r}{r}, \tag{2.36}$$

$$r = \sqrt{\frac{s}{n \log \frac{\pi}{2r}}} \leq \sqrt{\frac{s}{n \log \frac{\pi}{2t}}} = \sqrt{\frac{s}{n(\frac{1}{4}\log n + \log \frac{\pi}{2})}} \leq \sqrt{\frac{4s}{n \log \frac{\pi n}{2}}}. \tag{2.37}$$

Therefore,

$$\int_0^t \frac{\mathrm{d}r}{r}\, r^d e^{-nr^2 \log \frac{\pi}{2r}} \leq \frac{4^{d/2}}{c(n \log \frac{\pi n}{2})^{d/2}} \int_0^{nt^2 \log \frac{\pi}{2t}} \frac{\mathrm{d}s}{s}\, s^{d/2} e^{-s} \leq \frac{4^{d/2}\Gamma(\frac{d}{2})}{c(n \log \frac{\pi n}{2})^{d/2}}, \tag{2.38}$$

as required. □

3. Analysis for the Two-Point Function

In this section, we use the lace expansion (1.2) to prove Theorem 1.4. First, in Sect. 3.1, we summarize some known facts, including the precise statement of the lace expansion for the two-point function. Then, in Sect.3.2, we prove the infrared bound (1.35) by using convolution bounds on power functions with log corrections (Lemma 3.5) and bounds on the lace-expansion coefficients (Lemma 3.6). The proofs of those two lemmas follow, in Sects. 3.3–3.4, respectively. Finally, in Sect. 3.5, we prove the asymptotic behavior (1.36) and complete the proof of Theorem 1.4.

3.1. List of known facts. The following four propositions hold independently of the value of $\alpha > 0$.

Proposition 3.1 (Lemma 2.2 of [13]). *For every $x \in \mathbb{Z}^d$, $G_p(x)$ is nondecreasing and continuous in $p < p_c$ for SAW, and in $p \le p_c$ for percolation and the Ising model. The continuity up to $p = p_c$ for SAW is also valid if $G_p(x)$ is uniformly bounded in $p < p_c$.*

Proposition 3.2 (Lemma 2.3 of [13]). *For every $p < p_c$ and $x \in \mathbb{Z}^d$,*

$$G_p(x) \le S_p(x), \qquad pD(x) \le G_p(x) - \delta_{o,x} \le (pD * G_p)(x). \tag{3.1}$$

Proposition 3.3 (Lemma 2.4 of [13]). *For every $p < p_c$, there is a $K_p = K_p(\alpha, d, L) < \infty$ such that, for any $x \in \mathbb{Z}^d$,*

$$G_p(x) \le K_p |||x|||_L^{-d-\alpha}. \tag{3.2}$$

Proposition 3.4 ([9] for SAW; [17] for percolation; [26] for the Ising model). *There are model-dependent nonnegative functions on $\mathbb{Z}^d$, $\{\pi_p^{(n)}\}_{n=0}^\infty$ ($\pi_p^{(0)} \equiv 0$ for SAW) and $\{R_p^{(n)}\}_{n=1}^\infty$, such that, for every integer $n \ge 0$,*

$$G_p = \begin{cases} \delta + (pD_{\neq} + \pi_p^{(\le n)}) * G_p + (-1)^{n+1} R_p^{(n+1)} & [\textit{SAW}], \\ \pi_p^{(\le n)} + \pi_p^{(\le n)} * pD_{\neq} * G_p + (-1)^{n+1} R_p^{(n+1)} & [\textit{percolation \& Ising}], \end{cases} \tag{3.3}$$

where the spatial variables are omitted (e.g., G_p for $G_p(x)$, δ for $\delta_{o,x}$) and[1]

$$D_{\neq} = D - D(o)\delta, \qquad \pi_p^{(\le n)} = \sum_{j=0}^n (-1)^j \pi_p^{(j)}. \tag{3.4}$$

Moreover, the remainder term obeys the following bound:

$$R_p^{(n+1)} \le \begin{cases} \pi_p^{(n+1)} * G_p & [\textit{SAW}], \\ \pi_p^{(n)} * pD * G_p & [\textit{percolation \& Ising}]. \end{cases} \tag{3.5}$$

Before proceeding to the next subsection, we derive the unified expression (1.2) from (3.3). To do so, we first assume $p < p_c$ and $\sum_j \|\pi_p^{(j)}\|_1 < \infty$, which has been verified for $\alpha \neq 2$, $d > d_c$ and $L \gg 1$ in [13] and is verified in the next subsection for $\alpha = 2$, $d \ge d_c$ and $L \gg 1$. Then, by (3.5), we can take the $n \to \infty$ limit to obtain

$$G_p = \begin{cases} \delta + (pD_{\neq} + \pi_p) * G_p & [\text{SAW}], \\ \pi_p + \pi_p * pD_{\neq} * G_p & [\text{percolation \& Ising}], \end{cases} \tag{3.6}$$

[1] The recursion equation [13, (1.11)] is correct for percolation and the Ising model, but not quite for SAW, as long as $D(o) > 0$. To deal with such D, the definition [13, (1.13)] of Π_p needs slight modification. See (3.10) below.

where $\pi_p = \lim_{n\to\infty} \pi_p^{(\le n)}$. For percolation and the Ising model, if $pD(o)\|\pi_p\|_1 < 1$ (also verified for $\alpha \ne 2$, $d > d_c$ and $L \gg 1$ in [13], and for $\alpha = 2$, $d \ge d_c$ and $L \gg 1$ in the next subsection), then

$$\begin{aligned}
G_p &= \pi_p + \pi_p * pD * G_p - pD(o)\pi_p * \underbrace{G_p}_{\text{replace}} \\
&= \pi_p + \pi_p * pD * G_p - pD(o)\pi_p * \Big(\pi_p + \pi_p * pD * G_p - pD(o)\pi_p * G_p\Big) \\
&= \big(\pi_p - pD(o)\pi_p^{*2}\big) + \big(\pi_p - pD(o)\pi_p^{*2}\big) * pD * G_p + \big(-pD(o)\big)^2 \pi_p^{*2} * \underbrace{G_p}_{\text{replace}} \\
&= \big(\pi_p - pD(o)\pi_p^{*2}\big) + \big(\pi_p - pD(o)\pi_p^{*2}\big) * pD * G_p \\
&\quad + \big(-pD(o)\big)^2 \pi_p^{*2} * \Big(\pi_p + \pi_p * pD * G_p - pD(o)\pi_p * G_p\Big) \\
&\ \ \vdots \\
&= \Pi_p + \Pi_p * pD * G_p,
\end{aligned} \tag{3.7}$$

where

$$\Pi_p = \pi_p + \sum_{n=1}^{\infty} \big(-pD(o)\big)^n \pi_p^{*(n+1)}. \tag{3.8}$$

