Local Well-Posedness for Boltzmann's Equation and the Boltzmann Hierarchy via Wigner Transform

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Abstract: We use the dispersive properties of the linear Schrödinger equation to prove local well-posedness results for the Boltzmann equation and the related Boltzmann hierarchy, set in the spatial domain $\mathbb{R}^d$ for $d \geq 2$. The proofs are based on the use of the (inverse) Wigner transform along with the spacetime Fourier transform. The norms for the initial data f_0 are weighted versions of the Sobolev spaces $L_v^2 H_x^\alpha$ with $\alpha \in \left(\frac{d-1}{2}, \infty\right)$. Our main results are local well-posedness for the Boltzmann equation for cutoff Maxwell molecules and hard spheres, as well as local well-posedness for the Boltzmann hierarchy for cutoff Maxwell molecules (but not hard spheres); the latter result holds without any factorization assumption for the initial data.

1. Introduction

Boltzmann's equation is an evolutionary partial differential equation (PDE) which describes the behavior of a dilute gas of identical particles in a specific scaling limit. The equation describes the time evolution of a density function $f(t, x, v) \geq 0$, where $x, v \in \mathbb{R}^d$ are the position and velocity of a typical particle.

The Cauchy problem for Boltzmann's equation is one of the fundamental mathematical problems in kinetic theory and it may be written in the following form:

$$(\partial_t + v \cdot \nabla_x) f(t, x, v) = Q(f, f)(t, x, v) \tag{1}$$

$$f(0, x, v) = f_0(x, v) \tag{2}$$

where the *collision operator* Q is defined as follows:

$$Q(f, f)(t, x, v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} d\omega dv_2 \mathbf{b}\left(|v - v_2|, \omega \cdot \frac{v - v_2}{|v - v_2|}\right) \\ \times \left(f(t, x, v^*)f(t, x, v_2^*) - f(t, x, v)f(t, x, v_2)\right). \tag{3}$$

The *collision kernel* **b** is a function which depends on the physical interaction between particles; pre-collisional and post-collisional velocities are related by the following involutive transformation, for $v, v_2 \in \mathbb{R}^d$ and fixed $\omega \in \mathbb{S}^{d-1}$:

$$v^* = v + (\omega \cdot (v_2 - v))\, \omega$$
$$v_2^* = v_2 - (\omega \cdot (v_2 - v))\, \omega.$$

The most general known solutions of Boltzmann's equation are the *renormalized solutions* of [7], which exist globally in time for arbitrary data f_0 having finite mass, second moments and entropy:

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x, v) \left(1 + |v|^2 + |x|^2 + \log(1 + f_0(x, v))\right) dx\, dv < +\infty. \qquad (4)$$

However, renormalized solutions have many limitations; for instance, they are not known to solve the Boltzmann equation in the usual distributional sense (which makes them difficult to manipulate), nor are they known to be unique, nor are they known to conserve energy. A different and very fruitful line of investigation considers solutions close to an equilibrium distribution of fixed temperature, see e.g. [1,8,13,14,22,23]. These solutions exist globally in time and enjoy uniqueness and continuous dependence in appropriate functional spaces; however, the theory only applies in a small neighborhood of equilibrium.

In the paper at hand, we investigate the *local* theory of well-posedness, and will leave questions addressing global well-posedness for future work. In particular, we intend to prove existence, uniqueness and continuous dependence of solutions, locally in time and for large data, with regularity as low as possible. See [2,3,16] for some existing theories of local solutions for Boltzmann's equation. We especially refer to Remark 1 of of [2], which provides (in the case of Grad cut-off) a large data local well-posedness result which parallels our Theorem 2.2 when $\alpha > \frac{d}{2}$ in $d = 3$.[1] Our main intention, however, is not to investigate optimal regularity spaces for solving Boltzmann's equation. Rather, we intend to demonstrate the close connection between Boltzmann's equation and nonlinear Schrödinger equations (NLS) in the density matrix formulation; this connection has been recognized implicitly for some time, but we wish to make it quite explicit, and to the best of our knowledge, this is the first time such an explicit connection has been established.[2] The local well-posedness theory for NLS is by now very mature and it is our hope that some tools which have been useful for NLS will turn out to be applicable to the corresponding problem for Boltzmann's equation. If the theory can be made precise enough, it may turn out to be useful for such problems as global well-posedness or the derivation of Boltzmann's equation from deterministic particle systems.

Besides providing a new approach to proving local well-posedness for Boltzmann's equation, we will also prove new results concerning the *Boltzmann hierarchy* for at least some collision kernels. The Boltzmann hierarchy is an infinite hierarchy of coupled PDE which describes a gas of infinitely many particles, possibly accounting for correlations between particles. For some class of collisional kernels, the Boltzmann hierarchy appears in the derivation of Boltzmann's equation from classical system of many particles.

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[1] We are able to prove a *conditional* local well-posedness result when f_0 is in a weighted version of $L_v^2 H_x^\alpha$ with $\alpha > \frac{d-1}{2}$ (here conditional means that uniqueness only holds assuming some auxiliary estimate satisfied by the constructed solution). It is conceivable that the uniqueness is unconditional when $\alpha > \frac{d}{2}$, cf. [2], but we do not pursue this issue.

[2] We emphasize that we *do not* make use of any semiclassical limit.

Roughly speaking, the marginals $f_N^{(k)}(t)$, $1 \leq k \leq N$, of the N-particle probability density f_N (which solves Liouville's equation from classical mechanics), satisfy a system of N coupled linear PDEs referred to as a *BBGKY Hierarchy*. Deriving the BBGKY hierarchy from first principles is in itself a nontrivial task in the case of hard spheres (we note that the hard sphere Boltzmann hierarchy is *not treated* in the present work, due to technical challenges). The Boltzmann hierarchy (for a given interparticle potential) is obtained from the BBGKY hierarchy in a suitable limit as $N \to \infty$, under the Boltzmann–Grad scaling (i.e. gas particles have a finite, non-zero mean free propagation time). We refer the reader to [12,19–21], and references therein, for further details. The Boltzmann hierarchy and Boltzmann equation are connected by the fact that the Boltzmann hierarchy admits a class of factorized solutions with each factor being a solution to the Boltzmann equation.

The classical local well-posedness result for the Boltzmann hierarchy is due to Lanford [19], who assumes L^∞ bounds on the initial data. Our results establish local well-posedness in a functional setting much different than Lanford's; in particular, we can work with spaces that do not embed locally into L^∞ in any variable. Unfortunately, we cannot report any new results concerning the Boltzmann hierarchy for hard spheres; this is the topic of ongoing research.

Remark. An important aspect of the Boltzmann hierarchy is that typical results one can expect to prove depend strongly on whether the marginals $f^{(k)}$, $k = 1, 2, 3, \ldots$, are assumed to be true marginals of some underlying exchangeable distribution (cf. de Finetti theorems). Under such assumptions, one can usually expect to prove optimal regularity results, but the situation is quite different without de Finetti type assumptions. We note that, for finite N in the Boltzmann–Grad limit, it is usually very hard to make optimal use of exchangeability of particles (but see [4,20] for results in this direction). Hence, it is of interest to find local well-posedness results for the Boltzmann hierarchy that do not rely on exchangeability. In the work at hand, we do not rely on exchangeability in our treatment of the Boltzmann hierarchy.

The idea at the heart of our proofs is to take the inverse Wigner transform of Boltzmann's equation (resp. the Boltzmann hierarchy). The transport operator

$$(\partial_t + v \cdot \nabla_x)$$

is transformed into the linear Schrödinger operator

$$\left(i \partial_t + \frac{1}{2} \left(\Delta_x - \Delta_{x'} \right) \right),$$

and the nonlinear operator $Q(f, f)$ becomes a new operator $B(\gamma, \gamma)$. This provides us with the possibility to prove a bilinear estimate of a similar type as the one proved by Klainerman and Machedon [18]. Subsequently, we can employ an iteration method inspired by the one developed in [6]; these methods were originally devised for proving the local well-posedness of the Gross–Pitaevskii hierarchy. In this paper, we implement them at the level of the transformed Boltzmann equation, as well as at the level of the transformed Boltzmann hierarchy.[3] The main point that we make here is that the transformed Boltzmann equation is closely related to a nonlinear Schrödinger equation,

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[3] In which case we also use the boardgame combinatorial argument as presented by Klainerman and Machedon [18], which is a reformulation of the combinatorial methods of Erdös, Schlein and Yau [9–11].

and the transformed Boltzmann hierarchy is closely related to the Gross–Pitaevskii hierarchy, with nonlinearities capturing the information about the interaction between particles encoded in the Boltzmann collision kernels. Based on this point of view, we develop tools and employ techniques for local well-posedness inspired by tools and techniques that have been recently introduced in the context of the Gross–Pitaevskii hierarchy.

Organization of the paper. Section 2 provides a detailed description of the main results proven in this paper, using the Wigner transform. Section 3 gives the proof of a crucial proposition which is used to prove all our results, and constitutes the main technical contribution of the paper. Section 4 is devoted to the proof of local well-posedness for the Boltzmann equation; this result extends to cutoff Maxwell molecules, hard spheres, and variable hard sphere models. Section 5 gives a brief outline of the proof of local well-posedness for the Boltzmann hierarchy, including the case of cutoff Maxwell molecules (but not hard spheres).

2. Main Results

2.1. Notation and preliminaries. Given a function $f(x, v) \in L^2_{x,v}$ we define its inverse Wigner transform $\gamma(x, x') \in L^2_{x,x'}$ by the following formula:

$$\gamma(x, x') = \int_{\mathbb{R}^d} f\left(\frac{x + x'}{2}, v\right) e^{iv \cdot (x - x')} dv. \tag{5}$$

The inverse of the inverse Wigner transform is the usual Wigner transform, namely:

$$f(x, v) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \gamma\left(x + \frac{y}{2}, x - \frac{y}{2}\right) e^{-iv \cdot y} dy. \tag{6}$$

All of our main results will be stated in terms of γ; in particular, if we say $f(t)$ satisfies Boltzmann's equation, we mean that $\gamma(t)$ solves the Duhamel formula associated with the inverse Wigner transform of the Boltzmann equation.

Remark. Note that if $\gamma(x, x') = \overline{\gamma(x', x)}$ for all $x, x' \in \mathbb{R}^d$, then f is everywhere real-valued; the converse also holds. In particular, it is easy to check on the inverse Wigner side that f is real-valued. It is much less simple to determine whether f is non-negative, and this is an issue we do not address in the present work.

Throughout the paper, we will assume that $0 \leq \mathbf{b} \in L^\infty_A$ for some $A \in [0, 1]$, where we have defined

$$\|\mathbf{b}\|_{L^\infty_A} = \sup_{u \in \mathbb{R}^d, \, \omega \in \mathbb{S}^{d-1}} \frac{\left|\mathbf{b}\left(|u|, \omega \cdot \frac{u}{|u|}\right)\right|}{1 + |u|^A}. \tag{7}$$

We will require the Fourier transform of the collision kernel, which is written

$$\hat{\mathbf{b}}^\omega(\xi) = \int_{\mathbb{R}^d} \mathbf{b}\left(|u|, \omega \cdot \frac{u}{|u|}\right) e^{-iu \cdot \xi} du. \tag{8}$$

Note that $\hat{\mathbf{b}}^\omega$ is a tempered distribution in general. Special cases include $\mathbf{b} \equiv 1$ with $A = 0$ (Maxwell molecules with angular cut-off), $\mathbf{b} = [\omega \cdot u]_+$ with $A = 1$ (hard

spheres), and $0 < A < 1$ for variable hard sphere models. Not all results will apply for the full range $A \in [0, 1]$.

We introduce the weighted Sobolev spaces which define our functional setting. Let $\hat{\gamma}$ denote the Fourier transform of γ:

$$\hat{\gamma}(\xi, \xi') = \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-ix \cdot \xi} e^{-ix' \cdot \xi'} \gamma(x, x') dx dx'. \tag{9}$$

Then, for any $\alpha, \beta, \kappa \geq 0$, and any $\sigma > 0$,

$$\left\| \gamma(x, x') \right\|_{H^{\alpha, \beta, \sigma, \kappa}} = \left\| \langle \xi + \xi' \rangle^\alpha \langle \xi - \xi' \rangle^\beta e^{\kappa \langle \xi - \xi' \rangle^{\frac{1}{\sigma}}} \hat{\gamma}(\xi, \xi') \right\|_{L^2_{\xi, \xi'}}. \tag{10}$$

Note that this norm is equivalent to the following norm for the classical densities:

$$\left\| \langle 2v \rangle^\beta e^{\kappa \langle 2v \rangle^{\frac{1}{\sigma}}} (1 - \Delta_x)^{\frac{\alpha}{2}} f(x, v) \right\|_{L^2_{x, v}}. \tag{11}$$

Remark. We emphasize that we can allow $\kappa = 0$ for some of our results, e.g. the case of cutoff Maxwell molecules. We always require $\kappa > 0$ in the case of hard spheres.

2.2. Warm-up: free transport.
We present a few brief remarks on the free transport equation before turning to our main results. The main point we wish to make is that if $f(t, x, v)$ solves the equation

$$(\partial_t + v \cdot \nabla_x) f = 0 \tag{12}$$

then the inverse Wigner transform $\gamma(t, x, x')$ satisfies the following linear Schrödinger equation:

$$\left(i \partial_t + \frac{1}{2} (\Delta_x - \Delta_{x'}) \right) \gamma(t, x, x') = 0. \tag{13}$$

We emphasize that this correspondence does not rely on any semiclassical limit.

Example 2.1. If $\gamma(t, x, x') = e^{ik \cdot (x - x')}$ for some $k \in \mathbb{R}^d$, then γ solves (13) and $f(t, x, v) = c\delta(v - k)$ solves (12).

Example 2.2. If $\gamma(t, x, x') = |t|^{-d} e^{i(|x|^2 - |x'|^2)/(2t)}$, then γ solves (13) and $f(t, x, v) = c\delta(x - vt)$ solves (12).

Example 2.3. If $f(t, x, v) = \delta(x - vt)\delta(v - v_0)$, for a fixed $v_0 \in \mathbb{R}^d$, then f solves (12); moreover, the classical state (position and velocity) is known exactly. In any case, γ exists as a distribution; for any $u(t, x, x') \in \mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$ we have

$$\int_{\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d} \gamma(t, x, x') u(t, x, x') dt dx dx' \\ = c \int_{\mathbb{R} \times \mathbb{R}^d} e^{2iv_0 \cdot z} u(t, v_0 t + z, v_0 t - z) dz dt. \tag{14}$$

Equivalently, $\gamma(t, x, x') = \delta\left(\frac{x + x'}{2} - v_0 t \right) e^{iv_0 \cdot (x - x')}$. If $v_0 = 0$ then γ obviously solves (13); by a Galilean shift, γ solves (13) for arbitrary $v_0 \in \mathbb{R}^d$. Therefore, the "fundamental

solution" for (12) transforms into a solution of (13) under the inverse Wigner transform. We conclude that *any* classical state (evolving under free transport) can be represented by a distribution $\gamma(t, x, x')$ (evolving via a linear Schrödinger equation). Let us also point out that the inverse Wigner transform regarded as a map $f \in L^2_{x,v} \mapsto \gamma \in L^2_{x,x'}$ is an isometric isomorphism; and, these spaces are preserved by either (12) or (13) respectively. Hence the equivalence of (12) and (13) is reflected at the L^2 level of regularity.

Remark. If $\phi(t, x)$ solves the Schrödinger equation

$$\left(i\partial_t + \frac{1}{2}\Delta_x\right)\phi(t, x) = 0 \tag{15}$$

then the function $\gamma(t, x, x') = \phi(t, x)\overline{\phi(t, x')}$ solves (13) and the Wigner transform f solves (12) (though f in this case *need not* be non-negative).

We now prove the equivalence of (12) and (13) at the L^2 level of regularity. (The same result holds if f, γ are tempered distributions, and the proof is the same.)

Lemma 2.1. *Suppose*

$$f \in L^1\left([0, T], L^2(\mathbb{R}^d_x \times \mathbb{R}^d_v)\right) \tag{16}$$

and let

$$\gamma \in L^1\left([0, T], L^2(\mathbb{R}^d_x \times \mathbb{R}^d_{x'})\right) \tag{17}$$

denote the inverse Wigner transform of f. Then f solves

$$(\partial_t + v \cdot \nabla_x) f = 0 \tag{18}$$

in the sense of distributions, if and only if γ solves

$$\left(i\partial_t + \frac{1}{2}\left(\Delta_x - \Delta_{x'}\right)\right)\gamma(t, x, x') = 0 \tag{19}$$

in the sense of distributions.

