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Global Instability in the Restricted Planar Elliptic Three Body Problem

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Abstract: The restricted planar elliptic three body problem (RPETBP) describes the motion of a massless particle (a comet or an asteroid) under the gravitational field of two massive bodies (the primaries, say the Sun and Jupiter) revolving around their center of mass on elliptic orbits with some positive eccentricity. The aim of this paper is to show the existence of orbits whose angular momentum performs arbitrary excursions in a large region. In particular, there exist diffusive orbits, that is, with a large variation of angular momentum. The leading idea of the proof consists in analyzing parabolic motions of the comet. By a well-known result of McGehee, the union of future (resp. past) parabolic orbits is an analytic manifold $\mathcal{P}^+$ (resp. $\mathcal{P}^-$). In a properly chosen coordinate system these manifolds are stable (resp. unstable) manifolds of a manifold at infinity $\mathcal{P}_\infty$, which we call the manifold at parabolic infinity. On $\mathcal{P}_\infty$ it is possible to define two scattering maps, which contain the map structure of the homoclinic trajectories to it, i.e. orbits parabolic both in the future and the past. Since the inner dynamics inside $\mathcal{P}_\infty$ is trivial, two different scattering maps are used. The combination of these two scattering maps permits the design of the desired diffusive pseudo-orbits. Using shadowing techniques and these pseudo orbits we show the existence of true trajectories of the RPETBP whose angular momentum varies in any predetermined fashion.

1. Main Result and Methodology

The restricted planar elliptic three body problem (RPETBP) describes the motion q of a massless particle (a *comet*) under the gravitational field of two massive bodies (the *primaries*, say the *Sun* and *Jupiter*) with mass ratio μ revolving around their center of mass on elliptic orbits with eccentricity ϵ_{J}. In this paper we search for trajectories of

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motion which show a large variation of the angular momentum $G = q \times \dot{q}$. In other words, we search for global instability ("diffusion" is the term usually used) in the angular momentum of this problem. Notice that for $\mu = 0$ the angular momentum is a first integral.

If the eccentricity of Jupiter vanishes, the primaries revolve along circular orbits, and such diffusion is not possible, since the restricted (planar) circular three body problem (RPCTBP) is governed by an autonomous Hamiltonian with two degrees of freedom. This is not the case for the RPETBP, which is a 2+1/2 degrees-of-freedom Hamiltonian system with time-periodic Hamiltonian. Our main result is the following.

Theorem 1. *There exist two constants* $C > 0$, $c > 0$ *such that for any* $0 < \epsilon_J < c/C$ *there is* $\mu^* = \mu^*(C, c, \epsilon_J) > 0$[1] *such that for any* $0 < \mu < \mu^*$ *and any* $C \leq G_1^* < G_2^* \leq c/\epsilon_J$ *there exists a trajectory of the RPETBP such that* $G(0) < G_1^*$, $G(T) > G_2^*$ *for some* $T > 0$.

This result will be a consequence of Theorem 13, where it is also shown the existence of trajectories of motion such that their angular momentum performs arbitrary excursions along the region $C \leq G_1^* < G_2^* \leq c/\epsilon_J$. Comments about the values C and c can be found in Remark 9.

1.1. Previous works. Let us recall related results about oscillatory motions and diffusion for the RPCTBP or the RPETBP. They hold close to a region when there is some kind of hyperbolicity in the Three Body Problem, like the Euler libration points [LMS85,CZ11,DGR13,DGR16], collisions [Bol06], the parabolic infinity [GK13,LS80,Xia93,Xia92,Moe07,Mos01,MP94,MS14] or near mean motion resonances [FGKR16]. Aubry-Mather theory was used to study oscillatory motions and instabilities not close to parabolic motions [GK11].

Among these papers, two were very influential for our computations: the first one is [LS80], where the method of steepest descent was used along special complex paths to compute several integrals, and the second is [MP94], where asymptotic formulae for a scattering map on the infinity manifold for large values of $\epsilon_J G$ is computed. We also believe [GMMS16] to be very important in the future, since the proof of transversal manifolds of the infinity manifold is established for the RPCTBP for any $\mu \in (0, 1/2]$.

It is worth mentioning the paper [Bol06] where the existence of trajectories with diffusion of G was proven assuming small $0 < \varepsilon$ and $0 < \mu \ll \varepsilon$. Diffusive trajectories in [Bol06] are of a very different nature: G travels in a bounded interval while trajectories come close to collisions. In the present paper G is very large, and trajectories come near infinity. However the idea of the proof in [Bol06] is similar: after regularization of collisions there appears a normally hyperbolic symplectic invariant manifold M with trivial inner dynamics and it is possible to define several scattering maps which give rise to diffusive trajectories.

1.2. Comments on the proof: a parabolic infinity and scattering maps. Concerning the proof of our main result, let us first notice that, for a non-zero mass parameter small enough ($0 < \mu \ll 1/2$) and zero eccentricity ($\epsilon_J = 0$), the RPCTBP is non-integrable.

[1] The upper bound on μ^* can be improved in the sense that for $\epsilon_J \leq c/G_2^*$ we can choose $\mu^* = \mu^*(C, c, c/G_2^*)$.

Although for large G it is very close to integrable, since its chaotic zones have a size which is exponentially small for large G, more precisely, of size $O(\exp(-G^3/3))$ (see [LS80, GMMS16]). This phenomenon adds the first difficulty in proving the global instability of the angular momentum G in the RPETBP for large values of G.

The framework for proving our result consists in considering the motion close to the parabolic orbits of the Kepler problem that takes place when the mass parameter μ is zero. To this end we study the *manifold at parabolic infinity*, which turns out to be an invariant object *topologically equivalent to a normally hyperbolic invariant manifold* (TNHIM), in the sense that it is an invariant manifold of fixed points which, even if it is not normally hyperbolic, it has stable and unstable manifolds which consist of the union of the stable and unstable manifolds of its fixed points as proved in [GMMSS17].

More concretely, recall that a motion of the comet $q(t)$ is called *future (resp. past) parabolic* if $\lim_{t\to+\infty}|q(t)| = \infty$ and $\lim_{t\to+\infty}\dot{q}(t) = 0$ (resp. $t \to +\infty$ is replaced by $t \to -\infty$). For the RPCTBP McGehee [McG73] proved (see [GMMSS17] for the RPETBP) that the set of future (resp. past) parabolic motions, denoted $\mathcal{P}_\mu^+$ (resp. $\mathcal{P}_\mu^-$), is an immersed analytic manifold. The intersection $\mathcal{P}_\mu^+ \cap \mathcal{P}_\mu^-$ consists of orbits both future and past parabolic. For $\mu = 0$ we have that $\mathcal{P}_0^+ = \mathcal{P}_0^-$ and they correspond to parabolic motions of the Kepler problem (between the Sun and the comet). These manifolds are stable and unstable manifolds of the manifold at the parabolic infinity denoted $\mathcal{P}_\infty$. The infinity manifold $\mathcal{P}_\infty$ is independent of μ and turns out to be topologically equivalent to a normally hyperbolic invariant manifold (TNHIM).

On this TNHIM $\mathcal{P}^\infty$, it is possible to define two *scattering maps* [DLS00, DLS08], which contain the map structure of the homoclinic trajectories to $\mathcal{P}^\infty$. A non-canonical symplectic structure still persists close to $\mathcal{P}^\infty$ and extends naturally to a b^3-symplectic structure in the sense of [Sco16, KMS16]). Therefore, on $\mathcal{P}^\infty$, it is possible to define a symplectic scattering map, which contains the map structure of the homoclinic trajectories to the TNHIM. Unfortunately, the inner dynamics within $\mathcal{P}^\infty$ is trivial, so it cannot be combined with the scattering map to produce pseudo-orbits adequate for diffusion, and adds a second difficulty. Because of this, in this paper we introduce the use of *two* different scattering maps whose combination produces the desired diffusive pseudo-orbits. It is worth remarking that this strategy of combining several scattering maps to get diffusing orbits have been already applied to several problems [Bol06, DGR16, DS17, DS18]. Using the results in [GMMSS17] we prove the existence of orbits of the system shadowing diffusive pseudo-orbits.

The main issue of computing the two scattering maps consists in evaluating the *Melnikov potential* (37) associated to the TNHIM $\mathcal{P}^\infty$. The main difficulty comes from the fact that its size is exponentially small for a large angular momentum G, so it is necessary to perform very accurate estimates for its Fourier coefficients. Such computations are performed in Sect. 6, see Theorem 8, and they involve a careful treatment of several Fourier expansions, as well as the computation of several integrals using the method of steepest descent along adequate complex paths, playing both with the eccentric and the true anomaly. To guarantee the convergence of the Fourier series, we have to assume that G is large enough ($G \geq C$, $C = 32$), and ϵ_J small enough ($G\epsilon_\mathrm{J} \leq c$, $c = 1/8$). For a larger value of C and a smaller value of c, one can ensure that the dominant part of the Melnikov potential consists on four harmonics, from which it is possible to compute the existence of two functionally independent scattering maps (see Remark 9) which are globally defined in the manifold of parabolic infinity $\mathcal{P}_\infty$.

The combination of these two scattering maps permits the design of the desired diffusive pseudo-orbits, under the assumption of a mass μ very small compared to the eccentricity ($0 < \mu < \mu^*$, see (64)). Shadowing these pseudo-orbits by true trajectories of the system is done using the results of [GMMSS17].

It is worth noticing that since *all the diffusive trajectories* found in this paper shadow ellipses close to parabolas of the Kepler problem, that is, with a very large semi-major axis, their energy is close to zero. The orientation of their semi-major axis (precession) changes only slightly at each revolution.

1.3. Other parabolic regimes. The case of arbitrary eccentricity $0 < \epsilon_{\mathrm{J}} < 1$ and arbitrary mass parameter $0 < \mu < 1$ remains open in this paper. As it turns out, the case $\epsilon_{\mathrm{J}} G \approx 1$ involves the analysis of an infinite number of dominant Fourier coefficients of the Melnikov potential, whereas for the case $\epsilon_{\mathrm{J}} G > 1$, the qualitative properties of the Melnikov function should be known without using its Fourier expansion. Larger values of the mass parameter μ than those considered in this paper involve improving the estimates of the error terms of the splitting of separatrices in complex domains, as is usual when the splitting of separatrices is exponentially small. The computation of the explicit trajectories from the pseudo-orbits found in this paper needs a suitable shadowing result given in [GMMSS17], which involves the translation to TNHIM of the usual shadowing techniques for NHIM.

1.4. Plan of the paper. The plan of this paper is as follows. In Sect. 2 we introduce the equations of the RPETBP, as well as the McGehee coordinates to be used to study the motion close to infinity. In Sect. 3 we recall the geometry of the Kepler problem, i.e. when $\mu = 0$, close to the *parabolic infinity manifold* and its associated separatrix. Next, in Sect. 4, we study the transversal intersection of the invariant manifolds for the RPETBP, as well as the *scattering map* associated, which depend on the *Melnikov potential* of the problem, whose detailed computation is deferred to Sect. 6. The global instability is proven in Sect. 5, using the computation of the Melnikov potential, and is based on the computation of *two different* scattering maps, whose combination gives rise to a heteroclinic chain of periodic orbits with increasing angular momentum and, finally to trajectories with diffusing angular momentum.

2. Setting of the Problem

Fix a coordinate reference system with the origin at the center of mass and call q_{S} and q_{J} the position of the primaries, then under the classical assumptions regarding time units, distance and masses normalization, the motion q of a massless particle under Newton's law of universal gravitation is given by

$$\frac{d^2q}{dt^2} = (1-\mu)\frac{q_{\mathrm{S}} - q}{|q_{\mathrm{S}} - q|^3} + \mu\frac{q_{\mathrm{J}} - q}{|q_{\mathrm{J}} - q|^3} \tag{1}$$

where $1 - \mu$ is the mass of the particle at q_{S} and μ the mass of the particle at q_{J}. Introducing the conjugate momentum $p = dq/dt$ and the self-potential function

$$U_\mu(q, t; \epsilon_{\mathrm{J}}) = \frac{1-\mu}{|q - q_{\mathrm{S}}|} + \frac{\mu}{|q - q_{\mathrm{J}}|}, \tag{2}$$

then the Eq. (1) can be rewritten as a 2+1/2 degree-of-freedom Hamiltonian system with time-periodic Hamiltonian

$$H_\mu(q, p, t; \epsilon_J) = \frac{p^2}{2} - U_\mu(q, t; \epsilon_J). \tag{3}$$

In the (planar) RPETBP, the two primaries are assumed to be revolving around their center of mass on elliptic orbits with eccentricity ϵ_J, unaffected by the motion of the comet q. In polar coordinates $q = \rho(\cos\alpha, \sin\alpha)$, the equations of motion of the primaries are

$$q_S = \quad \mu r(\cos f, \sin f) \qquad q_J = -(1-\mu)r(\cos f, \sin f). \tag{4}$$

By the first Kepler's law the distance r between the primaries [Win41, p. 195] can be written as a function $r = r(f, \epsilon_J)$

$$r = \frac{1-\epsilon_J^2}{1+\epsilon_J \cos f} \tag{5}$$

where $f = f(t, \epsilon_J)$ is the so called *true anomaly*, which satisfies [Win41, p. 203]

$$\frac{df}{dt} = \frac{(1+\epsilon_J \cos f)^2}{(1-\epsilon_J^2)^{3/2}}. \tag{6}$$

Taking into account the expression (4) for the motion of the primaries, we can write explicitly the denominators of the self-potential function (2)

$$\begin{aligned} |q - q_S|^2 &= \rho^2 - 2\mu r\rho\cos(\alpha - f) + \mu^2 r^2, \\ |q - q_J|^2 &= \rho^2 + 2(1-\mu) r\rho\cos(\alpha - f) + (1-\mu)^2 r^2. \end{aligned} \tag{7}$$

We now perform a standard polar-canonical change of variables $(q, p) \longmapsto (\rho, \alpha, P_\rho, P_\alpha)$

$$q = (\rho\cos\alpha, \rho\sin\alpha), \quad p = \left(P_\rho\cos\alpha - \frac{P_\alpha}{\rho}\sin\alpha, P_\rho\sin\alpha + \frac{P_\alpha}{\rho}\cos\alpha\right)$$

to Hamiltonian (3). The equations of motion in the new coordinates are the associated to the Hamiltonian

$$H_\mu^*(\rho, \alpha, P_\rho, P_\alpha, t; \epsilon_J) = \frac{P_\rho^2}{2} + \frac{P_\alpha^2}{2\rho^2} - U_\mu^*(\rho, \alpha, t; \epsilon_J) \tag{8}$$

with a self-potential U_μ^*

$$U_\mu^*(\rho, \alpha, t; \epsilon_J) = U_\mu(\rho\cos\alpha, \rho\sin\alpha, t; \epsilon_J).$$

From now on we will write

$$G = P_\alpha, \qquad y = P_\rho,$$

so that Hamiltonian (8) becomes

$$H_\mu^*(\rho, \alpha, y, G, t; \epsilon_J) = \frac{y^2}{2} + \frac{G^2}{2\rho^2} - U_\mu^*(\rho, \alpha, t; \epsilon_J). \tag{9}$$

Remark 2. In the (planar) circular case $\epsilon_J = 0$ (RTBP), it is clear from Eqs. (5) and (6) that $r = 1$ and $f = t$, and that the expressions for the distances (7) between the primaries depend on the time t and the angle α just through their difference $\alpha - t$. As a consequence, $U_\mu^*(\rho, \alpha, t; 0)$, as well as $H_\mu^*(\rho, \alpha, y, G, t; 0)$, depend also on t and α just through the same difference $\alpha - t$, called the sinodic angle. This implies that the Jacobi constant $H^* + G$ is a first integral of the system.