For SAW, if $pD(o) + \|\pi_p\|_1 < 1$ (also verified for $\alpha \ne 2$, $d > d_c$ and $L \gg 1$ in [13], and for $\alpha = 2$, $d \ge d_c$ and $L \gg 1$ in the next subsection), then

$$\begin{aligned}
G_p &= \delta + pD * G_p + \big(-pD(o)\delta + \pi_p\big) * \underbrace{G_p}_{\text{replace}} \\
&= \delta + pD * G_p + \big(-pD(o)\delta + \pi_p\big) * \Big(\delta + pD * G_p + \big(-pD(o)\delta + \pi_p\big) * G_p\Big) \\
&= \Big(\delta + \big(-pD(o)\delta + \pi_p\big)\Big) + \Big(\delta + \big(-pD(o)\delta + \pi_p\big)\Big) * pD * G_p \\
&\quad + \big(-pD(o)\delta + \pi_p\big)^{*2} * \underbrace{G_p}_{\text{replace}} \\
&\ \ \vdots \\
&= \Pi_p + \Pi_p * pD * G_p,
\end{aligned} \tag{3.9}$$

where

$$\Pi_p = \delta + \sum_{n=1}^{\infty} \big(-pD(o)\delta + \pi_p\big)^{*n}. \tag{3.10}$$

3.2. Proof of the infrared bound (1.35). Let $\alpha = 2$, $d \ge d_c$ and

$$\lambda = \sup_{x \ne o} \frac{S_1(x)}{\|x\|_L^{2-d} / \log \|\frac{x}{L}\|_1} = O(L^{-2}). \tag{3.11}$$

Define

$$g_p = p \vee \sup_{x \neq o} \frac{G_p(x)}{\lambda |||x|||_L^{2-d} / \log |||\frac{x}{L}|||_1}. \tag{3.12}$$

We will show that g_p satisfies the following three properties:

(i) g_p is continuous (and nondecreasing) in $p \in [1, p_c)$.
(ii) $g_1 \leq 1$.
(iii) If $\lambda \ll 1$ (i.e., $L \gg 1$), then $g_p \leq 3$ implies $g_p \leq 2$ for every $p \in (1, p_c)$.

Notice that the above properties readily imply $G_p(x) \leq 2\lambda |||x|||_L^{2-d} / \log |||\frac{x}{L}|||_1$ for all $x \neq o$ and $p < p_c$ (≤ 2). By Proposition 3.1, we can extend this bound up to p_c, which completes the proof of (1.35).

It remains to prove the properties (i)–(iii).

Proof of (i). It suffices to show that $\sup_{x \neq o} G_p(x) / |||x|||_L^{2-d} / \log |||\frac{x}{L}|||_1$ is continuous in $p \in [1, p_0]$ for every fixed $p_0 \in (1, p_c)$. First, by the monotonicity of $G_p(x)$ in $p \leq p_0$ and using Proposition 3.3, we have

$$\begin{aligned} \frac{G_p(x)}{|||x|||_L^{2-d} / \log |||\frac{x}{L}|||_1} &\leq \frac{G_{p_0}(x)}{|||x|||_L^{2-d} / \log |||\frac{x}{L}|||_1} \leq \frac{K_{p_0} |||x|||_L^{-d-2}}{|||x|||_L^{2-d} / \log |||\frac{x}{L}|||_1} \\ &= \frac{K_{p_0}}{|||x|||_L^{4} / \log |||\frac{x}{L}|||_1}. \end{aligned} \tag{3.13}$$

On the other hand, for any $x_0 \neq o$ with $D(x_0) > 0$, there is an $R = R(p_0, x_0) < \infty$ such that, for all $|x| \geq R$,

$$\frac{K_{p_0}}{|||x|||_L^{4} / \log |||\frac{x}{L}|||_1} \leq \frac{D(x_0)}{|||x_0|||_L^{2-d} / \log |||\frac{x_0}{L}|||_1}. \tag{3.14}$$

Moreover, by using $p \geq 1$ and the lower bound of the second inequality in (3.1), we have

$$D(x_0) \leq pD(x_0) \leq G_p(x_0). \tag{3.15}$$

As a result, for any $p \in [1, p_0]$, we obtain

$$\sup_{x \neq o} \frac{G_p(x)}{|||x|||_L^{2-d} / \log |||\frac{x}{L}|||_1} = \frac{G_p(x_0)}{|||x_0|||_L^{2-d} \log |||\frac{x_0}{L}|||_1} \vee \max_{x: 0 < |x| < R} \frac{G_p(x)}{|||x|||_L^{2-d} / \log |||\frac{x}{L}|||_1}. \tag{3.16}$$

Since $G_p(x)$ is continuous in p (cf., Proposition 3.1) and the maximum of finitely many continuous functions is continuous, we can conclude that g_p is continuous in $p \in [1, p_0]$, as required. □

Proof of (ii). By Proposition 3.2 and the definition (3.11) of λ, we readily obtain

$$g_1 = 1 \vee \sup_{x \neq o} \frac{G_1(x)}{\lambda |||x|||_L^{2-d} / \log |||\frac{x}{L}|||_1} \leq 1 \vee \sup_{x \neq o} \frac{S_1(x)}{\lambda |||x|||_L^{2-d} / \log |||\frac{x}{L}|||_1} \leq 1, \tag{3.17}$$

as required. □

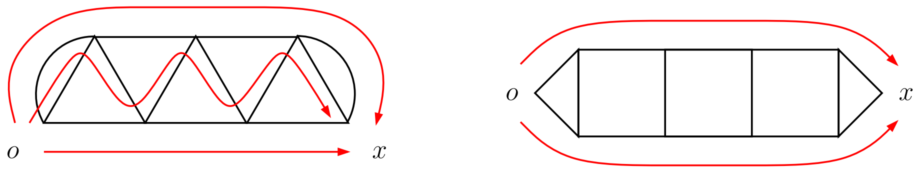


Fig. 1. Examples of the lace-expansion diagrams for SAW and the Ising model (left) and percolation (right). The factor ℓ in (3.19) is the number of disjoint paths (in red) from o to x using different sets of line segments

Proof of (iii). This is the most involved part among (i)–(iii), and here we use the lace expansion. To evaluate the lace-expansion coefficients, we use the following bounds on convolutions of power functions with log corrections, whose proof is deferred to Sect. 3.5.

