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Quasi Modules for the Quantum Affine Vertex Algebra in Type A

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Dedicated to Mirko Primc on the occasion of his 70th birthday

Abstract: We consider the quantum affine vertex algebra $\mathcal{V}_c(\mathfrak{gl}_N)$ associated with the rational R-matrix, as defined by Etingof and Kazhdan. We introduce certain subalgebras $\mathrm{A}_c(\mathfrak{gl}_N)$ of the completed double Yangian $\widetilde{\mathrm{DY}}_c(\mathfrak{gl}_N)$ at the level $c \in \mathbb{C}$, associated with the reflection equation, and we employ their structure to construct examples of quasi $\mathcal{V}_c(\mathfrak{gl}_N)$-modules. Finally, we use the quasi module map, together with the explicit description of the center of $\mathcal{V}_c(\mathfrak{gl}_N)$, to obtain formulae for families of central elements in the completed algebra $\widetilde{\mathrm{A}}_c(\mathfrak{gl}_N)$.

1. Introduction

In order to describe integrable systems with the boundary conditions, E. K. Sklyanin introduced in [23] the *reflection algebras*, a class of algebras associated with R-matrix $R(u)$ which are defined by the *reflection equation*

$$R_{12}(u-v)B_1(u)R_{12}(u+v)B_2(v) = B_2(v)R_{12}(u+v)B_1(u)R_{12}(u-v). \tag{1}$$

We explain the precise meaning of (1) in Sect. 2.2. His approach was motivated by Cherednik's treatment of factorized scattering with reflection [1]. Furthermore, Sklyanin constructed an analogue of the *quantum determinant* and described the *algebraic Bethe ansatz*; see [23]. Later on, different classes of algebras defined via relations of the form similar to or same as (1) were extensively studied; see, e.g., [11,12,17,19,22].

In this paper, we consider a certain family of reflection algebras associated with the Yang R-matrix, studied by Molev and Ragoucy [19], which are coideal subalgebras of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$. We introduce the subalgebra $\mathrm{A}_c(\mathfrak{gl}_N)$ of the h-adically completed double Yangian $\widetilde{\mathrm{DY}}_c(\mathfrak{gl}_N)$ at the level $c \in \mathbb{C}$ which, roughly speaking, consists of two reflection algebras. Motivated by the correspondence, indicated in [4], between the $\mathcal{S}$-*locality* (see (2.19) below) and the commutation relation for the quantum current which appeared in work of Reshetikhin and Semenov-Tian-Shansky [22] and which resembles the form of the reflection equation, we investigate algebras $\mathrm{A}_c(\mathfrak{gl}_N)$ using the theory of quantum VOAs.

The notion of *quantum vertex operator algebra* (*quantum VOA*) was introduced by Etingof and Kazhdan [4]. *Quantum affine VOA* can be associated with rational, trigonometric and elliptic R-matrix; see [4]. In the rational case, the double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$ over $\mathbb{C}[[h]]$ can be used to define the quantum VOA structure on its vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$ at the level $c \in \mathbb{C}$. The theory of quantum vertex algebras was further developed and generalized by Li; see, e.g., [14,15] and references therein. In particular, certain more general objects, such as *h-adic nonlocal vertex algebras* and their *quasi modules*, were introduced and studied in [15]. The main result of this paper is a construction of the quasi module map $Y_{\mathcal{W}_c(\mathfrak{gl}_N)}$ on $\mathcal{V}_{2c}(\mathfrak{gl}_N)$, so that the *vacuum module* $\mathcal{W}_c(\mathfrak{gl}_N)$ for the algebra $\mathrm{A}_c(\mathfrak{gl}_N)$ acquires a quasi $\mathcal{V}_{2c}(\mathfrak{gl}_N)$-module structure.

We use the quasi module map to obtain further information on the algebra $\mathrm{A}_c(\mathfrak{gl}_N)$. In our previous paper [9], coauthored with N. Jing, A. Molev, and F. Yang, the center $\mathfrak{z}(\mathcal{V}_c(\mathfrak{gl}_N))$ of the quantum VOA $\mathcal{V}_c(\mathfrak{gl}_N)$ was described by providing explicit formulae for its algebraically independent topological generators, thus establishing the quantum analogue of the Feigin–Frenkel theorem in type A; see [2,3,5]. By considering the image of the center $\mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$, with respect to the quasi module map $Y_{\mathcal{W}_{-N/2}(\mathfrak{gl}_N)}$, we find explicit formulae for families of central elements in the completed algebra $\widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N)$, which are, due to the *fusion procedure* originated in the work of A. Jucys [10], parametrized by arbitrary partitions with at most N parts. For $c \neq -N$ we obtain only one family of central elements in $\widetilde{\mathrm{A}}_{c/2}(\mathfrak{gl}_N)$, which, roughly speaking, coincide with the coefficients of the product of two *Sklyanin determinants* (i.e. with the coefficients of the product of four quantum determinants); see [19,23]. In the end, we employ these central elements to obtain invariants of the vacuum module $\mathcal{W}_c(\mathfrak{gl}_N)$.

2. Reflection Algebras

In this section, we recall the definition of the double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$ over $\mathbb{C}[[h]]$; see [8]. Next, we follow [19] to introduce a certain class of reflection algebras. We employ their structure to define subalgebra $\mathrm{A}_c(\mathfrak{gl}_N)$ of the h-adically completed double Yangian $\widetilde{\mathrm{DY}}_c(\mathfrak{gl}_N)$ at the level $c \in \mathbb{C}$, which will be our main point of interest in this paper.

2.1. Double Yangian for $\mathfrak{gl}_N$. Let $N \geqslant 2$ be an integer and let h be a formal parameter. Denote by $R(u)$ the *Yang R-matrix* over $\mathbb{C}[[h]]$ defined by

$$R(u) = 1 - hPu^{-1}, \tag{1.2}$$

where 1 is the identity and P is the permutation operator in $\mathbb{C}^N \otimes \mathbb{C}^N$, $P\colon x \otimes y \mapsto y \otimes x$. R-matrix (1.2) satisfies the *Yang–Baxter equation*

$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u). \tag{1.3}$$

Both sides of (1.3) are operators on the triple tensor product $(\mathbb{C}^N)^{\otimes 3}$ and subscripts indicate the copies of $\mathbb{C}^N$ on which $R(u)$ acts, for example, $R_{12}(u) = R(u) \otimes 1$ and $R_{23}(v) = 1 \otimes R(v)$. Let $g(u)$ be the unique series in $1 + u^{-1}\mathbb{C}[[u^{-1}]]$ satisfying

$$g(u+N) = g(u)(1 - u^{-2}). \tag{1.4}$$

The R-matrix $\overline{R}(u) = \overline{R}_{12}(u) = g(u/h)R(u)$ possesses the *crossing symmetry* properties,

$$\left(\overline{R}_{12}(u)^{-1}\right)^{t_1} \overline{R}_{12}(u+hN)^{t_1} = 1 \quad \text{and} \quad \left(\overline{R}_{12}(u)^{-1}\right)^{t_2} \overline{R}_{12}(u+hN)^{t_2} = 1, \tag{1.5}$$

where t_i denotes the transposition applied on the tensor factor $i = 1, 2$; and the *unitarity property*

$$\overline{R}_{12}(u)\overline{R}_{12}(-u) = 1, \tag{1.6}$$

see, e.g., [9, Sect. 2] for more details.

The *double Yangian* $\mathrm{DY}(\mathfrak{gl}_N)$ for $\mathfrak{gl}_N$ is defined as the associative algebra over $\mathbb{C}[[h]]$ generated by the central element C and the elements $t_{ij}^{(\pm r)}$, where $i, j = 1, \dots N$ and $r = 1, 2, \dots$, subject to the following defining relations (see [8]),

$$R\big(u-v\big)\,T_1(u)\,T_2(v) = T_2(v)\,T_1(u)\,R\big(u-v\big), \tag{1.7}$$

$$R\big(u-v\big)\,T_1^+(u)\,T_2^+(v) = T_2^+(v)\,T_1^+(u)\,R\big(u-v\big), \tag{1.8}$$

$$\overline{R}\big(u-v+hC/2\big)\,T_1(u)\,T_2^+(v) = T_2^+(v)\,T_1(u)\,\overline{R}\big(u-v-hC/2\big). \tag{1.9}$$

The elements $T(u)$ and $T^+(u)$ in $\operatorname{End}\mathbb{C}^N \otimes \mathrm{DY}(\mathfrak{gl}_N)[[u^{\pm 1}]]$ are defined by

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u) \qquad \text{and} \qquad T^+(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}^+(u),$$

where the e_{ij} are the matrix units, and the series $t_{ij}(u)$ and $t_{ij}^+(u)$ are given by

$$t_{ij}(u) = \delta_{ij} + h\sum_{r=1}^{\infty} t_{ij}^{(r)} u^{-r} \qquad \text{and} \qquad t_{ij}^+(u) = \delta_{ij} - h\sum_{r=1}^{\infty} t_{ij}^{(-r)} u^{r-1}.$$

We use the subscript to indicate a copy of the matrix in the tensor product algebra $(\operatorname{End}\mathbb{C}^N)^{\otimes m} \otimes \mathrm{DY}(\mathfrak{gl}_N)$, so that, for example,

$$T_k(u) = \sum_{i,j=1}^{N} 1^{\otimes(k-1)} \otimes e_{ij} \otimes 1^{\otimes(m-k)} \otimes t_{ij}(u). \tag{1.10}$$

In particular, we have $m = 2$ in defining relations (1.7)–(1.9).

The *Yangian* $\mathrm{Y}(\mathfrak{gl}_N)$ is the subalgebra of $\mathrm{DY}(\mathfrak{gl}_N)$ generated by the elements $t_{ij}^{(r)}$, $i, j = 1, \dots, N, r = 1, 2, \dots$. The *dual Yangian* $\mathrm{Y}^+(\mathfrak{gl}_N)$ is the subalgebra of the double Yangian $\mathrm{DY}(\mathfrak{gl}_N)$ generated by the elements $t_{ij}^{(-r)}$, $i, j = 1, \dots, N$, $r = 1, 2, \dots$. For any complex number c denote by $\mathrm{DY}_c(\mathfrak{gl}_N)$ the *double Yangian at the level* c, i.e. the quotient of the algebra $\mathrm{DY}(\mathfrak{gl}_N)$ by the ideal generated by the element $C - c$.

Recall that the *h-adic topology* on an arbitrary $\mathbb{C}[[h]]$-module V is the topology generated by the basis $v + h^n V$, $v \in V$, $n \in \mathbb{Z}_{\geqslant 1}$. The *vacuum module* $\mathcal{V}_c(\mathfrak{gl}_N)$ *at the level* c over the double Yangian is the h-adic completion of the quotient of the algebra $\mathrm{DY}_c(\mathfrak{gl}_N)$ by the left ideal generated by all elements $t_{ij}^{(r)}$, $r = 1, 2, \dots$, i.e. the h-adic completion of

$$\mathrm{DY}_c(\mathfrak{gl}_N)/\mathrm{DY}_c(\mathfrak{gl}_N)\big\langle t_{ij}^{(r)}\ :\ i, j = 1, \dots, N,\ r = 1, 2, \dots\big\rangle. \tag{1.11}$$

By the Poincaré–Birkhoff–Witt theorem for the double Yangian, see [9, Theorem 2.2], the vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$ is isomorphic, as a $\mathbb{C}[[h]]$-module, to the h-adically completed dual Yangian $\widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)$.

2.2. Algebra $\mathrm{A}_c(\mathfrak{gl}_N)$. We now proceed as in [19] to introduce the reflection algebras. Fix nonnegative integer $M \leqslant N$. Let $G = (g_{ij})_{i,j=1}^N$ be the diagonal matrix of order N,

$$G = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_N), \tag{1.12}$$

where $\varepsilon_1 = \dots = \varepsilon_M = 1$ and $\varepsilon_{M+1} = \dots = \varepsilon_N = -1$. Let c be a fixed complex number. Consider the series

$$B^+(u) = \sum_{i,j=1}^{N} e_{ij} \otimes b_{ij}^+(u) \in \operatorname{End}\mathbb{C}^N \otimes \widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)[[u]] \quad \text{and} \tag{1.13}$$

$$B(u) = \sum_{i,j=1}^{N} e_{ij} \otimes b_{ij}(u) \in \operatorname{End}\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]] \tag{1.14}$$

defined by

$$B^+(u) = T^+(u)\, G\, T^+(-u)^{-1} \qquad \text{and} \qquad B(u) = T(u+hc)\, G\, T(-u)^{-1}. \tag{1.15}$$

We can write the matrix entries of (1.13) and (1.14) as

$$b_{ij}^+(u) = g_{ij} - h\sum_{r=1}^{\infty} b_{ij}^{(-r)} u^{r-1} \qquad \text{and} \qquad b_{ij}(u) = g_{ij} + h\sum_{r=1}^{\infty} b_{ij}^{(r)} u^{-r}$$

for some elements $b_{ij}^{(-r)} \in \widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)$ and $b_{ij}^{(r)} \in \mathrm{Y}(\mathfrak{gl}_N)$.

Series (1.15) satisfy the *unitary condition*

$$B^+(u)B^+(-u) = 1 \qquad \text{and} \qquad B(u)B(-u-hc) = 1. \tag{1.16}$$

Furthermore, using (1.7)–(1.9) and $R(u)G_1R(v)G_2 = G_2R(v)G_1R(u)$ one can easily verify that the following *reflection relations* hold for the elements of the h-adically completed double Yangian $\widehat{\mathrm{DY}}_c(\mathfrak{gl}_N)$ at the level $c \in \mathbb{C}$:

$$R(u-v)B_1^+(u)R(u+v)B_2^+(v) = B_2^+(v)R(u+v)B_1^+(u)R(u-v), \tag{1.17}$$

$$R(u-v)B_1(u)R(u+v+hc)B_2(v) = B_2(v)R(u+v+hc)B_1(u)R(u-v), \tag{1.18}$$

$$\begin{aligned}&\overline{R}(u-v+3hc/2)B_1(u)\overline{R}(u+v-hc/2)B_2^+(v)\\ &\quad = B_2^+(v)\overline{R}(u+v+3hc/2)B_1(u)\overline{R}(u-v-hc/2).\end{aligned} \tag{1.19}$$

As in Sect. 2.1, the subscripts in (1.17)–(1.19) indicate a copy of the matrix in the tensor product algebra $(\operatorname{End}\mathbb{C}^N)^{\otimes 2} \otimes \widehat{\mathrm{DY}}_c(\mathfrak{gl}_N)$; recall (1.10).

For $i, j = 1, \dots, N$ and $r = 1, 2, \dots$ let $\mathrm{A}'_c(\mathfrak{gl}_N)$ be the subalgebra of $\widehat{\mathrm{DY}}_c(\mathfrak{gl}_N)$ generated by the elements $b_{ij}^{(-r)}$ and $b_{ij}^{(r)}$, let $\mathrm{B}'^{+}(\mathfrak{gl}_N)$ be the subalgebra of the h-adically completed dual Yangian $\widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)$ generated by the elements $b_{ij}^{(-r)}$ and let $\mathrm{B}'_c(\mathfrak{gl}_N)$ be the subalgebra of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ generated by the elements $b_{ij}^{(r)}$.

Remark 2.1. By setting $h = 1$ in the algebra $\mathrm{B}'_0(\mathfrak{gl}_N)$ we obtain the reflection algebra $\mathcal{B}(N, N-M)$ over $\mathbb{C}$, as defined in [19].

As in [15], for an arbitrary $\mathbb{C}[[h]]$-submodule V of $\widehat{\mathrm{DY}}_c(\mathfrak{gl}_N)$ we define

$$[V] = \left\{v \in \widehat{\mathrm{DY}}_c(\mathfrak{gl}_N) \,:\, h^n v \in V \text{ for some } n \geqslant 0\right\}. \tag{1.20}$$

Finally, consider the following subalgebras of $\widehat{\mathrm{DY}}_c(\mathfrak{gl}_N)$:

$$\mathrm{A}_c(\mathfrak{gl}_N) = [\mathrm{A}'_c(\mathfrak{gl}_N)], \quad \mathrm{B}_c(\mathfrak{gl}_N) = [\mathrm{B}'_c(\mathfrak{gl}_N)] \quad \text{and} \quad \mathrm{B}^+(\mathfrak{gl}_N) = [\mathrm{B}'^+(\mathfrak{gl}_N)].$$

Clearly, the following inclusions hold:

$$\mathrm{A}_c(\mathfrak{gl}_N) \subset \widehat{\mathrm{DY}}_c(\mathfrak{gl}_N), \quad \mathrm{B}_c(\mathfrak{gl}_N) \subset \mathrm{Y}(\mathfrak{gl}_N) \quad \text{and} \quad \mathrm{B}^+(\mathfrak{gl}_N) \subset \widehat{\mathrm{Y}}^+(\mathfrak{gl}_N).$$

Moreover, due to [15, Lemma 3.5], the induced topology on $\mathrm{A}_c(\mathfrak{gl}_N)$, $\mathrm{B}_c(\mathfrak{gl}_N)$ and $\mathrm{B}^+(\mathfrak{gl}_N)$ from $\widehat{\mathrm{DY}}_c(\mathfrak{gl}_N)$ coincides with the h-adic topology on these algebras.

