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On the First Critical Field in the Three Dimensional Ginzburg–Landau Model of Superconductivity

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Abstract: The Ginzburg–Landau model is a phenomenological description of superconductivity. A crucial feature of type-II superconductors is the occurrence of vortices, which appear above a certain value of the strength of the applied magnetic field called the first critical field. In this paper we estimate this value, when the Ginzburg–Landau parameter is large, and we characterize the behavior of the Meissner solution, the unique vortexless configuration that globally minimizes the Ginzburg–Landau energy below the first critical field. In addition, we show that beyond this value, for a certain range of the strength of the applied field, there exists a unique Meissner-type solution that locally minimizes the energy.

1. Introduction

1.1. Problem and background. Superconductors are certain metals and alloys, which, when cooled down below a critical (typically very low) temperature, lose their resistivity, which allows permanent currents to circulate without loss of energy. Superconductivity was discovered by H. Kamerlingh Onnes in 1911. As a phenomenological description of this phenomenon, Ginzburg and Landau [GL50] introduced in 1950 the Ginzburg–Landau model of superconductivity, which has been proven to effectively predict the behavior of superconductors and that was subsequently justified as a limit of the Bardeen–Cooper–Schrieffer (BCS) quantum theory [BCS57]. It is a model of great importance in physics, with Nobel prizes awarded for it to Abrikosov, Ginzburg, and Landau.

The Ginzburg–Landau functional, which models the state of a superconducting sample in an applied magnetic field, assuming that the temperature is fixed and below the critical one, is

$$GL_\varepsilon(u, A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 + \frac{1}{2}\int_{\mathbb{R}^3} |H - H_{\text{ex}}|^2.$$

Here

- Ω is a bounded domain of $\mathbb{R}^3$, that we assume to be simply connected with C^2 boundary.
- $u : \Omega \to \mathbb{C}$ is called the *order parameter*. Its modulus squared (the density of Cooper pairs of superconducting electrons in the BCS quantum theory) indicates the local state of the superconductor: where $|u|^2 \approx 1$ the material is in the superconducting phase, where $|u|^2 \approx 0$ in the normal phase.
- $A : \mathbb{R}^3 \to \mathbb{R}^3$ is the electromagnetic vector potential of the induced magnetic field $H = \operatorname{curl} A$.
- ∇_A denotes the covariant gradient $\nabla - iA$.
- $H_{\text{ex}} : \mathbb{R}^3 \to \mathbb{R}^3$ is a given external (or applied) magnetic field.
- $\varepsilon > 0$ is the inverse of the *Ginzburg–Landau parameter* usually denoted κ, a non-dimensional parameter depending only on the material. We will be interested in the regime of small ε, corresponding to extreme type-II superconductors.

A key physical feature of this type of superconductors is the occurrence of *vortices* (similar to those in fluid mechanics, but quantized), in the presence of an applied magnetic field. They correspond to the regions where $|u|$ vanishes, and since u is complex-valued they carry a nonzero integer topological degree. Vortices become co-dimension 2 topological singularities in the limit $\varepsilon \to 0$, and are the crucial objects of interest in the analysis of the model.

There are three main critical values or critical fields H_{c_1}, H_{c_2}, and H_{c_3} of the strength of the applied field H_{ex}, for which phase transitions occur.

- Below $H_{c_1} = O(|\log \varepsilon|)$, the superconductor is everywhere in its superconducting phase, i.e. $|u|$ is uniformly close to 1, and the applied field is expelled by the material due to the occurrence of supercurrents near $\partial\Omega$. This phenomenon is known as the *Meissner effect*.
- At H_{c_1}, the first vortice(s) appear and the applied field penetrates the superconductor through the vortice(s).
- Between H_{c_1} and H_{c_2}, the superconducting and normal phases coexist in the sample. As the strength of the applied field increases, so does the number of vortices. The vortices repel each other, while the external magnetic field confines them inside the sample.
- At $H_{c_2} = O\left(\frac{1}{\varepsilon^2}\right)$, the superconductivity is lost in the bulk of the sample.
- Between H_{c_2} and H_{c_3}, superconductivity persists only near the boundary.
- Above $H_{c_3} = O\left(\frac{1}{\varepsilon^2}\right)$, the applied magnetic field completely penetrates the sample and the superconductivity is lost, i.e. $u = 0$.

The Ginzburg–Landau model is known to be a $\mathbb{U}(1)$-gauge theory. This means that all the meaningful physical quantities are invariant under the gauge transformations

$$u \mapsto ue^{i\phi}, \quad A \mapsto A + \nabla\phi,$$

where ϕ is any real-valued function in $H^2_{\text{loc}}(\mathbb{R}^3)$. The Ginzburg–Landau energy and its associated free energy

$$F_\varepsilon(u, A) = \frac{1}{2}\int_\Omega |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 + |\operatorname{curl} A|^2$$

are gauge invariant, as well as the density of superconducting Cooper pairs $|u|^2$, the induced magnetic field H, and the vorticity, defined, for any sufficiently regular configuration (u, A), as

$$\mu(u, A) = \operatorname{curl}(iu, \nabla_A u) + \operatorname{curl} A,$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{C}$ identified with $\mathbb{R}^2$ i.e. $(a, b) = \frac{\overline{a}b+a\overline{b}}{2}$. This quantity is the gauge-invariant version of the Jacobian determinant of u and is the analogue of the vorticity of a fluid. For further physics background on the model, we refer to [Tin96, DG99].

The main purpose of this paper is to give a precise estimate of H_{c_1} and to characterize the behavior of global minimizers of GL_ε below this value in three dimensions. The analysis of H_{c_2} or higher applied fields requires completely different techniques. The interested reader can refer to [GP99, FH10, FK13, FKP13] and references therein.

The first critical field is (rigorously) defined by the fact that below H_{c_1} global minimizers of the Ginzburg–Landau functional do not have vortices, while they do for applied fields whose strength is higher than H_{c_1}. In the 2D setting, Sandier and Serfaty (see [Ser99a, SS00, SS03, SS07]) provided an expansion of the first critical field, up to an error $o(1)$ as $\varepsilon \to 0$, and rigorously characterized the behavior of global minimizers of the Ginzburg–Landau functional below and near this value. Conversely, in three dimensions much less is known. Recently Baldo, Jerrard, Orlandi, and Soner [BJOS13], via a Γ-convergence argument, provided the asymptotic leading order value of the first critical field as $\varepsilon \to 0$ (see also [BJOS12] for related results). In short, in a uniform applied field, i.e. when $H_{\mathrm{ex}} = h_\varepsilon \vec{e}$, where $h_\varepsilon \geq 0$ and $\vec{e} \in \mathbb{R}^3$ is a fixed unit vector, they proved that if $(u_\varepsilon, A_\varepsilon)$ minimizes $GL_\varepsilon(u_\varepsilon, A_\varepsilon)$ then there exists a measure μ_0 such that

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{|\log \varepsilon|} \to \mu_0 \quad \text{as } \varepsilon \to 0$$

in weak sense (the precise type of convergence can be found in [BJOS13, Proposition 1]). Moreover, there exists a constant H^* such that if $\lim_{\varepsilon\to 0} \frac{h_\varepsilon}{|\log \varepsilon|} < H^*$ then $\mu_0 \equiv 0$, while $\mu_0 \not\equiv 0$ if $\lim_{\varepsilon\to 0} \frac{h_\varepsilon}{|\log \varepsilon|} > H^*$. This result gives H_{c_1} up to an error $o(|\log \varepsilon|)$ as $\varepsilon \to 0$ and agrees with previous work by Alama, Bronsard, and Montero [ABM06] in the special case when Ω is a ball. An intermediate situation, when the superconducting sample is a thin shell, was treated in [Con11].

Before stating our results, let us recall the three dimensional ε-level estimates for the Ginzburg–Landau functional provided by the author in [Rom19]. These tools will play a crucial role in this paper.

Theorem 1.1. *For any $m, n, M > 0$ there exist $C, \varepsilon_0 > 0$ depending only on m, n, M, and $\partial\Omega$, such that, for any $\varepsilon < \varepsilon_0$, if $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq M|\log \varepsilon|^m$ then there exists a polyhedral 1-dimensional current ν_ε such that*

(1) ν_ε/π is integer multiplicity,
(2) $\partial\nu_\varepsilon = 0$ relative to Ω,
(3) $\operatorname{supp}(\nu_\varepsilon) \subset S_{\nu_\varepsilon} \subset \overline{\Omega}$ with $|S_{\nu_\varepsilon}| \leq C|\log \varepsilon|^{-q}$, where $q(m, n) := \frac{3}{2}(m + n)$,
(4) $\displaystyle\int_{S_{\nu_\varepsilon}} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2 + |\operatorname{curl} A_\varepsilon|^2 \geq |\nu_\varepsilon|(\Omega)\left(\log\frac{1}{\varepsilon} - C\log\log\frac{1}{\varepsilon}\right) - \frac{C}{|\log \varepsilon|^n}$,
(5) and for any $\gamma \in (0, 1]$ there exists a constant C_γ depending only on γ and $\partial\Omega$, such that

$$\|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C_T^{0,\gamma}(\Omega)^*} \leq C_\gamma \frac{F_\varepsilon(u_\varepsilon, A_\varepsilon) + 1}{|\log \varepsilon|^{q\gamma}}.$$

Here and in the rest of the paper, $C_T^{0,\gamma}(\Omega)$ denotes the space of vector fields $\Phi \in C^{0,\gamma}(\Omega)$ such that $\Phi \times \nu = 0$ on $\partial\Omega$, where ν is the outer unit normal to $\partial\Omega$. The symbol * denotes its dual space.

1.2. Main results. Throughout this article we assume that $H_{\mathrm{ex}} \in L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ satisfies $\operatorname{div} H_{\mathrm{ex}} = 0$ in $\mathbb{R}^3$. In particular, we deduce that there exists a vector-potential $A_{\mathrm{ex}} \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$\operatorname{curl} A_{\mathrm{ex}} = H_{\mathrm{ex}} \quad \text{and} \quad \operatorname{div} A_{\mathrm{ex}} = 0 \text{ in } \mathbb{R}^3.$$

Let $h_{\mathrm{ex}} := \|H_{\mathrm{ex}}\|_{L^2(\Omega,\mathbb{R}^3)}$. We define $H_{0,\mathrm{ex}} := h_{\mathrm{ex}}^{-1} H_{\mathrm{ex}}$ and assume that this vector field is Hölder continuous in Ω with Hölder exponent $\beta \in (0, 1]$ and Hölder norm bounded independently of ε. In particular, note that $\|H_{0,\mathrm{ex}}\|_{L^2(\Omega,\mathbb{R}^3)} = 1$. We also set $A_{0,\mathrm{ex}} := h_{\mathrm{ex}}^{-1} A_{\mathrm{ex}}$.

We remark that the divergence-free assumption on the applied magnetic field is in accordance with the fact that magnetic monopoles do not exist in Maxwell's electromagnetism theory.

The natural space for the minimization of GL_ε in three dimensions is $H^1(\Omega, \mathbb{C}) \times [A_{\mathrm{ex}} + H_{\mathrm{curl}}]$, where

$$H_{\mathrm{curl}} := \{A \in H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3) \mid \operatorname{curl} A \in L^2(\mathbb{R}^3, \mathbb{R}^3)\}.$$

Let us also introduce the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3, \mathbb{R}^3)$, which is defined as the completion of $C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ with respect to the norm $\|\nabla(\cdot)\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}$. We observe that, by Sobolev embedding, there exists a constant $C > 0$ such that

$$\|A\|_{L^6(\mathbb{R}^3,\mathbb{R}^3)} \le C\|\nabla A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} \tag{1.1}$$

for any $A \in \dot{H}^1(\mathbb{R}^3, \mathbb{R}^3)$. Moreover, by [KS91, Proposition 2.4], we have

$$\dot{H}^1(\mathbb{R}^3, \mathbb{R}^3) = \{A \in L^6(\mathbb{R}^3, \mathbb{R}^3) \mid \nabla A \in L^2(\mathbb{R}^3, \mathbb{R}^3)\}.$$

It is also convenient to define the subspace

$$\dot{H}^1_{\operatorname{div}=0} := \{A \in \dot{H}^1(\mathbb{R}^3, \mathbb{R}^3) \mid \operatorname{div} A = 0 \text{ in } \mathbb{R}^3\}.$$

In this subspace, one has

$$\|A\|_{\dot{H}^1_{\operatorname{div}=0}} := \|\nabla A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} = \|\operatorname{curl} A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}. \tag{1.2}$$

Let us now define a special vortexless configuration that turns out to be a good approximation of the so-called *Meissner solution*, i.e. the vortexless global minimizer of the Ginzburg–Landau energy below the first critical field, which, as we shall see, is unique up to a gauge transformation. By recalling that any vector field $A \in H^1(\Omega, \mathbb{R}^3)$ can be decomposed as (see Lemma 2.2)

$$\begin{cases} A = \operatorname{curl} B_A + \nabla\phi_A & \text{in } \Omega \\ B_A \times \nu = 0 & \text{on } \partial\Omega \\ \nabla\phi_A \cdot \nu = A \cdot \nu & \text{on } \partial\Omega \end{cases}$$

with $B_A \in \{B \in H^2(\Omega, \mathbb{R}^3) \mid \operatorname{div} B = 0 \text{ in } \Omega\}$ and $\phi_A \in \{\phi \in H^2(\Omega) \mid \int_\Omega \phi_A = 0\}$, we consider the pair $(u_0, h_{\text{ex}} A_0)$, where $u_0 = e^{ih_{\text{ex}}\phi_{A_0}}$ and A_0 is the unique minimizer (in a suitable space) of the functional

$$J(A) = \frac{1}{2}\int_\Omega |\operatorname{curl} B_A|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl}(A - A_{0,ex})|^2.$$

This special configuration satisfies the following properties:

- $GL_\varepsilon(u_0, h_{\text{ex}} A_0) = h_{\text{ex}}^2 J(A_0)$.
- $|u_0| = 1$ and $\mu(u_0, h_{\text{ex}} A_0) = 0$ in Ω.
- $H_0 = \operatorname{curl} A_0$ satisfies the usually called *London equation*

$$\operatorname{curl}^2(H_0 - H_{0,\text{ex}}) + H_0 \chi_\Omega = 0 \quad \text{in } \mathbb{R}^3,$$

  where χ_Ω denotes the characteristic function of Ω.
- The divergence-free vector field $B_0 = B_{A_0} \in C_T^{2,\beta}(\Omega, \mathbb{R}^3)$ satisfies

$$\begin{cases} -\Delta B_0 + B_0 = H_{0,\text{ex}} & \text{in } \Omega \\ B_0 \times \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

  This vector field is the analog of the function ξ_0, considered by Sandier and Serfaty in the analysis of the first critical field in 2D (see [Ser99a, Ser99b, SS00, SS03, SS07]). We shall see that B_0 plays an important role in our three dimensional analysis.