Lemma 3.5. *For $a_1 \geq b_1 > 0$ with $a_1 + b_1 \geq d$, and for $a_2, b_2 \geq 0$ with $a_2 \geq b_2$ when $a_1 = b_1$, there is an L-independent constant $C = C(d, a_1, a_2, b_1, b_2) < \infty$ such that*

$$\sum_{y\in\mathbb{Z}^d} \frac{\|\!|x-y|\!\|_L^{-a_1}}{(\log \|\!|\frac{x-y}{L}|\!\|_1)^{a_2}} \frac{\|\!|y|\!\|_L^{-b_1}}{(\log \|\!|\frac{y}{L}|\!\|_1)^{b_2}} \leq \frac{C\,\|\!|x|\!\|_L^{-b_1}}{(\log \|\!|\frac{x}{L}|\!\|_1)^{b_2}} \times \begin{cases} L^{d-a_1} & [a_1 > d], \\ \log\log \|\!|\frac{x}{L}|\!\|_1 & [a_1 = d,\ a_2 = 1], \\ (\log \|\!|\frac{x}{L}|\!\|_1)^{0\vee(1-a_2)} & [a_1 = d,\ a_2 \neq 1], \\ \|\!|x|\!\|_L^{d-a_1} & [a_1 < d,\ a_1 + b_1 > d], \\ \|\!|x|\!\|_L^{b_1} (\log \|\!|\frac{x}{L}|\!\|_1)^{0\vee(1-a_2)} & [a_1 < d,\ a_1 + b_1 = d,\ a_2 + b_2 > 1]. \end{cases} \tag{3.18}$$

Assuming $g_p \leq 3$ and Lemma 3.5, we prove in Sect. 3.4 the following bounds on the lace-expansion coefficients $\{\pi_p^{(n)}\}_{n=0}^{\infty}$ (recall that $\pi_p^{(0)} \equiv 0$ for SAW) in (3.3).

Lemma 3.6. *Let (cf., (1.34) for the definition of m)*

$$\ell = \frac{m+1}{m-1} = \begin{cases} 3 & [\textit{SAW \& Ising}], \\ 2 & [\textit{percolation}]. \end{cases} \tag{3.19}$$

Suppose $g_p \leq 3$ and $p < p_c$. Under the same condition as in Theorem 1.4, we have

$$(pD * G_p)(x) \leq O(\lambda) \frac{\|\!|x|\!\|_L^{2-d}}{\log \|\!|\frac{x}{L}|\!\|_1} \qquad [x \in \mathbb{Z}^d]. \tag{3.20}$$

Moreover, for SAW,

$$\pi_p^{(j)}(x) \leq \begin{cases} O(L^{-d})\delta_{o,x} & [j = 1], \\ O(\lambda)^{j+1} \dfrac{\|\!|x|\!\|_L^{3(2-d)}}{(\log \|\!|\frac{x}{L}|\!\|_1)^3} & [j \geq 2], \end{cases} \tag{3.21}$$

and for the Ising model and percolation,

$$\pi_p^{(j)}(x) \le \begin{cases} O(L^{-d})^j \delta_{o,x} + O(\lambda)^2 \dfrac{|\!|\!|x|\!|\!|_L^{\ell(2-d)}}{(\log |\!|\!|\frac{x}{L}|\!|\!|_1)^\ell} & [j=0,1], \\ O(\lambda)^j \dfrac{|\!|\!|x|\!|\!|_L^{\ell(2-d)}}{(\log |\!|\!|\frac{x}{L}|\!|\!|_1)^\ell} & [j \ge 2]. \end{cases} \tag{3.22}$$

Consequently, we have $\sum_j \|\pi_p^{(j)}\|_1 < \infty$ for $d \ge d_c$ and $L \gg 1$. Then, by using (3.5) for $p < p_c$, we obtain $\lim_{n\to\infty} \|R_p^{(n)}\|_1 = 0$ and (3.6) with

$$|\pi_p(x)| \le \begin{cases} O(L^{-d})\delta_{o,x} + O(\lambda^3) \dfrac{|\!|\!|x|\!|\!|_L^{3(2-d)}}{(\log |\!|\!|\frac{x}{L}|\!|\!|_1)^3} & [\text{SAW}], \\ \big(1 + O(L^{-d})\big)\delta_{o,x} + O(\lambda^2) \dfrac{|\!|\!|x|\!|\!|_L^{\ell(2-d)}}{(\log |\!|\!|\frac{x}{L}|\!|\!|_1)^\ell} & [\text{Ising \& percolation}]. \end{cases} \tag{3.23}$$

This implies that, for SAW (cf., (3.10)),

$$\underbrace{pD(o)}_{O(L^{-d})}\delta_{o,x} + |\pi_p(x)| \le O(L^{-d})\delta_{o,x} + O(\lambda^3)\frac{|\!|\!|x|\!|\!|_L^{3(2-d)}}{(\log |\!|\!|\frac{x}{L}|\!|\!|_1)^3}, \tag{3.24}$$

hence $pD(o) + \|\pi_p\|_1 < 1$ for $d \ge d_c$ and $L \gg 1$. Also, for the Ising model and percolation (cf., (3.8)), since $L^{-d} = O(\lambda)$ for $d \ge 2$,

$$pD(o)|\pi_p(x)| \le O(L^{-d})\delta_{o,x} + O(\lambda^3)\frac{|\!|\!|x|\!|\!|_L^{\ell(2-d)}}{(\log |\!|\!|\frac{x}{L}|\!|\!|_1)^\ell}, \tag{3.25}$$

hence $pD(o)\|\pi_p\|_1 < 1$ for $d \ge d_c$ and $L \gg 1$. We note that, for all three models,

$$\ell(2-d) = -d - 2 - (\ell - 1)(d - d_c). \tag{3.26}$$

By repeated applications of (3.24) and Lemma 3.5, $\Pi_p(x)$ for SAW obeys the bound

$$\begin{aligned} |\Pi_p(x) - \delta_{o,x}| &\le \sum_{n=1}^\infty \big(pD(o)\delta + |\pi_p|\big)^{*n}(x) \\ &\le \underbrace{\sum_{n=1}^\infty O(L^{-d})^n}_{O(L^{-d})} \delta_{o,x} \\ &\quad + \sum_{n=1}^\infty \sum_{j=1}^n \binom{n}{j} O(L^{-d})^{n-j} O(\lambda^3)^j \underbrace{\frac{O(L^{-2-2(d-4)})^{j-1} |\!|\!|x|\!|\!|_L^{3(2-d)}}{(\log |\!|\!|\frac{x}{L}|\!|\!|_1)^3}}_{\because\,\text{Lemma 3.5}} \\ &\le O(L^{-d})\delta_{o,x} + O(\lambda^3)\frac{|\!|\!|x|\!|\!|_L^{3(2-d)}}{(\log |\!|\!|\frac{x}{L}|\!|\!|_1)^3} \underbrace{\sum_{n=1}^\infty n\Big(O(L^{-d}) + \underbrace{O(\lambda^3 L^{-2-2(d-4)})}_{O(L^{-2d})}\Big)^{n-1}}_{O(1)}. \end{aligned} \tag{3.27}$$