2.1. McGehee coordinates. To study the behavior of orbits near infinity, we make the McGehee [McG73] non-canonical change of variables

$$\rho = \frac{2}{x^2} \tag{10}$$

for $x > 0$. This brings the infinity $\rho = \infty$ to the origin $x = 0$ (and extends naturally to a b^3-symplectic structure in the sense of [Sco16,KMS16]; other related examples can be found in [DKM17,BDM+18]).

In these McGehee coordinates, the equations associated to the Hamiltonian (8) become

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{4}x^3 y & \frac{dy}{dt} &= \frac{1}{8}G^2x^6 - \frac{x^3}{4}\frac{\partial \mathcal{U}_\mu}{\partial x} \\ \frac{d\alpha}{dt} &= \frac{1}{4}x^4 G & \frac{dG}{dt} &= \frac{\partial \mathcal{U}_\mu}{\partial \alpha}, \end{aligned} \tag{11}$$

where the self-potential $\mathcal{U}_\mu$ is given by

$$\mathcal{U}_\mu(x, \alpha, t; \epsilon_{\mathrm{J}}) = U_\mu^*(2/x^2, \alpha, t; \epsilon_{\mathrm{J}}) = \frac{x^2}{2}\left(\frac{1-\mu}{\sigma_S} + \frac{\mu}{\sigma_J}\right) \tag{12}$$

with

$$|q - q_{\mathrm{S}}|^2 = \sigma_{\mathrm{S}}^2 = 1 - \mu r x^2 \cos(\alpha - f) + \frac{1}{4}\mu^2 r^2 x^4,$$
$$|q - q_{\mathrm{J}}|^2 = \sigma_{\mathrm{J}}^2 = 1 + (1-\mu) r x^2 \cos(\alpha - f) + \frac{1}{4}(1-\mu)^2 r^2 x^4.$$

It is important to notice that the true anomaly f is present in these equations, so that the equation for f given in (6) should be added to have the complete description of the dynamics.

2.1.1. The symplectic structure. Under McGehee change of variables (10), the canonical form $d\rho \wedge dy + d\alpha \wedge dG$ is transformed to

$$\omega = -\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG$$

which, on $x > 0$, is a (non-canonical) symplectic form. Therefore, expressing the Hamiltonian (9) in the McGehee coordinates

$$\mathcal{H}_\mu(x, \alpha, y, G, t; \epsilon_{\mathrm{J}}) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \mathcal{U}_\mu(x, \alpha, t; \epsilon_{\mathrm{J}}), \tag{13}$$

the Eq. (11) can be written as

$$\begin{aligned} \frac{dx}{dt} &= -\frac{x^3}{4}\left(\frac{\partial \mathcal{H}_\mu}{\partial y}\right) & \frac{dy}{dt} &= -\frac{x^3}{4}\left(-\frac{\partial \mathcal{H}_\mu}{\partial x}\right) \\ \frac{d\alpha}{dt} &= \frac{\partial \mathcal{H}_\mu}{\partial G} & \frac{dG}{dt} &= -\frac{\partial \mathcal{H}_\mu}{\partial \alpha}. \end{aligned} \tag{14}$$

Equivalently, we can write the Eq. (14) as $dz/dt = \{z, \mathcal{H}_\mu\}$ in terms of the Poisson bracket

$$\{f, g\} = -\frac{x^3}{4}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) + \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial G} - \frac{\partial f}{\partial G}\frac{\partial g}{\partial \alpha}.$$

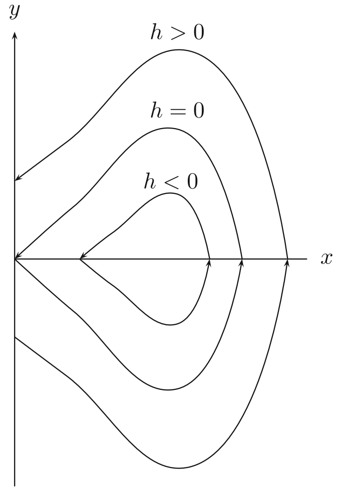

Fig. 1. Level curves of $\mathcal{H}_0$ in the $(x \geq 0, y)$ plane, for fixed $G > 0$

3. Geometry of the Kepler Problem ($\mu = 0$)

3.1. The manifold at parabolic infinity. For $\mu = 0$ and $G > 0$, the Hamiltonian (13) becomes Duffing Hamiltonian (see Fig. 1):

$$\mathcal{H}_0(x, y, G) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \mathcal{U}_0(x) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \frac{x^2}{2}$$

and is a first integral, since the system is autonomous. Moreover, $\mathcal{H}_0$ is also independent of ϵ_{J} and α. Its associated equations are

$$\frac{dx}{dt} = -\frac{1}{4}x^3 y \qquad \frac{dy}{dt} = \frac{1}{8}G^2 x^6 - \frac{1}{4}x^4$$
$$\frac{d\alpha}{dt} = \frac{1}{4}x^4 G \qquad \frac{dG}{dt} = 0 \tag{15}$$

where it is clear that G is a conserved quantity, which will be restricted to the case $G > 0$ from now on, that is, $G \in \mathbb{R}_+$. The phase space, including the invariant locus $x = 0$ is given by $(x, \alpha, y, G) \in \mathbb{R}_{\geq 0} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+$. From Eq. (15) it is clear that

$$\mathcal{E}_\infty = \{z = (x = 0, \alpha, y, G) \in \mathbb{R}_{\geq 0} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+\}$$

is the set of equilibrium points of system (15). Moreover, for any fixed $\alpha \in \mathbb{T}$, $G \in \mathbb{R}$,

$$\Lambda_{\alpha,G} = \{(0, \alpha, 0, G)\}$$

is a parabolic equilibrium point, which is topologically equivalent to a saddle point, since it possesses stable and unstable 1-dimensional invariant manifolds. The union of such points is the 2-dimensional manifold of equilibrium points

$$\Lambda_\infty = \bigcup_{\alpha,G} \Lambda_{\alpha,G},$$

which was previously denoted as $\mathcal{P}_\infty$.

As we will deal with a time-periodic Hamiltonian, it is natural to work in the extended phase space

$$\tilde{z} = (z, s) = (x, \alpha, y, G, s) \in \mathbb{R}_{\geq 0} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{T}$$

just by writing s instead of t in the Hamiltonian and adding the equation

$$\frac{ds}{dt} = 1$$

to systems (14) and (15). We write now the extended version of the invariant sets we have defined so far. For any $\alpha \in \mathbb{T}$, $G \in \mathbb{R}$, the set

$$\tilde{\Lambda}_{\alpha,G} = \{\tilde{z} = (0, \alpha, 0, G, s), s \in \mathbb{T}\}$$

is a 2π-periodic orbit with motion determined by $ds/dt = 1$. The union of such periodic orbits is the 3-dimensional invariant manifold (the parabolic *infinity manifold*)

$$\tilde{\Lambda}_\infty = \bigcup_{\alpha,G} \tilde{\Lambda}_{\alpha,G} = \{(0, \alpha, 0, G, s),\ (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\} \simeq \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}, \quad (16)$$

which is *topologically equivalent to a normally hyperbolic invariant manifold* (TNHIM).

Parameterizing the points in $\tilde{\Lambda}_\infty$ by

$$\tilde{\mathbf{x}}_0 = \tilde{\mathbf{x}}_0(\alpha, G, s) = (\mathbf{x}_0(\alpha, G), s) = (0, \alpha, 0, G, s) \in \tilde{\Lambda}_\infty \simeq \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}$$

the inner dynamics on $\tilde{\Lambda}_\infty$ is trivial, since it is given by the dynamics on each periodic orbit $\tilde{\Lambda}_{\alpha,G}$:

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{x}}_0) = (0, \alpha, 0, G, s + t) = (\mathbf{x}_0(\alpha, G), s + t) = \tilde{\mathbf{x}}_0(\alpha, G, s + t), \quad (17)$$

where we denote by $\tilde{\phi}_{t,\mu}$ the flow of system (14) in the extended phase space.

3.2. The scattering map. In the region of the phase space with positive angular momentum G, let us now look at the homoclinic orbits to the previously introduced invariant objects.

The equilibrium points $\Lambda_{\alpha,G}$ have stable and unstable 1-dimensional invariant manifolds which coincide:

$$\begin{aligned}\gamma_{\alpha,G} &= W^{\mathrm{u}}(\Lambda_{\alpha,G}) = W^{\mathrm{s}}(\Lambda_{\alpha,G}) \\ &= \left\{z = (x, \hat{\alpha}, y, G),\ \mathcal{H}_0(x, y, G) = 0, \hat{\alpha} = \alpha - G \int_{\mathcal{H}_0=0} \frac{x}{y} dx\right\},\end{aligned}$$

whereas the 2-dimensional manifold of equilibrium points Λ_∞ has stable and unstable 3-dimensional invariant manifolds which coincide and are given by

$$\gamma = W^{\mathrm{u}}(\Lambda_\infty) = W^{\mathrm{s}}(\Lambda_\infty) = \{z = (x, \alpha, y, G),\ \mathcal{H}_0(x, y, G) = 0\}.$$

The surface

$$\begin{aligned}\tilde{\gamma}_{\alpha,G} &= W^{\mathrm{u}}(\tilde{\Lambda}_{\alpha,G}) = W^{\mathrm{s}}(\tilde{\Lambda}_{\alpha,G}) \\ &= \left\{ \tilde{z} = (x, \hat{\alpha}, y, G, s), s \in \mathbb{T},\ \mathcal{H}_0(x, y, G) = 0, \hat{\alpha} = \alpha - G \int_{\mathcal{H}_0=0} \frac{x}{y} dx \right\}\end{aligned} \tag{18}$$

is a 2-dimensional homoclinic manifold to the periodic orbit $\tilde{\Lambda}_{\alpha,G}$ in the extended phase space. The 4-dimensional stable and unstable manifolds of the infinity manifold $\tilde{\Lambda}_\infty$ coincide along the 4-dimensional homoclinic invariant manifold (the *separatrix*), which is just the union of the homoclinic surfaces $\tilde{\gamma}_{\alpha,G}$:

$$\begin{aligned}\tilde{\gamma} &= W^{\mathrm{u}}(\tilde{\Lambda}_\infty) = W^{\mathrm{s}}(\tilde{\Lambda}_\infty) = \bigcup_{\alpha,G} \tilde{\gamma}_{\alpha,G} \\ &= \{\tilde{z} = (x, \alpha, y, G, s), (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T},\ \mathcal{H}_0(x, \alpha, y, G) = 0\}.\end{aligned}$$

Due to the presence of the factor $-x^3/4$ in front of Eq. (15), it is more convenient to parameterize the separatrix $\tilde{\gamma}_{\alpha,G}$, given in (18), by the solutions of the Hamiltonian flow contained in $\mathcal{H}_0 = 0$ in some time τ satisfying (see [MP94])

$$\frac{dt}{d\tau} = \frac{2G}{x^2}. \tag{19}$$

In this way, the homoclinic solution to the periodic orbit $\tilde{\Lambda}_{\alpha,G}$ of system (15) can be written as

$$x_{\mathrm{h}}(t; G) = \frac{2}{G(1+\tau^2)^{1/2}} \tag{20a}$$

$$\begin{aligned}\alpha_{\mathrm{h}}(t; \alpha, G) &= \alpha + \pi + 2 \arctan \tau \\ y_{\mathrm{h}}(t; G) &= \frac{2\tau}{G(1+\tau^2)} \\ G_{\mathrm{h}}(t; G) &= G \\ s_{\mathrm{h}}(t; s) &= s + t,\end{aligned} \tag{20b}$$

where α and G are free parameters and the relation between t and τ is

$$t = \frac{G^3}{2}\left(\tau + \frac{\tau^3}{3}\right), \tag{21}$$

which is equivalent to (19) on $\mathcal{H}_0$. From the expressions above, we see that the convergence along the separatrix to the infinity manifold is power-like in τ and t:

$$x_{\mathrm{h}}, y_{\mathrm{h}}, \frac{\alpha - \alpha_{\mathrm{h}} + \pi}{G} \sim \frac{2}{G\tau} \sim \frac{2}{\sqrt[3]{\pm 6t}}, \quad \tau, t \to \pm\infty. \tag{22}$$

We now introduce the notation

$$\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\sigma, \alpha, G, s) = (\mathbf{z}_0(\sigma, \alpha, G), s) = (x_{\mathrm{h}}(\sigma; G), \alpha_{\mathrm{h}}(\sigma; \alpha, G), y_{\mathrm{h}}(\sigma; G), G, s) \in \tilde{\gamma} \tag{23}$$

so that we can parameterize any surface $\tilde{\gamma}_{\alpha,G}$ as

$$\tilde{\gamma}_{\alpha,G} = \{\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\sigma, \alpha, G, s) = (\mathbf{z}_0(\sigma, \alpha, G), s),\ \sigma \in \mathbb{R}, s \in \mathbb{T}\}.$$

and we can parameterize the 4-dimensional separatrix as

$$\tilde{\gamma} = W(\tilde{\Lambda}_\infty) = \{\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\sigma, \alpha, G, s) = (\mathbf{z}_0(\sigma, \alpha, G), s), \sigma \in \mathbb{R}, G \in \mathbb{R}_+, (\alpha, s) \in \mathbb{T}^2\}.$$