In addition, this pair allows us to split the Ginzburg–Landau energy of a given configuration (u, A). More precisely, by writing $u' = u_0^{-1} u$ and $A' = A - h_{\text{ex}} A_0$, one can prove that (see Proposition 3.1)

$$GL_\varepsilon(u, A) = h_{\text{ex}}^2 J(A_0) + F_\varepsilon(u', A') + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A'|^2 - h_{\text{ex}} \int_\Omega \mu(u', A') \wedge B_0 + R_0,$$

where $R_0 = o(1)$, in particular, when h_{ex} is bounded above by a positive power of $|\log \varepsilon|$. Let us emphasize that one of the achievements of this paper is to find the right pair $(u_0, h_{\text{ex}} A_0)$ to split the energy, which then allows to implement (almost) the same strategies as in 2D.

By combining this splitting with the optimal ε-level estimates of Theorem 1.1, we find

$$GL_\varepsilon(u, A) \geq h_{\text{ex}}^2 J(A_0) + \frac{1}{2}|\nu'_\varepsilon|(\Omega)\left(\log\frac{1}{\varepsilon} - C\log\log\frac{1}{\varepsilon}\right) - h_{\text{ex}} \int_\Omega \nu'_\varepsilon \wedge B_0 + o(1),$$

where ν'_ε denotes the 1-current associated to (u', A') by Theorem 1.1. By construction of ν'_ε (see [Rom19, Section 5.2]), we can write

$$\nu'_\varepsilon = \sum_{i \in I_\varepsilon} 2\pi \Gamma_i^\varepsilon,$$

where the sum is understood in the sense of currents, I_ε is a finite set of indices, and Γ_i^ε is an oriented Lipschitz curve in Ω with multiplicity 1. Each of these curves, which are non-necessarily distinct, does not self intersect and is either a loop contained in Ω or has two different endpoints on $\partial\Omega$. We will denote by X the class of Lipschitz curves, seen as 1-currents, described here.

Inserting this expression in the previous inequality, allows us to heuristically derive the leading order of the first critical field:

$$H^0_{c_1} := \frac{1}{2\|B_0\|_*}|\log\varepsilon|,$$

where

$$\|B_0\|_* := \sup_{\Gamma\in X}\frac{1}{|\Gamma|(\Omega)}\int_\Omega \Gamma\wedge B_0.^1 \tag{1.3}$$

We may now state our first result, that characterizes the behavior of global minimizers of GL_ε below $H^0_{c_1}$. In the 2D setting, an analogous result was proved by Sandier and Serfaty (see [SS00, Theorem 1]).

Theorem 1.2. *There exist constants* $\varepsilon_0, K_0 > 0$ *such that for any* $\varepsilon < \varepsilon_0$ *and* $h_{\text{ex}} \le H^0_{c_1} - K_0 \log|\log\varepsilon|$, *the global minimizers* $(u_\varepsilon, A_\varepsilon)$ *of* GL_ε *in* $H^1(\Omega,\mathbb{C})\times[A_{\text{ex}}+H_{\text{curl}}]$ *are vortexless configurations such that, as* $\varepsilon\to 0$,

- $\|1-|u_\varepsilon|\|_{L^\infty(\Omega,\mathbb{C})} = o(1)$,
- $\|\mu(u_\varepsilon, A_\varepsilon)\|_{C^{0,\gamma}_T(\Omega)^*} = o(1)$ *for any* $\gamma\in(0,1]$, *and*
- $h^2_{\text{ex}}J(A_0) + o(1) \le GL_\varepsilon(u_\varepsilon, A_\varepsilon) \le h^2_{\text{ex}}J(A_0)$.

It is important to mention that in the proof of this theorem we use the fact that solutions of the Ginzburg–Landau equations (see Sect. 2.3), in the Coulomb gauge, satisfy a clearing-out result proved by Chiron [Chi05]. Roughly speaking, this states that if the energy of a solution in a ball (with center in $\overline{\Omega}$) intersected with Ω is sufficiently small, then $|u|$ is uniformly away from 0 in a ball of half radius intersected with Ω. The proof given by Chiron relies on monotonicity formulas, and is very much inspired by previous work by Bethuel, Orlandi, and Smets [BOS04]. The interested reader can refer to [Riv95,LR99,LR01,BBO01,SS17] for results in the same spirit.

Our second result provides bounds from above and below for the first critical field in three dimensions.

Theorem 1.3. *There exist constants* $\varepsilon_0, K_0 > 0$ *such that for any* $\varepsilon < \varepsilon_0$ *we have*

$$H^0_{c_1} - K_0\log|\log\varepsilon| \le H_{c_1}.$$

Moreover, if there exists a multiplicity 1 *rectifiable* 1*-current* Γ_1 *with* $\partial\Gamma_1 = 0$ *relative to* Ω *such that*

$$\|B_0\|_* = \frac{1}{|\Gamma_1|(\Omega)}\int_\Omega \Gamma_1\wedge B_0,$$

then there exist constants $\varepsilon_1, K_1 > 0$ *such that for any* $\varepsilon < \varepsilon_1$ *we have*

$$H_{c_1} \le H^0_{c_1} + K_1.$$

Remark 1.1. *In the special case* $\Omega = B(0,R)$ *and* $H_{0,\text{ex}} = \hat{z}$ *in* $B(0,R)$, $\|B_0\|_*$ *is achieved by the vertical diameter seen as a* 1*-current with multiplicity* 1 *and oriented in the direction of positive z axis; see Proposition* 4.1. *In particular, in this case the hypothesis of this theorem is satisfied by a curve which belongs to X.*

[1] The notation used here is explained in the preliminaries (see Sect. 2).

Remark 1.2. *These inequalities show that indeed $H_{c_1}^0$ is the leading order of H_{c_1} as $\varepsilon \to 0$. Of course this agrees with the previously mentioned result by Baldo, Jerrard, Orlandi, and Soner. The author strongly believes that, as $\varepsilon \to 0$,*

$$H_{c_1} = H_{c_1}^0 + O(1).$$

To prove this result, one needs to avoid the uncertainty of order $O(\log|\log\varepsilon|)$ in the statement of Theorem 1.2. To accomplish this, it is crucially important to characterize, near the first critical field, the behavior of the vorticity $\mu(u, A)$ of global minimizers of GL_ε. We plan to address this problem in future work.

Our next result shows that beyond the first critical field there exists a locally minimizing vortexless configuration. A similar result was proved by Serfaty in 2D (see [Ser99b, Theorem 1]).

Theorem 1.4. *Let $\alpha \in \left(0, \frac{1}{3}\right)$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ if $h_{\text{ex}} \leq \varepsilon^{-\alpha}$ then there exists a vortexless configuration $(u_\varepsilon, A_\varepsilon) = (u_0 u'_\varepsilon, h_{\text{ex}} A_0 + A'_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + \dot{H}^1_{\text{div}=0}]$, which locally minimizes GL_ε in $H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ and satisfies the following properties as $\varepsilon \to 0$:*

(1) $\|1 - |u_\varepsilon|\|_{L^\infty(\Omega,\mathbb{C})} = o(1)$.
(2) $h_{\text{ex}}^2 J(A_0) + o(1) \leq GL_\varepsilon(u_\varepsilon, A_\varepsilon) \leq h_{\text{ex}}^2 J(A_0)$.
(3) The configuration $(u'_\varepsilon, A'_\varepsilon)$ satisfies

$$\inf_{\theta \in [0,2\pi]} \|u'_\varepsilon - e^{i\theta}\|_{H^1(\Omega,\mathbb{C})} + \|A'_\varepsilon\|_{\dot{H}^1_{\text{div}=0}} = o(1).$$

(4) Up to a gauge transformation, $(u_\varepsilon, A_\varepsilon)$ converges to $(u_0, h_{\text{ex}} A_0)$. More precisely, we have

$$\inf_{\theta \in [0,2\pi]} \|u_\varepsilon - e^{i\theta} u_0\|_{H^1(\Omega,\mathbb{C})} + \|A_\varepsilon - h_{\text{ex}} A_0\|_{\dot{H}^1_{\text{div}=0}} = o(1).$$

Let us point out that in Remark 5.1 we explain why we require $\alpha < \frac{1}{3}$.

Our last result concerns the uniqueness, up to a gauge transformation, of locally minimizing vortexless configurations.

Theorem 1.5. *Let $\alpha, c \in (0, 1)$. There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$, if $h_{\text{ex}} \leq \varepsilon^{-\alpha}$ then a configuration $(u, A) = (u_0 u', h_{\text{ex}} A_0 + A')$ which locally minimizes GL_ε in $H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ and satisfies $|u| \geq c$ and $F_\varepsilon(u', A') \leq \varepsilon^{1+\delta}$ for some $\delta > 0$, is unique up to a gauge transformation.*

Remark 1.3. *The assumption that $F_\varepsilon(u', A') \leq \varepsilon^{1+\delta}$ for some $\delta > 0$ plays a crucial role in the proof of this result. In Proposition A.2, we prove that if $\alpha \in \left(0, \frac{1}{4}\right)$ then this condition is implied by the other assumptions of this theorem provided that $GL_\varepsilon(u, A) \leq GL_\varepsilon(u_0, h_{\text{ex}} A_0) = h_{\text{ex}}^2 J(A_0)$, i.e. uniqueness holds without assuming that $F_\varepsilon(u', A') \leq \varepsilon^{1+\delta}$ for some $\delta > 0$ if the Ginzburg–Landau energy of the vortexless local minimizer is below the energy of $(u_0, h_{\text{ex}} A_0)$. We observe that this condition is satisfied by the locally minimizing solution of Theorem 1.4.*

Let us also note that if $\alpha \geq \frac{1}{4}$ then the strategy of the proof of Proposition A.2 fails. For this reason, we are able to guaranty the uniqueness of the locally minimizing vortexless configuration of Theorem 1.4 only if $\alpha < \frac{1}{4}$.

Finally, let us emphasize that this uniqueness result allows to conclude that the locally minimizing configuration of Theorem 1.4 is, indeed, up to a gauge transformation, the unique global minimizer of the Ginzburg–Landau energy below the first critical field. Therefore Theorem 1.4, in particular, provides a detailed characterization of the behavior of the Meissner solution.

Thus, we prove that below the first critical field, up to a gauge transformation, the Meissner solution is the unique global minimizer of GL_ε. Beyond this value, at least up to $h_{\text{ex}} = o(\varepsilon^{-\frac{1}{3}})$, a Meissner-type solution continues to exists as a local minimizer of the Ginzburg–Landau energy. This solution is unique, up to a gauge transformation, at least up to $h_{\text{ex}} = o(\varepsilon^{-\frac{1}{4}})$. Since this branch of vortexless solutions remains stable, in the process of raising h_{ex} vortices should not appear at H_{c_1}, but rather at a critical value of h_{ex} called the *superheating field* H_{sh}, at which the Meissner-type solution becomes unstable. It is expected that $H_{\text{sh}} = O(\varepsilon^{-1})$. The interested reader can refer to [Xia16] and references therein for further details.

Outline of the paper. The rest of the paper is organized as follows. In Sect. 2 we introduce some basic quantities and notation, describe two Hodge-type decompositions, and present some classical results in Ginzburg–Landau theory. In Sect. 3 we define the approximation of the Meissner solution, split the Ginzburg–Landau energy, and prove Theorem 1.2. In Sect. 4 we present the proof of Theorem 1.3 and compute $\|B_0\|_*$ in a special case. Section 5 contains the proof of Theorem 1.4 and Section 6 the proof of Theorem 1.5. Appendix A is devoted to prove some improved estimates for locally minimizing configurations, that allow to obtain the uniqueness of the Meissner-type solution of Theorem 1.4 for $\alpha < \frac{1}{4}$, as a consequence of Theorem 1.5.

2. Preliminaries

2.1. Some definitions and notation. We define the superconducting current of a pair $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ as the 1-form

$$j(u, A) = (iu, d_A u) = \sum_{k=1}^{3} (iu, \partial_k u - iA_k u) dx_k.$$

It is related to the vorticity $\mu(u, A)$ of a configuration (u, A) through

$$\mu(u, A) = dj(u, A) + dA.$$

This quantity can be seen as a 1-current, which is defined through its action on 1-forms by the relation

$$\mu(u, A)(\phi) = \int_\Omega \mu(u, A) \wedge \phi.$$

We recall that the boundary of a 1-current T relative to a set Θ, is the 0-current ∂T defined by

$$\partial T(\phi) = T(d\phi)$$

for all smooth compactly supported 0-form ϕ defined in Θ. In particular, $\mu(u, A)$ has zero boundary relative to Ω. We denote by $|T|(\Theta)$ the mass of a 1-current T in Θ.

2.2. Hodge-type decompositions. Next, we provide a decomposition of vector fields in H_{curl}.

Lemma 2.1. *Every vector field* $A \in H_{\text{curl}}$ *can be decomposed as*

$$A = \operatorname{curl} \mathcal{B} + \nabla \Phi,$$

where $\mathcal{B}, \operatorname{curl} \mathcal{B} \in \dot{H}^1_{\text{div}=0}$ *and* $\Phi \in H^2_{\text{loc}}(\mathbb{R}^3)$.

Proof. First, let us observe that there exists a function $\Phi_1 \in H^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$\Delta \Phi_1 = \operatorname{div} A \in L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3).$$

Second, we consider the problem

$$\begin{cases} \operatorname{curl}^2 B = \operatorname{curl} A \in L^2(\mathbb{R}^3, \mathbb{R}^3) \\ \quad \operatorname{div} B = 0. \end{cases}$$

By observing that $\operatorname{curl}^2 B = -\Delta B$, [KS91, Theorem 1] provides the existence of a solution $\mathcal{B} \in \dot{H}^1_{\text{div}=0}$ to this problem such that $\operatorname{curl} \mathcal{B} \in \dot{H}^1_{\text{div}=0}$.

Finally, by noting that

$$\operatorname{curl}(A - \nabla \Phi_1 - \operatorname{curl} \mathcal{B}) = \operatorname{div}(A - \nabla \Phi_1 - \operatorname{curl} \mathcal{B}) = 0,$$

we deduce that

$$A - \nabla \Phi_1 - \operatorname{curl} \mathcal{B} = \nabla \Phi_2$$

for some harmonic function $\Phi_2 \in H^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$. By writing $\Phi = \Phi_1 + \Phi_2$, we obtain the result. □

We now recall a decomposition of vector fields in $H^1(\Omega, \mathbb{R}^3)$. The proof of this result can be found in [BBO01, Appendix A].