We now introduce some new notation in order to write the more general form of relations (1.17)–(1.19). For positive integers n, m and the families of variables $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$ set

$$\begin{aligned} \overline{R}_{ij} &= \overline{R}_{ij}(u_i - v_{j-n}) \quad \text{and} \\ \underline{\overline{R}}_{ij} &= \overline{R}_{ij}(u_i + v_{j-n}), \qquad i = 1, \ldots, n, \; j = n+1, \ldots, n+m. \end{aligned}$$

Consider the functions with values in the space $(\operatorname{End}\mathbb{C}^N)^{\otimes n} \otimes (\operatorname{End}\mathbb{C}^N)^{\otimes m}$

$$\overline{R}^{12}_{nm}(u|v) = \prod_{i=1,\ldots,n}^{\longrightarrow} \; \prod_{j=n+1,\ldots,n+m}^{\longleftarrow} \overline{R}_{ij} \qquad \text{and} \qquad \underline{\overline{R}}^{12}_{nm}(u|v) = \prod_{i=1,\ldots,n}^{\longrightarrow} \; \prod_{j=n+1,\ldots,n+m}^{\longrightarrow} \underline{\overline{R}}_{ij} \tag{1.21}$$

with the arrows indicating the order of the factors. The functions $R^{12}_{nm}(u|v)$ and $\underline{R}^{12}_{nm}(u|v)$ corresponding to R-matrix (1.2) can be defined analogously. Introduce the series

$$\underline{B}^+_n(u) = \prod_{i=1,\ldots,n}^{\longrightarrow} \left(B^+_i(u_i)\overline{R}_{i\,i+1}(u_i + u_{i+1}) \ldots \overline{R}_{in}(u_i + u_n)\right) \quad \text{and} \tag{1.22}$$

$$\underline{B}_n(u) = \prod_{i=1,\ldots,n}^{\longrightarrow} \left(B_i(u_i)\overline{R}_{i\,i+1}(u_i + u_{i+1} + hc) \ldots \overline{R}_{in}(u_i + u_n + hc)\right). \tag{1.23}$$

For a family of variables $u = (u_1, \ldots, u_n)$ and $\alpha \in \mathbb{C}$ we will often denote the families $(u_1 + \alpha h, \ldots, u_n + \alpha h)$ and $(\alpha u_1, \ldots, \alpha u_n)$ by $u + \alpha h$ and αu respectively. We also adopt the superscript notation for multiple tensor products of the form

$$(\operatorname{End}\mathbb{C}^N)^{\otimes n} \otimes (\operatorname{End}\mathbb{C}^N)^{\otimes m} \otimes (\operatorname{End}\mathbb{C}^N)^{\otimes k} \otimes \mathrm{A}_c(\mathfrak{gl}_N) \otimes \mathrm{A}_c(\mathfrak{gl}_N) \otimes \mathrm{A}_c(\mathfrak{gl}_N).$$

Expressions like $\underline{B}^{+14}_n(u)$ or $\underline{B}^{35}_k(w)$, where $w = (w_1, \ldots, w_k)$, will be understood as the respective operators $\underline{B}^+_n(u)$ or $\underline{B}_k(w)$, whose non-identity components belong to the corresponding tensor factors. In particular, the non-identity components of $\underline{B}^{35}_k(w)$ belong to the factors $n + m + 1$, $n + m + 2$, $\ldots$, $n + m + k$ and $n + m + k + 2$. This notation is employed in the next proposition, which can be proved using (1.17)–(1.19) and Yang–Baxter equation (1.3).

Proposition 2.2. *For any positive integers n and m the following equalities hold on* $(\operatorname{End}\mathbb{C}^N)^{\otimes n} \otimes (\operatorname{End}\mathbb{C}^N)^{\otimes m} \otimes \mathrm{A}_c(\mathfrak{gl}_N)$*:*

$$R_{nm}^{12}(u|v)\underline{B}_n^{+13}(u)\underline{R}_{nm}^{12}(u|v)\underline{B}_m^{+23}(v) = \underline{B}_m^{+23}(v)\underline{R}_{nm}^{12}(u|v)\underline{B}_n^{+13}(u)R_{nm}^{12}(u|v), \quad (1.24)$$

$$R_{nm}^{12}(u|v)\underline{B}_n^{13}(u)\underline{R}_{nm}^{12}(u+hc|v)\underline{B}_m^{23}(v) = \underline{B}_m^{23}(v)\underline{R}_{nm}^{12}(u+hc|v)\underline{B}_n^{13}(u)R_{nm}^{12}(u|v), \quad (1.25)$$

$$\overline{R}_{nm}^{12}(u+3hc/2|v)\underline{B}_n^{13}(u)\underline{\overline{R}}_{nm}^{12}(u-hc/2|v)\underline{B}_m^{+23}(v) \\ = \underline{B}_m^{+23}(v)\underline{\overline{R}}_{nm}^{12}(u+3hc/2|v)\underline{B}_n^{13}(u)\overline{R}_{nm}^{12}(u-hc/2|v). \quad (1.26)$$

Our next goal is to derive Proposition 2.4, which will be useful in what follows. First, note that by applying the transposition t_1 on the first and t_2 on the second equality in (1.5) we get

$$^{rl}\left(\overline{R}_{12}(u)^{-1}\right) \cdot \overline{R}_{12}(u+hN) = 1 \quad \text{and} \quad ^{lr}\left(\overline{R}_{12}(u)^{-1}\right) \cdot \overline{R}_{12}(u+hN) = 1, \quad (1.27)$$

where the superscript rl (lr) in (1.27) indicates that the first tensor factor of $\overline{R}_{12}(u)^{-1}$ is applied from the right (left) while the second tensor factor of $\overline{R}_{12}(u)^{-1}$ is applied from the left (right). One can generalize *ordered products* (1.27) in an obvious way. For example,

$$K^{(n,m)} = \prod_{i=1,\ldots,n}^{\longleftarrow} \prod_{j=n+1,\ldots,n+m}^{\longleftarrow} \overline{R}_{ij}(u_i + v_{j-n} - hc/2 - hN)^{-1} \quad (1.28)$$

satisfies

$$^{rl}\left(K^{(n,m)}\right) \cdot \underline{\overline{R}}_{nm}^{12}(u-hc/2|v) = 1, \quad (1.29)$$

where superscript rl in (1.29) indicates that the tensor factors of $K^{(n,m)}$ corresponding to the first index $i = 1, \ldots, n$ in (1.28) are applied from the right in reversed order, while the tensor factors corresponding to the second index $j = n+1, \ldots, n+m$ in (1.28) are applied from the left.

Example 2.3. Set $K_{ij} = \overline{R}_{ij}(u_i+v_{j-n}-hc/2-hN)^{-1}$ and $S_{ij} = R_{ij}(u_i+v_{j-n}-hc/2)$. We briefly explain how to verify (1.29) for $n = m = 2$; the general case can be proved analogously. First, due to (1.21) and (1.28), on $(\operatorname{End}\mathbb{C}^N)^{\otimes 2} \otimes (\operatorname{End}\mathbb{C}^N)^{\otimes 2}$ we have

$$\underline{\overline{R}}_{22}^{12}(u-hc/2|v) = S_{13}S_{14}S_{23}S_{24} \quad \text{and} \quad K^{(2,2)} = K_{24}K_{23}K_{14}K_{13}.$$

The element $^{rl}\left(K^{(2,2)}\right) \cdot \underline{\overline{R}}_{22}^{12}(u-hc/2|v)$ can be written as

$$^{rl}(K_{23}) \cdot \left(^{rl}(K_{24}) \cdot \left(^{rl}(K_{13}) \cdot \left(^{rl}(K_{14}) \cdot (S_{13}S_{14}S_{23}S_{24})\right)\right)\right). \quad (1.30)$$

By the first equality in (1.27) we have

$$^{rl}(K_{14}) \cdot (S_{13}S_{14}S_{23}S_{24}) = S_{13}\left(^{rl}(K_{14}) \cdot S_{14}\right)S_{23}S_{24} = S_{13}S_{23}S_{24}.$$

Next, as before, by the first equality in (1.27) we have

$$^{rl}(K_{13}) \cdot (S_{13}S_{23}S_{24}) = \left(^{rl}(K_{13}) \cdot S_{13}\right)S_{23}S_{24} = S_{23}S_{24}.$$

Hence, (1.30) is equal to $^{rl}(K_{23}) \cdot \left(^{rl}(K_{24}) \cdot (S_{23}S_{24})\right)$. By repeating the same arguments two more times, we finally obtain $^{rl}\left(K^{(2,2)}\right) \cdot \underline{\overline{R}}_{22}^{12}(u-hc/2|v) = 1$, as required.

Observe that, due to (1.27), the element

$$L^{(n,m)} = \prod_{i=n+1,\ldots,n+m-1}^{\longleftarrow} \prod_{j=i+1,\ldots,n+m}^{\longleftarrow} \overline{R}_{ij}(v_{i-n} + v_{j-n} - hN)^{-1} \tag{1.31}$$

satisfies

$$^{rl}\Big(L^{(n,m)}\Big) \cdot \underline{B}_m^{+23}(v) = B_{n+1}^+(v_1)B_{n+2}^+(v_2)\ldots B_{n+m}^+(v_m), \tag{1.32}$$

where, as before, superscript rl in (1.32) indicates that the tensor factors of $L^{(n,m)}$ corresponding to the first index $i = n+1,\ldots,n+m-1$ in (1.31) are applied from the right in reversed order, while the tensor factors corresponding to the second index $j = i+1,\ldots,n+m$ in (1.31) are applied from the left. Relation (1.26), together with (1.29) and (1.32), implies

Proposition 2.4. *The following equality holds on* $(\operatorname{End}\mathbb{C}^N)^{\otimes n} \otimes (\operatorname{End}\mathbb{C}^N)^{\otimes m} \otimes \mathrm{A}_c(\mathfrak{gl}_N)$*:*

$$\begin{aligned}\underline{B}_n^{13}(u)\, B_{n+1}^+(v_1)B_{n+2}^+(v_2)\ldots B_{n+m}^+(v_m) = {}^{rl}\Big(L^{(n,m)}\Big) \cdot \Big({}^{rl}\Big(K^{(n,m)}\Big) \\ \cdot \Big(\overline{R}_{nm}^{12}(u+3hc/2|v)^{-1}\underline{B}_m^{+23}(v)\underline{\overline{R}}_{nm}^{12}(u+3hc/2|v)\underline{B}_n^{13}(u)\overline{R}_{nm}^{12}(u-hc/2|v)\Big)\Big).\end{aligned} \tag{1.33}$$

Denote by $\mathbf{1}$ the image of the unit $1 \in \mathrm{DY}_c(\mathfrak{gl}_N)$ in the quotient (1.11). Let $\mathcal{W}_c'(\mathfrak{gl}_N)$ be the $\mathrm{B}^+(\mathfrak{gl}_N)$-submodule of $\mathcal{V}_c(\mathfrak{gl}_N)$ generated by $\mathbf{1}$. Introduce the *vacuum module* $\mathcal{W}_c(\mathfrak{gl}_N)$ over the algebra $\mathrm{A}_c(\mathfrak{gl}_N)$ as the h-adic completion of $\mathcal{W}_c'(\mathfrak{gl}_N)$. Observe that $\mathcal{W}_c(\mathfrak{gl}_N)$ is closed under the action of $\mathrm{B}_c(\mathfrak{gl}_N)$, so it possesses a structure of an $\mathrm{A}_c(\mathfrak{gl}_N)$-module. Indeed, by applying (1.33) with $n = 1$ on $\mathbf{1}$ and using

$$B(u)\,\mathbf{1} = T(u+hc)\,G\,T(-u)^{-1}\,\mathbf{1} = T(u+hc)\,G\,\mathbf{1} = G\,\mathbf{1}, \tag{1.34}$$

we obtain

$$\begin{aligned}B_1(u_1)\, B_2^+(v_1)B_3^+(v_2)\ldots B_{m+1}^+(v_m)\,\mathbf{1} = {}^{rl}\Big(L^{(1,m)}\Big) \cdot \Big({}^{rl}\Big(K^{(1,m)}\Big) \\ \cdot \Big(\overline{R}_{1m}^{12}(u_1+3hc/2|v)^{-1}\underline{B}_m^{+23}(v)\underline{\overline{R}}_{1m}^{12}(u_1+3hc/2|v)\,G_1\,\overline{R}_{1m}^{12}(u_1-hc/2|v)\,\mathbf{1}\Big)\Big),\end{aligned} \tag{1.35}$$

so it remains to observe that all coefficients of the right hand side in (1.35) belong to $\mathcal{W}_c(\mathfrak{gl}_N)$.

By the Poincaré–Birkhoff–Witt theorem for the double Yangian, see [9, Theorem 2.2], the $\mathbb{C}[[h]]$-modules $\mathcal{W}_c'(\mathfrak{gl}_N)$ and $\mathrm{B}^+(\mathfrak{gl}_N)$ are isomorphic. Hence, in particular, the completion $\mathcal{W}_c(\mathfrak{gl}_N)$ is topologically free, i.e. separated, torsion-free and h-adically complete.

3. Quasi Modules for *h*-Adic Nonlocal Vertex Algebras

In this section, we study h-adic nonlocal vertex algebras and their quasi modules, as defined by Li [15], and we establish some technical results on their center, which will be useful in Sect. 4. Next, we recall Etingof–Kazhdan's definition [4] of quantum VOA structure on the vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$, $c \in \mathbb{C}$. Finally, we construct quasi modules for the quantum VOA $\mathcal{V}_{2c}(\mathfrak{gl}_N)$ on the $\mathbb{C}[[h]]$-module $\mathcal{W}_c(\mathfrak{gl}_N)$.

3.1. Quasi modules. Let us recall the notion of quasi module for h-adic nonlocal vertex algebra; see [15]. The tensor products in the next two definitions are h-adically completed.

Definition 3.1. An *h-adic nonlocal vertex algebra* is a triple $(V, Y, \mathbf{1})$, where V is a topologically free $\mathbb{C}[[h]]$-module, $\mathbf{1}$ is a distinguished element of V (*vacuum vector*) and

$$\begin{aligned} Y\colon V \otimes V &\to V((z))[[h]] \\ v \otimes w &\mapsto Y(z)(v \otimes w) = Y(v, z)w = \sum_{r\in\mathbb{Z}} v_r w z^{-r-1} \end{aligned}$$

is a $\mathbb{C}[[h]]$-module map which satisfies the *weak associativity* property: For any integer $n \geqslant 0$ and elements $u, v, w \in V$ there exists an integer $r \geqslant 0$ such that

$$(z_0+z_2)^r Y(v, z_0+z_2)Y(w, z_2)u - (z_0+z_2)^r Y\big(Y(v, z_0)w, z_2\big)u \in h^n V[[z_0^{\pm 1}, z_2^{\pm 1}]]; \tag{2.1}$$

and the following conditions hold:

$$Y(v, z)\mathbf{1} \in V[[z]], \quad \lim_{z\to 0} Y(v, z)\mathbf{1} = v \quad \text{and} \quad Y(\mathbf{1}, z)v = v \qquad \text{for any } v \in V.$$

Definition 3.2. Let $(V, Y, \mathbf{1})$ be an h-adic nonlocal vertex algebra. *Quasi V-module* is a pair (W, Y_W), where W is a topologically free $\mathbb{C}[[h]]$-module and

$$\begin{aligned} Y_W(z)\colon V \otimes W &\to W((z))[[h]] \\ v \otimes w &\mapsto Y_W(z)(v \otimes w) = Y_W(v, z)w = \sum_{r\in\mathbb{Z}} v_r w z^{-r-1} \end{aligned}$$

is a $\mathbb{C}[[h]]$-module map which satisfies the following: For any integer $n \geqslant 0$ and elements $u, v \in V$, $w \in W$ there exists a nonzero polynomial $p(x_1, x_2)$ in $\mathbb{C}[x_1, x_2]$ such that

$$\begin{aligned} &p(z_0+z_2, z_2)Y_W(u, z_0+z_2)Y_W(v, z_2)w \\ &\quad - p(z_0+z_2, z_2)Y_W\big(Y(u, z_0)v, z_2\big)w \in h^n W[[z_0^{\pm 1}, z_2^{\pm 1}]]; \end{aligned} \tag{2.2}$$

and for any $w \in W$ we have $Y_W(\mathbf{1}, z)w = w$.