Proof. Assume that

$$(\partial_t + v \cdot \nabla_x) f = g. \tag{20}$$

Using the definition of the inverse Wigner transform we have

$$\begin{aligned}
i\partial_t\gamma(t, x, x') &= \int_{\mathbb{R}^d} i\partial_t f\left(t, \frac{x+x'}{2}, v\right) e^{iv\cdot(x-x')}dv \\
&= \int_{\mathbb{R}^d} i\left(-v \cdot \nabla_x f + g\right)\left(t, \frac{x+x'}{2}, v\right) e^{iv\cdot(x-x')}dv.
\end{aligned} \tag{21}$$

Let us focus on the transport term, $v \cdot \nabla_x f$. We have

$$
\int_{\mathbb{R}^d} i \left(-v \cdot \nabla_x f\right) \left(t, \frac{x + x'}{2}, v\right) e^{iv \cdot (x - x')} dv
$$

$$
= -i \int_{\mathbb{R}^d} v \cdot (\nabla_x + \nabla_{x'}) \left[f\left(t, \frac{x + x'}{2}, v\right) \right] e^{iv \cdot (x - x')} dv
$$

$$
= - \int_{\mathbb{R}^d} (\nabla_x + \nabla_{x'}) \left[f\left(t, \frac{x + x'}{2}, v\right) \right] \cdot i v e^{iv \cdot (x - x')} dv
$$

$$
= - (\nabla_x + \nabla_{x'}) \cdot \int_{\mathbb{R}^d} f\left(t, \frac{x + x'}{2}, v\right) i v e^{iv \cdot (x - x')} dv
$$

$$
= - (\nabla_x + \nabla_{x'}) \cdot \int_{\mathbb{R}^d} f\left(t, \frac{x + x'}{2}, v\right) \frac{\nabla_x - \nabla_{x'}}{2} e^{iv \cdot (x - x')} dv \tag{22}
$$

$$
= -\frac{1}{2} (\nabla_x + \nabla_{x'}) \cdot (\nabla_x - \nabla_{x'}) \int_{\mathbb{R}^d} f\left(t, \frac{x + x'}{2}, v\right) e^{iv \cdot (x - x')} dv
$$

$$
= -\frac{1}{2} (\Delta_x - \Delta_{x'}) \int_{\mathbb{R}^d} f\left(t, \frac{x + x'}{2}, v\right) e^{iv \cdot (x - x')} dv
$$

$$
= -\frac{1}{2} (\Delta_x - \Delta_{x'}) \gamma(t, x, x').
$$

Therefore,

$$
\left(i \partial_t + \frac{1}{2} (\Delta_x - \Delta_{x'})\right) \gamma(t, x, x') = \int_{\mathbb{R}^d} g\left(t, \frac{x + x'}{2}, v\right) e^{iv \cdot (x - x')} dv. \tag{23}
$$

But g vanishes identically if and only if its inverse Wigner transform vanishes identically. Therefore, γ solves (13) if and only if f solves (12). $\quad\square$

2.3. The main result for the Boltzmann equation. It is possible to compute explicitly the equation satisfied by γ if the Wigner transform f is smooth with rapid decay and satisfies Boltzmann's equation, (1); see Corollaries A.3 and A.4 in "Appendix A". The result is as follows:

$$
\left(i \partial_t + \frac{1}{2} (\Delta_x - \Delta_{x'})\right) \gamma(t) = B(\gamma(t), \gamma(t)) \tag{24}
$$

$$
B(\gamma_1, \gamma_2) = B^+(\gamma_1, \gamma_2) - B^-(\gamma_1, \gamma_2) \tag{25}
$$

$$
B^-(\gamma_1, \gamma_2)(x, x') = \frac{i}{2^{2d} \pi^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} dz \hat{\mathbf{b}}^\omega \left(\frac{z}{2}\right) \times
$$

$$
\times \gamma_1 \left(x - \frac{z}{4}, x' + \frac{z}{4}\right) \gamma_2 \left(\frac{x + x'}{2} + \frac{z}{4}, \frac{x + x'}{2} - \frac{z}{4}\right) \tag{26}
$$

$$
B^+(\gamma_1, \gamma_2)(x, x') = \frac{i}{2^{2d} \pi^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} dz \hat{\mathbf{b}}^\omega \left(\frac{z}{2}\right)
$$

$$
\times \gamma_1 \left(x - \frac{1}{2} P_\omega(x - x') - \frac{R_\omega(z)}{4}, x'\right)
$$

$$+\frac{1}{2}P_\omega(x-x')+\frac{R_\omega(z)}{4}\Bigg)$$

$$\times\gamma_2\left(\frac{x+x'}{2}+\frac{1}{2}P_\omega(x-x')+\frac{R_\omega(z)}{4},\frac{x+x'}{2}\right.$$

$$\left.-\frac{1}{2}P_\omega(x-x')-\frac{R_\omega(z)}{4}\right) \tag{27}$$

where we define

$$P_\omega(x)=(\omega\cdot x)\,\omega \tag{28}$$
$$R_\omega(x)=(\mathbb{I}-2P_\omega)\,(x) \tag{29}$$

and $\mathbb{I}(x)=x$. Solutions of Boltzmann's equation (in the γ formulation) are understood using Duhamel's formula:

$$\gamma(t)=e^{\frac{1}{2}it\Delta_\pm}\gamma(0)-i\int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_\pm}B(\gamma(t_1),\gamma(t_1))dt_1. \tag{30}$$

Here $\Delta_\pm=\Delta_x-\Delta_{x'}$.

Remark. At first glance it may appear that Boltzmann equation is more complicated in the γ formulation than in the classical formulation. However, in some sense it is actually simpler. This is most readily seen when $\mathbf{b}\equiv 1$, in which case $\hat{\mathbf{b}}^\omega$ becomes a δ-function concentrated at $z=0$. Then the collision integral $B(\gamma,\gamma)$ involves only an angular integration over $\omega\in\mathbb{S}^{d-1}$, whereas the classical collision integral $Q(f,f)$ contains an angular integration as well as an integral over $v_2\in\mathbb{R}^d$.

We are now ready to state our first main result.

Theorem 2.2. *Suppose $A\in[0,1]$, $\alpha\in\left(\frac{d-1}{2},\infty\right)$, $\beta\in(d,\infty)$, $\kappa\in(0,\infty)$, and additionally $\frac{1}{\sigma}\in(\max(0,2A-1),2]$; fix any $\lambda\in(0,\infty)$. Consider the Boltzmann equation* (24) *with $\mathbf{b}\in L_A^\infty$. For any $\gamma_0\in H^{\alpha,\beta,\sigma,\kappa}$ there exists a unique solution $\gamma(t)$ of Boltzmann's equation on a small time interval $[0,T]$ such that*

$$\left\|\,\|\gamma(t)\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t}}\,\right\|_{L_{t\in[0,T]}^\infty}<\infty \tag{31}$$

and

$$\left\|\,\|B(\gamma(t),\gamma(t))\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t}}\,\right\|_{L_{t\in[0,T]}^1}<\infty \tag{32}$$

both hold, and $\gamma(0)=\gamma_0$. Moreover, for some $r\in[0,1)$ we have the following: if $\|\gamma_0\|_{H^{\alpha,\beta,\sigma,\kappa}}\le M$ then for all small enough T depending only on $\alpha,\beta,\kappa,\sigma,\lambda$ and M, there holds:

$$T^{\frac{1}{2}(1-r)}\left\|\,\|\gamma(t)\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t}}\,\right\|_{L_{t\in[0,T]}^\infty}+\left\|\,\|B\left(\gamma(t),\gamma(t)\right)\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t}}\,\right\|_{L_{t\in[0,T]}^1}$$

$$\le C(M,\alpha,\beta,\sigma,\kappa,\lambda)\times T^{\frac{1}{2}(1-r)}\|\gamma_0\|_{H^{\alpha,\beta,\sigma,\kappa}}. \tag{33}$$

If $A\in\left[0,\frac{1}{2}\right)$ then we may take $\lambda=0$ and $\kappa\in[0,\infty)$ and the same results hold, with the same restrictions on α,β,σ.

Remark. If $A=0$ it is possible to optimize the proof of Theorem 2.2 and obtain the same result, with $\lambda=0$, for any $\kappa\in[0,\infty)$, $\frac{1}{\sigma}\in(0,2]$, and $\alpha,\beta\in\left(\frac{d-1}{2},\infty\right)$. We omit the details.

2.4. Comparison with a nonlinear Schrödinger equation. The direct comparison between nonlinear Schrödinger equations, e.g.

$$(i\partial_t + \Delta)\,\varphi = \pm|\varphi|^2\varphi \tag{34}$$

and Boltzmann's equation is typically not meaningful because Boltzmann's equation allows a richer set of scaling symmetries. In particular, for Boltzmann's equation it is possible to scale x and v independently by making an appropriate choice of timescale and correctly accounting for the form of the collision kernel $\mathbf{b}$.[4]

Nevertheless, an interesting informal comparison can be drawn in the case of a constant collision kernel, $\mathbf{b} \equiv$ cst. (i.e. Maxwell molecules). Indeed, up to a linear change of variables, the Wigner transform is simply a Fourier transform in v; then, as is well-known by Bobylev's formula, the dimension of the collision integral is reduced from $2d - 1$ to $d - 1$ (but only in the case of Maxwell molecules). Moreover the remaining integral over $\omega \in \mathbb{S}^{d-1}$ occurs on a compact set (modulo scaling by the factor $|x - x'|$), so the formal algebraic structure is very much like that of a power nonlinearity. (Of course B^+ features important convolutive structure which is absent from NLS; however, we are not necessarily bound to exploit the convolution, nor does this aspect play a role in our proof.)

In the case of the loss term B^- (again assuming $\mathbf{b} \equiv$ cst.), the analogy is quite dramatic.[5] Indeed in that case we have from (26) the result

$$B^-(\gamma, \gamma) = i\rho_\gamma\left(\frac{x + x'}{2}\right)\gamma(x, x') \tag{35}$$

where $\rho_\gamma(x) = \gamma(x, x)$ is the spatial density. On the other hand, for φ a solution of cubic NLS (34), the corresponding density matrix $\gamma(t, x, x') = \varphi(t, x)\overline{\varphi(t, x')}$ solves

$$(i\partial_t + \Delta_\pm)\,\gamma = \pm B_{\mathrm{NLS}}(\gamma, \gamma) \tag{36}$$

where $\Delta_\pm = \Delta_x - \Delta_{x'}$ and

$$B_{\mathrm{NLS}}(\gamma, \gamma)(x, x') = [\rho_\gamma, \gamma](x, x') = (\rho_\gamma(x) - \rho_\gamma(x'))\,\gamma(x, x') \tag{37}$$

and $\rho_\gamma(x) = \gamma(x, x)$ as before.

The analogy extends even more deeply for $\mathbf{b} \equiv$ cst., since we directly transfer techniques from [6,18] which have been previously used to solve equations like (36) via bilinear Strichartz estimates. Indeed, the key *analytical*[6] estimates from [6,18] imply that for any nice enough data $\gamma_0(x, x')$,

$$\left\|\langle\nabla_x\rangle^\alpha\,\langle\nabla_{x'}\rangle^\alpha\,B_{\mathrm{NLS}}\left(e^{it\Delta_\pm}\gamma_0, e^{it\Delta_\pm}\gamma_0\right)\right\|_{L^2_{t\in\mathbb{R}}L^2_{x,x'}} \le C\left\|\langle\nabla_x\rangle^\alpha\,\langle\nabla_{x'}\rangle^\alpha\,\gamma_0\right\|^2_{L^2_{x,x'}} \tag{38}$$

whenever $d \ge 2$ and $\alpha > \frac{d-1}{2}$. The estimate (38) is false when $0 \le \alpha < \frac{d-1}{2}$ by scaling (even if $\gamma_0 = |\varphi_0\rangle\langle\varphi_0|$), whereas the borderline estimate $\alpha = \frac{d-1}{2}\,(= 1)$ is true when

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[4] This fact does not contradict the correspondence under the Wigner transform, because general kinetic symmetries will destroy the factorized structure $\gamma = |\varphi\rangle\langle\varphi|$.

[5] For simplicity, we will ignore the distinction between $B^-(\gamma_1, \gamma_2)$ and $B^-(\gamma_2, \gamma_1)$.

[6] Analytical, as opposed to combinatorial.

$d = 3$ by [18]. As a direct consequence of (38), using the methods of [6], one deduces local well-posedness of (36) for the norm

$$\left\| \langle\nabla_x\rangle^\alpha \langle\nabla_{x'}\rangle^\alpha \gamma(t) \right\|_{L_T^\infty L_{x,x'}^2} + \left\| \langle\nabla_x\rangle^\alpha \langle\nabla_{x'}\rangle^\alpha \partial_t \left\{ e^{-it\Delta_\pm} \gamma(t) \right\} \right\|_{L_T^1 L_{x,x'}^2} \quad (39)$$

whenever

$$\left\| \langle\nabla_x\rangle^\alpha \langle\nabla_{x'}\rangle^\alpha \gamma_0 \right\|_{L_{x,x'}^2} < \infty \quad (40)$$

and $\alpha > \frac{d-1}{2}$. Note that the exponent $\frac{d-1}{2}$ is critical for the *proof*, and one draws no conclusion about the solvability of (36) when $\alpha \in \left(\frac{d-2}{2}, \frac{d-1}{2} \right)$ (nor is it clear to us what the threshold for *fully general (non-physical) solutions* of (36) should be, e.g. if γ_0 is non-Hermitian).

In the case of Boltzmann's equation with $\mathbf{b} = \mathrm{cst.}$, we obtain by Proposition 3.1 (and the remarks immediately thereafter) the estimates

$$\left\| \langle D_+\rangle^\alpha \langle D_-\rangle^\alpha B^\pm \left(e^{it\Delta_\pm}\gamma_0, e^{it\Delta_\pm}\gamma_0 \right) \right\|_{L_{t\in\mathbb{R}}^2 L_{x,x'}^2} \leq C \left\| \langle D_+\rangle^\alpha \langle D_-\rangle^\alpha \gamma_0 \right\|_{L_{x,x'}^2}^2 \quad (41)$$

where $D_+ = \nabla_x + \nabla_{x'}$ and $D_- = \nabla_x - \nabla_{x'}$, whenever $d \geq 2$ and $\alpha > \frac{d-1}{2}$. The estimate (41) is false when $0 \leq \alpha < \frac{d-1}{2}$ by scaling (nor is improvement possible in this case by decoupling the exponents for $\langle D_+\rangle$ and $\langle D_-\rangle$ from each other). We conclude that if $\mathbf{b} \equiv \mathrm{cst.}$ then Boltzmann's equation is locally well-posed in the space

$$\left\| \langle D_+\rangle^\alpha \langle D_-\rangle^\alpha \gamma(t) \right\|_{L_T^\infty L_{x,x'}^2} + \left\| \langle D_+\rangle^\alpha \langle D_-\rangle^\alpha \partial_t \left\{ e^{-it\Delta_\pm} \gamma(t) \right\} \right\|_{L_T^1 L_{x,x'}^2} \quad (42)$$

whenever

$$\left\| \langle D_+\rangle^\alpha \langle D_-\rangle^\alpha \gamma_0 \right\|_{L_{x,x'}^2} < \infty \quad (43)$$

and $\alpha > \frac{d-1}{2}$.

The proof of (41) (for both B^+ and B^-) is structurally identical to the proof of (38) given in [18], with only small changes where needed. Such a direct functional comparison between NLS and Boltzmann is only possible thanks to the Wigner transform. Moreover, it suggests a way to quantify the role of dispersive effects for Boltzmann's equation (with a constant collision kernel), by comparison with the corresponding NLS problem. Indeed, despite incredible strides in kinetic theory, many fundamental questions remain completely open even in the simplest possible case, $\mathbf{b} \equiv \mathrm{cst.}$

Remark. The most natural way to generalize the strategy of [6,18] is to draw from the theory of $X^{s,b}$ spaces (which are robust enough to approach "optimal" results in principle); however, this does not seem to lead in a straightforward way to improvements of the regularity for local well-posedness of (36).

Remark. Note that when $\alpha = \alpha_c = \frac{d-2}{2}$ the homogeneous version of the norm (43) defines a scale invariant space ($x \mapsto \lambda x, v \mapsto v/\mu$, arbitrary $\lambda, \mu > 0$) for Boltzmann's equation with $\mathbf{b} \equiv \mathrm{cst.}$ We refer the reader to [3] for *existence only* with a Maxwell-molecule-type kernel, given small data in the scaling critical space $L_{x,v}^2 \left(\mathbb{R}_x^2 \times \mathbb{R}_v^2 \right)$ when $d = 2$. Also, $\alpha_c = \frac{d-2}{2}$ is the scaling critical regularity for the cubic NLS (34); it is likewise scaling critical for (36) when $(x, x') \mapsto (\lambda x, \lambda x')$, but no fully general LWP theorem seems to be known for (36) when $\alpha \approx \alpha_c$.

2.5. The main result for the Boltzmann hierarchy. We now turn to the Boltzmann hierarchy. The Boltzmann hierarchy is an infinite sequence of coupled PDEs describing the evolution of densities $f^{(k)}(t, x_1, \ldots, x_k, v_1, \ldots, v_k)$ for $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$. The densities $f^{(k)}$ are assumed to be symmetric with respect to interchange of particle indices. The Boltzmann hierarchy arises as an intermediate equation in the derivation of Boltzmann's equation from an underlying Hamiltonian evolution of many particles, [12,17,19]. We use the notation $X_k = (x_1, \ldots, x_k)$ and, for $i \le j$, $X_{i:j} = (x_i, x_{i+1}, \ldots, x_j)$, and similarly for V_k and $V_{i:j}$. For each $k \in \mathbb{N}$, the kth equation of the Boltzmann hierarchy is written:

$$\left(\partial_t + V_k \cdot \nabla_{X_k}\right) f^{(k)}(t, X_k, V_k) = C_{k+1} f^{(k+1)}(t, X_k, V_k) \tag{44}$$

where the collision operator C_{k+1} is split into gain and loss parts:

$$C_{k+1} f^{(k+1)} = \sum_{i=1}^{k} C_{i,k+1}^{+} f^{(k+1)} - \sum_{i=1}^{k} C_{i,k+1}^{-} f^{(k+1)}. \tag{45}$$

The gain term is written

$$C_{i,k+1}^{+} f^{(k+1)}(t, X_k, V_k)$$
$$= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} dv_{k+1} d\omega \mathbf{b} \left(|v_{k+1} - v_i|, \omega \cdot \frac{v_{k+1} - v_i}{|v_{k+1} - v_i|}\right)$$
$$\times f^{(k+1)}(t, x_1, \ldots, x_i, \ldots, x_k, x_i, v_1, \ldots, v_i^*, \ldots, v_k, v_{k+1}^*) \tag{46}$$

where

$$v_i^* = v_i + P_\omega (v_{k+1} - v_i)$$
$$v_{k+1}^* = v_{k+1} - P_\omega (v_{k+1} - v_i). \tag{47}$$

Similarly for the loss term we have

$$C_{i,k+1}^{-} f^{(k+1)}(t, X_k, V_k)$$
$$= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} dv_{k+1} d\omega \mathbf{b} \left(|v_{k+1} - v_i|, \omega \cdot \frac{v_{k+1} - v_i}{|v_{k+1} - v_i|}\right)$$
$$\times f^{(k+1)}(t, x_1, \ldots, x_i, \ldots, x_k, x_i, v_1, \ldots, v_i, \ldots, v_k, v_{k+1}). \tag{48}$$

Note carefully that the collision operators $C_{i,k+1}^{\pm}$ involve the evaluation of $f^{(k+1)}$ along the hypersurface $x_{k+1} = x_i$.

