Commun. Math. Phys. 366, 1025–1067 (2019)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-019-03328-4

Communications in **Mathematical Physics**

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Realizations of Simple Affine Vertex Algebras and Their Modules: The Cases $\widehat{sl(2)}$ and $\widehat{osp(1,2)}$

Dražen Adamović

Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička 30, 10 000 Zagreb, Croatia.
E-mail: adamovic@math.hr

Received: 5 February 2018 / Accepted: 25 October 2018
Published online: 2 February 2019 –

Abstract: We study the embeddings of the simple admissible affine vertex algebras $V_k(sl(2))$ and $V_k(osp(1,2))$, $k \notin \mathbb{Z}_{\geq 0}$, into the tensor product of rational Virasoro and $N = 1$ Neveu–Schwarz vertex algebra with lattice vertex algebras. By using these realizations we construct a family of weight, logarithmic, and Whittaker $\widehat{sl(2)}$ and $\widehat{osp(1,2)}$-modules. As an application, we construct all irreducible degenerate Whittaker modules for $V_k(sl(2))$.

Contents

1. Introduction

Let $V^k(\mathfrak{g})$ denotes the universal affine vertex algebra of level k associated to a simple finite-dimensional Lie super algebra $\mathfrak{g}$. Let $J^k(\mathfrak{g})$ be the maximal ideal in $V^k(\mathfrak{g})$ and $V_k(\mathfrak{g}) = V^k(\mathfrak{g})/J^k(\mathfrak{g})$ its simple quotient. The representation theory of $V_k(\mathfrak{g})$ depends on the structure of the maximal ideal $J^k(\mathfrak{g})$. One sees that a $V^k(\mathfrak{g})$-module M is a module for the simple vertex algebra $V_k(\mathfrak{g})$ if and only if $J^k(\mathfrak{g}).M = 0$. Such approach can be applied for a construction and classification of modules in the category $\mathcal{O}$ and in the category of weight modules (cf. [10,24,25,32,57,59]). But it seems that for a construction of logarithmic, indecomposable and Whittaker modules one needs different methods.

In this paper, we explore the possibility that a simple affine vertex algebra can be realized as a vertex subalgebra of the tensor product:

$$V_k(\mathfrak{g}) \subset W(\mathfrak{g}) \otimes \Pi_{\mathfrak{g}}(0) \tag{1}$$

where $W(\mathfrak{g})$ is a $\mathcal{W}$-algebra associated to $\mathfrak{g}$ and $\Pi_{\mathfrak{g}}(0)$ is a lattice type vertex algebra. This can be treated as an inverse of the quantum Hamiltonian reduction (cf. [62]).

In this moment we cannot prove that such inclusion exists in general, but we present a proof of (1) in the cases $\mathfrak{g} = sl(2)$ and $\mathfrak{g} = osp(1, 2)$. Let us describe our results in more detail. Let $V^{Vir}(d_{p,p'}, 0)$ and $V^{ns}(c_{p,q}, 0)$ denote the universal Virasoro and N $= 1$ Neveu–Schwarz vertex algebras with central charges: $d_{p,p'} = 1 - \frac{6(p-p')^2}{pp'}$ and $c_{p,q} = 3/2 - \frac{3(p-q)^2}{pq}$. Their simple quotients are denoted by $L^{Vir}(d_{p,p'}, 0)$ and $L^{ns}(c_{p,q}, 0)$. Let $\Pi(0) = M(1) \otimes \mathbb{C}[\mathbb{Z}c]$ and $\Pi^{1/2}(0) = M(1) \otimes \mathbb{C}[\mathbb{Z}\frac{c}{2}]$ be the vertex algebras of lattice type associated to the lattice of $L = \mathbb{Z}c + \mathbb{Z}d$, with products

$$\langle c, c\rangle = \langle d, d\rangle = 0, \quad \langle c, d\rangle = 2.$$

Let F be the fermionic vertex algebra of central charge $c = 1/2$ associated to a neutral fermion field.

We prove:

Theorem 1.1. *There are non-trivial homomorphisms of simple admissible affine vertex algebras:*

1. $\Phi_1 : V_k(sl(2)) \to L^{Vir}(d_{p,p'}, 0) \otimes \Pi(0)$ *where* $k + 2 = \frac{p}{p'}$ *such that* $p, p' \geq 2$, $(p, p') = 1$,
2. $\Phi_2 : V_k(osp(1,2)) \to L^{ns}(c_{p,q}, 0) \otimes F \otimes \Pi^{1/2}(0)$, *where* $k + 3/2 = \frac{p}{2q}$, *such that* $p, q \in \mathbb{Z}$, $p, q \geq 2$, $(\frac{p-q}{2}, q) = 1$.

Let us discuss some application of previous theorem in the case $V_k(sl(2))$:

- We show in Sect. 7 that all relaxed highest weight modules for the admissible vertex algebra $V_k(sl(2))$ have the form

$$L^{Vir}(d_{p,p'}, h) \otimes \Pi_{(-1)}(\lambda)$$

  where $L^{Vir}(d_{p,p'}, h)$ is an irreducible $L^{Vir}(d_{p,p'}, 0)$-module and $\Pi_{(-1)}(\lambda)$ is a weight $\Pi(0)$-module. These modules were first detected in [10] by using the theory of Zhu's algebras. We also show that the character of $L^{Vir}(d_{p,p'}, h) \otimes \Pi_{-1}(\lambda)$ coincides with the Creutzig–Ridout character formula presented in [32] and proved recently in [45].

We should also say that a similar realization of irreducible relaxed highest weight modules were presented in [5, Section 9] in the case of critical level for $A_1^{(1)}$ and in [6, Corollary 7] in the case of affine Lie algebra $A_2^{(1)}$ at level $k = -3/2$.

- We prove in Sect. 8 that a family of degenerate Whittaker modules for $V_k(sl(2))$ have the form

$$L^{Vir}(d_{p,p'}, h) \otimes \Pi_\lambda$$

  where Π_λ is a Whittaker $\Pi(0)$-modules. This result is the final step in the classification and realization of Whittaker $A_1^{(1)}$-modules (all other Whittaker $A_1^{(1)}$-modules were realized in [9]). But our present result implies that affine admissible vertex algebra $V_k(sl(2))$ admits a family of Whittaker modules. One can expect a similar result in general.
- In Sect. 9 we present a vertex-algebraic construction of logarithmic modules by using the methods from [13] and the expressions for screening operators from [38, Section 5]. We prove that the admissible vertex algebra $V_k(sl(2))$, for arbitrary admissible $k \notin \mathbb{Z}_{\geq 0}$, admits logarithmic modules $\widetilde{\mathcal{M}_{r,s}^{\ell,\pm}}(\lambda)$ of nilpotent rank two (cf. Corollaries 9.6, 9.7). These logarithmic modules were previously constructed only for levels $k = -1/2$ and $k = -4/3$ (cf. [13,42,51,56]).
- We present in Sect. 10 a realization of the simple affine vertex algebra $\mathcal{W}_{k'}(spo(2,3), f_\theta)$ with central charge $c = -3/2$. It is realized on the tensor product of the simple super-triplet vertex algebra $SW(1)$ (introduced by the author and Milas [11]) and a rank one lattice vertex algebra. As a consequence, we give a direct proof that the parafermion vertex algebra $K(sl(2), -\frac{2}{3})$ is a $\mathbb{Z}_2$-orbifold of a super-singlet vertex algebra, also introduced in [11].

We should mention that a different approach based on the extension theory was recently presented in [26].

Some applications in the case $V_k(osp(1,2))$ will be presented in our forthcoming paper [8]. Let us note here that we have the following realization at the critical level. We introduce a vertex algebra V^{ns}_{crit} which is freely generated by G^{crit} and T, such that T is central and the following λ-bracket relation holds:

$$[G^{cri}_{\lambda} G^{cri}] = 2T - \lambda^2.$$

We prove:

Theorem 1.2. *Let $k = -3/2$. There is non-trivial homomorphism of vertex algebras:*

$$\overline{\Phi} : V^k(osp(1,2)) \to V^{ns}_{crit} \otimes F \otimes \Pi^{1/2}(0)$$

such that T is a central element of $V^k(osp(1,2))$.

In Remarks 7, 8 and 12 we discuss how one prove irreducibility of modules of the type

- $L^{Vir}(d_{p,p'}, h) \otimes \Pi_{(-1)}(\lambda)$ for $V_k(sl(2))$,
- $L^{\mathcal{R}}(c_{p,q}, h)^{\pm} \otimes M^{\pm} \otimes \Pi^{1/2}_{(-1)}(\lambda)$ for $V_k(osp(1,2))$,

by using new results on characters of relaxed highest weight modules from [45].

In our forthcoming papers we plan to investigate a higher rank generalizations of the result discussed above.

2. Preliminaries

In the paper, we assume that the reader is familiar with basic concepts in the vertex algebra theory such as modules and intertwining operators. In this section, we recall the definition of the logarithmic modules for vertex operator algebras and construction of logarithmic modules. We also recall how we can extend vertex operator algebra V by its module M and get extended vertex algebra $\mathcal{V} = V \oplus M$, and construct $\mathcal{V}$-modules. This construction is important for the construction of logarithmic modules for affine vertex algebra $V_k(sl(2))$ in Sect. 9.

2.1. Logarithmic modules. Let us recall the definition of the logarithmic module of a vertex operator algebra. More informations on the theory of logarithmic modules for vertex operator algebras can be found in the papers [16,29,43,52,54].

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra, and (M, Y_M) be its weak module. Then the components of the field

$$Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$$

defines on M the structure of a module for the Virasoro algebra.

Definition 2.1. A weak module (M, Y_M) for the vertex operator algebra $(V, Y, 1, \omega)$ is called a logarithmic module if it admits the following decomposition

$$M = \coprod_{r \in \mathbb{C}} M_r, \quad M_r = \{v \in M \mid (L(0) - r)^k = 0 \text{ for some } k \in \mathbb{Z}_{>0}\}.$$

If M is a logarithmic module, we say that it has a nilpotent rank $m \in \mathbb{Z}_{\geq 1}$ if

$$(L(0) - L_{ss}(0))^m = 0, \quad (L(0) - L_{ss}(0))^{m-1} \neq 0,$$

where $L_{ss}(0)$ is the semisimple part of $L(0)$.

Now we shall recall the construction of logarithmic modules from [13]. Let $v \in V$ such that

$$[v_n, v_m] = 0 \quad \forall n, m \in \mathbb{Z}, \tag{2}$$
$$L(n)v = \delta_{n,0} v \quad \forall\, n \in \mathbb{Z}_{\geq 0}, \tag{3}$$

so that v is of conformal weight one.

Define

$$\Delta(v, z) = z^{v_0} \exp\left(\sum_{n=1}^{\infty} \frac{v_n}{-n}(-x)^{-n}\right).$$

The following result was proved in [13, Theorems 2.1 and 2.2].

Theorem 2.2. [13] *Assume that V is a vertex operator algebra and that $v \in V$ satisfies conditions (2) and (3). Let $\overline{V}$ be the vertex subalgebra of V such that $\overline{V} \subseteq Ker_V v_0$.*

1. *Assume that (M, Y_M) is a weak V-module. Define the pair $(\widetilde{M}, \widetilde{Y}_{\widetilde{M}})$ such that*

$$\widetilde{M} = M \quad \textit{as a vector space},$$
$$\widetilde{Y}_{\widetilde{M}}(a, x) = Y_M(\Delta(v, x)a, x) \quad \textit{for } a \in \overline{V}.$$

Then $(\widetilde{M}, \widetilde{Y}_{\widetilde{M}})$ is a weak $\overline{V}$-module.

2. *Assume that (M, Y_M) is a V-module such that $L(0)$ acts semisimply on M. Then $(\widetilde{M}, \widetilde{Y}_{\widetilde{M}})$ is logarithmic $\overline{V}$-module if and only if v_0 does not act semisimply on M. On $\widetilde{M}$ we have:*

$$\widetilde{L(0)} = L(0) + v_0.$$

2.2. Extended vertex algebra $\mathcal{V} = V \oplus M$. Let $(V, Y_V, \mathbf{1}, \omega)$ be a vertex operator algebra and (M, Y_M) a V-module having integral weights with respect to $L(0)$, where $L(n) = \omega_{n+1}$. Let $\mathcal{V} = V \oplus M$. Define

$$Y_{\mathcal{V}}(v_1 + w_1, z)(v_2 + w_2) = Y_V(v_1, z)v_2 + Y_M(v_1, z)w_2 + e^{zL(-1)} Y_M(v_2, -z)w_1,$$

where $v_1, v_2 \in V$, $w_1, w_2 \in M$. Then by [48] $(\mathcal{V}, Y_{\mathcal{V}}, \mathbf{1}, \omega)$ is a vertex operator algebra. The following lemma gives a method for a construction of a family of $\mathcal{V}$-modules.

Lemma 2.3. [15] *Assume that (M_2, Y_{M_2}) and (M_3, Y_{M_3}) be V-modules, and let $\mathcal{Y}(\cdot, z)$ be an intertwining operator of type $\binom{M_3}{M\ M_2}$ with integral powers of z. Then $(M_2 \oplus M_3, Y_{M_2 \oplus M_3})$ is a $\mathcal{V}$-module, where the vertex operator is given by*

$$Y_{M_2 \oplus M_3}(v + w)(w_2 + w_3) = Y_{M_2}(v, z)w_2 + Y_{M_3}(v, z)w_3 + \mathcal{Y}(w, z)w_2$$

for $v \in V$, $w \in M$, $w_i \in M_i$, $i = 1, 2$.

Remark 1. Note that the vertex operator algebra $\mathcal{V}$ is not simple, and that the module $M_2 \oplus M_3$ is also not simple. Moreover, the module structure on $M_2 \oplus M_3$ is, in general, not unique.

2.3. Fusion rules for the minimal Virasoro vertex operator algebras. Now we review some results on Virasoro vertex operator algebras, their fusion rules and intertwining operators. More details can be found in [41,65].

Let $V^{Vir}(c,0)$ be the universal Virasoro vertex operator algebra of central charge c, and let $L^{Vir}(c,0)$ be it simple quotient (cf. [41]). Let

$$d_{p,p'} = 1 - 6\frac{(p-p')^2}{pp'}, \quad p, p' \in \mathbb{Z}_{\geq 2},\ (p,p') = 1.$$

The Virasoro vertex algebra $L^{Vir}(d_{p,p'},0)$ is rational (cf. [65]) and its irreducible modules are $\{L^{Vir}(d_{p,p'},h) \mid h \in \mathcal{S}_{p,p'}\}$ where

$$\mathcal{S}_{p,p'} = \{h^{r,s}_{p,p'} = \frac{(sp-rp')^2 - (p-p')^2}{4pp'} \mid 1 \leq r \leq p-1, 1 \leq s \leq p'-1\}.$$

Let us denote the highest weight vector in $L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'})$ by $v_{r,s}$.

The fusion rules for $L^{Vir}(d_{p,p'},0)$-modules are

$$\begin{aligned} &L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \boxtimes L^{Vir}(d_{p,p'}, h^{r',s'}_{p,p'}) \\ &= \sum_{r'',s''} \begin{bmatrix} (r'',s'') \\ (r,s)\ (r',s') \end{bmatrix}_{(p,p')} L^{Vir}(d_{p,p'}, h^{r'',s''}_{p,p'}), \end{aligned}$$

where the fusion coefficient $\begin{bmatrix} (r'',s'') \\ (r,s)\ (r',s') \end{bmatrix}_{(p,p')}$ is equal to the dimension of the vector space of all intertwining operators of the type

$$\binom{L^{Vir}(d_{p,p'}, h^{r'',s''}_{p,p'})}{L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'})\ \ L^{Vir}(d_{p,p'}, h^{r',s'}_{p,p'})}.$$

The fusion coefficient $\begin{bmatrix} (r'',s'') \\ (r,s)\ (r',s') \end{bmatrix}_{(p,p')}$ is 0 or 1. For explicit formula see [65], [30, Section 7].

Let $\mathcal{Y}(\cdot,z)$ be a non-trivial intertwining operator of type

$$\binom{L^{Vir}(d_{p,p'}, h_3)}{L^{Vir}(d_{p,p'}, h_1)\ \ L^{Vir}(d_{p,p'}, h_2)}, \quad (h_i \in S_{p,p'}).$$

Then, for every $v \in L^{Vir}(d_{p,p'}, h_1)$ we have

$$\mathcal{Y}(v,z) = \sum_{r \in \Delta + \mathbb{Z}} v_r z^{-r-1}$$

where $\Delta = h_1 + h_2 - h_3$. Let v_{h_i} be the highest weight vector in $L^{Vir}(d_{p,p'}, h_i)$, $i = 1,2,3$. Then one can show that

$$(v_{h_1})_{\Delta-1} v_{h_2} = C v_{h_3}, \quad (v_{h_1})_{\Delta+n} v_{h_2} = 0$$

where $C \neq 0$, $n \in \mathbb{Z}_{\geq 0}$.

3. Wakimoto Modules for $\widehat{sl(2)}$

In this section, we first recall the construction of the Wakimoto modules for $\widehat{sl(2)}$ (cf. [34,64]). Then, by using the embedding of the Weyl vertex algebra into a lattice vertex algebra (also called FMS bosonization) we show that the universal affine vertex algebra $V^k(sl(2))$ can be embedded into the tensor product of a Virasoro vertex algebra with a vertex algebra $\Pi(0)$ of a lattice type. This result is stated in Proposition 3.1, which is a vertex-algebraic interpretation of the result of Semikhatov from [61].

3.1. Weyl vertex algebra W. Recall that the Weyl algebra *Weyl* is an associative algebra with generators $a(n), a^*(n)$ $(n \in \mathbb{Z})$ and relations

$$[a(n), a^*(m)] = \delta_{n+m,0}, \quad [a(n), a(m)] = [a^*(m), a^*(n)] = 0 \quad (n, m \in \mathbb{Z}). \tag{4}$$

Let W denotes the simple *Weyl*–module generated by the cyclic vector $\mathbf{1}$ such that

$$a(n)\mathbf{1} = a^*(n+1)\mathbf{1} = 0 \quad (n \geq 0).$$

As a vector space $W \cong \mathbb{C}[a(-n), a^*(-m) \mid n > 0,\ m \geq 0]$. There is a unique vertex algebra $(W, Y, \mathbf{1})$ where the vertex operator map is $Y : W \to \mathrm{End}(W)[[z, z^{-1}]]$ such that

$$Y(a(-1)\mathbf{1}, z) = a(z), \quad Y(a^*(0)\mathbf{1}, z) = a^*(z),$$
$$a(z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1}, \quad a^*(z) = \sum_{n\in\mathbb{Z}} a^*(n) z^{-n}.$$

3.2. The Heisenberg vertex algebra $M_\delta(\kappa, 0)$. Let $\mathfrak{h} = \mathbb{C}\delta$ be the 1-dimensional commutative Lie algebra with a symmetric bilinear form defined by $(\delta, \delta) = 1$, and $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c$ be its affinization. Set $\delta(n) = \delta \otimes t^n$. Let $M_\delta(\kappa, 0)$ denotes the simple $\widehat{\mathfrak{h}}$-module of level $\kappa \neq 0$ generated by the vector $\mathbf{1}$ such that $\delta(n)\mathbf{1} = 0 \quad \forall n \geq 0$. As a vector space $M_\delta(\kappa, 0) = \mathbb{C}[\delta(n) \mid n \leq -1]$.