The motion on $\tilde{\gamma}$ is given by

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{z}}_0) = \tilde{\mathbf{z}}_0(\sigma + t, \alpha, G, s + t) = (\mathbf{z}_0(\sigma + t, \alpha, G), s + t) \tag{24}$$

and by Eqs. (20), (21) the following asymptotic formula follows:

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{z}}_0) - \tilde{\phi}_{t,0}(\tilde{\mathbf{x}}_0) = (\mathbf{z}_0(\sigma + t, \alpha, G), s + t) - (\mathbf{x}_0(\alpha, G), s + t) \xrightarrow[t\to\pm\infty]{} 0. \tag{25}$$

The *scattering map* $\widetilde{S}$ describes the homoclinic orbits to the infinity manifold $\tilde{\Lambda}_\infty$ (defined in (16)) to itself. Given $\tilde{\mathbf{x}}_-, \tilde{\mathbf{x}}_+ \in \tilde{\Lambda}_\infty$, we define

$$\widetilde{S}_\mu(\tilde{\mathbf{x}}_-) := \tilde{\mathbf{x}}_+$$

if there exists $\tilde{\mathbf{z}}^* \in W^{\mathrm{u}}_\mu(\tilde{\Lambda}_\infty) \cap W^s_\mu(\tilde{\Lambda}_\infty)$ such that

$$\tilde{\phi}_{t,\mu}(\tilde{\mathbf{z}}^*) - \tilde{\phi}_{t,\mu}(\tilde{\mathbf{x}}_\pm) \to 0 \quad \text{for } t \to \pm\infty.$$

In the case $\mu = 0$ the asymptotic relation (25) implies $\widetilde{S}_0(\tilde{\mathbf{x}}_0) = \tilde{\mathbf{x}}_0$ so that that the scattering map $\widetilde{S}_0 : \tilde{\Lambda}_\infty \longrightarrow \tilde{\Lambda}_\infty$ is the identity.

4. Invariant Manifolds for the RPETBP ($\mu > 0$)

4.1. The parabolic infinity manifold. In order to analyse the structure of system (14), we will write $\mathcal{H}_\mu$ given in (13) as

$$\mathcal{H}_\mu(x, \alpha, y, G, s; \epsilon_{\mathrm{J}}) = \mathcal{H}_0(x, y, G) - \mu\Delta\mathcal{U}_\mu(x, \alpha, s; \epsilon_{\mathrm{J}}) \tag{26}$$

where we have written $\mathcal{U}_\mu$ in (12) as

$$\mathcal{U}_\mu(x, \alpha, s; \epsilon_{\mathrm{J}}) = \mathcal{U}_0(x) + \mu\Delta\mathcal{U}_\mu(x, \alpha, s; \epsilon_{\mathrm{J}}) = \frac{x^2}{2} + \mu\Delta\mathcal{U}_\mu(x, \alpha, s; \epsilon_{\mathrm{J}}),$$

and we proceed to study the dynamics as a perturbation of the limit case $\mu = 0$. From (12),

$$\begin{aligned}\Delta\mathcal{U}_0(x, \alpha, s; \epsilon_{\mathrm{J}}) &= \lim_{\mu\to 0} \Delta\mathcal{U}_\mu(x, \alpha, s; \epsilon_{\mathrm{J}}) \\ &= \frac{x^2}{\left[4 + x^4r^2 + 4x^2 r\cos(\alpha - f)\right]^{1/2}} - \left(\frac{x^2}{2}\right)^2 r\cos(\alpha - f) - \frac{x^2}{2}\end{aligned} \tag{27}$$

where $r = r(f, \epsilon_{\mathrm{J}})$ and $f = f(s, \epsilon_{\mathrm{J}})$ are given, respectively, in (5–6).

For $\mu > 0$, it is clear from Eq. (14) that the set $\mathcal{E}_\infty$ remains invariant and, therefore, so does the infinity manifold $\tilde{\Lambda}_\infty$, being again a TNHIM, and all the periodic orbits $\tilde{\Lambda}_{\alpha,G}$ also persist. The inner dynamics on $\tilde{\Lambda}_\infty$ is the same as in the case $\mu = 0$, so that the parametrization $\tilde{\mathbf{x}}_0$ as well as its trivial dynamics (17) remain the same.

4.2. The scattering map. From [McG73,GMMSS17] we know that $W^{\mathrm{s}}_{\mu}(\tilde{\Lambda}_{\infty})$ and $W^{\mathrm{u}}_{\mu}(\tilde{\Lambda}_{\infty})$ exist for μ small enough and are 4-dimensional in the extended phase space. The existence of a scattering map will depend on the transversal intersection between these two manifolds.

Let us take an arbitrary $\tilde{\mathbf{z}}_0 = (\mathbf{z}_0, s) = (\mathbf{z}_0(\sigma, \alpha, G), s) \in \tilde{\gamma}$ as in (23). Now, we have to construct points in $W^{\mathrm{s}}_{\mu}(\tilde{\Lambda}_{\infty})$ and $W^{\mathrm{u}}_{\mu}(\tilde{\Lambda}_{\infty})$ to measure the distance between them. It is clear from the definition of $\tilde{\gamma}$ that

$$\tilde{\mathbf{v}} = (\nabla \mathcal{H}_0(\mathbf{z}_0), 0)$$

is orthogonal to $\tilde{\gamma} = W^{\mathrm{u}}(\tilde{\Lambda}_{\infty}) = W^{\mathrm{s}}(\tilde{\Lambda}_{\infty})$ at $\tilde{\mathbf{z}}_0$ and then if the normal bundle to $\tilde{\gamma}$ is denoted by

$$N(\tilde{\mathbf{z}}_0) = \{\tilde{\mathbf{z}}_0 + \lambda\, \tilde{\mathbf{v}}, \lambda \in \mathbb{R}\}$$

we have that, if μ is small enough, there exist unique points $\tilde{\mathbf{z}}^{\mathrm{s,u}}_{\mu} = (z^{\mathrm{s,u}}_{\mu}, s)$ such that

$$\{\tilde{\mathbf{z}}^{\mathrm{s,u}}_{\mu}\} = W^{\mathrm{s,u}}_{\mu}(\tilde{\Lambda}_{\infty}) \cap N(\tilde{\mathbf{z}}_0). \tag{28}$$

The distance we want to compute between $W^{\mathrm{s}}_{\mu}(\tilde{\Lambda}_{\infty})$ and $W^{\mathrm{u}}_{\mu}(\tilde{\Lambda}_{\infty})$ is the signed magnitude given by

$$d(\tilde{\mathbf{z}}_0, \mu) = \mathcal{H}_0(\tilde{\mathbf{z}}^{\mathrm{u}}_{\mu}) - \mathcal{H}_0(\tilde{\mathbf{z}}^{s}_{\mu}).$$

We now introduce the *Melnikov potential* (see [DG00,DLS06])

$$\mathcal{L}(\alpha, G, s; \epsilon_{\mathrm{J}}) = \int_{-\infty}^{\infty} \Delta\mathcal{U}_0(x_{\mathrm{h}}(t; G), \alpha_{\mathrm{h}}(t; \alpha, G), s + t; \epsilon_{\mathrm{J}})\, dt, \tag{29}$$

where $\Delta\mathcal{U}_0$ is defined in (27). Thanks to the asymptotic behavior (22) of the solutions along the separatrix and of the self potential close to the parabolic infinity manifold

$$\Delta\mathcal{U}_0(x, \alpha, s; \epsilon_{\mathrm{J}}) = O(x^4) \quad \text{as} \quad x \to 0$$

this integral is absolutely convergent, and will be computed in detail in Sect. 6.

Proposition 3. *Given* $(\alpha, G, s) \in \mathbb{T} \times \mathbb{R}^{+} \times \mathbb{T}$*, assume that the function*

$$\sigma \in \mathbb{R} \longmapsto \mathcal{L}(\alpha, G, s - \sigma; \epsilon_J) \in \mathbb{R} \tag{30}$$

has a non-degenerate critical point $\sigma^* = \sigma^*(\alpha, G, s; \epsilon_J)$*. Then, there exists* $\mu^* = \mu^*(G, \epsilon_J)$*, such that for* $0 < \mu < \mu^*$*, close to the point* $\tilde{\mathbf{z}}^*_0 = (\mathbf{z}_0(\sigma^*, \alpha, G), s) \in \tilde{\gamma}$ *(see the parameterization in* (23)*), there exists a locally unique point*

$$\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}^*(\sigma^*, \alpha, G, s; \epsilon_J, \mu) \in W^{s}_{\mu}(\tilde{\Lambda}_{\infty}) \pitchfork W^{u}_{\mu}(\tilde{\Lambda}_{\infty}) \pitchfork N(\tilde{\mathbf{z}}^*_0)$$

of the form

$$\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}^*_0 + O(\mu).$$

Also, there exist unique points $\tilde{\mathbf{x}}_{\pm} = (0, \alpha_{\pm}, 0, G_{\pm}, s) = (0, \alpha, 0, G, s) + O(\mu) \in \tilde{\Lambda}_{\infty}$ *such that*

$$\tilde{\phi}_{t,\mu}(\tilde{\mathbf{z}}^*) - \tilde{\phi}_{t,\mu}(\tilde{\mathbf{x}}_{\pm}) \longrightarrow 0 \quad \textit{for } t \to \pm\infty.$$

Moreover, we have

$$G_{+} - G_{-} = \mu \frac{\partial \mathcal{L}}{\partial \alpha}(\alpha, G, s - \sigma^*(\alpha, G, s; \epsilon_J)) + O(\mu^2). \tag{31}$$

Proof. From the Eq. (23) we know that any point $\tilde{\mathbf{z}}_0 \in \tilde{\gamma}$ has the form

$$\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\sigma, \alpha, G, s).$$

As in (28), we consider

$$\tilde{\mathbf{z}}_\mu^{\mathrm{s,u}} = (\mathbf{z}_\mu^{\mathrm{s,u}}, s) \in W_\mu^{\mathrm{s,u}}(\tilde{\Lambda}_\infty) \cap N(\tilde{\mathbf{z}}_0),$$

and we look for $\tilde{\mathbf{z}}_0$ such that $\tilde{\mathbf{z}}_\mu^{\mathrm{s}} = \tilde{\mathbf{z}}_\mu^{\mathrm{u}}$. There must exist points $\tilde{\mathbf{x}}_\pm = (\mathbf{x}_\pm, s) \in \tilde{\Lambda}_\infty$ such that

$$\tilde{\phi}_{t,\mu}(\tilde{\mathbf{z}}_\mu^{\mathrm{s,u}}) - \tilde{\phi}_{t,\mu}(\tilde{\mathbf{x}}_\pm) \xrightarrow[t\to\pm\infty]{} 0, \tag{32}$$

moreover $\tilde{\phi}_{t,\mu}(\tilde{\mathbf{z}}_\mu^{\mathrm{s,u}}) - \tilde{\phi}_{t,0}(\tilde{\mathbf{z}}_0) = O(\mu)$ for $\pm t \geq 0$ (see [McG73,GMMSS17]). Since $\mathcal{H}_0$ does not depend on time, by (26) and the chain rule we have that

$$\frac{d}{dt}\mathcal{H}_0(\tilde{\phi}_{t,\mu}(\tilde{\mathbf{z}}_\mu^{\mathrm{s,u}})) = \{\mathcal{H}_0, \mathcal{H}_\mu\}(\tilde{\phi}_{t,\mu}(\tilde{\mathbf{z}}_\mu^{\mathrm{s,u}}); \epsilon_{\mathrm{J}}) = -\mu\{\mathcal{H}_0, \Delta\mathcal{U}_\mu\}(\tilde{\phi}_{t,\mu}(\tilde{\mathbf{z}}_\mu^{\mathrm{s,u}}); \epsilon_{\mathrm{J}}).$$

Since $\mathcal{H}_0 = 0$ in $\tilde{\Lambda}_\infty$, using (32) and the trivial dynamics on $\tilde{\Lambda}_\infty$ we obtain

$$\mathcal{H}_0(\tilde{\mathbf{z}}_\mu^{\mathrm{s,u}}) = -\mu\int_0^{\pm\infty}\{\mathcal{H}_0, \Delta\mathcal{U}_\mu\}(\tilde{\phi}_{t,\mu}(\tilde{\mathbf{z}}_\mu^{\mathrm{s,u}}); \epsilon_{\mathrm{J}})\, dt.$$

Taylor expanding in μ and using the notation (24)

$$\begin{aligned}\mathcal{H}_0(\tilde{\mathbf{z}}_\mu^{\mathrm{u}}) - \mathcal{H}_0(\tilde{\mathbf{z}}_\mu^{\mathrm{s}}) &= \mu\int_{-\infty}^{\infty}\{\mathcal{H}_0, \Delta\mathcal{U}_0\}(\tilde{\phi}_{t,0}(\tilde{\mathbf{z}}_0); \epsilon_{\mathrm{J}})\, dt + O(\mu^2)\\ &= \mu\int_{-\infty}^{\infty}\{\mathcal{H}_0, \Delta\mathcal{U}_0\}(\mathbf{z}_0(\sigma + t, \alpha, G), s + t; \epsilon_{\mathrm{J}})\, dt + O(\mu^2).\end{aligned} \tag{33}$$

On the other hand, from (29)

$$\mathcal{L}(\alpha, G, s; \epsilon_{\mathrm{J}}) = \int_{-\infty}^{\infty}\Delta\mathcal{U}_0(x_{\mathrm{h}}(\nu - s; G), \alpha_{\mathrm{h}}(\nu - s; \alpha, G), \nu; \epsilon_{\mathrm{J}})\, d\nu$$

and then

$$\frac{\partial\mathcal{L}}{\partial s}(\alpha, G, s; \epsilon_{\mathrm{J}}) = -\int_{-\infty}^{\infty}\{\Delta\mathcal{U}_0, \mathcal{H}_0\}(\mathbf{z}_0(\nu - s, \alpha, G), \nu; \epsilon_{\mathrm{J}})\, d\nu$$

so that

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial s}(\alpha, G, s - \sigma; \epsilon_{\mathrm{J}}) &= \int_{-\infty}^{\infty}\{\mathcal{H}_0, \Delta\mathcal{U}_0\}(\mathbf{z}_0(\nu - s + \sigma, \alpha, G), \nu; \epsilon_{\mathrm{J}})\, d\nu\\ &= \int_{-\infty}^{\infty}\{\mathcal{H}_0, \Delta\mathcal{U}_0\}(\mathbf{z}_0(t + \sigma, \alpha, G), s + t; \epsilon_{\mathrm{J}})\, dt\end{aligned} \tag{34}$$

and therefore, from (33) and (34)

$$d(\tilde{\mathbf{z}}_0, \mu) = \mathcal{H}_0(\tilde{\mathbf{z}}_\mu^{\mathrm{u}}) - \mathcal{H}_0(\tilde{\mathbf{z}}_\mu^{\mathrm{s}}) = \mu\frac{\partial\mathcal{L}}{\partial s}(\alpha, G, s - \sigma; \epsilon_{\mathrm{J}}) + O(\mu^2). \tag{35}$$

For a non-zero small enough μ, it is clear by the Implicit Function Theorem that a non degenerate critical value σ^* of the function (30) gives rise to a homoclinic point $\tilde{\mathbf{z}}^*$

to $\tilde{\Lambda}_\infty$ where the manifolds $W^{\mathrm{s}}_\mu(\tilde{\Lambda}_\infty)$ and $W^{\mathrm{u}}_\mu(\tilde{\Lambda}_\infty)$ intersect transversally and has the desired form $\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}^*_0 + O(\mu)$.