In exactly the same manner as for the Boltzmann equation, we define the Wigner tranform and its inverse for multiple particles:

$$\gamma^{(k)}(t, X_k, X_k') = \int_{\mathbb{R}^{dk}} f^{(k)}\left(t, \frac{X_k + X_k'}{2}, V_k\right) e^{iV_k \cdot (X_k - X_k')} dV_k \tag{49}$$

$$f^{(k)}(t, X_k, V_k) = \frac{1}{(2\pi)^{dk}} \int_{\mathbb{R}^{dk}} \gamma^{(k)}\left(t, X_k + \frac{Y_k}{2}, X_k - \frac{Y_k}{2}\right) e^{-iV_k \cdot Y_k} dY_k. \tag{50}$$

The Fourier transform of $\gamma^{(k)}$ is written

$$\hat{\gamma}^{(k)}(\xi_1, \ldots, \xi_k, \xi_1', \ldots, \xi_k')$$
$$= \int e^{-i \sum_{i=1}^{k} x_i \cdot \xi_i} e^{-i \sum_{i=1}^{k} x_i' \cdot \xi_i'} \gamma^{(k)}(X_k, X_k') dX_k dX_k'. \tag{51}$$

Let us define the weighted Sobolev spaces $H_k^{\alpha,\beta,\sigma,\kappa}$ for $\alpha, \beta, \kappa \geq 0$ and $\sigma > 0$:

$$
\begin{aligned}
&\left\| \gamma^{(k)}(X_k, X_k') \right\|_{H_k^{\alpha,\beta,\sigma,\kappa}} \\
&= \left\| \prod_{i=1}^{k} \left\{ \langle \xi_i + \xi_i' \rangle^{\alpha} \langle \xi_i - \xi_i' \rangle^{\beta} e^{\kappa \langle \xi_i - \xi_i' \rangle^{\frac{1}{\sigma}}} \right\} \right. \\
&\qquad \left. \times \hat{\gamma}^{(k)}(\xi_1, \ldots, \xi_k, \xi_1', \ldots, \xi_k') \right\|_{L^2_{\xi_1,\ldots,\xi_k,\xi_1',\ldots,\xi_k'}}
\end{aligned}
\tag{52}
$$

These norms are equivalent (up to a factor like C^k) to the following norms for classical densities:

$$
\left\| \prod_{i=1}^{k} \left\{ \langle 2v_i \rangle^{\beta} e^{\kappa \langle 2v_i \rangle^{\frac{1}{\sigma}}} \left(1 - \Delta_{x_i}\right)^{\frac{\alpha}{2}} \right\} f^{(k)}(X_k, V_k) \right\|_{L^2_{X_k, V_k}}.
\tag{53}
$$

If $\Gamma = \left\{ \gamma^{(k)} \right\}_{k \in \mathbb{N}}$ and $\xi > 0$ then we further define

$$
\| \Gamma \|_{\mathcal{H}_\xi^{\alpha,\beta,\sigma,\kappa}} = \sum_{k \in \mathbb{N}} \xi^k \left\| \gamma^{(k)}(X_k, V_k) \right\|_{H_k^{\alpha,\beta,\sigma,\kappa}}.
\tag{54}
$$

Note that $\gamma \in H^{\alpha,\beta,\sigma,\kappa}$ if and only if $\Gamma = \left\{ \gamma^{\otimes k} \right\}_{k \in \mathbb{N}} \in \mathcal{H}_\xi^{\alpha,\beta,\sigma,\kappa}$ for some (arbitrary) $\xi > 0$.

The inverse Wigner transform of the Boltzmann hierarchy is: (see Propositions A.1 and A.2 in "Appendix A")

$$
\left(i \partial_t + \frac{1}{2} \left(\Delta_{X_k} - \Delta_{X_k'} \right) \right) \gamma^{(k)}(t, X_k, X_k') = B_{k+1} \gamma^{(k+1)}(t, X_k, X_k')
\tag{55}
$$

$$
B_{k+1} \gamma^{(k+1)} = \sum_{i=1}^{k} \left(B_{i,k+1}^{+} \gamma^{(k+1)} - B_{i,k+1}^{-} \gamma^{(k+1)} \right)
\tag{56}
$$

$$
\begin{aligned}
&B_{i,k+1}^{-} \gamma^{(k+1)}(t, X_k, X_k') \\
&= \frac{i}{2^{2d} \pi^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} dz \, \hat{\mathbf{b}}^{\omega} \left(\frac{z}{2} \right) \\
&\quad \times \gamma^{(k+1)} \left(t, X_{1:(i-1)}, x_i - \frac{z}{4}, X_{(i+1):k}, \frac{x_i + x_i'}{2} + \frac{z}{4}, \right. \\
&\qquad\qquad \left. X_{1:(i-1)}', x_i' + \frac{z}{4}, X_{(i+1):k}', \frac{x_i + x_i'}{2} - \frac{z}{4} \right)
\end{aligned}
\tag{57}
$$

$$B_{i,k+1}^+ \gamma^{(k+1)}(t, X_k, X_k')$$

$$= \frac{i}{2^{2d}\pi^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} dz \, \hat{\mathbf{b}}^\omega\left(\frac{z}{2}\right)$$

$$\times \gamma^{(k+1)}\left(t, X_{1:(i-1)}, x_i - \frac{1}{2}P_\omega(x_i - x_i') - \frac{R_\omega(z)}{4}, X_{(i+1):k},\right.$$

$$\frac{x_i + x_i'}{2} + \frac{1}{2}P_\omega(x_i - x_i') + \frac{R_\omega(z)}{4},$$

$$X_{1:(i-1)}', x_i' + \frac{1}{2}P_\omega(x_i - x_i') + \frac{R_\omega(z)}{4}, X_{(i+1):k}',$$

$$\left.\frac{x_i + x_i'}{2} - \frac{1}{2}P_\omega(x_i - x_i') - \frac{R_\omega(z)}{4}\right).$$

$$(58)$$

Solutions of the Boltzmann hierarchy are understood using Duhamel's formula: for all $k \in \mathbb{N}$,

$$\gamma^{(k)}(t) = e^{\frac{1}{2}it\Delta_\pm^{(k)}} \gamma^{(k)}(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_\pm^{(k)}} B_{k+1}\gamma^{(k+1)}(t_1)dt_1. \tag{59}$$

Here $\Delta_\pm^{(k)} = \Delta_{X_k} - \Delta_{X_k'}$. We further define $B\Gamma = \left\{B_{k+1}\gamma^{(k+1)}\right\}_{k\in\mathbb{N}}$.

We are ready to state our second main result.

Theorem 2.3. *Suppose* $\mathbf{b} \in L_A^\infty$ *with* $A \in \left[0, \frac{1}{2}\right)$, $\alpha \in \left(\frac{d-1}{2}, \infty\right)$, $\beta \in (d, \infty)$, $\kappa \in [0, \infty)$, *and* $\frac{1}{\sigma} \in (0, 2]$. *Assume* $\Gamma_0 = \left\{\gamma_0^{(k)}\right\}_{k\in\mathbb{N}} \in \mathcal{H}_{\xi_1}^{\alpha,\beta,\sigma,\kappa}$ *where* $\xi_1 \in (0, 1)$, *and further assume that the functions* $\gamma_0^{(k)}$ *are symmetric under particle interchange. Then there exists* $T > 0$ *and* $0 < \xi_2 < \xi_1$ *such that there exists a unique solution* $\Gamma(t)$ *of the Boltzmann hierarchy* (55) *for* $t \in [0, T]$ *with* $\|\Gamma(t)\|_{L_{t\in[0,T]}^\infty \mathcal{H}_{\xi_2}^{\alpha,\beta,\sigma,\kappa}} < \infty$ *and* $\|B\Gamma(t)\|_{L_{t\in[0,T]}^1 \mathcal{H}_{\xi_2}^{\alpha,\beta,\sigma,\kappa}} < \infty$, *and* $\Gamma(0) = \Gamma_0$. *Moreover, the following estimate holds:*

$$\|\Gamma\|_{L_{t\in[0,T]}^\infty \mathcal{H}_{\xi_2}^{\alpha,\beta,\sigma,\kappa}} + \|B\Gamma\|_{L_{t\in[0,T]}^1 \mathcal{H}_{\xi_2}^{\alpha,\beta,\sigma,\kappa}} \leq C \|\Gamma_0\|_{\mathcal{H}_{\xi_1}^{\alpha,\beta,\sigma,\kappa}} \tag{60}$$

where C *depends on* $T, d, \xi_1, \xi_2, \alpha, \beta, \sigma, \kappa$.

Remark. If $A = 0$ it is possible to optimize the proof of Theorem 2.3 and obtain the same result for any $\kappa \in [0, \infty)$, $\frac{1}{\sigma} \in (0, 2]$, and $\alpha, \beta \in \left(\frac{d-1}{2}, \infty\right)$. We omit the details.

2.6. Interpretation of the Boltzmann hierarchy. Extending Theorem 2.3 to the full range $A \in [0, 1]$ would require revising the boardgame argument as presented in [18] to be compatible with time-dependent weights, as in Theorem 2.2. Unfortunately this seems to be technically out of reach at the present time; indeed, it seems to be an interesting open question to determine whether the hard sphere Boltzmann hierarchy is in fact locally well-posed for data $\Gamma(0) \in \mathcal{H}_\xi^{\alpha,\beta,\sigma,\kappa}$ with a suitable choice of parameters.

Since we cannot (at present) extend our well-posedness result to the hard sphere Boltzmann hierarchy ($A = 1$), the reader will rightfully question why we study the Boltzmann hierarchy at all. After all, the hard sphere interaction is the *only* interaction with Grad cut-off that is physically relevant (and all our results assume the Grad cut-off). In particular, at present, we have nothing to offer in the context of Lanford's theorem, even

at the level of the Boltzmann hierarchy. Nevertheless, the Boltzmann hierarchy always has an interpretation in the context of *statistical solutions* of the Boltzmann equation. (See [5] for a formal discussion of statistical solutions.) Under suitable regularity assumptions, if π_t is a statistical solution of Boltzmann's equation, then

$$f^{(k)}(t) = \int h^{\otimes k} d\pi_t(h) \tag{61}$$

is a solution of the Boltzmann hierarchy (for any interaction, physical or not). Conversely, suppose the functions $f^{(k)}(t)$ (assumed smooth and growing at most exponentially in k), which solve the Boltzmann hierarchy, define the joint distribution of some exchangeable sequence of random variables $(x_1, v_1), (x_2, v_2), \ldots$. In that case, the Hewitt-Savage theorem guarantees the existence of a unique underlying π_t which must be a statistical solution of Boltzmann's equation [15].

3. The Key Proposition

The proofs of Theorems 2.2 and 2.3 will rely on the following proposition:

Proposition 3.1. *Suppose $A \in [0, 1]$, $\alpha \in \left(\frac{d-1}{2}, \infty\right)$, $\beta \in (d, \infty)$, and $\frac{1}{\sigma} \in (\max(0, 2A - 1), 2]$. Then for any $r \in [0, 1)$ such that $\frac{r}{\sigma} \geq \max(0, 2A - 1 + \delta)$ for a small $\delta > 0$ we have for all $\kappa_0 > \kappa > 0$, any $1 \leq i \leq k$, and any $\gamma_0^{(k+1)} \in H_{k+1}^{\alpha, \beta, \sigma, \kappa_0}$ the following estimates:*

$$\left\| B_{i,k+1}^{\pm} \left[e^{\frac{1}{2}it\left(\Delta_{X_{k+1}} - \Delta_{X'_{k+1}}\right)} \gamma_0^{(k+1)} \right] \right\|_{L_t^2 H_k^{\alpha, \beta, \sigma, \kappa}}$$

$$\leq C(\alpha, \beta, \sigma, r) \|\mathbf{b}\|_{L_A^\infty} \left(1 + (\kappa_0 - \kappa)^{-\frac{1}{2}r}\right) \left\| \gamma_0^{(k+1)} \right\|_{H_{k+1}^{\alpha, \beta, \sigma, \kappa_0}}. \tag{62}$$

Moreover, if $A \in \left[0, \frac{1}{2}\right)$, $\alpha \in \left(\frac{d-1}{2}, \infty\right)$, $\beta \in (d, \infty)$, and $\frac{1}{\sigma} \in (0, 2]$, then for any $\kappa_0 \geq \kappa \geq 0$, any $1 \leq i \leq k$, and any $\gamma_0^{(k+1)} \in H_{k+1}^{\alpha, \beta, \sigma, \kappa_0}$ the following estimates hold:

$$\left\| B_{i,k+1}^{\pm} \left[e^{\frac{1}{2}it\left(\Delta_{X_{k+1}} - \Delta_{X'_{k+1}}\right)} \gamma_0^{(k+1)} \right] \right\|_{L_t^2 H_k^{\alpha, \beta, \sigma, \kappa}}$$

$$\leq C(\alpha, \beta, \sigma) \|\mathbf{b}\|_{L_A^\infty} \left\| \gamma_0^{(k+1)} \right\|_{H_{k+1}^{\alpha, \beta, \sigma, \kappa_0}}. \tag{63}$$

Remark. Note that the second part of Proposition 3.1 formally follows from the first part by setting $r = 0$. In fact we will only prove the first part since the second part follows after trivial changes to the proof.

Remark. If $A = 0$ it is possible to optimize the proof of Proposition 3.1 and obtain (63) for any $\kappa_0 \geq \kappa \geq 0$, $\frac{1}{\sigma} \in (0, 2]$, and $\alpha, \beta \in \left(\frac{d-1}{2}, \infty\right)$. We omit the details.

Proof of Proposition 3.1.

Loss term. Consider a typical part of the loss term, e.g. $B_{1,k+1}^{-} \gamma^{(k+1)}$:

$$B_{1,k+1}^{-}\gamma^{(k+1)}(t, X_k, X_k') = \frac{i}{2^{2d}\pi^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} dz \hat{\mathbf{b}}^\omega \left(\frac{z}{2}\right)$$

$$\times \gamma^{(k+1)} \left(t, x_1 - \frac{z}{4}, X_{2:k}, \frac{x_1 + x_1'}{2} + \frac{z}{4}, x_1' + \frac{z}{4}, X_{2:k}', \frac{x_1 + x_1'}{2} - \frac{z}{4}\right). \quad (64)$$

We will fix some initial data $\gamma_0^{(k+1)}(X_{k+1}, X_{k+1}')$ and consider the following function:

$$B_{1,k+1}^{-} \left[e^{\frac{1}{2}it\left(\Delta_{X_{k+1}} - \Delta_{X_{k+1}'}\right)}\gamma_0^{(k+1)}\right](t, X_k, X_k'). \quad (65)$$

The spacetime Fourier transform of a function $F(t, X_k, X_k')$ is

$$\tilde{F}(\tau, \xi_1, \ldots, \xi_k, \xi_1', \ldots, \xi_k')$$
$$= \int dt dX_k dX_k' e^{-it\tau} e^{-i\sum_{i=1}^k x_i \cdot \xi_i} e^{-i\sum_{i=1}^k x_i' \cdot \xi_i'} F(t, X_k, X_k'). \quad (66)$$

The spacetime Fourier transform of $e^{\frac{1}{2}it(\Delta_{X_{k+1}} - \Delta_{X_{k+1}'})}\gamma_0^{(k+1)}$ is, up to a constant depending on k,

$$\hat{\gamma}_0^{(k+1)}(\xi_1, \ldots, \xi_{k+1}, \xi_1', \ldots, \xi_{k+1}')\delta \left(\tau + \frac{1}{2}\sum_{i=1}^{k+1} |\xi_i|^2 - \frac{1}{2}\sum_{i=1}^{k+1} |\xi_i'|^2\right). \quad (67)$$

We also have

$$\left(B_{1,k+1}^{-}\left[e^{\frac{1}{2}it\left(\Delta_{X_{k+1}} - \Delta_{X_{k+1}'}\right)}\gamma_0^{(k+1)}\right]\right)^{\sim}(\tau, \xi_1, \ldots, \xi_k, \xi_1', \ldots, \xi_k')$$

$$= \text{cst.} \int d\eta d\eta' \delta \left(\tau + \frac{1}{2}\left|\xi_1 - \frac{\eta + \eta'}{2}\right|^2 - \frac{1}{2}\left|\xi_1' - \frac{\eta + \eta'}{2}\right|^2 \right.$$

$$+ \frac{1}{2}|\eta|^2 - \frac{1}{2}|\eta'|^2 + \frac{1}{2}\sum_{2 \le i \le k} (|\xi_i|^2 - |\xi_i'|^2)\Bigg)$$

$$\times \mathbf{b} \left(\frac{|-\xi_1 + \xi_1' + \eta - \eta'|}{2}, \omega \cdot \frac{-\xi_1 + \xi_1' + \eta - \eta'}{|-\xi_1 + \xi_1' + \eta - \eta'|}\right)$$

$$\times \hat{\gamma}_0^{(k+1)}\left(\xi_1 - \frac{\eta + \eta'}{2}, \xi_2, \ldots, \xi_k, \eta, \xi_1' - \frac{\eta + \eta'}{2}, \xi_2', \ldots, \xi_k', \eta'\right). \quad (68)$$

The constant is uniformly bounded in k. Now we simply bound the collision kernel $\mathbf{b}$ using $\|\mathbf{b}\|_{L_A^\infty}$ to yield:

$$\left| \left(B_{1,k+1}^{-} \left[e^{\frac{1}{2}it\left(\Delta_{X_{k+1}} - \Delta_{X'_{k+1}}\right)} \gamma_0^{(k+1)} \right] \right)^{\sim} (\tau, \xi_1, \ldots, \xi_k, \xi'_1, \ldots, \xi'_k) \right|$$

$$\lesssim \|\mathbf{b}\|_{L_A^\infty} \int d\eta d\eta' \left(\langle \xi_1 - \xi'_1 \rangle^A + \langle \eta - \eta' \rangle^A \right)$$

$$\times \delta \left(\tau + \frac{1}{2} \left| \xi_1 - \frac{\eta + \eta'}{2} \right|^2 - \frac{1}{2} \left| \xi'_1 - \frac{\eta + \eta'}{2} \right|^2 \right.$$

$$\left. + \frac{1}{2} |\eta|^2 - \frac{1}{2} |\eta'|^2 + \frac{1}{2} \sum_{2 \leq i \leq k} (|\xi_i|^2 - |\xi'_i|^2) \right)$$

$$\times \left| \hat{\gamma}_0^{(k+1)} \left(\xi_1 - \frac{\eta + \eta'}{2}, \xi_2, \ldots, \xi_k, \eta, \xi'_1 - \frac{\eta + \eta'}{2}, \xi'_2, \ldots, \xi'_k, \eta' \right) \right|. \qquad (69)$$