There is a unique vertex algebra $(M_\delta(\kappa, 0), Y, \mathbf{1})$ generated by the Heisenberg field $Y(\delta(-1)\mathbf{1}, z) = \delta(z) = \sum_{n\in\mathbb{Z}} \delta(n) z^{-n-1}$ such that

$$[\delta(n), \delta(m)] = \kappa n \delta_{n+m,0} \quad (n, m \in \mathbb{Z}).$$

Vector $\omega = \left(\frac{1}{2\kappa}\delta(-1)^2 + a\delta(-2)\right)\mathbf{1}$ is a conformal vector of central charge $1 - \frac{12a^2}{\kappa}$.

For $r \in \mathbb{C}$, let $M_\delta(\kappa, r)$ denotes the irreducible $M_\delta(\kappa, 0)$-module generated by the highest weight vector v_r such that

$$\delta(n) v_r = r \delta_{n,0} v_r \quad (n \geq 0).$$

We can consider lattice $D_r = \mathbb{Z}(\frac{r\delta}{\kappa})$ and the generalized lattice vertex algebra $V_{D_r} := M_\delta(\kappa, 0) \otimes \mathbb{C}[D_r]$ (cf. [33]). We have:

$$M_\delta(\kappa, r) = M_\delta(\kappa, 0).e^{\frac{r\delta}{\kappa}}.$$

Then, the restriction of the vertex operator $Y(e^{\frac{r\delta}{\kappa}}, z)$ on $M_\delta(\kappa, 0)$ can be considered as a map $M_\delta(\kappa, 0) \to M_\delta(\kappa, r)[[z, z^{-1}]]$.

3.3. The Wakimoto module $W_{k,\mu}$. Assume that $k \neq -2$ and $\mu \in \mathbb{C}$. Let

$$W_{k,\mu} = W \otimes M_\delta(2(k+2), \mu).$$

Then, $W_{k,0}$ has the structure of a vertex algebra and $W_{k,\mu}$ is a $W_{k,0}$-module.

Let $V^k(sl(2))$ be the universal vertex algebra of level k associated to the affine Lie algebra $\widehat{sl(2)}$. There is a injective homomorphism of vertex algebras $\Phi : V^k(sl(2)) \to W_{k,0}$ generated by

$$\begin{aligned} e(z) &= a(z);\\ h(z) &= -2 : a^*(z)a(z) : +\delta(z);\\ f(z) &= - : a^*(z)^2 a(z) : +k\partial_z a^*(z) + a^*(z)\delta(z). \end{aligned}$$

The screening operator is $Q = \mathrm{Res}_z : a(z)Y(e^{-\frac{1}{k+2}\delta}, z) := (a(-1)e^{-\frac{1}{k+2}\delta})_0$ (cf. [34]).

3.4. Bosonization. Let H be the lattice

$$H = \mathbb{Z}\alpha + \mathbb{Z}\beta,\ \langle\alpha, \alpha\rangle = -\langle\beta, \beta\rangle = 1, \quad \langle\alpha, \beta\rangle = 0,$$

and $V_H = M_{\alpha,\beta}(1) \otimes \mathbb{C}[L]$ the associated lattice vertex algebra, where $M_{\alpha,\beta}(1)$ denotes the Heisenberg vertex algebra generated by α and β.

The Weyl vertex algebra W can be realized as a subalgebra of V_H generated by

$$a = e^{\alpha+\beta},\ a^* = -\alpha(-1)e^{-\alpha-\beta}.$$

This gives a realization of the universal affine vertex algebra $V_k(sl(2))$ as a subalgebra of $V_H \otimes M_\delta(2(k+2), 0)$ generated by

$$e = e^{\alpha+\beta}, \tag{5}$$

$$h = -2\beta(-1) + \delta(-1) \tag{6}$$

$$f = \Big[(k+1)(\alpha(-1)^2 - \alpha(-2)) + (k+2)\alpha(-1)\beta(-1) - \alpha(-1)\delta(-1)\Big]e^{-\alpha-\beta}. \tag{7}$$

Screening operators are (cf. [34, Section 7]):

$$Q = Res_z(Y(e^{\alpha+\beta-\frac{1}{k+2}\delta}, z), \quad \widetilde{Q} = \mathrm{Res}_z Y(e^{-(k+2)(\alpha+\beta)+\delta}, z). \tag{8}$$

3.5. Embedding of $V^k(sl(2))$ into vertex algebra $V^{Vir}(d_k, 0) \otimes \Pi(0)$. We shall first define new generators of the Heisenberg vertex algebra $M_{\alpha,\beta}(1) \otimes M_\delta(2(k+2))$. Let

$$\gamma = \alpha + \beta - \tfrac{1}{k+2}\delta, \quad \mu = -\beta + \tfrac{1}{2}\delta, \quad \nu = -\tfrac{k}{2}\alpha - \tfrac{k+2}{2}\beta + \tfrac{1}{2}\delta.$$

Then

$$\langle\gamma, \gamma\rangle = \frac{2}{k+2}, \quad \langle\mu, \mu\rangle = -\langle\nu, \nu\rangle = \frac{k}{2},$$

and all other products are zero. For our calculation, it is useful to notice that

$$\alpha = \nu + \tfrac{k+2}{2}\gamma,$$

$$\beta = -\tfrac{k+2}{2}\gamma + \tfrac{2}{k}\mu - \tfrac{k+2}{k}\nu,$$
$$\delta = -(k+2)\gamma + \tfrac{2(k+2)}{k}\mu - \tfrac{2(k+2)}{k}\nu.$$

Let $M(1) := M_{\mu,\nu}(1)$ be the Heisenberg vertex algebra generated by μ and ν. Consider the rank one lattice $\mathbb{Z}c \subset M(1)$ where $c = \frac{2}{k}(\mu - \nu)$. Then

$$\Pi(0) := M(1) \otimes \mathbb{C}[\mathbb{Z}c]$$

has the structure of a vertex algebra. Some properties of $\Pi(0)$ will be discussed in Sect. 4.

Let $M_\gamma(\frac{2}{k+2})$ be the Heisenberg vertex algebra generated by γ.

We obtain the following expression for the generators of $V^k(sl(2))$:

$$e = e^{\frac{2}{k}(\mu-\nu)}, \tag{9}$$
$$h = 2\mu(-1) \tag{10}$$
$$f = \Big[\tfrac{1}{4}(k+2)^2\gamma(-1)^2 - \tfrac{1}{2}(k+1)(k+2)\gamma(-2) -\nu(-1)^2 - (k+1)\nu(-2)\Big] e^{-\frac{2}{k}(\mu-\nu)}. \tag{11}$$

Set

$$\omega^{(k)} = \left(\frac{k+2}{4}\gamma(-1)^2 - \frac{k+1}{2}\gamma(-2)\right)\mathbf{1}.$$

Then

$$f = \Big[(k+2)\omega^{(k)} - \nu(-1)^2 - (k+1)\nu(-2)\Big] e^{-\frac{2}{k}(\mu-\nu)}.$$

Note that $\omega^{(k)}$ generates the universal Virasoro vertex algebra $V^{Vir}(d_k, 0)$ where $d_k = 1-6\frac{(k+1)^2}{(k+2)}$, which is realized as a subalgebra of the Heisenberg vertex algebra $M_\gamma(\frac{2}{k+2}, 0)$.

As usual we set $L(n) = \omega_{n+1}$ and denote the Virasoro field by $L(z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$.

We get the following result:

Proposition 3.1. [61] *Let ω be the conformal vector in $V^{Vir}(d_k, 0)$. There is a injective homomorphism of vertex algebras*

$$\Phi : V^k(sl(2)) \to V^{Vir}(d_k, 0) \otimes \Pi(0) \subset M_\gamma(\tfrac{2}{k+2}, 0) \otimes \Pi(0)$$

such that

$$e \mapsto e^{\frac{2}{k}(\mu-\nu)}, \tag{12}$$
$$h \mapsto 2\mu(-1), \tag{13}$$
$$f \mapsto \Big[(k+2)\omega - \nu(-1)^2 - (k+1)\nu(-2)\Big] e^{-\frac{2}{k}(\mu-\nu)}. \tag{14}$$

Remark 2. The realization in Proposition 3.1 had first obtained by Semikhatov [61] using slightly different notations.

A critical level version of this proposition was obtained in [9]. Let $M_T(0) = \mathbb{C}[T(-n), n \geq 2]$ be the commutative vertex algebra generated by the commutative field

$$T(z) = \sum_{n \leq -2} T(n) z^{-n-2}.$$

Proposition 3.2. *Let $k = -2$. There is a injective homomorphism of vertex algebras*

$$\Phi : V^k(sl(2)) \to M_T(0) \otimes \Pi(0)$$

such that

$$e \mapsto e^{\frac{2}{k}(\mu-\nu)}, \tag{15}$$

$$h \mapsto 2\mu(-1), \tag{16}$$

$$f \mapsto \Big[T(-2) - \nu(-1)^2 - (k+1)\nu(-2)\Big] e^{-\frac{2}{k}(\mu-\nu)}. \tag{17}$$

4. Some $\Pi(0)$-Modules

In this section we study vertex algebra $\Pi(0)$ which is associated to an isotropic rank two lattice $L = \mathbb{Z}c + \mathbb{Z}d$.

Lattice L is realized as $L = \mathbb{Z}c + \mathbb{Z}d \subset M(1)$, where $c = \alpha + \beta = \frac{2}{k}(\mu - \nu)$, and $d = \mu + \nu$. Then

$$\langle c, c \rangle = \langle d, d \rangle = 0, \quad \langle c, d \rangle = 2.$$

The vertex algebra $\Pi(0) = M(1) \otimes \mathbb{C}[\mathbb{Z}c]$ is generated by $c(-1), d(-1), u = e^c, u^{-1} = e^{-c}$. Its representation theory was studied in [27].

4.1. Weight $\Pi(0)$*-modules and their characters.* Let us recall some steps in the construction of $\Pi(0)$-modules. Let $\mathcal{A}$ be the associative algebra generated by d, e^{nc}, where $n \in \mathbb{Z}$ and relations

$$[d, e^{nc}] = 2ne^{nc}, \quad e^{nc}e^{mc} = e^{(n+m)c}, \quad (n, m \in \mathbb{Z}).$$

(We use the convention $e^0 = 1$). By using results from [27, Section 4] we see that for any $\mathcal{A}$-module U and any $r \in \mathbb{Z}$, there exists a unique $\Pi(0)$-module structure on the vector space

$$\mathcal{L}_r(U) = U \otimes M(1)$$

such that $c(0) \equiv r\mathrm{Id}$ on $\mathcal{L}_r(U)$. Moreover $\mathcal{L}_r(U)$ is irreducible $\Pi(0)$-module if and only if U is irreducible $\mathcal{A}$-module. By using this method one can construct the weight $\Pi(0)$-modules from [27]. (see Proposition 4.1).

In the present paper we shall need the following simple current extension of $\Pi(0)$:

$$\Pi^{1/2}(0) = M(1) \otimes \mathbb{C}[\mathbb{Z}\tfrac{c}{2}] = \Pi(0) \oplus \Pi(0)e^{\frac{c}{2}}.$$

$\Pi^{1/2}(0)$ is again the vertex algebra of the same type and it is generated by $c(-1), d(-1), u^{1/2} = e^{c/2}, u^{-1/2} = e^{-c/2}$. Note that $g = \exp[\pi i \mu(0)]$ is an automorphism of order two of the vertex algebra $\Pi^{1/2}(0)$ and that $g = \mathrm{Id}$ on $\Pi(0)$.

In order to construct $\Pi^{1/2}(0)$-modules, we need to consider a slightly larger associative algebra. Let $\mathcal{A}^{1/2}$ be the associative algebra generated by d, e^{nc}, $n \in \frac{1}{2}\mathbb{Z}$, and relations

$$[d, e^{nc}] = 2ne^{nc}, \quad e^{nc}e^{mc} = e^{(n+m)c} \quad (n, m \in \frac{1}{2}\mathbb{Z}).$$

For any $\mathcal{A}^{1/2}$-module U' and any $r \in \mathbb{Z}$, there exists a unique (g-twisted) $\Pi^{1/2}(0)$-module structure on the vector space

$$\mathcal{L}_r(U') = U' \otimes M(1)$$

such that $c(0) \equiv r\mathrm{Id}$ on $\mathcal{L}_r(U')$. Module $\mathcal{L}_r(U')$ is untwisted if r is even and g-twisted if r is odd. We omit details, since arguments are completely analogous to those of [27]. In this way we get a realization of a family of irreducible modules for the vertex algebras $\Pi(0)$ and $\Pi^{1/2}(0)$.

Proposition 4.1. 1. [27] *For every $r \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$, $\Pi_{(r)}(\lambda) := \Pi(0)e^{r\mu+\lambda c}$ is an irreducible $\Pi(0)$-module on which $c(0)$ acts as $r\mathrm{Id}$.*

2. *Assume that $r \in \mathbb{Z}$ is even (resp. odd) and $\lambda \in \mathbb{C}$. Then $\Pi^{1/2}_{(r)}(\lambda) := \Pi^{1/2}(0)e^{r\mu+\lambda c}$ is an irreducible untwisted (resp. g-twisted) $\Pi^{1/2}(0)$-module on which $c(0)$ acts as $r\mathrm{Id}$.*

As usual for a vector V is a vertex algebra V we define

$$\Delta(v, z) = z^{v_0} \exp\left(\sum_{n=1}^{\infty} \frac{v_n}{-n}(-z)^{-n}\right).$$

The following lemma follows from [49, Proposition 3.4].

Lemma 4.2. *For $\ell, r \in \mathbb{Z}$ we have*

$$(\Pi_{(\ell+r)}(\lambda), Y_{\Pi_{(\ell+r)}(\lambda)}(\cdot, z)) \cong (\Pi_{(r)}(\lambda), Y_{\Pi_{(r)}(\lambda)}(\Delta(\ell\mu, z)\cdot, z)).$$

We also have the following important observation which essentially follows from the analysis of $\Pi_{(-1)}(\lambda)$ as a module for the Heisenberg–Virasoro vertex algebra at level zero [21].[1]

Lemma 4.3. *The operator e^c_0 acts injectively on $\Pi_{(-1)}(\lambda)$.*

Let $k \in \mathbb{C}$. Now we shall fix the Heisenberg and the Virasoro vector in $\Pi(0)$, and calculate the character of the weight $\Pi(0)$-modules.

Vector

$$\omega_{\Pi(0)} = \frac{1}{2}c(-1)d(-1) - \frac{1}{2}d(-2) + \frac{k}{4}c(-2) \tag{18}$$

is a Virasoro vector in the vertex algebra $\Pi(0)$ of central charge $\overline{c} = 6k+2$. The Virasoro field is $\overline{L}(z) = \sum_{n\in\mathbb{Z}} \overline{L}(n)z^{-n-2}$.

Define the Heisenberg vector $h = 2\mu = \frac{k}{2}c + d$ and the corresponding field $h(z) = \sum_{n\in\mathbb{Z}} h(n)z^{-n-2}$. Then $[h(n), h(m)] = 2k\delta_{n+m,0}$.

[1] It was proved in [21] that $\Pi_{(-1)}(\lambda)$ is a direct sum of irreducible modules for the Heisenberg–Virasoro vertex algebra at level zero, and e^c_0 is a homomorphism which acts non-trivially on each irreducible component.

Definition 4.4. A module M for the vertex algebra $\Pi(0)$ (resp. $\Pi^{1/2}(0)$) is called weight if the operators $\overline{L}(0)$ and $h(0)$ act semisimply on M.

Assume that M is a weight module for the vertex algebra $\Pi(0)$ (resp. $\Pi^{1/2}(0)$) having finite-dimensional weight spaces for $(\overline{L}(0), h(0))$. Then we can define the character of M:

$$\mathrm{ch}[M](q,z) = \mathrm{Tr}_M q^{\overline{L}(0)-\overline{c}/24} z^{h(0)}.$$

Proposition 4.5. 1. *For every $\lambda \in \mathbb{C}$, $\Pi_{(-1)}(\lambda)$ is a $\mathbb{Z}_{\geq 0}$-graded weight $\Pi(0)$-module with character*

$$ch[\Pi_{(-1)}(\lambda)](q,z) = \frac{z^{-k+2\lambda}\delta(z^2)}{\eta(\tau)^2}.$$

2. *For every $\lambda \in \mathbb{C}$, $\Pi^{1/2}_{(-1)}(\lambda)$ is a $\mathbb{Z}_{\geq 0}$-graded weight $\Pi^{1/2}(0)$-module with character*

$$ch[\Pi^{1/2}_{(-1)}(\lambda)](q,z) = \frac{z^{-k+2\lambda}\delta(z)}{\eta(\tau)^2}$$

where $\delta(z) = \sum_{\ell \in \mathbb{Z}} z^{\ell}$, $\eta(\tau) = q^{1/24} \prod_{n \geq 1}(1-q^n)$.

Proof. Set $M = \Pi_{(-1)}(\lambda)$. First we notice that M is a $\mathbb{Z}_{\geq 0}$-graded $M = \bigoplus_{m=0}^{\infty} M(m)$ such that

$$M(0) \cong \mathrm{span}_{\mathbb{C}}\{e^{-\mu+(\lambda+j)c} \mid j \in \mathbb{Z}\},$$
$$\overline{L}(0)|M(0) \equiv \frac{k}{4}\mathrm{Id}, \quad h(0)e^{-\mu+(\lambda+j)c} = (-k+2(\lambda+j))e^{-\mu+(\lambda+j)c}.$$

Now we have

$$\mathrm{ch}[\Pi_{(-1)}(\lambda)](q,z) = \mathrm{Tr}_{\Pi_{(-1)}(\lambda)} q^{\overline{L}(0)-\overline{c}/24} z^{h(0)}$$
$$= q^{-\frac{k}{4}-\frac{1}{12}} q^{\frac{k}{4}} \frac{z^{-k+2\lambda}\delta(z^2)}{\prod_{n=1}^{\infty}(1-q^n)^2} = \frac{z^{-k+2\lambda}\delta(z^2)}{\eta(\tau)^2}.$$

This proves the assertion (1). The proof of (2) is analogous. □

4.2. Whittaker $\Pi(0)$-modules. The construction of Whittaker modules for the vertex algebra $\Pi(0)$-modules were presented in [9, Section 11]. We considered $\mathcal{A}$-module U_{λ} generated by vector v_1 such that

$$e^{nc} v_1 = \lambda^n v_1 \quad (n \in \mathbb{Z}).$$

Note that as a vector space $U_{\lambda} \cong \mathbb{C}[d]$ with the free action of d. Then we proved that $\Pi_{\lambda} = U_{\lambda} \otimes M(1)$ has that structure of an irreducible Whittaker $\Pi(0)$-module with the Whittaker vector $w_{\lambda} = 1 \otimes v_1$.