Similarly, by repeated applications of (3.23), (3.25) and Lemma 3.5, $\varPi_p(x)$ for the Ising model and percolation obeys the bound

$$\begin{aligned} |\varPi_p(x) - \delta_{o,x}| &\le |\pi_p(x) - \delta_{o,x}| + \Bigg(|\pi_p| * \underbrace{\sum_{n=1}^{\infty} \big(pD(o)|\pi_p|\big)^{*n}}_{\le \text{ RHS of (3.27)}}\Bigg)(x) \\ &\le O(L^{-d})\delta_{o,x} + O(\lambda^2)\frac{|||x|||_L^{3(2-d)}}{(\log |||\frac{x}{L}|||_1)^3}. \end{aligned} \tag{3.28}$$

By weakening the $O(\lambda^3)$ term in the right-most expression of (3.27) to $O(\lambda^2)$, $\varPi_p(x)$ for all three models enjoys the unified bound

$$|\varPi_p(x) - \delta_{o,x}| \le O(L^{-d})\delta_{o,x} + O(\lambda^2)\frac{|||x|||_L^{\ell(2-d)}}{(\log |||\frac{x}{L}|||_1)^\ell}. \tag{3.29}$$

As a result,

$$|\hat{\varPi}_p(0) - 1| \le O(L^{-d}) + O(\lambda^2)\underbrace{\sum_x \frac{|||x|||_L^{\ell(2-d)}}{(\log |||\frac{x}{L}|||_1)^\ell}}_{O(L^{-2-(\ell-1)(d-d_c)})} = O(L^{-d}), \tag{3.30}$$

and

$$|\hat{\varPi}_p(0) - \hat{\varPi}_p(k)| \le O(\lambda^2)|k|^2 \underbrace{\sum_x |x|^2 \frac{|||x|||_L^{\ell(2-d)}}{(\log |||\frac{x}{L}|||_1)^\ell}}_{O(L^{-(\ell-1)(d-d_c)})} \le O(\lambda^2)|k|^2. \tag{3.31}$$

Now we are back to the proof of (iii). First, by summing both sides of (1.2) over x and solve the resulting equation for χ_p, we have

$$\chi_p = \hat{\varPi}_p(0) + \hat{\varPi}_p(0)p\chi_p = \frac{\hat{\varPi}_p(0)}{1 - p\hat{\varPi}_p(0)}. \tag{3.32}$$

Since $\chi_p < \infty$ (because $p < p_c$) and $\hat{\varPi}_p(0) = 1 + O(L^{-d}) > 0$ for large L, we obtain

$$p\hat{\varPi}_p(0) \in (0, 1), \tag{3.33}$$

which implies $p < \hat{\varPi}_p(0)^{-1} = 1 + O(L^{-d}) \le 2$, as required.

Next, we investigate $G_p(x)$. By repeated applications of (1.2) for N times, we have

$$\begin{aligned} G_p(x) &= \varPi_p(x) + (\varPi_p * pD * G_p)(x) \\ &= \varPi_p(x) + (\varPi_p * pD * \varPi_p)(x) + \big((\varPi_p * pD)^{*2} * G_p\big)(x) \\ &\vdots \\ &= \Bigg(\varPi_p * \sum_{n=0}^{N-1} (pD * \varPi_p)^{*n}\Bigg)(x) + \big((\varPi_p * pD)^{*N} * G_p\big)(x). \end{aligned} \tag{3.34}$$

Notice that, by (3.26), (3.29) and Lemma 3.5, there are finite constants C, C', C'' such that

$$(\Pi_p * D)(x) \geq (1 - CL^{-d})D(x) - C'\lambda^2 \sum_y \frac{\|\!|y|\!\|_L^{\ell(2-d)}}{(\log \|\!|\frac{y}{L}|\!\|_1)^\ell} D(x-y)$$
$$\geq (1 - CL^{-d} - C''\lambda^3)D(x), \tag{3.35}$$

which is positive for all x, if L is large enough (see Remark 3.7 below). Therefore,

$$\mathcal{D}(x) = \frac{(\Pi_p * D)(x)}{\hat{\Pi}_p(0)} \tag{3.36}$$

is a probability distribution that satisfies Assumptions 1.1–1.2 (see computations below). By this observation, we can take the limit

$$0 \leq \big((\Pi_p * pD)^{*N} * G_p\big)(x) = \underbrace{\big(p\hat{\Pi}_p(0)\big)^N}_{\in(0,1)} \underbrace{(\mathcal{D}^{*N} * G_p)(x)}_{\leq \chi_p} \xrightarrow[N\to\infty]{} 0, \tag{3.37}$$

so that

$$G_p(x) = \bigg(\Pi_p * \sum_{n=0}^{\infty} \big(p\hat{\Pi}_p(0)\big)^n \mathcal{D}^{*n}\bigg)(x) = (\Pi_p * \mathcal{S}_{p\hat{\Pi}_p(0)})(x), \tag{3.38}$$

where $\mathcal{S}_q$ is the random-walk Green function generated by the 1-step distribution $\mathcal{D}$ with fugacity $q \in [0, 1]$, for which (1.19) holds. By (3.29) and Lemma 3.5, we obtain that, for $x \neq o$,

$$G_p(x) \leq \big(1 + O(L^{-d})\big)\mathcal{S}_1(x) + \underbrace{\sum_{y(\neq o)} \frac{O(\lambda^2)\|\!|y|\!\|_L^{\ell(2-d)}}{(\log \|\!|\frac{y}{L}|\!\|_1)^\ell} \bigg(\delta_{y,x} + \frac{O(\lambda)\|\!|x-y|\!\|_L^{2-d}}{\log \|\!|\frac{x-y}{L}|\!\|_1}\bigg)}_{O(\lambda^4)\|\!|x|\!\|_L^{2-d}/\log \|\!|\frac{x}{L}|\!\|_1}. \tag{3.39}$$

Suppose $\mathcal{S}_1(x) \leq (1 + O(\lambda^3))S_1(x)$ holds for all x. Then, for $x \neq o$,

$$G_p(x) \leq \big(1 + O(\lambda^3)\big)S_1(x) + O(\lambda^4)\frac{\|\!|x|\!\|_L^{2-d}}{\log \|\!|\frac{x}{L}|\!\|_1} \overset{L\gg 1}{\leq} 2\lambda \frac{\|\!|x|\!\|_L^{2-d}}{\log \|\!|\frac{x}{L}|\!\|_1}, \tag{3.40}$$

as required.