Lemma 2.2. *There exists a constant* $C = C(\Omega)$ *such that for every* $A \in H^1(\Omega, \mathbb{R}^3)$ *there exist a unique vector field* $B_A \in \{B \in H^2(\Omega, \mathbb{R}^3) \mid \operatorname{div} B = 0 \text{ in } \Omega\}$ *and a unique function* $\phi_A \in \{\phi \in H^2(\Omega) \mid \int_\Omega \phi_A = 0\}$ *satisfying*

$$\begin{cases} \quad A = \operatorname{curl} B_A + \nabla \phi_A & \text{in } \Omega \\ B_A \times \nu = 0 & \text{on } \partial\Omega \\ \nabla \phi_A \cdot \nu = A \cdot \nu & \text{on } \partial\Omega. \end{cases}$$

Moreover,

$$\|B_A\|_{H^2(\Omega,\mathbb{R}^3)} \leq C \|\operatorname{curl} A\|_{L^2(\Omega,\mathbb{R}^3)} \quad \text{and} \quad \|\phi_A\|_{H^2(\Omega)} \leq C \|A\|_{H^1(\Omega,\mathbb{R}^3)}.$$

2.3. Ginzburg–Landau equations.

Definition 2.1 (Critical point of GL_ε). *We say that* $(u, A) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ *is a critical point of* GL_ε *if for every smooth configuration* (v, B) *with* B *compactly supported in* $\mathbb{R}^3$ *we have*

$$\frac{d}{dt} GL_\varepsilon(u + tv, A + tB)|_{t=0} = 0.$$

We now present the Euler-Lagrange equations satisfied by critical points of GL_ε. This is a well-known result, but for the sake of completeness we prove it here.

Proposition 2.1 (Ginzburg–Landau equations). *If* $(u, A) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ *is a critical point of* GL_ε *then* (u, A) *satisfies the system of equations*

$$\begin{cases} -(\nabla_A)^2 u = \frac{1}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega \\ \operatorname{curl}(H - H_{\text{ex}}) = (iu, \nabla_A u)\chi_\Omega & \text{in } \mathbb{R}^3 \\ \nabla_A u \cdot \nu = 0 & \text{on } \partial\Omega \\ [H - H_{\text{ex}}] \times \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{GL}$$

where χ_Ω *is the characteristic function of* Ω, $[\,\cdot\,]$ *denotes the jump across* $\partial\Omega$, $\nabla_A u \cdot \nu = \sum_{j=1}^3 (\partial_j u - iA_j u)\nu_j$, *and the covariant Laplacian* $(\nabla_A)^2$*is defined by*

$$(\nabla_A)^2 u = (\operatorname{div} - iA\cdot)\nabla_A u.$$

Proof. We have

$$\frac{d}{dt} GL_\varepsilon(u + tv, A)|_{t=0} = \int_\Omega (\nabla_A u, \nabla_A v) - \frac{1}{\varepsilon^2} \int_\Omega (u, v)(1 - |u|^2).$$

By noting that

$$(\nabla_A u, \nabla_A v) = \operatorname{div}(\nabla_A u, v) - ((\nabla_A)^2 u, v),$$

where $(\nabla_A u, v) = ((\partial_1 u - iA_1 u, v), (\partial_2 u - iA_2 u, v), (\partial_3 u - iA_3 u, v))$, and by integrating by parts, we obtain

$$\frac{d}{dt} GL_\varepsilon(u + tv, A)|_{t=0} = \int_{\partial\Omega} (\nabla_A u \cdot \nu, v) - \int_\Omega ((\nabla_A)^2 u, v) - \frac{1}{\varepsilon^2} \int_\Omega (u, v)(1 - |u|^2).$$

Since this is true for any v, we find

$$-(\nabla_A)^2 u = \frac{1}{\varepsilon^2} u(1 - |u|^2) \text{ in } \Omega \quad \text{and} \quad \nabla_A u \cdot \nu = 0 \text{ on } \partial\Omega.$$

On the other hand, we have

$$\frac{d}{dt} GL_\varepsilon(u, A + tB)|_{t=0} = -\int_\Omega (iBu, \nabla_A u) + \int_{\mathbb{R}^3} (H - H_{\text{ex}}) \cdot \operatorname{curl} B = 0.$$

By integration by parts, we get

$$\frac{d}{dt} GL_\varepsilon(u, A + tB)|_{t=0} = -\int_\Omega (iu, \nabla_A u) \cdot B + \int_{\mathbb{R}^3} \operatorname{curl}(H - H_{\text{ex}}) \cdot B = 0. \tag{2.1}$$

We deduce that

$$\operatorname{curl}(H - H_{\text{ex}}) = (iu, \nabla_A u)\chi_\Omega \quad \text{in } \mathbb{R}^3.$$

By testing this equation against $B\chi_\Omega$ and integrating by parts, we find

$$\int_\Omega (H - H_{\text{ex}}) \cdot \operatorname{curl} B - \int_{\partial\Omega} ((H - H_{\text{ex}}) \times \nu) \cdot B - \int_\Omega (iu, \nabla_A u) \cdot B = 0.$$

Now, by testing against $B\chi_{\mathbb{R}^3 \setminus \Omega}$ and integrating by parts, we get

$$\int_{\mathbb{R}^3 \setminus \Omega} (H - H_{\text{ex}}) \cdot \operatorname{curl} B + \int_{\partial(\mathbb{R}^3 \setminus \Omega)} ((H - H_{\text{ex}}) \times \nu) \cdot B = 0.$$

Thus

$$\int_{\partial\Omega} ([H - H_{\text{ex}}] \times \nu) \cdot B = 0,$$

which implies that $[H - H_{\text{ex}}] \times \nu = 0$ on $\partial\Omega$. □

Remark 2.1. *By taking* $B = \operatorname{curl} X$ *in* (2.1) *with* $X \in C_0^\infty(\Omega, \mathbb{R}^3)$ *and integrating by parts, we find*

$$-\int_\Omega (\mu(u, A) - H) \cdot X + \int_\Omega \operatorname{curl}^2(H - H_{\text{ex}}) \cdot X = 0.$$

Doing the same with $X \in C_0^\infty(\mathbb{R}^3 \setminus \Omega, \mathbb{R}^3)$*, we get*

$$\int_{\mathbb{R}^3 \setminus \Omega} \operatorname{curl}^2(H - H_{\text{ex}}) \cdot X = 0.$$

We then deduce that H *and* $\mu(u, A)$ *satisfy (in the sense of currents) the London equation*

$$\operatorname{curl}^2(H - H_{\text{ex}}) + H\chi_\Omega = \mu(u, A)\chi_\Omega. \tag{2.2}$$

We will come back to this equation later on.

2.4. Minimization of GL_ε.

Proposition 2.2. *The minimum of* GL_ε *over* $H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ *is achieved.*

Proof. Let $\{(\tilde{u}_n, \tilde{A}_n)\}_n$ be a minimizing sequence for GL_ε in $H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$. Lemma 2.1 yields a gauge transformed sequence $\{(u_n, A_n)\}_n$ such that $A_n \in [A_{\text{ex}} + \dot{H}^1_{\text{div}=0}]$. In particular, we have that $GL_\varepsilon(\tilde{u}_n, \tilde{A}_n) = GL_\varepsilon(u_n, A_n)$ and

$$\|\nabla(A_n - A_{\text{ex}})\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)} = \|\operatorname{curl}(A_n - A_{\text{ex}})\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)}.$$

Using the bound $GL_\varepsilon(u_n, A_n) \le C$, where C is independent of n, we find that

$$\|1 - |u_n|^2\|_{L^2(\Omega, \mathbb{C})}, \ \|\nabla_{A_n} u_n\|_{L^2(\Omega, \mathbb{C}^3)}, \ \text{and} \quad \|\operatorname{curl}(A_n - A_{\text{ex}})\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)}$$

are bounded independently of n. Therefore, by recalling (1.1), we deduce that $A_n - A_{\text{ex}}$ is bounded in $\dot{H}^1(\mathbb{R}^3, \mathbb{R}^3)$. Because $\{u_n\}_n$ is bounded in $L^4(\Omega)$ we find that $\{iA_n u_n\}_n$

is bounded in $L^2(\Omega, \mathbb{C}^3)$. By noting that $\nabla u_n = \nabla_{A_n} u_n + i A_n u_n$, we conclude that u_n is bounded in $H^1(\Omega, \mathbb{C})$.

We may then extract a subsequence, still denoted $\{(u_n, A_n)\}_n$, such that $\{(u_n, A_n - A_{\text{ex}})\}_n$ converges to some $(u, A - A_{\text{ex}})$ weakly in $H^1(\Omega, \mathbb{C}) \times \dot{H}_{\text{div}=0}$ and, by compact Sobolev embedding, strongly in every $L^q(\Omega, \mathbb{C}) \times L^q(\Omega, \mathbb{R}^3)$ for $q < 6$.

Let us now show that (u, A) is a minimizer of GL_ε. By strong $L^4(\Omega, \mathbb{C})$ convergence,

$$\liminf_n \|1 - |u_n|^2\|_{L^2(\Omega,\mathbb{C})} = \|1 - |u|^2\|_{L^2(\Omega,\mathbb{C})}.$$

Also, by weak $\dot{H}^1(\mathbb{R}^3, \mathbb{R}^3)$ convergence, we have

$$\begin{aligned}\liminf_n \|\operatorname{curl}(A_n - A_{\text{ex}})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} &= \liminf_n \|\nabla(A_n - A_{\text{ex}})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} \\ &\geq \|\nabla(A - A_{\text{ex}})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} = \|\operatorname{curl}(A - A_{\text{ex}})\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}.\end{aligned}$$

Moreover, standard arguments show that

$$\begin{aligned}\liminf_n \|\nabla_{A_n} u_n\|^2_{L^2(\Omega,\mathbb{C}^3)} &= \liminf_n \|\nabla u_n\|^2_{L^2(\Omega,\mathbb{C}^3)} - 2\int_\Omega (\nabla u_n, i A_n u_n) + \|A_n u_n\|_{L^2(\Omega,\mathbb{C}^3)} \\ &\geq \|\nabla_A u\|_{L^2(\Omega,\mathbb{C}^3)}.\end{aligned}$$

Hence

$$\liminf_n GL_\varepsilon(u_\varepsilon, A_\varepsilon) \geq GL_\varepsilon(u, A).$$

□

3. Global Minimizers Below $H^0_{c_1}$

3.1. An approximation of the Meissner solution. Next, we find a configuration $(u_0, h_{\text{ex}} A_0)$ with $|u_0| = 1$ and which satisfies (2.2) with zero right-hand side. As mentioned in the introduction, this turns out to be a good approximation of the Meissner solution, the vortexless configuration which minimizes GL_ε below the first critical field.

Let us consider a configuration of the form $(e^{i\phi_0}, h_{\text{ex}} A_0)$ with $\phi_0 \in H^2(\Omega)$ and $A_0 \in A_{0,\text{ex}} + \dot{H}^1_{\text{div}=0}$. Observe that, by using Lemma 2.2 and letting $u_0 := e^{i\phi_0}$, we have

$$\begin{aligned}GL_\varepsilon(u_0, h_{\text{ex}} A_0) =& \frac{1}{2}\int_\Omega |\nabla\phi_0 - h_{\text{ex}}(\operatorname{curl} B_{A_0} + \nabla\phi_{A_0})|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |h_{\text{ex}} \operatorname{curl} A_0 - H_{\text{ex}}|^2 \\ =& \frac{1}{2}\int_\Omega |\nabla(\phi_0 - h_{\text{ex}}\phi_{A_0})|^2 + h_{\text{ex}}^2 |\operatorname{curl} B_{A_0}|^2 + \frac{h_{\text{ex}}^2}{2}\int_{\mathbb{R}^3} |\operatorname{curl}(A_0 - A_{0,\text{ex}})|^2.\end{aligned}$$

By choosing $\phi_0 = h_{\text{ex}}\phi_{A_0}$, we obtain

$$GL_\varepsilon(u_0, h_{\text{ex}} A_0) = \frac{h_{\text{ex}}^2}{2}\int_\Omega |\operatorname{curl} B_{A_0}|^2 + \frac{h_{\text{ex}}^2}{2}\int_{\mathbb{R}^3} |\operatorname{curl}(A_0 - A_{0,\text{ex}})|^2 =: h_{\text{ex}}^2 J(A_0).$$

We let A_0 to be the minimizer of J in the space $\left(A_{0,\text{ex}} + \dot{H}^1_{\text{div}=0}, \|\cdot\|_{\dot{H}^1_{\text{div}=0}}\right)$, whose existence and uniqueness follows by noting that J is continuous, coercive, and strictly convex in this Hilbert space (recall (1.1) and (1.2)). We also let $H_0 = \operatorname{curl} A_0$ and here and in the rest of the paper we use the notation $B_0 := B_{A_0}$.

Let us observe that, by minimality of A_0 and Lemma 2.2, we have

$$J(A_0) \leq J(A_{0,\mathrm{ex}}) = \frac{1}{2}\int_\Omega |\operatorname{curl} B_{A_{0,\mathrm{ex}}}|^2 \leq C\int_\Omega |\operatorname{curl} A_{0,\mathrm{ex}}|^2 = C\int_\Omega |H_{0,ex}|^2 = C. \tag{3.1}$$

One can easily check that, for any $A \in \dot{H}^1_{\mathrm{div}=0}$, we have

$$\int_\Omega \operatorname{curl} B_0 \cdot \operatorname{curl} B_A + \int_{\mathbb{R}^3} (H_0 - H_{0,\mathrm{ex}}) \cdot \operatorname{curl} A = 0.$$

Because

$$\int_\Omega \operatorname{curl} B_0 \cdot \nabla\phi_A = \int_\Omega B_0 \cdot \operatorname{curl} \nabla\phi_A - \int_{\partial\Omega} (B_0 \times \nu) \cdot \nabla\phi_A = 0,$$

we have

$$\int_\Omega \operatorname{curl} B_0 \cdot A + \int_{\mathbb{R}^3} (H_0 - H_{0,\mathrm{ex}}) \cdot \operatorname{curl} A = 0. \tag{3.2}$$

Moreover, Lemma 2.1 implies that this equality also holds for any $A \in H_{\mathrm{curl}}$.