Similarly, we can construct Whittaker modules for the vertex algebra $\Pi^{1/2}(0)$. Consider $\mathcal{A}^{1/2}$-module $U_{\lambda}^{1/2} = \mathbb{C}[d(0)]v_2$ such that

$$e^{nc} v_2 = \lambda^n v_2 \quad (n \in \tfrac{1}{2}\mathbb{Z}).$$

Then $U_{\lambda}^{1/2} \otimes M(1)$ has the structure of an irreducible g-twisted $\Pi^{1/2}(0)$-module.

Proposition 4.6. 1. [9, Theorem 11.1] *For every* $\lambda \in \mathbb{C}\setminus\{0\}$ *there is an irreducible* $\Pi(0)$*-module* Π_λ *so that* $c(0)$ *acts on* Π_λ *as* $-Id$ *and that* Π_λ *is generated by cyclic vector* w_λ *satisfying*

$$e_0^c w_\lambda = \lambda w_\lambda, \quad e_0^{-c} w_\lambda = \frac{1}{\lambda} w_\lambda.$$

As a vector space, $\Pi_\lambda = M(1) \otimes \mathbb{C}[d(0)]$.

2. *For every* $\lambda \in \mathbb{C}\setminus\{0\}$ Π_λ *has the structure of an irreducible* g*-twisted* $\Pi^{1/2}(0)$*-module generated by cyclic vector* w_λ *such that*

$$e_0^{c/2} w_\lambda = \sqrt{\lambda} w_\lambda, \quad e_0^{-c/2} w_\lambda = \frac{1}{\sqrt{\lambda}} w_\lambda.$$

Remark 3. Note that in [9, Theorem 11.1], the operator e_0^c is denoted by $a(0)$.

Remark 4. The operator $\bar{L}(0)$ acts semi-simply on Π_λ. But the action of $h(0)$ is not diagonalizable. This can be easily seen from the action of $h(0)$ on top component $\Pi_\lambda(0) \cong U_\lambda$:

$$h(0) \equiv \frac{k}{2} c(0) + d(0) = -\frac{k}{2} + d(0)$$

which is not diagonalizable.

5. Realization of the Admissible Affine Vertex Algebra $V_k(sl(2))$

In this section, we use the realization from Proposition 3.1 and get a realization of the admissible affine vertex algebra $V_k(sl(2))$.

Assume now that k is admissible and $k \notin \mathbb{Z}$. Then

$$k + 2 = \frac{p'}{p}, \quad d_k = 1 - 6\frac{(p - p')^2}{pp'} = d_{p,p'}.$$

Let $L^{Vir}(d_{p,p'}, 0)$ be the simple rational vertex operator algebra of central charge $d_{p,p'}$ (cf. Sect. 2.3).

Let now $\varphi = p'\gamma$. Since $\langle \varphi, \varphi \rangle = \frac{2p'^2}{k+2} = 2pp'$, we set $M_\varphi(2pp', 0) = M_\gamma(\frac{2}{k+2}, 0)$ and

$$\omega^{(k)} = \left(\frac{1}{4pp'} \varphi(-1)^2 + \frac{p - p'}{2pp'} \varphi(-2) \right) \mathbf{1}.$$

The universal vertex algebra $V^{Vir}(d_{p,p'}, 0)$ is not simple and it contains a non-trivial ideal generated by singular vector $\Omega^{Vir}_{p,p'}$ of conformal weight $(p-1)(p'-1)$. Moveover,

$$L^{Vir}(d_{p,p'}, 0) = \frac{V^{Vir}(d_{p,p'}, 0)}{U(Vir).\Omega^{Vir}_{p,p'}}$$

is a simple vertex algebra (minimal model). The singular vector $\Omega^{Vir}_{p,p'}$ can be constructed in the free–field realization using screening operators.

Proposition 5.1. [58,63] *There exist a unique, up to a scalar factor, Vir-homomorphism*

$$\Phi^{Vir}_{p,p'} : M_{\varphi}(2pp', 0).e^{-\frac{p'-1}{p'}\varphi} \to M_{\varphi}(2pp', 0)$$
$$e^{-\frac{p'-1}{p'}\varphi} \mapsto \Omega^{Vir}_{p,p'}.$$

There is a cycle $\Delta_{p'-1}$ and a non-trivial scalar c_{p-1} such that $\Phi^{Vir}_{p,p'}$ can be be represented as

$$\frac{1}{c_{p'-1}} \int_{\Delta_{p'-1}} Y(e^{\frac{\varphi}{p'}}, z_1) \cdots Y(e^{\frac{\varphi}{p'}}, z_{p'-1}) dz_1 \cdots dz_{p'-1}.$$

Then $\omega_{p,p'} = \omega^{(k)} + U(Vir).\Omega^{Vir}_{p,p'}$ is the conformal vector in $L^{Vir}(d_{p,p'}, 0)$.

The universal affine vertex algebra $V^k(sl(2))$ also contains a non-trivial maximal ideal generated by $\widehat{sl(2)}$-singular vector $\Omega^{sl(2)}_k$ of conformal weight $p(p'-1)$. Moreover,

$$V_k(sl(2)) = \frac{V^k(sl_2)}{U(\widehat{sl(2)}).\Omega^{sl(2)}_k}$$

is a simple, admissible vertex algebra. Let ω_{sug} denotes the Sugawara Virasoro vector in $V_k(sl(2))$ od central charge $\frac{3k}{k+2}$. The singular vector $\Omega^{sl(2)}_k$ can be also constructed using screening operators. The proof was presented in [59, Theorem 3.1] for $\widehat{sl(2)}$ and in [23, Proposition 6.14] in a more general setting (see also [24] for some applications).

Proposition 5.2. [23,59] *There exist a unique, up to a scalar factor, $\widehat{sl(2)}$-homomorphism*

$$\Phi^{sl(2)}_k : W_{k,2(p'-1)} \to W_{k,0}$$
$$e^{-\frac{p'-1}{p'}\varphi + (p'-1)(\alpha+\beta)} \mapsto \Omega^{sl_2}_k.$$

By [59, Theorem 3.1] $\Phi^{sl(2)}_k$ can be represent as

$$\frac{1}{c_{p'-1}} \int_{\Delta_{p'-1}} U(z_1) \cdots U(z_{p'-1}) dz_1 \cdots dz_{p'-1},$$

where $U(z) = Y(a(-1)e^{-\frac{\delta}{k+2}}, z)$ and the cycle $\Delta_{p'-1}$ is as in Proposition 5.1. But, since $U(z) = Y(e^{\frac{\varphi}{p'}}, z)$, we get the following consequence:

Corollary 5.3. $\Phi^{sl(2)}_k$ *can be extended to a $\widehat{sl(2)}$-homomorphism*

$$M_{\varphi}(2pp', 0).e^{-\frac{p'-1}{p'}\varphi} \otimes \Pi(0) \to M_{\varphi}(2pp', 0) \otimes \Pi(0)$$

such that $\Phi^{sl(2)}_k = \Phi^{Vir}_{p,p'} \otimes Id$ and $\Omega^{sl_2}_k = \Omega^{Vir}_{p,p'} \otimes e^{(p'-1)c}$.

Example 5.4. Let us illustrate the above analysis in the simplest case $p' = 2$. Then we have that $k+2 = \frac{2}{p}$ where p is odd natural number, $p \geq 3$. Moreover, we have $\langle \varphi, \varphi \rangle = 4p$. The construction of the Virasoro singular vectors was obtained in [14] by using lattice vertex algebras.

The singular vector in $V^k(sl(2))$ is given by

$$\begin{aligned} Qe^{\frac{\delta}{k+2}} &= (a(-1)e^{-\frac{\delta}{k+2}})_0 e^{\frac{\delta}{k+2}} \\ &= S_{p-1}(\alpha + \beta - \frac{\delta}{k+2})a(-1)\mathbf{1} \\ &= S_{p-1}\left(\frac{\varphi}{2}\right) e^{\alpha+\beta} \\ &= Qe^{-\frac{\varphi}{2}} \otimes e^{\alpha+\beta} = \Omega^{Vir}_{p,2} \otimes e^c. \end{aligned}$$

Here, $S_n(\gamma)$ denotes the n-th Schur polynomial in $(\gamma(-1), \gamma(-2), \dots)$. In particular, $Qe^{-\frac{\varphi}{2}} = S_{p-1}(\frac{\varphi}{2})$ is a singular vector in $V^{Vir}(d_{p,2}, 0) \subset M_\varphi(4p)$ (cf. [14]).

Finally, we get the realization of $V_k(sl(2))$:

Theorem 5.5. *There exist a non-trivial $\widehat{sl(2)}$-homomorphism*

$$\overline{\Phi} : V_k(sl(2)) \to L^{Vir}(d_{p,p'}, 0) \otimes \Pi(0)$$

defined by the relations (12)–(14)*. Moreover,*

$$\overline{\Phi}(\omega_{sug}) = \omega_{p,p'} + \frac{1}{k}\mu(-1)^2 - \frac{1}{k}\nu(-1)^2 - \nu(-2) \tag{19}$$

$$= \omega_{p,p'} + \frac{1}{2}c(-1)d(-1) - \frac{1}{2}d(-2) + \frac{k}{4}c(-2). \tag{20}$$

Proof. We have constructed homomorphism $\Phi : V^k(sl(2)) \to V^{Vir}(d_{p,p'}, 0) \otimes \Pi(0)$ and showed in Corollary 5.3 that $\Phi(\Omega_k^{sl(2)}) = \Omega^{Vir}_{p,p'} \otimes e^{(p'-1)c}$. The claim follows. □

In what follows, we identify ω_{sug} with $\overline{\Phi}(\omega_{sug})$ and denote the Sugawara Virasoro field by

$$L_{sug}(z) = \sum_{n \in \mathbb{Z}} L_{sug}(n) z^{-n-2}, \quad L_{sug}(n) = (\omega_{sug})_{n+1}.$$

Remark 5. Note that $\overline{\Phi}(\omega_{sug}) = \omega_{p,p'} + \omega_{\Pi(0)}$, where $\omega_{\Pi(0)}$ is a Virasoro vector in the vertex algebra $\Pi(0)$ of central charge $6k+2$ given by (18). In particular, we have

$$c_{sug} = \frac{3k}{k+2} = d_{p,p'} + 6k + 2.$$

6. Realization of Ordinary $V_k(sl(2))$-Modules and Their Intertwining Operators

In this section, we present a realization of irreducible, ordinary $V_k(sl(2))$-modules, i.e., the $V_k(\mathfrak{g})$-modules having finite-dimensional $L_{sug}(0)$-eigenspaces. It was proved in [30], that the category of ordinary $V_k(sl(2))$-modules at the admissible level, denoted by $\mathcal{O}_{k,ord}$, form a braided tensor category with the tensor product bifunctor $\boxtimes_{\mathcal{P}(1)}$.

In this section, we show that the intertwining operators among ordinary $V_k(\mathfrak{g})$-modules can be constructed from the intertwining operators for the minimal Virasoro vertex algebra.

Recall [10] that the set

$$\{\mathcal{L}_s := L_{A_1}((k+1-s)\Lambda_0 + (s-1)\Lambda_1) \mid s = 1, \dots, p'-1\}$$

provides all irreducible, ordinary $V_k(\mathfrak{g})$-modules.

Proposition 6.1. *Let $s \in \mathbb{Z}$, $1 \le s \le p'-1$. We have*

$$\mathcal{L}_s \cong V_k(sl(2)).(v_{1,s} \otimes e^{\frac{s-1}{2}c}) \subset L^{Vir}(d_{p,p'}, h^{1,s}_{p,p'}) \otimes \Pi^{1/2}(0).$$

Proof. Since $L^{Vir}(d_{p,p'}, h^{1,s}_{p,p'}) \otimes \Pi^{1/2}(0)$ is a $V_k(sl(2))$-module, it suffices to show that $w_s = v_{1,s} \otimes e^{\frac{s-1}{2}c}$ is a singular vector for $\widehat{sl(2)}$. For $n \ge 0$ we have:

$$e(n)w_s = f(n+2)w_s = 0, \; h(n)w_s = (s-1)\delta_{n,0}w_s.$$

It remains to prove that $f(1)w_s = 0$. Since

$$\begin{aligned}(w_s)_1 f &= (k+2)(v_{1,s})_1\omega_{p,p'} \otimes e^{\frac{s-1}{2}c-c} - v_{1,s} \otimes e_1^{\frac{s-1}{2}c}(\nu(-1)^2 + (k+1)\nu(-2))e^{-c} \\ &= \left((k+2)h^{1,s}_{p,p'} - \frac{(s-1)^2}{4} + \frac{(k+1)(s-1)}{2}\right) v_{1,s} \otimes e^{\frac{s-1}{2}c-c} = 0, \end{aligned} \tag{21}$$

we get $f(1)w_s = -[(w_s)_{-1}, f(1)]\mathbf{1} = (w_s)_1 f = 0$. The proof follows. □

The following fusion rules result was proved in [30]:

$$\mathcal{L}_{s_1} \boxtimes_{\mathcal{P}(1)} \mathcal{L}_{s_2} = \bigoplus_{s_3=1}^{p'-1} N^{s_3}_{s_1,s_2} \mathcal{L}_{s_3}, \tag{22}$$

where the fusion coefficient is

$$N^{s_3}_{s_1,s_2} := \begin{cases} 1 & \text{if } |s_2 - s_1| + 1 \le s_3 \le \min\{s_1+s_2+1, 2p'-s_1-s_2-1\} \\ & \quad s_1+s_2+s_3 \text{ odd} \\ 0 & \text{otherwise} \end{cases}.$$

Moreover, $N^{s_3}_{s_1,s_2} = \begin{bmatrix} (1,s_3) \\ (1,s_1)\;(1,s_2) \end{bmatrix}_{(p,p')}$ coincides with the fusion coefficient for the Virasoro minimal models (cf. Sect. 2.3):

$$L^{Vir}(d_{p,p'}, h^{1,s_1}_{p,p'}) \boxtimes_{\mathcal{P}(1)} L^{Vir}(d_{p,p'}, h^{1,s_2}_{p,p'}) = \bigoplus_{s_3=1}^{p'-1} N^{s_3}_{s_1,s_2} L^{Vir}(d_{p,p'}, h^{1,s_3}_{p,p'}).$$

By using our realization we can construct all intertwining operators appearing in the fusion rules (22) as follows. Let $\mathcal{Y}_1(\cdot, z)$ be a non-trivial intertwining operator of the type

$$\binom{L^{Vir}(d_{p,p'}, h^{1,s_3}_{p,p'})}{L^{Vir}(d_{p,p'}, h^{1,s_1}_{p,p'}) \;\; L^{Vir}(d_{p,p'}, h^{1,s_2}_{p,p'})}$$

for the Virasoro vertex operator algebra $L^{Vir}(d_{p,p'}, 0)$. We can tensor it with the vertex operator map $Y_{\Pi^{1/2}(0)}(\cdot, z)$ for the vertex algebra $\Pi^{1/2}(0)$, and obtain the intertwining operator $\mathcal{Y} = \mathcal{Y}_1 \otimes Y_{\Pi^{1/2}(0)}$ of type

$$\binom{L^{Vir}(d_{p,p'}, h^{1,s_3}_{p,p'}) \otimes \Pi^{1/2}(0)}{L^{Vir}(d_{p,p'}, h^{1,s_1}_{p,p'}) \otimes \Pi^{1/2}(0) \;\; L^{Vir}(d_{p,p'}, h^{1,s_2}_{p,p'}) \otimes \Pi^{1/2}(0)} \tag{23}$$

in the category of $V_k(sl(2))$-modules. Intertwining operator corresponding to the fusion rules (22) can be obtained by restricting the above intertwining operators.

Proposition 6.2. *Assume that $N^{s_3}_{s_1,s_2} = 1$. Then there is a non-trivial intertwining operator of type*

$$\binom{\mathcal{L}_{s_3}}{\mathcal{L}_{s_1} \;\; \mathcal{L}_{s_2}},$$

realizead as a restriction of the intertwining operator (23).

Proof. Note that $\mathcal{L}_{s_i} = V_k(sl(2)).w_{s_i}$, where $w_{s_i} = v_{1,s_i} \otimes e^{\frac{s_i-1}{2}c}$, $i = 1, 2, 3$. By restricting $\mathcal{Y}(\cdot, z)$ on $\mathcal{L}_{s_1} \otimes \mathcal{L}_{s_2}$, we get a non-trivial intertwining operator of type $\binom{\mathcal{M}_{s_3}}{\mathcal{L}_{s_1} \;\; \mathcal{L}_{s_2}}$, where $M_{s_3} = V_k(sl(2)).v_{s_3} \otimes e^{\frac{s_1+s_2-2}{2}c}$. Note that $s_1 + s_2 - s_3 - 1 \in 2\mathbb{Z}_{\geq 0}$. Then

$$e(-1)^{\frac{s_1+s_2-s_3-1}{2}} w_{s_3} = e_{-1}^{\frac{s_1+s_2-s_3-1}{2}c} \left(v_{s_3} \otimes e^{\frac{s_3-1}{2}c} \right) = v_{s_3} \otimes e^{\frac{s_1+s_2-2}{2}c}.$$

This shows that $M_{s_3} \subseteq \mathcal{L}_{s_3}$, and since $\mathcal{L}_{s_3}$ is irreducible, we have that $\mathcal{L}_{s_3} = M_{s_3}$. Thus, we have constructed a non-trivial intertwining operator of type $\binom{\mathcal{L}_{s_3}}{\mathcal{L}_{s_1} \;\; \mathcal{L}_{s_2}}$. The proof follows. □

7. Explicit Realization of Relaxed Highest Weight $V_k(sl(2))$-Modules

We say that a $\mathbb{Z}_{\geq 0}$-graded $V^k(sl(2))$-module $M = \bigoplus_{m=0}^{\infty} M(m)$ is a relaxed highest weight module if the following conditions hold:

- Each graded component $M(m)$ is an eigespace for $L_{sug}(0)$;
- $M = V^k(sl(2)).M(0)$;
- $M(0)$ is an irreducible weight $sl(2)$-module which is neither highest nor lowest weight $sl(2)$-module.