It remains to show $\mathcal{S}_1(x) \leq (1 + O(\lambda^3))S_1(x)$ for all x. This is not so hard to verify, as explained now. First, by (3.35) and its opposite inequality with all negative signs replaced by positive signs,

$$\bigg|\frac{\mathcal{D}(x)}{D(x)} - 1\bigg| = O(\lambda^3). \tag{3.41}$$

Also, by (3.30)–(3.31) and (1.10),

$$\frac{1 - \hat{\mathcal{D}}(k)}{1 - \hat{D}(k)} \overset{(3.36)}{=} 1 + \underbrace{\frac{\hat{D}(k)}{\hat{\Pi}_p(0)}}_{1+O(L^{-d})} \underbrace{\frac{\hat{\Pi}_p(0) - \hat{\Pi}_p(k)}{1 - \hat{D}(k)}}_{O(\lambda^3)} = 1 + O(\lambda^3). \tag{3.42}$$

Similarly,

$$\frac{1-\hat{\mathcal{D}}(k)}{|k|^2\log(1/|k|)} = \underbrace{\frac{1-\hat{D}(k)}{|k|^2\log(1/|k|)}}_{\to v_2,\ \because (1.4)} + \underbrace{\frac{\hat{D}(k)}{\hat{\Pi}_p(0)}\frac{\hat{\Pi}_p(0)-\hat{\Pi}_p(k)}{|k|^2\log(1/|k|)}}_{\to 0,\ \because (3.31)} \xrightarrow[|k|\to 0]{} v_2. \tag{3.43}$$

Therefore, for L large enough, $\mathcal{D}$ satisfies all (1.9)–(1.12) with the same constants as D (modulo $O(\lambda^3)$ terms). Similar analysis can be applied to show that $\mathcal{D}$ also satisfies (1.13) with the same constant as D. As a result, we can get $\mathcal{S}_1(x) \le (1+O(\lambda^3))S_1(x)$ for all x. This completes the proof of (iii), hence the proof of the infrared bound (1.35). □

Remark 3.7. The above proof works as long as $\alpha \le 2 + (\ell-1)(d-d_{\mathrm{c}})$ (cf., (3.35)), then we can define the probability distribution (3.36) by taking L sufficiently large. For short-range models investigated in [15,16,26], on the other hand, since α is regarded as an arbitrarily large number, there is no way for (3.35) to be nonnegative for every x. In this case, we may have to introduce a quite delicate function $E_{p,q,r}(x)$ as in [13,16] that is required to satisfy some symmetry conditions. Since we do not need such a function for all $\alpha \le 2$ and $d \ge d_{\mathrm{c}}$, the analysis explained in this subsection is much easier and more transparent than the previous one in [13,16]. This is also related to the reason why the multiplicative constant A in the asymptotic expression (1.41) becomes 1 for $\alpha \le 2$.

3.3. Convolution bounds on power functions with log corrections. In this subsection, we prove Lemma 3.5. First, we rewrite the sum in (3.18) as

$$\begin{aligned}
\sum_{y\in\mathbb{Z}^d} \frac{|||x-y|||_L^{-a_1}}{(\log|||\frac{x-y}{L}|||_1)^{a_2}} \frac{|||y|||_L^{-b_1}}{(\log|||\frac{y}{L}|||_1)^{b_2}} &= \sum_{y:|x-y|\le|y|} \frac{|||x-y|||_L^{-a_1}}{(\log|||\frac{x-y}{L}|||_1)^{a_2}} \frac{|||y|||_L^{-b_1}}{(\log|||\frac{y}{L}|||_1)^{b_2}} \\
&\quad + \sum_{y:|x-y|>|y|} \frac{|||x-y|||_L^{-a_1}}{(\log|||\frac{x-y}{L}|||_1)^{a_2}} \frac{|||y|||_L^{-b_1}}{(\log|||\frac{y}{L}|||_1)^{b_2}} \\
&= \sum_{y:|x-y|\le|y|} \bigg(\frac{|||x-y|||_L^{-a_1}}{(\log|||\frac{x-y}{L}|||_1)^{a_2}} \frac{|||y|||_L^{-b_1}}{(\log|||\frac{y}{L}|||_1)^{b_2}} \\
&\quad + \frac{|||x-y|||_L^{-b_1}}{(\log|||\frac{x-y}{L}|||_1)^{b_2}} \frac{|||y|||_L^{-a_1}}{(\log|||\frac{y}{L}|||_1)^{a_2}} \bigg).
\end{aligned} \tag{3.44}$$

Notice that the ratio of the second term to the first term in the parentheses, which is

$$\frac{\frac{|||x-y|||_L^{-b_1}}{(\log|||\frac{x-y}{L}|||_1)^{b_2}} \frac{|||y|||_L^{-a_1}}{(\log|||\frac{y}{L}|||_1)^{a_2}}}{\frac{|||x-y|||_L^{-a_1}}{(\log|||\frac{x-y}{L}|||_1)^{a_2}} \frac{|||y|||_L^{-b_1}}{(\log|||\frac{y}{L}|||_1)^{b_2}}} = \bigg(\frac{|||x-y|||_L}{|||y|||_L}\bigg)^{a_1-b_1} \bigg(\frac{\log|||\frac{x-y}{L}|||_1}{\log|||\frac{y}{L}|||_1}\bigg)^{a_2-b_2}, \tag{3.45}$$

is bounded above by an L-independent constant $C \in [1, \infty)$ as long as $a_1 > b_1$, or $a_1 = b_1$ and $a_2 \geq b_2$. Therefore,

$$\sum_{y \in \mathbb{Z}^d} \frac{|||x-y|||_L^{-a_1}}{(\log |||\frac{x-y}{L}|||_1)^{a_2}} \frac{|||y|||_L^{-b_1}}{(\log |||\frac{y}{L}|||_1)^{b_2}} \leq 2C \sum_{y:|x-y| \leq |y|} \frac{|||x-y|||_L^{-a_1}}{(\log |||\frac{x-y}{L}|||_1)^{a_2}} \frac{|||y|||_L^{-b_1}}{(\log |||\frac{y}{L}|||_1)^{b_2}}. \tag{3.46}$$

Now we consider the following cases separately: (a) $a_1 > d$, (b) $a_1 = d$, (c) $a_1 < d$ and $a_1 + b_1 \geq d$.