Let W be a $\mathbb{C}[[h]]$-module. For any $a, b \in W[[z_0^{\pm 1}, z_1^{\pm 1}, \ldots]]$ and $n \geqslant 0$ we will write

$$a \underset{h^n}{\sim} b \qquad \text{if} \qquad a - b \in h^n W[[z_0^{\pm 1}, z_1^{\pm 1}, \ldots]].$$

Lemma 3.3. *Let V be an h-adic nonlocal vertex algebra and let W be a quasi V-module. Suppose that the elements $a, b \in V$ and $w_1, w_2 \in W$ satisfy*

$$[Y_W(a, z_1), Y_W(b, z_2)]w_i = 0 \quad \textit{for } i = 1, 2. \tag{2.3}$$

Then, for any integers p, t, n, $n \geqslant 0$, there exist scalars $\alpha_{r,s} \in \mathbb{C}$, which do not depend on $i = 1, 2$, such that

$$(a_p b)_t w_i \underset{h^n}{\sim} \sum_{r,s\in\mathbb{Z}} \alpha_{r,s} a_r b_s w_i \quad \textit{for } i = 1, 2. \tag{2.4}$$

Proof. Fix integers $p, t, n, n \geqslant 0$. By (2.2), there exist nonzero polynomials $p_i(x_1, x_2)$ in $\mathbb{C}[x_1, x_2]$, where $i = 1, 2$, such that

$$p_i(z_0 + z_2, z_2) Y_W(a, z_0 + z_2) Y_W(b, z_2) w_i \underset{h^n}{\sim} p_i(z_0 + z_2, z_2) Y_W\big(Y(a, z_0)b, z_2\big) w_i. \tag{2.5}$$

Consider the left hand side in (2.5). Due to (2.3), there exist an integer $l \geqslant 0$ such that

$$(z_0 + z_2)^l z_2^l Y_W(a, z_0 + z_2) Y_W(b, z_2) w_i = X_i(z_0, z_2) + h^n Z_i(z_0, z_2), \quad i = 1, 2, \tag{2.6}$$

for some $X_i(z_0, z_2) \in W[[z_0, z_2]][h]$ and $Z_i(z_0, z_2) \in W((z_0))((z_2))[[h]]$. Indeed, we can set $l = \max\{l_1, l_2, k_1, k_2\}$, where l_i and k_i are chosen so that the expression

$$z_1^{l_i} z_2^{k_i} Y_W(a, z_1) Y_W(b, z_2) w_i = z_1^{l_i} z_2^{k_i} Y_W(b, z_2) Y_W(a, z_1) w_i, \quad i = 1, 2,$$

possesses only nonnegative powers of the variables z_1, z_2 modulo h^n. Equality (2.6) implies that there exist scalars $\beta_{r,s} \in \mathbb{C}$, which do not depend on $i = 1, 2$, such that the coefficient of $z_0^{-p-1} z_2^{-t-1}$ in $X_i(z_0, z_2)$ is equal to

$$\sum_{r,s} \beta_{r,s} a_r b_s w_i \quad \bmod h^n \qquad \text{for } i = 1, 2. \tag{2.7}$$

By combining relations (2.5) and (2.6) we obtain

$$p_i(z_0 + z_2, z_2) X_i(z_0, z_2) \underset{h^n}{\sim} p_i(z_0 + z_2, z_2)(z_0 + z_2)^l z_2^l Y_W\big(Y(a, z_0)b, z_2\big) w_i, \quad i = 1, 2. \tag{2.8}$$

The left hand side in (2.8), as well as $X_i(z_0, z_2)$, possesses only nonnegative powers of the variables z_0 and z_2, while the right hand side in (2.8), as well as the expression $Y_W\big(Y(a, z_0)b, z_2\big) w_i$, belongs to $W((z_2))((z_0))[[h]]$. Hence, we can multiply (2.8) by the inverse of the polynomial $p(z_2 + z_0, z_2)$ in $\mathbb{C}((z_2))((z_0))$, thus getting

$$X_i(z_0, z_2) \underset{h^n}{\sim} (z_0 + z_2)^l z_2^l Y_W\big(Y(a, z_0)b, z_2\big) w_i, \quad i = 1, 2. \tag{2.9}$$

Next, we multiply (2.9) by the inverse of the polynomial $(z_2 + z_0)^l z_2^l$ in $\mathbb{C}((z_2))((z_0))$, which gives us

$$\left((z_2 + z_0)^l z_2^l\right)^{-1} \cdot X_i(z_0, z_2) \underset{h^n}{\sim} Y_W\big(Y(a, z_0)b, z_2\big) w_i, \quad i = 1, 2. \tag{2.10}$$

In particular, the coefficients of $z_0^{-p-1} z_2^{-t-1}$ in (2.10) coincide modulo h^n. Recall (2.7). Clearly, there exist scalars $\alpha_{r,s} \in \mathbb{C}$, which do not depend on $i = 1, 2$, such that the coefficient of $z_0^{-p-1} z_2^{-t-1}$ on the left hand side in (2.10) equals

$$\sum_{r,s \in \mathbb{Z}} \alpha_{r,s} a_r b_s w_i \quad \bmod h^n \qquad \text{for } i = 1, 2.$$

Since the coefficient of z_0^{-p-1} on the right hand side in (2.10) equals $Y_W(a_p b, z_2) w_i$, by taking the coefficient of z_2^{-t-1} we obtain

$$\sum_{r,s \in \mathbb{Z}} \alpha_{r,s} a_r b_s w_i \underset{h^n}{\sim} (a_p b)_t w_i, \quad i = 1, 2,$$

as required. □

As with quantum VOAs in [9], we can introduce the center of an h-adic nonlocal vertex algebra V in analogy with vertex algebra theory; see, e.g., [7, Chap. 2]. Define the *center* of V as the $\mathbb{C}[[h]]$-submodule

$$\mathfrak{z}(V) = \{v \in V \ : \ w_r v = 0 \text{ for all } w \in V \text{ and } r \geqslant 0\}.$$

It is worth noting that, in contrast with the vertex algebra theory, the center of an h-adic nonlocal vertex algebra does not need to be commutative; see [9, Proposition 4.2].

Proposition 3.4. *Let V be an h-adic nonlocal vertex algebra and let W be a quasi V-module. Suppose that the center $\mathfrak{z}(V)$ is a commutative associative algebra, with respect to the product $a \cdot b := a_{-1}b$ for $a, b \in \mathfrak{z}(V)$. Furthermore, assume that the algebra $\mathfrak{z}(V)$ is topologically generated, with respect to the h-adic topology, by some family $\Phi \subseteq \mathfrak{z}(V)$.*

(a) *If $[Y_W(a, z_1), Y_W(b, z_2)] = 0$ for all $a, b \in \Phi$, then*

$$[Y_W(a, z_1), Y_W(b, z_2)] = 0 \qquad \textit{for all } a, b \in \mathfrak{z}(V). \tag{2.11}$$

1. (b) *If $\psi : W \to W$ is a $\mathbb{C}[[h]]$-module map satisfying $[Y_W(a, z), \psi] = 0$ for all $a \in \Phi$, then*

$$[Y_W(a, z), \psi] = 0 \qquad \textit{for all } a \in \mathfrak{z}(V). \tag{2.12}$$

Proof. Let a, b, c be elements of the center $\mathfrak{z}(V)$ such that the pairs (a, b), (b, c) and (a, c) satisfy (2.11). In order to prove (a), it is sufficient to verify that the pair $(a \cdot b, c) = (a_{-1}b, c)$ satisfies (2.11). Fix $w \in W$ and integers $p, t, n, n \geqslant 0$. Due to our assumption, the pair (a, b) satisfies (2.11), so Lemma 3.3 implies that there exist scalars $\alpha_{r,s} \in \mathbb{C}$ such that

$$(a \cdot b)_t c_p w \underset{h^n}{\sim} \sum_{r,s \in \mathbb{Z}} \alpha_{r,s} a_r b_s c_p w \qquad \text{and} \qquad (a \cdot b)_t w \underset{h^n}{\sim} \sum_{r,s \in \mathbb{Z}} \alpha_{r,s} a_r b_s w. \tag{2.13}$$

Since the pairs (a, c) and (b, c) satisfy (2.11), we have $[b_s, c_p] = [a_r, c_p] = 0$ on W, so by relations in (2.13) we have

$$\begin{aligned}(a \cdot b)_t c_p w \underset{h^n}{\sim} \sum_{r,s \in \mathbb{Z}} \alpha_{r,s} a_r b_s c_p w &= \sum_{r,s \in \mathbb{Z}} \alpha_{r,s} a_r c_p b_s w \\ &= \sum_{r,s \in \mathbb{Z}} \alpha_{r,s} c_p a_r b_s w \underset{h^n}{\sim} c_p (a \cdot b)_t w.\end{aligned}$$

Hence we proved $(a \cdot b)_t c_p w \underset{h^n}{\sim} c_p (a \cdot b)_t w$. Since n was arbitrary, we conclude that

$$(a \cdot b)_t c_p w = c_p (a \cdot b)_t w.$$

Finally, since integers p, t and element $w \in W$ were arbitrary, this gives us

$$[Y_W(a \cdot b, z_1), Y_W(c, z_2)] = 0,$$

as required. Statement (b) can be proved analogously. $\square$

3.2. Vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$ as a quantum VOA. Let n and m be positive integers. For the families of variables $u=(u_1,\ldots,u_n)$ and $v=(v_1,\ldots,v_m)$ and the variable z consider the functions with values in $(\operatorname{End}\mathbb{C}^N)^{\otimes n}\otimes(\operatorname{End}\mathbb{C}^N)^{\otimes m}$

$$\overline{R}_{nm}^{12}(u|v|z)=\prod_{i=1,\ldots,n}^{\longrightarrow}\ \prod_{j=n+1,\ldots,n+m}^{\longleftarrow}\overline{R}_{ij}(z+u_i-v_{j-n}),\tag{2.14}$$

$$\underline{\overline{R}}_{nm}^{12}(u|v|z)=\prod_{i=1,\ldots,n}^{\longrightarrow}\ \prod_{j=n+1,\ldots,n+m}^{\longrightarrow}\overline{R}_{ij}(z+u_i+v_{j-n}).\tag{2.15}$$

The functions $R_{nm}^{12}(u|v|z)$ and $\underline{R}_{nm}^{12}(u|v|z)$ corresponding to R-matrix (1.2) can be defined analogously. In (2.14)–(2.15), as well as in the rest of the paper, we use the common expansion convention: expressions of the form $(a_1z_1+\cdots+a_nz_n)^k$, where $a_i\in\mathbb{C}, a_i\neq 0$ and $k<0$, are expanded in negative powers of the variable appearing on the left, e.g.,

$$(z_1-z_2)^{-1}=\sum_{l\geqslant 0}\frac{z_2^l}{z_1^{l+1}}\in\mathbb{C}[z_1^{-1}][[z_2]]\quad\text{and}\quad(-z_2+z_1)^{-1}$$
$$=-\sum_{l\geqslant 0}\frac{z_1^l}{z_2^{l+1}}\in\mathbb{C}[z_2^{-1}][[z_1]].$$

In particular, (2.14)–(2.15) contain only nonnegative powers of the variables u_i and v_{j-n}.

Define the following operators on $(\operatorname{End}\mathbb{C}^N)^{\otimes n}\otimes\mathcal{V}_c(\mathfrak{gl}_N)$:

$$T_n^+(u|z)=T_1^+(z+u_1)\ldots T_n^+(z+u_n)\quad\text{and}\quad T_n(u|z)=T_1(z+u_1)\ldots T_n(z+u_n).$$

Using (1.7)–(1.9), one can easily verify the following equations for the operators on $(\operatorname{End}\mathbb{C}^N)^{\otimes n}\otimes(\operatorname{End}\mathbb{C}^N)^{\otimes m}\otimes\mathcal{V}_c(\mathfrak{gl}_N)$, originally given in [4], which employ the superscript notation introduced prior to Proposition 2.2.

$$R_{nm}^{12}(u|v|z-w)T_n^{+13}(u|z)T_m^{+23}(v|w)=T_m^{+23}(v|w)T_n^{+13}(u|z)R_{nm}^{12}(u|v|z-w),\tag{2.16}$$

$$R_{nm}^{12}(u|v|z-w)T_n^{13}(u|z)T_m^{23}(v|w)=T_m^{23}(v|w)T_n^{13}(u|z)R_{nm}^{12}(u|v|z-w),\tag{2.17}$$

$$\begin{aligned}&\overline{R}_{nm}^{12}(u|v|z-w+hc/2)T_n^{13}(u|z)T_m^{+23}(v|w)\\&\qquad=T_m^{+23}(v|w)T_n^{13}(u|z)\overline{R}_{nm}^{12}(u|v|z-w-hc/2).\end{aligned}\tag{2.18}$$

The next theorem, which is due to Etingof and Kazhdan [4], introduces the structure of quantum VOA on the vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$. Roughly speaking, quantum VOA, as defined in [4], is an h-adic nonlocal vertex algebra $(V,Y,\mathbf{1})$ equipped with the $\mathbb{C}[[h]]$-module map $\mathcal{S}(z)\colon V\otimes V\to V\otimes V\otimes\mathbb{C}((z))$ (with the tensor products being h-adically completed) satisfying the *$\mathcal{S}$-locality*: For any integer $n\geqslant 0$ and elements $v,w\in V$ there exists an integer $k\geqslant 0$ such that for any $u\in V$

$$\begin{aligned}&(z_1-z_2)^kY(z_1)\big(1\otimes Y(z_2)\big)\big(\mathcal{S}(z_1-z_2)(v\otimes w)\otimes u\big)\\&\quad-(z_1-z_2)^kY(z_2)\big(1\otimes Y(z_1)\big)(w\otimes v\otimes u)\in h^nV[[z_1^{\pm 1},z_2^{\pm 1}]];\end{aligned}\tag{2.19}$$

and several other properties. In this paper, we only use $\mathcal{S}$-locality (2.19) and the underlying structure of an h-adic nonlocal vertex algebra on $\mathcal{V}_c(\mathfrak{gl}_N)$, so we omit the original definition of quantum VOA.

Theorem 3.5. *For any $c \in \mathbb{C}$ there exists a unique structure of quantum VOA on $\mathcal{V}_c(\mathfrak{gl}_N)$ such that the vacuum vector is $\mathbf{1} \in \mathcal{V}_c(\mathfrak{gl}_N)$, the vertex operator map is given by*

$$Y\big(T_n^+(u|0)\,\mathbf{1}, z\big) = T_n^+(u|z)\,T_n(u|z + hc/2)^{-1} \tag{2.20}$$

and the map $\mathcal{S}(z)$ is defined by the relation

$$\begin{aligned}\mathcal{S}^{34}(z)\Big(&\overline{R}_{nm}^{12}(u|v|z)^{-1}T_m^{+24}(v|0)\overline{R}_{nm}^{12}(u|v|z-hc)T_n^{+13}(u|0)(\mathbf{1}\otimes\mathbf{1})\Big)\\ &= T_n^{+13}(u|0)\overline{R}_{nm}^{12}(u|v|z+hc)^{-1}T_m^{+24}(v|0)\overline{R}_{nm}^{12}(u|v|z)(\mathbf{1}\otimes\mathbf{1})\end{aligned} \tag{2.21}$$

for operators on $(\mathrm{End}\,\mathbb{C}^N)^{\otimes n} \otimes (\mathrm{End}\,\mathbb{C}^N)^{\otimes m} \otimes \mathcal{V}_c(\mathfrak{gl}_N) \otimes \mathcal{V}_c(\mathfrak{gl}_N)$.

3.3. Vacuum module $\mathcal{W}_c(\mathfrak{gl}_N)$ as a quasi $\mathcal{V}_{2c}(\mathfrak{gl}_N)$-module. Consider the operators on $(\mathrm{End}\,\mathbb{C}^N)^{\otimes n} \otimes \mathcal{W}_c(\mathfrak{gl}_N)$ given by

$$\underline{B}_n^+(u|z) = \prod_{i=1,\dots,n}^{\longrightarrow} \big(B_i^+(z+u_i)\overline{R}_{i\,i+1}(2z+u_i+u_{i+1})\dots\overline{R}_{in}(2z+u_i+u_n)\big) \quad \text{and}$$

$$\underline{B}_n(u|z) = \prod_{i=1,\dots,n}^{\longrightarrow} \big(B_i(z+u_i)\overline{R}_{i\,i+1}(2z+u_i+u_{i+1}+hc)\dots\overline{R}_{in}(2z+u_i+u_n+hc)\big).$$

The next proposition can be proved by using Proposition 2.2.

Proposition 3.6. *Let n and m be positive integers. The following equalities hold for the operators on* $(\mathrm{End}\,\mathbb{C}^N)^{\otimes n} \otimes (\mathrm{End}\,\mathbb{C}^N)^{\otimes m} \otimes \mathcal{W}_c(\mathfrak{gl}_N)$*:*

$$\begin{aligned}&R_{nm}^{12}(u|v|z-w)\underline{B}_n^{+13}(u|z)\underline{R}_{nm}^{12}(u|v|z+w)\underline{B}_m^{+23}(v|w)\\ &\quad= \underline{B}_m^{+23}(v|w)\underline{R}_{nm}^{12}(u|v|z+w)\underline{B}_n^{+13}(u|z)R_{nm}^{12}(u|v|z-w),\end{aligned} \tag{2.22}$$

$$\begin{aligned}&R_{nm}^{12}(u|v|z-w)\underline{B}_n^{13}(u|z)\underline{R}_{nm}^{12}(u|v|z+w+hc)\underline{B}_m^{23}(v|w)\\ &\quad= \underline{B}_m^{23}(v|w)\underline{R}_{nm}^{12}(u|v|z+w+hc)\underline{B}_n^{13}(u|z)R_{nm}^{12}(u|v|z-w),\end{aligned} \tag{2.23}$$

$$\begin{aligned}&\overline{R}_{nm}^{12}(u|v|z-w+3hc/2)\underline{B}_n^{13}(u|z)\underline{\overline{R}}_{nm}^{12}(u|v|z+w-hc/2)\underline{B}_m^{+23}(v|w)\\ &\quad= \underline{B}_m^{+23}(v|w)\underline{\overline{R}}_{nm}^{12}(u|v|z+w+3hc/2)\underline{B}_n^{13}(u|z)\overline{R}_{nm}^{12}(u|v|z-w-hc/2).\end{aligned} \tag{2.24}$$

The following theorem is our main result.