Consider now the solution of system (14) in the extended phase space represented by $\tilde{\phi}_{t,\mu}(\tilde{\mathbf{z}}^*)$. By the Fundamental Theorem of Calculus and (26) we have

$$\begin{aligned} G_+ - G_- &= -\int_{-\infty}^{\infty} \frac{\partial \mathcal{H}_\mu}{\partial \alpha}(\tilde{\phi}_{t,\mu}(\tilde{\mathbf{z}}^*))\,dt = \mu \int_{-\infty}^{\infty} \frac{\partial \Delta \mathcal{U}_\mu}{\partial \alpha}(\tilde{\phi}_{t,\mu}(\tilde{\mathbf{z}}^*);\epsilon_{\mathrm{J}})\,dt \\ &= \mu \int_{-\infty}^{\infty} \frac{\partial \Delta \mathcal{U}_0}{\partial \alpha}(\tilde{\phi}_{t,0}(\tilde{\mathbf{z}}^*_0);\epsilon_{\mathrm{J}})\,dt + O(\mu^2) \\ &= \mu \int_{-\infty}^{\infty} \frac{\partial \Delta \mathcal{U}_0}{\partial \alpha}(\mathbf{z}_0(\sigma^* + t, \alpha, G), s + t;\epsilon_{\mathrm{J}})\,dt + O(\mu^2) \\ &= \mu \frac{\partial \mathcal{L}}{\partial \alpha}(\alpha, G, s - \sigma^*;\epsilon_{\mathrm{J}}) + O(\mu^2). \end{aligned}$$

□

Remark 4. From (35) it is clear that, to apply the implicit function Theorem, we need that $\mu \ll \mu^*$, where

$$\mu^* = O\left(\frac{\partial^2}{\partial s^2}\left(\mathcal{L}\left(\alpha, G, s - \sigma^*(\alpha, G, s;\epsilon_{\mathrm{J}});\epsilon_{\mathrm{J}}\right)\right)\right).$$

We will give more precise information about μ^* after the computation of the Melnikov function given in Theorem 8, where we will see that it is exponentially small for large G.

Once we have found a critical point $\sigma^* = \sigma^*(\alpha, G, s;\epsilon_{\mathrm{J}})$ of (30) on a domain of (α, G, s), we can define the *reduced Poincaré function* (see [DLS06])

$$\mathcal{L}^*(\alpha, G;\epsilon_{\mathrm{J}}) := \mathcal{L}(\alpha, G, s - \sigma^*;\epsilon_{\mathrm{J}}) = \mathcal{L}(\alpha, G, s^*;\epsilon_{\mathrm{J}}) \tag{36}$$

with $s^* = s - \sigma^*$. Note that the reduced Poincaré function does not depend on the s chosen, since by Proposition 3

$$\frac{\partial}{\partial s}\left(\mathcal{L}\left(\alpha, G, s - \sigma^*(\alpha, G, s;\epsilon_{\mathrm{J}});\epsilon_{\mathrm{J}}\right)\right) \equiv 0.$$

Note also that if the function (30) in Proposition 3 has different non degenerate critical points there will exist different scattering maps.

The next Proposition gives an approximation of the scattering map in the general case $\mu > 0$.

Proposition 5. *The associated scattering map* $(\alpha_+, G_+, s_+) = \widetilde{S}_\mu(\alpha, G, s)$ *for any non degenerate critical point* σ^* *of the function defined in* (30) *is given by*

$$(\alpha, G, s) \longmapsto \left(\alpha - \mu \frac{\partial \mathcal{L}^*}{\partial G}(\alpha, G;\epsilon_J) + O(\mu^2), G + \mu \frac{\partial \mathcal{L}^*}{\partial \alpha}(\alpha, G;\epsilon_J) + O(\mu^2), s\right)$$

where $\mathcal{L}^*$ *is the Poincaré reduced function introduced in* (36).

Proof. By hypothesis we have a non degenerate critical point σ^* of (30). By definition (36), Proposition 3 gives

$$G_+ - G = \mu \frac{\partial \mathcal{L}^*}{\partial \alpha}(\alpha, G) + O(\mu^2).$$

as well as $G_- = G + O(\mu)$ to get the correspondence between G_+ and G_- that were looking for.

The companion equation to (31)

$$\alpha_+ - \alpha = -\mu \frac{\partial \mathcal{L}^*}{\partial G}(\alpha, G) + O(\mu^2)$$

follows from the fact that the scattering map is of the form $\widetilde{S}_\mu(\alpha, G, s) = (S_\mu(\alpha, G, s), s)$ and, for each fixed $s \in \mathbb{T}$, S_μ is symplectic.

Indeed, this is a standard result for a scattering map associated to a NHIM, and is proven in [DLS08, Theorem 8]. For what concerns our scattering map defined on a TNHIM, the only difference is that the stable contraction (expansion) along $W_\mu^{s,u}(\tilde{\Lambda}_\infty)$ is power-like (22) instead of exponential with respect to time. Therefore, we only have to check that Proposition 10 in [DLS08] still holds, namely that Area $\left(\phi_{t,\mu}(\mathcal{R})\right) \to 0$ when $t \to 0$ for every 2-cell $\mathcal{R}$ in $W_\mu^{s}(\tilde{\Lambda}_\infty)$ parameterized by $R : [0, 1] \times [0, 1] \to W_\mu^{s}(\tilde{\Lambda}_\infty)$ in such a way that $R(t_1, t_2) \in W_\mu^{s}(\tilde{\Lambda}_\infty)$, $R(0, t_2) \in \tilde{\Lambda}_\infty$. But this is a direct consequence of the fact that the stable coordinates contract at least by $C/\sqrt[3]{t}$ (see (22)) and the coordinates along Λ_∞ do not expand at all. □

Remark 6. In the (planar) circular case $\epsilon_J = 0$ (RTBP), $\Delta\mathcal{U}_\mu(x, \alpha, s; \epsilon_J)$ depends on the time s and the angle α just through their difference $\alpha - s$, see Remark 2. From

$$\frac{\partial \Delta\mathcal{U}_\mu}{\partial \alpha}(x, \alpha, s; 0) = -\frac{\partial \Delta\mathcal{U}_\mu}{\partial s}(x, \alpha, s; 0)$$

one readily obtains

$$\frac{\partial \mathcal{L}}{\partial s}(\alpha, G, s; 0) = -\frac{\partial \mathcal{L}}{\partial \alpha}(\alpha, G, s; 0)$$

and, therefore,

$$\frac{\partial \mathcal{L}}{\partial \alpha}(\alpha, G, s - \sigma^*; \epsilon_J) = -\frac{\partial \mathcal{L}}{\partial s}(\alpha, G, s - \sigma^*; 0) = 0$$

and consequently the reduced Poincaré function $\mathcal{L}^*$ does not depend on α, and $G_+ = G_- + O(\mu^2)$.

But indeed $G_+ \equiv G_-$ in the circular case, since there exists the first integral provided by the Jacobi constant $C_J = \mathcal{H}_\mu + G$ and as $\mathcal{H}_\mu = 0$ on $\tilde{\Lambda}_\infty$, $G_+ = G_-$. Therefore, in the circular case there is no possibility to find diffusive orbits studying the intersection of $W_\mu^{s}(\tilde{\Lambda}_\infty)$ and $W_\mu^{u}(\tilde{\Lambda}_\infty)$ since any scattering map preserves the angular momentum.

5. Global Diffusion in the RPETBP

We have already the tools to derive the scattering maps to the infinity manifold $\tilde{\Lambda}_\infty$, namely, Proposition 3 to find transversal homoclinic orbits to $\tilde{\Lambda}_\infty$ and Proposition 5 to give their expressions. Both of them rely on computations on the Melnikov potential $\mathcal{L}$. Inserting in the Melnikov potential introduced in (29) the expression for $\Delta\mathcal{U}_0$ in (27), we get

$$\mathcal{L}(\alpha, G, s; \epsilon_{\mathrm{J}}) = \int_{-\infty}^{\infty} \left[\frac{x_{\mathrm{h}}^2}{\left[4 + x_{\mathrm{h}}^4 r^2 + 4x_{\mathrm{h}}^2 r \cos(\alpha_{\mathrm{h}} - f)\right]^{1/2}} + \left(\frac{x_{\mathrm{h}}^2}{2}\right)^2 r \cos(\alpha_{\mathrm{h}} - f) - \frac{x_{\mathrm{h}}^2}{2} \right] dt \tag{37}$$

where x_{h} and α_{h}, coordinates of the homoclinic orbit defined in (20), are evaluated at t, whereas r and f, defined in (5) and (6), are evaluated at $s + t$.

To evaluate the above Melnikov potential, we will compute its Fourier coefficients with respect to the angular variables α, s. Since x_{h} and r are even functions of t and f and α_{h} are odd, $\mathcal{L}$ is an even function of the angular variables α, s: $\mathcal{L}(-\alpha, G, -s; \epsilon_{\mathrm{J}}) = \mathcal{L}(\alpha, G, s; \epsilon_{\mathrm{J}})$, and therefore $\mathcal{L}$ has a Fourier Cosine series with real coefficients $L_{q,k}$:

$$\mathcal{L} = \sum_{q\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} L_{q,k} \mathrm{e}^{i(qs+k\alpha)} = L_{0,0} + 2\sum_{k\geq 1} L_{0,k} \cos k\alpha + 2\sum_{q\geq 1} \sum_{k\in\mathbb{Z}} L_{q,k} \cos(qs + k\alpha). \tag{38}$$

The concrete computation of the Fourier coefficients of the Melnikov potential (37) will be carried out in Sect. 6. In that section, the following general bounds will be obtained in Proposition 20 and Lemma 35:

Proposition 7. *Let $G \geq 32$, $q \geq 1$, $k \geq 2$ and $\ell \geq 0$. Then $|L_{q,\ell}| \leq B_{q,\ell}$ and $|L_{0,\ell}| \leq B_{0,\ell}$, where*

$$\begin{aligned}
B_{q,0} &= 2^{9+q}\, e^{2q} \epsilon_J^q G^{-3/2} e^{-qG^3/3} \\
B_{q,1} &= 2^{11}\, e^{2q} \frac{\epsilon_J^{q+1}}{\sqrt{1-\epsilon_J^2}} G^{-7/2} e^{-qG^3/3} \\
B_{q,-1} &= 2^{9+q}\, e^{2q} \epsilon_J^{|1-q|} G^{-1/2} e^{-qG^3/3} \\
B_{q,k} &= 2^{2k+5} e^{2q} \frac{\epsilon_J^{q+k}}{(\sqrt{1-\epsilon_J^2})^k} G^{-2k-1/2} e^{-qG^3/3} \\
B_{q,-k} &= 2^{5+q+2k}\, e^{2q}\, \epsilon_J^{|k-q|} G^{k-1/2} e^{-qG^3/3} \\
B_{0,\ell} &= 2^{8+2\ell}\, \epsilon_J^\ell\, G^{-2\ell-3}.
\end{aligned} \tag{39}$$

Directly from this Proposition, we first see that the harmonics $L_{q,\ell}$ are exponentially small for large G and $q \geq 1$, so it will be convenient to split the Fourier expansion (38) as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \cdots = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{\geq 2} \tag{40}$$

where

$$\mathcal{L}_0(\alpha, G; \epsilon_J) = L_{0,0} + 2\sum_{k\geq 1} L_{0,k}\cos k\alpha,$$
$$\mathcal{L}_q(\alpha, G, s; \epsilon_J) = 2\sum_{k\in\mathbb{Z}} L_{q,k}\cos(qs + k\alpha), \qquad q \geq 1. \tag{41}$$

The function $\mathcal{L}_0$ does not depend on the angle s as it contains the harmonics of $\mathcal{L}$ of order 0 in s, which are of finite order in terms of G, $\mathcal{L}_1$ the harmonics of first order, which are of order $e^{-qG^3/3}$, and all the harmonics of $\mathcal{L}_q$ for $q \geq 2$ are much exponentially smaller for large G than those of $\mathcal{L}_1$, so we will estimate $\mathcal{L}_0$ and $\mathcal{L}_1$ and bound $\mathcal{L}_{\geq 2}$.