(a) Let $a_1 > d$. Since $|x-y| \leq |y|$ implies $|y| \geq \frac{1}{2}|x|$, and since

$$|||\tfrac{x}{2}|||_L \geq \frac{1}{2}|||x|||_L, \qquad \log |||\tfrac{x}{2L}|||_1 \geq \frac{\log \frac{\pi}{2}}{\log \pi} \log |||\tfrac{x}{L}|||_1, \tag{3.47}$$

we obtain

$$\sum_{y:|x-y| \leq |y|} \frac{|||x-y|||_L^{-a_1}}{(\log |||\frac{x-y}{L}|||_1)^{a_2}} \frac{|||y|||_L^{-b_1}}{(\log |||\frac{y}{L}|||_1)^{b_2}} \leq \frac{O(1)|||x|||_L^{-b_1}}{(\log |||\frac{x}{L}|||_1)^{b_2}} \underbrace{\sum_{y \in \mathbb{Z}^d} \frac{|||x-y|||_L^{-a_1}}{(\log |||\frac{x-y}{L}|||_1)^{a_2}}}_{O(L^{d-a_1})}. \tag{3.48}$$

(b) Let $a_1 = d$. First we split the sum as

$$\sum_{y:|x-y| \leq |y|} = \sum_{\substack{y:|x-y| \leq |y| \\ (|y| \leq \frac{3}{2}|x|)}} + \sum_{\substack{y:|x-y| \leq |y| \\ (|y| > \frac{3}{2}|x|)}}. \tag{3.49}$$

For the first sum, since $|x-y| \leq |y|$ implies $|y| \geq \frac{1}{2}|x|$ (so that (3.47) holds), and since

$$\log |||\tfrac{3x}{2L}|||_1 \leq \frac{\log \frac{3\pi}{4}}{\log \frac{\pi}{2}} \log |||\tfrac{x}{L}|||_1, \qquad \log \log |||\tfrac{3x}{2L}|||_1 \leq \frac{\log \log \frac{3\pi}{4}}{\log \log \frac{\pi}{2}} \log \log |||\tfrac{x}{L}|||_1, \tag{3.50}$$

we obtain

$$\begin{aligned} &\sum_{\substack{y:|x-y| \leq |y| \\ (|y| \leq \frac{3}{2}|x|)}} \frac{|||x-y|||_L^{-d}}{(\log |||\frac{x-y}{L}|||_1)^{a_2}} \frac{|||y|||_L^{-b_1}}{(\log |||\frac{y}{L}|||_1)^{b_2}} \\ &\leq \frac{O(1)|||x|||_L^{-b_1}}{(\log |||\frac{x}{L}|||_1)^{b_2}} \sum_{y:|x-y| \leq \frac{3}{2}|x|} \frac{|||x-y|||_L^{-d}}{(\log |||\frac{x-y}{L}|||_1)^{a_2}} \\ &\leq \frac{O(1)|||x|||_L^{-b_1}}{(\log |||\frac{x}{L}|||_1)^{b_2}} \times \begin{cases} 1 & [a_2 > 1], \\ \log \log |||\frac{x}{L}|||_1 & [a_2 = 1], \\ (\log |||\frac{x}{L}|||_1)^{1-a_2} & [a_2 < 1]. \end{cases} \end{aligned} \tag{3.51}$$

For the second sum in (3.49), since $|y| > \frac{3}{2}|x|$ implies $|x-y| \ge \frac{1}{3}|y|$, and since

$$\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \tfrac{y}{3} \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_L \ge \frac{1}{3} |||y|||_L, \qquad \log \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \tfrac{y}{3L} \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_1 \ge \frac{\log \frac{\pi}{2}}{\log \frac{3\pi}{2}} \log \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \tfrac{y}{L} \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_1, \tag{3.52}$$

we obtain

$$\sum_{\substack{y:|x-y|\le|y| \\ (|y|>\frac{3}{2}|x|)}} \frac{|||x-y|||_L^{-d}}{(\log |||\frac{x-y}{L}|||_1)^{a_2}} \frac{|||y|||_L^{-b_1}}{(\log |||\frac{y}{L}|||_1)^{b_2}} \le \frac{O(1)}{(\log |||\frac{x}{L}|||_1)^{a_2+b_2}} \underbrace{\sum_{y:|y|>\frac{3}{2}|x|} |||y|||_L^{-d-b_1}}_{O(1)|||x|||_L^{-b_1}}, \tag{3.53}$$

which is smaller than (3.51).

(c) Let $a_1 < d$ and $a_1 + b_1 \ge d$. Similarly to the case (b), we split the sum as in (3.49) and evaluate each sum by using (3.47) and (3.52). Then, by discarding the log-dumping term $(\log |||\frac{x-y}{L}|||_1)^{-a_2}$, the first sum in (3.49) is bounded as

$$\sum_{\substack{y:|x-y|\le|y| \\ (|y|\le\frac{3}{2}|x|)}} \frac{|||x-y|||_L^{-a_1}}{(\log |||\frac{x-y}{L}|||_1)^{a_2}} \frac{|||y|||_L^{-b_1}}{(\log |||\frac{y}{L}|||_1)^{b_2}} \le \frac{O(1)|||x|||_L^{-b_1}}{(\log |||\frac{x}{L}|||_1)^{b_2}} \underbrace{\sum_{y:|x-y|\le\frac{3}{2}|x|} |||x-y|||_L^{-a_1}}_{O(1)|||x|||_L^{d-a_1}} \tag{3.54}$$

while the second sum in (3.49) is bounded as

$$\begin{aligned} \sum_{\substack{y:|x-y|\le|y| \\ (|y|>\frac{3}{2}|x|)}} \frac{|||x-y|||_L^{-a_1}}{(\log |||\frac{x-y}{L}|||_1)^{a_2}} \frac{|||y|||_L^{-b_1}}{(\log |||\frac{y}{L}|||_1)^{b_2}} &\overset{\because |x-y|\ge\frac{1}{3}|y|}{\le} O(1) \sum_{y:|y|>\frac{3}{2}|x|} \frac{|||y|||_L^{-a_1-b_1}}{(\log |||\frac{y}{L}|||_1)^{a_2+b_2}} \\ &\le \frac{O(1)|||x|||_L^{d-a_1-b_1}}{(\log |||\frac{x}{L}|||_1)^{a_2+b_2}} \times \begin{cases} 1 & [a_1+b_1>d], \\ \log |||\frac{x}{L}|||_1 & [a_1+b_1=d,\ a_2+b_2>1], \end{cases} \end{aligned} \tag{3.55}$$

which is smaller (resp., larger) than (3.54) if $a_2 > 1$ (resp., $a_2 < 1$). This completes the proof of Lemma 3.5. □