Theorem 3.7. *For any $c \in \mathbb{C}$ there exists a unique structure of quasi $\mathcal{V}_{2c}(\mathfrak{gl}_N)$-module on the vacuum module $\mathcal{W}_c(\mathfrak{gl}_N)$ such that*

$$Y_{\mathcal{W}_c(\mathfrak{gl}_N)}(T_n^+(u|0)\,\mathbf{1}, z) = \underline{B}_n^+(u|z)\underline{B}_n(u|z+hc/2)^{-1}. \tag{2.25}$$

Proof. Set $\mathcal{W}_c = \mathcal{W}_c(\mathfrak{gl}_N)$. We first prove that map (2.25) is well-defined. It is sufficient to verify that $a \mapsto Y_{\mathcal{W}_c}(a, z)$ maps the ideal of relations (1.8) to itself since, due to Poincaré–Birkhoff–Witt theorem for the double Yangian [9, Theorem 2.2], $\mathrm{Y}^+(\mathfrak{gl}_N)$ is isomorphic to the algebra generated by the elements $t_{ij}^{(-r)}$, where $r = 1, 2\dots$ and

$i, j = 1, \dots, N$, subject to (1.8). Set $\widehat{R}_{k\,k+1} = R_{k\,k+1}(u_k - u_{k+1})$, where $1 \leqslant k < n$. Relation (1.8) implies

$$\begin{aligned}&\widehat{R}_{k\,k+1} T_n^+(u|0)\,\mathbf{1}\\&\quad = T_1^+(u_1)\dots T_{k-1}^+(u_{k-1}) T_{k+1}^+(u_{k+1}) T_k^+(u_k) T_{k+2}^+(u_{k+2})\dots T_n^+(u_n)\,\mathbf{1}\,\widehat{R}_{k\,k+1}.\end{aligned} \tag{2.26}$$

Set $\widetilde{R}^z_{ij} = \overline{R}_{ij}(2z + u_i + u_j)$ and $\widetilde{R}^{z+hc}_{ij} = \overline{R}_{ij}(2z + u_i + u_j + 2hc)$. Due to Yang–Baxter equation (1.3) and unitarity (1.6), for any indices $1 \leqslant j < k < k+1 < l \leqslant n$ we have

$$\widehat{R}_{k\,k+1}\widetilde{R}^z_{jk}\widetilde{R}^z_{j\,k+1} = \widetilde{R}^z_{j\,k+1}\widetilde{R}^z_{jk}\widehat{R}_{k\,k+1} \quad \text{and} \quad \widehat{R}_{k\,k+1}\widetilde{R}^z_{kl}\widetilde{R}^z_{k+1\,l} = \widetilde{R}^z_{k+1\,l}\widetilde{R}^z_{kl}\widehat{R}_{k\,k+1}. \tag{2.27}$$

Relation (2.22), together with (2.27), implies

$$\begin{aligned}&\widehat{R}_{k\,k+1}\underline{B}^+_n(u|z) = \underline{B}^+_{n,k\leftrightarrow k+1}(u|z)\widehat{R}_{k\,k+1}, \qquad \text{where}\\&\underline{B}^+_{n,k\leftrightarrow k+1}(u|z) = \overrightarrow{\prod_{i=1,\dots,k-1}} \left(B_i^+(z+u_i)\widetilde{R}^z_{i\,i+1}\dots\widetilde{R}^z_{i\,k-1}\widetilde{R}^z_{i\,k+1}\widetilde{R}^z_{i\,k}\widetilde{R}^z_{i\,k+2}\dots\widetilde{R}^z_{i\,n}\right)\\&\quad\cdot\left(B_{k+1}^+(z+u_{k+1})\widetilde{R}^z_{k+1\,k}\widetilde{R}^z_{k+1\,k+2}\dots\widetilde{R}^z_{k+1\,n}\right)\cdot\left(B_k^+(z+u_k)\widetilde{R}^z_{k\,k+2}\dots\widetilde{R}^z_{k\,n}\right)\\&\quad\cdot\overrightarrow{\prod_{i=k+2,\dots,n}}\left(B_i^+(z+u_i)\widetilde{R}^z_{i\,i+1}\dots\widetilde{R}^z_{i\,n}\right).\end{aligned} \tag{2.28}$$

Next, due to Yang–Baxter equation (1.3) and unitarity (1.6) we have

$$\widehat{R}_{k\,k+1}(\widetilde{R}^{z+hc}_{k+1\,l})^{-1}(\widetilde{R}^{z+hc}_{kl})^{-1} = (\widetilde{R}^{z+hc}_{kl})^{-1}(\widetilde{R}^{z+hc}_{k+1\,l})^{-1}\widehat{R}_{k\,k+1}, \tag{2.29}$$

$$\widehat{R}_{k\,k+1}(\widetilde{R}^{z+hc}_{j\,k+1})^{-1}(\widetilde{R}^{z+hc}_{jk})^{-1} = (\widetilde{R}^{z+hc}_{jk})^{-1}(\widetilde{R}^{z+hc}_{j\,k+1})^{-1}\widehat{R}_{k\,k+1} \tag{2.30}$$

for $1 \leqslant j < k < k+1 < l \leqslant n$. Relation (2.23), together with (2.29)–(2.30), implies

$$\begin{aligned}&\widehat{R}_{k\,k+1}\underline{B}_n(u|z+hc/2)^{-1} = \underline{B}_{n,k\leftrightarrow k+1}(u|z+hc/2)^{-1}\widehat{R}_{k\,k+1}, \qquad \text{where}\\&\underline{B}_{n,k\leftrightarrow k+1}(u|z+hc/2) = \overrightarrow{\prod_{i=1,\dots,k-1}}\Bigg(B_i(z+u_i+hc/2)\widetilde{R}^{z+hc}_{i\,i+1}\dots\widetilde{R}^{z+hc}_{i\,k-1}\\&\quad\cdot\widetilde{R}^{z+hc}_{i\,k+1}\widetilde{R}^{z+hc}_{i\,k}\widetilde{R}^{z+hc}_{i\,k+2}\dots\widetilde{R}^{z+hc}_{i\,n}\Bigg)\\&\quad\cdot\left(B_{k+1}(z+u_{k+1}+hc/2)\widetilde{R}^{z+hc}_{k+1\,k}\widetilde{R}^{z+hc}_{k+1\,k+2}\dots\widetilde{R}^{z+hc}_{k+1\,n}\right)\\&\quad\cdot\left(B_k(z+u_k+hc/2)\widetilde{R}^{z+hc}_{k\,k+2}\dots\widetilde{R}^{z+hc}_{k\,n}\right)\\&\quad\cdot\overrightarrow{\prod_{i=k+2,\dots,n}}\left(B_i(z+u_i+hc/2)\widetilde{R}^{z+hc}_{i\,i+1}\dots\widetilde{R}^{z+hc}_{i\,n}\right).\end{aligned} \tag{2.31}$$

Finally, by applying the map $a \mapsto Y_{\mathcal{W}_c}(a, z)$ on the left hand side of (2.26), we obtain

$$\widehat{R}_{k\,k+1}\underline{B}^+_n(u|z)\underline{B}_n(u|z+hc/2)^{-1}.$$

By (2.28) and (2.31) this is equal to

$$\underline{B}^+_{n,k\leftrightarrow k+1}(u|z)\underline{B}_{n,k\leftrightarrow k+1}(u|z+hc/2)^{-1}\widehat{R}_{k\,k+1}. \tag{2.32}$$

However, (2.32) coincides with the image of the right hand side in (2.26), with respect to the map $a \mapsto Y_{\mathcal{W}_c}(a, z)$, so we conclude that $Y_{\mathcal{W}_c}(z)$ is well-defined.

It is clear that (2.25) determines the map $Y_{\mathcal{W}_c}(z)$ uniquely. Our next goal is to show that the image of $Y_{\mathcal{W}_c}(z)$ belongs to $\mathcal{W}_c((z))[[h]]$. Relation (2.24) implies

$$\begin{aligned}&\overline{R}_{nm}^{12}(u|v|z+2hc)\underline{B}_n^{13}(u|z+hc/2)\underline{\overline{R}}_{nm}^{12}(u|v|z)\underline{B}_m^{+23}(v)\\ &\quad=\underline{B}_m^{+23}(v)\underline{\overline{R}}_{nm}^{12}(u|v|z+2hc)\underline{B}_n^{13}(u|z+hc/2)\overline{R}_{nm}^{12}(u|v|z).\end{aligned} \tag{2.33}$$

Observe that

$$\underline{B}_n^{13}(u|z+hc/2)^{-1}\,\mathbf{1} = \widehat{G}\,\mathbf{1} \quad \text{for} \quad \widehat{G} = G_1 \dots G_n,$$

so, by using (1.27) and (2.33) and arguing as in the proof of Equality (1.33) we get

$$\begin{aligned}&Y_{\mathcal{W}_c}(T_n^+(u|0)\,\mathbf{1}, z)B_{n+1}^+(v_1)\dots B_{n+m}^+(v_m)\,\mathbf{1}\\ &\quad=\underline{B}_n^{+13}(u|z)\underline{B}_n^{13}(u|z+hc/2)^{-1}B_{n+1}^+(v_1)\dots B_{n+m}^+(v_m)\,\mathbf{1}\\ &\quad=\underline{B}_n^{+13}(u|z)\Big(^{rl}(L)\cdot\Big(^{rl}(K)\\ &\qquad\cdot\Big(\overline{\underline{R}}_{nm}^{12}(u|v|z)\underline{B}_m^{+23}(v)\overline{R}_{nm}^{12}(u|v|z)^{-1}\,\widehat{G}\,\mathbf{1}\,\underline{\overline{R}}_{nm}^{12}(u|v|z+2hc)^{-1}\Big)\Big)\Big),\end{aligned} \tag{2.34}$$

where $K = \overline{R}_{nm}^{12}(u|v|z+2hc+hN)$ and $L = L^{(n,m)}$ is given by (1.31). Recall that the R-matrix $\overline{R}(x)$ belongs to $(\operatorname{End}\mathbb{C}^N)[x^{-1}][[h]]$. Therefore, the right hand side of (2.34) is a Taylor series in the variables $u_1, \dots, u_n, v_1, \dots, v_m$ and h such that the coefficient of each monomial $u_1^{a_1}\dots u_n^{a_n}v_1^{b_1}\dots v_m^{b_m}h^b$ possesses only finitely many negative powers of the variable z. This implies that the image of $Y_{\mathcal{W}_c}(z)$ belongs to $\mathcal{W}_c((z))[[h]]$.

The property $Y_{\mathcal{W}_c}(\mathbf{1}, z) = 1_{\mathcal{W}_c}$ is clear, so it remains to prove (2.2). Consider the second summand in (2.2). By applying the vertex operator map $Y(z_0)$ for the quantum VOA $\mathcal{V}_{2c}(\mathfrak{gl}_N)$, as defined in Theorem 3.5, on the series[1]

$$T_n^{+13}(u|0)\overline{R}_{nm}^{12}(u|v|z_0+2hc)^{-1}T_m^{+24}(v|0)(\mathbf{1}\otimes\mathbf{1}), \tag{2.35}$$

whose coefficients belong to $(\operatorname{End}\mathbb{C}^N)^{\otimes n}\otimes(\operatorname{End}\mathbb{C}^N)^{\otimes m}\otimes\mathcal{V}_{2c}(\mathfrak{gl}_N)\otimes\mathcal{V}_{2c}(\mathfrak{gl}_N)$, we get

$$T_n^{+13}(u|z_0)T_n^{13}(u|z_0+hc)^{-1}\overline{R}_{nm}^{12}(u|v|z_0+2hc)^{-1}T_m^{+23}(v|0)\,\mathbf{1}\,.$$

Due to (2.18) at the level $2c$ and $T_n^{13}(u|z_0+hc)^{-1}\,\mathbf{1} = \mathbf{1}$ this equals to

$$T_n^{+13}(u|z_0)T_m^{+23}(v|0)\,\mathbf{1}\,\overline{R}_{nm}^{12}(u|v|z_0)^{-1}, \tag{2.36}$$

which is a series with coefficients in $(\operatorname{End}\mathbb{C}^N)^{\otimes n}\otimes(\operatorname{End}\mathbb{C}^N)^{\otimes m}\otimes\mathcal{V}_{2c}(\mathfrak{gl}_N)$. Finally, by applying the map $a \mapsto Y_{\mathcal{W}_c}(a, z_2)$ on (2.36) we get

$$\underline{B}_{n+m}^+(x|z_2)\underline{B}_{n+m}(x|z_2+hc/2)^{-1}\overline{R}_{nm}^{12}(u|v|z_0)^{-1}, \tag{2.37}$$

where x denotes the $n+m$ variables $x = (z_0+u_1, \dots, z_0+u_n, v_1, \dots, v_m)$.

[1] It is possible (and perhaps more natural) to prove (2.2) by starting from $T_n^{+13}(u|0)T_m^{+24}(v|0)(\mathbf{1}\otimes\mathbf{1})$ instead of (2.35). However, this requires the use of ordered products, as defined in Sect. 2.2, thus making the calculations seemingly more complicated, even though the proof remains analogous.

Let us consider the first summand in (2.2). By applying $Y_{\mathcal{W}_c}(z_0+z_2)(1\otimes Y_{\mathcal{W}_c}(z_2))$ on (2.35) we obtain

$$\underline{B}_n^{+13}(u|z_0+z_2)\underline{B}_n^{13}(u|z_0+z_2+hc/2)^{-1} \cdot \overline{R}_{nm}^{12}(u|v|z_0+2hc)^{-1}\underline{B}_m^{+23}(v|z_2)\underline{B}_m^{23}(v|z_2+hc/2)^{-1}. \tag{2.38}$$

Using (2.24) we can express $\underline{B}_n^{13}(u|z_0+z_2+hc/2)^{-1}\overline{R}_{nm}^{12}(u|v|z_0+2hc)^{-1}\underline{B}_m^{+23}(v|z_2)$ as

$$\overline{\underline{R}}_{nm}^{12}(u|v|z_0+2z_2)\underline{B}_m^{+23}(v|z_2) \cdot \overline{R}_{nm}^{12}(u|v|z_0)^{-1}\underline{B}_n^{13}(u|z_0+z_2+hc/2)^{-1}\overline{\underline{R}}_{nm}^{12}(u|v|z_0+2z_2+2hc)^{-1},$$

so that (2.38) is equal to

$$\underline{B}_n^{+13}(u|z_0+z_2)\overline{\underline{R}}_{nm}^{12}(u|v|z_0+2z_2)\underline{B}_m^{+23}(v|z_2)\overline{R}_{nm}^{12}(u|v|z_0)^{-1} \cdot \underline{B}_n^{13}(u|z_0+z_2+hc/2)^{-1}\overline{\underline{R}}_{nm}^{12}(u|v|z_0+2z_2+2hc)^{-1}\underline{B}_m^{23}(v|z_2+hc/2)^{-1}. \tag{2.39}$$

Finally, we rewrite (2.39) using (2.23), thus getting

$$\underline{B}_n^{+13}(u|z_0+z_2)\overline{\underline{R}}_{nm}^{12}(u|v|z_0+2z_2)\underline{B}_m^{+23}(v|z_2)\underline{B}_m^{23}(v|z_2+hc/2)^{-1} \cdot \overline{\underline{R}}_{nm}^{12}(u|v|z_0+2z_2+2hc)^{-1}\underline{B}_n^{13}(u|z_0+z_2+hc/2)^{-1}\overline{R}_{nm}^{12}(u|v|z_0)^{-1}. \tag{2.40}$$

Expressions (2.37) and (2.40) are not equal, even though they do coincide when viewed as Taylor series in the variables $u_1,\dots,u_n,v_1,\dots,v_m,h$ whose coefficients are rational functions in z_0,z_2. Indeed, due to our expansion convention, the operators and R-matrices in (2.37), whose arguments contain both the variables z_0 and z_2, should be expanded in nonnegative powers of z_0, while the same operators and R-matrices in (2.40) should be expanded in nonnegative powers of z_2. Fix an integer $k\geqslant 0$ and an element $w\in\mathcal{W}_c$. Apply both (2.37) and (2.40) on w and denote the resulting expressions by $P(u,v,z_0,z_2)$ and $S(u,v,z_0,z_2)$ respectively. Then, for any choice of integers $a_1,\dots,a_n\geqslant 0$ and $b_1,\dots,b_m\geqslant 0$ there exist an integer $r\geqslant 0$ such that the coefficients of $u_1^{a_1}\dots u_n^{a_n}v_1^{b_1}\dots v_m^{b_m}$ in

$$(z_0+z_2)^r(z_0+2z_2)^r P(u,v,z_0,z_2) \quad\text{and}\quad (z_0+z_2)^r(z_0+2z_2)^r S(u,v,z_0,z_2)$$

coincide modulo h^k, which implies (2.2). □

The map $Y_{\mathcal{W}_c(\mathfrak{gl}_N)}(z)$ satisfies the following "twisted" $\mathcal{S}$-locality property; cf. [13, 16].

Proposition 3.8. *For any $u,v\in\mathcal{V}_{2c}(\mathfrak{gl}_N)$ and integer $k\geqslant 0$ there exists an integer $r\geqslant 0$ such that for any $w\in\mathcal{W}_c(\mathfrak{gl}_N)$*

$$\begin{aligned}&(z_1^2-z_2^2)^r\, Y_{\mathcal{W}_c(\mathfrak{gl}_N)}(z_1)\big(1\otimes Y_{\mathcal{W}_c(\mathfrak{gl}_N)}(z_2)\big)\big(\mathcal{S}(z_1-z_2)(u\otimes v)\otimes w\big)\\&\quad-(z_1^2-z_2^2)^r\, Y_{\mathcal{W}_c(\mathfrak{gl}_N)}(z_2)\big(1\otimes Y_{\mathcal{W}_c(\mathfrak{gl}_N)}(z_1)\big)(v\otimes u\otimes w)\\&\quad\in h^k\mathcal{W}_c(\mathfrak{gl}_N)[[z_1^{\pm1},z_2^{\pm1}]].\end{aligned} \tag{2.41}$$

Proof. Set $\mathcal{W}_c = \mathcal{W}_c(\mathfrak{gl}_N)$. Consider the first summand in (2.41) and set $z = z_1 - z_2$. Notice that the variable z_1 appears on the left in $z = z_1 - z_2$, so the negative powers of z should be expanded in negative powers of z_1. By applying $\mathcal{S}(z)$ at the level $2c$, as defined in (2.21), on the last two tensor factors of the expression

$$\overline{R}_{nm}^{12}(u|v|z)^{-1} T_m^{+24}(v|0) \overline{R}_{nm}^{12}(u|v|z-2hc) T_n^{+13}(u|0)(\mathbf{1}\otimes\mathbf{1}), \tag{2.42}$$

whose coefficients belong to $(\operatorname{End}\mathbb{C}^N)^{\otimes(n+m)} \otimes \mathcal{V}_{2c}(\mathfrak{gl}_N)^{\otimes 2}$, we get

$$T_n^{+13}(u|0) \overline{R}_{nm}^{12}(u|v|z+2hc)^{-1} T_m^{+24}(v|0) \overline{R}_{nm}^{12}(u|v|z)(\mathbf{1}\otimes\mathbf{1}). \tag{2.43}$$

Next, we apply $Y_{\mathcal{W}_c}(z_1)(1\otimes Y_{\mathcal{W}_c}(z_2))$ on (2.43), thus getting

$$\begin{aligned}&\underline{B}_n^{+13}(u|z_1)\underline{B}_n^{13}(u|z_1+hc/2)^{-1}\overline{R}_{nm}^{12}(u|v|z+2hc)^{-1}\\ &\qquad\cdot \underline{B}_m^{+23}(v|z_2)\underline{B}_m^{23}(v|z_2+hc/2)^{-1}\overline{R}_{nm}^{12}(u|v|z).\end{aligned} \tag{2.44}$$

We may now proceed as in calculation (2.38)–(2.40) and prove that (2.44) equals

$$\begin{aligned}&\underline{B}_n^{+13}(u|z_1)\underline{\overline{R}}_{nm}^{12}(u|v|z_1+z_2)\underline{B}_m^{+23}(v|z_2)\\ &\quad\cdot \underline{B}_m^{23}(v|z_2+hc/2)^{-1}\underline{\overline{R}}_{nm}^{12}(u|v|z_1+z_2+2hc)^{-1}\underline{B}_n^{13}(u|z_1+hc/2)^{-1}.\end{aligned} \tag{2.45}$$