Let us observe that, for any $A \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$, by integration by parts, we have

$$\int_\Omega \operatorname{curl} B_0 \cdot A + \int_{\mathbb{R}^3} \operatorname{curl}(H_0 - H_{0,\mathrm{ex}}) \cdot A = 0.$$

Therefore, A_0 satisfies the Euler-Lagrange equation

$$\operatorname{curl}(H_0 - H_{0,\mathrm{ex}}) + \operatorname{curl} B_0 \chi_\Omega = 0 \quad \text{in } \mathbb{R}^3. \tag{3.3}$$

In addition, it is easy to see that the boundary condition $[H_0 - H_{0,\mathrm{ex}}] \times \nu = 0$ on $\partial\Omega$ holds.

Arguing as in Remark 2.1, we find

$$\operatorname{curl}^2(H_0 - H_{0,\mathrm{ex}}) + H_0\chi_\Omega = 0 \quad \text{in } \mathbb{R}^3,$$

namely (up to multiplying by h_{ex}) (2.2) with $\mu(u_0, A_0) = 0$.

On the other hand, by integration by parts, for any vector field $B \in C_0^\infty(\Omega, \mathbb{R}^3)$, we have

$$\int_\Omega B_0 \cdot \operatorname{curl} B + \int_\Omega (H_0 - H_{0,\mathrm{ex}}) \cdot \operatorname{curl} B = 0.$$

Besides, for any function $\phi \in C_0^\infty(\Omega)$, we have

$$\int_\Omega (B_0 + (H_0 - H_{0,\mathrm{ex}})) \cdot \nabla\phi = -\int_\Omega \operatorname{div}(B_0 + (H_0 - H_{0,\mathrm{ex}}))\phi = 0.$$

Then, given any vector field $A \in C_0^\infty(\Omega, \mathbb{R}^3)$, by taking $B = B_A$ and $\phi = \phi_A$ in the previous equalities, we find

$$\int_\Omega (B_0 + (H_0 - H_{0,\mathrm{ex}})) \cdot (\operatorname{curl} B_A + \nabla\phi_A) = \int_\Omega (B_0 + (H_0 - H_{0,\mathrm{ex}})) \cdot A = 0.$$

Hence, the divergence-free vector field B_0 weakly solves the problem

$$\begin{cases} -\Delta B_0 + B_0 = H_{0,\mathrm{ex}} & \text{in } \Omega \\ B_0 \times \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Remark 3.1. *Since we assume that $\|H_{0,\mathrm{ex}}\|_{C^{0,\beta}(\Omega,\mathbb{R}^3)} \leq C$, by standard elliptic regularity theory, we deduce that $B_0 \in C_T^{2,\beta}(\Omega,\mathbb{R}^3)$ with $\|B_0\|_{C_T^{2,\beta}(\Omega,\mathbb{R}^3)} \leq C$ for some constant independent of ε. In addition, if the applied field is taken to be uniform in Ω, i.e. if $H_{0,\mathrm{ex}}$ is a fixed unit vector in Ω, then B_0 depends on the domain Ω only.*

3.2. Energy-splitting. Next, by using the approximation of the Meissner solution, we present a splitting of GL_ε.

Proposition 3.1. *For any $(u,A) \in H^1(\Omega,\mathbb{C}) \times [A_{\mathrm{ex}} + H_{\mathrm{curl}}]$, letting $u = u_0 u'$ and $A = h_{\mathrm{ex}} A_0 + A'$, where $(u_0, h_{\mathrm{ex}} A_0)$ is the approximation of the Meissner solution, we have*

$$GL_\varepsilon(u,A) = h_{\mathrm{ex}}^2 J(A_0) + F_\varepsilon(u',A') + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A'|^2 - h_{\mathrm{ex}} \int_\Omega \mu(u',A') \wedge B_0 + R_0, \tag{3.5}$$

where $F_\varepsilon(u',A')$ is the free energy of the configuration $(u',A') \in H^1(\Omega,\mathbb{C}) \times H_{\mathrm{curl}}$, i.e.

$$F_\varepsilon(u',A') = \frac{1}{2}\int_\Omega |\nabla_{A'} u'|^2 + \frac{1}{2\varepsilon^2}(1-|u'|^2)^2 + |\operatorname{curl} A'|^2$$

and

$$R_0 = \frac{h_{\mathrm{ex}}^2}{2}\int_\Omega (|u|^2 - 1)|\operatorname{curl} B_0|^2.$$

In particular, $|R_0| \leq C\varepsilon h_{\mathrm{ex}}^2 E_\varepsilon(|u|)^{\frac{1}{2}}$ with $E_\varepsilon(|u|) = \frac{1}{2}\int_\Omega |\nabla |u||^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2$.

Proof. One immediately checks that $A' \in H_{\mathrm{curl}}$. Since $u' = u_0^{-1} u = e^{-ih_{\mathrm{ex}}\phi_0} u$ and $\phi_0 \in H^2(\Omega)$, by Sobolev embedding we deduce that $u' \in H^1(\Omega,\mathbb{C})$.

Writing $u = u_0 u'$ and $A = h_{\mathrm{ex}} A_0 + A'$ and plugging them into $GL_\varepsilon(u,A)$, we obtain

$$GL_\varepsilon(u,A) = \frac{1}{2}\int_\Omega |\nabla_{A'} u' - i h_{\mathrm{ex}} \operatorname{curl} B_0 u'|^2 + \frac{1}{2\varepsilon^2}(1-|u'|^2)^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl} A' + h_{\mathrm{ex}}(H_0 - H_{0,\mathrm{ex}})|^2.$$

By expanding the square terms, we get

$$GL_\varepsilon(u,A) = \frac{1}{2}\int_\Omega |\nabla_{A'} u'|^2 + h_{\mathrm{ex}}^2 |\operatorname{curl} B_0|^2 |u'|^2 - 2h_{\mathrm{ex}}(\nabla_{A'} u', iu') \cdot \operatorname{curl} B_0 + \frac{1}{2\varepsilon^2}(1-|u'|^2)^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl} A'|^2 + h_{\mathrm{ex}}^2 |H_0 - H_{0,\mathrm{ex}}|^2 + 2h_{\mathrm{ex}} \operatorname{curl} A' \cdot (H_0 - H_{0,\mathrm{ex}}).$$

Observe that, by (3.2), we have

$$\int_{\mathbb{R}^3} \operatorname{curl} A' \cdot (H_0 - H_{0,\mathrm{ex}}) = -\int_\Omega A' \cdot \operatorname{curl} B_0.$$

Therefore, grouping terms and writing $|u'|^2$ as $1 + (|u'|^2 - 1)$, we find

$$GL_\varepsilon(u, A) = h_{\text{ex}}^2 J(A_0) + F_\varepsilon(u', A') + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A'|^2 \\ -h_{\text{ex}} \int_\Omega (j(u', A') + A') \cdot \operatorname{curl} B_0 + R_0.$$

Then, an integration by parts yields

$$\int_\Omega (j(u', A') + A') \cdot \operatorname{curl} B_0 = \int_\Omega \mu(u', A') \wedge B_0 - \int_{\partial\Omega} (j(u', A') + A') \cdot (B_0 \times \nu).$$

By using the boundary condition $B_0 \times \nu = 0$ on $\partial\Omega$, we find (3.5). The inequality for R_0 follows directly from the Cauchy-Schwarz inequality. □

Remark 3.2. *Let $\varphi \in C_T^{0,1}(\Omega)$ be a 1-form. Observe that, by gauge invariance and by integration by parts, we have*

$$\int_\Omega \mu(u, A) \wedge \varphi = \int_\Omega \mu(u', A' + h_{\text{ex}} \operatorname{curl} B_0) \wedge \varphi \\ = \int_\Omega \mu(u', A') \wedge \varphi + h_{\text{ex}}(1 - |u|^2) \operatorname{curl} B_0 \cdot \operatorname{curl} \varphi.$$

Then, the Cauchy-Schwarz inequality yields

$$\|\mu(u, A) - \mu(u', A')\|_{C_T^{0,1}(\Omega)^*} \le C\varepsilon h_{\text{ex}} E_\varepsilon(|u|)^{\frac{1}{2}}.$$

Moreover, arguing as in the proof of the vorticity estimate in Theorem 1.1 for $\gamma \in (0, 1)$ (see [Rom19, Section 8]), we conclude that, for any $\gamma \in (0, 1)$,

$$\|\mu(u, A) - \mu(u', A')\|_{C_T^{0,\gamma}(\Omega)^*} \le C\left(F_\varepsilon(u, A) + F_\varepsilon(u', A')\right)^{1-\gamma} (\varepsilon h_{\text{ex}} E_\varepsilon(|u|)^{\frac{1}{2}})^\gamma.$$

3.3. Proof of Theorem 1.2.

Proof. Proposition 3.1 yields

$$GL_\varepsilon(u_\varepsilon, A_\varepsilon) \ge h_{\text{ex}}^2 J(A_0) + F_\varepsilon(u'_\varepsilon, A'_\varepsilon) - h_{\text{ex}} \int_\Omega \mu(u'_\varepsilon, A'_\varepsilon) \wedge B_0 + o(\varepsilon^{\frac{1}{2}}), \tag{3.6}$$

where $(u_\varepsilon, A_\varepsilon) = (u_0 u'_\varepsilon, h_{\text{ex}} A_0 + A'_\varepsilon)$.

Step 1. Estimating $F_\varepsilon(u'_\varepsilon, A'_\varepsilon)$. By minimality, we have

$$\inf_{(u,A)\in H^1(\Omega,\mathbb{C})\times[A_{\text{ex}}+H_{\text{curl}}]} GL_\varepsilon(u, A) \le GL_\varepsilon(u_0, h_{\text{ex}} A_0) = h_{\text{ex}}^2 J(A_0). \tag{3.7}$$

On the other hand, by gauge invariance, we get

$$F_\varepsilon(u'_\varepsilon, A'_\varepsilon) = F_\varepsilon(u_\varepsilon, A_\varepsilon - h_{\text{ex}} \operatorname{curl} B_0) \le 2F_\varepsilon(u_\varepsilon, A_\varepsilon) + 2F_\varepsilon(1, h_{\text{ex}} \operatorname{curl} B_0) \\ \le 2F_\varepsilon(u_\varepsilon, A_\varepsilon) + Ch_{\text{ex}}^2,$$

which combined with (3.7) and (3.1) implies that $F_\varepsilon(u'_\varepsilon, A'_\varepsilon) \le M|\log\varepsilon|^2$. We may then apply Theorem 1.1 (with n large enough) to obtain

$$F_\varepsilon(u'_\varepsilon, A'_\varepsilon) - h_{\mathrm{ex}} \int_\Omega \mu(u'_\varepsilon, A'_\varepsilon) \wedge B_0 \ge \frac{1}{2}|\nu'_\varepsilon|(\Omega)\left(\log\frac{1}{\varepsilon} - C\log\log\frac{1}{\varepsilon}\right) - h_{\mathrm{ex}}\int_\Omega \nu'_\varepsilon \wedge B_0 + o(|\log\varepsilon|^{-2}),$$

where $C > 0$ is a universal constant and ν'_ε denotes the polyhedral 1-dimensional current associated to the configuration $(u'_\varepsilon, A'_\varepsilon)$ by Theorem 1.1. By noting that

$$\int_\Omega \nu'_\varepsilon \wedge B_0 \le |\nu'_\varepsilon|(\Omega)\|B_0\|_*, \tag{3.8}$$

we find

$$F_\varepsilon(u'_\varepsilon, A'_\varepsilon) - h_{\mathrm{ex}} \int_\Omega \mu(u'_\varepsilon, A'_\varepsilon) \wedge B_0 \ge \frac{1}{2}|\nu'_\varepsilon|(\Omega)\left(\log\frac{1}{\varepsilon} - C\log\log\frac{1}{\varepsilon} - 2\|B_0\|_* h_{\mathrm{ex}}\right) + o(|\log\varepsilon|^{-2}).$$

Writing $h_{\mathrm{ex}} = H^0_{c_1} - K_0\log|\log\varepsilon|$ with $H^0_{c_1} = \frac{1}{2\|B_0\|_*}|\log\varepsilon|$, we get

$$GL_\varepsilon(u_\varepsilon, A_\varepsilon) \ge h^2_{\mathrm{ex}} J(A_0) + \frac{1}{2}|\nu'_\varepsilon|(\Omega)\,(2\|B_0\|_* K_0 - C)\log\log\frac{1}{\varepsilon} + o(|\log\varepsilon|^{-2}).$$

Combining with (3.7), we deduce that

$$o(|\log\varepsilon|^{-2}) \ge |\nu'_\varepsilon|(\Omega)\,(2\|B_0\|_* K_0 - C)\log\log\frac{1}{\varepsilon}.$$

Therefore, by letting $K_0 := (2\|B_0\|_*)^{-1}C + 1$, we deduce that $|\nu'_\varepsilon|(\Omega) = o(|\log\varepsilon|^{-2})$. In particular, from the vorticity estimate in Theorem 1.1 and (3.8), we deduce that $h_{\mathrm{ex}}\int_\Omega \mu(u'_\varepsilon, A'_\varepsilon) \wedge B_0 = o(|\log\varepsilon|^{-1})$. Therefore, inserting in (3.6) and using (3.7), we are led to

$$F_\varepsilon(u'_\varepsilon, A'_\varepsilon) + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A'_\varepsilon|^2 \le o(|\log\varepsilon|^{-1}). \tag{3.9}$$

In particular, we deduce that $GL_\varepsilon(u_\varepsilon, A_\varepsilon) = h^2_{\mathrm{ex}} J(A_0) + o(|\log\varepsilon|^{-1})$.