3.4. Bounds on the lace-expansion coefficients. In this subsection, we prove Lemma 3.6. Suppose that $g_p \le 3$ and $p < p_c$. Since $G_p(y) = \delta_{o,y} + G_p(y)\mathbb{1}_{\{y\ne o\}}$ for all three models, we have

$$(D * G_p)(x) = D(x) + \sum_{y\ne o} D(x-y)\, G_p(y). \tag{3.56}$$

The first term is easy, because

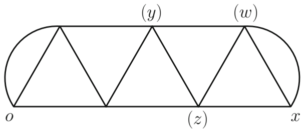
$$D(x) = \frac{O(L^2)}{|||x|||_L^{d+2}} = O(\lambda) \frac{\log |||\frac{x}{L}|||_1}{|||\frac{x}{L}|||_1^4} \frac{|||x|||_L^{2-d}}{\log |||\frac{x}{L}|||_1} \le O(\lambda) \frac{|||x|||_L^{2-d}}{\log |||\frac{x}{L}|||_1}. \tag{3.57}$$

For the second term in (3.56), we use $g_p \leq 3$ and Lemma 3.5 as

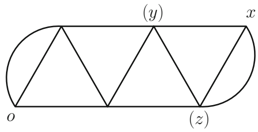
$$\sum_{y \neq o} D(x-y)\, G_p(y) \leq \sum_{y \in \mathbb{Z}^d} \frac{O(L^2)}{|||x-y|||_L^{d+2}} \frac{3\lambda |||y|||_L^{2-d}}{\log |||\frac{y}{L}|||_1} \leq O(\lambda) \frac{|||x|||_L^{2-d}}{\log |||\frac{x}{L}|||_1}. \tag{3.58}$$

This completes the proof of (3.20).

To prove (3.21)–(3.22), we repeatedly apply Lemma 3.5 to the diagrammatic bounds on $\pi_p(x)$ in [16,26]. For example, the lace-expansion diagram in Figure 1 for SAW and the Ising model can be bounded as follows. Suppose for now that each line segment, say, from x to y, represents $3\lambda |||x-y|||_L^{2-d} / \log |||\frac{x-y}{L}|||_1$, i.e., the assumed bound on the nonzero two-point function. Then, by using Lemma 3.5 (to perform the sum over w), we can show that, for $d \geq 4$,



$$\leq 4^{d-1} \left(3\lambda \frac{\log \pi}{\log \frac{\pi}{2}}\right)^2 C$$

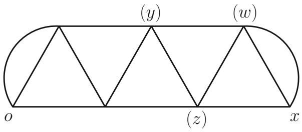


, (3.59)

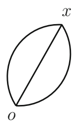
where the indicies in the parentheses are summed over $\mathbb{Z}^d$. This is due to the following computation: for $d \geq 4$,

$$\begin{aligned}
&\sum_w \frac{3\lambda |||y-w|||_L^{2-d}}{\log |||\frac{y-w}{L}|||_1} \frac{3\lambda |||z-w|||_L^{2-d}}{\log |||\frac{z-w}{L}|||_1} \left(\frac{3\lambda |||w-x|||_L^{2-d}}{\log |||\frac{w-x}{L}|||_1}\right)^2 \\
&\quad \times \left(\mathbb{1}_{\{|y-w| \leq |w-x|\}} + \mathbb{1}_{\{|y-w| \geq |w-x|\}}\right)\left(\mathbb{1}_{\{|z-w| \leq |w-x|\}} + \mathbb{1}_{\{|z-w| \geq |w-x|\}}\right) \\
&\leq \sum_w \frac{3\lambda |||y-w|||_L^{2-d}}{\log |||\frac{y-w}{L}|||_1} \frac{3\lambda |||z-w|||_L^{2-d}}{\log |||\frac{z-w}{L}|||_1} \\
&\quad \times \left(\frac{3\lambda |||w-x|||_L^{2-d}}{\log |||\frac{w-x}{L}|||_1}\right)^2 \underbrace{\mathbb{1}_{\{|y-w| \leq |w-x|\}}}_{\Rightarrow |w-x| \geq \frac{1}{2}|y-x|} \underbrace{\mathbb{1}_{\{|z-w| \leq |w-x|\}}}_{\Rightarrow |w-x| \geq \frac{1}{2}|z-x|} \\
&\quad + [3 \text{ other cases}] \\
&\leq (3\lambda)^2 \frac{3\lambda |||\frac{y-x}{2}|||_L^{2-d}}{\log |||\frac{y-x}{2L}|||_1} \frac{3\lambda |||\frac{z-x}{2}|||_L^{2-d}}{\log |||\frac{z-x}{2L}|||_1} \\
&\quad \times \underbrace{\left(\underbrace{\sum_w \frac{|||y-w|||_L^{2-d}}{\log |||\frac{y-w}{L}|||_1} \frac{|||z-w|||_L^{2-d}}{\log |||\frac{z-w}{L}|||_1}}_{\leq C |||y-z|||_L^{2-d} / \log |||\frac{y-z}{L}|||_1} + [3 \text{ other cases}]\right)}_{\leq 4C} \\
&\leq 4^{d-1} \left(3\lambda \frac{\log \pi}{\log \frac{\pi}{2}}\right)^2 C \frac{3\lambda |||y-x|||_L^{2-d}}{\log |||\frac{y-x}{L}|||_1} \frac{3\lambda |||z-x|||_L^{2-d}}{\log |||\frac{z-x}{L}|||_1}.
\end{aligned} \tag{3.60}$$

By repeated application of the above inequality, we will end up with



$$\leq \left(4^{d-1}\left(3\lambda \frac{\log \pi}{\log \frac{\pi}{2}}\right)^2 C\right)^5$$



$$\leq \underbrace{\left(4^{d-1}\left(3\lambda \frac{\log \pi}{\log \frac{\pi}{2}}\right)^2 C\right)^5 (3\lambda)^3}_{O(\lambda)^{13}} \frac{|||x|||_L^{3(2-d)}}{(\log |||\frac{x}{L}|||_1)^3}, \tag{3.61}$$

which is smaller than (3.21)–(3.22), by a factor $O(\lambda)^5$ for SAW, in particular. This is because, in fact, not every line segment is nonzero. The situation for the Ising model and percolation is harder, because most of the line segments can be zero-length, which do not have small factors of λ. However, the convolution $pD * G_p$ shows up repeatedly, which has a small factor of λ, as in (3.20). This also provides a bound on the main contribution from $\pi_p^{(1)}(x)$, as

$$(pD * G_p)(o)\delta_{o,x} \leq 3\left(D(o) + \sum_{y \neq o} \frac{O(L^2)}{|||y|||_L^{d+2}} \frac{3\lambda |||y|||_L^{2-d}}{\log |||\frac{y}{L}|||_1}\right)\delta_{o,x} = O(L^{-d})\delta_{o,x}. \tag{3.62}$$