We want to estimate the following integral, for suitable $\alpha, \beta, \kappa, \sigma > 0$:

$$I^-(\alpha, \beta, \kappa, \sigma) = \int d\tau d\xi_1 \ldots d\xi_k d\xi'_1 \ldots d\xi'_k$$

$$\times \prod_{i=1}^k \left\{ \langle \xi_i + \xi'_i \rangle^{2\alpha} \langle \xi_i - \xi'_i \rangle^{2\beta} e^{2\kappa \langle \xi_i - \xi'_i \rangle^{\frac{1}{\sigma}}} \right\}$$

$$\times \left| \left(B_{1,k+1}^{-} \left[e^{\frac{1}{2}it\left(\Delta_{X_{k+1}} - \Delta_{X'_{k+1}}\right)} \gamma_0^{(k+1)} \right] \right)^{\sim} \right|^2. \qquad (70)$$

To start, observe that

$$I^-(\alpha, \beta, \kappa, \sigma) \lesssim \|\mathbf{b}\|_{L_A^\infty}^2 \int d\tau d\xi_1 \ldots d\xi_k d\xi'_1 \ldots d\xi'_k d\eta_1 d\eta'_1 d\eta_2 d\eta'_2$$

$$\times \prod_{i=1}^k \left\{ \langle \xi_i + \xi'_i \rangle^{2\alpha} \langle \xi_i - \xi'_i \rangle^{2\beta} e^{2\kappa \langle \xi_i - \xi'_i \rangle^{\frac{1}{\sigma}}} \right\}$$

$$\times \left(\langle \xi_1 - \xi'_1 \rangle^A + \langle \eta_1 - \eta'_1 \rangle^A \right) \left(\langle \xi_1 - \xi'_1 \rangle^A + \langle \eta_2 - \eta'_2 \rangle^A \right)$$

$$\times \delta \left(\tau + \frac{1}{2} \left| \xi_1 - \frac{\eta_1 + \eta'_1}{2} \right|^2 - \frac{1}{2} \left| \xi'_1 - \frac{\eta_1 + \eta'_1}{2} \right|^2 \right.$$

$$\left. + \frac{1}{2} |\eta_1|^2 - \frac{1}{2} |\eta'_1|^2 + \frac{1}{2} \sum_{2 \leq i \leq k} (|\xi_i|^2 - |\xi'_i|^2) \right)$$

$$\times \delta \left(\tau + \frac{1}{2} \left| \xi_1 - \frac{\eta_2 + \eta'_2}{2} \right|^2 - \frac{1}{2} \left| \xi'_1 - \frac{\eta_2 + \eta'_2}{2} \right|^2 \right.$$

$$\left. + \frac{1}{2} |\eta_2|^2 - \frac{1}{2} |\eta'_2|^2 + \frac{1}{2} \sum_{2 \leq i \leq k} (|\xi_i|^2 - |\xi'_i|^2) \right)$$

$$\times \left| \hat{\gamma}_0^{(k+1)} \left(\xi_1 - \frac{\eta_1 + \eta_1'}{2}, \xi_2, \ldots, \xi_k, \eta_1, \xi_1' - \frac{\eta_1 + \eta_1'}{2}, \xi_2', \ldots, \xi_k', \eta_1' \right) \right|$$

$$\times \left| \hat{\gamma}_0^{(k+1)} \left(\xi_1 - \frac{\eta_2 + \eta_2'}{2}, \xi_2, \ldots, \xi_k, \eta_2, \xi_1' - \frac{\eta_2 + \eta_2'}{2}, \xi_2', \ldots, \xi_k', \eta_2' \right) \right|.$$

Let $\kappa_0 > \kappa$, then multiply and divide the integrand by the following factor:

$$\prod_{j=1}^{2} \left\{ \begin{array}{c} \langle \xi_1 + \xi_1' - \eta_j - \eta_j' \rangle^\alpha \langle \xi_1 - \xi_1' \rangle^\beta \, e^{\kappa_0 \langle \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}}} \\ \times \langle \eta_j + \eta_j' \rangle^\alpha \langle \eta_j - \eta_j' \rangle^\beta \, e^{\kappa_0 \langle \eta_j - \eta_j' \rangle^{\frac{1}{\sigma}}} \end{array} \right\}. \tag{71}$$

Then group terms together and apply Cauchy–Schwarz *pointwise* under the integral sign. We obtain two different terms that are equal due to symmetry under re-labeling coordinates; hence,

$$I^- \lesssim \|\mathbf{b}\|_{L_A^\infty}^2 \int d\tau d\xi_1 \ldots d\xi_k d\xi_1' \ldots d\xi_k' d\eta_1 d\eta_1' d\eta_2 d\eta_2'$$

$$\times \frac{\langle \xi_1 + \xi_1' \rangle^{2\alpha} \langle \xi_1 - \xi_1' \rangle^{2\beta} \left(\langle \xi_1 - \xi_1' \rangle^{2A} + \langle \eta_1 - \eta_1' \rangle^{2A} \right)}{\langle \xi_1 + \xi_1' - \eta_1 - \eta_1' \rangle^{2\alpha} \langle \xi_1 - \xi_1' \rangle^{2\beta} \langle \eta_1 + \eta_1' \rangle^{2\alpha} \langle \eta_1 - \eta_1' \rangle^{2\beta}}$$

$$\times \frac{e^{2\kappa \langle \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}}}}{e^{2\kappa_0 \langle \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}}} e^{2\kappa_0 \langle \eta_1 - \eta_1' \rangle^{\frac{1}{\sigma}}}}$$

$$\times \delta \left(\tau + \frac{1}{2} \left| \xi_1 - \frac{\eta_1 + \eta_1'}{2} \right|^2 - \frac{1}{2} \left| \xi_1' - \frac{\eta_1 + \eta_1'}{2} \right|^2 \right.$$

$$\left. + \frac{1}{2} |\eta_1|^2 - \frac{1}{2} |\eta_1'|^2 + \frac{1}{2} \sum_{2 \leq i \leq k} (|\xi_i|^2 - |\xi_i'|^2) \right)$$

$$\times \delta \left(\tau + \frac{1}{2} \left| \xi_1 - \frac{\eta_2 + \eta_2'}{2} \right|^2 - \frac{1}{2} \left| \xi_1' - \frac{\eta_2 + \eta_2'}{2} \right|^2 \right.$$

$$\left. + \frac{1}{2} |\eta_2|^2 - \frac{1}{2} |\eta_2'|^2 + \frac{1}{2} \sum_{2 \leq i \leq k} (|\xi_i|^2 - |\xi_i'|^2) \right)$$

$$\times \langle \xi_1 + \xi_1' - \eta_2 - \eta_2' \rangle^{2\alpha} \langle \xi_1 - \xi_1' \rangle^{2\beta} e^{2\kappa_0 \langle \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}}}$$

$$\times \langle \eta_2 + \eta_2' \rangle^{2\alpha} \langle \eta_2 - \eta_2' \rangle^{2\beta} e^{2\kappa_0 \langle \eta_2 - \eta_2' \rangle^{\frac{1}{\sigma}}}$$

$$\times \prod_{2 \leq i \leq k} \left\{ \langle \xi_i + \xi_i' \rangle^{2\alpha} \langle \xi_i - \xi_i' \rangle^{2\beta} e^{2\kappa_0 \langle \xi_i - \xi_i' \rangle^{\frac{1}{\sigma}}} \right\}$$

$$\times \left| \hat{\gamma}_0^{(k+1)} \left(\xi_1 - \frac{\eta_2 + \eta_2'}{2}, \xi_2, \ldots, \xi_k, \eta_2, \xi_1' - \frac{\eta_2 + \eta_2'}{2}, \xi_2', \ldots, \xi_k', \eta_2' \right) \right|^2.$$

The integral completely factorizes in the following way:

$$I^- \leq \int d\tau d\xi_1 \ldots d\xi_k d\xi_1' \ldots d\xi_k' \left(\int d\eta_1 d\eta_1' \ldots \right) \left(\int d\eta_2 d\eta_2' \ldots \right)$$

$$\leq \left(\sup_{\tau, \xi_i, \xi_i'} \int d\eta_1 d\eta_1' \ldots \right) \times \int d\tau d\xi_1 \ldots d\xi_k d\xi_1' \ldots d\xi_k' \left(\int d\eta_2 d\eta_2' \ldots \right).$$

Finally we are able to conclude that if the following integral,

$$\int d\eta d\eta' \delta \left(\tau + \frac{1}{2} \left| \xi_1 - \frac{\eta + \eta'}{2} \right|^2 - \frac{1}{2} \left| \xi_1' - \frac{\eta + \eta'}{2} \right|^2 \right.$$

$$\left. + \frac{1}{2} |\eta|^2 - \frac{1}{2} |\eta'|^2 + \frac{1}{2} \sum_{2 \leq i \leq k} (|\xi_i|^2 - |\xi_i'|^2) \right)$$

$$\times \frac{\langle \xi_1 + \xi_1' \rangle^{2\alpha} \left(\langle \xi_1 - \xi_1' \rangle^{2A} + \langle \eta - \eta' \rangle^{2A} \right)}{\langle \xi_1 + \xi_1' - \eta - \eta' \rangle^{2\alpha} \langle \eta + \eta' \rangle^{2\alpha} \langle \eta - \eta' \rangle^{2\beta}} e^{-2(\kappa_0 - \kappa) \langle \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}}} \tag{72}$$

is bounded *uniformly* with respect to $\tau, \xi_1, \ldots, \xi_k, \xi_1', \ldots, \xi_k'$, then the following estimate holds:

$$\left\| B_{1,k+1}^- \left[e^{\frac{1}{2} it \left(\Delta_{X_{k+1}} - \Delta_{X_{k+1}'} \right)} \gamma_0^{(k+1)} \right] \right\|_{L_t^2 H_k^{\alpha, \beta, \sigma, \kappa}}$$

$$\leq C(\alpha, \beta, \sigma, \kappa, \kappa_0) \| \mathbf{b} \|_{L_A^\infty} \left\| \gamma_0^{(k+1)} \right\|_{H_{k+1}^{\alpha, \beta, \sigma, \kappa_0}}. \tag{73}$$

Let us make the change of variables $w = \frac{\eta + \eta'}{2}$, $z = \frac{\eta - \eta'}{2}$ in (72); then, up to a constant, the integral becomes:

$$\int dw dz \delta \left(\tau + \frac{1}{2} |\xi_1 - w|^2 - \frac{1}{2} |\xi_1' - w|^2 + \frac{1}{2} |w + z|^2 - \frac{1}{2} |w - z|^2 \right.$$

$$\left. + \frac{1}{2} \sum_{2 \leq i \leq k} \left(|\xi_i|^2 - |\xi_i'|^2 \right) \right)$$

$$\times \frac{\langle \xi_1 + \xi_1' \rangle^{2\alpha} \left(\langle \xi_1 - \xi_1' \rangle^{2A} + \langle 2z \rangle^{2A} \right)}{\langle \xi_1 + \xi_1' - 2w \rangle^{2\alpha} e^{2(\kappa_0 - \kappa) \langle \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}}} \langle 2w \rangle^{2\alpha} \langle 2z \rangle^{2\beta}}. \tag{74}$$

This is the same as:

$$K = \int dw dz \delta \left(\tau + \frac{1}{2} \sum_{i=1}^k \left(|\xi_i|^2 - |\xi_i'|^2 \right) - (\xi_1 - \xi_1' - 2z) \cdot w \right)$$

$$\times \frac{\langle \xi_1 + \xi_1' \rangle^{2\alpha} \left(\langle \xi_1 - \xi_1' \rangle^{2A} + \langle 2z \rangle^{2A} \right)}{\langle \xi_1 + \xi_1' - 2w \rangle^{2\alpha} e^{2(\kappa_0 - \kappa) \langle \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}}} \langle 2w \rangle^{2\alpha} \langle 2z \rangle^{2\beta}}. \tag{75}$$

Hence, one way to parametrize the integral is to let $z \in \mathbb{R}^d$ be arbitrary and let w range over a codimension one hyperplane in $\mathbb{R}^d$; the hyperplane is determined by τ, ξ, ξ', z. We have:

$$K \leq \int_{\mathbb{R}^d} dz \frac{\langle \xi_1 - \xi_1' \rangle^{2A} + \langle 2z \rangle^{2A}}{|\xi_1 - \xi_1' - 2z| \, e^{2(\kappa_0 - \kappa)\langle \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}}} \langle 2z \rangle^{2\beta}} \int_P \frac{dS(w) \langle \xi_1 + \xi_1' \rangle^{2\alpha}}{\langle \xi_1 + \xi_1' - 2w \rangle^{2\alpha} \langle 2w \rangle^{2\alpha}} \quad (76)$$

where $dS(w)$ is the induced surface measure on a hyperplane $P \subset \mathbb{R}^d$, given explicitly by

$$P = \left\{ w \in \mathbb{R}^d \,\middle|\, \tau + \frac{1}{2} \sum_{i=1}^{k} \left(|\xi_i|^2 - |\xi_i'|^2 \right) - \left(\xi_1 - \xi_1' - 2z \right) \cdot w = 0 \right\}. \quad (77)$$

In order to show the uniform boundedness of K with respect to $\tau, \xi_1, \dots, \xi_k, \xi_1', \dots, \xi_k'$, it suffices to prove the uniform boundedness of the following three quantities with respect to $W \in \mathbb{R}^d$:

$$I_1 = \sup_{P \subset \mathbb{R}^d : \dim P = d-1} \int_P dS(w) \frac{\langle W \rangle^{2\alpha}}{\langle W - w \rangle^{2\alpha} \langle w \rangle^{2\alpha}} \quad (78)$$

$$I_2 = \int_{\mathbb{R}^d} dz \frac{1}{|W - z| \langle z \rangle^{2\beta - 2A}} \quad (79)$$

$$I_3 = \int_{\mathbb{R}^d} dz \frac{\langle W \rangle^{2A}}{e^{2(\kappa_0 - \kappa)\langle W \rangle^{\frac{1}{\sigma}}} |W - z| \langle z \rangle^{2\beta}}. \quad (80)$$

Note that in the expression for I_1, P is an arbitrary hyperplane of codimension one in $\mathbb{R}^d$.

We begin with I_3; clearly the integral over the set $|z - W| < 1$ is uniformly bounded in W if $\beta \geq A$. Therefore it suffices to bound the following integral:

$$I_3' = \int_{\mathbb{R}^d} dz \frac{\langle W \rangle^{2A}}{e^{2(\kappa_0 - \kappa)\langle W \rangle^{\frac{1}{\sigma}}} \langle W - z \rangle \langle z \rangle^{2\beta}}. \quad (81)$$

We have the following inequality:

$$e^{2(\kappa - \kappa_0)\langle W \rangle^{\frac{1}{\sigma}}} \geq 1 + 2(\kappa_0 - \kappa) \langle W \rangle^{\frac{1}{\sigma}} \\ \gtrsim (\kappa_0 - \kappa)^r \langle W \rangle^{\frac{r}{\sigma}} \quad (82)$$

where $0 \leq r \leq 1$. Since $\frac{1}{\sigma} > \max(0, (2A - 1))$, we can always find an $r \in [0, 1)$ such that $\frac{r}{\sigma} \geq \max(0, (2A - 1))$. For any such value of r, we have:

$$I_3' \lesssim (\kappa_0 - \kappa)^{-r} \int_{\mathbb{R}^d} dz \frac{\langle W \rangle}{\langle W - z \rangle \langle z \rangle^{2\beta}}. \quad (83)$$

Splitting the integral into the regions $|z| < \frac{1}{2}|W|$, $|z| > 2|W|$, and $\frac{1}{2}|W| \leq |z| \leq 2|W|$, we are able to show that $I_3 \lesssim (\kappa_0 - \kappa)^{-r}$ uniformly in W as long as $\beta > \frac{d+1}{2}$, $\frac{1}{\sigma} > \max(0, (2A - 1))$, and $r \in [0, 1)$ is such that $\frac{r}{\sigma} \geq \max(0, (2A - 1))$.

Let us turn to I_2; clearly, the integral over the set $|z - W| < 1$ is uniformly bounded in W if $\beta \geq A$. Therefore, it suffices to bound the following integral uniformly in W:

$$I_2' = \int_{\mathbb{R}^d} dz \frac{1}{\langle W - z \rangle \langle z \rangle^{2\beta - 2A}}. \tag{84}$$

For any $A \in [0, 1]$, this integral is automatically bounded, uniformly in W, if $\beta > \frac{d+2}{2}$.

Finally we turn to I_1:

$$I_1 = \sup_{P \subset \mathbb{R}^d : \dim P = d-1} \int_P dS(w) \frac{\langle W \rangle^{2\alpha}}{\langle W - w \rangle^{2\alpha} \langle w \rangle^{2\alpha}}. \tag{85}$$

We consider separately the regions $|w| < \frac{1}{2}|W|$, $|w| > 2|W|$, and $\frac{1}{2}|W| \leq |w| \leq 2|W|$; we find that the integral is uniformly bounded in W and P as long as $\alpha > \frac{d-1}{2}$.