To this end, it will be necessary to sum the series in (41). From the bounds $B_{q,k}$ in (39) for the harmonics $L_{q,k}$ we get the quotients

$$\frac{B_{q,k+1}}{B_{q,k}} = \frac{4}{G^2}\frac{\epsilon_J}{\sqrt{1-\epsilon_J^2}} \text{ for } k \geq 2, \quad \frac{B_{q,-(k+1)}}{B_{q,-k}} = 4\epsilon_J G \text{ for } k \geq q,$$
$$\frac{B_{0,\ell+1}}{B_{0,\ell}} = \frac{4\epsilon_J}{G^2} \text{ for } \ell \geq 0. \tag{42}$$

To guarantee the convergence of the Fourier series of $\mathcal{L}_q$, we impose the following conditions

$$G > \sqrt{\frac{4\epsilon_J}{\sqrt{1-\epsilon_J^2}}} \quad \text{and} \quad \epsilon_J G < 1/4.$$

This is one of the reasons why we are going to restrict ourselves to the region $G \geq C$ large enough and $\epsilon_J G \leq c$ small enough along this paper to get the diffusive orbits.

Among the harmonics $L_{0,\ell}$ of 0 order in s, by (42), the harmonic $L_{0,0}$ appears to be the dominant one, but we will also estimate $L_{0,1}$ to get information about the variable α, and bound the rest of harmonics $L_{0,\ell}$ for $\ell \geq 2$. Among the harmonics of first order $L_{1,k}$, again by (42), the five harmonics $L_{1,k}$ for $|k| \leq 2$ are the only candidates to be the dominant ones, but the quotients from (39)

$$\frac{B_{1,2}}{B_{1,-1}} = \frac{\epsilon_J^3}{2(1-\epsilon_J^2)G^4}, \qquad \frac{B_{1,1}}{B_{1,-1}} = \frac{2\epsilon_J^2}{\sqrt{1-\epsilon_J^2}G^3}, \qquad \frac{B_{1,0}}{B_{1,-1}} = \frac{\epsilon_J}{G} = \frac{\epsilon_J G}{G^2},$$

indicate that $L_{1,-1}$ and $L_{1,-2}$ appear to be the two dominant harmonics of order 1. Nevertheless, as we will need to use two different scatterig maps, the coefficient $L_{1,-3}$ will be necessary to check that both scatterig maps are independent. Summarizing, to compute the series (38) we compute only the five harmonics $L_{0,0}$, $L_{0,1}$, $L_{1,-1}$, $L_{1,-2}$ and $L_{1,-3}$, and bound all the rest, providing the following result, whose proof will also be carried out in Sect. 6.

Theorem 8. *For $G \geq 32$, $\epsilon_J G \leq 1/8$, the Melnikov potential (37) is given by*

$$\mathcal{L}(\alpha, G, s; \epsilon_J) = \mathcal{L}_0(\alpha, G; \epsilon_J) + \mathcal{L}_1(\alpha, G, s; \epsilon_J) + \mathcal{L}_{\geq 2}(\alpha, G, s; \epsilon_J) \tag{43}$$

with

$$\mathcal{L}_0(\alpha, G; \epsilon_J) = L_{0,0} + L_{0,1}\cos\alpha + \mathcal{E}_0(\alpha, G; \epsilon_J)$$
$$\mathcal{L}_1(\alpha, G, s; \epsilon_J) = 2L_{1,-1}\cos(s-\alpha) + 2L_{1,-2}\cos(s-2\alpha) + 2L_{1,-3}\cos(s-3\alpha) + \mathcal{E}_1(\alpha, G, s; \epsilon_J), \tag{44}$$

where the four harmonics above are given by

$$L_{0,0} = L_{0,0}(G;\epsilon_J) = \frac{\pi}{2G^3}(1+E_{0,0}) \tag{45}$$

$$L_{0,1} = L_{0,1}(G;\epsilon_J) = -\frac{15\pi\epsilon_J}{8G^5}(1+E_{0,1}) \tag{46}$$

$$2\,L_{1,-1} = 2\,L_{1,-1}(G;\epsilon_J) = \sqrt{\frac{\pi}{8G}}e^{-G^3/3}(1+E_{1,-1}) \tag{47}$$

$$2\,L_{1,-2} = 2\,L_{1,-2}(G;\epsilon_J) = -3\sqrt{2\pi}\epsilon_J G^{3/2}e^{-G^3/3}(1+E_{1,-2}) \tag{48}$$

$$2\,L_{1,-3} = 2\,L_{1,-3}(G;\epsilon_J) = \frac{19}{8}\sqrt{2\pi}\epsilon_J^2 G^{5/2}e^{-G^3/3}(1+E_{1,-3}) \tag{49}$$

and the error functions satisfy

$$\begin{aligned}
|E_{0,0}| &\le 2^{12}G^{-4} + 2^2\,49\,\epsilon_J^2 \\
|E_{0,1}| &\le 2^{13}G^{-4} + \epsilon_J^2 \\
|E_{1-1}| &\le 2^{21}G^{-1} + 2\,49\,\epsilon_J^2 \\
|E_{1,-2}| &\le 2^{17}G^{-1} + \frac{49}{3}\epsilon_J \\
|E_{1,-3}| &\le 2^{17}G^{-1} + 15\epsilon_J \\
|\mathcal{E}_0| &\le 2^{14}\,\epsilon_J^2 G^{-7} \\
|\mathcal{E}_1| &\le 2^{18}\epsilon_J e^{-G^3/3}\left[\epsilon_J^2 G^{7/2} + G^{-3/2}\right] \qquad (50) \\
\left|\mathcal{L}_{\ge 2}\right| &\le 2^{28}G^{3/2}e^{-2G^3/3}. \qquad (51)
\end{aligned}$$

Remark 9. To estimate properly the first harmonics $L_{0,0}$, $L_{0,1}$, $L_{1,-1}$, $L_{1,-2}$, $L_{1,-3}$ we will need to take $G > C$, with C big enough and $\epsilon_J G < c$ with c small enough to ensure that the corresponding relative errors $E_{i,j}$ are smaller than one, say $|E_{i,j}| \le 1/2$. This is the main reason why we have to enlarge the constant $C = 32$ given in Theorem 8.

The function $\mathcal{L}_1$ introduced in (41) with $q = 1$, contains only harmonics of first order in s, so we can write it as a cosine function in s. Introducing the parameters (depending on G and ϵ_J)

$$p := -\frac{L_{1,-2}}{L_{1,-1}} = 12\epsilon_J G^2\frac{1+E_{1,-2}}{1+E_{1,-1}} =: 12\epsilon_J G^2(1+E_p) \tag{52}$$

$$q := -\frac{L_{1,-3}}{L_{1,-2}} = \frac{19}{24}\epsilon_J G\frac{1+E_{1,-3}}{1+E_{1,-2}} =: \frac{19}{24}\epsilon_J G(1+E_q) \tag{53}$$

with

$$E_p, E_q = O(\epsilon_J, G^{-1})$$

in the expression (41) of $\mathcal{L}_1$, we can write

$$\begin{aligned}
\mathcal{L}_1 &= 2L_{1,-1}\left(\sum_{k\in\mathbb{Z}}\frac{L_{1,k}}{L_{1,-1}}\cos(s+k\alpha)\right) \\
&= 2L_{1,-1}\left(\cos(s-\alpha) - p\cos(s-2\alpha) + qp\cos(s-3\alpha)\right.
\end{aligned}$$

$$+\sum_{k\neq -1,-2,-3}\frac{L_{1,k}}{L_{1,-1}}\cos(s+k\alpha)\Bigg)$$
$$=2L_{1,-1}\Re\left(e^{i(s-\alpha)}\left(1-pe^{-i\alpha}+qpe^{-2i\alpha}+\sum_{k\neq -1,-2,-3}\frac{L_{1,k}}{L_{1,-1}}e^{i(k+1)\alpha}\right)\right)$$
$$=2L_{1,-1}\Re\left(e^{i(s-\alpha)}Be^{-i\theta}\right)=2L_{1,-1}B\cos(s-\alpha-\theta), \tag{54}$$

where $B=B(\alpha,G;\epsilon_J)\geq 0$ and $-\theta=-\theta(\alpha,G;\epsilon_J)\in[-\pi,\pi)$ are the modulus and the argument of the complex expression

$$1-pe^{-i\alpha}+qpe^{-2i\alpha}+\sum_{k\neq -1,-2,-3}\frac{L_{1,k}}{L_{1,-1}}e^{i(k+1)\alpha}=:Be^{-i\theta}. \tag{55}$$

Writing also in polar form the quotient of the sum in (55) by the parameter p introduced in (52)

$$Ee^{-i\phi}:=\sum_{k\neq -1,-2,-3}\frac{L_{1,k}}{pL_{1,-1}}e^{i(k+1)\alpha}=-\sum_{k\neq -1,-2,-3}\frac{L_{1,k}}{L_{1,-2}}e^{i(k+1)\alpha}$$
$$=q\sum_{k\neq -1,-2,-3}\frac{L_{1,k}}{L_{1,-3}}e^{i(k+1)\alpha},$$

with $E=E(\alpha,G;\epsilon_J)\geq 0$ and $-\phi=-\phi(\alpha,G;\epsilon_J)\in[-\pi,\pi)$, Eq. (55) for B and θ reads now as

$$Be^{-i\theta}=1-pe^{-i\alpha}+qpe^{-2i\alpha}+pEe^{-i\phi} \tag{56}$$

or, equivalently, as the couple of real equations

$$B\cos\theta=1-p\cos\alpha+qp\cos 2\alpha+pE\cos\phi=1-qp-(1-2q\cos\alpha)p\cos\alpha$$
$$+pE\cos\phi \tag{57}$$
$$-B\sin\theta=p\sin\alpha-qp\sin 2\alpha-pE\sin\phi=(1-2q\cos\alpha)p\sin\alpha-pE\sin\phi. \tag{58}$$

One can also obtain explicit formulas for B:

$$B^2=1+p^2(1+q^2+E^2)$$
$$+\;2p\left(-(1+qp)\cos\alpha+q\cos(2\alpha)+E\cos\phi+pE\cos(\phi+\alpha)+pqE\cos(\phi-2\alpha)\right). \tag{59}$$

The function $E=E(\alpha,G;\epsilon_J)$ is small, since, by (50), (48) and (52), if $G>C$ is large enough and $G\epsilon_J<c$ is small enough (see Remark 9),

$$|E|\leq\frac{|\mathcal{E}_1|}{|L_{1,-2}|}\leq\frac{2^{19}\epsilon_J(\epsilon_J^2G^{7/2}+G^{-3/2})}{\frac{3}{2}\sqrt{2\pi}\epsilon_J G^{3/2}}=\frac{2^{20}}{3\sqrt{2\pi}}(\epsilon_J^2G^2+G^{-3})=O\left(G^{-3},\epsilon_J^2G^2\right), \tag{60}$$

with an analogous bound for its derivative with respect to α.

From expression (54), $\mathcal{L}_1$ is a genuine cosine function in s (non identically zero) as long as $B>0$. If we first consider the case $E=0$ in the Eqs. (57–58) defining B, it follows that $B=0$ only for $\alpha=0$ and $1-p+qp=0$, or $p=1/(1-q)$, that is, for $G\simeq(12\epsilon_J)^{-1/2}$ (see (52–53)). A totally analogous property holds when E is taken into account:

Lemma 10. *There exists $C > 32$ and $c < 1/8$ such that, for $G \geq C$ and $\epsilon_J G < c$, then $B(\alpha, G; \epsilon_J) > 0$ except for $\alpha = 0$ and $\sum_{k\in\mathbb{Z}} L_{1,k} = 0$.*

Remark 11. $\sum_{k\in\mathbb{Z}} L_{1,k} = 0 \iff 1 - p + qp + \sum_{k\neq -1,-2,-3} \frac{L_{1,k}}{L_{1,-1}} = 0 \iff p = \frac{1 + \sum_{k\neq -1,-2,-3} \frac{L_{1,k}}{L_{1,-1}}}{1-q}$.

Proof. For $B = 0$, Eq. (58) reads as

$$\sin\alpha = f(\alpha). \tag{61}$$

where $f(\alpha) = f(\alpha, G; \epsilon_J) := E \sin\phi/(1 - 2q\cos\alpha)$. By (53) and (60), if $G > C$ is large enough and $G\epsilon_J < c$ is small enough, we have that $f^2 + (\partial f/\partial\alpha)^2 < 1$, and therefore there are exactly two simple solutions of Eq. (61) in the interval $[-\pi/2, 3\pi/2]$; one is $\alpha^*_{0,+} \in (-\pi/2, \pi/2)$ obtained as a fixed point of the contraction $\alpha = \arcsin(f(\alpha, G; \epsilon_J))$, and a second $\alpha^*_{0,-} \in (\pi/2, 3\pi/2)$ fixed point of the contraction $\alpha = \pi - \arcsin(f(\alpha, G; \epsilon_J))$. Taking a closer look at Eq. (56), we see that if α changes to $-\alpha$, then $-\phi, -\theta, B, E$ are solutions of (56) or, in other words, ϕ, θ are odd functions of α and B, E even. Therefore, $\alpha = 0, \pi$ are the unique solutions of Eq. (58) for $B = 0$. Substituting $\alpha = 0, \pi$ in (57) for $B = 0$, only $\alpha = 0$ provides a positive p, which is then given by $p = 1 + qp + pE = (1 + \sum_{k\neq -1,-2,-3} L_{1,k}/L_{1,-1})/(1-q)$. □

We are now in position to find critical points of the function $s \mapsto \mathcal{L}(\alpha, G, s; \epsilon_J)$. To this end we will check that $s \mapsto \mathcal{L}(\alpha, G, s; \epsilon_J)$ is indeed a *cosine-like* function, that is, with a non-degenerate maximum (minimum) and no other critical points. By Theorem 8, the dominant part of the Melnikov potential $\mathcal{L}$ is given by $\mathcal{L}_0 + \mathcal{L}_1$. By Eq. (43) and the bounds for the error term, for $G > C$ big enough and $G\epsilon_J < c$ small enough, the critical points in the variable s are well approximated by the critical points of the function $\mathcal{L}_0 + \mathcal{L}_1$ (in fact of $\mathcal{L}_1$ because $\mathcal{L}_0$ does not depend on s) and therefore will be close to $s - \alpha - \theta = 0, \pi \pmod{2\pi}$ thanks to expression (54). For this purpose, we introduce

$$\mathcal{L}_1^* = \mathcal{L}_1^*(\alpha, G; \epsilon_J) = 2L_{1,-1}B \tag{62}$$

where $B = B(\alpha, G; \epsilon_J)$ is given in (55) (see also (59)) and $L_{1,-1}$ is the harmonic computed in (47). By (54) and (62), the function $\mathcal{L}_1$ can thus be written as a cosine function in s

$$\mathcal{L}_1(\alpha, G, s; \epsilon_J) = \mathcal{L}_1^*(\alpha, G; \epsilon_J)\cos(s - \alpha - \theta),$$

and differentiating the Melnikov potential (43) with respecto to s we get

$$\frac{\partial\mathcal{L}}{\partial s} = -\mathcal{L}_1^*\sin(s - \alpha - \theta) + \frac{\partial\mathcal{L}_{\geq 2}}{\partial s} = 0 \iff \sin(s - \alpha - \theta) = \frac{1}{\mathcal{L}_1^*}\frac{\partial\mathcal{L}_{\geq 2}}{\partial s}$$

which is a equation of the form (61) for $s - \alpha - \theta$ instead of α and $f = \left(\partial\mathcal{L}_{\geq 2}/\partial s\right)/\mathcal{L}_1^* = \left(\partial\mathcal{L}_{\geq 2}/\partial s\right)/(2L_{1,-1}B)$. Therefore, as long as $f^2 + (\partial f/\partial\alpha)^2 < 1$, there exist exactly two non-degenerate critical points $s^*_\pm$ of the function $s \mapsto \mathcal{L}(\alpha, G, s; \epsilon_J)$.