Let us also observe that, since $|\nu'_\varepsilon|(\Omega) = o(|\log\varepsilon|^{-2})$, from the vorticity estimate in Theorem 1.1 and Remark 3.2, one immediately deduces that, for any $\gamma \in (0, 1]$,

$$\|\mu(u_\varepsilon, A_\varepsilon)\|_{C^{0,\gamma}_T(\Omega)^*} \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.10}$$

Step 2. Applying a clearing out result. To prove that $\|1 - |u_\varepsilon|\|_{L^\infty(\Omega,\mathbb{C})} \to 0$ as $\varepsilon \to 0$, we use a clearing out result. Let us define

$$v_\varepsilon := e^{-i\varphi_\varepsilon} u'_\varepsilon \quad \text{and} \quad X_\varepsilon := A'_\varepsilon - \nabla\varphi_\varepsilon,$$

where φ_ε satisfies

$$\begin{cases} \Delta\varphi_\varepsilon = \operatorname{div} A'_\varepsilon & \text{in } \Omega \\ \nabla\varphi_\varepsilon \cdot \nu = A'_\varepsilon \cdot \nu & \text{on } \partial\Omega. \end{cases}$$

This implies that X_ε is in the Coulomb gauge, i.e. it satisfies

$$\begin{cases} \operatorname{div} X_\varepsilon = 0 & \text{in } \Omega \\ X_\varepsilon \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Since the configuration $(u_\varepsilon, A_\varepsilon)$ minimizes GL_ε in $H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$, it satisfies the Ginzburg–Landau equations (GL). By observing that the configurations $(u_\varepsilon, A_\varepsilon)$ and $(v_\varepsilon, X_\varepsilon + h_{\text{ex}} \operatorname{curl} B_0)$ are gauge equivalent in Ω, we deduce that v_ε satisfies

$$\begin{cases} -(\nabla_{X_\varepsilon + h_{\text{ex}} \operatorname{curl} B_0})^2 v_\varepsilon = \dfrac{1}{\varepsilon^2} v_\varepsilon(1 - |v_\varepsilon|^2) & \text{in } \Omega \\ \nabla_{X_\varepsilon + h_{\text{ex}} \operatorname{curl} B_0} v_\varepsilon \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Expanding the covariant Laplacian, and using (3.11) and $\operatorname{curl} B_0 \cdot \nu = 0$ on $\partial\Omega$, which follows from $B_0 \times \nu = 0$ on $\partial\Omega$, one can rewrite this problem in the form

$$\begin{cases} -\Delta v_\varepsilon + i|\log\varepsilon| c(x) \cdot \nabla v_\varepsilon + |\log\varepsilon|^2 d(x) v_\varepsilon = \dfrac{1}{\varepsilon^2} v_\varepsilon(1 - |v_\varepsilon|^2) & \text{in } \Omega \\ \nabla v_\varepsilon \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.12}$$

where

$$c(x) := \frac{2(X_\varepsilon + h_{\text{ex}} \operatorname{curl} B_0)}{|\log\varepsilon|} \quad \text{and} \quad d(x) := \frac{|X_\varepsilon + h_{\text{ex}} \operatorname{curl} B_0|^2}{|\log\varepsilon|^2}.$$

By Remark 3.1 and by standard elliptic regularity theory for solutions of the Ginzburg–Landau equations in the Coulomb gauge, we have

$$\|c\|_{L^\infty(\Omega,\mathbb{R}^3)}, \|\nabla c\|_{L^\infty(\Omega,\mathbb{R}^{3\times 3})}, \|d\|_{L^\infty(\Omega)}, \|\nabla d\|_{L^\infty(\Omega)} \le \Lambda_0 \tag{3.13}$$

for some constant $\Lambda_0 > 0$ independent of ε.

In addition, by gauge invariance, we have

$$F(u'_\varepsilon, A'_\varepsilon) = F_\varepsilon(v_\varepsilon, X_\varepsilon).$$

Since $(v_\varepsilon, X_\varepsilon)$ is in the Coulomb gauge, we have

$$E_\varepsilon(v_\varepsilon) := F_\varepsilon(v_\varepsilon, 0) \le C F_\varepsilon(v_\varepsilon, X_\varepsilon)$$

for some universal constant $C > 0$. We define $a_\varepsilon(x) = 1 - d(x)\varepsilon^2 |\log\varepsilon|^2$ and observe that

$$\tilde{E}_\varepsilon(v_\varepsilon) := \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon(x) - |v_\varepsilon|^2)^2 \le E_\varepsilon(v_\varepsilon) + O(\varepsilon|\log\varepsilon|^2).$$

This combined with (3.9), implies that

$$\tilde{E}_\varepsilon(v_\varepsilon) = o(|\log\varepsilon|^{-1}). \tag{3.14}$$

Finally, from (3.11), (3.12), (3.13), and (3.14), we conclude that all the hypotheses of [Chi05, Theorem 3] are fulfilled, and therefore

$$\|1 - |u_\varepsilon|\|_{L^\infty(\Omega,\mathbb{C})} = \|1 - |v_\varepsilon|\|_{L^\infty(\Omega,\mathbb{C})} \to 0 \quad \text{as } \varepsilon \to 0.$$

It is worth mentioning that one can also obtain (3.10) from the improved vorticity estimate in Proposition A.1. The proof is complete. □

4. The First Critical Field

Let us recall that, given a fixed $\varepsilon > 0$, the first critical field is defined as the value $H_{c_1} = H_{c_1}(\varepsilon)$ such that if $h_{\rm ex} < H_{c_1}$ and $(u_\varepsilon, A_\varepsilon)$ is a minimizer of GL_ε then $|u_\varepsilon| > 0$ in Ω, while if $h_{\rm ex} > H_{c_1}$ and $(u_\varepsilon, A_\varepsilon)$ minimizes GL_ε then u_ε must vanish in Ω.

Before giving the proof of Theorem 1.3, let us state a well-known result.

Lemma 4.1. *Let Γ be a multiplicity 1 rectifiable 1-current with $\partial\Gamma = 0$ relative to Ω. There exist constants $C_1, \varepsilon_1 > 0$ such that, for any $\varepsilon < \varepsilon_1$, there exists $v_\varepsilon \in H^1(\Omega, \mathbb{C})$ such that*

$$F_\varepsilon(v_\varepsilon, 0) \leq \pi|\Gamma|(\Omega)|\log\varepsilon| + C_1$$

and

$$\|\mu(v_\varepsilon, 0) - 2\pi\Gamma\|_{C_0^{0,1}(\Omega)^*} = o(|\log\varepsilon|^{-1}). \tag{4.1}$$

We refer the reader to the proof of Theorem 1.1 (ii) in [ABO05, Section 4] for a proof of this result. It is worth mentioning that the construction of v_ε relies on the existence of a map provided in [ABO03, Theorem 5.10]. Let us also point out that, arguing as in the proof of [JMS04, Proposition 3.2], one can replace the space $C_0^{0,1}(\Omega)^*$ by $C_T^{0,1}(\Omega)^*$ in the vorticity estimate (4.1). We will use this version of the result in the following proof.

Proof of Theorem 1.3. Theorem 1.2 immediately implies that

$$H_{c_1}^0 - K_0 \log|\log\varepsilon| \leq H_{c_1}.$$

It remains to prove that $H_{c_1} \leq H_{c_1}^0 + K_1$, for some constant K_1 sufficiently large. Given $K > 0$, let us assume towards a contradiction that $h_{\rm ex} = H_{c_1}^0 + K$ and $(u_\varepsilon, A_\varepsilon)$ minimizes GL_ε in $H^1(\Omega, \mathbb{C}) \times [A_{\rm ex} + H_{\rm curl}]$[2] with $|u_\varepsilon| > 0$.

Step 1. Estimating $GL_\varepsilon(u_\varepsilon, A_\varepsilon)$. We write $(u_\varepsilon, A_\varepsilon) = (u_0 u'_\varepsilon, h_{\rm ex}A_0 + A'_\varepsilon)$, where $(u_0, h_{\rm ex}A_0)$ is the approximation of the Meissner solution. Since $|u'_\varepsilon| = |u_\varepsilon| > 0$, we deduce that the 1-dimensional current ν'_ε associated to $(u'_\varepsilon, A'_\varepsilon)$ by Theorem 1.1 vanishes identically, and therefore, by taking n large enough, we have

$$\|\mu(u'_\varepsilon, A'_\varepsilon)\|_{C_T^{0,1}(\Omega)^*} \leq \frac{C}{|\log\varepsilon|^2}.$$

The energy-splitting (3.5) then yields

$$\begin{aligned} GL_\varepsilon(u_\varepsilon, A_\varepsilon) &= h_{\rm ex}^2 J(A_0) + F_\varepsilon(u'_\varepsilon, A'_\varepsilon) + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A'_\varepsilon|^2 + o(|\log\varepsilon|^{-1}) \\ &\geq h_{\rm ex}^2 J(A_0) + o(|\log\varepsilon|^{-1}). \end{aligned}$$

But since $(u_\varepsilon, A_\varepsilon)$ minimizes GL_ε, we have

$$GL_\varepsilon(u_\varepsilon, A_\varepsilon) \leq GL_\varepsilon(u_0, h_{\rm ex}A_0) = h_{\rm ex}^2 J(A_0).$$

Combining these inequalities, we find

$$GL_\varepsilon(u_\varepsilon, A_\varepsilon) = h_{\rm ex}^2 J(A_0) + o(|\log\varepsilon|^{-1}).$$

[2] This in particular implies that $(u_\varepsilon, A_\varepsilon)$ satisfies the Ginzburg–Landau equations (GL) and therefore u_ε is continuous.

Step 2. Definition of a vortex configuration. To reach a contradiction, we will show that there exists a configuration $(u_1^\varepsilon, A_1^\varepsilon)$, whose vorticity concentrates along the multiplicity 1 rectifiable 1-current Γ_1 with $\partial\Gamma_1 = 0$ relative to Ω that satisfies

$$\|B_0\|_* = \frac{1}{|\Gamma_1|(\Omega)} \int_\Omega \Gamma_1 \wedge B_0, \tag{4.2}$$

such that if $h_{\mathrm{ex}} \ge H_{c_1}^0 + K$ then $GL_\varepsilon(u_1^\varepsilon, A_1^\varepsilon) < GL_\varepsilon(u_\varepsilon, A_\varepsilon)$, provided $K \ge K_1$ for some constant K_1 independent of ε.

Lemma 4.1 with $\Gamma = \Gamma_1$ provides the existence of $v_\varepsilon \in H^1(\Omega, \mathbb{C})$ such that

$$F_\varepsilon(v_\varepsilon, 0) \le \pi|\Gamma_1|(\Omega)|\log\varepsilon| + C_1 \tag{4.3}$$

for some constant $C_1 > 0$ independent of ε, and

$$\|\mu(v_\varepsilon, 0) - 2\pi\Gamma_1\|_{C_T^{0,1}(\Omega)^*} = o(|\log\varepsilon|^{-1}). \tag{4.4}$$

Now, we let $(u_1^\varepsilon, A_1^\varepsilon)$ be defined by

$$u_1^\varepsilon = u_0 v_\varepsilon, \quad A_1^\varepsilon = h_{\mathrm{ex}} A_0.$$

Proposition 3.1 yields

$$GL_\varepsilon(u_1^\varepsilon, A_1^\varepsilon) = h_{\mathrm{ex}}^2 J(A_0) + F_\varepsilon(v_\varepsilon, 0) - h_{\mathrm{ex}} \int_\Omega \mu(v_\varepsilon, 0) \wedge B_0 + R_0. \tag{4.5}$$

From (4.2) and (4.4), we get

$$\int_\Omega \mu(v_\varepsilon, 0) \wedge B_0 = 2\pi\|B_0\|_*|\Gamma_1|(\Omega) + o(|\log\varepsilon|^{-1}).$$

Inserting this and (4.3) into (4.5), we are led to

$$GL_\varepsilon(u_1^\varepsilon, A_1^\varepsilon) \le h_{\mathrm{ex}}^2 J(A_0) + \pi|\Gamma_1|(\Omega)|\log\varepsilon| + C_1 - 2\pi\|B_0\|_* h_{\mathrm{ex}}|\Gamma_1|(\Omega) + o(h_{\mathrm{ex}}|\log\varepsilon|^{-1}).$$

Step 3. Contradiction. Writing $h_{\mathrm{ex}} = H_{c_1}^0 + K$ with $H_{c_1}^0 = \dfrac{1}{2\|B_0\|_*}|\log\varepsilon|$, we get

$$\begin{aligned} GL_\varepsilon(u_1^\varepsilon, A_1^\varepsilon) &\le h_{\mathrm{ex}}^2 J(A_0) + \pi|\Gamma_1|(\Omega)|\log\varepsilon| + C_1 - \pi|\Gamma_1|(\Omega)\left(|\log\varepsilon| + 2\|B_0\|_* K\right) + o(1) \\ &= h_{\mathrm{ex}}^2 J(A_0) + C_1 - 2\pi\|B_0\|_* K|\Gamma_1|(\Omega) + o(1). \end{aligned}$$

By choosing $K_1 := (2\pi\|B_0\|_*|\Gamma_1|(\Omega))^{-1} C_1 + 1$, we deduce that, for any $K \ge K_1$,

$$GL_\varepsilon(u_1^\varepsilon, A_1^\varepsilon) \le h_{\mathrm{ex}}^2 J(A_0) - 1 + o(1) < GL_\varepsilon(u_\varepsilon, A_\varepsilon).$$

Therefore, provided $K \ge K_1$, this contradicts the fact that $(u_\varepsilon, A_\varepsilon)$ globally minimizes GL_ε. Thus

$$H_{c_1} \le H_{c_1}^0 + K_1.$$

□

Remark 4.1. *The isoperimetric inequality allows to prove that if* $\Gamma \in X$ *has small length then the ratio* $\dfrac{\int_\Omega \Gamma \wedge B_0}{|\Gamma|(\Omega)}$ *is small.*

Indeed, if Γ *is a loop contained in* Ω *then, by Stokes' theorem, we have*

$$\int_\Omega \Gamma \wedge B_0 = \int_{S_\Gamma} \operatorname{curl} B_0,$$

where S_Γ *denotes a surface with least area among those whose boundary is* Γ*, i.e. a solution to the associated Plateau's problem. By the isoperimetric inequality, we have*

$$\int_{S_\Gamma} |\operatorname{curl} B_0| \le \|\operatorname{curl} B_0\|_{L^\infty(\Omega,\mathbb{R}^3)} \operatorname{Area}(S_\Gamma) \le C|\Gamma|(\Omega)^2.$$

On the other hand, if both different endpoints of Γ *belong to* $\partial\Omega$*, we consider the geodesic connecting the endpoints of* Γ *on* $\partial\Omega$*, oriented accordingly to the orientation of* Γ*. We then denote by* $\tilde{\Gamma}$ *the loop formed by the union of* Γ *and this geodesic. Since* $B_0 \times \nu = 0$ *on* $\partial\Omega$*, by Stokes' theorem, we have*

$$\int_\Omega \Gamma \wedge B_0 = \int_{S_\Gamma} \operatorname{curl} B_0,$$

where S_Γ *denotes a surface with least area among those whose boundary is* $\tilde{\Gamma}$*. Arguing as above, we conclude that*

$$\int_{S_\Gamma} |\operatorname{curl} B_0| \le \|\operatorname{curl} B_0\|_{L^\infty(\Omega,\mathbb{R}^3)} \operatorname{Area}(S_\Gamma) \le C \operatorname{Length}(\tilde{\Gamma})^2 \le C(\partial\Omega)|\Gamma|(\Omega)^2.$$

Therefore,

$$\frac{1}{|\Gamma|(\Omega)} \int_\Omega |\Gamma \wedge B_0| \le C|\Gamma|(\Omega),$$

from which the assertion follows.