The subspace $M(0)$ is usually called the top component of $M(0)$ (although it has lowest conformal weight with respect to $L_{sug}(0)$).

By using the classification of irreducible $V_k(sl(2))$-modules from [10] (see also [59]), we conclude that any irreducible $\mathbb{Z}_{\geq 0}$-graded $V_k(sl(2))$-module belongs to one of the following series:

1. The ordinary modules $\mathcal{L}_s$ (cf. Proposition 6.1) with lowest conformal weight $h^{1,s}_{p,p'} + \frac{s-1}{2}$.
2. The $\mathbb{Z}_{\geq 0}$ graded $V_k(sl(2))$-modules $D^{\pm}_{r,s}$, $1 \leq r \leq p-1$, $1 \leq s \leq p'-1$, where
   - $D^+_{r,s}$ is an irreducible $\mathbb{Z}_{\geq 0}$-graded $V_k(sl(2))$-module such that $D^+_{r,s}(0)$ is an irreducible highest weight $sl(2)$-module with highest weight $\mu_{r,s} = (s-1-(k+2)r)\omega_1$, where ω_1 is the fundamental weight for $sl(2)$.
   - $D^-_{r,s}$ is an irreducible $\mathbb{Z}_{\geq 0}$-graded $V_k(sl(2))$-module such that $D^-_{r,s}(0)$ is an irreducible lowest weight $sl(2)$-module with lowest weight $-\mu_{r,s}$.
3. Relaxed highest weight modules $M = \bigoplus_{m=0}^{\infty} M(m)$, such that the top component $M(0)$ has conformal weight $h^{r,s}_{p,p'} + k/4$.

In this section we construct a family of relaxed highest weight modules for $V_k(sl(2))$. These modules also appeared in [25,32,35,39,57,59,60]. In this section, we shall explicitly construct these modules and see from the realization that their characters are given by the Creutzig–Ridout character formulas [32] (see also [45]).

7.1. Realization of relaxed $V_k(sl(2))$-modules. For every $\lambda \in \mathbb{C}$ and $r, s \in \mathbb{Z}$, $0 < r < p$, $0 < s < p'$ we define the $L^{Vir}(d_{p,p'}, 0) \otimes \Pi(0)$-module

$$\mathcal{E}^{\lambda}_{r,s} = L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{-1}(\lambda) = L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi(0).e^{-\mu+\lambda\frac{2}{k}(\mu-\nu)}.$$

Let $\ell \in \mathbb{Z}$ and π_ℓ be the (spectral flow) automorphism of $V_k(sl(2))$ defined by

$$\pi_\ell(e(n)) = e(n+\ell), \quad \pi_\ell(f(n)) = f(n-\ell), \quad \pi_\ell(h(n)) = h(n) + \ell k \delta_{n,0}.$$

By using the realization of $V_k(sl(2))$ one can see that the spectral-flow automorphism π_ℓ can be realized as the lattice element $e^{\ell\mu}$ acting on $\Pi(0)$-modules:

Proposition 7.1. *We have:* $\pi_\ell(\mathcal{E}^{\lambda}_{r,s}) = L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{\ell-1}(\lambda)$.

Proof. Using [5, Proposition 2.1] we get that if $(M, Y_M(\cdot, z))$ is a $V_k(sl(2))$-module then

$$(\pi_\ell(M), Y_{\pi_\ell(M))}(\cdot, z)) := (M, Y_M(\Delta(\tfrac{\ell h}{2}, z)\cdot, z)).$$

Using Lemma 4.2 we get

$$(\Pi_{\ell-1}(\lambda), Y_{\Pi_{\ell-1}}(\cdot, z)) = (\Pi_{-1}(\lambda), Y_{\Pi_{-1}(\lambda)}(\Delta(\ell\mu, z)\cdot, z))$$

which implies $\pi_\ell(\mathcal{E}^{\lambda}_{r,s}) = L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{\ell-1}(\lambda)$. The proof follows. □

Let

$$E^{\lambda}_{r,s} = v_{r,s} \otimes e^{-\mu+\lambda\frac{2}{k}(\mu-\nu)} = v_{r,s} \otimes e^{\beta-\delta/2+\lambda(\alpha+\beta)}.$$

Then $E^{\lambda}_{r,s}$ is a primary vector with conformal weight $k/4 + h^{r,s}_{p,p'}$, i.e.

$$L_{sug}(n)E^{\lambda}_{r,s} = (k/4 + h^{r,s}_{p,p'})\delta_{n,0}E^{\lambda}_{r,s}, \quad (n \geq 0). \tag{24}$$

The $sl(2)$ action on these vectors is as follows:

$$e(0)E^{\lambda}_{r,s} = E^{\lambda+1}_{r,s}, \tag{25}$$

$$h(0)E^{\lambda}_{r,s} = (-k + 2\lambda)E^{\lambda}_{r,s}, \tag{26}$$

$$\begin{aligned} f(0)E^{\lambda}_{r,s} &= \left((k+2)h^{r,s}_{p,p'} - \lambda^2 + \lambda(k+1)\right) E^{\lambda-1}_{r,s} \\ &= \left(\frac{(sp - rp')^2}{4p^2} - (\lambda - \frac{p'-p}{2p})^2\right) E^{\lambda-1}_{r,s}. \end{aligned} \tag{27}$$

Remark 6. Note that $f(0)E^{\lambda}_{r,s} = 0$ iff $\lambda = \lambda^{\pm}_{r,s}$ where $\lambda^{\pm}_{r,s} = \frac{p'-p}{2p} \pm \frac{sp-rp'}{2p}$. It is also important to notice that $\lambda^{+}_{r,s} = \frac{s-1}{2} - \frac{r-1}{2}(k+2)$, $\lambda^{-}_{r,s} = \lambda^{+}_{p-r,p'-s}$.

If $\lambda = \lambda^{+}_{r,s}$, then $\mathcal{E}^{\lambda}_{r,s}$ is an indecomposable $\mathbb{Z}_{\geq 0}$-graded $V_k(sl(2))$-module which appears in the non-split extension

$$0 \to D^{-}_{p-r,p'-s} \to \mathcal{E}^{\lambda}_{r,s} \to D^{+}_{r,s} \to 0 \tag{28}$$

where $D^{\pm}_{r,s}$ are irreducible $V_k(sl(2))$-modules described above. This extension was also constructed in [32] (see [32, Section 4] and their formula (4.3)). In Sect. 9, we shall see that indecomposable modules $\mathcal{E}^{\lambda}_{r,s}$ appear in the construction of logarithmic modules.

Assume that $\lambda \notin \lambda^{\pm}_{r,s} + \mathbb{Z}$.

Theorem 7.2. *We have:*

1. $\mathcal{E}^{\lambda}_{r,s}$ *is* $\mathbb{Z}_{\geq 0}$*-graded* $V_k(sl(2))$*-module.*
2. *The top component is* $\mathcal{E}^{\lambda}_{r,s}(0) = span_{\mathbb{C}}\{E^{\lambda+j}_{r,s} \mid j \in \mathbb{Z}\}$ *and it has conformal weight* $k/4 + h^{r,s}_{p,p'}$*. If* $\lambda \notin (\lambda^{\pm}_{r,s} + \mathbb{Z})$*, then* $\mathcal{E}^{\lambda}_{r,s}(0)$ *is an irreducible* $sl(2)$*-module.*
3. *The character of* $\mathcal{E}^{\lambda}_{r,s}$ *is given by*

$$ch[\mathcal{E}^{\lambda}_{r,s}](q,z) = z^{-k+2\lambda}\chi_{r,s}(q)\frac{\delta(z^2)}{\eta(\tau)^2},$$

where $\chi_{r,s}(q) = ch[L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'})](q)$, $\delta(z^2) = \sum_{\ell\in\mathbb{Z}} z^{2\ell}$, $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$.

Proof. By using (24) we see that $\mathcal{E}^{\lambda}_{r,s}$ is $\mathbb{Z}_{\geq 0}$-graded, and by (25)–(27) we get that the top component $\mathcal{E}^{\lambda}_{r,s}(0)$ is an irreducible $sl(2)$-module. This proves assertions (1) and (2). Recall that $c_{sug} = \frac{3k}{k+2} = d_{p,p'} + 2 + 6k$ (see Remark 5). Using Proposition 4.5 our explicit realization gives the following character formula:

$$\begin{aligned}\mathrm{ch}[\mathcal{E}^{\lambda}_{r,s}](q,z) &= \mathrm{Tr}_{\mathcal{E}^{\lambda}_{r,s}} q^{L_{sug}(0)-c_{sug}/24} z^{h(0)}\\ &= \mathrm{Tr}_{L^{Vir}(d_{p,p'},h^{r,s}_{p,p'})} q^{L(0)-d_{p,p'}/24} \mathrm{ch}_{\Pi_{(-1)}(\lambda)}(q,z)\\ &= \mathrm{ch}[L^{Vir}(d_{p,p'},h^{r,s}_{p,p'})](q)\cdot \frac{z^{-k+2\lambda}\delta(z^2)}{\eta(\tau)^2}.\end{aligned}$$

The proof follows. □

7.2. Irreducibility of relaxed $V_k(sl(2))$-modules. We will now discuss the irreducibility of relaxed $V_k(sl(2))$-modules $\mathcal{E}^{\lambda}_{r,s}$. We shall present a proof of irreducibility in generic cases (cf. Proposition 7.4) which uses our realization and the representation theory of the vertex operator algebra $V_k(sl(2))$.

Lemma 7.3. *Assume that $M=\bigoplus_{n\in\mathbb{Z}_{\geq 0}} M(n)$ is an irreducible $\mathbb{Z}_{\geq 0}$-graded $V_k(sl(2))$-module such that $M(0)$ is an irreducible, infinite-dimensional weight $sl(2)$-module. Then M is isomorphic to a subquotient of $\mathcal{E}^{\lambda}_{r,s}$ for certain $1\leq r\leq p'-1$, $1\leq s\leq p-1$, $\lambda\in\mathbb{C}$ and*

$$L_{sug}(0)\equiv (h^{r,s}_{p,p'}+k/4)Id \quad on\ M(0).$$

Proof. If $M(0)$ is an irreducible highest (resp. lowest) weight module for $sl(2)$, then the classification of irreducible $V_k(sl(2))$-modules gives that $M\cong D^{+}_{r,s}$ (resp. $M\cong D^{-}_{p-r,p'-s}$.). By Remark 6, M can be realized as a submodule or a quotient of the indecomposable module $\mathcal{E}^{\lambda}_{r,s}$. If M is an irreducible relaxed $V_k(sl(2))$-module, then $M(0)\cong\mathcal{E}^{\lambda}_{r,s}(0)$ for certain r,s,λ, and therefore M is isomorphic to a (quotient) of $E^{\lambda}_{r,s}$. The proof follows. □

Remark 7. Modules $\mathcal{E}^{\lambda}_{r,s}$ are irreducible for $\lambda\notin\lambda^{\pm}_{r,s}+\mathbb{Z}$. This follows from the fact that they have the same characters as irreducible quotients of relaxed Verma modules presented by Creutzig and Ridout [32]. We should mention that a new proof of irreducibility of a large family of relaxed highest weight modules is presented in new paper [45] using Mathieu' s coherent families.[2] Sato [60] presented a proof of irreducibility of certain typical modules for the $N=2$ superconformal algebra which are related to the relaxed $\widehat{sl(2)}$-modules via the anti Kazama–Suzuki mapping [2,39].

Proposition 7.4. *Let r_0,s_0 such that $1\leq r_0\leq p'-1$, $1\leq s_0\leq p-1$ and $\lambda\notin(\lambda^{\pm}_{r_0,s_0}+\mathbb{Z})$. Assume that*

$$h-h^{r_0,s_0}_{p,p'}\notin\mathbb{Z}_{>0}\qquad \forall h\in\mathcal{S}_{p,p'}. \tag{29}$$

Then $\mathcal{E}^{\lambda}_{r_0,s_0}$ is an irreducible $V_k(sl(2))$-module. In particular, $\mathcal{E}^{\lambda}_{r_0,s_0}$ is irreducible if $h^{r_0,s_0}_{p,p'}$ is maximal in the set $\mathcal{S}_{p,p'}$.

Proof. Assume that $\mathcal{E}^{\lambda}_{r_0,s_0}$ is reducible. By Lemma 4.3 the operator $e(0)=e^c_0$ acts injectively on the module $\mathcal{E}^{\lambda}_{r_0,s_0}$, and therefore there are no submodules of $\mathcal{E}^{\lambda}_{r_0,s_0}$ with

2 Talk presented by K. Kawasetsu at the conference *Affine, vertex and W-algebras, Rome, December 11–15, 2017*.

finite-dimensional $L_{sug}(0)$-eigenspaces. Since the top component $\mathcal{E}^{\lambda}_{r_0,s_0}(0)$ is an irreducible $sl(2)$-module, we conclude that $\mathcal{E}^{\lambda}_{r_0,s_0}$ has a non-trivial $\mathbb{Z}_{\geq 0}$-graded irreducible subquotient $M = \bigoplus_{m=0}^{\infty} M(m)$, such that $\dim M(0) = \infty$. By Lemma 7.3, the conformal weight of $M(0)$ is $h + k/4$, such that

$$h \in \mathcal{S}_{p,p'},\ h > h^{r_0,s_0}_{p,p'},\ h - h^{r_0,s_0}_{p,p'} \in \mathbb{Z}_{>0}.$$

This contradicts the choice of (r_0, s_0). The claim holds. □

Remark 8. A new result on characters of irreducible relaxed $V_k(sl(2))$ from [45, Theorem 5.2] directly implies that $\mathcal{E}^{\lambda}_{r,s}$ is irreducible for all $1 \leq r \leq p'-1$, $1 \leq s \leq p-1$ and $\lambda \notin (\lambda^{\pm}_{r,s} + \mathbb{Z})$.

7.3. Realization of the intertwining operators among relaxed modules. We will now compare our realization with conjectural fusion rules from [32].

Note that in the realization we have $\pi_\ell(\mathcal{E}^{\lambda}_{r,s}) = L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{\ell-1}(\lambda)$ (cf. Proposition 7.1). One can show that if $\begin{bmatrix} & (r'', s'') \\ (r,s) & (r',s') \end{bmatrix}_{(p,p')} = 1$, we can realize the intertwining operator of the following type

$$\begin{pmatrix} & \pi_{\ell+\ell'-1}(\mathcal{E}^{\lambda+\lambda'}_{r'',s''}) \\ \pi_\ell(\mathcal{E}^{\lambda}_{r,s}) & \pi_{\ell'}(\mathcal{E}^{\lambda'}_{r',s'}) \end{pmatrix}. \tag{30}$$

- There is a small difference between notation in [32] where relaxed modules were denoted by $\mathcal{E}_{\mu,\Delta_{r,s}}$, where $\Delta_{r,s} = h^{r,s}_{p,p'}$. Precise relation is $\mathcal{E}^{\lambda}_{r,s} = \mathcal{E}_{2\lambda-k,\Delta_{r,s}}$.
- Based on the Grothendieck fusion rules [32, Propositions 13 and 18] and [26, Proposition 2.17], it is conjectured that the following fusion rules holds in the category of weight $V_k(sl(2))$-modules:

$$\begin{aligned} &\pi_\ell(\mathcal{E}_{\mu,\Delta_{r,s}}) \boxtimes \pi_{\ell'}(\mathcal{E}_{\mu',\Delta_{r,s}}) \\ &= \sum_{r'',s''} \begin{bmatrix} & (r'', s'') \\ (r,s) & (r',s') \end{bmatrix}_{(p,p')} \left(\pi_{\ell+\ell'+1}(\mathcal{E}_{\mu+\mu'-k,\Delta_{r'',s''}}) + \pi_{\ell+\ell'-1}(\mathcal{E}_{\mu+\mu'+k,\Delta_{r'',s''}})\right) \\ &\quad + \sum_{r'',s''} \left(\begin{bmatrix} & (r'', s'') \\ (r,s) & (r',s'-1) \end{bmatrix}_{(p,p')} \right. \\ &\quad \left. + \begin{bmatrix} & (r'', s'') \\ (r,s) & (r',s'+1) \end{bmatrix}_{(p,p')} \right) \pi_{\ell+\ell'}(\mathcal{E}_{\mu+\mu',\Delta_{r'',s''}}). \end{aligned} \tag{31}$$

In free field realization, we can only construct an intertwining operator of the type (30) which is in the terminology of [32]:

$$\begin{pmatrix} & \pi_{\ell+\ell'-1}(\mathcal{E}_{\mu+\mu'+k,\Delta_{r'',s''}}) \\ \pi_\ell(\mathcal{E}_{\mu,\Delta_{r,s}}) & \pi_{\ell'}(\mathcal{E}_{\mu',\Delta_{r,s}}) \end{pmatrix}.$$

The construction of other three type of intertwining operators is still an open problem.

We should mention that the cases of the collapsing levels $k = -1/2$ and $k = -4/3$ are very interesting, since then $V_k(sl(2))$ is related with the triplet vertex algebras $W(p)$

for $p = 2, 3$ (cf. [4,55]). The fusion rules for these vertex algebras are also related to the (conjectural) fusion rules for the singlet vertex algebra $\mathcal{M}(p)$ (cf. [17,28]). By [32], the fusion rules for $k = -1/2$ are

$$\begin{aligned}&\pi_{\ell}(\mathcal{E}_{\mu,-1/8}) \boxtimes \pi_{\ell'}(\mathcal{E}_{\mu',-1/8})\\ &= \pi_{\ell+\ell'-1}(\mathcal{E}_{\mu+\mu'-1/2,-1/8}) + \pi_{\ell+\ell'+1}(\mathcal{E}_{\mu+\mu'+1/2,-1/8}).\end{aligned} \tag{32}$$

In our forthcoming paper [20], we shall study the fusion rules (32).

8. Whittaker Modules for $V_k(sl(2))$

In this section, we extend result from [9] and construct all degenerate Whittaker modules at an arbitrary admissible level. As a consequence, we will see that admissible affine vertex algebra $V_k(sl(2))$ contains $\mathbb{Z}_{\geq 0}$-graded modules of the Whittaker type.

Let us first recall some notation from [9].

For a $(\lambda, \mu) \in \mathbb{C}^2$, let $\widetilde{Wh}_{\widehat{sl_2}}(\lambda, \mu, k)$ denotes the universal Whittaker module at level k which is generated by the Whittaker vector $w_{\lambda,\mu,k}$ satisfying

$$e(n)w_{\lambda,\mu,k} = \delta_{n,0}\lambda w_{\lambda,\mu,k} \quad (n \in \mathbb{Z}_{\geq 0}), \tag{33}$$

$$f(m)w_{\lambda,\mu,k} = \delta_{m,1}\mu w_{\lambda,\mu,k} \quad (m \in \mathbb{Z}_{\geq 1}). \tag{34}$$

If $\mu \cdot \lambda \neq 0$, then the Whittaker module is called non-degenerate. It was proved in [9] that at the non-critical level the universal non-degenerate Whittaker module is automatically irreducible.