To summarize, as long as $\alpha > \frac{d-1}{2}$, $\beta > d$, $\frac{1}{\sigma} > \max(0, (2A - 1))$, and $r \in [0, 1)$ is chosen such that $\frac{r}{\sigma} \geq \max(0, (2A - 1))$, then for all $\kappa_0 > \kappa > 0$:

$$\left\| B_{i,k+1}^- \left[e^{\frac{1}{2} it \left(\Delta_{X_{k+1}} - \Delta_{X_{k+1}'} \right)} \gamma_0^{(k+1)} \right] \right\|_{L_t^2 H_k^{\alpha, \beta, \sigma, \kappa}}$$
$$\leq C(\alpha, \beta, \sigma, r) \|\mathbf{b}\|_{L_A^\infty} \left(1 + (\kappa_0 - \kappa)^{-\frac{1}{2}r} \right) \left\| \gamma_0^{(k+1)} \right\|_{H_{k+1}^{\alpha, \beta, \sigma, \kappa_0}}. \tag{86}$$

Gain term. Consider a typical part of the gain term, e.g. $B_{1,k+1}^+ \gamma^{(k+1)}$:

$$B_{1,k+1}^+ \gamma^{(k+1)}(t, X_k, X_k')$$
$$= \frac{i}{2^{2d} \pi^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} dz \hat{\mathbf{b}}^\omega \left(\frac{z}{2} \right)$$
$$\times \gamma^{(k+1)} \left(t, x_1 - \frac{1}{2} P_\omega(x_1 - x_1') - \frac{R_\omega(z)}{4}, X_{2:k}, \right.$$
$$\frac{x_1 + x_1'}{2} + \frac{1}{2} P_\omega(x_1 - x_1') + \frac{R_\omega(z)}{4},$$
$$x_1' + \frac{1}{2} P_\omega(x_1 - x_1') + \frac{R_\omega(z)}{4}, X_{2:k}',$$
$$\left. \frac{x_1 + x_1'}{2} - \frac{1}{2} P_\omega(x_1 - x_1') - \frac{R_\omega(z)}{4} \right). \tag{87}$$

The spacetime Fourier transform of the function

$$B_{1,k+1}^+ \left[e^{\frac{1}{2} it \left(\Delta_{X_{k+1}} - \Delta_{X_{k+1}'} \right)} \gamma_0^{(k+1)} \right] (t, X_k, X_k') \tag{88}$$

is the following, up to a constant:

$$\int_{\mathbb{S}^{d-1}} d\omega \int d\eta_1 d\eta_1' d\eta_2 d\eta_2'$$

$$\times \delta \left(\tau + \frac{1}{2}|\eta_1|^2 - \frac{1}{2}|\eta_1'|^2 + \frac{1}{2}|\eta_2|^2 - \frac{1}{2}|\eta_2'|^2 + \frac{1}{2} \sum_{2 \le i \le k} (|\xi_i|^2 - |\xi_i'|^2) \right)$$

$$\times \delta \left(-\xi_1 + \eta_1 + \frac{\eta_2 + \eta_2'}{2} - \frac{1}{2}P_\omega(\eta_1 - \eta_1') + \frac{1}{2}P_\omega(\eta_2 - \eta_2') \right)$$

$$\times \delta \left(-\xi_1' + \eta_1' + \frac{\eta_2 + \eta_2'}{2} + \frac{1}{2}P_\omega(\eta_1 - \eta_1') - \frac{1}{2}P_\omega \cdot (\eta_2 - \eta_2') \right)$$

$$\times \mathbf{b} \left(\frac{|-\eta_1 + \eta_1' + \eta_2 - \eta_2'|}{2}, \omega \cdot \frac{R_\omega(-\eta_1 + \eta_1' + \eta_2 - \eta_2')}{|-\eta_1 + \eta_1' + \eta_2 - \eta_2'|} \right)$$

$$\times \hat{\gamma}_0^{(k+1)} \left(\eta_1, \xi_2, \ldots, \xi_k, \eta_2, \eta_1', \xi_2', \ldots, \xi_k', \eta_2' \right). \tag{89}$$

This is bounded by $\|\mathbf{b}\|_{L_A^\infty}$ times the following integral:

$$\int_{\mathbb{S}^{d-1}} d\omega \int d\eta_1 d\eta_1' d\eta_2 d\eta_2' \left(\langle \eta_1 - \eta_1' \rangle^A + \langle \eta_2 - \eta_2' \rangle^A \right)$$

$$\times \delta \left(\tau + \frac{1}{2}|\eta_1|^2 - \frac{1}{2}|\eta_1'|^2 + \frac{1}{2}|\eta_2|^2 - \frac{1}{2}|\eta_2'|^2 + \frac{1}{2} \sum_{2 \le i \le k} (|\xi_i|^2 - |\xi_i'|^2) \right)$$

$$\times \delta \left(-\xi_1 + \eta_1 + \frac{\eta_2 + \eta_2'}{2} - \frac{1}{2}P_\omega(\eta_1 - \eta_1') + \frac{1}{2}P_\omega(\eta_2 - \eta_2') \right)$$

$$\times \delta \left(-\xi_1' + \eta_1' + \frac{\eta_2 + \eta_2'}{2} + \frac{1}{2}P_\omega(\eta_1 - \eta_1') - \frac{1}{2}P_\omega(\eta_2 - \eta_2') \right)$$

$$\times \left| \hat{\gamma}_0^{(k+1)} \left(\eta_1, \xi_2, \ldots, \xi_k, \eta_2, \eta_1', \xi_2', \ldots, \xi_k', \eta_2' \right) \right|. \tag{90}$$

Introduce the change of variables $w_1 = \frac{\eta_1 + \eta_1'}{2}$, $z_1 = \frac{\eta_1 - \eta_1'}{2}$, $w_2 = \frac{\eta_2 + \eta_2'}{2}$, $z_2 = \frac{\eta_2 - \eta_2'}{2}$. Then (90) becomes

$$\int_{\mathbb{S}^{d-1}} d\omega \int dw_1 dz_1 dw_2 dz_2 \left(\langle 2z_1 \rangle^A + \langle 2z_2 \rangle^A \right)$$

$$\times \delta \left(\tau + \frac{1}{2}|w_1 + z_1|^2 - \frac{1}{2}|w_1 - z_1|^2 + \frac{1}{2}|w_2 + z_2|^2 - \frac{1}{2}|w_2 - z_2|^2 \right.$$

$$\left. + \frac{1}{2} \sum_{2 \le i \le k} (|\xi_i|^2 - |\xi_i'|^2) \right) \tag{91}$$

$$\times \delta \left(-\xi_1 + w_1 + z_1 + w_2 - P_\omega(z_1 - z_2) \right)$$

$$\times \delta \left(-\xi_1' + w_1 - z_1 + w_2 + P_\omega(z_1 - z_2) \right)$$

$$\times \left| \hat{\gamma}_0^{(k+1)} \left(w_1 + z_1, \xi_2, \ldots, \xi_k, w_2 + z_2, w_1 - z_1, \xi_2', \ldots, \xi_k', w_2 - z_2 \right) \right|.$$

Introduce yet another change of variables $r_1 = \frac{w_1 + w_2}{2}$, $s_1 = \frac{w_1 - w_2}{2}$, $r_2 = \frac{z_1 + z_2}{2}$, $s_2 = \frac{z_1 - z_2}{2}$. Then (91) becomes

$$
\int_{\mathbb{S}^{d-1}} d\omega \int dr_1 ds_1 dr_2 ds_2 \left(\langle 2(r_2 + s_2) \rangle^A + \langle 2(r_2 - s_2) \rangle^A \right)
$$

$$
\times \delta \left(\tau + \frac{1}{2} |r_1 + s_1 + r_2 + s_2|^2 - \frac{1}{2} |r_1 + s_1 - r_2 - s_2|^2 \right.
$$

$$
+ \frac{1}{2} |r_1 - s_1 + r_2 - s_2|^2 - \frac{1}{2} |r_1 - s_1 - r_2 + s_2|^2
$$

$$
\left. + \frac{1}{2} \sum_{2 \le i \le k} (|\xi_i|^2 - |\xi_i'|^2) \right) \tag{92}
$$

$$
\times \delta \left(-\xi_1 + 2r_1 + r_2 + R_\omega(s_2) \right)
$$

$$
\times \delta \left(-\xi_1' + 2r_1 - r_2 - R_\omega(s_2) \right)
$$

$$
\times \left| \hat{\gamma}_0^{(k+1)} \left(r_1 + s_1 + r_2 + s_2, \xi_2, \ldots, \xi_k, r_1 - s_1 + r_2 - s_2, \right. \right.
$$

$$
\left. \left. r_1 + s_1 - r_2 - s_2, \xi_2', \ldots, \xi_k', r_1 - s_1 - r_2 + s_2 \right) \right|.
$$

Replace r_1 with $\frac{r_1}{2}$ throughout:

$$
\int_{\mathbb{S}^{d-1}} d\omega \int dr_1 ds_1 dr_2 ds_2 \left(\langle 2(r_2 + s_2) \rangle^A + \langle 2(r_2 - s_2) \rangle^A \right)
$$

$$
\times \delta \left(\tau + \frac{1}{2} \left| \frac{r_1}{2} + s_1 + r_2 + s_2 \right|^2 - \frac{1}{2} \left| \frac{r_1}{2} + s_1 - r_2 - s_2 \right|^2 \right.
$$

$$
+ \frac{1}{2} \left| \frac{r_1}{2} - s_1 + r_2 - s_2 \right|^2 - \frac{1}{2} \left| \frac{r_1}{2} - s_1 - r_2 + s_2 \right|^2
$$

$$
\left. + \frac{1}{2} \sum_{2 \le i \le k} (|\xi_i|^2 - |\xi_i'|^2) \right) \tag{93}
$$

$$
\times \delta \left(-\xi_1 + r_1 + r_2 + R_\omega(s_2) \right)
$$

$$
\times \delta \left(-\xi_1' + r_1 - r_2 - R_\omega(s_2) \right)
$$

$$
\times \left| \hat{\gamma}_0^{(k+1)} \left(\frac{r_1}{2} + s_1 + r_2 + s_2, \xi_2, \ldots, \xi_k, \frac{r_1}{2} - s_1 + r_2 - s_2, \right. \right.
$$

$$
\left. \left. \frac{r_1}{2} + s_1 - r_2 - s_2, \xi_2', \ldots, \xi_k', \frac{r_1}{2} - s_1 - r_2 + s_2 \right) \right|.
$$

Finally perform the change of variables $\zeta_1 = r_1 + r_2$, $\zeta_2 = r_1 - r_2$:

$$
\int_{\mathbb{S}^{d-1}} d\omega \int d\zeta_1 d\zeta_2 ds_1 ds_2 \left(\langle \zeta_1 - \zeta_2 + 2s_2 \rangle^A + \langle \zeta_1 - \zeta_2 - 2s_2 \rangle^A \right)
$$

$$
\times \delta \left(\tau + \frac{1}{2} \left| \frac{3\zeta_1}{4} - \frac{\zeta_2}{4} + s_1 + s_2 \right|^2 - \frac{1}{2} \left| -\frac{\zeta_1}{4} + \frac{3\zeta_2}{4} + s_1 - s_2 \right|^2 \right.
$$

$$
+ \frac{1}{2} \left| \frac{3\zeta_1}{4} - \frac{\zeta_2}{4} - s_1 - s_2 \right|^2 - \frac{1}{2} \left| -\frac{\zeta_1}{4} + \frac{3\zeta_2}{4} - s_1 + s_2 \right|^2
$$

$$+\frac{1}{2}\sum_{2\le i\le k}(|\xi_i|^2-|\xi_i'|^2)\Bigg)$$

$$\times\delta\left(-\xi_1+\zeta_1+R_\omega(s_2)\right)$$
$$\times\delta\left(-\xi_1'+\zeta_2-R_\omega(s_2)\right)$$
$$\times\left|\hat\gamma_0^{(k+1)}\left(\frac{3\zeta_1}{4}-\frac{\zeta_2}{4}+s_1+s_2,\xi_2,\ldots,\xi_k,\frac{3\zeta_1}{4}-\frac{\zeta_2}{4}-s_1-s_2,\right.\right.$$
$$\left.\left.\frac{-\zeta_1}{4}+\frac{3\zeta_2}{4}+s_1-s_2,\xi_2',\ldots,\xi_k',-\frac{\zeta_1}{4}+\frac{3\zeta_2}{4}-s_1+s_2\right)\right|. \tag{94}$$

Now we can integrate out the variables ζ_1, ζ_2 to obtain:

$$\int_{\mathbb{S}^{d-1}}d\omega\int ds_1 ds_2\left(\left\langle 4s_2^\|+\xi_1-\xi_1'\right\rangle^A+\left\langle -4s_2^\perp+\xi_1-\xi_1'\right\rangle^A\right)$$

$$\times\delta\left(\tau+\frac{1}{2}\left|s_1+2s_2^\|+\frac{3\xi_1-\xi_1'}{4}\right|^2-\frac{1}{2}\left|s_1-2s_2^\|+\frac{3\xi_1'-\xi_1}{4}\right|^2\right.$$

$$+\frac{1}{2}\left|-s_1-2s_2^\perp+\frac{3\xi_1-\xi_1'}{4}\right|^2-\frac{1}{2}\left|-s_1+2s_2^\perp+\frac{3\xi_1'-\xi_1}{4}\right|^2$$

$$+\frac{1}{2}\sum_{2\le i\le k}(|\xi_i|^2-|\xi_i'|^2)\Bigg)$$

$$\times\left|\hat\gamma_0^{(k+1)}\left(s_1+2s_2^\|+\frac{3\xi_1-\xi_1'}{4},\xi_2,\ldots,\xi_k,-s_1-2s_2^\perp+\frac{3\xi_1-\xi_1'}{4},\right.\right.$$

$$\left.\left.s_1-2s_2^\|+\frac{3\xi_1'-\xi_1}{4},\xi_2',\ldots,\xi_k',-s_1+2s_2^\perp+\frac{3\xi_1'-\xi_1}{4}\right)\right| \tag{95}$$

where $s_2^\|=P_\omega(s_2)$ and $s_2^\perp=(\mathbb{I}-P_\omega)(s_2)$.

We want to estimate the following integral, for suitable $\alpha,\beta,\kappa,\sigma>0$:

$$I^+(\alpha,\beta,\kappa,\sigma)=\int d\tau d\xi_1\ldots d\xi_k d\xi_1'\ldots d\xi_k'$$

$$\times\prod_{i=1}^k\left\{\langle\xi_i+\xi_i'\rangle^{2\alpha}\langle\xi_i-\xi_i'\rangle^{2\beta}\,e^{2\kappa\langle\xi_i-\xi_i'\rangle^{\frac{1}{\sigma}}}\right\}$$

$$\times\left|\left(B_{1,k+1}^+\left[e^{\frac{1}{2}it\left(\Delta_{x_{k+1}}-\Delta'_{x_{k+1}}\right)}\gamma_0^{(k+1)}\right]\right)^\sim\right|^2. \tag{96}$$

Reasoning as for the loss term, if we can show that the following integral

$$\int_{\mathbb{S}^{d-1}}d\omega\int ds_1 ds_2$$

$$\times\delta\left(\tau+\frac{1}{2}\left|s_1+2s_2^\|+\frac{3\xi_1-\xi_1'}{4}\right|^2-\frac{1}{2}\left|s_1-2s_2^\|+\frac{3\xi_1'-\xi_1}{4}\right|^2+\right.$$

$$+ \frac{1}{2} \left| -s_1 - 2s_2^{\perp} + \frac{3\xi_1 - \xi_1'}{4} \right|^2 - \frac{1}{2} \left| -s_1 + 2s_2^{\perp} + \frac{3\xi_1' - \xi_1}{4} \right|^2$$

$$+ \sum_{2 \leq i \leq k} (|\xi_i|^2 - |\xi_i'|^2) \Bigg)$$

$$\times \frac{\langle \xi_1 + \xi_1' \rangle^{2\alpha} \langle \xi_1 - \xi_1' \rangle^{2\beta} \left(\langle 4s_2^{\|} + \xi_1 - \xi_1' \rangle^{2A} + \langle -4s_2^{\perp} + \xi_1 - \xi_1' \rangle^{2A} \right)}{\langle 2s_1 + \frac{\xi_1 + \xi_1'}{2} \rangle^{2\alpha} \langle 4s_2^{\|} + \xi_1 - \xi_1' \rangle^{2\beta} \langle -2s_1 + \frac{\xi_1 + \xi_1'}{2} \rangle^{2\alpha} \langle -4s_2^{\perp} + \xi_1 - \xi_1' \rangle^{2\beta}}$$

$$\times e^{-2(\kappa_0 - \kappa)\langle 4s_2^{\|} + \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}}} e^{-2(\kappa_0 - \kappa)\langle -4s_2^{\perp} + \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}}}$$

$$\times e^{2\kappa \left(\langle \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}} - \langle 4s_2^{\|} + \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}} - \langle -4s_2^{\perp} + \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}} \right)} \tag{97}$$

is bounded uniformly in $\tau, \xi_1, \dots, \xi_k, \xi_1', \dots, \xi_k'$, then we will have the following estimate:

$$\left\| B_{1,k+1}^+ \left[e^{\frac{1}{2} i t \left(\Delta_{x_{k+1}} - \Delta_{x_{k+1}'} \right)} \gamma_0^{(k+1)} \right] \right\|_{L_t^2 H_k^{\alpha, \beta, \sigma, \kappa}}$$

$$\leq C(\alpha, \beta, \sigma, \kappa, \kappa_0) \|\mathbf{b}\|_{L_A^\infty} \left\| \gamma_0^{(k+1)} \right\|_{H_{k+1}^{\alpha, \beta, \sigma, \kappa_0}}. \tag{98}$$

Before proceeding further, we must eliminate the most dangerous contribution in (97), which is the following *exponential* factor:

$$e^{2\kappa \left(\langle \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}} - \langle 4s_2^{\|} + \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}} - \langle -4s_2^{\perp} + \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}} \right)}. \tag{99}$$

We will show that this factor is in fact bounded by 1, as long as $\sigma \geq \frac{1}{2}$. Indeed for $\sigma \geq \frac{1}{2}$ we have:

$$\langle \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}} - \langle 4s_2^{\|} + \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}} - \langle -4s_2^{\perp} + \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}}$$

$$\leq \langle \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}} - \langle (\xi_1 - \xi_1')^{\perp} \rangle^{\frac{1}{\sigma}} - \langle (\xi_1 - \xi_1')^{\|} \rangle^{\frac{1}{\sigma}}$$

$$\leq \left(\langle (\xi_1 - \xi_1')^{\|} \rangle^2 + \langle (\xi_1 - \xi_1')^{\perp} \rangle^2 \right)^{\frac{1}{2\sigma}} - \langle (\xi_1 - \xi_1')^{\perp} \rangle^{\frac{1}{\sigma}} - \langle (\xi_1 - \xi_1')^{\|} \rangle^{\frac{1}{\sigma}}$$

$$\leq 0. \tag{100}$$

We now deal with the other exponential factors in (97), namely:

$$e^{-2(\kappa_0 - \kappa)\langle 4s_2^{\|} + \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}}} e^{-2(\kappa_0 - \kappa)\langle -4s_2^{\perp} + \xi_1 - \xi_1' \rangle^{\frac{1}{\sigma}}}. \tag{101}$$