Moreover, this property extends to the class of multiplicity 1 *rectifiable* 1*-currents* Γ *with* $\partial\Gamma = 0$ *relative to* Ω*, since the action of a* 1*-current in this class on a vector field can be seen as oriented integration over a countable family of Lipschitz curves in* X*. In particular, since* $\|B_0\|_* > 0$*, we deduce that* $|\Gamma_1|(\Omega) \ge C > 0$*, where* Γ_1 *is the* 1*-current that appears in the statement of Theorem* 1.3 *and C is a constant that depends on* B_0 *and* Ω *only.*

Let us now study $\|B_0\|_*$ in a special case.

Proposition 4.1. *Consider the special case* $\Omega = B(0, R)$ *and* $H_{0,\mathrm{ex}} = \hat{z}$ *in* $B(0, R)$*. Then, if* S_1 *denotes the vertical diameter seen as a* 1*-current with multiplicity* 1 *and oriented in the direction of positive z axis, we have*

$$\|B_0\|_* = \frac{1}{|S_1|(\Omega)} \int_\Omega S_1 \wedge B_0 = \frac{1}{2R} \int_{-R}^{R} B_0(0,0,z) \cdot \hat{z} dz = \frac{3}{2}\left(1 - \frac{1}{\sinh R} \int_0^R \frac{\sinh r}{r} dr\right).$$

Moreover, S_1 *is the only curve in* X *achieving the maximum in* (1.3)*.*

Proof. We use some ideas from [ABM06].

Step 1. Explicit computation of B_0. When $\Omega = B(0, R)$ and $H_{0,\mathrm{ex}} = \hat{z}$ in $B(0, R)$, the solution to (3.4) can be explicitly computed (see [Lon50]). By using spherical coordinates (r, θ, ϕ), where r is the Euclidean distance from the origin, θ is the azimuthal angle, and ϕ is the polar angle, we have

$$B_0 = -\frac{3R}{r^2 \sinh R}\left(\cosh r - \frac{\sinh r}{r}\right)\cos\phi \hat{r}$$
$$-\frac{3R}{2r^2 \sinh R}\left(\cosh r - \frac{1+r^2}{r}\sinh r\right)\sin\phi\hat{\phi} - c\hat{z},$$

where $c = \dfrac{3}{2R \sinh R}\left(\cosh R - \dfrac{1+R^2}{R}\sinh R\right)$. In particular, we observe that B_0 does not depend on the azimuthal angle and therefore it is constant along $\hat{\theta}$.

Step 2. Dimension reduction. Let $\Gamma \in X$ with $\int_{B(0,R)} \Gamma \wedge B_0 > 0$. We will project it along the azimuthal angle onto $B(0, R)^{2\mathrm{D},+} := \{(x, z) \in \mathbb{R}^2 \mid x^2 + z^2 < R^2,\ x \geq 0\}$. For this, we consider the map $q : B(0, R) \subset \mathbb{R}^3 \to B(0, R)^{2\mathrm{D},+}$ defined by

$$q(r, \theta, \phi) = (r \sin\phi, r\cos\phi),$$

and we let

$$\Gamma_{2\mathrm{D}} := q \circ \Gamma.$$

It is easy to check that $\partial\Gamma_{2\mathrm{D}} = 0$ relative to $B(0, R)^{2\mathrm{D}}$,

$$\int_{B(0,R)} \Gamma \wedge B_0 = \int_{B(0,R)^{2\mathrm{D},+}} \Gamma_{2\mathrm{D}} \wedge B_0, \quad \text{and} \quad |\Gamma_{2\mathrm{D}}|(B(0, R)^{2\mathrm{D},+}) \leq |\Gamma|(B(0, R)).$$

Therefore

$$\frac{1}{|\Gamma|(\Omega)}\int_{B(0,R)} \Gamma \wedge B_0 \leq \frac{1}{|\Gamma_{2\mathrm{D}}|(\Omega)}\int_{B(0,R)^{2\mathrm{D},+}} \Gamma_{2\mathrm{D}} \wedge B_0.$$

Even though $\Gamma_{2\mathrm{D}}$ does not necessarily belong to X, we can decompose

$$\Gamma_{2\mathrm{D}} = \sum_{i\in I}\Gamma_i,$$

where the sum is understood in the sense of currents, I is a finite set of indices, and $\Gamma_i \in X$ for all $i \in I$. In particular,

$$\int_{B(0,R)^{2\mathrm{D},+}} \Gamma_{2\mathrm{D}} \wedge B_0 \leq \sum_{i\in I}|\Gamma_i|(B(0, R)^{2\mathrm{D},+})\|B_0\|_* = |\Gamma_{2\mathrm{D}}|(B(0, R)^{2\mathrm{D},+})\|B_0\|_*.$$

We deduce that in order to compute $\|B_0\|_*$ it is enough to consider Lipschitz curves $\Gamma \in X$ contained in $B(0, R)^{2\mathrm{D},+}$ with $\int_{B(0,R)} \Gamma \wedge B_0 > 0$. From now on we consider Γ of this form.

Step 3. Application of Stokes' theorem. If Γ has both endpoints on $\partial B(0, R) \cap \partial B(0, R)^{2\mathrm{D},+}$, we then define $\tilde{\Gamma}$ as the loop formed by the union of Γ and the curve lying on $\partial B(0,R)\cap\partial B(0,R)^{2D,+}$ which connects the endpoints of Γ oriented accordingly to the orientation of Γ. Since $B_0\times\nu=0$ on $\partial B(0,R)$, the Stokes' theorem yields

$$\int_{B(0,R)^{2D,+}}\Gamma\wedge B_0=\int_{B(0,R)^{2D,+}}\tilde{\Gamma}\wedge B_0=\int_{S_\Gamma}\operatorname{curl}B_0\cdot\hat{y}, \tag{4.6}$$

where S_Γ is the surface enclosed by $\tilde{\Gamma}$. Of course if Γ is a loop contained in $B(0,R)^{2D,+}$ then the Stokes' theorem gives

$$\int_{B(0,R)^{2D,+}}\Gamma\wedge B_0=\int_{S_\Gamma}\operatorname{curl}B_0\cdot\hat{y},$$

where S_Γ is the surface enclosed by Γ.

An explicit computation gives

$$\operatorname{curl}B_0\cdot\hat{y}=\frac{3R}{2\sinh R}\left(\cosh r-\frac{\sinh r}{r}\right)\frac{\sin\phi}{r}\geq 0\quad\text{in } B(0,R)^{2D,+}. \tag{4.7}$$

In what follows we use the notation

$$f(r):=\frac{3R}{2\sinh R}\left(\cosh r-\frac{\sinh r}{r}\right).$$

Step 4. Estimate for curves with endpoints on $\partial B(0,R)\cap\partial B(0,R)^{2D,+}$. For $a,b\in[0,\pi]$ with $a<\pi-b$ let us define

$$S_{a,b}:=\{(r,\phi)\mid 0\leq r\leq R,\ a\leq\phi\leq\pi-b\}.$$

We let ϕ_1,ϕ_2 be the maximum angles for which $S_\Gamma\subset S_{\phi_1,\phi_2}$. From (4.6) and (4.7), we deduce that

$$\begin{aligned}\int_{B(0,R)^{2D,+}}\Gamma\wedge B_0\leq\int_{S_{\phi_1,\phi_2}}\operatorname{curl}B_0\cdot\hat{y}&=\int_0^R\int_{\phi_1}^{\pi-\phi_2}f(r)\sin\phi\, d\phi\, dr\\&=(\cos\phi_1+\cos\phi_2)\int_0^R f(r)dr.\end{aligned}$$

On the other hand, by definition of ϕ_1,ϕ_2, S_Γ intersects the rays $\{(r,\phi_1)\mid 0\leq r\leq R\}$ and $\{(r,\phi_2)\mid 0\leq r\leq R\}$. Since the endpoints of Γ belong to $\partial B(0,R)\cap\partial B(0,R)^{2D,+}$, a simple geometric argument shows that

$$|\Gamma|(B(0,R)^{2D,+})\geq d((R,\phi_1),(R,\phi_2)).$$

The law of cosines yields $d((R,\phi_1),(R,\phi_2))=R\sqrt{2(1-\cos(\pi-\phi_1-\phi_2))}$. Hence

$$\frac{1}{|\Gamma|(B(0,R)^{2D,+})}\int_{B(0,R)^{2D,+}}\Gamma\wedge B_0\leq\frac{\cos\phi_1+\cos\phi_2}{\sqrt{2(1-\cos(\pi-\phi_1-\phi_2))}}\frac{\int_0^R f(r)dr}{R}.$$

We now estimate the right-hand side of this inequality. Let us observe that

$$\cos\phi_1+\cos\phi_2=2\cos\left(\frac{\phi_1+\phi_2}{2}\right)\cos\left(\frac{\phi_1-\phi_2}{2}\right)$$

and

$$\cos(\pi-\phi_1-\phi_2)=\cos(\phi_1+\phi_2)=\cos^2\left(\frac{\phi_1+\phi_2}{2}\right)-\sin^2\left(\frac{\phi_1+\phi_2}{2}\right)$$
$$=2\cos^2\left(\frac{\phi_1+\phi_2}{2}\right)-1.$$

Using $0\le\frac{\phi_1+\phi_2}{2}<\frac{\pi}{2}$, we deduce that

$$\frac{\cos\phi_1+\cos\phi_2}{\sqrt{2(1-\cos(\pi-\phi_1-\phi_2))}}=\cos\left(\frac{\phi_1-\phi_2}{2}\right)\le 1,$$

with equality if and only if $\phi_1=\phi_2$. Therefore

$$\frac{1}{|\Gamma|(B(0,R)^{2\mathrm{D},+})}\int_{B(0,R)^{2\mathrm{D},+}}\Gamma\wedge B_0\le\frac{\int_0^R f(r)dr}{R}=\frac{3}{2}\left(1-\frac{1}{\sinh R}\int_0^R\frac{\sinh r}{r}dr\right)$$
$$=\frac{1}{2R}\int_{B(0,R)}S_1\wedge B_0.$$

Besides, from the previous computations we easily deduce that the inequality is strict if $\Gamma\neq S_1$.

Step 5. Estimate for loops in $B(0,R)^{2\mathrm{D},+}$. Let us define $0<r_0<R$ as the minimum radius such that

$$S_\Gamma\subset B(0,r_0)^{2\mathrm{D},+}.$$

In particular, $S_\Gamma\cap(\partial B(0,r_0)\cap\partial B(0,r_0)^{2\mathrm{D},+})\neq\emptyset$. We can then use the estimate provided in the previous step and conclude that

$$\frac{1}{|\Gamma|(B(0,R)^{2\mathrm{D},+})}\int_{B(0,R)^{2\mathrm{D},+}}\Gamma\wedge B_0\le\frac{3}{2}\left(1-\frac{1}{\sinh r_0}\int_0^{r_0}\frac{\sinh r}{r}dr\right).$$

One can check that the function $t\to\frac{1}{\sinh t}\int_0^t\frac{\sinh r}{r}dr$ is strictly decreasing in $[0,\infty)$ and therefore

$$\frac{1}{|\Gamma|(B(0,R)^{2\mathrm{D},+})}\int_{B(0,R)^{2\mathrm{D},+}}\Gamma\wedge B_0<\frac{3}{2}\left(1-\frac{1}{\sinh R}\int_0^R\frac{\sinh r}{r}dr\right).$$

This concludes the proof of the proposition. □

5. A Meissner-Type Solution Beyond the First Critical Field

In this section, we present the proof of Theorem 1.4.

Proof. **Step 1. Existence of a locally minimizing vortexless configuration.** Let us introduce the set

$$U = \left\{ (u, A) \in H^1(\Omega, \mathbb{C}) \times [A_{\rm ex} + H_{\rm curl}] \mid F_\varepsilon(u', A') < \varepsilon^{\frac{2}{3}} \right\},$$

where $u' = u_0^{-1} u$ and $A' = A - h_{\rm ex} A_0$. Consider a minimizing sequence $\{(\tilde{u}_n, \tilde{A}_n)\}_n \in U$. Lemma 2.1 yields a gauge transformed sequence $\{(u_n, A_n)\}_n \in H^1(\Omega, \mathbb{C}) \times [A_{\rm ex} + \dot{H}^1_{\rm div=0}]$ that, in particular, satisfies $F_\varepsilon(u_n', A_n') = F_\varepsilon(\tilde{u}_n', \tilde{A}_n') < \varepsilon^{\frac{2}{3}}$. Then arguing as in Proposition 2.2, we deduce that (up to subsequence) $\{(u_n, A_n - A_{\rm ex})\}_n$ converges to some $(u, A - A_{\rm ex})$ weakly in $H^1(\Omega, \mathbb{C}) \times \dot{H}^1_{\rm div=0}$. Arguing again as in Proposition 2.2, we find

$$F_\varepsilon(u', A') \leq \liminf_n F_\varepsilon(u_n', A_n') \quad \text{and} \quad GL_\varepsilon(u, A) \leq \liminf_n GL_\varepsilon(u_n, A_n).$$

Hence, $(u, A) \in \overline{U} \cap H^1(\Omega, \mathbb{C}) \times [A_{\rm ex} + \dot{H}^1_{\rm div=0}]$ minimizes GL_ε over $\overline{U}$.

Let us now prove that $(u, A) \in U$. We consider, for $\delta = \delta(\varepsilon) = c_1 \varepsilon^{\frac{1}{3}}$ and ε sufficiently small, the grid $\mathfrak{G}(b_\varepsilon, R_0, \delta)$ associated to (u', A') by [Rom19, Lemma 2.1] with $\gamma = -\frac{2}{3}$. In particular, using the same notation as in this lemma, we have

$$\begin{aligned} & |u_\varepsilon| > 5/8 \quad \text{on } \mathfrak{R}_1(\mathfrak{G}(b_\varepsilon, R_0, \delta)), \\ & I_\varepsilon^1 := \int_{\mathfrak{R}_1(\mathfrak{G}(b_\varepsilon, R_0, \delta))} e_\varepsilon(u', A') d\mathcal{H}^1 \leq C\delta^{-2} F_\varepsilon(u', A') \leq C\varepsilon^{-\frac{2}{3}} \varepsilon^{\frac{2}{3}}, \\ & I_\varepsilon^2 := \int_{\mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, R_0, \delta))} e_\varepsilon(u', A') d\mathcal{H}^2 \leq C\delta^{-1} F_\varepsilon(u', A') \leq C\varepsilon^{-\frac{1}{3}} \varepsilon^{\frac{2}{3}}, \end{aligned} \tag{5.1}$$

where C is a universal constant.