But in the degenerate case when $\mu = 0$, $\widetilde{Wh}_{\widehat{sl(2)}}(\lambda, 0, k)$ is reducible and it contains a non-trivial submodule

$$M_{\widehat{sl(2)}}(\lambda, 0, k, a) := \widetilde{Wh}_{\widehat{sl(2)}}(\lambda, 0, k)/U(\widehat{sl(2)}).(L_{sug}(0) - a)w_{\lambda,\mu,k} \quad (a \in \mathbb{C}).$$

Let $Wh_{\widehat{sl(2)}}(\lambda, 0, k, a)$ be the simple quotient of $M_{\widehat{sl(2)}}(\lambda, 0, k, a)$.

We have the following result.

Theorem 8.1. *For all $k, h, \lambda \in \mathbb{C}$, $\lambda \neq 0$ we have:*

$$Wh_{\widehat{sl(2)}}(\lambda, 0, k, h + k/4) \cong L^{Vir}(d_k, h) \otimes \Pi_\lambda.$$

Proof. The proof will use [9, Lemma 10.2] which says that Π_λ is an irreducible $\widehat{\mathfrak{b}}_1$-module, where $\widehat{\mathfrak{b}}_1$ is a Lie subalgebra of $\widehat{sl(2)}$ generated by $e(n), h(n), n \in \mathbb{Z}$.

On $L^{Vir}(d_k, h)$ we have the weight decomposition:

$$\begin{aligned}L^{Vir}(d_k, h) &= \bigoplus_{m \in \mathbb{Z}_{\geq 0}} L^{Vir}(d_k, h)_{h+m}, \quad L^{Vir}(d_k, h)_{h+m}\\ &= \{v \in L^{Vir}(d_k, h) \mid L(0)v = (h+m)v\}.\end{aligned}$$

Let v_h be the highest weight vector in $L^{Vir}(d_k, h)$, and define $\widetilde{w}_{\lambda,0,k} = v_h \otimes w_\lambda$. Since

$$e(n)\widetilde{w}_{\lambda,0,k} = \delta_{n,0}\lambda w_{\lambda,0,k} \quad (n \in \mathbb{Z}_{\geq 0}), \tag{35}$$

$$f(m)\widetilde{w}_{\lambda,0,k} = 0 \quad (m \in \mathbb{Z}_{\geq 1}), \tag{36}$$

$$L_{sug}(n)\widetilde{w}_{\lambda,0,k} = \delta_{n,0}(h + k/4)w_{\lambda,0,k} \quad (n \in \mathbb{Z}_{\geq 0}) \tag{37}$$

we conclude that $\widehat{W} = U(\widehat{sl(2)}).\widetilde{w}_{\lambda,0,k}$ is a certain quotient of $M_{\widehat{sl(2)}}(\lambda, 0, k, h + k/4)$.

Let us first prove that $\widehat{W} = L^{Vir}(d_k, h) \otimes \Pi_\lambda$. It suffices to prove that for every $m \in \mathbb{Z}_{\geq 0}$ we have that

$$v \otimes w \in \widehat{W} \quad \forall v \in L^{Vir}(d_k, h)_{h+m}, \quad \forall w \in \Pi_\lambda. \tag{38}$$

For $m = 0$, the claim follows by using the irreducibility of Π_λ as a $\widehat{\mathfrak{b}}_1$-module. Assume now that $v' \otimes w \in \widehat{W}$ for all $v' \in L^{Vir}(d_k, h)_{h+m'}$ such that $m' < m$ and all $w \in \Pi_\lambda$. Let $v \in L^{Vir}(d_k, h)_{h+m}$. It suffices to consider homogeneous vectors

$$v = L(-n_0)L(-n_1)\cdots L(-n_s)v_h, \quad n_0 \geq \cdots \geq n_s \geq 1, \ n_0 + \cdots + n_s = m.$$

Then by inductive assumption we have that $L(-n_1)\cdots L(-n_s)v_h \otimes w \in \widehat{W}$ for all $w \in \Pi_\lambda$. By using the formulae for the action of $f(m)$, $m \in \mathbb{Z}$, we get

$$f(-n_0)(L(-n_1)\cdots L(-n_s)v_h \otimes w_\lambda) = AL(-n_0)L(-n_1)\cdots L(-n_s)v_h \otimes w_\lambda + z$$

where $A \neq 0$ and

$$z = \sum_i v_i \otimes w_i, \quad v_i \in L^{Vir}(d_k, h)_{h+m_i'}, \ m_i' < m, \ w_i \in \Pi_\lambda.$$

By using inductive assumption we get that $z \in \widehat{W}$, and therefore $v \otimes w_\lambda \in \widehat{W}$. Using the fact that Π_λ is an irreducible $\widehat{\mathfrak{b}}_1$-module, we get that $v \otimes w \in \widehat{W}$ or every $w \in \Pi_\lambda$. The claim (38) now follows by induction.

Now the irreducibility result will be a consequence of the following claim:

$$\begin{aligned} v \otimes w \text{ is cyclic vector in } L^{Vir}(d_k, h) \otimes \Pi_\lambda \quad \forall v \\ \in L^{Vir}(d_k, h)_{h+m}, \ m \in \mathbb{Z}_{\geq 0}, \ \forall w \in \Pi_\lambda. \end{aligned} \tag{39}$$

For $m = 0$, the claim (39) follows by using irreducibility of Π_λ as a $\widehat{\mathfrak{b}}_1$-module and (38). Assume now that $v \in L^{Vir}(d_k, h)_{h+m}$ for $m > 0$. Then there is m_0, $0 < m_0 \leq m$ so that $L(m_0)v \neq 0$ and $L(m_0)v \in L^{Vir}(c_k, h)_{h+m-m_0}$. Since

$$f(m_0)(v \otimes w_\lambda) = (k+2)\lambda L(m_0)v \otimes w_\lambda,$$

by induction we have that $L(m_0)v \otimes w_\lambda$ is a cyclic vector. So $v \otimes w_\lambda$ is also cyclic. By using again the irreducibility of Π_λ as $\widehat{\mathfrak{b}}_1$-module, we see that $v \otimes w$ is cyclic for all $w \in \Pi_\lambda$. The proof follows. □

As a consequence, we shall describe the structure of simple Whittaker module $Wh_{\widehat{sl_2}}(\lambda, 0, k, a)$ at admissible levels, and show that these modules are $V_k(sl_2)$-modules.

Theorem 8.2. *Assume that k is admissible, non-integral, and $\lambda \neq 0$. Then we have:*

1. $Wh_{\widehat{sl(2)}}(\lambda, 0, k, a) \cong L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_\lambda$, *where* $a = h^{r,s}_{p,p'} + k/4$.
2. *The set*

$$\{Wh_{\widehat{sl(2)}}(\lambda, 0, k, h + k/4) \mid h \in \mathcal{S}_{p,p'}\}$$

provides all irreducible $\mathbb{Z}_{\geq 0}$-graded $V_k(sl(2))$-modules which are Whittaker $\widehat{sl(2)}$-modules.

Proof. Since $L^{Vir}(d_{p,p'},h)$ for $h \in \mathcal{S}_{p,p'}$ is a $L^{Vir}(d_{p,p'},0)$-module, we conclude that $L^{Vir}(d_{p,p'},h)\otimes\Pi_\lambda$ is a $L^{Vir}(d_{p,p'},0)\otimes\Pi(0)$-module and therefore a $V_k(sl(2))$-module.

Assume that $Wh_{\widehat{sl(2)}}(\lambda,0,k,h+k/4)$ is a $V_k(sl(2))$-module. We proved in Theorem 8.1 that $Wh_{\widehat{sl(2)}}(\lambda,0,k,h+k/4)\cong L^{Vir}(d_{p,p'},h)\otimes\Pi_\lambda$ for certain $h\in\mathbb{C}$ and that $L_{sug}(0)$ acts on lowest weight component as $h+\frac{k}{4}$. By using description of Zhu's algebra for $V_k(sl(2))$ (cf. [10,59]) we see that $L(0)$ must act on lowest component as $h\cdot \mathrm{Id}$ for $h\in\mathcal{S}_{p,p'}$. Therefore, $Wh_{\widehat{sl(2)}}(\lambda,0,k,h+k/4)\cong L^{Vir}(d_{p,p'},h)\otimes\Pi_\lambda$ for $h\in\mathcal{S}_{p,p'}$. The proof follows. □

9. Screening Operators and Logarithmic Modules for $V_k(sl(2))$

This section gives a vertex-algebraic interpretation of the construction of logarithmic modules from [38, Section 5]. By using the embedding of $V_k(sl(2))$ in the vertex algebra $L^{Vir}(d_{p,p'},0)\otimes\Pi(0)\subset L^{Vir}(d_{p,p'},0)\otimes V_L$, we are able to use methods [13] to construct logarithmic modules for admissible affine vertex algebra $V_k(sl(2))$. Formula for the screening operator S appeared in [38]. In the case $k=-\frac{4}{3}$, the construction of logarithmic modules reconstructs modules from [13, Section 8] and [42].

Note that the basic definitions and constructions related with logarithmic modules were discussed in Sect. 2.

9.1. Screening operators. First we notice that $L^{Vir}(d_{p,p'},h^{2,1}_{p,p'})$ is an irreducible $L^{Vir}(d_{p,p'},0)$-module generated by lowest weight vector $v_{2,1}$ of conformal weight

$$h^{2,1}:=h^{2,1}_{p,p'}=\frac{3p'-2p}{4p}=\frac{3}{4}k+1.$$

Let us now consider $L^{Vir}(c_{p,p'},0)\otimes\Pi(0)$-module

$$\mathcal{M}_{2,1}=L^{Vir}(d_{p,p'},0)\otimes\Pi(0).(v_{2,1}\otimes e^{\nu})=L^{Vir}(d_{p,p'},h^{2,1}_{p,p'})\otimes\Pi_{(1)}(-\tfrac{k}{2}).$$

Note that $\mathcal{M}_{2,1}$ has integral weights with respect to $L_{sug}(0)$. Using construction from [49], which was reviewed in Sect. 2.2, we have the extended vertex algebra

$$\mathcal{V}=L^{Vir}(d_{p,p'},0)\otimes\Pi(0)\bigoplus\mathcal{M}_{2,1}.$$

Note also

$$\begin{aligned}
L(-2)v_{2,1}&=\frac{1}{k+2}L(-1)^2v_{2,1}.\\
[L(n),(v_{2,1})_m]&=((h^{2,1}-1)(n+1)-m)(v_{2,1})_{m+n}\qquad(m,n\in\mathbb{Z}).\\
[L(-2),(v_{2,1})_{-1}]&=(2-h^{2,1})(v_{2,1})_{-3}\\
[L(-2),(v_{2,1})_{0}]&=(1-h^{2,1})(v_{2,1})_{-2}\\
[L(-2),(v_{2,1})_{1}]&=-h^{2,1}(v_{2,1})_{-1}
\end{aligned}$$

Let $s=v_{2,1}\otimes e^{\nu}$. By using formula (19) we get

$$L_{sug}(n)s=\delta_{n,0}s\quad(n\ge 0).$$

Therefore

$$S = s_0 = \text{Res}_z Y(s, z)$$

commute with the action of the Virasoro algebra $L_{sug}(n)$, $n \in \mathbb{Z}$.

We want to see that S commutes with $\widehat{sl(2)}$-action. The arguments for claim were essentially presented in [38]. The following lemma can be proved by direct calculation in lattice vertex algebras.

Lemma 9.1. [38] *We have*

$$\begin{aligned} s_2 f &= 2(k+1)v_{2,1} \otimes e^{\nu - \frac{2}{k}(\mu-\nu)} \\ s_1 f &= kL(-1)v_{2,1} \otimes e^{\nu - \frac{2}{k}(\mu-\nu)} + (k+2)v_{2,1} \otimes \nu(-1)e^{\nu - \frac{2}{k}(\mu-\nu)} \\ s_0 f &= Sf = 0. \end{aligned}$$

Proposition 9.2. [38] *We have:*

$$[S, \widehat{sl(2)}] = 0,$$

i.e., S is a scrrening operator.

Proof. Since

$$s_n e = s_n h = 0 \quad (n \geq 0),$$

we get $[S, e(n)] = [S, h(n)] = 0$.

By using Lemma 9.1 we get $[S, f(n)] = (Sf)(n) = 0$. The claim follows. □

9.2. Construction of logarithmic modules for $V_k(sl(2))$.

Lemma 9.3. *Assume that $\ell \in \mathbb{Z}$, $1 \leq s \leq p'-1$, $1 \leq r \leq p-2$ and*

$$\lambda \equiv \lambda^+_{r,s} = \frac{1}{2}\left(s - 1 - (k+2)(r-1)\right) \quad mod(\mathbb{Z}). \tag{40}$$

Then we have:

1. $\mathcal{M}^{\ell,+}_{r,s}(\lambda) = L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{(\ell)}(\lambda) \bigoplus L^{Vir}(d_{p,p'}, h^{r+1,s}_{p,p'}) \otimes \Pi_{(\ell+1)}(-\frac{k}{2}+\lambda)$ *is a $\mathcal{V}$-module.*
2. $S^2 = 0$ *on* $\mathcal{M}^{\ell,+}_{r,s}(\lambda)$.
3. *Let* $\lambda = \lambda^+_{r,s} - 1$. *Then*

$$S(v_{r,s} \otimes e^{\ell\mu + \lambda c}) = C v_{r+1,s} \otimes e^{(\ell+1)\mu + (\lambda - k/2)c},$$

where $C \neq 0$. In particular, $S \neq 0$ on $\mathcal{M}^{\ell,+}_{r,s}(\lambda)$.

Proof. In general, $L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{(\ell)}(\lambda)$ is a $\mathbb{Z}$-graded module whose conformal weights are congruent mod($\mathbb{Z}$) to $h^{r,s}_{p,p'} + \frac{1}{4}(k\ell^2 + 4(\ell+1)\lambda)$. By direct calculation we see that

$$h^{r+1,s}_{p,p'} + \frac{1}{4}((\ell+1)^2 k + 4(\ell+2)(\lambda - \frac{k}{2})) \equiv h^{r,s}_{p,p'} + \frac{1}{4}(\ell^2 k + 4(\ell+1)\lambda) \qquad \mathrm{mod}(\mathbb{Z})$$

if and only if (40) holds. Therefore, we conclude $L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{(\ell)}(\lambda)$ and $L^{Vir}(d_{p,p'}, h^{r,s+1}_{p,p'}) \otimes \Pi_{(\ell+1)}(-\frac{k}{2} + \lambda)$ have conformal weights congruent mod($\mathbb{Z}$) if and only if (40) holds.

Let $\mathcal{Y}_1(\cdot, z)$ be the non-trivial intertwining operator of type

$$\binom{L^{Vir}(d_{p,p'}, h^{r+1,s}_{p,p'})}{L^{Vir}(d_{p,p'}, h^{2,1}_{p,p'}) \;\; L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'})}.$$

Then

$$\mathcal{Y}_1(v, z) = \sum_{r \in \mathbb{Z}+\Delta} v_r z^{-r-1} \quad (v \in L^{Vir}(c_{p,p'}, h^{2,1}_{p,p'}))$$

where

$$\Delta = h^{2,1}_{p,p'} + h^{r,s}_{p,p'} - h^{r+1,s}_{p,p'} = \frac{1}{2}((s-1) - (r-1)(k+2)) = \lambda^{+}_{r,s}.$$

In particular, we have

$$(v_{2,1})_{\Delta-1} v_{r,s} = C_2 v_{r+1,s} \quad (C_2 \neq 0), (v_{2,1})_{\Delta+n} v_{r,s} = 0 \quad (n \in \mathbb{Z}_{\geq 0}).$$

Let $\mathcal{Y}_2(\cdot, z)$ be the non-trivial intertwining operator of type

$$\binom{\Pi_{(\ell+1)}(-\frac{k}{2} + \lambda)}{\Pi_{(1)}(-\frac{k}{2}) \;\; \Pi_{(\ell)}(\lambda)}.$$

Then

$$\mathcal{Y}_2(v, z) = \sum_{r \in \mathbb{Z}-\lambda} v_r z^{-r-1} \quad (v \in \Pi_{(1)}(-\tfrac{k}{2}) = \Pi(0).e^{\nu}).$$

In particular, we have

$$\begin{aligned} e^{\nu}_{-\lambda-1} e^{\ell\mu+\lambda c} &= C_1 e^{(\ell+1)\mu+(\lambda-k/2)c} \quad (C_1 \neq 0) \\ e^{\nu}_{-\lambda-n-1} e^{\ell\mu+\lambda c} &\neq 0, \quad e^{\nu}_{-\lambda+n} e^{\ell\mu+\lambda c} = 0 \quad (n \in \mathbb{Z}_{\geq 0}). \end{aligned}$$

We conclude that there is an non-trivial intertwining operator $\mathcal{Y} = \mathcal{Y}_1 \otimes \mathcal{Y}_2$ of type

$$\binom{L^{Vir}(c_{p,p'}, h^{r+1,s}_{p,p'}) \otimes \Pi_{(\ell+1)}(-\frac{k}{2} + \lambda)}{L^{Vir}(c_{p,p'}, h^{2,1}_{p,p'}) \otimes \Pi_{(1)}(-\frac{k}{2}) \;\; L^{Vir}(c_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{(\ell)}(\lambda)}$$

with integral powers of z. Now, the assertion (1) follows by applying Lemma 2.3.

By construction we have $S^2 = 0$ on $\widetilde{\mathcal{M}^{\ell,+}_{r,s}}(\lambda)$, so (2) holds.

For $\lambda = \lambda_{r,s}^{+} - 1$ we have

$$\begin{aligned} S v_{r,s} \otimes e^{\ell\mu+\lambda c} &= (v_{2,1})_{\Delta-1} v_{r,s} \otimes e^{\nu}_{-\lambda-1} e^{\ell\mu+\lambda c} \\ &= C_1 \cdot C_2 v_{r+1,s} \otimes e^{(\ell+1)\mu+(\lambda-k/2)c} \neq 0. \end{aligned} \tag{41}$$

The proof follows. $\square$

By using the Virasoro intertwining operator of type

$$\binom{L^{Vir}(d_{p,p'}, h^{r-1,s}_{p,p'})}{L^{Vir}(d_{p,p'}, h^{2,1}_{p,p'}) \;\; L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'})}$$

and an analogous proof to that of Lemma 9.3 we get:

Lemma 9.4. *Assume that $\ell \in \mathbb{Z}$, $1 \le s \le p'-1$, $2 \le r \le p-1$ and*

$$\lambda \equiv \lambda^{-}_{r,s} = -\frac{1}{2}\left(s+1-(k+2)(r+1)\right) \quad mod(\mathbb{Z}). \tag{42}$$

Then we have:

1. $\mathcal{M}^{\ell,-}_{r,s}(\lambda) = L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{(\ell)}(\lambda) \bigoplus L^{Vir}(d_{p,p'}, h^{r-1,s}_{p,p'}) \otimes \Pi_{(\ell+1)}(-\frac{k}{2}+\lambda)$ *is a $\mathcal{V}$-module.*
2. $S^2 = 0$ *on* $\mathcal{M}^{\ell,-}_{r,s}(\lambda)$.
3. *Let* $\lambda = \lambda^{-}_{r,s} - 1$. *Then*

$$S(v_{r,s} \otimes e^{\ell\mu+\lambda c}) = C v_{r-1,s} \otimes e^{(\ell+1)\mu+(\lambda-k/2)c},$$

where $C \neq 0$.