This completes the sketch proof of Lemma 3.6. □

3.5. Proof of the asymptotic behavior (1.36). First we recall (3.38). Since $\chi_p = \hat{\Pi}_p(0)/(1 - p\hat{\Pi}_p(0))$ diverges as $p \uparrow p_c$, while $\hat{\Pi}_p(0) = 1 + O(L^{-d})$ uniformly in $p < p_c$, we have $p_c\hat{\Pi}_{p_c}(0) = 1$. Therefore,

$$G_{p_c}(x) = \Pi_{p_c} * \mathcal{S}_1(x) = \underbrace{\hat{\Pi}_{p_c}(0)}_{1/p_c} \mathcal{S}_1(x) + \sum_{y \neq o} \Pi_{p_c}(y)\big(\mathcal{S}_1(x-y) - \mathcal{S}_1(x)\big). \tag{3.63}$$

The asymptotic expression of $\mathcal{S}_1(x)$ is the same as that of $S_1(x)$. This can be shown by following the proof of (1.20) and using the limit (3.43).

To investigate the error term in (3.63), we first split the sum as

$$\sum_{y \neq o} = \sum_{y:0<|y|\leq\frac{1}{3}|x|} + \sum_{y:|x-y|\leq\frac{1}{3}|x|} + \sum_{y:|y|\wedge|x-y|>\frac{1}{3}|x|} \equiv \sum_y{}' + \sum_y{}'' + \sum_y{}'''. \tag{3.64}$$

For $\sum_y''$, since $|x-y| \le \frac{1}{3}|x|$ implies $\frac{2}{3}|x| \le |y|$ (so that a similar inequality to (3.47) or (3.52) holds), we have that, for large $|x|$,

$$\begin{aligned}
&\Big|\sum_y{}'' \Pi_{p_c}(y)\big(\mathcal{S}_1(x-y)-\mathcal{S}_1(x)\big)\Big| \\
&\quad \overset{(3.29)}{\le} \frac{O(\lambda^2)|x|^{\ell(2-d)}}{(\log|x|)^\ell} \underbrace{\sum_{y:|x-y|\le\frac{1}{3}|x|} \big(\mathcal{S}_1(x-y)+\mathcal{S}_1(x)\big)}_{O(\lambda)|x|^2/\log|x|} \\
&\quad = O(\lambda^3)\frac{|x|^{-d-(\ell-1)(d-d_c)}}{(\log|x|)^{\ell+1}}. \qquad (3.65)
\end{aligned}$$

Similarly, for $\sum_y'''$ for large $|x|$,

$$\begin{aligned}
\Big|\sum_y{}''' \Pi_{p_c}(y)\big(\mathcal{S}_1(x-y)-\mathcal{S}_1(x)\big)\Big| &\overset{(3.29)}{\le} \frac{O(\lambda)|x|^{2-d}}{\log|x|} \underbrace{\sum_{y:|y|>\frac{1}{3}|x|} \frac{O(\lambda^2)|y|^{\ell(2-d)}}{(\log|y|)^\ell}}_{O(\lambda^2)|x|^{-2-(\ell-1)(d-d_c)}/(\log|x|)^\ell} \\
&= O(\lambda^3)\frac{|x|^{-d-(\ell-1)(d-d_c)}}{(\log|x|)^{\ell+1}}. \qquad (3.66)
\end{aligned}$$

It remains to investigate $\sum_y'$. For that, we first use (1.20) and the $\mathbb{Z}^d$-symmetry of Π_{p_c} to obtain that, for large $|x|$,

$$\begin{aligned}
&\Big|\sum_y{}' \Pi_{p_c}(y)\big(\mathcal{S}_1(x-y)-\mathcal{S}_1(x)\big)\Big| \\
&\quad \le \underbrace{\frac{\gamma_2}{v_2}}_{O(\lambda)} \sum_y{}' |\Pi_{p_c}(y)| \left|\frac{1}{2}\left(\frac{|x+y|^{2-d}}{\log|x+y|}+\frac{|x-y|^{2-d}}{\log|x-y|}\right)-\frac{|x|^{2-d}}{\log|x|}\right| \\
&\qquad + \underbrace{\sum_{y:0<|y|\le\frac{1}{3}|x|} |\Pi_{p_c}(y)|}_{O(\lambda^3)} \frac{O(\lambda)|x|^{2-d}}{(\log|x|)^{1+\epsilon}}. \qquad (3.67)
\end{aligned}$$

Then, by Taylor's theorem,

$$\begin{aligned}
|x\pm y|^{2-d} &= |x|^{2-d}\left(1\pm 2\frac{x\cdot y}{|x|^2}+\frac{O(|y|^2)}{|x|^2}\right)^{(2-d)/2} \\
&= |x|^{2-d}\left(1\pm(2-d)\frac{x\cdot y}{|x|^2}+\frac{O(|y|^2)}{|x|^2}\right), \qquad (3.68)
\end{aligned}$$

$$\log|x\pm y| = \log|x| + \log\frac{|x\pm y|}{|x|} = \log|x| \pm \frac{x\cdot y}{|x|^2}+\frac{O(|y|^2)}{|x|^2}, \qquad (3.69)$$

which implies

$$\left|\frac{1}{2}\left(\frac{|x+y|^{2-d}}{\log|x+y|}+\frac{|x-y|^{2-d}}{\log|x-y|}\right)-\frac{|x|^{2-d}}{\log|x|}\right|\leq\frac{O(|y|^2)|x|^{-d}}{\log|x|}. \tag{3.70}$$

Therefore, the first term on the right-hand side of (3.67) is bounded by

$$\frac{O(\lambda)|x|^{-d}}{\log|x|}\underbrace{\sum_{y:0<|y|\leq\frac{1}{3}|x|}\frac{O(\lambda^2)|||y|||_L^{-d-(\ell-1)(d-d_c)}}{(\log|||\frac{y}{L}|||_1)^\ell}}_{O(\lambda^2)}=O(\lambda^3)\frac{|x|^{-d}}{\log|x|}. \tag{3.71}$$

This completes the proof of Theorem 1.4. □

Acknowledgments. The work of AS was supported by JSPS KAKENHI Grant Number 18K03406. The work of LCC was supported by the Grant MOST 107-2115-M-004-004-MY2. We are grateful to the Institute of Mathematics and Mathematics Research Promotion Center (MRPC) of Academia Sinica, as well as the National Center for Theoretical Sciences (NCTS) at National Taiwan University, for providing us with comfortable working environment in multiple occasions. We would also like to thank the referees for their comments to improve presentation of this paper.

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Communicated by H. Spohn