Now, by estimate (47) for $L_{1,-1}$, bound (51) for $\mathcal{L}_{\geq 2}$ and Lemma 10, it turns out that $f^2 + (\partial f/\partial\alpha)^2 < 1$ happens outside of a neighborhood of size $O\left(G^{3/2}\mathrm{e}^{-G^3/3}\right)$ of the point

$$(\alpha = 0, G = G^*) \text{ where } G^* \approx (12\epsilon_{\mathrm{J}})^{-1/2} \text{ is such that } p = \frac{1 + \sum_{k\neq -1,-2,-3} \frac{L_{1,k}}{L_{1,-1}}}{1-q}. \tag{63}$$

Let us recall now that the Melnikov function $\mathcal{L}$ (see (40), (41)), as well as its terms $\mathcal{L}_q$ are all expressed as Fourier Cosine series in the angles α and s, or equivalently, they are even functions of (α, s). Consequently, $\partial\mathcal{L}_q/\partial s$ is an odd function of (α, s), and it is easy to check that each critical point $s^*_{\pm}$ is an odd function of α. Moreover, using the Fourier Sine expansion of $\partial\mathcal{L}_q/\partial s$, one sees that if s is a critical point of $s \mapsto \mathcal{L}(\alpha, G, s; \epsilon_{\mathrm{J}})$, $s + \pi$ too, so $s^*_- = s^*_+ + \pi$. We state all this in the following Proposition.

Proposition 12. *Let $\mathcal{L}$ be the Melnikov potential given in* (43)*. There exists $C > 32$ and $c < 1/8$ such that, for $G \geq C$ and $\epsilon_J G < c$, except for a neighborhood of size $O\left(G^{3/2}e^{-G^3/3}\right)$ of the point $(\alpha = 0, G = G^*)$ given in* (63)*, $s \mapsto \mathcal{L}(\alpha, G, s; \epsilon_J)$ is a* cosine-like *function, and its critical points are given by*

$$s^*_+ = s^*_+(\alpha, G; \epsilon_J) = \alpha + \theta + \varphi^*, \qquad s^*_- = s^*_- + \pi = \alpha + \theta + \pi + \varphi^*$$

where $\theta = \theta(\alpha, G; \epsilon_J)$ is given in (55) *and $\varphi^* = O\left(G^{3/2}e^{-G^3/3}\right)$.*

From the Proposition above we know that there exist s^*_+ and $s^*_- = s^*_+ + \pi$, non-degenerate critical points of $s \mapsto \mathcal{L}(\alpha, G, s; \epsilon_{\mathrm{J}})$. Therefore, applying Proposition 3 and Remark 4, we know that $W^u(\tilde{\Lambda}_\infty)$ intersects transversally $W^s(\tilde{\Lambda}_\infty)$ if $0 < \mu \ll \mu^*$ with

$$\mu^* = O\left(\frac{\partial^2}{\partial s^2}\left(\mathcal{L}\left(\alpha, G, s^*; \epsilon_{\mathrm{J}}\right)\right)\right).$$

Using Theorem 8 and (62), we see that it is enough to impose that:

$$|\mu| \ll \left|\mathcal{L}^*_1\right| = 2|L_{1,-1}B| = O\left(G^{-1/2}\mathrm{e}^{-G^3/3}\right),$$

that is, μ exponentially small for large G in the region $C \leq G \leq c/\epsilon_{\mathrm{J}}$ which a fortiori is satisfied for

$$0 < \mu \ll \mu^* = \mathrm{e}^{-(c/\epsilon_{\mathrm{J}})^3/3}. \tag{64}$$

We will see that this is the relation between the eccentricity and the mass parameter that we need to guarantee that our main result, Theorem 1, holds. This kind of relation is typical in problems with exponentially small splitting, when the bound of the remainder, here $O(\mu^2)$, is obtained through a direct application of the Melnikov method for the real system. To get better estimates for this remainder, one needs to bound this remainder for complex values of the parameter t or τ of the parameterization (20) of the unperturbed separatrix. Such approach has recently been used for in the RPCTBP in [GMMS16] and it is likely to work in the RPETBP, allowing us to consider any $\mu \in (0, 1/2]$, that is, imposing no restrictions on the mass parameter, although this is not the purpose of this paper which focuses on the geometric mechanism that gives rise to diffusive orbits.

We are now in position to define two different scattering maps $\tilde{S}^{\pm}$. By Proposition 5, we begin by defining two different reduced Poincaré functions (36)

$$\begin{aligned}\mathcal{L}^*_{\pm}(\alpha, G; \epsilon_J) &= \mathcal{L}(\alpha, G, s^*_{\pm}; \epsilon_J)\\ &= \mathcal{L}_0(\alpha, G; \epsilon_J) \pm \mathcal{L}^*_1(\alpha, G; \epsilon_J) + \mathcal{E}_{\pm}(\alpha, G; \epsilon_J).\end{aligned}$$

By the symmetry properties of $\mathcal{L}_q(\alpha, G, s; \epsilon_J)$ (see (41)), it turns out that each $(\mathcal{L}^*_q)_{\pm}(\alpha, G; \epsilon_J) = \mathcal{L}(\alpha, G, s^*_{\pm}; \epsilon_J)$ is an even function of α. Moreover, since $s^*_- = s^*_+ + \pi$, one has that $(\mathcal{L}^*_q)_- = (-1)^q(\mathcal{L}^*_q)_+$, so we can write the reduced Poincaré map as

$$\mathcal{L}^*_{\pm} = \mathcal{L}_0 \pm \mathcal{L}^*_1 + \mathcal{L}^*_2 \pm \mathcal{L}^*_3 + \mathcal{L}^*_4 \pm \cdots \tag{65}$$

with $\mathcal{L}^*_q = (\mathcal{L}^*_q)_+$.

From the expression for the scattering map given in Proposition 5 we can define two different scattering maps $\widetilde{S}_{\pm}(\alpha, G, s) = (S_{\pm}(\alpha, G, s), s)$, where

$$S_{\pm}(\alpha, G, s) = \left(\alpha - \mu\frac{\partial \mathcal{L}^*_{\pm}}{\partial G}(\alpha, G; \epsilon_J) + O(\mu^2), G + \mu\frac{\partial \mathcal{L}^*_{\pm}}{\partial \alpha}(\alpha, G; \epsilon_J) + O(\mu^2)\right). \tag{66}$$

These two scattering maps are different since they depend on the two reduced Poincaré-Melnikov potentials $\mathcal{L}^*_{\pm}$. From their expression (66), the scattering maps $S_{\pm}$ follow closely the level curves of the Hamiltonians $\mathcal{L}^*_{\pm}$. More precisely, up to $O(\mu^2)$ terms, $S_{\pm}$ is given by the time $-\mu$ map of the Hamiltonian flow of Hamiltonian $\mathcal{L}^*_{\pm}$. The $O(\mu^2)$ remainder will be negligible as long as

$$|\mu| \ll \left|\frac{\partial \mathcal{L}^*_{\pm}}{\partial G}\right|, \left|\frac{\partial \mathcal{L}^*_{\pm}}{\partial \alpha}\right|,$$

which is already true for $\mu \ll \mu^*$ in (64).

We want to show now that the foliations of $\mathcal{L}^*_{\pm}$ = constant are different, since this will imply that the scattering maps $S_{\pm}$ are different. Even more, we will design a mechanism in which we will determine the places in the plane (α, G) where we will change from one scattering map to the other, obtaining trajectories with increasing angular momentum G. To check that the level curves of $\mathcal{L}^*_+$ and $\mathcal{L}^*_-$ are different, and indeed transversal, we only need to check that their Poisson bracket is not zero. Since $\mathcal{L}^*_+$ and $\mathcal{L}^*_-$ are even functions of α, their Poisson bracket $\{\mathcal{L}^*_+, \mathcal{L}^*_+\}$ will be an odd function of α, so we already know that it will have a factor $\sin\alpha$. Using Eq. (65) we can write

$$\{\mathcal{L}^*_+, \mathcal{L}^*_-\} = \{\mathcal{L}_0 + \mathcal{L}^*_1 + \mathcal{L}^*_2 + \cdots, \mathcal{L}_0 - \mathcal{L}^*_1 + \mathcal{L}^*_2 - \cdots\} = -2\{\mathcal{L}_0, \mathcal{L}^*_1\} + \mathcal{E}_3 \tag{67}$$

where $\mathcal{E}_3$ contains only Poisson brackets of odd order

$$\mathcal{E}_3 = -2\left(\{\mathcal{L}_0, \mathcal{L}^*_3\} + \{\mathcal{L}^*_1, \mathcal{L}^*_2\}\right) - 2\sum_{q\ \text{odd}\geq 5}\sum_{q=0}^{[q/2]}\{\mathcal{L}^*_{q'}, \mathcal{L}^*_{q-q'}\}.$$

Therefore, by formula (41) defining $\mathcal{L}_q$ and the bounds (39) for the harmonics $L_{q,k}$, the error term $\mathcal{E}_3 = O\left(\mathrm{e}^{-G^3}\right)$ is much exponentially smaller for large G than $\{\mathcal{L}_0, \mathcal{L}^*_1\}$, which is $O\left(\mathrm{e}^{-G^3/3}\right)$ and we now compute.

By differentiating $\mathcal{L}_0$, using (44) and bounds (45) and (46), one easily obtains:

$$\frac{\partial \mathcal{L}_0}{\partial \alpha} = \frac{15\pi\epsilon_J}{8G^5}\sin\alpha\left(1 + O(G^{-4}, \epsilon_J G^{-2}, \epsilon_J^2 G^{-1})\right)$$
$$\frac{\partial \mathcal{L}_0}{\partial G} = -\frac{3\pi}{2G^4}\left(1 + O(G^{-4}, \epsilon_J^2)\right) + \frac{75\pi\epsilon_J}{8G^6}\cos\alpha\left(1 + O(G^{-4}, \epsilon_J^2)\right).$$

With respect to $\mathcal{L}_1^* = 2L_{1,-1}B$, we will use (55) and the definitions of p, q in (52–53) which give

$$\mathcal{L}_1^* e^{-i\theta} = 2L_{1,-1} + 2L_{1,-2}e^{-i\alpha} + 2L_{1,-3}e^{-2i\alpha} + \mathcal{E}_1^*,$$

where the error term $\mathcal{E}_1^*$ contains a factor $\sin\alpha$ and satisfies the same bound as $\mathcal{E}_1$ in (50):

$$\mathcal{E}_1^* = \sum_{k\neq -1,-2,-3} L_{1,k}e^{i(k+1)\alpha} = O\left(\epsilon_J G^{-\frac{3}{2}}, \epsilon_J^3 G^{\frac{7}{2}}\right)e^{-G^3/3}.$$

Taking into account the expressions for $L_{1,-1}$, $L_{1,-2}$, $L_{1,-3}$ given in (47–49), after a straightforward computation, we arrive at

$$\frac{\partial \mathcal{L}_1^*}{\partial \alpha} = \frac{1}{B}\frac{\partial B}{\partial \alpha}\mathcal{L}_1^* = \frac{p\mathcal{L}_1^*\sin\alpha}{B^2}\left(1 + qp - 2q\cos\alpha + O\left(G^{-3}, \epsilon_J G^{-1}, \epsilon_J^2 G^2, (\epsilon_J G)^3 G\right)\right)$$
$$\frac{\partial \mathcal{L}_1^*}{\partial G} = -G^2(1 + O(G^{-1})\mathcal{L}_1^*.$$

Using these computations we arrive at

$$\{\mathcal{L}_0, \mathcal{L}_1^*\} = -\frac{15\pi\epsilon_J\mathcal{L}_1^* d\sin\alpha}{8G^3B^2} \tag{68}$$

with

$$d := B^2\left(1 + O\left(G^{-1}\right)\right) - \frac{4p}{5\epsilon_J G}\left(1 + qp - 2q\cos\alpha + O\left(G^{-3}, \epsilon_J G^{-1}, \epsilon_J^2 G^2, \epsilon_J^3 G^4\right)\right)$$
$$\times\left(1 - \frac{25\epsilon_J}{4G^2}\cos\alpha\left(1 + O\left(G^{-4}, \epsilon_J^2\right)\right) + O\left(G^{-4}, \epsilon_J^2\right)\right). \tag{69}$$

5.1. Strategy for diffusion. The previous computations (67), (68) as well as Lemma 10 tell us that the level curves of $\mathcal{L}_+^*$ and $\mathcal{L}_-^*$ are transversal in the region $G \geq C > 32$ and $\epsilon_J G \leq c < 1/8$, except for the three curves $\alpha = 0$, $\alpha = \pi$ and $d = 0$, which are transversal to any of these level curves of $\mathcal{L}_+^*$ and $\mathcal{L}_-^*$, see Fig. 2. Indeed, this is clear for the lines $\alpha = 0$ and $\alpha = \pi$, and the same happens for the curve $d = 0$ using the expression of d given in (69) which implies

$$G = \left(\frac{2}{11\epsilon_J^2}\right)^{1/3}\left(1 + K\epsilon_J^{1/3}\cos\alpha + O\left(\epsilon_J^{2/3}\right)\right)$$

with $K \neq 0$.