The next result shows that at a non-integral admissible levels, logarithmic modules always exist. By applying Theorem 2.2 and taking $v = s$ we get:

Proposition 9.5. *Assume that $(\mathcal{M}, \mathcal{Y}_{\mathcal{M}})$ is any $\mathcal{V}$-module. Then*

$$(\widetilde{\mathcal{M}}, \widetilde{\mathcal{Y}_{\mathcal{M}}}(\cdot, z)) := (\mathcal{M}, \mathcal{Y}_{\mathcal{M}}(\Delta(s, z)\cdot, z))$$

is a $V_k(sl(2))$-module such that

$$\widetilde{L}_{sug}(0) = L_{sug}(0) + S.$$

In particular, $\widetilde{\mathcal{V}}$ is a logarithmic $V_k(sl(2))$-module of $\widetilde{L}_{sug}(0)$ nilpotent rank two.

Proof. First claim follows directly by applying Theorem 2.2.
The assertion (1) follows from the following observations:

- $L_{sug}(0)$ acts semi-simply on $\mathcal{V}$, and $\widetilde{L}_{sug}(0) - L_{sug}(0) = S$ on $\widetilde{\mathcal{V}}$.
- By construction $S(L^{Vir}(d_{p,p'}, 0) \otimes \Pi(0)) \subset \mathcal{M}_{2,1}$, $S(\mathcal{M}_{2,1}) = 0$, so $S^2 = 0$ on $\mathcal{V}$. Since $S\nu(-1)\mathbf{1} = \frac{k}{2} v_{2,1} \otimes e^{\nu} \neq 0$ we have

$$(\widetilde{L}_{sug}(0) - L_{sug}(0)) \neq 0, \; (\widetilde{L}_{sug}(0) - L_{sug}(0))^2 = 0 \quad \text{on } \widetilde{\mathcal{V}}.$$

$\square$

Using Lemma 9.3 we obtain:

Corollary 9.6. *Assume that $\ell \in \mathbb{Z}$, $1 \le s \le p' - 1$, $1 \le r \le p - 2$ and $\lambda = \lambda^+_{r,s}$. Then we have:*

1. $\widetilde{\mathcal{M}^{\ell,+}_{r,s}}(\lambda)$ *is a logarithmic $V_k(sl(2))$-module of $\widetilde{L}_{sug}(0)$ nilpotent rank two,*
2. *The logarithmic module $\widetilde{\mathcal{M}^{\ell,+}_{r,s}}(\lambda)$ appears in the following non-split extension of weight $V_k(sl(2))$-modules:*

$$0 \to L^{Vir}(d_{p,p'}, h^{r+1,s}_{p,p'}) \otimes \Pi_{(\ell+1)}\left(-\frac{k}{2} + \lambda\right) \to \widetilde{\mathcal{M}^{\ell,+}_{r,s}}(\lambda)$$
$$\to L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{(\ell)}(\lambda) \to 0.$$

Proof. Lemma 9.3 gives that $S \neq 0$ and $S^2 = 0$ on $\mathcal{M}^{\ell}_{r,s}(\lambda)$.
This implies that

$$(\widetilde{L}_{sug}(0) - L_{sug}(0)) \neq 0, \; (\widetilde{L}_{sug}(0) - L_{sug}(0))^2 = 0 \quad \text{on} \quad \widetilde{\mathcal{M}^{\ell,+}_{r,s}}(\lambda),$$

which proves the assertion (1).

Note that $L^{Vir}(d_{p,p'}, h^{r+1,s}_{p,p'}) \otimes \Pi_{(\ell+1)}(-\frac{k}{2} + \lambda)$ is a submodule of $\widetilde{\mathcal{M}^{\ell,+}_{r,s}}(\lambda)$ on which S acts trivially.
Therefore, $L^{Vir}(d_{p,p'}, h^{r+1,s}_{p,p'}) \otimes \Pi_{(\ell+1)}(-\frac{k}{2} + \lambda)$ is a weight submodule. The quotient module is isomorphic to the weight module $L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{(\ell)}(\lambda)$.

Since $\widetilde{\mathcal{M}^{\ell,+}_{r,s}}(\lambda)$ is non-weight module by (1), we have that the sequence in (2) does not split. The proof follows. □

Similarly, using Lemma 9.4 we obtain:

Corollary 9.7. *Assume that $\ell \in \mathbb{Z}$, $1 \le s \le p' - 1$, $2 \le r \le p - 1$ and $\lambda = \lambda^-_{r,s}$. Then we have:*

1. $\widetilde{\mathcal{M}^{\ell,-}_{r,s}}(\lambda)$ *is a logarithmic $V_k(sl(2))$-module of $\widetilde{L}_{sug}(0)$ nilpotent rank two,*
2. *The logarithmic module $\widetilde{\mathcal{M}^{\ell,-}_{r,s}}(\lambda)$ appears in the following non-split extension of weight $V_k(sl(2))$-modules:*

$$0 \to L^{Vir}(d_{p,p'}, h^{r-1,s}_{p,p'}) \otimes \Pi_{(\ell+1)}\left(-\frac{k}{2} + \lambda\right) \to \widetilde{\mathcal{M}^{\ell,-}_{r,s}}(\lambda)$$
$$\to L^{Vir}(d_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{(\ell)}(\lambda) \to 0.$$

10. A Realization of the $N = 3$ Superconformal Vertex Algebra $W_{k'}(spo(2, 3), f_\theta)$ for $k' = -1/3$

The cases $k \in \{-1/2, -4/3\}$ were already studied in the literature. In these cases the quantum Hamiltonian reduction maps $V_k(sl(2))$ to the trivial vertex algebra, and therefore the affine vertex algebra $V_k(sl(2)$ is realizead as a vertex subalgebra of $\Pi(0)$. In the case $k = -1/2$, $V_k(sl(2))$ admits a realization as a subalgebra of the Weyl vertex algebra and it is also related with the triplet vertex algebra $\mathcal{W}(2)$ with central charge $c = -2$ (cf. [37,55]). In [4], the author related $V_{-4/3}(sl(2))$ with the triplet vertex algebra $\mathcal{W}(3)$ at central charge $c = -7$.

Let $k = -2/3$. Then $V_k(sl(2))$ is realized as a subalgebra of $L^{Vir}(d_{3,4}, 0) \otimes \Pi(0)$. But $L^{Vir}(d_{3,4}, 0)$ is exactly the even subalgebra of the fermionic vertex superalgebra $F = L^{Vir}(d_{3,4}, 0) \oplus L^{Vir}(d_{3,4}, \frac{1}{2})$ generated by the odd field $\Psi(z) = \sum_{m\in\mathbb{Z}} \Psi(m + \frac{1}{2})z^{-m-1}$ (see Sect. 11). The Virasoro vector is $\omega_F = \frac{1}{2}\Psi(-\frac{3}{2})\Psi(-\frac{1}{2})\mathbf{1}$. Let $\gamma = \frac{2}{k}\nu$, $\varphi = \frac{2}{k}\mu$. Then

$$\langle\gamma, \gamma\rangle = -\langle\varphi, \varphi\rangle = 3.$$

Let $D = \mathbb{Z}\gamma$. Then $V_D = M_\gamma(1) \otimes \mathbb{C}[D]$ is the lattice vertex superalgebra, where $M_\gamma(1)$ is the Heisenberg vertex algebra generated by γ and $\mathbb{C}[D]$ the group algebra of the lattice D. The screening operator S is then expressed as

$$S = \mathrm{Res}_z : \Psi(z)e^{\nu}(z) := \mathrm{Res}_z : \Psi(z)e^{-\frac{1}{3}\gamma}(z) : .$$

Define also

$$Q = \mathrm{Res}_z : \Psi(z)e^{\gamma}(z) : .$$

We have:

Proposition 10.1. [11]

1. $SW(1) \cong Ker_{F\otimes V_D}S$ *is isomorphic to the* $N = 1$ *super-triplet vertex algebra at central charge* $c = -5/2$ *strongly generated by*

$$X = e^{-\gamma},\ H = QX,\ Y = Q^2X,\ \widehat{X} = \Psi(-1/2)X,\ \widehat{H} = Q\widehat{X},\ \widehat{Y} = Q^2\widehat{X}$$

and superconformal vector $\tau = \frac{1}{\sqrt{3}}\left(\Psi(-\frac{1}{2})\gamma(-1) + 2\Psi(-\frac{3}{2})\right)\mathbf{1}$ *and corresponding conformal vector*

$$\omega_{N=1} = \frac{1}{2}\tau_0\tau = \frac{1}{6}\left(\gamma(-1)^2 + 2\gamma(-2)\right)\mathbf{1} + \frac{1}{2}\Psi\left(-\frac{3}{2}\right)\Psi\left(-\frac{1}{2}\right)\mathbf{1}.$$

2. $\overline{SM(1)} \cong Ker_{F\otimes M_\gamma(1)}S$ *is isomorphic to the* $N = 1$ *super singlet vertex algebra at central charge* $c = -5/2$ *strongly generated by* $\tau, \omega_{N=1}, H, \widehat{H}$.

Consider the lattice vertex algebra $F_{-3} = V_{\mathbb{Z}\varphi}$. We shall now see that the admissible affine vertex algebra $V_{-2/3}(sl(2))$ is a vertex subalgebra of $SW(1) \otimes F_{-3}$. Note that $\gamma(0) - \varphi(0)$ acts semisimply on $SW(1) \otimes F_{-3}$ and we have the following vertex algebra

$$\mathcal{U} = \{v \in SW(1) \otimes F_{-3} |\ (\gamma(0) - \varphi(0))v = 0\}.$$

Following [44], we identify the $N = 3$ superconformal vertex algebra with affine W-algebra $W_{k'}(spo(2, 3), f_\theta)$. By applying results on conformal embeddings from [19, Theorem 6.8 (12)], we see that the vertex algebra $W_{k'}(spo(2, 3), f_\theta)$ for $k' = -1/3$ is isomorphic to the simple current extension of $V_{-2/3}(sl(2))$:

$$W_{k'}(spo(2, 3), f_\theta) = L_{A_1}\left(-\frac{2}{3}\Lambda_0\right) \bigoplus L_{A_1}\left(-\frac{8}{3}\Lambda_0 + 2\Lambda_1\right).$$

Theorem 10.2. *We have:*

1. $\mathcal{U} \cong W_{k'}(spo(2, 3), f_\theta)$ *for* $k' = -1/3$.
2. $Com(M_h(1), W_{k'}(spo(2, 3), f_\theta)) \cong \overline{SM(1)}$.

3. *Ker*$(sl(2), -\frac{2}{3}) \cong \overline{SM(1)}^0$, *where* $\overline{SM(1)}^0$ *is even subalgebra of the supersinglet vertex algebra* $\overline{SM(1)}$.

Proof. Since

$$Y = Q^2 X = \left(-6\Psi\left(-\frac{3}{2}\right)\Psi\left(-\frac{1}{2}\right) + \gamma(-1)^2 - \gamma(-2)\right)e^{\gamma}$$
$$= -9\left((k+2)\omega_F - \nu(-1)^2 - (k+1)\nu(-2)\right)e^{\gamma} \quad (k = -2/3)$$

we have that

$$e = X \otimes e^{\varphi} = e^{\varphi-\gamma},$$
$$h = -\frac{2}{3}\varphi,$$
$$f = -\frac{1}{9}Y \otimes e^{-\varphi} = -\frac{1}{9}Q^2 e^{-\varphi-\gamma},$$
$$\omega_{sug} = \omega_{N=1} - \frac{1}{6}\varphi(-1)^2\mathbf{1}.$$

This implies that $V_{-2/3}(sl(2))$ is a vertex subalgebra of $\mathcal{U}$. Therefore, $\mathcal{U}$ is a $V_{-2/3}(sl(2))$-module which is $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded $\mathcal{U} = \bigoplus_{m\in\frac{1}{2}\mathbb{Z}_{\geq 0}}\mathcal{U}_m$ with respect to $L_{sug}(0)$. One directly sees that $\mathcal{U}_{1/2} = \{0\}$ and that $\mathcal{U}_{3/2} = \text{span}_{\mathbb{C}}\{\widehat{X} \otimes e^{-\varphi}, \tau, \widehat{Y} \otimes e^{-\varphi}\}$. Then $\mathcal{U}_{3/2}$ generates a $V_{-2/3}(sl(2))$-module isomorphic to $L_{A_1}(-\frac{8}{3}\Lambda_0 + 2\Lambda_1)$. Since $\mathcal{U}$ is completely reducible as $V_{-2/3}(sl(2))$-module we easily conclude that

$$\mathcal{U} \cong L_{A_1}\left(-\frac{2}{3}\Lambda_0\right)\bigoplus L_{A_1}\left(-\frac{8}{3}\Lambda_0 + 2\Lambda_1\right).$$

Since $\mathcal{U}$ is simple and the extension of $V_{-2/3}(sl(2))$ by its simple current module $L_{A_1}(-\frac{8}{3}\Lambda_0 + 2\Lambda_1)$ is unique, we get the assertion (1). Assertion (2) follows from

$$\text{Com}(M_h(1), W_{k'}(spo(2,3), f_\theta)) = \{v \in SW(1) \otimes F_{-3} \mid \varphi(n)v = (\gamma(0) - \varphi(0))v = 0,\ n \geq 0\} \cong \text{Ker}_{SW(1)}\gamma(0) = \overline{SM(1)}.$$

(3) easily follows from (2). □

11. Realization of $V_k(osp(1,2))$

A free field realization of $\widehat{osp(1,2)}$ of the Wakimoto type was presented in [36]. In this section, we study an explicit realization of affine vertex algebras associated to $\widehat{osp(1,2)}$ which generalize realizations for $\widehat{sl(2)}$ from previous sections. Since the quantum Hamiltonian reduction of $V^k(osp(1,2))$ is the $N = 1$ Neveu–Schwarz vertex algebra $V^{ns}(c_k, 0)$ where $c_k = \frac{3}{2} - 12\frac{(k+1)^2}{2k+3}$ (cf. [44, Section 8.2]), one can expect that inverse of the quantum Hamoltonian reduction (assuming that it should exist) gives a realization of the form $V^{ns}(c_k, 0) \otimes \mathcal{F}$, where $\mathcal{F}$ is a certain vertex algebra of free-fields. In this section, we show that for $\mathcal{F}$ one can take the tensor product of the fermionic vertex algebra F at central charge $1/2$ and the lattice type vertex algebra $\Pi^{1/2}(0)$ introduced in Sect. 4.

11.1. Affine vertex algebra $V_k(osp(1,2))$. Recall that $\mathfrak{g} = osp(1,2)$ is the simple complex Lie superalgebra with basis $\{e, f, h, x, y\}$ such that the even part $\mathfrak{g}^0 = \text{span}_{\mathbb{C}}\{e, f, h\}$ and the odd part $\mathfrak{g}^1 = \text{span}_{\mathbb{C}}\{x, y\}$. The anti-commutation relations are given by

$$\begin{aligned}
&[e,f] = h,\ [h,e] = 2e,\ [h,f] = -2f \\
&[h,x] = x,\ [e,x] = 0,\ [f,x] = -y \\
&[h,y] = -y,\ [e,y] = -x\ [f,y] = 0 \\
&\{x,x\} = 2e,\ \{x,y\} = h,\ \{y,y\} = -2f.
\end{aligned}$$

Choose the non-degenerate super-symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$ such that non-trivial products are given by

$$(e,f) = (f,e) = 1,\ (h,h) = 2,\ (x,y) = -(y,x) = 2.$$

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] + \mathbb{C}K$ be the associated affine Lie superalgebra, and $V^k(\mathfrak{g})$ (resp. $V_k(\mathfrak{g})$) the associated universal (resp. simple) affine vertex algebra. As usually, we identify $x \in \mathfrak{g}$ with $x(-1)\mathbf{1}$.

11.2. Clifford vertex algebras. Consider the Clifford algebra Cl with generators $\Psi_i(r)$, $r \in \frac{1}{2} + \mathbb{Z}$, $i = 1, \cdots, n$ and relations

$$\{\Psi_i(r), \Psi_j(s)\} = \delta_{r+s,0}\delta_{i,j}, \quad (r, s \in \tfrac{1}{2} + \mathbb{Z},\ 1 \leq i, j \leq n).$$

Then the fields

$$\Psi_i(z) = \sum_{m \in \mathbb{Z}} \Psi_i(m + \tfrac{1}{2}) z^{-m-1} \quad (i = 1, \ldots, n)$$

generate on $F_n = \bigwedge(\Psi_i(-n - 1/2) \mid n \in \mathbb{Z}_{\geq 0})$ a unique structure of the vertex superalgebra with conformal vector

$$\omega_{F_n} = \sum_{i=1}^{n} \frac{1}{2} \Psi_i(-\tfrac{3}{2})\Psi_i(-\tfrac{1}{2})\mathbf{1}$$

of central charge $n/2$. Let $F = F_1$. Then F is a vertex operator superalgebra of central charge $c = d_{3,4} = 1/2$. Moreover $F = F^{even} \oplus F^{odd}$ and

$$F^{even} = L^{Vir}(d_{3,4}, 0), \quad F^{even} = L^{Vir}(d_{3,4}, 1/2).$$

Let σ be the canonical automorphism of F of order two. The vertex algebra F has precisely two irreducible σ-twisted modules $M^{\pm}$. Twisted modules can be also constructed explicitly as an exterior algebra

$$M^{\pm} = \bigwedge (\Psi(-n) \mid n \in \mathbb{Z}_{>0})$$

which is an irreducible module for twisted Clifford algebra CL^{tw} with generators $\Psi(r)$, $r \in \mathbb{Z}$, and relations

$$\{\Psi(r), \Psi_j(s)\} = \delta_{r+s,0}, \quad (r, s \in \mathbb{Z}).$$

$\Psi(0)$ acts on top component of $M^{\pm}$ as $\pm\frac{1}{\sqrt{2}}\text{Id}$.