Since $\frac{1}{\sigma} > \max(0, 2A - 1)$, we can always find $r \in [0, 1)$ such that $\frac{r}{\sigma} \geq \max(0, 2A - 1 + \delta)$ for a small $\delta > 0$. Since $e^u \geq 1 + u \gtrsim u^r$ for $u > 0$, we find that if $\kappa < \kappa_0$ then (101) is bounded above by the following quantity:

$$(\kappa_0 - \kappa)^{-r}$$

$$\times \min\left(\left\langle 4s_2^{\parallel} + \xi_1 - \xi_1'\right\rangle^{-\max(0, 2A-1+\delta)}, \left\langle -4s_2^{\perp} + \xi_1 - \xi_1'\right\rangle^{-\max(0, 2A-1+\delta)}\right).$$

(102)

The integral (97) is now bounded by the following integral, if $\delta > 0$ is sufficiently small depending on A, σ and r: (note that this follows from the previous paragraph by considering separately $0 \le A < \frac{1}{2}$ and $\frac{1}{2} \le A \le 1$)

$$(\kappa_0 - \kappa)^{-r} \int_{\mathbb{S}^{d-1}} d\omega \int ds_1 ds_2$$

$$\times \delta\left(\tau + \frac{1}{2}\sum_{i=1}^{k}\left(|\xi_i|^2 - |\xi_i'|^2\right) + \left(4s_1 - R_\omega(\xi_1 + \xi_1')\right) \cdot s_2\right)$$

$$\times \frac{\left\langle \xi_1 + \xi_1'\right\rangle^{2\alpha}\left\langle \xi_1 - \xi_1'\right\rangle^{2\beta}\left(\left\langle 4s_2^{\parallel} + \xi_1 - \xi_1'\right\rangle^{1-\delta} + \left\langle -4s_2^{\perp} + \xi_1 - \xi_1'\right\rangle^{1-\delta}\right)}{\left\langle 2s_1 + \frac{\xi_1+\xi_1'}{2}\right\rangle^{2\alpha}\left\langle 4s_2^{\parallel} + \xi_1 - \xi_1'\right\rangle^{2\beta}\left\langle -2s_1 + \frac{\xi_1+\xi_1'}{2}\right\rangle^{2\alpha}\left\langle -4s_2^{\perp} + \xi_1 - \xi_1'\right\rangle^{2\beta}}.$$

(103)

This is in turn equivalent to the following integral:

$$(\kappa_0 - \kappa)^{-r}$$

$$\times \int_{\mathbb{S}^{d-1}} d\omega \int ds_2 \frac{\left\langle \xi_1 - \xi_1'\right\rangle^{2\beta}\left(\left\langle 4s_2^{\parallel} + \xi_1 - \xi_1'\right\rangle^{1-\delta} + \left\langle -4s_2^{\perp} + \xi_1 - \xi_1'\right\rangle^{1-\delta}\right)}{|4s_2|\left\langle 4s_2^{\parallel} + \xi_1 - \xi_1'\right\rangle^{2\beta}\left\langle -4s_2^{\perp} + \xi_1 - \xi_1'\right\rangle^{2\beta}}$$

(104)

$$\times \int_P dS(s_1) \frac{\left\langle \xi_1 + \xi_1'\right\rangle^{2\alpha}}{\left\langle 2s_1 + \frac{\xi_1+\xi_1'}{2}\right\rangle^{2\alpha}\left\langle -2s_1 + \frac{\xi_1+\xi_1'}{2}\right\rangle^{2\alpha}}$$

where $P \subset \mathbb{R}^d$ is the following codimension one hyperplane:

$$P = \left\{ s_1 \in \mathbb{R}^d \,\middle|\, \tau + \frac{1}{2}\sum_{i=1}^{k}\left(|\xi_i|^2 - |\xi_i'|^2\right) + \left(4s_1 - R_\omega(\xi_1 + \xi_1')\right) \cdot s_2 = 0\right\}.$$

(105)

Therefore we only need to show the boundedness of the following three quantities uniformly in ξ_1, ξ_1', τ:

$$I_1 = \sup_{P \subset \mathbb{R}^d : \dim P = d-1} \int_P dS(s) \frac{\left\langle \xi_1 + \xi_1'\right\rangle^{2\alpha}}{\left\langle 2s + \frac{\xi_1+\xi_1'}{2}\right\rangle^{2\alpha}\left\langle -2s + \frac{\xi_1+\xi_1'}{2}\right\rangle^{2\alpha}}$$

(106)

$$I_2 = \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\left\langle \xi_1 - \xi_1'\right\rangle^{2\beta}}{|4s|\left\langle 4s^{\parallel} + \xi_1 - \xi_1'\right\rangle^{2\beta-1+\delta}\left\langle -4s^{\perp} + \xi_1 - \xi_1'\right\rangle^{2\beta}}$$

(107)

$$I_3 = \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\left\langle \xi_1 - \xi_1'\right\rangle^{2\beta}}{|4s|\left\langle 4s^{\parallel} + \xi_1 - \xi_1'\right\rangle^{2\beta}\left\langle -4s^{\perp} + \xi_1 - \xi_1'\right\rangle^{2\beta-1+\delta}}.$$

(108)

Let us first consider the integral I_2; in what follows we will assume that $\beta > \frac{d}{2}$. Clearly, I_2 is equivalent to the following quantity:

$$I_2 \lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\langle \xi_1 - \xi_1' \rangle^{2\beta}}{|s| \langle s^{\parallel} + \xi_1 - \xi_1' \rangle^{2\beta-1+\delta} \langle s^{\perp} + \xi_1 - \xi_1' \rangle^{2\beta}}. \tag{109}$$

Setting $W = \xi_1 - \xi_1'$, this gives:

$$I_2 \lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\langle W \rangle^{2\beta}}{|s| \langle s^{\parallel} + W \rangle^{2\beta-1+\delta} \langle s^{\perp} + W \rangle^{2\beta}}. \tag{110}$$

Moreover, since the integral for $|s| \leq 1$ is obviously uniformly bounded in W, we may instead bound the following integral:

$$I_2' \lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\langle W \rangle^{2\beta}}{\langle s \rangle \langle s^{\parallel} + W \rangle^{2\beta-1+\delta} \langle s^{\perp} + W \rangle^{2\beta}}. \tag{111}$$

Since $|s^{\parallel}| \leq |s|$ we have:

$$I_2' \lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\langle W \rangle^{2\beta}}{\langle s^{\parallel} \rangle \langle s^{\parallel} + W \rangle^{2\beta-1+\delta} \langle s^{\perp} + W \rangle^{2\beta}}. \tag{112}$$

Therefore, for all large enough $|W|$,

$$\begin{aligned} I_2' &\lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \frac{\langle W \rangle^{2\beta}}{\langle s^{\parallel} \rangle \langle s^{\parallel} + W \rangle^{2\beta-1+\delta} \langle s^{\perp} + W \rangle^{2\beta}} \\ &= \int_{\mathbb{S}^{d-1}} d\omega \langle W \rangle^{2\beta} \left(\int \frac{ds^{\parallel}}{\langle s^{\parallel} \rangle \langle s^{\parallel} + W \rangle^{2\beta-1+\delta}} \right) \left(\int \frac{ds^{\perp}}{\langle s^{\perp} + W \rangle^{2\beta}} \right) \\ &\lesssim \int_{\mathbb{S}^{d-1}} d\omega \langle W \rangle^{2\beta} \left(\langle W \rangle^{-1} \langle W^{\perp} \rangle^{2-2\beta-\delta} \log \langle W \rangle \right) \left(\langle W^{\parallel} \rangle^{d-1-2\beta} \right). \end{aligned} \tag{113}$$

The integral over $s^{\perp}$ is estimated by a trivial computation, whereas the integral over $s^{\parallel}$ may be estimated by considering separately the regions $|s^{\parallel}| < \frac{1}{2}|W|$, $|s^{\parallel}| > 2|W|$, and $\frac{1}{2}|W| \leq |s^{\parallel}| \leq 2|W|$.

We find that I_2' obeys the following estimate:

$$I_2' \lesssim \int_{\mathbb{S}^{d-1}} d\omega \langle W \rangle^{2\beta-1+\frac{1}{2}\delta} \langle W^{\perp} \rangle^{2-2\beta-\delta} \langle W^{\parallel} \rangle^{d-1-2\beta}. \tag{114}$$

Then we have

$$\langle W \rangle^{2\beta-1+\frac{1}{2}\delta} \lesssim \langle W^{\parallel} \rangle^{2\beta-1+\frac{1}{2}\delta} + \langle W^{\perp} \rangle^{2\beta-1+\frac{1}{2}\delta}. \tag{115}$$

Hence, $I_2' \lesssim I_2'' + I_2'''$ where

$$I_2'' = \int_{\mathbb{S}^{d-1}} d\omega \langle W^{\perp} \rangle^{2-2\beta-\delta} \langle W^{\parallel} \rangle^{d-2+\frac{1}{2}\delta}. \tag{116}$$

$$I_2''' = \int_{\mathbb{S}^{d-1}} d\omega \left\langle W^\perp \right\rangle^{1-\frac{1}{2}\delta} \left\langle W^\parallel \right\rangle^{d-1-2\beta}. \tag{117}$$

Then for any δ sufficiently small and β sufficiently large ($\beta \geq d$ is easily sufficient for small δ), both I_2'' and I_2''' may be bounded using dyadic decompositions in the angular parameter ω, as follows: neglecting additive constants,

$$I_2'' \lesssim \sum_{k=1}^{\infty} \int_{\omega:2^{-k-1}|W^\parallel| \leq |W^\perp| < 2^{-k}|W^\parallel|} d\omega \left\langle W^\perp \right\rangle^{2-2\beta-\delta} \left\langle W^\parallel \right\rangle^{d-2+\frac{1}{2}\delta}$$
$$\lesssim \sum_{k=1}^{\infty} 2^{-k-1} \times (2^{-k})^{d-2} \times (2^{k+1})^{d-2+\frac{1}{2}\delta} < \infty \tag{118}$$

$$I_2''' \lesssim \sum_{k=1}^{\infty} \int_{\omega:2^{-k-1}|W^\perp| \leq |W^\parallel| < 2^{-k}|W^\perp|} d\omega \left\langle W^\perp \right\rangle^{1-\frac{1}{2}\delta} \left\langle W^\parallel \right\rangle^{d-1-2\beta}$$
$$\lesssim \sum_{k=1}^{\infty} 2^{-k-1} \times (2^{k+1})^{1-\frac{1}{2}\delta} < \infty. \tag{119}$$

The factor of $(2^{-k})^{d-2}$ in I_2'' comes from the Jacobian for spherical coordinates in $\mathbb{R}^d$.

Let us now consider the integral I_3, and assume $\beta > \frac{d}{2}$. Clearly, I_3 is equivalent to the following quantity:

$$I_3 \lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \, \frac{\left\langle \xi_1 - \xi_1' \right\rangle^{2\beta}}{|s| \left\langle s^\parallel + \xi_1 - \xi_1' \right\rangle^{2\beta} \left\langle s^\perp + \xi_1 - \xi_1' \right\rangle^{2\beta-1+\delta}}. \tag{120}$$

Setting $W = \xi_1 - \xi_1'$, this gives:

$$I_3 \lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \, \frac{\left\langle W \right\rangle^{2\beta}}{|s| \left\langle s^\parallel + W \right\rangle^{2\beta} \left\langle s^\perp + W \right\rangle^{2\beta-1+\delta}}. \tag{121}$$

Moreover, since the integral for $|s| \leq 1$ is obviously uniformly bounded in W, we may instead bound the following integral:

$$I_3' \lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \, \frac{\left\langle W \right\rangle^{2\beta}}{\left\langle s \right\rangle \left\langle s^\parallel + W \right\rangle^{2\beta} \left\langle s^\perp + W \right\rangle^{2\beta-1+\delta}}. \tag{122}$$

Since $|s^\parallel| \leq |s|$ we have:

$$I_3' \lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \, \frac{\left\langle W \right\rangle^{2\beta}}{\left\langle s^\parallel \right\rangle \left\langle s^\parallel + W \right\rangle^{2\beta} \left\langle s^\perp + W \right\rangle^{2\beta-1+\delta}}. \tag{123}$$

Therefore, for all large enough $|W|$,

$$I_3' \lesssim \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} ds \, \frac{\left\langle W \right\rangle^{2\beta}}{\left\langle s^\parallel \right\rangle \left\langle s^\parallel + W \right\rangle^{2\beta} \left\langle s^\perp + W \right\rangle^{2\beta-1+\delta}}$$
$$= \int_{\mathbb{S}^{d-1}} d\omega \left\langle W \right\rangle^{2\beta} \left(\int \frac{ds^\parallel}{\left\langle s^\parallel \right\rangle \left\langle s^\parallel + W \right\rangle^{2\beta}} \right) \left(\int \frac{ds^\perp}{\left\langle s^\perp + W \right\rangle^{2\beta-1+\delta}} \right) \tag{124}$$
$$\lesssim \int_{\mathbb{S}^{d-1}} d\omega \left\langle W \right\rangle^{2\beta} \left(\left\langle W \right\rangle^{-1} \left\langle W^\perp \right\rangle^{1-2\beta} \log \left\langle W \right\rangle \right) \left(\left\langle W^\parallel \right\rangle^{d-2\beta-\delta} \right).$$

As before, the integral over $s^\perp$ is estimated by a trivial computation, whereas the integral over $s^\parallel$ may be estimated by considering separately the regions $|s^\parallel| < \frac{1}{2}|W|$, $|s^\parallel| > 2|W|$, and $\frac{1}{2}|W| \le |s^\parallel| \le 2|W|$.

We find that I_3' obeys the following estimate:

$$I_3' \lesssim \int_{\mathbb{S}^{d-1}} d\omega \, \langle W \rangle^{2\beta - 1 + \frac{1}{2}\delta} \left\langle W^\perp \right\rangle^{1-2\beta} \left\langle W^\parallel \right\rangle^{d-2\beta-\delta}. \tag{125}$$

Then we have

$$\langle W \rangle^{2\beta - 1 + \frac{1}{2}\delta} \lesssim \left\langle W^\parallel \right\rangle^{2\beta - 1 + \frac{1}{2}\delta} + \left\langle W^\perp \right\rangle^{2\beta - 1 + \frac{1}{2}\delta}. \tag{126}$$

Hence $I_3' \lesssim I_3'' + I_3'''$ where

$$I_3'' = \int_{\mathbb{S}^{d-1}} d\omega \left\langle W^\perp \right\rangle^{1-2\beta} \left\langle W^\parallel \right\rangle^{d-1-\frac{1}{2}\delta} \tag{127}$$

$$I_3''' = \int_{\mathbb{S}^{d-1}} d\omega \left\langle W^\perp \right\rangle^{\frac{1}{2}\delta} \left\langle W^\parallel \right\rangle^{d-2\beta-\delta}. \tag{128}$$

Then for any sufficiently small δ and $\beta > d$, both I_3'' and I_3''' may be bounded using dyadic decompositions in the angular parameter ω, as follows: neglecting additive constants,

$$I_3'' \lesssim \sum_{k=1}^{\infty} \int_{\omega:2^{-k-1}|W^\parallel| \le |W^\perp| < 2^{-k}|W^\parallel|} d\omega \left\langle W^\perp \right\rangle^{1-2\beta} \left\langle W^\parallel \right\rangle^{d-1-\frac{1}{2}\delta}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-k-1} \times (2^{-k})^{d-2} \times (2^{k+1})^{d-1-\frac{1}{2}\delta} < \infty \tag{129}$$

$$I_3''' \lesssim \sum_{k=1}^{\infty} \int_{\omega:2^{-k-1}|W^\perp| \le |W^\parallel| < 2^{-k}|W^\perp|} d\omega \left\langle W^\perp \right\rangle^{\frac{1}{2}\delta} \left\langle W^\parallel \right\rangle^{d-2\beta-\delta}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-k-1} \times (2^{k+1})^{\frac{1}{2}\delta} < \infty. \tag{130}$$

The factor of $(2^{-k})^{d-2}$ in I_3'' comes from the Jacobian for spherical coordinates in $\mathbb{R}^d$.

We finally turn to I_1, which is clearly bounded by the following quantity:

$$I_1 \lesssim \sup_{W \in \mathbb{R}^d} \sup_{P \subset \mathbb{R}^d : \dim P = d-1} \int_P dS(s) \frac{\langle W \rangle^{2\alpha}}{\langle s \rangle^{2\alpha} \langle s + W \rangle^{2\alpha}}. \tag{131}$$

The integrals over $P \cap \left\{ |s| < \frac{1}{2}|W| \right\}$, $P \cap \{|s| > 2|W|\}$, and $P \cap \left\{ \frac{1}{2}|W| \le |s| \le 2|W| \right\}$ are each easily bounded uniformly in W as long as $\alpha > \frac{d-1}{2}$.