Thus, apart from these three curves, at any point in the plane (α, G) the slopes $dG/d\alpha$ of the level curves of $\mathcal{L}_+^*$ and $\mathcal{L}_-^*$ are different, and we are able to choose which level curve increases more the value of G, when both slopes are positive, or alternatively, to choose

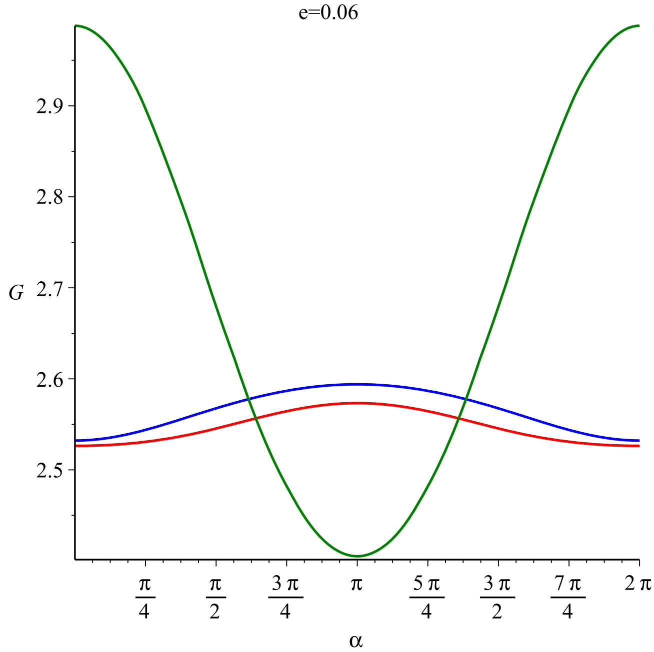

Fig. 2. Illustration of the level Sets of $\mathcal{L}_+^*$ ($\mathcal{L}_-^*$) in Blue (Red) and $d = 0$ in Green (color figure online)

the level curve which decreases less the value of G, when both slopes are negative (see Fig. 3). In the same way, we can find trajectories along which the angular momentum performs arbitrary excursions. More precisely, given an arbitrary finite sequence of values $G_i, i = 1, \ldots, n$ we can find trajectories which satisfy $G(T_i) = G_i, i = 1, \ldots, n$.

Strictly speaking, this mechanism given by the application of scattering maps produce indeed pseudo-orbits, that is, heteroclinic connections between different periodic orbits in the infinity manifold which are commonly known as transition chains after Arnold's pioneering work [Arn64]. The existence of true orbits of the system which follow closely these transition chains relies on shadowing methods, which are standard for partially hyperbolic periodic orbits (the so-called whiskered tori in the literature) lying on a normally hyperbolic invariant manifold (NHIM) [Moe02,Moe07,GL06,GLMS14]. Such shadowing methods are equally applicable in our case as it is proven in [GMMSS17], where we have an infinity manifold $\tilde{\Lambda}_\infty$ which is only topologically equivalent to a NHIM.

With all these elements, we can finally state our main result

Theorem 13. *Let $G_1^* < G_2^*$ large enough and $\epsilon_J > 0$, $\mu > 0$ small enough. More precisely $C \leq G_1^* < G_2^* \leq c/\epsilon_J$ and $0 < \mu < \mu^* = \frac{c}{C} e^{-(8\epsilon_J)^{-3}/3}$, for $C > 32$ large enough and $c < 1/8$ small enough. Then, for any finite sequence of values $G_i \in (G_1^*, G_2^*)$, $i = 1, \ldots, n$, there exists a trajectory of the RPETBP such that $G(T_i) = G_i$, $i = 1, \ldots, n$ for some $0 < T_i < T_{i+1}$. In particular, for any two values $G_1 < G_2 \in (G_1^*, G_2^*)$, there exists a trajectory such that $G(0) < G_1$, and $G(T) > G_2$ for some time $T > 0$.*

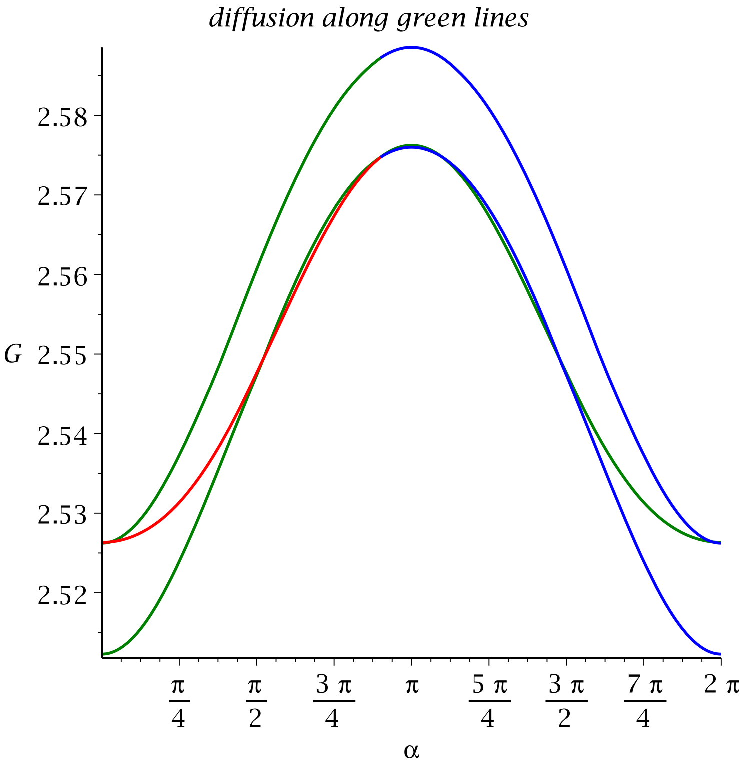

Fig. 3. Zone of diffusion: Level curves of $\mathcal{L}_+^*$ ($\mathcal{L}_-^*$) in blue (red) and diffusion trajectories in green (color figure online)

6. Computation of the Melnikov Potential: Proof of Theorem 8

The main difficulty to compute the Melnikov potential is that it is given by an integral (37) where the coordinates of the separatrix x_h and α_h are given implicitly (20) in terms of the time t through the variable τ (21), whereas r and f are given in terms of $s + t$ through the differential equation (6) defining the true anomaly f. To evaluate the above Melnikov potential, we will compute its Fourier Cosine series (38) in the angles s, α. We will detect that there are only five dominant harmonics, $L_{0,0}$, $L_{0,1}$, $L_{1,-1}$, $L_{1,-2}$, and $L_{1,-3}$, so we will estimate them and bound all the rest.

The plan of this proof is thus divided in different parts. In Sect. 6.1 we Fourier expand the Melnikov potential $\mathcal{L}$ to find that each of its harmonics $L_{q,k}$ is given by a series in terms of some constants $c_q^{n,m}$ and integrals $N(q, m, n)$. General upper bounds for these constants and integrals are given in Sect. 6.2, which provide the upper bounds $B_{q,k}$ for the harmonics $L_{q,k}$ announced in Proposition 7. Since the upper bounds $B_{q,k}$ are exponentially small for large G and $q \geq 1$, we split the Fourier expansion (38) as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{\geq 2}$$

where $\mathcal{L}_0$, $\mathcal{L}_1$ contain the harmonics of $\mathcal{L}$ of order 0, 1 in s, respectively, whereas $\mathcal{L}_{\geq 2}$ contain the harmonics of higher order, which are readily bounded. Section 6.3 contains an asymptotic expression for the integrals $N(q, m, n)$ which are necessary for the computation of $\mathcal{L}_1$. Finally, the Sects. 6.4 and 6.5 are devoted to the computation to the harmonics of $\mathcal{L}_1$, and $\mathcal{L}_0$, respectively, estimating, for each order, the two most dominant ones, and bounding all the rest.

6.1. Fourier expansion of the Melnikov potential. The next Proposition gives formulae for the Fourier coefficients (38) of the Melnikov potential (37). For any integer n, m, we will use the Fourier expansion of the function

$$r(f(t))^n \,\mathrm{e}^{imf(t)} = \sum_{q\in\mathbb{Z}} c_q^{n,m} \mathrm{e}^{iqt} \tag{70}$$

which can be found in [MP94] and [Win41, p. 204]. Since r is an even function and f is and odd function, one readily sees that the above coefficients are real and indeed they satisfy

$$c_{-q}^{n,-m} = c_q^{n,m} = \overline{c_q^{n,m}}.$$

Once these coefficients $c_q^{n,m}$ are introduced we can give explicit formulae for the Fourier coefficients of the Melnikov potential $\mathcal{L}$.

Proposition 14. *The Melnikov potential given in* (37) *or in* (38) *can be written as*

$$\mathcal{L} = \sum_{q\in\mathbb{Z}} L_q e^{iqs}, \quad \textit{where} \quad L_q = \sum_{k\in\mathbb{Z}} L_{q,k} e^{ik\alpha}, \tag{71}$$

with

$$\begin{aligned}
L_{q,0} &= \sum_{l\geq 1} c_q^{2l,0} N(q,l,l) \\
L_{q,1} &= \sum_{l\geq 2} c_q^{2l-1,-1} N(q,l-1,l) \\
L_{q,-1} &= \sum_{l\geq 2} c_q^{2l-1,1} N(q,l,l-1) \\
L_{q,k} &= \sum_{l\geq k} c_q^{2l-k,-k} N(q,l-k,l) \quad \textit{for } k\geq 2 \\
L_{q,-k} &= \sum_{l\geq k} c_q^{2l-k,k} N(q,l,l-k) \quad\quad \textit{for } k\geq 2
\end{aligned} \tag{72}$$

and

$$N(q,m,n) = \frac{2^{m+n}}{G^{2m+2n-1}} \binom{-1/2}{m}\binom{-1/2}{n} \int_{-\infty}^{\infty} \frac{e^{iq(\tau+\tau^3/3)\,G^3/2}}{(\tau-i)^{2m}(\tau+i)^{2n}} d\tau. \tag{73}$$

Proof. We write Melnikov potential (37) as:

$$\mathcal{L} = \mathcal{L}_{\mathrm{main}} + \int_{-\infty}^{\infty} \left[\left(\frac{x_{\mathrm{h}}^2}{2}\right)^2 r\cos(\alpha_{\mathrm{h}} - f) - \frac{x_{\mathrm{h}}^2}{2} \right] dt, \tag{74}$$

where

$$\mathcal{L}_{\mathrm{main}} = \int_{-\infty}^{\infty} \frac{x_{\mathrm{h}}^2}{\left[4 + x_{\mathrm{h}}^4 r^2 + 4x_{\mathrm{h}}^2 r\cos(\alpha_{\mathrm{h}} - f)\right]^{1/2}} dt$$

can be written as

$$\mathcal{L}_{\text{main}} = \int_{-\infty}^{\infty} \frac{x_{\text{h}}^2}{2} \left(1 + \frac{x_{\text{h}}^2}{2} r\left(f(t+s)\right) \mathrm{e}^{i(\alpha_{\text{h}} - f(t+s))}\right)^{-1/2} \cdot \left(1 + \frac{x_{\text{h}}^2}{2} r\left(f(t+s)\right) \mathrm{e}^{-i(\alpha_{\text{h}} - f(t+s))}\right)^{-1/2} dt.$$

Using the expansion for $z = \frac{x_{\text{h}}^2}{2} r\left(f(t+s)\right) \mathrm{e}^{\pm i(\alpha_{\text{h}} - f(t+s))}$

$$(1+z)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} \binom{-1/2}{l} z^l$$

which, by (20a), (5), is valid as long as $|z| = |x_{\text{h}}^2 r/2| \le 2(1+\epsilon_{\text{J}})/G^2 < 1$, we get

$$\mathcal{L}_{\text{main}} = \sum_{k \ge 0} \sum_{l \ge k} \tilde{L}_k^l + \sum_{k<0} \sum_{l \le k} \tilde{S}_k^l$$

where

$$\tilde{L}_k^l = \frac{1}{2^{2l-k+1}} \binom{-1/2}{l-k} \binom{-1/2}{l} \int_{-\infty}^{\infty} x_{\text{h}}^{4l-2k+2} \left[r(f(t+s))\right]^{2l-k} \mathrm{e}^{ik\alpha_{\text{h}}} \mathrm{e}^{-ikf(t+s)}\, dt; \quad 0 \le k \le l$$

$$\tilde{S}_k^l = \frac{1}{2^{-2l+k+1}} \binom{-1/2}{k-l} \binom{-1/2}{-l} \int_{-\infty}^{\infty} x_{\text{h}}^{-4l+2k+2} \left[r(f(t+s))\right]^{-2l+k} \mathrm{e}^{ik\alpha_{\text{h}}} \mathrm{e}^{-ikf(t+s)} dt; \quad l \le k \le -1.$$

With these expressions is easy to see that $\tilde{L}_0^0$ cancels out the last term in the integral (74) and that $\tilde{L}_1^1 + \tilde{S}_{-1}^{-1}$ cancels the cosine term, and so

$$\mathcal{L} = \sum_{l \ge 1} \tilde{L}_0^l + \sum_{l \ge 2} \tilde{L}_1^l + \sum_{l \le -2} \tilde{S}_{-1}^l + \sum_{k \ge 2} \sum_{l \ge k} \tilde{L}_k^l + \sum_{k \le -2} \sum_{l \le k} \tilde{S}_k^l. \tag{75}$$