As a $L^{Vir}(d_{3,4},0)$-module, we have that $M^{\pm} \cong L^{Vir}(d_{3,4},\frac{1}{16})$.

Note also that the character of $M^{\pm}$ is given by

$$\mathrm{ch}[M^{\pm}](q) = \frac{f_2(\tau)}{\sqrt{2}} = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1+q^n) \tag{43}$$

where $f_2(\tau) = \sqrt{2} q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1+q^n)$ is the Weber function.

11.3. The general case. Let $V^{ns}(c_{p,q},0)$ be the universal $N=1$ Neveu–Schwarz vertex superalgebra with central charge $c_{p,q} = \frac{3}{2}(1 - \frac{2(p-q)^2}{pq})$. Let $L^{ns}(c_{p,q},0)$ be its simple quotient. If

$$p,q \in \mathbb{Z},\ \ p,q \geq 2,\ \left(\frac{p-q}{2}, q\right) = 1, \tag{44}$$

then $L^{ns}(c_{p,q},0)$ is called a minimal $N=1$ Neveu–Schwarz vertex operator superalgebra. It is a rational vertex operator superalgebra [1].

Proposition 11.1. *Let $p,q \in \mathbb{Z}$, $p,q \neq 0$. We have:*

1. *Assume that $p,q,p+q \neq 0$. The Virasoro vertex operator algebra $V^{Vir}(d_{p,\frac{p+q}{2}},0) \otimes V^{Vir}(d_{\frac{p+q}{2},q},0)$ is conformally embedded in $V^{ns}(c_{p,q},0) \otimes F$ and $\omega_{p,q} + \omega_F = \omega_{p,\frac{p+q}{2}} + \omega_{\frac{p+q}{2},q}$ where*

$$\omega_{\frac{p+q}{2},q} = \frac{p}{p+q}\omega_{p,q} + i\frac{\sqrt{pq}}{p+q}G(-\tfrac{3}{2})\Psi\left(-\frac{1}{2}\right)\mathbf{1} + \frac{2q-p}{p+q}\omega_F$$
$$\omega_{p,\frac{p+q}{2}} = \frac{q}{p+q}\omega_{p,q} - i\frac{\sqrt{pq}}{p+q}G(-\tfrac{3}{2})\Psi\left(-\frac{1}{2}\right)\mathbf{1} + \frac{2p-q}{p+q}\omega_F$$

2. *Assume that $p+q=0$. Then*

$$t_{p,q} := \frac{1}{2}\left(-\omega_{p,q} - G\left(-\frac{3}{2}\right)\Psi\left(-\frac{1}{2}\right)\mathbf{1} + 3\omega_F\right) \in L^{ns}(c_{p,q},0) \otimes F$$

   is a commutative vector in the vertex algebra $L^{ns}(\frac{27}{2},0) \otimes F$. The vertex subalgebra generated by $t_{p,q}$ is isomorphic to the commutative vertex algebra $M_T(0)$.

3. *Assume that p,q satisfy condition (44). Then the rational Virasoro vertex operator algebra $L^{Vir}(d_{p,\frac{p+q}{2}},0) \otimes L^{Vir}(d_{\frac{p+q}{2},q},0)$ is conformally embedded in $L^{ns}(c_{p,q},0) \otimes F$.*

Proof. Assertions (1) follows by direct calculation. One can also directly show that $T = t_{p,q}$ is a commutative vector in $L^{ns}(\frac{27}{2},0) \otimes F$. Let $\langle T \rangle$ be the vertex subalgebra generated by T. By using fact that vectors

$$L(-n_1)\cdots L(-n_r)\mathbf{1} \qquad (n_1 \geq \cdots n_r \geq 2)$$

are linearly independent in $L^{ns}(\frac{27}{2},0)$ and expression for T one can easily show that the vectors

$$T(-n_1)\cdots T(-n_r)\mathbf{1} \qquad (n_1 \geq \cdots n_r \geq 2)$$

provide a basis of $\langle T \rangle$. So $\langle T \rangle \cong M_T(0)$.

Assertion (3) was proved in [3] (see also [47,50]). □

Theorem 11.2. *Assume that* $k + \frac{3}{2} = \frac{p}{2q} \neq 0$. *There exists a non-trivial vertex superalgebra homomorphism*

$$\overline{\Phi} : V^k(osp(1,2)) \to V^{ns}(c_{p,q}, 0) \otimes F \otimes \Pi^{1/2}(0)$$

such that

$$\begin{aligned}
e &\mapsto e^{\frac{2}{k}(\mu-\nu)},\\
h &\mapsto 2\mu(-1),\\
f &\mapsto \left[\Omega_{p,q} - \nu(-1)^2 - (k+1)\nu(-2)\right] e^{-\frac{2}{k}(\mu-\nu)}\\
x &\mapsto \sqrt{2}\Psi\left(-\frac{1}{2}\right) e^{\frac{1}{k}(\mu-\nu)}\\
y &\mapsto \sqrt{2}\left[-\frac{\sqrt{-2k-3}}{2}G(-3/2) + \Psi\left(-\frac{1}{2}\right)\nu(-1) + \frac{2k+1}{2}\Psi\left(-\frac{3}{2}\right)\right] e^{-\frac{1}{k}(\mu-\nu)},
\end{aligned}$$

where $\Omega_{p,q} = (k+2)\omega_{\frac{p+q}{2},q}$ *if* $k \neq -2$ *and* $\Omega_{p,q} = t_{p,q}$ *if* $k = -2$.

The proof of Theorem 11.2 will be presented in Sect. 12.

Theorem 11.3. *Assume that* $k + \frac{3}{2} = \frac{p}{2q} \neq 0$ *and that* p, q *satisfy condition* (44).

1. *There exists a non-trivial vertex superalgebra homomorphism*

$$\overline{\Phi} : V_k(osp(1,2)) \to L^{ns}(c_{p,q}, 0) \otimes F \otimes \Pi^{1/2}(0)$$

such that

$$\begin{aligned}
e &\mapsto e^{\frac{2}{k}(\mu-\nu)},\\
h &\mapsto 2\mu(-1),\\
f &\mapsto \left[(k+2)\omega_{\frac{p+q}{2},q} - \nu(-1)^2 - (k+1)\nu(-2)\right] e^{-\frac{2}{k}(\mu-\nu)}\\
x &\mapsto \sqrt{2}\Psi\left(-\frac{1}{2}\right) e^{\frac{1}{k}(\mu-\nu)}\\
y &\mapsto \sqrt{2}\left[-\frac{\sqrt{-2k-3}}{2}G(-3/2) + \Psi\left(-\frac{1}{2}\right)\nu(-1) + \frac{2k+1}{2}\Psi\left(-\frac{3}{2}\right)\right] e^{-\frac{1}{k}(\mu-\nu)},
\end{aligned}$$

2. $\omega_{p,\frac{p+q}{2}} \in Com(V_k(sl(2)), V_k(osp(1,2)))$.

Proof. 1. Using Theorem 11.2 we get a homomorphism $\widetilde{\Phi} : V^k(osp(1,2)) \to L^{ns}(c_{p,q}, 0) \otimes F \otimes \Pi^{1/2}(0)$. Then Proposition 11.1 implies that $\omega_{\frac{p+q}{2},q}$ generates a subalgebra of $L^{ns}(c_{p,q}, 0) \otimes F$ isomorphic to the minimal Virasoro vertex algebra $L^{Vir}(d_{\frac{p+q}{2},q}, 0)$. Therefore, Theorem 5.5 gives that e, f, h generate the simple admissible affine vertex algebra $V_k(sl(2))$.
At admissible level k, the vertex algebra $V^k(osp(1,2))$ contains a unique singular vector, i.e., the maximal ideal of $V^k(osp(1,2))$ is simple. So we have two possibilities:

$$\mathrm{Im}(\widetilde{\Phi}) = V^k(osp(1,2)) \quad \text{or} \quad \mathrm{Im}(\widetilde{\Phi}) = V_k(osp(1,2)).$$

But, if $\mathrm{Im}(\widetilde{\Phi}) = V^k(osp(1,2))$, then the subalgebra generated by the embedding $sl(2)$ into $osp(1,2)$ must be universal affine vertex algebra $V^k(sl(2))$. A contradiction. So $\mathrm{Im}(\Phi) = V_k(osp(1,2))$, and first assertion holds.

2. By using relation

$$x(-1)y - \omega_{sug}^{sl_2} - \frac{1}{2}h(-2) = -\frac{p}{q}\omega_{p,\frac{p+q}{2}}$$

we see that $\omega_{p,\frac{p+q}{2}} \in V_k(osp(1,2))$. Since $V_k(sl(2)) \subset L^{Vir}(d_{\frac{p+q}{2},q}, 0) \otimes \Pi(0)$ we get that $\omega_{p,\frac{p+q}{2}}$ commutes with the action of $V_k(sl(2))$. The claim (2) follows. □

Remark 9. T. Creutzig and A. Linshaw studied the coset $Com(V_k(sl(2)), V_k(osp(1,2))$, and proved in [31, Theorem 8.2] that if k is admissible, then the coset is isomorphic to a minimal Virasoro vertex algebra. This can be also directly proved from Theorem 11.3.

11.4. Realization of $V^k(osp(1,2))$ at the critical level. Let $M(0) = \mathbb{C}[b(-n)\ |n \geq 1]$ be the commutative vertex algebra generated by the field $b(z) = \sum_{n \leq -1} b(n)z^{-n-1}$.

Let NS_{cri} the infinite-dimensional Lie superalgebra with generators

$$C, T(n), G^{cri}\left(n + \frac{1}{2}\right) \quad (n \in \mathbb{Z})$$

such that $T(n)$, C are in the center and

$$\{G^{cri}(r), G^{cri}(s)\} = 2T(r+s) + \frac{r^2 - \frac{1}{4}}{3}\delta_{r+s,0}C \quad (r, s \in \tfrac{1}{2} + \mathbb{Z}).$$

Let V_{cri}^{ns} be the universal vertex superalgebra associated to NS_{cri} such that C acts as scalar $C = -3$. V_{cri}^{ns} is freely generated by odd field $G^{cri}(z) = \sum_{n \in \mathbb{Z}} G^{cri}(n + \frac{3}{2})z^{-n-2}$ and even vector $T(z) = \sum_{n\mathbb{Z}} T(n)z^{-n-2}$ such that T is in central and that the following λ-bracket relation holds:

$$[G_\lambda^{cri} G^{cri}] = 2T - \lambda^2.$$

V_{cri}^{ns} can by realized as the vertex subalgebra of $F_2 \otimes M(0)$ generated by

$$G^{cri} = b(-1)\Psi_2\left(-\frac{1}{2}\right) + \Psi_2\left(-\frac{3}{2}\right), \quad T = \frac{1}{2}(b(-1)^2 + b(-2)).$$

By direct calculation we get that

$$\omega_{1,2} = T(-2) + G^{cri}\left(-\frac{3}{2}\right)\Psi\left(-\frac{1}{2}\right) + 2\omega_F$$

is a Virasoro vector of central charge $c_{1,2} = -2$.

Theorem 11.4. 1. *Assume that* $k = -3/2$. *There exists a non-trivial homomorphism*

$$\overline{\Phi} : V^k(osp(1,2)) \to V_{cri}^{ns} \otimes F \otimes \Pi^{1/2}(0)$$

such that

$$\begin{aligned}
e &\mapsto e^{\frac{2}{k}(\mu-\nu)},\\
h &\mapsto 2\mu(-1),\\
f &\mapsto \Big[(k+2)\omega_{1,2} - \nu(-1)^2 - (k+1)\nu(-2)\Big] e^{-\frac{2}{k}(\mu-\nu)}\\
x &\mapsto \sqrt{2}\Psi\left(-\frac{1}{2}\right) e^{\frac{1}{k}(\mu-\nu)}\\
y &\mapsto \sqrt{2}\left[-\frac{i}{2}G^{cri}\left(-\frac{3}{2}\right) + \Psi\left(-\frac{1}{2}\right)\nu(-1) + \frac{2k+1}{2}\Psi\left(-\frac{3}{2}\right)\right] e^{-\frac{1}{k}(\mu-\nu)}.
\end{aligned}$$

2. $T = \frac{1}{2}G^{cri}(-\frac{1}{2})G^{cri}(-\frac{3}{2})\mathbf{1}$ *is a central element of* $V^k(osp(1,2))$.

Remark 10. In the same way as in [44, Section 8.2] one can show that for $k = -\frac{3}{2}$:

$$\mathcal{W}^k(osp(1,2), f_\theta) \cong V_{cri}^{ns}.$$

T. Arakawa proved in [22] that when $\mathfrak{g}$ is a Lie algebra and f nilpotent element, then

$$\mathfrak{Z}(\mathcal{W}^{-h^\vee}(\mathfrak{g}, f)) = \mathfrak{Z}(V^{-h^\vee}(\mathfrak{g})).$$

We believe that the results from [22] hold for $\mathfrak{g} = osp(1,2)$, which would prove the following:

- $\mathfrak{Z}(V^k(osp(1,2))) \cong \mathfrak{Z}(V_{cri}^{ns}) \cong M_T(0)$;
- the homomorphism $\overline{\Phi}$ from Theorem 11.4 is injective.

12. Proof of Theorem 11.2

We shall first prove an important technical lemma.

Lemma 12.1. *Let* $\bar{y} = \left[\Psi\left(-\frac{1}{2}\right)\nu(-1) + \frac{2k+1}{2}\Psi(-\frac{3}{2})\right] e^{-\frac{1}{k}(\mu-\nu)}$. *We have:*

1. $\bar{y}(2)\bar{y} = -\frac{1}{4}(2k+1)(4k+5)e^{-\frac{2}{k}(\mu-\nu)}$,
2. $\bar{y}(1)\bar{y} = \frac{(2k+1)(4k+5)}{4k}(\mu(-1) - \nu(-1))e^{-\frac{2}{k}(\mu-\nu)} = -\frac{(2k+1)(4k+5)}{8}De^{-\frac{2}{k}(\mu-\nu)}$,
3. $\bar{y}(0)\bar{y} = \left(\frac{2k+1}{4}\Psi(-\frac{3}{2})\Psi(-\frac{1}{2}) + \nu(-1)^2 + (k+1)\nu(-2) - \frac{(2k+1)(4k+5)}{4}S_2\left(\frac{\nu-\mu}{k}\right)\right)$
   $e^{-\frac{2}{k}(\mu-\nu)}$,

where $S_2(\gamma) = \frac{1}{2}(\gamma(-1)^2 + \gamma(-2))$.

Proof. Let $\overline{\tau} = \sqrt{\frac{-2}{k}}\left(\Psi(-\frac{1}{2})\nu(-1) + (k+1)\Psi(-\frac{3}{2})\right)\mathbf{1}$, $b^r = e^{\frac{r}{k}(\mu-\nu)}$. Then $\overline{\tau}$ generates a $N = 1$ superconformal vertex algebra of central charge $\overline{c} = \frac{3}{2k}(4(k+1)^2 + k)$. We have

$$\overline{\omega} = \frac{1}{2}\overline{\tau}_0\overline{\tau} = -\frac{1}{k}\left(\nu(-1)^2 + (k+1)\nu(-2)\right) + \omega_{fer}.$$

$$\bar{y}(p) = \sqrt{-\frac{k}{2}}(\tau_{-1}b^{-})_p = \sqrt{-\frac{k}{2}}\sum_{j=0}^{\infty}\tau_{-1-j}b^{-1}_{p+j} + b^{-1}_{-1-j+p}\tau_j.$$

By applying formulas

$$\begin{aligned}
\tau_n b^{-1} &= -\frac{1}{2}\sqrt{\frac{-2}{k}}\delta_{n,0}\Psi\left(-\frac{1}{2}\right)b^{-1} \qquad (n \geq 0),\\
\tau_j\tau_{-1}b^{-1} &= 0 \quad (j \geq 3),\\
\tau_2\tau_{-1}b^{-1} &= \left(\frac{4(k+1)^2+k}{k} - \frac{2k+3}{2k}\right)b^{-1} = \frac{8(k+1)^2-3}{2k}b^{-1}\\
\tau_1\tau_{-1}b^{-1} &= \frac{2}{k}\nu(-1)b^{-1}\\
\tau_0\tau_{-1}b^{-1} &= -\frac{2}{k}\left(\nu(-1)^2 + k\nu(-2)\right)b^{-1} + \Psi\left(-\frac{3}{2}\right)\Psi\left(-\frac{1}{2}\right)b^{-1}\\
&\quad -\frac{1}{k}\nu(-2)b^{-1} - \frac{2k+1}{2k}\Psi\left(-\frac{3}{2}\right)\Psi\left(-\frac{1}{2}\right)b^{-1}\\
&= -\frac{2}{k}\left(\nu(-1)^2 + \frac{2k+1}{2}\nu(-2)\right)b^{-1} - \frac{1}{2k}\Psi\left(-\frac{3}{2}\right)\Psi\left(-\frac{1}{2}\right)b^{-1}
\end{aligned}$$

we get

$$\begin{aligned}
\bar{y}(2)\bar{y} &= -\frac{k}{2}\left(b^{-1}_{-1}\tau_2\tau_{-1}b^{-1} + b^{-1}_0\tau_1\tau_{-1}b^{-1} + b^{-1}_1\tau_0\tau_{-1}b^{-1}\right)\\
&= -\frac{k}{2}\left(\frac{4(k+1)^2+k}{k} - \frac{2k+3}{2k} + \frac{1}{k} - \frac{k+1}{k}\right)b^{-2}\\
&= -\frac{1}{4}(2k+1)(4k+5)b^{-2}.\\
\bar{y}(1)\bar{y} &= -\frac{k}{2}\left(b^{-1}_{-2}\tau_2\tau_{-1}b^{-1} + b^{-1}_{-1}\tau_1\tau_{-1}b^{-1} + b^{-1}_0\tau_0\tau_{-1}b^{-1}\right)\\
&= -\frac{k}{2}(-\frac{8(k+1)^2-3}{2k^2}(\mu(-1)-\nu(-1))b^{-2}\\
&\quad +\frac{2}{k}\nu(-1)b^{-2} - \frac{1}{k^2}(\mu(-1)-\nu(-1))b^{-2}\\
&\quad +\frac{k+1}{k^2}(\mu(-1)-\nu(-1))b^{-2} - \frac{2}{k}\nu(-1)b^{-2})\\
&= \frac{(4k+5)(2k+1)}{4k}(\mu(-1)-\nu(-1))b^{-2} = -\frac{(2k+1)(4k+5)}{8}De^{-\frac{2}{k}(\mu-\nu)}.\\
\bar{y}(0)\bar{y} &= -\frac{k}{2}\left(b^{-1}_{-3}\tau_2\tau_{-1}b^{-1} + b^{-1}_{-2}\tau_1\tau_{-1}b^{-1} + b^{-1}_{-1}\tau_0\tau_{-1}b^{-1} + \tau_{-1}b^{-1}_0\tau_{-1}b^{-1}\right)
\end{aligned}$$

$$= -\frac{k}{2}\left(\frac{8(k+1)^2-3}{2k}S_2\left(\frac{\nu-\mu}{k}\right)b^{-2} - \frac{2}{k^2}\nu(-1)(\mu(-1)-\nu(-1))b^{-2}\right.$$
$$+\frac{1}{k}S_2\left(\frac{\nu-\mu}{k}\right)b^{-2} - \frac{2}{k}\left(\nu(-1)^2+\frac{2k+1}{2}\nu(-2)\right)b^{-2}$$
$$-\frac{1}{2k}\Psi\left(-\frac{3}{2}\right)\Psi\left(-\frac{1}{2}\right)b^{-2}$$
$$-\frac{k+1}{k}S_2\left(\frac{\nu-\mu}{k}\right)b^{-2} + \frac{2}{k^2}\nu(-1)(\mu(-1)-\nu(-1))b^{-2}$$
$$\left.-\frac{1}{k}\left(\nu(-2)+k\Psi\left(-\frac{3}{2}\right)\Psi\left(-\frac{1}{2}\right)b^{-2}\right)\right)$$
$$= -\frac{(2k+1)(4k+5)}{4}S_2\left(\frac{\nu-\mu}{k}\right)b^{-2} + \frac{2k+1}{4}\Psi\left(-\frac{3}{2}\right)\Psi\left(-\frac{1}{2}\right)b^{-2}$$
$$+(\nu(-1)^2+(k+1)\nu(-2))b^{-2}.$$

The proof follows. □

12.1. Proof of Theorem 11.2. First we notice that $c_{p,q} = -\frac{3}{2}\frac{(4k+5)(2k+1)}{2k+3}$.

Assume that $k \neq -2$. Since

$$e, f, h \in V^{Vir}(d_{\frac{p+q}{2},q}, 0) \otimes \Pi(0) \subset V^{ns}(c_{p,q}, 0) \otimes F \otimes \Pi^{1/2}(0),$$

then Proposition 3.1 implies that vectors e, f, h generate a vertex subalgebra of $V^{ns}(c_{p,q}, 0) \otimes F \otimes \Pi^{1/2}(0)$ isomorphic to $V^k(sl(2))$. In the case $k = -2$, vector $\Omega_{p,q}$ generates a commutative vertex algebra isomorphic to $M_T(0)$, and therefore Proposition 3.2 implies that e, f, h generate a quotient of $V^k(sl(2))$.