Let us consider the second summand in (2.41). First, by swapping tensor factors $n+m+1$ and $n+m+2$ in (2.42) we get

$$\overline{R}_{nm}^{12}(u|v|z)^{-1} T_m^{+23}(v|0) \overline{R}_{nm}^{12}(u|v|z-2hc) T_n^{+14}(u|0)(\mathbf{1}\otimes\mathbf{1}).$$

Next, by applying $Y_{\mathcal{W}_c}(z_2)(1\otimes Y_{\mathcal{W}_c}(z_1))$ we obtain

$$\begin{aligned}&\overline{R}_{nm}^{12}(u|v|z)^{-1}\underline{B}_m^{+23}(v|z_2)\underline{B}_m^{23}(v|z_2+hc/2)^{-1}\\ &\qquad\cdot \overline{R}_{nm}^{12}(u|v|z-2hc)\underline{B}_n^{+13}(u|z_1)\underline{B}_n^{13}(u|z_1+hc/2)^{-1}.\end{aligned} \tag{2.46}$$

We now want to apply relation (2.24) on (2.46). However, the factors

$$\overline{R}_{nm}^{12}(u|v|z)^{-1} \quad\text{and}\quad \overline{R}_{nm}^{12}(u|v|z-2hc), \qquad\text{where}\quad z=z_1-z_2, \tag{2.47}$$

in (2.46) should be expanded in nonnegative powers of z_2, while (2.24) requires for the R-matrices in (2.47) to be expanded in nonnegative powers of z_1. Fix an integer $k \geqslant 0$. For any choice of integers $a_1,\ldots,a_n \geqslant 0$ and $b_1,\ldots,b_m \geqslant 0$ there exist an integer $p \geqslant 0$ such that the coefficients of all monomials $u_1^{a_1'}\ldots u_n^{a_n'} v_1^{b_1'}\ldots v_m^{b_m'}$, where $0 \leqslant a_i' \leqslant a_i$ and $0 \leqslant b_j' \leqslant b_j$, in

$$(z_1-z_2)^p\,\overline{R}_{nm}^{12}(u|v|z_1-z_2)^{-1} \quad\text{and}\quad (z_1-z_2)^p\,\overline{R}_{nm}^{12}(u|v|z_1-z_2-2hc) \tag{2.48}$$

coincide with the corresponding coefficients in

$$(z_1-z_2)^p\,\overline{R}_{nm}^{12}(u|v|-z_2+z_1)^{-1} \quad\text{and}\quad (z_1-z_2)^p\,\overline{R}_{nm}^{12}(u|v|-z_2+z_1-2hc) \tag{2.49}$$

modulo h^k. Moreover, assume that the integer p is large enough, so that the coefficients of all monomials $u_1^{a_1'} \dots u_n^{a_n'} v_1^{b_1'} \dots v_m^{b_m'}$, where $0 \leqslant a_i' \leqslant a_i$ and $0 \leqslant b_j' \leqslant b_j$, in

$$(z_1+z_2)^p\, \underline{\overline{R}}_{nm}^{12}(u|v|z_1+z_2) \quad \text{and} \quad (z_1+z_2)^p\, \underline{\overline{R}}_{nm}^{12}(u|v|z_1+z_2+2hc)^{-1} \quad (2.50)$$

coincide with the corresponding coefficients in

$$(z_1+z_2)^p\, \underline{\overline{R}}_{nm}^{12}(u|v|z_2+z_1) \quad \text{and} \quad (z_1+z_2)^p\, \underline{\overline{R}}_{nm}^{12}(u|v|z_2+z_1+2hc)^{-1} \quad (2.51)$$

modulo h^k. By using (2.24) and unitarity (1.6) we obtain

$$\underline{B}_m^{23}(v|z_2+hc/2)^{-1}\overline{R}_{nm}^{12}(u|v|-z_2+z_1-2hc)\underline{B}_n^{+13}(u|z_1) = \underline{\overline{R}}_{nm}^{12}(u|v|z_2+z_1)$$
$$\cdot\, \underline{B}_n^{+13}(u|z_1)\overline{R}_{nm}^{12}(u|v|-z_2+z_1)\underline{B}_m^{23}(v|z_2+hc/2)^{-1}\underline{\overline{R}}_{nm}^{12}(u|v|z_2+z_1+2hc)^{-1}.$$

This implies, due to the fact that certain coefficients in (2.48) and (2.50) coincide with the corresponding coefficients in (2.49) and (2.51) modulo h^k, that the product of (2.46) and $(z_1^2-z_2^2)^{2p}$ coincides with

$$(z_1^2-z_2^2)^{2p}\, \overline{R}_{nm}^{12}(u|v|z)^{-1}\underline{B}_m^{+23}(v|z_2)\underline{\overline{R}}_{nm}^{12}(u|v|z_1+z_2)\underline{B}_n^{+13}(u|z_1)\overline{R}_{nm}^{12}(u|v|z)$$
$$\cdot\, \underline{B}_m^{23}(v|z_2+hc/2)^{-1}\underline{\overline{R}}_{nm}^{12}(u|v|z_1+z_2+2hc)^{-1}\underline{B}_n^{13}(u|z_1+hc/2)^{-1} \quad (2.52)$$

modulo h^k. Finally, we rewrite (2.52) using (2.22), thus getting

$$(z_1^2-z_2^2)^{2p}\, \underline{B}_n^{+13}(u|z_1)\underline{\overline{R}}_{nm}^{12}(u|v|z_1+z_2)\underline{B}_m^{+23}(v|z_2)$$
$$\cdot\, \underline{B}_m^{23}(v|z_2+hc/2)^{-1}\underline{\overline{R}}_{nm}^{12}(u|v|z_1+z_2+2hc)^{-1}\underline{B}_n^{13}(u|z_1+hc/2)^{-1}. \quad (2.53)$$

Since (2.53) is equal to the product of $(z_1^2-z_2^2)^{2p}$ and (2.45), we conclude that (2.41) holds. □

As with the operator $\mathcal{T}(z) = Y(T^+(0)\,\mathbf{1}, z)$, see [4, 2.1.4], the proof of Proposition 3.8 implies that the operator $\mathcal{B}(z) = Y_{\mathcal{W}_c(\mathfrak{gl}_N)}(T^+(0)\,\mathbf{1}, z)$ satisfies the (slightly modified version of the) quantum current commutation relation from [22]. More precisely, for any integer $n \geqslant 0$ there exist an integer $r \geqslant 0$ such that

$$(z_1^2-z_2^2)^r\, \mathcal{B}_1(z_1)\overline{R}_{12}(z_1-z_2+2hc)^{-1}\mathcal{B}_2(z_2)\overline{R}_{12}(z_1-z_2)$$
$$\underset{h^n}{\sim}\ (z_1^2-z_2^2)^r\, \overline{R}_{12}(z_1-z_2)^{-1}\mathcal{B}_2(z_2)\overline{R}_{12}(z_1-z_2-2hc)\mathcal{B}_1(z_1).$$

4. Image of the Center $\mathfrak{z}(\mathcal{V}_{2c}(\mathfrak{gl}_N))$

In this section, we employ map $Y_{\mathcal{W}_c(\mathfrak{gl}_n)}(z)$ to find explicit formulae for families of central elements in the completed algebra $\widetilde{\mathrm{A}}_c(\mathfrak{gl}_N)$. As a consequence, we obtain families of invariants of the vacuum module $\mathcal{W}_c(\mathfrak{gl}_n)$. Also, we show that the image of the center $\mathfrak{z}(\mathcal{V}_{2c}(\mathfrak{gl}_N))$, with respect to the map $a \mapsto Y_{\mathcal{W}_c(\mathfrak{gl}_n)}(a, z)$, is commutative.

4.1. Central elements of the completed algebra $\widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N)$. Let I_p for $p \geqslant 1$ denote the left ideal of the double Yangian $\mathrm{DY}_c(\mathfrak{gl}_N)$ at the level $c \in \mathbb{C}$, generated by all elements $t_{ij}^{(r)}$ with $r \geqslant p$. As in [9], define the completed double Yangian $\widetilde{\mathrm{DY}}_c(\mathfrak{gl}_N)$ at the level c as the h-adic completion of the inverse limit $\lim_{\longleftarrow} \mathrm{DY}_c(\mathfrak{gl}_N)/I_p$. Introduce the algebra $\widetilde{\mathrm{A}}_c(\mathfrak{gl}_N)$ as the h-adic completion of the inverse limit

$$\lim_{\longleftarrow} \mathrm{A}_c(\mathfrak{gl}_N)/(\mathrm{A}_c(\mathfrak{gl}_N) \cap I_p).$$

In order to employ certain results from [9], we briefly recall the fusion procedure for the rational R-matrix originated in [10]; see also [18, Sect. 6.4] for more details. Let μ be a Young diagram with n boxes, whose length is less than or equal to N, and let $\mathcal{U} = \mathcal{U}_\mu$ be a standard μ-tableau with entries $1, \ldots, n$. For $k = 1, \ldots, n$ define the contents c_k of $\mathcal{U}$ by $c_k = j - i$ if k occupies the box (i, j) of $\mathcal{U}$. Denote by $e_{\mathcal{U}}$ the primitive idempotent in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ of the symmetric group $\mathfrak{S}_n$, which is associated with $\mathcal{U}$ through the use of the orthonormal Young bases in the irreducible representations of $\mathfrak{S}_n$. The group $\mathfrak{S}_n$ acts on the space $(\mathbb{C}^N)^{\otimes n}$ by permuting the tensor factors. Denote by $\mathcal{E}_{\mathcal{U}}$ the image of $e_{\mathcal{U}}$ with respect to this action. By [10], the consecutive evaluations $u_1 = hc_1, \ldots, u_n = hc_n$ of the function

$$R(u_1, \ldots, u_n) := \prod_{1 \leqslant i < j \leqslant n} R_{ij}(u_i - u_j),$$

where the product is taken in the lexicographical order on the pairs (i, j), are well-defined. Furthermore, the result is proportional to $\mathcal{E}_{\mathcal{U}}$, i.e.

$$R(u_1, \ldots, u_n)\big|_{u_1 = hc_1}\big|_{u_2 = hc_2} \ldots \big|_{u_n = hc_n} = p(\mu)\, \mathcal{E}_{\mathcal{U}}, \tag{3.1}$$

where $p(\mu)$ denotes the product of all hook lengths of the boxes of μ.

Define the n-tuple $u_\mu = u_{\mathcal{U}_\mu}$ by

$$u_\mu = (u_1, \ldots, u_n), \qquad \text{where} \qquad u_k = u + hc_k \quad \text{for } k = 1, \ldots, n. \tag{3.2}$$

It was proved in [9] that all coefficients of the series

$$\mathbb{T}_\mu^+(u) = \mathrm{tr}_{1,\ldots,n}\, \mathcal{E}_{\mathcal{U}} T_1^+(u_1) \ldots T_n^+(u_n)\, \mathbf{1} \in \mathcal{V}_{-N}(\mathfrak{gl}_N)[[u]], \tag{3.3}$$

where the trace is taken over all n copies of $\mathrm{End}\,\mathbb{C}^N$ in (3.3), belong to the center $\mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$. The series $\mathbb{T}_\mu^+(u)$ does not depend on the choice of the standard μ-tableau $\mathcal{U}$; see [21]. The image of the constant term in (3.3), with respect to map (2.25), equals

$$Y_{\mathcal{W}_{-N/2}(\mathfrak{gl}_N)}(\mathbb{T}_\mu^+(0), u) = \mathrm{tr}_{1,\ldots,n}\, \mathcal{E}_{\mathcal{U}} \underline{B}_n^+(u_\mu) \underline{B}_n(u_\mu - hN/4)^{-1} \tag{3.4}$$

and belongs to $\mathrm{Hom}(\mathcal{W}_{-N/2}(\mathfrak{gl}_N), \mathcal{W}_{-N/2}(\mathfrak{gl}_N)((u))[[h]])$. All coefficients of series (3.4),

$$\widetilde{\mathbb{A}}_\mu(u) := \mathrm{tr}_{1,\ldots,n}\, \mathcal{E}_{\mathcal{U}} \underline{B}_n^+(u_\mu) \underline{B}_n(u_\mu - hN/4)^{-1} \tag{3.5}$$

can be also viewed as elements of the completed algebra $\widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N)$.

Consider the tensor product

$$\mathrm{End}\,\mathbb{C}^N \otimes (\mathrm{End}\,\mathbb{C}^N)^{\otimes n} \otimes \widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N), \tag{3.6}$$

where the $n+1$ copies of $\operatorname{End}\mathbb{C}^N$ are now labeled by $0,\dots,n$. It will be convenient to denote the tensor factors $\operatorname{End}\mathbb{C}^N$, $(\operatorname{End}\mathbb{C}^N)^{\otimes n}$ and $\widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N)$ in (3.6) by the superscripts 0, 1 and 2 respectively, so that, e.g., for the variable u_0 and variables (3.2) we have

$$\overline{\underline{R}}^{01}_{1n}(u_0|u_\mu) = \overline{R}_{01}(u_0+u_1)\dots\overline{R}_{0n}(u_0+u_n).$$

The arrow at the top of the symbol will indicate that the products are written in the opposite order, e.g.,

$$\overleftarrow{\underline{R}}^{01}_{1n}(u_0|u_\mu) = \overline{R}_{0n}(u_0+u_n)\dots\overline{R}_{01}(u_0+u_1).$$

Lemma 4.1. *The following equalities hold on* $\operatorname{End}\mathbb{C}^N\otimes(\operatorname{End}\mathbb{C}^N)^{\otimes n}\otimes\widetilde{\mathrm{DY}}_c(\mathfrak{gl}_N)$*:*

$$\mathcal{E}_{\mathcal{U}}\overline{R}^{01}_{1n}(u_0|u_\mu) = \overleftarrow{R}^{01}_{1n}(u_0|u_\mu)\mathcal{E}_{\mathcal{U}}, \qquad \mathcal{E}_{\mathcal{U}}\overline{R}^{01}_{1n}(u_0|u_\mu)^{-1} = \overleftarrow{R}^{01}_{1n}(u_0|u_\mu)^{-1}\mathcal{E}_{\mathcal{U}}, \tag{3.7}$$

$$\mathcal{E}_{\mathcal{U}}\overline{\underline{R}}^{01}_{1n}(u_0|u_\mu) = \overleftarrow{\underline{R}}^{01}_{1n}(u_0|u_\mu)\mathcal{E}_{\mathcal{U}}, \qquad \mathcal{E}_{\mathcal{U}}\overline{\underline{R}}^{01}_{1n}(u_0|u_\mu)^{-1} = \overleftarrow{\underline{R}}^{01}_{1n}(u_0|u_\mu)^{-1}\mathcal{E}_{\mathcal{U}}, \tag{3.8}$$

$$\mathcal{E}_{\mathcal{U}}\underline{B}^{+12}_n(u_\mu) = \overleftarrow{\underline{B}}^{+12}_n(u_\mu)\mathcal{E}_{\mathcal{U}}, \qquad \mathcal{E}_{\mathcal{U}}\underline{B}^{12}_n(u_\mu-hN/4)^{-1} = \overleftarrow{\underline{B}}^{12}_n(u_\mu-hN/4)^{-1}\mathcal{E}_{\mathcal{U}}, \tag{3.9}$$

$$\mathcal{E}_{\mathcal{U}}T^{+12}_n(u_\mu|0) = \overleftarrow{T}^{+12}_n(u_\mu|0)\mathcal{E}_{\mathcal{U}}, \qquad \mathcal{E}_{\mathcal{U}}\overleftarrow{T}^{+12}_n(-u_\mu|0)^{-1} = T^{+12}_n(-u_\mu|0)^{-1}\mathcal{E}_{\mathcal{U}}, \tag{3.10}$$

$$\mathcal{E}_{\mathcal{U}}\overleftarrow{T}^{12}_n(-u_\mu+hN/4|0) = T^{12}_n(-u_\mu+hN/4|0)\mathcal{E}_{\mathcal{U}}, \tag{3.11}$$

$$\mathcal{E}_{\mathcal{U}}T^{12}_n(u_\mu-3hN/4|0)^{-1} = \overleftarrow{T}^{12}_n(u_\mu-3hN/4|0)^{-1}\mathcal{E}_{\mathcal{U}}, \tag{3.12}$$

where $\mathcal{E}_{\mathcal{U}}$ *is applied on the tensor factors* $1,\dots,n$*, i.e.* $\mathcal{E}_{\mathcal{U}}$ *denotes the operator* $1\otimes\mathcal{E}_{\mathcal{U}}$ *on* $\operatorname{End}\mathbb{C}^N\otimes(\operatorname{End}\mathbb{C}^N)^{\otimes n}$.

Proof. The given equalities follow from fusion procedure (3.1) with the use of Yang–Baxter equation (1.3), unitarity (1.6) and relations (1.7)–(1.9) and (1.24)–(1.26). More details on the proof can be found in [9, Proof of Theorem 2.4] (for relations (3.7)–(3.8)), first part of the proof of Theorem 3.7 (for relations (3.9)) and in [6, Proof of Theorem 3.2] (for relations (3.10)–(3.12)). As an illustration, let us prove the first equality in (3.8). For the variables $v=(u+v_1,\dots,u+v_n)$ Yang–Baxter equation (1.3) implies

$$\prod_{1\leqslant i<j\leqslant n} R_{ij}(v_i-v_j)\cdot\overline{\underline{R}}^{01}_{1n}(u_0|v) = \overleftarrow{\underline{R}}^{01}_{1n}(u_0|v)\cdot\prod_{1\leqslant i<j\leqslant n} R_{ij}(v_i-v_j), \tag{3.13}$$

where the products are written in the lexicographical order on the pairs (i,j). By applying consecutive evaluations $v_1=hc_1,\dots,v_n=hc_n$ on (3.13) and using (3.1) we get $\mathcal{E}_{\mathcal{U}}\overline{\underline{R}}^{01}_{1n}(u_0|u_\mu) = \overleftarrow{\underline{R}}^{01}_{1n}(u_0|u_\mu)\mathcal{E}_{\mathcal{U}}$, as required. □

The following is our main result in this section. Its proof adapts the standard R-matrix techniques used with RTT relations, see, e.g., [6, Theorem 3.2], to the reflection algebra setting.