We claim that if ε is small enough then for each face ω of a cube of the grid, every connected component of $\{x \in \omega \mid |u'(x)| \leq 1/2\}$ has degree zero. Assume towards a contradiction that there exist a face ω and a connected component S_ω of $\{x \in \omega \mid |u'(x)| \leq 1/2\}$ whose degree $d_{S_\omega} = \deg(u'/|u'|, \partial S_\omega)$ is different from zero. By [Rom19, Lemma 4.1], a result adapted from [Jer99], we have

$$|d_{S_\omega}| \leq C \int_{S_\omega} |\nabla_{A'} u'|^2,$$

where C is a universal constant. Combining this with (5.1), we get

$$|d_{S_\omega}| \leq C I_\varepsilon^2 \leq C\varepsilon^{\frac{1}{3}},$$

and therefore if ε is sufficiently small we reach a contradiction.

We thus deduce that the 1-current ν_ε', which approximates well the vorticity $\mu(u', A')$, vanishes identically in $\overline{\Omega}$. Then, from the proof of Theorem 1.1 (see [Rom19, Section 8]), we find

$$\|\mu(u', A')\|_{C_T^{0,1}(\Omega)^*} \leq C\delta F_\varepsilon(u', A') + C\varepsilon(1 + I_\varepsilon^1 + I_\varepsilon^2) \leq C(\delta + \varepsilon\delta^{-2}) F_\varepsilon(u', A').$$

Let us now use Proposition 3.1. From the previous inequality and since $\alpha < \frac{1}{3}$, we have

$$\left| h_{\mathrm{ex}} \int_\Omega \mu(u', A') \wedge B_0 \right| \le C h_{\mathrm{ex}} (\delta + \varepsilon\delta^{-2}) F_\varepsilon(u', A') \le C \varepsilon^{\frac{1}{3}-\alpha} F_\varepsilon(u', A') = o(\varepsilon^{\frac{2}{3}}). \tag{5.2}$$

On the other hand

$$R_0 \le C\varepsilon h_{\mathrm{ex}}^2 E_\varepsilon(|u'|)^{\frac{1}{2}} \le C\varepsilon h_{\mathrm{ex}}^2 F_\varepsilon(u', A')^{\frac{1}{2}} \le C\varepsilon^{1-2\alpha}\varepsilon^{\frac{1}{3}} = o(\varepsilon^{\frac{2}{3}}).$$

The energy-splitting (3.5) then yields

$$GL_\varepsilon(u, A) = h_{\mathrm{ex}}^2 J(A_0) + F_\varepsilon(u', A') + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A'|^2 + o(\varepsilon^{\frac{2}{3}}).$$

But, since $(u_0, h_{\mathrm{ex}} A_0)$ belongs to U, we have

$$GL_\varepsilon(u, A) \le GL_\varepsilon(u_0, h_{\mathrm{ex}} A_0) = h_{\mathrm{ex}}^2 J(A_0).$$

We thus deduce that

$$F_\varepsilon(u', A') + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A'|^2 = o(\varepsilon^{\frac{2}{3}}), \tag{5.3}$$

and therefore $(u, A) \in U$ provided ε is small enough.

Now, since U is open in $H^1(\Omega, \mathbb{C}) \times [A_{\mathrm{ex}} + H_{\mathrm{curl}}]$, the minimizer (u, A) must be a critical point of GL_ε and therefore satisfies the Ginzburg–Landau equations (GL). Arguing as in the proof of Theorem 1.2, we deduce that (u, A) is a vortexless configuration such that

$$\|1 - |u|\|_{L^\infty(\Omega,\mathbb{C})} = \|1 - |u'|\|_{L^\infty(\Omega,\mathbb{C})} = o(1) \quad \text{as } \varepsilon \to 0.$$

We note that we have omitted in our notation the dependence on ε of the minimizer (u, A).

Step 2. Characterization of (u', A'). From (5.3), we have $\|\operatorname{curl} A'\|^2_{L^2(\mathbb{R}^3,\mathbb{R}^3)} = o(\varepsilon^{\frac{2}{3}})$, which combined with the fact that $A' = A - h_{\mathrm{ex}} A_0 \in \dot{H}^1_{\mathrm{div}=0}$ implies that

$$\|A'\|^2_{\dot{H}^1_{\mathrm{div}=0}} = o(\varepsilon^{\frac{2}{3}}).$$

Observe that

$$\int_\Omega |\nabla u'|^2 \le \int_\Omega |\nabla_{A'} u'|^2 + |A'|^2 |u'|^2.$$

Since $\|1 - |u'|\|_{L^\infty(\Omega,\mathbb{C})} = o(1)$ as $\varepsilon \to 0$ and $\|A'\|_{L^2(\Omega,\mathbb{R}^3)} \le C\|\operatorname{curl} A'\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}$, we deduce that

$$\int_\Omega |\nabla u'|^2 \le C\left(F_\varepsilon(u', A') + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A'|^2\right),$$

which combined with (5.3), gives

$$\int_\Omega |\nabla u'|^2 = o(\varepsilon^{\frac{2}{3}}). \tag{5.4}$$

On the other hand, using the Poincaré-Wirtinger inequality, we have

$$\int_\Omega |u' - \underline{u}'|^2 \le C \int_\Omega |\nabla u'|^2, \quad \text{where } \underline{u}' = \frac{1}{|\Omega|}\int_\Omega u'. \tag{5.5}$$

In addition, we have

$$\int_\Omega \big||u'| - |\underline{u}'|\big|^2 \le \int_\Omega |u' - \underline{u}'|^2$$

and

$$\int_\Omega (1 - |u'|)^2 \le \int_\Omega (1 - |u'|^2)^2 \le 4\varepsilon^2 F_\varepsilon(u', A') \le 4\varepsilon^{2+\frac{2}{3}}.$$

We deduce that $\|1 - |\underline{u}'|\|_{L^2(\Omega,\mathbb{C})} \le C\varepsilon^{1+\frac{1}{3}}$. But $\underline{u}'$ is a constant, thus $\underline{u}' = e^{i\theta_\varepsilon} + O(\varepsilon^{1+\frac{1}{3}})$ for some $\theta_\varepsilon \in [0, 2\pi]$. By combining with (5.5) and (5.4), we find

$$\int_\Omega |u' - e^{i\theta_\varepsilon}|^2 = o(\varepsilon^{\frac{2}{3}}). \tag{5.6}$$

Thus

$$\inf_{\theta\in[0,2\pi]} \|u' - e^{i\theta}\|_{H^1(\Omega,\mathbb{C})} \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.7}$$

In particular, by noting that $(e^{i\theta}, h_{\text{ex}} \operatorname{curl} B_0)$ is gauge equivalent to $(1, h_{\text{ex}} \operatorname{curl} B_0)$ in Ω for any $\theta \in [0, 2\pi]$, we deduce that (up a gauge transformation) the configuration $(u', A' + h_{\text{ex}} \operatorname{curl} B_0)$, which is gauge equivalent to (u, A) in Ω, converges in the $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$-norm to $(1, h_{\text{ex}} \operatorname{curl} B_0)$.

Step 3. (u, A) **globally approaches** $(u_0, h_{\text{ex}} A_0)$. Observe that, for any $\theta \in [0, 2\pi]$, we have

$$\int_\Omega |u - e^{i\theta} u_0|^2 = \int_\Omega |u' u_0 - e^{i\theta} u_0|^2 = \int_\Omega |u' - e^{i\theta}|^2$$

and

$$\int_\Omega |\nabla(u - e^{i\theta} u_0)|^2 \le \int_\Omega |\nabla u_0|^2 |u' - e^{i\theta}|^2 + \int_\Omega |\nabla u'|^2.$$

From (5.7), we deduce that

$$\inf_{\theta\in[0,2\pi]} \|u - e^{i\theta} u_0\|_{L^2(\Omega,\mathbb{C})} \to 0 \quad \text{as } \varepsilon \to 0.$$

Recall that $u_0 = e^{i h_{\text{ex}} \phi_0}$ and that A_0 satisfies the Euler-Lagrange equation (3.3). Since $\operatorname{curl}(H_0 - H_{0,\text{ex}}) = \operatorname{curl}^2(A_0 - A_{0,\text{ex}}) = -\Delta(A_0 - A_{0,\text{ex}})$, standard elliptic regularity theory implies that $\phi_0 = A_0 - \operatorname{curl} B_0 \in L^\infty(\Omega)$. Therefore

$$\int_\Omega |\nabla u_0|^2 |u' - e^{i\theta}|^2 \le h_{\text{ex}}^2 \|\nabla\phi_0\|^2_{L^\infty(\Omega)} \|u' - e^{i\theta}\|^2_{L^2(\Omega,\mathbb{C})}.$$

This combined with (5.6) for $\theta = \theta_\varepsilon$, yields

$$\int_\Omega |\nabla u_0|^2 |u' - e^{i\theta_\varepsilon}|^2 = o(\varepsilon^{-2\alpha}\varepsilon^{\frac{2}{3}}).$$

Since $\alpha < \frac{1}{3}$, the right-hand side converges to 0 as $\varepsilon \to 0$. Using once again (5.7), we obtain

$$\inf_{\theta\in[0,2\pi]} \int_\Omega |\nabla(u - e^{i\theta}u_0)|^2 \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence

$$\inf_{\theta\in[0,2\pi]} \|u - e^{i\theta}u_0\|_{H^1(\Omega,\mathbb{C})} \to 0 \quad \text{as } \varepsilon \to 0.$$

Besides, we have

$$\|A - h_{\text{ex}}A_0\|_{\dot{H}^1_{\text{div}=0}} = \|A'\|_{\dot{H}^1_{\text{div}=0}} \to 0 \quad \text{as } \varepsilon \to 0.$$

We have hence shown that, up to a gauge transformation in $\mathbb{R}^3$, the solution (u, A) converges in the $H^1(\Omega,\mathbb{C}) \times \dot{H}^1_{\text{div}=0}$-norm to $(u_0, h_{\text{ex}}A_0)$. In addition, up to a (different) gauge transformation in Ω, the solution approaches in the $H^1(\Omega,\mathbb{C}) \times H^1(\Omega,\mathbb{R}^3)$-norm the configuration $(1, h_{\text{ex}} \operatorname{curl} B_0)$. □

Remark 5.1. *The assumption $h_{\text{ex}} \leq \varepsilon^{-\alpha}$ for $\alpha < \frac{1}{3}$ is needed to prove that*

$$\left| h_{\text{ex}} \int_\Omega \mu(u', A') \wedge B_0 \right| \leq o(F_\varepsilon(u', A'));$$

see (5.2). If $\alpha \geq \frac{1}{3}$, we are not able to show this, and our strategy to prove that $(u, A) \in U$ then fails.

6. Uniqueness of Locally Minimizing Vortexless Configurations

In this section we prove Theorem 1.5. We follow the same strategy as in [Ser99b, Section 2].

Proof. First, let us observe that any pair $(\tilde{v}, \tilde{B}) \in H^1(\Omega,\mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ is gauge-equivalent to a pair $(v, B) \in H^1(\Omega,\mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ that satisfies

$$\begin{cases} \operatorname{div} B = 0 & \text{in } \Omega \\ B \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Indeed, by letting

$$v = e^{-i\varphi}\tilde{v} \quad \text{and} \quad B := \tilde{B} - \nabla\varphi,$$

where φ satisfies

$$\begin{cases} \Delta\varphi = \operatorname{div} \tilde{B} & \text{in } \Omega \\ \nabla\varphi \cdot \nu = \tilde{B} \cdot \nu & \text{on } \partial\Omega \end{cases}$$

and is extended to a function in $H^2(\mathbb{R}^3)$, we immediately verify that B satisfies (6.1). We say that (v, B) is in the Coulomb gauge.

Let us assume towards a contradiction that there are two distinct locally minimizing vortexless solutions $(u_j, A_j) = (u_0 u_j', h_{\text{ex}} A_0 + A_j')$ to (GL) with $(u_j, A_j) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$, $|u_j| \geq c$ for some $c \in (0, 1)$, and

$$F_\varepsilon(u_j', A_j') \leq C\varepsilon^{1+\delta} \quad \text{for } j = 1, 2,$$

for some $\delta > 0$. As we shall see, this estimate is crucial to prove the theorem.

By gauge invariance, we may assume that (u_j', A_j') is in the Coulomb gauge for $j = 1, 2$. Since $|u_j'| = |u_j| \geq c > 0$, we can write $u_j' = \eta_j e^{i\phi_j}$ in Ω for $j = 1, 2$. Note that the functions $\phi_0, \phi_1, \phi_2 \in H^2(\Omega)$ can be extended to functions in $H^2(\mathbb{R}^3)$. Therefore, for $j = 1, 2$, (u_j, A_j) is gauge equivalent to $(\eta_j, \tilde{A}_j)$ with

$$\tilde{A}_j = h_{\text{ex}}(A_0 - \nabla\phi_0) + A_j' - \nabla\phi_j.$$

Step 1. Estimating $\|\tilde{A}_j\|_{L^\infty(\Omega,\mathbb{R}^3)}$. Let us show that, for $j = 1, 2$, we have

$$\|\tilde{A}_j\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq o(\varepsilon^{-1}). \tag{6.2}$$

By gauge equivalence, $(u_j', h_{\text{ex}}(A_0 - \nabla\phi_0) + A_j')$ solves (GL). We observe that this pair is in the Coulomb gauge. Then, by standard elliptic regularity theory for solutions of the Ginzburg–Landau equations in the Coulomb gauge, we have

$$\|h_{\text{ex}} \operatorname{curl} B_0 + A_j'\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq Ch_{\text{ex}} \quad \text{and} \quad \|\nabla u_j'\|_{L^\infty(\Omega,\mathbb{C}^3)} \leq C\varepsilon^{-1}.$$

Since

$$\|\nabla\eta_j\|_{L^\infty(\Omega,\mathbb{R}^3)} + \|\nabla\phi_j\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq 2\|\nabla u_j'\|_{L^\infty(\Omega,\mathbb{C}^3)}, \tag{6.3}$$

and $h_{\text{ex}} = o(\varepsilon^{-1})$, we find

$$\|\tilde{A}_j\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq \|h_{\text{ex}} \operatorname{curl} B_0 + A_j'\|_{L^\infty(\Omega,\mathbb{R}^3)} + \|\nabla\phi_j\|_{L^\infty(\Omega,\mathbb{R}^3)} \leq C\varepsilon^{-1}. \tag{6.4}$$

We will now improve this estimate. By gauge equivalence, $(\eta_j, \tilde{A}_j)$ solves (GL). In particular, the second Ginzburg–Landau equation in Ω reads

$$\operatorname{curl}^2(\tilde{A}_j - A_{\text{ex}}) = -\eta_j^2 \tilde{A}_j.$$

This implies that $\operatorname{div}(\eta_j^2 \tilde{A}_j) = 0$ in Ω. In addition, the boundary condition $\nabla_{\tilde{A}_j}\eta_j \cdot \nu = 0$ on $\partial\Omega$, implies, in particular, that $\nabla\phi_j \cdot \nu = 0$ on $\partial\Omega$. Therefore, ϕ_j satisfies the elliptic problem

$$\begin{cases} \Delta\phi_j = \dfrac{2}{\eta_j}\nabla\eta_j \cdot \tilde{A}_j & \text{in } \Omega \\ \nabla\phi_j \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Because $\eta_j \geq c > 0$, we deduce that, for any $p > 1$,

$$\|\Delta\phi_j\|_{L^p(\Omega)} \leq C\|\tilde{A}_j\|_{L^\infty(\Omega,\mathbb{R}^3)}\|\nabla\eta_j\|_{L^p(\Omega,\mathbb{R}^3)} \leq C\varepsilon^{-1}\|\nabla\eta_j\|_{L^p(\Omega,\mathbb{R}^3)}, \tag{6.5}$$

where the last inequality is obtained by using (6.4).