To summarize, as long as $\alpha > \frac{d-1}{2}$, $\beta > d$, and $\max(0, 2A - 1) < \frac{1}{\sigma} \le 2$, then for $r \in [0, 1)$ such that $\frac{r}{\sigma} \ge \max(0, 2A - 1 + \delta)$ for a small $\delta > 0$ we have for any $\kappa_0 > \kappa > 0$ the following estimate:

$$\left\| B_{i,k+1}^+ \left[e^{\frac{1}{2}it\left(\Delta_{X_{k+1}} - \Delta_{X_{k+1}'} \right)} \gamma_0^{(k+1)} \right] \right\|_{L_t^2 H_k^{\alpha,\beta,\sigma,\kappa}}$$

$$\le C(\alpha, \beta, \sigma, r) \|\mathbf{b}\|_{L_A^\infty} (\kappa_0 - \kappa)^{-\frac{1}{2}r} \left\| \gamma_0^{(k+1)} \right\|_{H_{k+1}^{\alpha,\beta,\sigma,\kappa_0}}. \tag{132}$$

4. Proof of Theorem 2.2

Formally speaking, solutions of Boltzmann's equation are factorized solutions of the Boltzmann hierarchy, i.e. $\gamma^{(k)} = \gamma^{\otimes k}$. We use the notation

$$\Delta_{\pm}^{(k)} = \Delta_{X_k} - \Delta_{X_k'} \tag{133}$$

$$\Delta_{\pm} = \Delta_{\pm}^{(1)}. \tag{134}$$

Then if $B_{k+1} = \sum_{i=1}^{k} \left(B_{i,k+1}^{+} - B_{i,k+1}^{-} \right)$, the Boltzmann hierarchy in integral form reads as follows:

$$\gamma^{(k)}(t) = e^{\frac{1}{2}it\Delta_{\pm}^{(k)}} \gamma^{(k)}(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}^{(k)}} B_{k+1}\gamma^{(k+1)}(t_1)dt_1. \tag{135}$$

Let us assume $\gamma^{(k)} = \gamma^{\otimes k}$ for all $k \in \mathbb{N}$ and consider the Boltzmann hierarchy for $k = 1, 2$:

$$\gamma(t) = e^{\frac{1}{2}it\Delta_{\pm}} \gamma(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} B_2 \left(\gamma^{\otimes 2} \right)(t_1)dt_1 \tag{136}$$

$$(\gamma \otimes \gamma)(t) = e^{\frac{1}{2}it\Delta_{\pm}^{(2)}} (\gamma \otimes \gamma)(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}^{(2)}} B_3 \left(\gamma^{\otimes 3} \right)(t_1)dt_1. \tag{137}$$

Now we apply the operator B_2 to the second equation, thereby obtaining the following system:

$$\gamma(t) = e^{\frac{1}{2}it\Delta_{\pm}} \gamma(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} B_2 \left(\gamma^{\otimes 2} \right)(t_1)dt_1 \tag{138}$$

$$B_2 \left(\gamma^{\otimes 2} \right)(t)$$
$$= B_2 \left(e^{\frac{1}{2}it\Delta_{\pm}^{(2)}} \left(\gamma^{\otimes 2} \right)(0) \right) - i \int_0^t B_2 \left[e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}^{(2)}} B_3 \left(\gamma^{\otimes 3} \right)(t_1) \right] dt_1. \tag{139}$$

Let us observe that $B(\gamma_1, \gamma_2) = B_2(\gamma_1 \otimes \gamma_2)$. Therefore if we define $\zeta(t) = B(\gamma(t), \gamma(t))$ then we obtain the following system of equations for the pair (γ, ζ):

$$\gamma(t) = e^{\frac{1}{2}it\Delta_{\pm}} \gamma(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta(t_1)dt_1 \tag{140}$$

$$\zeta(t) = B \left(e^{\frac{1}{2}it\Delta_{\pm}} \gamma(0), e^{\frac{1}{2}it\Delta_{\pm}} \gamma(0) \right)$$
$$+ (-i) \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta(t_1) \right) dt_1$$
$$+ (-i) \int_0^t B \left(e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \zeta(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_{\pm}} \gamma(t_1) \right) dt_1. \tag{141}$$

We will solve this simultaneous system of equations for $(\gamma(t), \zeta(t))$ on a small time interval $[0, T]$ by Picard iteration, using the following norm:

$$\|(\gamma, \zeta)\| = T^{\frac{1}{2}(1-r)} \left\| \|\gamma(t)\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t}} \right\|_{L_T^{\infty}} + \left\| \|\zeta(t)\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t}} \right\|_{L_T^1}. \tag{142}$$

Here we have fixed some $r \in [0, 1)$ as in the statement of Proposition 3.1. The key result we will use is that Proposition 3.1 implies the following bilinear estimates:

$$
\left\| B \left(e^{\frac{1}{2} it \Delta_\pm} \gamma_{0,1}, e^{\frac{1}{2} it \Delta_\pm} \gamma_{0,2} \right) \right\|_{L_t^2 H^{\alpha,\beta,\sigma,\kappa_1}}
$$
$$
\leq C \left(1 + (\kappa_0 - \kappa_1)^{-\frac{1}{2}r} \right) \| \gamma_{0,1} \|_{H^{\alpha,\beta,\sigma,\kappa_0}} \| \gamma_{0,2} \|_{H^{\alpha,\beta,\sigma,\kappa_0}} . \tag{143}
$$

To set up the fixed point iteration, we fix the initial data $\gamma_0 \in H^{\alpha,\beta,\sigma,\kappa}$ and define the map $\Phi = (\Phi_1, \Phi_2)(\gamma, \zeta)$ as follows:

$$
[\Phi_1(\gamma, \zeta)](t) = e^{\frac{1}{2} it \Delta_\pm} \gamma_0 - i \int_0^t e^{\frac{1}{2} i (t - t_1) \Delta_\pm} \zeta(t_1) dt_1 \tag{144}
$$

$$
[\Phi_2(\gamma, \zeta)](t) = B \left(e^{\frac{1}{2} it \Delta_\pm} \gamma_0, e^{\frac{1}{2} it \Delta_\pm} \gamma_0 \right)
$$
$$
+ (-i) \int_0^t B \left(e^{\frac{1}{2} i (t - t_1) \Delta_\pm} \gamma(t_1), e^{\frac{1}{2} i (t - t_1) \Delta_\pm} \zeta(t_1) \right) dt_1
$$
$$
+ (-i) \int_0^t B \left(e^{\frac{1}{2} i (t - t_1) \Delta_\pm} \zeta(t_1), e^{\frac{1}{2} i (t - t_1) \Delta_\pm} \gamma(t_1) \right) dt_1. \tag{145}
$$

We wish to solve the equation $(\gamma, \zeta) = \Phi(\gamma, \zeta)$.

First, using (144) and the fact that the propagator $e^{\frac{1}{2} it \Delta_\pm}$ preserves the space $H^{\alpha,\beta,\sigma,\kappa}$, along with the embedding $H^{\alpha,\beta,\sigma,\kappa_0} \subset H^{\alpha,\beta,\sigma,\kappa_1}$ for $\kappa_0 > \kappa_1 > 0$, we easily obtain:

$$
\left\| \left\| [\Phi_1(\gamma, \zeta)](t) \right\|_{H^{\alpha,\beta,\sigma,\kappa - \lambda t}} \right\|_{L_T^\infty} \leq \| \gamma_0 \|_{H^{\alpha,\beta,\sigma,\kappa}} + \left\| \| \zeta(t) \|_{H^{\alpha,\beta,\sigma,\kappa - \lambda t}} \right\|_{L_T^1} . \tag{146}
$$

We now turn to Φ_2. We begin by estimating the first term on the right hand side of (145). We will use a dyadic decomposition in time:

$$
\left\| \left\| B \left(e^{\frac{1}{2} it \Delta_\pm} \gamma_0, e^{\frac{1}{2} it \Delta_\pm} \gamma_0 \right) \right\|_{H^{\alpha,\beta,\sigma,\kappa - \lambda t}} \right\|_{L_T^1}
$$
$$
= \sum_{m=0}^\infty \int_{2^{-m-1}T < t \leq 2^{-m}T} \left\| B \left(e^{\frac{1}{2} it \Delta_\pm} \gamma_0, e^{\frac{1}{2} it \Delta_\pm} \gamma_0 \right) \right\|_{H^{\alpha,\beta,\sigma,\kappa - \lambda t}} dt
$$
$$
\leq \sum_{m=0}^\infty \int_{2^{-m-1}T < t \leq 2^{-m}T} \left\| B \left(e^{\frac{1}{2} it \Delta_\pm} \gamma_0, e^{\frac{1}{2} it \Delta_\pm} \gamma_0 \right) \right\|_{H^{\alpha,\beta,\sigma,\kappa - \lambda 2^{-m-1}T}} dt.
$$

Now apply the Cauchy–Schwarz inequality, followed by (143). We implicitly assume $\lambda T < 1$, which is acceptable because we only want to address small times T in any case.

$$
\left\| \left\| B \left(e^{\frac{1}{2} it \Delta_\pm} \gamma_0, e^{\frac{1}{2} it \Delta_\pm} \gamma_0 \right) \right\|_{H^{\alpha,\beta,\sigma,\kappa - \lambda t}} \right\|_{L_T^1}
$$
$$
\leq \sum_{m=0}^\infty \left(2^{-m-1}T \right)^{\frac{1}{2}} \left\| B \left(e^{\frac{1}{2} it \Delta_\pm} \gamma_0, e^{\frac{1}{2} it \Delta_\pm} \gamma_0 \right) \right\|_{L_t^2 H^{\alpha,\beta,\sigma,\kappa - \lambda 2^{-m-1}T}}
$$
$$
\leq \sum_{m=0}^\infty \left(2^{-m-1}T \right)^{\frac{1}{2}} \frac{C}{\left(\lambda 2^{-m-1}T \right)^{\frac{1}{2}r}} \| \gamma_0 \|_{H^{\alpha,\beta,\sigma,\kappa}}^2
$$
$$
\leq C \lambda^{-\frac{1}{2}r} T^{\frac{1}{2}(1-r)} \left(\sum_{m=0}^\infty 2^{-\frac{1}{2}m(1-r)} \right) \| \gamma_0 \|_{H^{\alpha,\beta,\sigma,\kappa}}^2 .
$$

We now estimate the second term on the right hand side of (145); the third term is handled similarly. We will employ a dyadic decomposition in $t - t_1$ and apply Cauchy–Schwarz and (143) as before.

$$\left\| \left\| \int_0^t B\left(e^{\frac{1}{2}i(t-t_1)\Delta_\pm}\gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_\pm}\zeta(t_1)\right) dt_1 \right\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t}} \right\|_{L_T^1}$$

$$\leq \sum_{m=0}^\infty \int_0^T dt_1 \int_{t_1+2^{-m-1}T < t \leq t_1+2^{-m}T} dt$$
$$\times \left\| B\left(e^{\frac{1}{2}i(t-t_1)\Delta_\pm}\gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_\pm}\zeta(t_1)\right) \right\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t}}$$

$$\leq \sum_{m=0}^\infty \int_0^T dt_1 \int_{t_1+2^{-m-1}T < t \leq t_1+2^{-m}T} dt$$
$$\times \left\| B\left(e^{\frac{1}{2}i(t-t_1)\Delta_\pm}\gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_\pm}\zeta(t_1)\right) \right\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda(t_1+2^{-m-1}T)}}$$

$$\leq \sum_{m=0}^\infty \left(2^{-m-1}T\right)^{\frac{1}{2}} \int_0^T dt_1$$
$$\times \left\| B\left(e^{\frac{1}{2}i(t-t_1)\Delta_\pm}\gamma(t_1), e^{\frac{1}{2}i(t-t_1)\Delta_\pm}\zeta(t_1)\right) \right\|_{L_t^2 H^{\alpha,\beta,\sigma,\kappa-\lambda(t_1+2^{-m-1}T)}}$$

$$\leq \sum_{m=0}^\infty \left(2^{-m-1}T\right)^{\frac{1}{2}} \times C\left(\lambda 2^{-m-1}T\right)^{-\frac{1}{2}r}$$
$$\times \int_0^T dt_1 \|\gamma(t_1)\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t_1}} \|\zeta(t_1)\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t_1}}$$

$$\leq C\lambda^{-\frac{1}{2}r}T^{\frac{1}{2}(1-r)}\left(\sum_{m=0}^\infty 2^{-\frac{1}{2}m(1-r)}\right)$$
$$\times \left\| \|\gamma(t)\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t}} \right\|_{L_T^\infty} \left\| \|\zeta(t)\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t}} \right\|_{L_T^1}.$$

We can finally conclude the following estimate for Φ_2:

$$\left\| \|[\Phi_2(\gamma,\zeta)](t)\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t}} \right\|_{L_T^1} \leq C\lambda^{-\frac{1}{2}r}T^{\frac{1}{2}(1-r)}\left(\sum_{m=0}^\infty 2^{-\frac{1}{2}m(1-r)}\right) \tag{147}$$
$$\times \left(\|\gamma_0\|_{H^{\alpha,\beta,\sigma,\kappa}}^2 + \left\| \|\gamma(t)\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t}} \right\|_{L_T^\infty} \left\| \|\zeta(t)\|_{H^{\alpha,\beta,\sigma,\kappa-\lambda t}} \right\|_{L_T^1}\right)$$

Combining (146) and (147), and defining $C_{\lambda,r} = C\lambda^{-\frac{1}{2}r}\sum_{m=0}^\infty 2^{-\frac{1}{2}m(1-r)}$, we obtain:

$$\|\Phi(\gamma,\zeta)\| \leq T^{\frac{1}{2}(1-r)}\|\gamma_0\|_{H^{\alpha,\beta,\sigma,\kappa}} + C_{\lambda,r}T^{\frac{1}{2}(1-r)}\|\gamma_0\|_{H^{\alpha,\beta,\sigma,\kappa}}^2 \tag{148}$$
$$+ T^{\frac{1}{2}(1-r)}\|(\gamma,\zeta)\| + C_{\lambda,r}\|(\gamma,\zeta)\|^2.$$

By a completely analogous argument, we obtain the following continuity bound:

$$\|\Phi(\gamma_1,\zeta_1) - \Phi(\gamma_2,\zeta_2)\|$$
$$\leq \left(T^{\frac{1}{2}(1-r)} + 4C_{\lambda,r}(\|(\gamma_1,\zeta_1)\| + \|(\gamma_2,\zeta_2)\|)\right)\|(\gamma_1,\zeta_1) - (\gamma_2,\zeta_2)\|. \tag{149}$$

Combining (148) and (149), and applying the Banach fixed point theorem, we conclude the existence and uniqueness of a solution to the fixed point equation $(\gamma, \zeta) = \Phi(\gamma, \zeta)$ once T is chosen sufficiently small depending only on $\|\gamma_0\|_{H^{\alpha,\beta,\sigma,\kappa}}$. This gives us uniqueness under the assumption that $\|(\gamma, \zeta)\|$ is small, but in fact for any solution we can apply (148) and a standard continuity argument to conclude that $\|(\gamma, \zeta)\|$ is necessarily small if T is small, as long as it is finite for some positive T. The estimate (33) follows directly from (148).

Finally, we remark that if $A \in \left[0, \frac{1}{2}\right)$ then we may take $r = 0$, so that $C_{\lambda,r}$ loses its dependence on λ; hence, we are allowed to take $\lambda = 0$ and we can permit any $\kappa \in [0, \infty)$. The rest of the proof proceeds in exactly the same manner.

5. Proof of Theorem 2.3

Theorem 2.3 follows from Proposition 3.1 combined with the arguments of Chen and Pavlović [6], which in turn rely on the combinatorial arguments of Erdös and Schlein-Yau [9–11], in the boardgame representation given by Klainerman and Machedon [18]. We outline the proof here for the reader's convenience.

To begin, we point out that the Boltzmann hierarchy may be written in vector integral form as follows:

$$\Gamma(t) = e^{\frac{1}{2}it\hat{\Delta}_\pm}\Gamma(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\hat{\Delta}_\pm} B\Gamma(t_1)dt_1 \tag{150}$$

where $\hat{\Delta}_\pm\Gamma = \left\{\left(\Delta_{X_k} - \Delta_{X_k'}\right)\gamma^{(k)}\right\}_{k\in\mathbb{N}}$ and $B\Gamma = \left\{B_{k+1}\gamma^{(k+1)}\right\}_{k\in\mathbb{N}}$. We can apply B to both sides to yield a closed equation for $B\Gamma$:

$$B\Gamma(t) = B\left[e^{\frac{1}{2}it\hat{\Delta}_\pm}\Gamma(0)\right] - i \int_0^t B\left[e^{\frac{1}{2}i(t-t_1)\hat{\Delta}_\pm} B\Gamma(t_1)\right]dt_1. \tag{151}$$

Letting $\Xi = B\Gamma$, we conclude that the pair (Γ, Ξ) satisfies the following system of equations:

$$\Gamma(t) = e^{\frac{1}{2}it\hat{\Delta}_\pm}\Gamma(0) - i \int_0^t e^{\frac{1}{2}i(t-t_1)\hat{\Delta}_\pm} \Xi(t_1)dt_1 \tag{152}$$

$$\Xi(t) = B\left[e^{\frac{1}{2}it\hat{\Delta}_\pm}\Gamma(0)\right] - i \int_0^t B\left[e^{\frac{1}{2}i(t-t_1)\hat{\Delta}_\pm} \Xi(t_1)\right]dt_1 \tag{153}$$

and this system is equivalent to the original Boltzmann hierarchy.

Since (153) is a closed equation for Ξ, we proceed in two steps. First we solve (153) on a small time interval $[0, T]$ by Picard iteration; then, we establish that the right-hand side of (152) is indeed well-defined in the correct functional space. The proof proceeds by iterating the Duhamel formula (153), k times for the kth component, and applying the combinatorial methods of Erdös, Schlein, and Yau [9–11], expressed in boardgame form by Klainerman and Machedon [18]. Then we conclude by applying Proposition 3.1 inductively to bound all the terms (which are now $\mathcal{O}(C^k)$ in number instead of (even more than) $\mathcal{O}(k!)$ due to the combinatorial methods of [9–11]). The precise details are written out in [6] for the interested reader.

Remark. Note that if $A \geq \frac{1}{2}$, then the combinatorial methods of [9–11], and the reformulation in terms of a boardgame argument [18], all still apply at the formal level. However, the termwise estimates of [6] no longer apply uniformly across general re-ordering of collision times. This is simply not an issue when $A < \frac{1}{2}$ since no time-dependent loss of weight is required in that case. Most likely, if LWP holds at all for the hard sphere Boltzmann hierarchy for the spaces we consider, then completely new estimates (different from Proposition 3.1) will be required.

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Appendix A. Inverse Wigner Transform of the Boltzmann Equation and Boltzmann Hierarchy

We begin with the Boltzmann hierarchy.

Proposition A.1. *Let $f^{(k)} \in L^1\left([0, T], L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})\right)$ and let $\gamma^{(k)}$ denote the inverse Wigner transform of $f^{(k)}$. Then if*

$$\left(\partial_t + V_k \cdot \nabla_{X_k}\right) f^{(k)} = g^{(k)} \tag{154}$$

holds in the sense of distributions, then we have

$$\left(i\partial_t + \frac{1}{2}\left(\Delta_{X_k} - \Delta_{X'_k}\right)\right) \gamma^{(k)}(t, X_k, X'_k)$$
$$= i \int_{\mathbb{R}^{dk}} g^{(k)}\left(t, \frac{X_k + X'_k}{2}, V_k\right) e^{iV_k \cdot (X_k - X'_k)} dV_k \tag{155}$$

in the sense of distributions.