Theorem 4.2. *All coefficients of* $\widetilde{\mathbb{A}}_\mu(u)$ *belong to the center of the algebra* $\widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N)$.

Proof. We first prove that for the variable u_0 and variables (3.2) the following equality holds on $\operatorname{End}\mathbb{C}^N \otimes \widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N)$:

$$B(u_0)\widetilde{\mathbb{A}}_{\mu}(u) = \widetilde{\mathbb{A}}_{\mu}(u)B(u_0). \tag{3.14}$$

By applying $B_0(u_0)$ on (3.5) and using notation as in (3.6) we get

$$\operatorname{tr}_{1,\ldots,n} \mathcal{E}_{\mathcal{U}} B_0(u_0)\underline{B}_n^{+12}(u_\mu)\underline{B}_n^{12}(u_\mu - hN/4)^{-1}. \tag{3.15}$$

As with the proof of (1.33), we employ (1.26) and (1.27) to express (3.15) as

$$\begin{aligned}&\operatorname{tr}_{1,\ldots,n} \mathcal{E}_{\mathcal{U}} \left({}^{rl}\!\left(\overline{\underline{R}}{}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1}\right) \cdot \left(\overline{R}{}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1}\underline{B}_n^{+12}(u_\mu)\right.\right.\\ &\qquad \left.\left.\overline{\underline{R}}{}_{1n}^{01}(u_0 - 3hN/4|u_\mu)B_0(u_0)\overline{R}{}_{1n}^{01}(u_0 + hN/4|u_\mu)\underline{B}_n^{12}(u_\mu - hN/4)^{-1}\right)\right).\end{aligned} \tag{3.16}$$

Since $\mathcal{E}_{\mathcal{U}}^2 = \mathcal{E}_{\mathcal{U}}$, the second equality in (3.8) implies

$$\mathcal{E}_{\mathcal{U}}K = \mathcal{E}_{\mathcal{U}}^2 K = \mathcal{E}_{\mathcal{U}}\overleftarrow{K}\mathcal{E}_{\mathcal{U}} = \mathcal{E}_{\mathcal{U}}\overleftarrow{K}\mathcal{E}_{\mathcal{U}}^2 \qquad \text{for } K = \overline{\underline{R}}{}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1}. \tag{3.17}$$

By using (3.17) we can write (3.16) as

$$\begin{aligned}&\operatorname{tr}_{1,\ldots,n} \mathcal{E}_{\mathcal{U}} \left({}^{rl}\!\left(\overleftarrow{K}\right) \cdot \left(\mathcal{E}_{\mathcal{U}}^2\overline{R}{}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1}\underline{B}_n^{+12}(u_\mu)\right.\right.\\ &\qquad \left.\left.\overline{\underline{R}}{}_{1n}^{01}(u_0 - 3hN/4|u_\mu)B_0(u_0)\overline{R}{}_{1n}^{01}(u_0 + hN/4|u_\mu)\underline{B}_n^{12}(u_\mu - hN/4)^{-1}\right)\right).\end{aligned}$$

Due to the cyclic property of the trace, this equals to

$$\begin{aligned}&\operatorname{tr}_{1,\ldots,n} \mathcal{E}_{\mathcal{U}}\overline{R}{}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1}\underline{B}_n^{+12}(u_\mu)\overline{\underline{R}}{}_{1n}^{01}(u_0 - 3hN/4|u_\mu)\\ &\qquad B_0(u_0)\overline{R}{}_{1n}^{01}(u_0 + hN/4|u_\mu)\underline{B}_n^{12}(u_\mu - hN/4)^{-1}\mathcal{E}_{\mathcal{U}}\overleftarrow{K}\mathcal{E}_{\mathcal{U}}.\end{aligned} \tag{3.18}$$

By $\mathcal{E}_{\mathcal{U}}^2 = \mathcal{E}_{\mathcal{U}}$ and the second equality in (3.7) we have

$$\begin{aligned}\mathcal{E}_{\mathcal{U}}L &= \mathcal{E}_{\mathcal{U}}^2 L = \mathcal{E}_{\mathcal{U}}\overleftarrow{L}\mathcal{E}_{\mathcal{U}} = \mathcal{E}_{\mathcal{U}}\overleftarrow{L}\mathcal{E}_{\mathcal{U}}^2 = \mathcal{E}_{\mathcal{U}}^2 L\mathcal{E}_{\mathcal{U}}\\ &= \mathcal{E}_{\mathcal{U}}L\mathcal{E}_{\mathcal{U}} \quad \text{for } L = \overline{R}{}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1}.\end{aligned}$$

Therefore, using the cyclic property of the trace and $\mathcal{E}_{\mathcal{U}}^2 = \mathcal{E}_{\mathcal{U}}$, we can write (3.18) as

$$\begin{aligned}&\operatorname{tr}_{1,\ldots,n} \overline{R}{}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1}\mathcal{E}_{\mathcal{U}}\underline{B}_n^{+12}(u_\mu)\overline{\underline{R}}{}_{1n}^{01}(u_0 - 3hN/4|u_\mu)\\ &\qquad B_0(u_0)\overline{R}{}_{1n}^{01}(u_0 + hN/4|u_\mu)\underline{B}_n^{12}(u_\mu - hN/4)^{-1}\mathcal{E}_{\mathcal{U}}\overleftarrow{K}\mathcal{E}_{\mathcal{U}}.\end{aligned}$$

We now employ first equalities in (3.7) and (3.8), together with (3.9), to move the leftmost copy of $\mathcal{E}_{\mathcal{U}}$ to the right, which gives us:

$$\begin{aligned}&\operatorname{tr}_{1,\ldots,n} \overline{R}{}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1}\overleftarrow{\underline{B}}{}_n^{+12}(u_\mu)\overleftarrow{\overline{\underline{R}}}{}_{1n}^{01}(u_0 - 3hN/4|u_\mu)\\ &\qquad B_0(u_0)\overleftarrow{\overline{R}}{}_{1n}^{01}(u_0 + hN/4|u_\mu)\overleftarrow{\underline{B}}{}_n^{12}(u_\mu - hN/4)^{-1}\mathcal{E}_{\mathcal{U}}^2\overleftarrow{K}\mathcal{E}_{\mathcal{U}}.\end{aligned} \tag{3.19}$$

Using (3.17) and $\mathcal{E}_{\mathcal{U}}^2 = \mathcal{E}_{\mathcal{U}}$ we replace $\mathcal{E}_{\mathcal{U}}^2 \overleftarrow{K} \mathcal{E}_{\mathcal{U}}$ with $\mathcal{E}_{\mathcal{U}} K = \mathcal{E}_{\mathcal{U}} \underline{\overline{R}}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1}$ in (3.19). Next, we employ the first equalities in (3.7) and (3.8), together with (3.9), to move the remaining copy of $\mathcal{E}_{\mathcal{U}}$ to the left, thus getting

$$\begin{aligned}\operatorname{tr}_{1,\dots,n} \overline{R}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1} \mathcal{E}_{\mathcal{U}} \underline{B}_n^{+12}(u_\mu) \underline{\overline{R}}_{1n}^{01}(u_0 - 3hN/4|u_\mu) \\ B_0(u_0) \overline{R}_{1n}^{01}(u_0 + hN/4|u_\mu) \underline{B}_n^{12}(u_\mu - hN/4)^{-1} \underline{\overline{R}}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1}.\end{aligned} \tag{3.20}$$

By applying (1.25) on the last four factors in (3.20) and then by canceling the adjacent terms $\underline{\overline{R}}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{\pm 1}$ we obtain

$$\begin{aligned}\operatorname{tr}_{1,\dots,n} \overline{R}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1} \mathcal{E}_{\mathcal{U}} \underline{B}_n^{+12}(u_\mu) \underline{B}_n^{12} \\ (u_\mu - hN/4)^{-1} \overline{R}_{1n}^{01}(u_0 + hN/4|u_\mu) B_0(u_0).\end{aligned}$$

In order to prove (3.14), it is sufficient to verify that the expression

$$\operatorname{tr}_{1,\dots,n} \overline{R}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1} \mathcal{E}_{\mathcal{U}} \underline{B}_n^{+12}(u_\mu) \underline{B}_n^{12}(u_\mu - hN/4)^{-1} \overline{R}_{1n}^{01}(u_0 + hN/4|u_\mu) \tag{3.21}$$

is equal to $\widetilde{\mathbb{A}}_\mu(u)$. By the property $\operatorname{tr}_{1,\dots,n} XY = \operatorname{tr}_{1,\dots,n} X^{t_1 \dots t_n} Y^{t_1 \dots t_n}$ for

$$\begin{aligned}X &= \overline{R}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1} \mathcal{E}_{\mathcal{U}} \underline{B}_n^{+12}(u_\mu) \underline{B}_n^{12}(u_\mu - hN/4)^{-1} \quad \text{and} \\ Y &= \overline{R}_{1n}^{01}(u_0 + hN/4|u_\mu)\end{aligned}$$

we conclude that (3.21) is equal to

$$\begin{aligned}&\operatorname{tr}_{1,\dots,n} \left(\mathcal{E}_{\mathcal{U}} \underline{B}_n^{+12}(u_\mu) \underline{B}_n^{12}(u_\mu - hN/4)^{-1} \right)^{t_1 \dots t_n} Z, \qquad \text{where} \\ &Z = \left(\overline{R}_{1n}^{01}(u_0 - 3hN/4|u_\mu)^{-1} \right)^{t_1 \dots t_n} \overline{R}_{1n}^{01}(u_0 + hN/4|u_\mu)^{t_1 \dots t_n}.\end{aligned}$$

Finally, crossing symmetry property (1.5) implies $Z = 1$, so (3.14) clearly follows.

Consider the tensor product

$$(\operatorname{End} \mathbb{C}^N)^{\otimes n} \otimes \operatorname{End} \mathbb{C}^N \otimes \widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N), \tag{3.22}$$

where the $n+1$ copies of $\operatorname{End} \mathbb{C}^N$ are now labeled by $1, \dots, n+1$. It will be convenient to denote the tensor factors $(\operatorname{End} \mathbb{C}^N)^{\otimes n}$, $\operatorname{End} \mathbb{C}^N$ and $\widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N)$ in (3.22) by the superscripts 1,2 and 3 respectively.[2] Our next goal is to prove that for variables (3.2) and the variable u_{n+1} the following equality holds on $\operatorname{End} \mathbb{C}^N \otimes \widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N)$:

$$B^+(u_{n+1}) \widetilde{\mathbb{A}}_\mu(u) = \widetilde{\mathbb{A}}_\mu(u) B^+(u_{n+1}). \tag{3.23}$$

[2] We introduce the new labeling because the application of the original labels, as in (3.6), would require different, more appropriate notation. For example, notice that the R-matrices in the first part of the proof should be expanded in nonnegative powers of the variable u, while the R-matrices in the following, second part of the proof should be expanded in nonnegative powers of the variable u_{n+1}.

The proof of (3.23) is analogous to the proof of (3.14), so we only briefly sketch some details to take care of minor differences. First, by applying $B^{+}_{n+1}(u_{n+1})$ on (3.5) and using notation (3.22) we get

$$\operatorname{tr}_{1,\dots,n} \mathcal{E}_{\mathcal{U}} B^{+}_{n+1}(u_{n+1}) \underline{B}_n^{+13}(u_\mu) \underline{B}_n^{13}(u_\mu - hN/4)^{-1}. \tag{3.24}$$

As with the proof of (1.33), we employ (1.24) and (1.27) to express (3.24) as

$$\begin{aligned}\operatorname{tr}_{1,\dots,n} \mathcal{E}_{\mathcal{U}} \Big({}^{lr}\Big(\underline{\overline{R}}_{n1}^{12}(u_\mu - hN|u_{n+1})^{-1}\Big) \cdot \Big(\overline{R}_{n1}^{12}(u_\mu|u_{n+1}) \underline{B}_n^{+13}(u_\mu) \\ \underline{\overline{R}}_{n1}^{12}(u_\mu|u_{n+1}) B^{+}_{n+1}(u_{n+1}) \overline{R}_{n1}^{12}(u_\mu|u_{n+1})^{-1} \underline{B}_n^{13}(u_\mu - hN/4)^{-1}\Big)\Big).\end{aligned} \tag{3.25}$$

We may now proceed as in the first part of the proof and, using the properties of the primitive idempotent $\mathcal{E}_{\mathcal{U}}$, show that (3.25) is equal to

$$\begin{aligned}\operatorname{tr}_{1,\dots,n} \overline{R}_{n1}^{12}(u_\mu|u_{n+1}) \mathcal{E}_{\mathcal{U}} \underline{B}_n^{+13}(u_\mu) \underline{\overline{R}}_{n1}^{12}(u_\mu|u_{n+1}) \\ B^{+}_{n+1}(u_{n+1}) \overline{R}_{n1}^{12}(u_\mu|u_{n+1})^{-1} \underline{B}_n^{13}(u_\mu - hN/4)^{-1} \underline{\overline{R}}_{n1}^{12}(u_\mu - hN|u_{n+1})^{-1}.\end{aligned} \tag{3.26}$$

By applying (1.26) to the last four factors in (3.26) and then canceling the adjacent terms $\underline{\overline{R}}_{n1}^{12}(u_\mu|u_{n+1})^{\pm 1}$ we obtain

$$\operatorname{tr}_{1,\dots,n} \overline{R}_{n1}^{12}(u_\mu|u_{n+1}) \mathcal{E}_{\mathcal{U}} \underline{B}_n^{+13}(u_\mu) \underline{B}_n^{13}(u_\mu - hN/4)^{-1} \overline{R}_{n1}^{12}(u_\mu - hN|u_{n+1})^{-1} B^{+}_{n+1}(u_{n+1}).$$

Finally, in order to prove (3.23), it is sufficient to check that the expression

$$\operatorname{tr}_{1,\dots,n} \overline{R}_{n1}^{12}(u_\mu|u_{n+1}) \mathcal{E}_{\mathcal{U}} \underline{B}_n^{+13}(u_\mu) \underline{B}_n^{13}(u_\mu - hN/4)^{-1} \overline{R}_{n1}^{12}(u_\mu - hN|u_{n+1})^{-1}$$

is equal to $\widetilde{\mathbb{A}}_\mu(u)$. This can be done as in the first part of the proof, by employing crossing symmetry property (1.5) and unitarity (1.6).

The statement of the theorem now follows from (3.14) and (3.23). □

We now consider two special cases of Theorem 4.2. Denote by μ_n^{row} and μ_n^{col} the row diagram with n boxes and the column diagram with n boxes respectively. The unique idempotents corresponding to the standard μ_n^{row}-tableau and μ_n^{col}-tableau coincide with the images $H^{(n)}$ and $A^{(n)}$ of the symmetrizer and the anti-symmetrizer

$$h^{(n)} = \frac{1}{n!} \sum_{s \in \mathfrak{S}_n} s \qquad \text{and} \qquad a^{(n)} = \frac{1}{n!} \sum_{s \in \mathfrak{S}_n} \operatorname{sgn} s \cdot s$$

under the action of the symmetric group $\mathfrak{S}_n$ on $(\mathbb{C}^N)^{\otimes n}$. In this two cases, (3.5) becomes

$$\begin{aligned}\widetilde{\mathbb{A}}_{\mu_n^{\mathrm{row}}}(u) &= \operatorname{tr}_{1,\dots,n} H^{(n)} \underline{B}_n^{+}(u_+) \underline{B}_n(u_+ - hN/4)^{-1}, \\ \widetilde{\mathbb{A}}_{\mu_n^{\mathrm{col}}}(u) &= \operatorname{tr}_{1,\dots,n} A^{(n)} \underline{B}_n^{+}(u_-) \underline{B}_n(u_- - hN/4)^{-1},\end{aligned}$$

where

$$u_\pm = (u, u \pm h, \dots, u \pm (n-1)h). \tag{3.27}$$

Note that $u_+ = u_{\mu_n^{\text{row}}}$ and $u_- = u_{\mu_n^{\text{col}}}$; recall (3.2). Consider the series

$$\begin{aligned}\widetilde{\mathbb{B}}_{\mu_n^{\text{row}}}(u) &= \operatorname{tr}_{1,\dots,n} H^{(n)}\, T_n^+(u_+|0)\, \overleftarrow{T}_n^+(-u_+|0)^{-1}\\ &\qquad \overleftarrow{T}_n(-u_+ + hN/4|0)\, T_n(u_+ - 3hN/4|0)^{-1},\\ \widetilde{\mathbb{B}}_{\mu_n^{\text{col}}}(u) &= \operatorname{tr}_{1,\dots,n} A^{(n)}\, T_n^+(u_-|0)\, \overleftarrow{T}_n^+(-u_-|0)^{-1}\\ &\qquad \overleftarrow{T}_n(-u_- + hN/4|0)\, T_n(u_- - 3hN/4|0)^{-1}\end{aligned}$$

in $\widetilde{\mathrm{DY}}_{-N/2}(\mathfrak{gl}_N)[[u^{\pm 1}]]$, where, as before, the arrows indicate that the products are written in the opposite order, e.g., for $w = (w_1, \dots, w_n)$ we have $\overleftarrow{T}_n(w|0) = T_n(w_n) \dots T_1(w_1)$.

Corollary 4.3. *Suppose that the matrix G, given by (1.12), is equal to $\pm I$. Then all coefficients of $\widetilde{\mathbb{B}}_{\mu_n^{row}}(u)$ and $\widetilde{\mathbb{B}}_{\mu_n^{col}}(u)$ belong to the center of the algebra $\widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N)$.*

Proof. Let $G = \varepsilon I$ for $\varepsilon \in \{\pm 1\}$. For the family of variables $w = (w_1, \dots, w_n)$ we have

$$\underline{B}_n^+(w) = \varepsilon^n\, T_n^+(w|0) \left(\prod_{1 \leqslant i < j \leqslant n} \overline{R}_{ij}(w_i + w_j) \right) \overleftarrow{T}_n^+(-w|0)^{-1} \qquad \text{and}$$

$$\underline{B}_n(w - hN/4) = \varepsilon^n\, T_n(w - 3hN/4|0) \left(\prod_{1 \leqslant i < j \leqslant n} \overline{R}_{ij}(w_i + w_j - hN) \right) \overleftarrow{T}_n(-w + hN/4|0)^{-1},$$

where the products are taken in the lexicographical order on the pairs (i, j). Indeed, this easily follows from (1.7) and (1.8). Next, note that for any $1 \leqslant i < j \leqslant n$ there exist functions $f_{H^{(n)}}(z)$ and $f_{A^{(n)}}(z)$ in $\mathbb{C}[z^{-1}][[h]]$ satisfying

$$H^{(n)}\overline{R}_{ij}(z) = f_{H^{(n)}}(z) H^{(n)} \qquad \text{and} \qquad A^{(n)}\overline{R}_{ij}(z) = f_{A^{(n)}}(z) A^{(n)}.$$

Indeed, this follows from the form of Yang R-matrix (1.2) and the fact that for any transposition $p \in \mathfrak{S}_n$ we have $ph^{(n)} = h^{(n)}$ and $pa^{(n)} = \pm a^{(n)}$.