On the other hand, since $A_j' \cdot \nu = 0$ on $\partial\Omega$, we have

$$\|\nabla u_j'\|^2_{L^2(\Omega,\mathbb{C}^3)}, \ \|A_j'\|^2_{L^2(\Omega,\mathbb{R}^3)} \leq CF_\varepsilon(u_j', A_j') \leq C\varepsilon^{1+\delta}.$$

This implies that

$$\int_\Omega |\nabla \eta_j|^2 + \eta_j^2 |\nabla \phi_j|^2 = \int_\Omega |\nabla u'_j|^2 \le C\varepsilon^{1+\delta}.$$

In addition, by interpolation, for any $p > 1$, we have

$$\|\nabla \eta_j\|_{L^p(\Omega,\mathbb{R}^3)} \le C\|\nabla \eta_j\|_{L^\infty(\Omega,\mathbb{R}^3)}^{1-\frac{2}{p}} \|\nabla \eta_j\|_{L^2(\Omega,\mathbb{R}^3)}^{\frac{2}{p}}.$$

Combining the previous two inequalities with (6.3), yields

$$\|\nabla \eta_j\|_{L^p(\Omega,\mathbb{R}^3)} \le C\varepsilon^{-1+\frac{2}{p}} \varepsilon^{\frac{1+\delta}{p}} = C\varepsilon^{\frac{3+\delta-p}{p}}.$$

Combining with (6.5) for $p = 3 + \frac{\delta}{2} > 3$, we find

$$\|\Delta \phi_j\|_{L^p(\Omega)} \le C\varepsilon^{-1}\varepsilon^{\frac{\delta}{6+\delta}} = o(\varepsilon^{-1}).$$

By an elliptic estimate and Sobolev embedding, we then obtain

$$\|\nabla \phi_j\|_{L^\infty(\Omega,\mathbb{R}^3)} \le o(\varepsilon^{-1}).$$

Thus

$$\|\tilde{A}_j\|_{L^\infty(\Omega,\mathbb{R}^3)} \le \|h_{\rm ex} \operatorname{curl} B_0 + A'_j\|_{L^\infty(\Omega,\mathbb{R}^3)} + \|\nabla \phi_j\|_{L^\infty(\Omega,\mathbb{R}^3)} \le Ch_{\rm ex} + o(\varepsilon^{-1}) = o(\varepsilon^{-1}).$$

Step 2. Energy estimate. Let us prove that

$$Y := \frac{GL_\varepsilon(\eta_1, \tilde{A}_1) + GL_\varepsilon(\eta_2, \tilde{A}_2)}{2} - GL_\varepsilon\left(\frac{\eta_1+\eta_2}{2}, \frac{\tilde{A}_1+\tilde{A}_2}{2}\right) > 0.$$

First, observe that

$$\int_\Omega |\nabla_{\tilde{A}_j} \eta_j|^2 = \int_\Omega |\nabla \eta_j|^2 + \eta_j^2 |\tilde{A}_j|^2.$$

We write $Y = Y_0 + Y_1 + Y_2 + Y_3$ with

$$Y_0 = \frac{1}{2}\int_\Omega |\nabla \eta_1|^2 + |\nabla \eta_2|^2 - \int_\Omega \left|\nabla\left(\frac{\eta_1+\eta_2}{2}\right)\right|^2,$$

$$Y_1 = \frac{1}{2}\int_\Omega \eta_1^2|\tilde{A}_1|^2 + \eta_2^2|\tilde{A}_2|^2 - \int_\Omega \left(\frac{\eta_1+\eta_2}{2}\right)^2 \left|\frac{\tilde{A}_1+\tilde{A}_2}{2}\right|^2,$$

$$Y_2 = \frac{1}{2}\left(\frac{1}{4\varepsilon^2}\int_\Omega (1-\eta_1^2)^2 + (1-\eta_2^2)^2\right) - \frac{1}{4\varepsilon^2}\int_\Omega \left(1 - \left(\frac{\eta_1+\eta_2}{2}\right)^2\right)^2,$$

$$Y_3 = \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl} \tilde{A}_1 - H_{\rm ex}|^2 + |\operatorname{curl} \tilde{A}_2 - H_{\rm ex}|^2 - \int_{\mathbb{R}^3} \left|\operatorname{curl}\left(\frac{\tilde{A}_1+\tilde{A}_2}{2}\right) - H_{\rm ex}\right|^2.$$

Note that, by convexity, we have $Y_0, Y_3 \ge 0$.

On the other hand, arguing exactly as in the proof of [Ser99b, Lemma 2.5], we get

$$Y_1 = \frac{1}{16}\int_\Omega |\eta_1-\eta_2|^2|\tilde{A}_1+\tilde{A}_2|^2 + 4\eta_1^2|\tilde{A}_1-\tilde{A}_2|^2 \\ -(\eta_1-\eta_2)(\tilde{A}_1-\tilde{A}_2)\cdot\left(\tilde{A}_1(2\eta_1+4\eta_2)+\tilde{A}_2(6\eta_1+8\eta_2)\right)$$

and

$$Y_2 \geq \frac{3}{64\varepsilon^2}\int_\Omega (\eta_1-\eta_2)^2.$$

Let us prove that $Y_1 + Y_2 > 0$. We consider three cases.

- If $\eta_1 = \eta_2$, then

$$Y_1 + Y_2 \geq \int_\Omega 4\eta_1^2|\tilde{A}_1-\tilde{A}_2|^2 > 0.$$

- If $\tilde{A}_1 = \tilde{A}_2$, then $Y_1 \geq 0$. Therefore

$$Y_1 + Y_2 \geq Y_2 \geq \frac{3}{64\varepsilon^2}\int_\Omega (\eta_1-\eta_2)^2 > 0.$$

- If $\eta_1 \neq \eta_2$ and $\tilde{A}_1 \neq \tilde{A}_2$ then

$$Y_1 \geq \frac{1}{16}\int_\Omega |\eta_1-\eta_2|^2|\tilde{A}_1+\tilde{A}_2|^2 + 4\eta_1^2\left|\tilde{A}_1-\tilde{A}_2\right|^2 - |\eta_1-\eta_2||\tilde{A}_1-\tilde{A}_2|(6|\tilde{A}_1|+14|\tilde{A}_2|).$$

By the Cauchy-Schwarz inequality, we have

$$\int_\Omega |\eta_1-\eta_2||\tilde{A}_1-\tilde{A}_2|(6|\tilde{A}_1|+14|\tilde{A}_2|) \\ \leq 14(\|\tilde{A}_1\|_{L^\infty(\Omega,\mathbb{R}^3)} + \|\tilde{A}_2\|_{L^\infty(\Omega,\mathbb{R}^3)})\|\eta_1-\eta_2\|_{L^2(\Omega)}\|\tilde{A}_1-\tilde{A}_2\|_{L^2(\Omega,\mathbb{R}^3)},$$

which combined with (6.2), yields

$$\int_\Omega |\eta_1-\eta_2||\tilde{A}_1-\tilde{A}_2|(6|\tilde{A}_1|+14|\tilde{A}_2|) \leq o(\varepsilon^{-1})\|\eta_1-\eta_2\|_{L^2(\Omega)}\|\tilde{A}_1-\tilde{A}_2\|_{L^2(\Omega,\mathbb{R}^3)}.$$

On the other hand,

$$\int_\Omega \frac{1}{4}\eta_1^2|\tilde{A}_1-\tilde{A}_2|^2 + \frac{3}{64\varepsilon^2}(\eta_1-\eta_2)^2 \geq \frac{9}{32\varepsilon}\|\eta_1-\eta_2\|_{L^2(\Omega)}\|\tilde{A}_1-\tilde{A}_2\|_{L^2(\Omega,\mathbb{R}^3)}.$$

Hence, if ε is small enough then $Y_1 + Y_2 > 0$.

We have thus proved that $Y > 0$.

Step 3. Contradiction. Assume without loss of generality that

$$GL_\varepsilon(\eta_1,\tilde{A}_1) \leq GL_\varepsilon(\eta_2,\tilde{A}_2).$$

From the previous step, we have

$$GL_\varepsilon\left(\frac{\eta_1+\eta_2}{2},\frac{\tilde{A}_1+\tilde{A}_2}{2}\right) < \frac{GL_\varepsilon(\eta_1,\tilde{A}_1)+GL_\varepsilon(\eta_2,\tilde{A}_2)}{2} \leq GL_\varepsilon(\eta_2,\tilde{A}_2).$$

A standard argument then shows that, for any $t \in (0, 1)$,

$$GL_\varepsilon\left(t\eta_1 + (1-t)\eta_2, t\tilde{A}_1 + (1-t)\tilde{A}_2\right) < GL_\varepsilon(\eta_2, \tilde{A}_2),$$

contradicting the fact that $(\eta_2, \tilde{A}_2)$ is a local minimizer of the energy. Hence $(\eta_1, \tilde{A}_1) = (\eta_2, \tilde{A}_2)$. This concludes the proof. □

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A. Improved Estimates for Locally Minimizing Vortexless Configurations

Proposition A.1. *Let* $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ *with* u *continuous and* $|u| \geq c$ *for some* $c \in (0, 1)$. *Then*

$$\|\mu(u, A)\|_{C_T^{0,1}(\Omega,\mathbb{R}^3)^*} \leq C\varepsilon F_\varepsilon(u, A).$$

Proof. Let $\varphi \in C_T^{0,1}(\Omega, \mathbb{R}^3)$. By integration by parts, we have

$$\int_\Omega \mu(u, A) \wedge \varphi = \int_\Omega (j(u, A) + A) \cdot \operatorname{curl} \varphi.$$

Since $|u| \geq c > 0$, we can write $u = |u|e^{i\phi}$. A straightforward computation, shows that

$$j(u, A) + A = |u|^2 \nabla\phi + (1 - |u|^2)A = (1 - |u|^2)(A - \nabla\phi) + \nabla\phi.$$

Observe that, by integration by parts, we have $\int_\Omega \nabla\phi \cdot \operatorname{curl} \varphi = 0$. Then, from the Cauchy-Schwarz inequality, we deduce that

$$\left|\int_\Omega (j(u, A) + A) \cdot \operatorname{curl} \varphi\right| \leq \int_\Omega (1 - |u|^2)|A - \nabla\phi||\operatorname{curl} \varphi| \leq C\|\operatorname{curl} \varphi\|_{L^\infty(\Omega,\mathbb{R}^3)} \varepsilon F_\varepsilon(u, A).$$

Hence

$$\|\mu(u, A)\|_{C_T^{0,1}(\Omega,\mathbb{R}^3)^*} \leq C\varepsilon F_\varepsilon(u, A).$$

□

With this estimate at hand, we prove the following result.

Proposition A.2. *Denote* $(u_0, h_{\rm ex}A_0)$ *the approximation of the Meissner solution. Let* $(u, A) = (u_0u', h_{\rm ex}A_0 + A') \in H^1(\Omega, \mathbb{C}) \times [A_{\rm ex} + H_{\rm curl}]$ *with* u *continuous and* $|u| \geq c$ *for some* $c \in (0, 1)$. *If* $h_{\rm ex} \leq \varepsilon^{-\alpha}$ *for some* $\alpha \in \left(0, \frac{1}{4}\right)$ *and* $GL_\varepsilon(u, A) \leq GL_\varepsilon(u_0, h_{\rm ex}A_0)$ *then, for any* ε *sufficiently small, we have*

$$F_\varepsilon(u', A') + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A'|^2 \leq C\varepsilon^{1+\delta}$$

for some $\delta \in (0, 1)$.

Proof. Let us first observe that, since $GL_\varepsilon(u, A) \leq GL_\varepsilon(u_0, h_{\rm ex}A_0) = h_{\rm ex}^2 J(A_0)$, we have

$$F_\varepsilon(u', A') \leq Ch_{\rm ex}^2 \leq C\varepsilon^{-2\alpha} \tag{A.1}$$

for some constant $C > 0$. We will now use Proposition (3.1) to improve this estimate. By combining (3.5) with $GL_\varepsilon(u, A) \leq GL_\varepsilon(u_0, h_{\rm ex}A_0)$, we find

$$F_\varepsilon(u', A') + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A'|^2 \leq h_{\rm ex}\int_\Omega \mu(u', A') \wedge B_0 + C\varepsilon h_{\rm ex}^2 E_\varepsilon(|u'|)^{\frac{1}{2}}.$$

From Proposition A.1, $E_\varepsilon(|u'|) \leq F_\varepsilon(u', A')$, and (A.1) we deduce that

$$F_\varepsilon(u', A') + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A'|^2 \leq C\varepsilon h_{\rm ex} F_\varepsilon(u', A') + C\varepsilon h_{\rm ex}^2 F_\varepsilon(u', A')^{\frac{1}{2}} \leq C\varepsilon h_{\rm ex}^2 F_\varepsilon(u', A')^{\frac{1}{2}}. \tag{A.2}$$

Thus

$$\left(F_\varepsilon(u', A') + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A'|^2\right)^{\frac{1}{2}} \leq C\varepsilon h_{\rm ex}^2.$$

Combining with $h_{\rm ex} \leq \varepsilon^{-\alpha}$, we find

$$F_\varepsilon(u', A') + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega} |\operatorname{curl} A'|^2 \leq C\varepsilon^{1+\delta}.$$

with $\delta = 1 - 4\alpha > 0$. □

As a consequence, from Theorem 1.5, we obtain the uniqueness of the Meissner-type solution of Theorem 1.4 for $\alpha < \frac{1}{4}$.

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