Proof. We have

$$i\partial_t \gamma^{(k)}(t, X_k, X'_k) = i \int_{\mathbb{R}^{dk}} \left(\partial_t f^{(k)}\right)\left(t, \frac{X_k + X'_k}{2}, V_k\right) e^{iV_k \cdot (X_k - X'_k)} dV_k$$
$$= i \int_{\mathbb{R}^{dk}} \left(-V_k \cdot \nabla_{X_k} f^{(k)} + g^{(k)}\right)\left(t, \frac{X_k + X'_k}{2}, V_k\right) e^{iV_k \cdot (X_k - X'_k)} dV_k.$$

Consider the transport term alone.

$$i \int \left(-V_k \cdot \nabla_{X_k} f^{(k)}\right)\left(t, \frac{X_k + X'_k}{2}, V_k\right) e^{iV_k \cdot (X_k - X'_k)} dV_k$$
$$= -i \int V_k \cdot (\nabla_{X_k} + \nabla_{X'_k}) \left[f^{(k)}\left(t, \frac{X_k + X'_k}{2}, V_k\right)\right] e^{iV_k \cdot (X_k - X'_k)} dV_k$$
$$= -\int (\nabla_{X_k} + \nabla_{X'_k}) \left[f^{(k)}\left(t, \frac{X_k + X'_k}{2}, v\right)\right] \cdot i V_k e^{iV_k \cdot (X_k - X'_k)} dV_k$$

$$= -(\nabla_{X_k} + \nabla_{X_k'}) \cdot \int f^{(k)}\left(t, \frac{X_k + X_k'}{2}, V_k\right) i V_k e^{i V_k \cdot (X_k - X_k')} dV_k$$

$$= -(\nabla_{X_k} + \nabla_{X_k'}) \cdot \int f^{(k)}\left(t, \frac{X_k + X_k'}{2}, V_k\right) \frac{\nabla_{X_k} - \nabla_{X_k'}}{2} e^{i V_k \cdot (X_k - X_k')} dV_k$$

$$= -(\nabla_{X_k} + \nabla_{X_k'}) \cdot \frac{\nabla_{X_k} - \nabla_{X_k'}}{2} \int f^{(k)}\left(t, \frac{X_k + X_k'}{2}, V_k\right) e^{i V_k \cdot (X_k - X_k')} dV_k$$

$$= -\frac{1}{2}\left(\Delta_{X_k} - \Delta_{X_k'}\right) \gamma^{(k)}(t, X_k, X_k').$$

Therefore,

$$\left(i \partial_t + \frac{1}{2}\left(\Delta_{X_k} - \Delta_{X_k'}\right)\right) \gamma^{(k)}(t, X_k, X_k') =$$
$$= i \int_{\mathbb{R}^{dk}} g^{(k)}\left(t, \frac{X_k + X_k'}{2}, V_k\right) e^{i V_k \cdot (X_k - X_k')} dV_k. \tag{156}$$

$\square$

Proposition A.2. *Let* $f^{(k+1)}(X_{k+1}, V_{k+1})$ *be a Schwartz function, and let* $\gamma^{(k+1)}$ *denote its inverse Wigner transform. Then*

$$i \int C_{1,k+1}^- f^{(k+1)}\left(\frac{X_k + X_k'}{2}, V_k\right) e^{i V_k \cdot (X_k - X_k')} dV_k$$
$$= \frac{i}{2^{2d} \pi^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} dz \hat{\mathbf{b}}^\omega\left(\frac{z}{2}\right)$$
$$\times \gamma^{(k+1)}\left(x_1 - \frac{z}{4}, X_{2:k}, \frac{x_1 + x_1'}{2} + \frac{z}{4}, x_1' + \frac{z}{4}, X_{2:k}', \frac{x_1 + x_1'}{2} - \frac{z}{4}\right) \tag{157}$$

and

$$i \int C_{1,k+1}^+ f^{(k+1)}\left(\frac{X_k + X_k'}{2}, V_k\right) e^{i V_k \cdot (X_k - X_k')} dV_k$$
$$= \frac{i}{2^{2d} \pi^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} dz \hat{\mathbf{b}}^\omega\left(\frac{z}{2}\right)$$
$$\times \gamma^{(k+1)}\Bigg(x_1 - \frac{1}{2} P_\omega(x_1 - x_1') - \frac{R_\omega(z)}{4}, X_{2:k},$$
$$\frac{x_1 + x_1'}{2} + \frac{1}{2} P_\omega(x_1 - x_1') + \frac{R_\omega(z)}{4},$$
$$x_1' + \frac{1}{2} P_\omega(x_1 - x_1') + \frac{R_\omega(z)}{4}, X_{2:k}',$$
$$\frac{x_1 + x_1'}{2} - \frac{1}{2} P_\omega(x_1 - x_1') - \frac{R_\omega(z)}{4}\Bigg). \tag{158}$$

Proof. Consider the loss term.

$$i \int C^-_{1,k+1} f^{(k+1)} \left(\frac{X_k + X'_k}{2}, V_k \right) e^{i V_k \cdot (X_k - X'_k)} dV_k$$

$$= i \int dV_{k+1} d\omega \mathbf{b} \left(|v_{k+1} - v_1|, \omega \cdot \frac{v_{k+1} - v_1}{|v_{k+1} - v_1|} \right)$$

$$\times e^{i V_k \cdot (X_k - X'_k)} f^{(k+1)} \left(\frac{X_k + X'_k}{2}, \frac{x_1 + x'_1}{2}, V_k, v_{k+1} \right)$$

$$= i \frac{1}{(2\pi)^{d(k+1)}} \int dV_{k+1} dY_{k+1} d\omega \mathbf{b} \left(|v_{k+1} - v_1|, \omega \cdot \frac{v_{k+1} - v_1}{|v_{k+1} - v_1|} \right)$$

$$\times e^{i V_k \cdot (X_k - X'_k)} e^{-i V_{k+1} \cdot Y_{k+1}}$$

$$\times \gamma^{(k+1)} \left(\frac{X_k + X'_k + Y_k}{2}, \frac{x_1 + x'_1 + y_{k+1}}{2}, \frac{X_k + X'_k - Y_k}{2}, \frac{x_1 + x'_1 - y_{k+1}}{2} \right)$$

$$= i \frac{1}{(2\pi)^{2d}} \int dv_1 dv_{k+1} dy_1 dy_{k+1} d\omega \mathbf{b} \left(|v_{k+1} - v_1|, \omega \cdot \frac{v_{k+1} - v_1}{|v_{k+1} - v_1|} \right)$$

$$\times e^{i v_1 \cdot (x_1 - x'_1 - y_1)} e^{-i v_{k+1} \cdot y_{k+1}}$$

$$\times \gamma^{(k+1)} \left(\frac{x_1 + x'_1 + y_1}{2}, X_{2:k}, \frac{x_1 + x'_1 + y_{k+1}}{2}, \right.$$

$$\left. \frac{x_1 + x'_1 - y_1}{2}, X'_{2:k}, \frac{x_1 + x'_1 - y_{k+1}}{2} \right).$$

Use the change of variables $u_1 = \frac{1}{2}(v_{k+1} + v_1)$, $u_2 = \frac{1}{2}(v_{k+1} - v_1)$.

$$i \int C^-_{1,k+1} f^{(k+1)} \left(\frac{X_k + X'_k}{2}, V_k \right) e^{i V_k \cdot (X_k - X'_k)} dV_k$$

$$= i \frac{1}{2^d \pi^{2d}} \int du_1 du_2 dy_1 dy_{k+1} d\omega \mathbf{b} \left(2|u_2|, \omega \cdot \frac{u_2}{|u_2|} \right)$$

$$\times e^{i u_1 \cdot (x_1 - x'_1 - y_1 - y_{k+1})} e^{-i u_2 \cdot (x_1 - x'_1 - y_1 + y_{k+1})}$$

$$\times \gamma^{(k+1)} \left(\frac{x_1 + x'_1 + y_1}{2}, X_{2:k}, \frac{x_1 + x'_1 + y_{k+1}}{2}, \right.$$

$$\left. \frac{x_1 + x'_1 - y_1}{2}, X'_{2:k}, \frac{x_1 + x'_1 - y_{k+1}}{2} \right).$$

Now let $w = x_1 - x'_1 - y_1 - y_{k+1}$, $z = x_1 - x'_1 - y_1 + y_{k+1}$.

$$i \int C^-_{1,k+1} f^{(k+1)} \left(\frac{X_k + X'_k}{2}, V_k \right) e^{i V_k \cdot (X_k - X'_k)} dV_k$$

$$= i \frac{1}{(2\pi)^{2d}} \int du_1 du_2 dw dz d\omega \mathbf{b} \left(2|u_2|, \omega \cdot \frac{u_2}{|u_2|} \right) e^{i u_1 \cdot w} e^{-i u_2 \cdot z}$$

$$\times \gamma^{(k+1)} \left(x_1 - \frac{w + z}{4}, X_{2:k}, \frac{x_1 + x'_1}{2} - \frac{w - z}{4}, \right.$$

$$\left. x'_1 + \frac{w + z}{4}, X'_{2:k}, \frac{x_1 + x'_1}{2} + \frac{w - z}{4} \right).$$

Recall that

$$\hat{\mathbf{b}}^{\omega}(\xi) = \int_{\mathbb{R}^d} \mathbf{b}\left(|u|, \omega \cdot \frac{u}{|u|}\right) e^{-iu\cdot\xi} du. \tag{159}$$

Then we have

$$i \int C_{1,k+1}^{-} f^{(k+1)}\left(\frac{X_k + X_k'}{2}, V_k\right) e^{iV_k\cdot(X_k - X_k')} dV_k$$

$$= \frac{i}{2^{2d}\pi^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} dz \hat{\mathbf{b}}^{\omega}\left(\frac{z}{2}\right)$$

$$\times \gamma^{(k+1)}\left(x_1 - \frac{z}{4}, X_{2:k}, \frac{x_1 + x_1'}{2} + \frac{z}{4}, x_1' + \frac{z}{4}, X_{2:k}', \frac{x_1 + x_1'}{2} - \frac{z}{4}\right).$$

Now the gain term.

$$i \int C_{1,k+1}^{+} f^{(k+1)}\left(\frac{X_k + X_k'}{2}, V_k\right) e^{iV_k\cdot(X_k - X_k')} dV_k$$

$$= i \int dV_{k+1} d\omega \mathbf{b}\left(|v_{k+1} - v_1|, \omega \cdot \frac{v_{k+1} - v_1}{|v_{k+1} - v_1|}\right) e^{iV_k\cdot(X_k - X_k')}$$

$$\times f^{(k+1)}\left(\frac{x_1 + x_1'}{2}, \frac{X_{2:k} + X_{2:k}'}{2}, \frac{x_1 + x_1'}{2}, v_1^*, V_{2:k}, v_{k+1}^*\right)$$

$$= i \frac{1}{(2\pi)^{d(k+1)}} \int dV_{k+1} dY_{k+1} d\omega \mathbf{b}\left(|v_{k+1} - v_1|, \omega \cdot \frac{v_{k+1} - v_1}{|v_{k+1} - v_1|}\right)$$

$$\times e^{iV_k\cdot(X_k - X_k')} e^{-iV_{2:k}\cdot Y_{2:k}} e^{-i(v_1 + P_\omega(v_{k+1} - v_1))\cdot y_1} e^{-i(v_{k+1} - P_\omega(v_{k+1} - v_1))\cdot y_{k+1}}$$

$$\times \gamma^{(k+1)}\left(\frac{x_1 + x_1' + y_1}{2}, \frac{X_{2:k} + X_{2:k}' + Y_{2:k}}{2}, \frac{x_1 + x_1' + y_{k+1}}{2},\right.$$

$$\left.\frac{x_1 + x_1' - y_1}{2}, \frac{X_{2:k} + X_{2:k}' - Y_{2:k}}{2}, \frac{x_1 + x_1' - y_{k+1}}{2}\right)$$

$$= i \frac{1}{(2\pi)^{2d}} \int dv_1 dv_{k+1} dy_1 dy_{k+1} d\omega \mathbf{b}\left(|v_{k+1} - v_1|, \omega \cdot \frac{v_{k+1} - v_1}{|v_{k+1} - v_1|}\right)$$

$$\times e^{iv_1\cdot(x_1 - x_1')} e^{-i(v_1 + P_\omega(v_{k+1} - v_1))\cdot y_1} e^{-i(v_{k+1} - P_\omega(v_{k+1} - v_1))\cdot y_{k+1}}$$

$$\times \gamma^{(k+1)}\left(\frac{x_1 + x_1' + y_1}{2}, X_{2:k}, \frac{x_1 + x_1' + y_{k+1}}{2},\right.$$

$$\left.\frac{x_1 + x_1' - y_1}{2}, X_{2:k}', \frac{x_1 + x_1' - y_{k+1}}{2}\right).$$

Let $u_1 = \frac{1}{2}(v_{k+1} + v_1)$, $u_2 = \frac{1}{2}(v_{k+1} - v_1)$.

$$i \int C_{1,k+1}^{+} f^{(k+1)}\left(\frac{X_k + X_k'}{2}, V_k\right) e^{iV_k\cdot(X_k - X_k')} dV_k$$

$$= i \frac{1}{2^d \pi^{2d}} \int du_1 du_2 dy_1 dy_{k+1} d\omega \mathbf{b}\left(2|u_2|, \omega \cdot \frac{u_2}{|u_2|}\right)$$

$$\times e^{iu_1\cdot(x_1 - x_1' - y_1 - y_{k+1})} e^{-iu_2\cdot(x_1 - x_1' - R_\omega(y_1 - y_{k+1}))}$$

$$\times \gamma^{(k+1)} \left(\frac{x_1 + x_1' + y_1}{2}, X_{2:k}, \frac{x_1 + x_1' + y_{k+1}}{2}, \right.$$
$$\left. \frac{x_1 + x_1' - y_1}{2}, X_{2:k}', \frac{x_1 + x_1' - y_{k+1}}{2} \right).$$

Let $w = x_1 - x_1' - y_1 - y_{k+1}$, $z = x_1 - x_1' - R_\omega(y_1 - y_{k+1})$.

$$i \int C_{1,k+1}^+ f^{(k+1)} \left(\frac{X_k + X_k'}{2}, V_k \right) e^{i V_k \cdot (X_k - X_k')} dV_k$$
$$= i \frac{1}{(2\pi)^{2d}} \int du_1 du_2 dw dz d\omega \mathbf{b} \left(2|u_2|, \omega \cdot \frac{u_2}{|u_2|} \right) e^{i u_1 \cdot w} e^{-i u_2 \cdot z}$$
$$\times \gamma^{(k+1)} \left(x_1 - \frac{1}{2} P_\omega(x_1 - x_1') - \frac{w + R_\omega(z)}{4}, X_{2:k}, \right.$$
$$\frac{x_1 + x_1'}{2} + \frac{1}{2} P_\omega(x_1 - x_1') - \frac{w - R_\omega(z)}{4},$$
$$x_1' + \frac{1}{2} P_\omega(x_1 - x_1') + \frac{w + R_\omega(z)}{4}, X_{2:k}',$$
$$\left. \frac{x_1 + x_1'}{2} - \frac{1}{2} P_\omega(x_1 - x_1') + \frac{w - R_\omega(z)}{4} \right).$$

We have

$$i \int C_{1,k+1}^+ f^{(k+1)} \left(\frac{X_k + X_k'}{2}, V_k \right) e^{i V_k \cdot (X_k - X_k')} dV_k$$
$$= \frac{i}{2^{2d} \pi^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} dz \hat{\mathbf{b}}^\omega \left(\frac{z}{2} \right)$$
$$\times \gamma^{(k+1)} \left(x_1 - \frac{1}{2} P_\omega(x_1 - x_1') - \frac{R_\omega(z)}{4}, X_{2:k}, \right.$$
$$\frac{x_1 + x_1'}{2} + \frac{1}{2} P_\omega(x_1 - x_1') + \frac{R_\omega(z)}{4},$$
$$x_1' + \frac{1}{2} P_\omega(x_1 - x_1') + \frac{R_\omega(z)}{4}, X_{2:k}',$$
$$\left. \frac{x_1 + x_1'}{2} - \frac{1}{2} P_\omega(x_1 - x_1') - \frac{R_\omega(z)}{4} \right).$$

$\square$

We now turn to the Boltzmann equation. Here it suffices to notice that a solution f of the Boltzmann equation is just a *factorized* solution of the Boltzmann hierarchy, i.e. $f^{(k)} = f^{\otimes k}$. Using Propositions A.1 and A.2, we obtain:

Corollary A.3. *Let $f \in L^1 \left([0, T], L^2(\mathbb{R}^d \times \mathbb{R}^d) \right)$ and let γ denote the inverse Wigner transform of f. Then if*

$$(\partial_t + v \cdot \nabla_x) f = g \tag{160}$$

holds in the sense of distributions, then we have

$$\left(i\partial_t + \frac{1}{2} \left(\Delta_x - \Delta_{x'} \right) \right) \gamma = i \int_{\mathbb{R}^d} g \left(t, \frac{x + x'}{2}, v \right) e^{iv \cdot (x - x')} dv \qquad (161)$$

in the sense of distributions.

Corollary A.4. *Let $f(x, v)$ be a Schwartz function, and let γ denote its inverse Wigner transform. Then*

$$i \int Q(f, f) \left(\frac{x + x'}{2}, v \right) e^{iv \cdot (x - x')} dv$$

$$= \frac{i}{2^{2d} \pi^d} \int_{\mathbb{S}^{d-1}} d\omega \int_{\mathbb{R}^d} dz \hat{\mathbf{b}}^\omega \left(\frac{z}{2} \right)$$

$$\times \left\{ \gamma \left(x - \frac{1}{2} P_\omega (x - x') - \frac{R_\omega(z)}{4}, x' + \frac{1}{2} P_\omega (x - x') + \frac{R_\omega(z)}{4} \right) \right.$$

$$\times \gamma \left(\frac{x + x'}{2} + \frac{1}{2} P_\omega (x - x') + \frac{R_\omega(z)}{4}, \frac{x + x'}{2} - \frac{1}{2} P_\omega (x - x') - \frac{R_\omega(z)}{4} \right)$$

$$\left. - \gamma \left(x - \frac{z}{4}, x' + \frac{z}{4} \right) \gamma \left(\frac{x + x'}{2} + \frac{z}{4}, \frac{x + x'}{2} - \frac{z}{4} \right) \right\}. \qquad (162)$$

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