By combining these observations with fusion procedure (3.1) and equalities in (3.10)–(3.12), we conclude that there exist functions $\theta_n^{\text{row}}(z)$ and $\theta_n^{\text{col}}(z)$ in $\mathbb{C}[z^{-1}][[h]]$ such that

$$\widetilde{\mathbb{A}}_{\mu_n^{\text{row}}}(u) = \theta_n^{\text{row}}(u)\widetilde{\mathbb{B}}_{\mu_n^{\text{row}}}(u) \qquad \text{and} \qquad \widetilde{\mathbb{A}}_{\mu_n^{\text{col}}}(u) = \theta_n^{\text{col}}(u)\widetilde{\mathbb{B}}_{\mu_n^{\text{col}}}(u). \tag{3.28}$$

Therefore, all coefficients of $\widetilde{\mathbb{B}}_{\mu_n^{\text{row}}}(u)$ and $\widetilde{\mathbb{B}}_{\mu_n^{\text{col}}}(u)$ belong to the algebra $\widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N)$. Finally, (3.28) and Theorem 4.2 imply the statement of the corollary. $\square$

It is worth noting that the functions $\theta_n^{\text{row}}(z)$ and $\theta_n^{\text{col}}(z)$ can be computed explicitly, in terms of the function $g(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ defined by (1.4); cf. [19, Theorem 3.4].

4.2. Invariants of the vacuum module $\mathcal{W}_{-N/2}(\mathfrak{gl}_N)$. In this section we present some further consequences of Theorem 4.2. Let c be an arbitrary complex number. We can view $\mathcal{W}_c(\mathfrak{gl}_N)$ as a module for the algebra $\widetilde{\mathrm{A}}_c(\mathfrak{gl}_N)$. Recall (1.12) and define the *submodule of invariants* of $\mathcal{W}_c(\mathfrak{gl}_N)$ by

$$\mathfrak{z}(\mathcal{W}_c(\mathfrak{gl}_N)) = \left\{w \in \mathcal{W}_c(\mathfrak{gl}_N) \,:\, B(u)w = Gw\right\}.$$

Clearly, an element $w \in \mathcal{W}_c(\mathfrak{gl}_N)$ belongs to $\mathfrak{z}(\mathcal{W}_c(\mathfrak{gl}_N))$ if and only if

$$b_{ij}(u)w = \delta_{ij}\varepsilon_i w \qquad \text{for all } i, j = 1, \ldots, N,\ r = 1, 2, \ldots.$$

In particular, (1.34) implies that $\mathbf{1}$ is an element of $\mathfrak{z}(\mathcal{W}_c(\mathfrak{gl}_N))$. Consider the series

$$\mathbb{A}_\mu(u) := \widetilde{\mathbb{A}}_\mu(u)\,\mathbf{1} \in \mathcal{W}_{-N/2}(\mathfrak{gl}_N)[[u^{\pm 1}]]. \tag{3.29}$$

Denote by $\widehat{\mathrm{B}}^+(\mathfrak{gl}_N)$ the h-adic completion of the algebra $\mathrm{B}^+(\mathfrak{gl}_N)$. All coefficients of $\mathbb{A}_\mu(u)$ can be viewed as elements of $\widehat{\mathrm{B}}^+(\mathfrak{gl}_N)$.

Corollary 4.4. *All coefficients of the series $\mathbb{A}_\mu(u)$ belong to the submodule of invariants $\mathfrak{z}(\mathcal{W}_{-N/2}(\mathfrak{gl}_N))$. All coefficients of $\mathbb{A}_\mu(u) \in \widehat{\mathrm{B}}^+(\mathfrak{gl}_N)[[u^{\pm 1}]]$ pairwise commute.*

Proof. Using Theorem 4.2 and (3.29) we get

$$\begin{aligned} b_{ij}(v)\mathbb{A}_\mu(u) = b_{ij}(v)\widetilde{\mathbb{A}}_\mu(u)\,\mathbf{1} = \widetilde{\mathbb{A}}_\mu(u)b_{ij}(v)\,\mathbf{1} = \widetilde{\mathbb{A}}_\mu(u)\delta_{ij}\varepsilon_i\,\mathbf{1} \\ = \delta_{ij}\varepsilon_i\widetilde{\mathbb{A}}_\mu(u)\,\mathbf{1} = \delta_{ij}\varepsilon_i\mathbb{A}_\mu(u) \end{aligned}$$

for any $i, j = 1, \ldots, N$, which proves the first part of the corollary.

Let μ and ν be any two partitions having at most N parts. Using Theorem 4.2 we get

$$\widetilde{\mathbb{A}}_\mu(u)\widetilde{\mathbb{A}}_\nu(v)\,\mathbf{1} = \widetilde{\mathbb{A}}_\mu(u)\mathbb{A}_\nu(v)\,\mathbf{1} = \mathbb{A}_\nu(v)\widetilde{\mathbb{A}}_\mu(u)\,\mathbf{1} = \mathbb{A}_\nu(v)\mathbb{A}_\mu(u). \tag{3.30}$$

Since all coefficients of the series $\widetilde{\mathbb{A}}_\mu(u)$ and $\widetilde{\mathbb{A}}_\nu(v)$ commute, we can prove analogously that $\widetilde{\mathbb{A}}_\mu(u)\widetilde{\mathbb{A}}_\nu(v)\,\mathbf{1} = \mathbb{A}_\mu(u)\mathbb{A}_\nu(v)$, which, together with (3.30), implies $[\mathbb{A}_\mu(u), \mathbb{A}_\nu(v)] = 0$, as required. $\square$

Corollaries 4.3 and 4.4 imply

Corollary 4.5. *Let $G = \pm I$. All coefficients of the Taylor series*

$$\mathrm{tr}_{1,\ldots,n}\, H^{(n)}\, T_n^+(u_+|0)\, \overleftarrow{T}_n^{+}(-u_+|0)^{-1}\,\mathbf{1} \qquad \textit{and} \qquad \mathrm{tr}_{1,\ldots,n}\, A^{(n)}\, T_n^+(u_-|0)\, \overleftarrow{T}_n^{+}(-u_-|0)^{-1}\,\mathbf{1}$$

belong to the submodule of invariants $\mathfrak{z}(\mathcal{W}_{-N/2}(\mathfrak{gl}_N))$.

For any two partitions μ and ν which have at most N parts we have

$$[\widetilde{\mathbb{A}}_\mu(u), \widetilde{\mathbb{A}}_\nu(v)] = 0 \tag{3.31}$$

in the algebra $\widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N)$. Clearly, (3.31) remains true if we view $\widetilde{\mathbb{A}}_\mu(u)$ and $\widetilde{\mathbb{A}}_\nu(v)$ as operators on $\mathcal{W}_{-N/2}(\mathfrak{gl}_N)$. Applying the substitutions $u \leftrightarrow z_1 + u$ and $v \leftrightarrow z_2 + v$ we get

$$[\widetilde{\mathbb{A}}_\mu(z_1 + u), \widetilde{\mathbb{A}}_\nu(z_2 + v)] = 0 \quad \text{on } \mathcal{W}_{-N/2}(\mathfrak{gl}_N). \tag{3.32}$$

Note that (3.32) can be written as

$$[Y_{\mathcal{W}_{-N/2}(\mathfrak{gl}_N)}(\mathbb{T}_\mu^+(u), z_1), Y_{\mathcal{W}_{-N/2}(\mathfrak{gl}_N)}(\mathbb{T}_\nu^+(v), z_2)] = 0. \tag{3.33}$$

Theorem 4.6. *Let a be an element of the center $\mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$.*

(1) For any $b \in \mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$ we have

$$[Y_{\mathcal{W}_{-N/2}(\mathfrak{gl}_N)}(a, z_1), Y_{\mathcal{W}_{-N/2}(\mathfrak{gl}_N)}(b, z_2)] = 0. \tag{3.34}$$

(2) For any $x \in \widetilde{\mathrm{A}}_{-N/2}(\mathfrak{gl}_N)$

$$[Y_{\mathcal{W}_{-N/2}(\mathfrak{gl}_N)}(a, z), x] = 0 \quad \text{on } \mathcal{W}_{-N/2}(\mathfrak{gl}_N). \tag{3.35}$$

Proof. (1) Due to [9, Theorem 4.9], the center $\mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$ is a commutative associative algebra with respect to the product given by $a \cdot b = a_{-1}b$ for $a, b \in \mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$. Furthermore, it was proved therein that the algebra $\mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$ is topologically generated (with respect to the h-adic topology) by the elements $\Phi_m^{(r)}$, where $m = 1, \ldots, N$ and $r = 0, 1, \ldots$, defined by

$$\sum_{r=0}^{\infty} \Phi_m^{(r)} u^r := h^{-m} \sum_{k=0}^{m} (-1)^k \binom{N-k}{m-k} \mathbb{T}^+_{\mu_k^{\mathrm{col}}}(u) \in \mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))[[u]].$$

By (3.33) we conclude that (3.34) holds for any two elements a and b which belong to the family $\Phi_m^{(r)}$, $m = 1, \ldots, N$, $r = 0, 1, \ldots$. Finally, part (a) of Proposition 3.4 implies that (3.34) holds for any two elements $a, b \in \mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$.

(2) It suffices to observe that, due to Theorem 4.2, Equality (3.35) holds if $a = \Phi_m^{(r)}$ for some $m = 1, \ldots, N$ and $r = 0, 1, \ldots$. Hence, part (b) of Proposition 3.4 implies that (3.35) holds for any $a \in \mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$. □

Due to Theorem 4.6, for any $a \in \mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$ and $i, j = 1, \ldots, N$ we have

$$[b_{ij}(u), Y_{\mathcal{W}_{-N/2}(\mathfrak{gl}_N)}(a, z)] = 0 \quad \text{on } \mathcal{W}_{-N/2}(\mathfrak{gl}_N). \tag{3.36}$$

Hence, we can construct elements of $\mathfrak{z}(\mathcal{W}_{-N/2}(\mathfrak{gl}_N))$ as follows:

Corollary 4.7. *For any $a \in \mathfrak{z}(\mathcal{V}_{-N}(\mathfrak{gl}_N))$ and $w \in \mathfrak{z}(\mathcal{W}_{-N/2}(\mathfrak{gl}_N))$ all coefficients of the series $Y_{\mathcal{W}_{-N/2}(\mathfrak{gl}_N)}(a, z)w$ belong to the submodule of invariants $\mathfrak{z}(\mathcal{W}_{-N/2}(\mathfrak{gl}_N))$.*

Proof. Set $\mathcal{W}_{-N/2} = \mathcal{W}_{-N/2}(\mathfrak{g}_N)$. By employing (3.36) we get

$$\begin{aligned} b_{ij}(u) Y_{\mathcal{W}_{-N/2}}(a, z) w &= Y_{\mathcal{W}_{-N/2}}(a, z) b_{ij}(u) w \\ &= Y_{\mathcal{W}_{-N/2}}(a, z) \delta_{ij} \varepsilon_i w = \delta_{ij} \varepsilon_i Y_{\mathcal{W}_{-N/2}}(a, z) w \end{aligned}$$

for any $i, j = 1, \ldots, N$ and $w \in \mathfrak{z}(\mathcal{W}_{-N/2}(\mathfrak{gl}_N))$, as required. □

4.3. Central elements and invariants at the noncritical level. Let $c \neq -N/2$ be an arbitrary complex number. It is well known that all coefficients of *quantum determinants*

$$\operatorname{qdet} T^+(u) = \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn} \sigma \cdot t^+_{\sigma(1)\,1}(u) \ldots t^+_{\sigma(N)\,N}(u - (N-1)h) \in \widehat{\mathrm{Y}}^+(\mathfrak{gl}_N)[[u]], \tag{3.37}$$

$$\operatorname{qdet} T(u) = \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn} \sigma \cdot t_{\sigma(1)\,1}(u) \ldots t_{\sigma(N)\,N}(u - (N-1)h) \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]] \tag{3.38}$$

belong to the center of the algebra $\widetilde{\mathrm{DY}}_{2c}(\mathfrak{gl}_N)$; see, e.g., [9, Proposition 2.8]. Furthermore, if we identify the $\mathbb{C}[[h]]$-modules $\widetilde{\mathrm{Y}}^+(\mathfrak{gl}_N)$ and $\mathcal{V}_{2c}(\mathfrak{gl}_N)$, then all coefficients of $\mathrm{qdet}\, T^+(u)$ belong to the center $\mathfrak{z}(\mathcal{V}_{2c}(\mathfrak{gl}_N))$ of the quantum VOA $\mathcal{V}_{2c}(\mathfrak{gl}_N)$; see [9, Proposition 4.10].

Set $n = N$ in (3.27). The following equations in $(\mathrm{End}\,\mathbb{C}^N)^{\otimes N} \otimes \widetilde{\mathrm{DY}}_{2c}(\mathfrak{gl}_N)[[u^{\pm 1}]]$ hold:

$$A^{(N)} T_N^+(u_-|0) = A^{(N)} \mathrm{qdet}\, T^+(u) \quad \text{and} \quad A^{(N)} T_N(u_-|0) = A^{(N)} \mathrm{qdet}\, T(u), \tag{3.39}$$

see [18, Sect. 1] for more details. By applying quasi module map (2.25) on the constant term of (3.37), which is viewed as an element of the quantum VOA $\mathcal{V}_{2c}(\mathfrak{gl}_N)$, and by employing the first equality in (3.39), we obtain

$$Y_{\mathcal{W}_c(\mathfrak{gl}_N)}(\mathrm{qdet}\, T^+(0), u) = \mathrm{tr}_{1,\ldots,N} A^{(N)} \underline{B}_N^+(u_-) \underline{B}_N(u_- + hc/2)^{-1}. \tag{3.40}$$

Clearly, (3.40) belongs to $\mathrm{Hom}(\mathcal{W}_c(\mathfrak{gl}_N), \mathcal{W}_c(\mathfrak{gl}_N)((u))[[h]])$. However, we can view all coefficients of the series

$$\widetilde{\mathbb{A}}_c(u) := \mathrm{tr}_{1,\ldots,N} A^{(N)} \underline{B}_N^+(u_-) \underline{B}_N(u_- + hc/2)^{-1} \tag{3.41}$$

as elements of the algebra $\widetilde{\mathrm{A}}_c(\mathfrak{gl}_N)$, so that $\widetilde{\mathbb{A}}_c(u)$ is an element of $\widetilde{\mathrm{A}}_c(\mathfrak{gl}_N)[[u^{\pm 1}]]$.

Proposition 4.8. *Let $c \neq -N/2$ be an arbitrary complex number.*

(i) *All coefficients of $\widetilde{\mathbb{A}}_c(u)$ belong to the center of the algebra $\widetilde{\mathrm{A}}_c(\mathfrak{gl}_N)$.*

(ii) *For any $a, b \in \mathfrak{z}(\mathcal{V}_{2c}(\mathfrak{gl}_N))$ we have*

$$[Y_{\mathcal{W}_c(\mathfrak{gl}_N)}(a, z_1), Y_{\mathcal{W}_c(\mathfrak{gl}_N)}(b, z_2)] = 0.$$

(iii) *For any $a \in \mathfrak{z}(\mathcal{V}_{2c}(\mathfrak{gl}_N))$ and $x \in \widetilde{\mathrm{A}}_c(\mathfrak{gl}_N)$ we have*

$$[Y_{\mathcal{W}_c(\mathfrak{gl}_N)}(a, z), x] = 0 \quad \text{on } \mathcal{W}_c(\mathfrak{gl}_N).$$

(iv) *For any $a \in \mathfrak{z}(\mathcal{V}_{2c}(\mathfrak{gl}_N))$ and $w \in \mathfrak{z}(\mathcal{W}_c(\mathfrak{gl}_N))$ all coefficients of $Y_{\mathcal{W}_c(\mathfrak{gl}_N)}(a, z)w$ belong to the submodule of invariants $\mathfrak{z}(\mathcal{W}_c(\mathfrak{gl}_N))$.*

Proof. (i) Recall that $u_- = (u_1, \ldots, u_N) = (u, \ldots, u - (N-1)h)$, so the first equality in (3.39) can be written as

$$A^{(N)} T_1^+(u_1) \ldots T_N^+(u_N) = A^{(N)} \mathrm{qdet}\, T^+(u). \tag{3.42}$$

We now proceed as follows (cf. [19, Theorem 3.4]):

- Multiply (3.42) from the right by $T_N^+(u_N)^{-1} \ldots T_1^+(u_1)^{-1} (\mathrm{qdet}\, T^+(u))^{-1}$;
- Replace u with $-u + (N-1)h$;
- Conjugate the resulting equality by the permutation $(1, \ldots, N) \mapsto (N, \ldots, 1)$.