Next, we need to prove that for $n \geq 0$ the following relations hold:

$$h(n)x = \delta_{n,0}x,\ e(n)x = 0,\ x(n)f = \delta_{n,0}y \tag{45}$$
$$h(n)y = -\delta_{n,0}y,\ e(n)y = -\delta_{n,0}x,\ f(n)y=0 \tag{46}$$
$$x(n)x = 2\delta_{n,0}e,\ y(n)y = -2\delta_{n,0}y,\ x(0)y = h,\ x(1)y = 2k\mathbf{1}. \tag{47}$$

Let us first prove that $x(n)f = \delta_{n,0}y$ for $n \geq 0$. Clearly $x(n)f = 0$ for $n \geq 2$. We have:

$$x(1)f = \sqrt{2}\left(\Psi\left(\frac{3}{2}\right)\Omega_{p,q} + \frac{2k+1}{4}\Psi\left(-\frac{1}{2}\right)\right)e^{-\frac{1}{k}(\mu-\nu)}$$
$$= \sqrt{2}\left(-\frac{2k+1}{4}\Psi\left(\frac{3}{2}\right)\Psi\left(-\frac{3}{2}\right)\Psi\left(-\frac{1}{2}\right) + \frac{2k+1}{4}\Psi\left(-\frac{1}{2}\right)\right)e^{-\frac{1}{k}(\mu-\nu)}=0,$$
$$x(0)f = \sqrt{2}\left(\Psi\left(\frac{1}{2}\right)\Omega_{p,q} + \frac{2k+1}{4}\Psi\left(-\frac{3}{2}\right)\right)e^{-\frac{1}{k}(\mu-\nu)}$$
$$+\sqrt{2}\Psi\left(\frac{3}{2}\right)\Omega_{p,q}\frac{1}{k}(\mu-\nu)(-1)e^{-\frac{1}{k}(\mu-\nu)}$$
$$+\sqrt{2}\frac{2k+1}{4}\Psi\left(-\frac{1}{2}\right)\frac{1}{k}(\mu-\nu)(-1)e^{-\frac{1}{k}(\mu-\nu)}+\sqrt{2}\Psi\left(-\frac{1}{2}\right)\nu(-1)e^{-\frac{1}{k}(\mu-\nu)}$$

$$= \sqrt{2}\left[-\frac{\sqrt{-2k-3}}{2}G(-3/2) + \Psi\left(-\frac{1}{2}\right)\nu(-1) + \frac{2k+1}{2}\Psi\left(-\frac{3}{2}\right)\right]e^{-\frac{1}{k}(\mu-\nu)} = y.$$

By using an easy calculation we get:

$$\begin{aligned} x(1)y &= 2\left(\frac{2k+1}{2} - \frac{1}{2}\right)\mathbf{1} = 2k\mathbf{1}, \\ x(0)y &= 2k\frac{1}{k}(\mu-\nu)(-1) + 2\nu(-1) = 2\mu(-1) = h, \\ e(0)y &= \delta_{n,0}x, \\ x(n)x &= 2\delta_{n,0}e. \end{aligned}$$

Finally, we will check relation $y(n)y = -2\delta_{n,0}$. Clearly, $y(n)y = 0$ for $n \geq 3$. For the cases $n = 0, 1, 2$ we need to use Lemma 12.1. We have:

$$\begin{aligned} y(2)y &= (-\frac{1}{2}(2k+1)(4k+5) - \frac{2k+3}{3}c_{p,q})e^{-\frac{2}{k}(\mu-\nu)} = 0, \\ y(1)y &= \left(\frac{(2k+1)(4k+5)}{2k} + \frac{2k+3}{3k}c_{p,q}\right)(\mu(-1) - \nu(-1))e^{-\frac{2}{k}(\mu-\nu)} = 0, \\ y(0)y &= \left(-\frac{(2k+1)(4k+5)}{4} - \frac{2k+3}{3}c_{p,q}\right)S_2\left(\frac{\nu-\mu}{k}\right)e^{-\frac{2}{k}(\mu-\nu)} \\ &\quad + \left(-(2k+3)\omega_{p,q} + (2k+1)\omega_F - i\sqrt{2k+3}G\left(-\frac{3}{2}\right)\Psi\left(-\frac{1}{2}\right)\right)e^{-\frac{2}{k}(\mu-\nu)} \\ &\quad + \left(2(\nu(-1)^2 + (k+1)\nu(-2))\right)e^{-\frac{2}{k}(\mu-\nu)} \\ &= -2\left((k+2)\omega_{p,\frac{p+q}{2}} - (\nu(-1)^2 + (k+1)\nu(-2))\right)e^{-\frac{2}{k}(\mu-\nu)} = -2f. \end{aligned}$$

In this way we have checked relations (45)–(47). This finishes the proof of Theorem. □

13. Example: Weight and Whitaker Modules for $k = -5/4$

As we have seen in previous sections (see also [7,9,21]) for the analysis of weight, Whittaker and logarithmic modules, the explicit free-field realization is very useful.

The realization of $V_k(osp(1,2))$ is simpler in the cases when $L^{ns}(c_{p,q}, 0)$ is a 1-dimensional vertex algebra, and therefore $V_k(osp(1,2))$ can be realizaed on the vertex algebra $F \otimes \Pi^{1/2}(0)$. This happens only in the cases $k = -\frac{1}{2}$ and $k = -\frac{5}{4}$. In the case $k = -\frac{1}{2}$, $V_k(osp(1,2))$ can be realized on the tensor product of the Weyl vertex algebra W with the fermionic vertex algebra F of central chargce $c = 1/2$. But this is essentially known in the literature, as a special case of the realization of $V_{-1/2}(osp(1,2n))$ (cf. [37]).

In this section, we specialize our realization to the case $k = -5/4$. We get a realization of the vertex algebra $V_k(osp(1,2)$, which was investigated by D. Ridout, J. Snadden and S. Wood [57] by using different methods. It is also important to notice that the vertex algebra $V_k(osp(1,2))$ is a simple current extension of $V_k(sl(2))$:

$$V_{-\frac{5}{4}}(osp(1,2)) = L_{A_1}\left(-\frac{5}{4}\Lambda_0\right) + L_{A_1}\left(-\frac{9}{4}\Lambda_0 + \Lambda_1\right),$$

which can be also proved from our realization. Then $k+\frac{3}{2}=\frac{p}{2q}$ for $p=2, q=4$. Since $c_{p,q}=0$, we have that $L^{ns}(c_{p,q},0)$ is a 1-dimensional vertex algebra.

Remark 11. Note that $k=-h^\vee/6-1$ is a collapsing level for $\mathfrak{g}=osp(1,2)$ [18]. In this case we have the realization inside the free field algebra $F\otimes\Pi^{1/2}(0)$ without any $\mathcal{W}$-algebra.

We have the following realization of $V_k(osp(2,1))$.

Corollary 13.1. *Assume that* $k=-\frac{5}{4}$.

1. *There exists a non-trivial vertex superalgebra homomorphism*

$$\overline{\Phi}: V_k(osp(1,2)) \to F\otimes\Pi^{1/2}(0)$$

such that

$$\begin{aligned}
e &\mapsto e^{\frac{2}{k}(\mu-\nu)},\\
h &\mapsto 2\mu(-1),\\
f &\mapsto \left[(k+2)\omega_{3,4}-\nu(-1)^2-(k+1)\nu(-2)\right]e^{-\frac{2}{k}(\mu-\nu)}\\
x &\mapsto \sqrt{2}\Psi\left(-\frac{1}{2}\right)e^{\frac{1}{k}(\mu-\nu)}\\
y &\mapsto \sqrt{2}\left[\Psi\left(-\frac{1}{2}\right)\nu(-1)+\frac{2k+1}{2}\Psi\left(-\frac{3}{2}\right)\right]e^{-\frac{1}{k}(\mu-\nu)},
\end{aligned}$$

where $\omega_{3,4}=\frac{1}{2}\Psi(-\frac{3}{2})\Psi(-\frac{1}{2})\mathbf{1}$.

2. *Assume that U (resp. U^{tw}) is any untwisted (resp. g-twisted) $\Pi^{1/2}(0)$-module. Then*
   - *$F\otimes U$ and $M^\pm\otimes U^{tw}$ are untwisted $V_k(osp(1,2))$-modules.*
   - *$F\otimes U^{tw}$ and $M^\pm\otimes U$ are Ramond twisted $V_k(osp(1,2))$-modules.*

A classification of irreducible untwisted and twisted $V_k(osp(2,1))$-modules were obtained [57, Theorem 9] by using Zhu's algebra approach. All representations can be constructed using our free-field realization. Maybe most interesting examples are relaxed highest weight $V_k(osp(2,1))$-modules. We shall consider here only Neveu–Schwarz sector, i.e, non-twisted $V_k(osp(2,1))$-modules.

Consider the $\sigma\otimes g$-twisted module $F\otimes\Pi^{1/2}(0)$-module $\mathcal{F}^\lambda := M^\pm\otimes\Pi^{(1/2)}_{(-1)}(\lambda)$ for $\lambda\in\mathbb{C}$. Then $\mathcal{F}^\lambda$ is an untwisted $V_k(osp(1,2))$-module. As in Sect. 7 we define $E^\lambda_{1,2}=\mathbf{1}^\pm\otimes e^{-\mu+\lambda\frac{2}{k}(\mu-\nu)}$. Then the action of $osp(1,2)$ is given by

$$\begin{aligned}
e(0)E^\lambda_{1,2} &= E^{\lambda+1}_{1,2},\\
h(0)E^\lambda_{1,2} &= (-k+2\lambda)E^\lambda_{1,2},\\
f(0)E^\lambda_{1,2} &= \left(\frac{1}{16}-\left(\lambda+\frac{1}{8}\right)^2\right)E^{\lambda-1}_{1,2} = \left(\frac{3}{8}+\lambda\right)\left(\frac{1}{8}-\lambda\right)E^{\lambda-1}_{1,2}\\
x(0)E^\lambda_{1,2} &= \pm E^{\lambda+\frac{1}{2}}_{1,2}\\
y(0)E^\lambda_{1,2} &= \mp\left(\frac{3}{8}+\lambda\right)E^{\lambda-\frac{1}{2}}_{1,2}.
\end{aligned}$$

Moreover, we have

$$L_{sug}(n)E^{\lambda}_{1,2} = -\frac{1}{4}\delta_{n,0}E^{\lambda}_{1,2} \quad (n \geq 0).$$

Theorem 13.2. *Assume that* $\lambda \notin \frac{1}{8} + \frac{1}{2}\mathbb{Z}$*. Then* $\mathcal{F}^{\lambda}$ *is an irreducible* $\mathbb{Z}_{\geq 0}$*-graded* $V_k(osp(1,2))$*-module whose character is*

$$ch[\mathcal{F}^{\lambda}](q,z) = Tr_{\mathcal{F}^{\lambda}} q^{L_{sug}(0)-c/24} z^{h(0)} = z^{2\lambda-k}\frac{f_2(\tau)\delta(z)}{\sqrt{2}\eta(\tau)^2},$$

where $f_2(\tau) = \sqrt{2}q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1+q^n)$*. (In the terminology of* [57]*,* $\mathcal{F}^{\lambda}$ *corresponds to* $\mathcal{C}_{\Lambda,0}$ *where* $\Lambda = 2\lambda + \frac{5}{4}$*).*

Proof. Note first that $\mathcal{F}^{\lambda}$ is $\mathbb{Z}_{\geq 0}$-graded and that its lowest component is $\mathcal{F}^{\lambda}(0) = \mathrm{span}_{\mathbb{C}}\{E^{\lambda+i}_{1,2}, \quad i \in \frac{1}{2}\mathbb{Z}\}$. The $osp(1,2)$-action obtained above implies that $\mathcal{F}^{\lambda}(0)$ is irreducible for $\lambda \notin \frac{1}{8}+\frac{1}{2}\mathbb{Z}$. By using realization, we see that as $V_k(sl(2))$-module we have $\mathcal{F}^{\lambda} = \mathcal{E}^{\lambda}_{1,2} \oplus \mathcal{E}^{\lambda+1/2}_{1,2}$, where $\mathcal{E}^{r}_{1,2} = L^{Vir}(d_{3,4}, \frac{1}{16}) \otimes \Pi_{-1}(r)$. By Proposition 7.4, $\mathcal{E}^{r}_{1,2}$ is irreducible for $r \notin \frac{1}{8} + \frac{1}{2}\mathbb{Z}$. Therefore, $\mathcal{F}^{\lambda}$ is a direct sum of two irreducible $V_k(sl(2))$-modules, which easily gives irreducibility result since $V_k(osp(1,2))$ is a simple current extension of $V_k(sl(2))$. The character formula follows directly from the realization, character formula for $\Pi^{1/2}_{(-1)}(\lambda)$ (cf. Proposition 4.5) and (43):

$$\mathrm{ch}[\mathcal{F}^{\lambda}](q,z) = \mathrm{ch}[M](q)\mathrm{ch}[\Pi^{1/2}_{(-1)}(\lambda)](q) = z^{2\lambda-k}\frac{f_2(\tau)\delta(z)}{\sqrt{2}\eta(\tau)^2}. \tag{48}$$

□

We also have the following result on the irreducibility of some Whittaker modules.

Corollary 13.3. *We have:* $M^{\pm} \otimes \Pi_{\lambda}$ *is irreducible* $V_k(osp(1,2))$*-module.*

Proof. $M^{\pm} \otimes \Pi_{\lambda}$ is a $V_k(osp(1,2))$-module by Corollary 13.1 (2). The irreducibility follows from the fact that $M^{\pm} \otimes \Pi_{\lambda}$ is, as a $V_k(sl(2))$-module, isomorphic to the Whittaker module $L^{Vir}(d_{3,4}, \frac{1}{16}) \otimes \Pi_{\lambda}$, which is irreducible. □

Remark 12. A generalization of modules constructed above is as follows. Let $L^{\mathcal{R}}(c,h)^{\pm}$ be the irreducible Ramond twisted modules for the simple $N=1$ Neveu–Schwarz vertex algebra $L^{ns}(c,0)$ (cf. [12,40,53]). For an arbitrary admissible level k, we have the following family of $\mathbb{Z}_{\geq 0}$-graded relaxed and Whittaker $V_k(osp(1,2))$-modules:

$$L^{\mathcal{R}}(c_{p,q},h)^{\pm} \otimes M^{\pm} \otimes \Pi^{1/2}_{(-1)}(\lambda), \quad L^{\mathcal{R}}(c_{p,q},h)^{\pm} \otimes M^{\pm} \otimes \Pi_{\lambda}.$$

The irreducibility of modules $L^{\mathcal{R}}(c_{p,q},h)^{\pm} \otimes M^{\pm} \otimes \Pi^{1/2}_{-1}(\lambda)$ can be proved by using character formulas for irreducible relaxed $\widehat{osp(1,2)}$-modules from [45, Theorem 8.2].

Let $h = \Delta_{r,s}$ be given by formula (8.11) in [45]. Using Proposition 4.5, we see that the character of the module $L^{\mathcal{R}}(c_{p,q},h)^{\pm} \otimes M^{\pm} \otimes \Pi^{1/2}_{(-1)}(\lambda)$ is given by

$$\begin{aligned}&\mathrm{ch}[L^{\mathcal{R}}(c_{p,q},h)^{\pm}](q)\mathrm{ch}[M^{\pm}](q)\mathrm{ch}[\Pi^{1/2}_{(-1)}(\lambda)](q,z)\\&=\mathrm{ch}[L^{\mathcal{R}}(c_{p,q},h)^{\pm}](q)\,f_2(\tau)\frac{z^{2\lambda-k}}{\sqrt{2}\eta(\tau)^2}\delta(z),\end{aligned}$$

which coincides with the character of the admissible relaxed $V_k(osp(1,2))$-module ${}^{\mathrm{NS}}\mathcal{E}_{2\lambda-k,q_{r,s}}$ in [45, Theorem 8.2].

Acknowledgements. The author is partially supported by the Croatian Science Foundation under the Project 2634 and by the QuantiXLie Centre of Excellence, a Project cofinanced by the Croatian Government and European Union through the European Regional Development Fund—the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004). We would like to thank T. Creutzig, A. Milas, G. Radobolja, D. Ridout, and S. Wood for valuable discussions. Finally, we thank the referee for a careful reading of the paper and helpful comments.

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Communicated by C. Schweigert