This gives us

$$A^{(N)} \mathrm{qdet}\, T^+(-u + (N-1)h)^{-1} = A^{(N)} T_1^+(-u_1)^{-1} \ldots T_N^+(-u_N)^{-1}. \tag{3.43}$$

Starting from the second equality in (3.39), one can similarly prove

$$A^{(N)} \mathrm{qdet}\, T(-u + (N-1)h - hc/2) = A^{(N)} T_N(-u_N - hc/2) \ldots T_1(-u_1 - hc/2). \tag{3.44}$$

By employing (3.39), (3.43) and (3.44) and arguing as in the proof of Corollary 4.3, we can express $\widetilde{\mathbb{A}}_c(u)$ as

$$\widetilde{\mathbb{A}}_c(u) = \theta_c(u)\,\text{qdet}\,T^+(u)\,(\text{qdet}\,T^+(-u+(N-1)h))^{-1} \\ \cdot\ \text{qdet}\,T(-u+(N-1)h-hc/2)\,(\text{qdet}\,T(u+3hc/2))^{-1} \tag{3.45}$$

for some function $\theta_c(z)$ in $\mathbb{C}[z^{-1}][[h]]$.[3] Since the coefficients of quantum determinants belong to the center of the double Yangian, we conclude by (3.45) that the coefficients of $\widetilde{\mathbb{A}}_c(u)$ belong to the center of the algebra $\widetilde{\mathrm{A}}_c(\mathfrak{gl}_N)$.

(ii)–(iv) By [9, Proposition 4.10], the center $\mathfrak{z}(\mathcal{V}_{2c}(\mathfrak{gl}_N))$ is a commutative associative algebra with respect to the product given by $a\cdot b = a_{-1}b$ for $a,b\in\mathfrak{z}(\mathcal{V}_{2c}(\mathfrak{gl}_N))$. Furthermore, it was proved therein, that the algebra $\mathfrak{z}(\mathcal{V}_{2c}(\mathfrak{gl}_N))$ is topologically generated (with respect to the h-adic topology) by the elements $d_0, d_1, \ldots$, which are defined by

$$\text{qdet}\,T^+(u) = 1 - h(d_0 + d_1u + d_2u^2 + \cdots).$$

Therefore, statements (ii)–(iv) can be verified using Proposition 3.4, in the same way as their critical level counterparts. □

Consider the series

$$S^+(u) = \text{qdet}\,T^+(u)\,(\text{qdet}\,T^+(-u+(N-1)h))^{-1} \in \widehat{\mathrm{B}}^+(\mathfrak{gl}_N)[[u]],$$
$$S^{(c)}(u) = \text{qdet}\,T(u+hc)\,(\text{qdet}\,T(-u+(N-1)h))^{-1} \in \mathrm{B}_c(\mathfrak{gl}_N)[[u^{-1}]].$$

By part (i) of Proposition 4.8 and (3.45), all coefficients of $S^+(u)S^{(c)}(u+hc/2)^{-1}$ belong to the center of the algebra $\widetilde{\mathrm{A}}_c(\mathfrak{gl}_N)$. Moreover, by applying the given expression on $\mathbf{1}\in\mathcal{W}_c(\mathfrak{gl}_N)$ and employing part (iv) of Proposition 4.8, we see that all coefficients of the series $S^+(u)\,\mathbf{1}$ belong to the submodule of invariants $\mathfrak{z}(\mathcal{W}_c(\mathfrak{gl}_N))$.

Remark 4.9. Let $h=1$ and $c=0$. The series $S^{(0)}(u)$ coincides, modulo the multiplicative factor from $\mathbb{C}(u)$, with the *Sklyanin determinant* sdet $B(u)$, whose odd coefficients are algebraically independent and generate the center of the reflection algebra $\mathcal{B}(N, N-M)$; see [19, Theorem 3.4].

4.4. Discussion. In this section, we mainly discuss the classical limit of the algebra $\mathrm{A}_c(\mathfrak{gl}_N)$. In particular, we clarify a somewhat peculiar situation that a quasi module structure for the quantum VOA of level c is constructed on the vacuum module of level $c/2$; recall Theorem 3.7. Also, in Theorem 4.2, certain large families of central elements are given for the completed algebra $\widetilde{\mathrm{A}}_c(\mathfrak{gl}_N)$ at the level $c=-N/2$. However, in view of the Feigin–Frenkel theorem [5], see also [7] for more details, one would expect such a construction at the level $c=-N$. For simplicity we consider the case $G=I$.

Recall that the affine Lie algebra $\widehat{\mathfrak{gl}}_N = \mathfrak{gl}_N\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}C$ is defined by relations

$$[e_{ij}(z_1), e_{kl}(z_2)] = \left(\delta_{jk}e_{il}(z_1) - \delta_{il}e_{kj}(z_1)\right) z_2^{-1}\delta\left(\frac{z_1}{z_2}\right) \\ - C\left(\delta_{jk}\delta_{il} - \frac{\delta_{ij}\delta_{kl}}{N}\right)\frac{\partial}{\partial z_1} z_2^{-1}\delta\left(\frac{z_1}{z_2}\right), \tag{3.46}$$

[3] Observe that, in contrast with the proof of Corollary 4.3, we no longer need the assumption $G=\pm I$ because the image of the anti-symmetrizer $A^{(N)}$ on $(\mathbb{C}^N)^{\otimes N}$ is one-dimensional.

where $i, j, k, l = 1, \ldots, N$, C is a central element, $\delta(z)$ is the delta function

$$\delta(z) = \sum_{r\in\mathbb{Z}} z^r \quad \text{and} \quad e_{ij}(z) = \sum_{r\in\mathbb{Z}} (e_{ij} \otimes t^r)\, z^{-r-1} \quad \text{for all} \quad i, j = 1, \ldots, N.$$

Commutation relations (3.46) are equivalent to

$$\begin{aligned}[\widehat{e}_{ij}(u), \widehat{e}_{kl}(v)] = 2\left(\delta_{jk}\widehat{e}_{il}(u) - \delta_{il}\widehat{e}_{kj}(u)\right) v^{-1}\delta\left(\frac{u^2}{v^2}\right)& \\ - 2C\left(\delta_{jk}\delta_{il} - \frac{\delta_{ij}\delta_{kl}}{N}\right)\frac{\partial}{\partial u} v^{-1}\delta\left(\frac{u^2}{v^2}\right)&, \end{aligned} \tag{3.47}$$

where $i, j, k, l = 1, \ldots, N$ and

$$\widehat{e}_{ij}(u) = 2u e_{ij}(u^2) \in \widehat{\mathfrak{gl}}_N \otimes u\mathbb{C}[u^{\pm 2}].$$

Indeed, by multiplying (3.46) with $z_1 z_2$, then replacing the variables (z_1, z_2) with (u^2, v^2) and finally, multiplying the expression with $4u^{-1}v^{-1}$ we obtain (3.47).

Introduce the elements

$$t^+(u) = \frac{1 - T^+(u)}{h} \quad \text{and} \quad t(u) = \frac{T(u) - 1}{h}$$

in $\operatorname{End}\mathbb{C}^N \otimes \mathrm{DY}_c(\mathfrak{gl}_N)[[u^{\pm 1}]]$. As with $\widehat{e}_{ij}(u)$, the series

$$\overline{b}^+(u) := \sum_{i,j=1}^N e_{ij} \otimes \overline{b}^+_{ij}(u) := t^+(u) - t^+(-u),$$

$$\overline{b}(u) := \sum_{i,j=1}^N e_{ij} \otimes \overline{b}_{ij}(u) := t(u) - t(-u)$$

posses only odd powers of the variable u. Using (1.15) one can easily verify that

$$B^+(u) = 1 - h\overline{b}^+(u) \mod h^2 \qquad \text{and} \qquad B(u) = 1 + h\overline{b}(u) \mod h^2.$$

Let us regard relations (1.17)–(1.19) as Taylor series with respect to the parameter h. The coefficients of $h^0 = 1$ and h on the left and the right hand sides of these relations cancel, so that each relation (1.17)–(1.19) can be written in the form $h^2 X = 0$ for some X in $(\operatorname{End}\mathbb{C}^N)^{\otimes 2} \otimes \mathrm{A}_c(\mathfrak{gl}_N)[[u^{\pm 1}, v^{\pm 1}]]$. By considering the matrix entries $e_{ij} \otimes e_{kl}$ in the equalities $X\,|_{h=0} = 0$, where $X\,|_{h=0}$ denotes the evaluation of X at $h = 0$, we find

$$\begin{aligned}[\overline{b}^+_{ij}(u), \overline{b}^+_{kl}(v)] = -2\left(\delta_{il}\overline{b}^+_{kj}(u) - \delta_{jk}\overline{b}^+_{il}(u)\right) u^{-1}\delta_1\left(\frac{v}{u}\right)& \\ - 2\left(\delta_{jk}\overline{b}^+_{il}(v) - \delta_{il}\overline{b}^+_{kj}(v)\right) u^{-1}\delta_0\left(\frac{v}{u}\right)&, \end{aligned} \tag{3.48}$$

$$\begin{aligned}[\overline{b}_{ij}(u), \overline{b}_{kl}(v)] = 2\left(\delta_{il}\overline{b}_{kj}(u) - \delta_{jk}\overline{b}_{il}(u)\right) u^{-1}\delta_1\left(\frac{v}{u}\right)& \\ + 2\left(\delta_{jk}\overline{b}_{il}(v) - \delta_{il}\overline{b}_{kj}(v)\right) u^{-1}\delta_0\left(\frac{v}{u}\right)&, \end{aligned} \tag{3.49}$$

$$[\overline{b}_{ij}(u), \overline{b}^+_{kl}(v)] = -2\left(\delta_{il}\overline{b}^+_{kj}(v) - \delta_{jk}\overline{b}^+_{il}(v)\right) u^{-1}\delta_0\left(\frac{v}{u}\right)$$

$$- 2\left(\delta_{il}\overline{b}_{kj}(u) - \delta_{jk}\overline{b}_{il}(u)\right) u^{-1}\delta_1\Big(\frac{v}{u}\Big)$$
$$+ 4c\left(\delta_{jk}\delta_{il} - \frac{\delta_{ij}\delta_{kl}}{N}\right)\frac{\partial}{\partial v}u^{-1}\delta_0\Big(\frac{v}{u}\Big), \tag{3.50}$$

where

$$\delta_0(z) = \sum_{r=0}^{\infty} z^{2r} \quad \text{and} \quad \delta_1(z) = \sum_{r=0}^{\infty} z^{2r+1}.$$

Relations (3.48) for $i, j, k, l = 1, \ldots, N$ coincide with defining relations for the algebra $U(\mathfrak{gl}_N \otimes t^{-1}\mathbb{C}[t^{-1}])$ and relations (3.49) for $i, j, k, l = 1, \ldots, N$ coincide with defining relations for the algebra $U(\mathfrak{gl}_N \otimes \mathbb{C}[t])$. Furthermore, by combining (3.48)–(3.50) one can show that the series

$$\widehat{b}(u) \coloneqq \sum_{i,j=1}^{N} e_{ij} \otimes \widehat{b}_{ij}(u) \coloneqq \overline{b}^{+}(u) + \overline{b}(u)$$

satisfies

$$[\widehat{b}_{ij}(u), \widehat{b}_{kl}(v)] = 2\left(\delta_{jk}\widehat{b}_{il}(u) - \delta_{il}\widehat{b}_{kj}(u)\right) v^{-1}\delta\left(\frac{u^2}{v^2}\right)$$
$$- 4c\left(\delta_{jk}\delta_{il} - \frac{\delta_{ij}\delta_{kl}}{N}\right)\frac{\partial}{\partial u}v^{-1}\delta\left(\frac{u^2}{v^2}\right) \tag{3.51}$$

for all $i, j, k, l = 1, \ldots, N$. Finally, notice that (3.51) coincides with defining relations (3.47) for the affine Lie algebra $\widehat{\mathfrak{gl}}_N$ at the level $2c$. Therefore, even though the algebra $\mathrm{A}_c(\mathfrak{gl}_N)$ was introduced as a subalgebra of the double Yangian at the level c, it seems more consistent (with respect to the classical theory) to think of $\mathrm{A}_c(\mathfrak{gl}_N)$ as an algebra of level $2c$. Consequently, the corresponding $\mathrm{A}_c(\mathfrak{gl}_N)$-module $\mathcal{W}_c(\mathfrak{gl}_N)$ may be regarded as a vacuum module of level $2c$.

The underlying vector space of the universal affine vertex algebra $V_c(\mathfrak{gl}_N)$ of level c, which is associated with the affine Lie algebra $\widehat{\mathfrak{gl}}_N$, coincides with $U(\mathfrak{gl}_N \otimes t^{-1}\mathbb{C}[t^{-1}])$; see, e.g., [7] for details. As demonstrated in (3.51), defining relations for $\widehat{\mathfrak{gl}}_N$ can be obtained from (1.17)–(1.19) at $h = 0$ and, more specifically, defining relations for $U(\mathfrak{gl}_N \otimes t^{-1}\mathbb{C}[t^{-1}])$ can be obtained from (1.17) at $h = 0$. Hence, the explicit formulae in [2,3] for the invariants of the vacuum module $V_{-N}(\mathfrak{gl}_N)$ at the critical level $-N$, i.e. the formulae for the complete set of Segal–Sugawara vectors for the vertex algebra $V_{-N}(\mathfrak{gl}_N)$, suggest that the invariants found in Corollary 4.5 might exhaust the whole submodule of invariants $\mathfrak{z}(\mathcal{W}_{-N/2}(\mathfrak{gl}_N))$.

The classical limit of the dual Yangian $\mathrm{Y}^{+}(\mathfrak{gl}_N)$ coincides with the algebra $U(\mathfrak{gl}_N \otimes t^{-1}\mathbb{C}[t^{-1}])$. In particular, the classical limits of the elements $t_{ij}^{(-r)}$ are equal to $e_{ij} \otimes t^{-r}$ for all $i, j = 1, \ldots, N$ and $r \geqslant 1$; see [9, Sect. 2.1] for details. Hence, the classical limit of the quasi module map $Y_{\mathcal{W}_{c/2}(\mathfrak{gl}_N)}(z)$, as defined in (2.25), maps the elements $e_{ij}(u)\,\mathbf{1}$, where $\mathbf{1}$ is the vacuum vector in $V_c(\mathfrak{gl}_N)$, to the action of the elements $\widehat{b}_{ij}(z + u)$. In view of Etingof–Kazhdan's work [4], where the quantum VOA in type A is obtained as a formal deformation of the corresponding universal affine vertex algebra, it might be interesting to further investigate the classical analogue of the quasi module from Theorem 3.7.

In the end, it is worth mentioning that the quantum VOA in type A (for the rational R-matrix) can be also obtained by a *coinvariant construction*. This was found by Etingof and Kazhdan in [4, Sect. 3], where they generalized the construction of the affine vertex operator algebras, which relies on considering conformal blocks on the projective line $\mathbb{P}^1$ for the Wess–Zumino–Witten model, to the quantum case. Perhaps finding a coinvariant construction of the quasi module $\mathcal{W}_c(\mathfrak{gl}_N)$ might also present an interesting research direction.

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References

1. Cherednik, I.V.: Factorizing particles on a half line and root systems. Theor. Math. Phys. **61**, 977–983 (1984)
2. Chervov, A.V., Molev, A.I.: On higher order Sugawara operators. Int. Math. Res. Not. 1612–1635 (2009)
3. Chervov, A., Talalaev, D.: Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence. arXiv:hep-th/0604128
4. Etingof, P., Kazhdan, D.: Quantization of Lie bialgebras V. Sel. Math. (N.S.) **6**, 105–130 (2000)
5. Feigin, B., Frenkel, E.: Affine Kac–Moody algebras at the critical level and Gelfand–Dikii algebras. Int. J. Mod. Phys. A **7**(Suppl. 1A), 197–215 (1992)
6. Frappat, L., Jing, N., Molev, A., Ragoucy, E.: Higher Sugawara operators for the quantum affine algebras of type A. Commun. Math. Phys. **345**, 631–657 (2016)
7. Frenkel, E.: Langlands Correspondence for Loop Groups, Cambridge Studies in Advanced Mathematics, 103. Cambridge University Press, Cambridge (2007)
8. Iohara, K.: Bosonic representations of Yangian double $DY_\hbar(\mathfrak{g})$ with $\mathfrak{g} = \mathfrak{gl}_N, \mathfrak{sl}_N$. J. Phys. A **29**, 4593–4621 (1996)
9. Jing, N., Kožić, S., Molev, A., Yang, F.: Center of the quantum affine vertex algebra in type A. J. Algebra **496**, 138–186 (2018)
10. Jucys, A.: On the Young operators of the symmetric group. Lietuvos Fizikos Rinkinys **6**, 163–180 (1966)
11. Kulish, P.P., Sklyanin, E.K.: Algebraic structures related to reflection equations. J. Phys. A **25**, 5963–5975 (1992)
12. Kuznetsov, V.B., Jørgensen, M.F., Christiansen, P.L.: New boundary conditions for integrable lattices. J. Phys. A **28**, 4639–4654 (1995)
13. Li, H.-S.: Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules. Contemp. Math. **193**, 203–236 (1996)
14. Li, H.-S.: Modules-at-infinity for quantum vertex algebras. Commun. Math. Phys. **282**, 819–864 (2008)
15. Li, H.-S.: $\hbar$-Adic quantum vertex algebras and their modules. Commun. Math. Phys. **296**, 475–523 (2010)
16. Li, H.-S., Tan, S., Wang, Q.: Twisted modules for quantum vertex algebras. J. Pure Appl. Algebra **214**, 201–220 (2010)
17. Mintchev, M., Ragoucy, E., Sorba, P.: Spontaneous symmetry breaking in the $gl(N) - NLS$ hierarchy on the half line. J. Phys. A **34**, 8345–8364 (2001)
18. Molev, A.: Yangians and classical Lie algebras, Mathematical Surveys and Monographs, 143. American Mathematical Society, Providence, RI (2007)
19. Molev, A.I., Ragoucy, E.: Representations of reflection algebras. Rev. Math. Phys. **14**, 317–342 (2002)
20. Noumi, M.: Macdonald's symmetric polynomials as zonal spherical functions on quantum homogeneous spaces. Adv. Math. **123**, 16–77 (1996)
21. Okounkov, A.: Quantum immanants and higher Capelli identities. Transform. Groups **1**, 99–126 (1996)
22. Reshetikhin, N.Y., Semenov-Tian-Shansky, M.A.: Central extensions of quantum current groups. Lett. Math. Phys. **19**, 133–142 (1990)
23. Sklyanin, E.K.: Boundary conditions for integrable quantum systems. J. Phys. A **21**, 2375–2389 (1988)

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