Now we perform the change of variable

$$t = \frac{G^3}{2}\left(\tau + \frac{\tau^3}{3}\right), \qquad dt = \frac{G^3}{2}(1+\tau^2)\, d\tau$$

introduced in (21), and we use the formulae for x_{h} and α_{h} given in (20a) and (20b). In particular we will use that

$$x_{\text{h}}^2 = \frac{4}{G^2(1+\tau^2)}, \qquad x_{\text{h}}^2\, dt = 2G\, d\tau, \qquad \mathrm{e}^{i\alpha_{\text{h}}} = \frac{\tau - i}{\tau + i} \mathrm{e}^{i\alpha},$$

and the expansion in Fourier series given in (70) to obtain

$$\tilde{L}_k^l = \mathrm{e}^{ik\alpha} \frac{2^{2l-k}}{G^{4l-2k-1}} \binom{-1/2}{l} \binom{-1/2}{l-k} \sum_{q \in \mathbb{Z}} \mathrm{e}^{iqs} c_q^{2l-k,-k} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{iq(\tau+\tau^3/3)\, G^3/2}}{(\tau-i)^{2(l-k)}(\tau+i)^{2l}} d\tau$$

$$= \mathrm{e}^{ik\alpha}\sum_{q\in\mathbb{Z}}\mathrm{e}^{iq\,s}c_q^{2l-k,-k}N(q,l-k,l),\qquad 0\le k\le l; \tag{76}$$

$$\tilde{S}_k^l = \mathrm{e}^{ik\alpha}\frac{2^{-2l+k}}{G^{-4l+2k-1}}\binom{-1/2}{-l}\binom{-1/2}{k-l}\sum_{q\in\mathbb{Z}}\mathrm{e}^{iq\,s}c_q^{-2l+k,-k}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{iq(\tau+\tau^3/3)\,G^3/2}}{(\tau-i)^{-2l}(\tau+i)^{2(k-l)}}d\tau$$

$$= \mathrm{e}^{ik\alpha}\sum_{q\in\mathbb{Z}}\mathrm{e}^{iq\,s}c_q^{-2l+k,-k}N(q,-l,k-l),\qquad l\le k\le -1. \tag{77}$$

Substituting now Eqs. (76) and (77) into the expansion (75) we get

$$\begin{aligned}\mathcal{L} = &\sum_{q\in\mathbb{Z}}\mathrm{e}^{iqs}\sum_{l\ge1}c_q^{2l,0}N(q,l,l)+\sum_{q\in\mathbb{Z}}\mathrm{e}^{i(qs+\alpha)}\sum_{l\ge2}c_q^{2l-1,-1}N(q,l-1,l)\\ &+\sum_{q\in\mathbb{Z}}\mathrm{e}^{i(qs-\alpha)}\sum_{l\le-2}c_q^{-2l-1,1}N(q,-l,-l-1)\\ &+\sum_{q\in\mathbb{Z}}\sum_{k\ge2}\mathrm{e}^{i(qs+k\alpha)}\sum_{l\ge k}c_q^{2l-k,-k}N(q,l-k,l)\\ &+\sum_{q\in\mathbb{Z}}\sum_{k\le-2}\mathrm{e}^{i(qs+k\alpha)}\sum_{l\le k}c_q^{-2l+k,-k}N(q,-l,k-l).\end{aligned}$$

Changing now the indexes $l\to -l$ and $k\to -k$ in the third and fifth terms we obtain the desired formulae (72) for the Fourier coefficients $L_{q,k}$. □

6.2. General upper bounds. In view of Proposition 14 and formulae (72), to compute the dominant part of the Melnikov potential and obtain effective bounds of the errors we will need to estimate the constants $c_q^{n,m}$ defined in (70) and the integrals $N(q,m,n)$ defined in (73) for $q\ge 0$ and only for indexes m,n satisfying $n\ge 0$, $m\le n+1$. Alternatively to (5), it will be very convenient to express the distance r between the primaries as

$$r = 1-\epsilon_{\mathrm{J}}\cos E \tag{78}$$

in terms of the *eccentric anomaly* E, given by the Kepler equation [Win41, p. 194]

$$t = E-\epsilon_{\mathrm{J}}\sin E. \tag{79}$$

We obtain a general upper bound for the constants $c_q^{n,m}$, where the dominant order in ϵ_{J} appears explicitly.

Proposition 15. *Let $n,m,q\in\mathbb{Z}$, $n,q\ge 0$, $m\le n+1$. Then the Fourier coefficients $c_q^{n,m}$ defined in (70) satisfy*

$$\left|c_q^{n,m}\right| \le \begin{cases} 2^{q+n+1}e^{q\sqrt{1-\epsilon_J^2}}\epsilon_J^{|m-q|} & m\ge 0\\ 2^{n+1}e^{q\sqrt{1-\epsilon_J^2}}\dfrac{\epsilon_J^{q-m}}{(1-\epsilon_J^2)^{-m/2}} & m\le -1.\end{cases} \tag{80}$$

Proof. In the integral formula for the Fourier coefficients

$$c_q^{n,m} = \frac{1}{2\pi}\int_0^{2\pi} r^n \mathrm{e}^{imf}\mathrm{e}^{-iqt}dt \tag{81}$$

we change the variable of integration to the eccentric anomaly (79) ($dt = r\,dE$) to get

$$c_q^{n,m} = \frac{1}{2\pi}\int_0^{2\pi} \left(r\mathrm{e}^{if}\right)^m r^{n+1-m}\mathrm{e}^{-iqt}dE. \tag{82}$$

To compute $c_q^{n,m}$ from (82) we will use the identity (see [Win41, p. 202])

$$\left(r\mathrm{e}^{if}\right)^{\frac{1}{2}} = a\mathrm{e}^{iE/2} - \frac{\epsilon_\mathrm{J}}{2a}\mathrm{e}^{-iE/2}, \qquad a = \frac{\sqrt{1+\epsilon_\mathrm{J}}+\sqrt{1-\epsilon_\mathrm{J}}}{2}$$

which readily implies

$$r\mathrm{e}^{if} = a^2\mathrm{e}^{iE} - \epsilon_\mathrm{J} + \frac{\epsilon_\mathrm{J}^2}{4a^2}\mathrm{e}^{-iE} = (a\mathrm{e}^{iE/2} - \frac{\epsilon_\mathrm{J}}{2a}\mathrm{e}^{-iE/2})^2, \tag{83a}$$

$$a^2 + \frac{\epsilon_\mathrm{J}^2}{4a^2} = 1, \quad a^2 - \frac{\epsilon_\mathrm{J}^2}{4a^2} = \sqrt{1-\epsilon_\mathrm{J}^2}, \quad a^4 + \frac{\epsilon_\mathrm{J}^2}{16a^4} = 1-\epsilon_\mathrm{J}^2,$$

$$a^4 - \frac{\epsilon_\mathrm{J}^2}{16a^4} = \sqrt{1-\epsilon_\mathrm{J}^2}. \tag{83b}$$

To bound the integral (81) for $m \geq 0$ we will consider two different cases: $0 \leq q \leq m$ and $0 \leq m < q$. Let us first consider the case $0 \leq q \leq m$. By the analyticity and periodicity of the integral we change the path of integration from $\Im(E) = 0$ to $\Im E = \ln(2a^2/\epsilon_\mathrm{J})$

$$E = u + i\ln\left(\frac{2a^2}{\epsilon_\mathrm{J}}\right) \qquad u \in [0, 2\pi] \tag{84}$$

so that

$$\mathrm{e}^{iE} = \mathrm{e}^{iu - \ln(2a^2/\epsilon_\mathrm{J})} = \frac{\epsilon_\mathrm{J}}{2a^2}\mathrm{e}^{iu}$$

and then, by (78), (79) and (83a), (83b),

$$r\mathrm{e}^{if} = \frac{\epsilon_\mathrm{J}}{2}\mathrm{e}^{iu} - \epsilon_\mathrm{J} + \frac{\epsilon_\mathrm{J}}{2}\mathrm{e}^{-iu} = \epsilon_\mathrm{J}(\cos u - 1)$$

$$r = 1 - \frac{\epsilon_\mathrm{J}}{2}\left(\frac{\epsilon_\mathrm{J}}{2a^2}\mathrm{e}^{iu} + \frac{2a^2}{\epsilon_\mathrm{J}}\mathrm{e}^{-iu}\right) = 1 - \frac{\epsilon_\mathrm{J}^2}{4a^2}\mathrm{e}^{iu} - a^2\mathrm{e}^{-iu}$$

$$= 1 - \left(\frac{\epsilon_\mathrm{J}^2}{4a^2} + a^2\right)\cos u + i\left(a^2 - \frac{\epsilon_\mathrm{J}^2}{4a^2}\right)\sin u = 1 - \cos u + i\sqrt{1-\epsilon_\mathrm{J}^2}\sin u$$

$$\mathrm{e}^{-it} = \frac{2a^2\mathrm{e}^{-iu}}{\epsilon_\mathrm{J}}\exp\left(\frac{\epsilon_\mathrm{J}^2}{4a^2}\mathrm{e}^{iu} - a^2\mathrm{e}^{-iu}\right) = \frac{2a^2\mathrm{e}^{-iu}}{\epsilon_\mathrm{J}}\exp\left(-\sqrt{1-\epsilon_\mathrm{J}^2}\cos u + i\sin u\right).$$

Therefore, along the complex path (84) we have the following bounds

$$\left|r\mathrm{e}^{if}\right| = \epsilon_\mathrm{J}(1-\cos u) \leq 2\epsilon_\mathrm{J}$$

$$|r| = \sqrt{(1-\cos u)^2 + \left(1-\epsilon_J^2\right)\sin^2 u} = \sqrt{2\,(1-\cos u) - \epsilon_J^2 \sin^2 u} \le 2$$
$$\left|e^{-it}\right| = \frac{2a^2}{\epsilon_J}\exp\left(-\sqrt{1-\epsilon_J^2}\cos u\right) \le \frac{2a^2}{\epsilon_J}e^{\sqrt{1-\epsilon_J^2}}.$$

Since $2a^2 \le 2$, substituting these bounds in (82) we find directly the desired result (80) for $0 \le q \le m$.

For the the case $m < q$ we now perform the change of the integration variable through

$$E = v - i\ln\left(\frac{2a^2}{\epsilon_J}\right), \qquad v \in [0, 2\pi] \tag{85}$$

so that

$$e^{iE} = e^{iv + \ln(2a^2/\epsilon_J)} = \frac{2a^2}{\epsilon_J}e^{iv}$$

and then again, by (78), (79) and (83a), (83b),

$$\begin{aligned} re^{if} &= \frac{2a^4}{\epsilon_J}e^{iv} - \epsilon_J + \frac{\epsilon_J^3}{8a^4}e^{-iv} \\ &= \frac{2}{\epsilon_J}\left(\left(a^4 + \frac{\epsilon_J^4}{16a^4}\right)\cos v - \frac{\epsilon_J^2}{2} + i\left(a^4 - \frac{\epsilon_J^4}{16a^4}\right)\sin v\right) \\ &= \frac{2}{\epsilon_J}\left(\cos v - \frac{\epsilon_J^2}{2}(1+\cos v) + i\sqrt{1-\epsilon_J^2}\sin v\right) \\ r &= 1 - a^2 e^{iv} - \frac{\epsilon_J^2}{4a^2}e^{-iv} = 1 - \cos v - i\sqrt{1-\epsilon_J^2}\sin v \\ e^{-it} &= \frac{\epsilon_J e^{-iv}}{2a^2}\exp\left(a^2 e^{iv} - \frac{\epsilon_J^2}{4a^2}e^{-iv}\right) = \frac{\epsilon_J e^{-iv}}{2a^2}\exp\left(\sqrt{1-\epsilon_J^2}\cos v + i\sin v\right). \end{aligned}$$

Therefore

$$|re^{if}|^2 = \frac{2}{\epsilon_J}\left(1 - \frac{\epsilon_J^2(\cos v + 1)}{2}\right)$$

and consequently, using that $2a^2 \ge 1$, along the complex path (85) we have the following bounds

$$\begin{aligned} \frac{2}{\epsilon_J}(1-\epsilon_J^2)^{1/2} \le \left|re^{if}\right| \le \frac{2}{\epsilon_J}(1+\epsilon_J^2)^{1/2} \le \frac{4}{\epsilon_J}, \quad |r| \le 2, \\ \left|e^{-it}\right| \le \frac{\epsilon_J}{2a^2}e^{\sqrt{1-\epsilon_J^2}} \le \epsilon_J e^{\sqrt{1-\epsilon_J^2}}. \end{aligned} \tag{86}$$

Substituting the above upper bounds (86) in (82) we find the desired result (80) for $0 \le m < q$. In the case $m \le -1$ we use the above lower bounds for $|re^{if}|$ to get (80). □

As we can see from Eq. (72) the Fourier coefficients of the Melnikov potential $\mathcal{L}$ depend also on the function $N(q,m,n)$ defined in (73), so to bound (or to compute) these Fourier coefficients we need to bound (or to compute) $N(q,m,n)$.

Introducing the integral

$$I(q,m,n)=\int_{-\infty}^{\infty}\frac{\mathrm{e}^{iqG^3(\tau+\tau^3/3)/2}}{(\tau-i)^{2m}(\tau+i)^{2n}}d\tau$$

$N(q,m,n)$ can be written as

$$N(q,m,n)=\frac{2^{m+n}}{G^{2m+2n-1}}\binom{-1/2}{m}\binom{-1/2}{n}I(q,m,n).$$

We will denote

$$h(\tau)=i\left(\frac{\tau^3}{3}+\tau\right) \tag{87}$$

the variable term in the exponencial of the integral, so that

$$I(q,m,n)=\int_{-\infty}^{\infty}\frac{\mathrm{e}^{qG^3h(\tau)/2}}{(\tau-i)^{2m}(\tau+i)^{2n}}d\tau. \tag{88}$$

Since the integral $I(q,m,n)$ involves an exponential function with a large parameter G^3 in front of the exponent, we will apply the method of steepest descent [Erd56, §2.5–6]. In particular on a complex path with $\Im(h(\tau))=0$. So, let us define the path (see Fig. 4):

$$\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4\cup\Gamma_5 \tag{89}$$

where $0<\varepsilon<1$, τ^* is a point such that $\Im(h(\tau^*))=0$ that will be fixed in Lemma 23 as $|\tau^*-i|=1/2$, and

$$\begin{aligned}
\Gamma_1&=\{\tau\in\mathbb{C}:\Im(h(\tau))=0\}\cap\{\tau\in\mathbb{C}:\Re(\tau)\le\Re(-\bar\tau^*)\}\\
\Gamma_5&=\{\tau\in\mathbb{C}:\Im(h(\tau))=0\}\cap\{\tau\in\mathbb{C}:\Re(\tau)\ge\Re(\tau^*)\}\\
\Gamma_2&=\{\tau\in\mathbb{C}:\Im(h(\tau))=0\}\cap\{\tau\in\mathbb{C}:\Re(-\bar\tau^*)\le\Re(\tau)\le 0\}\cap\{\tau\in\mathbb{C}:|\tau-i|\ge c\,\varepsilon\}\\
\Gamma_4&=\{\tau\in\mathbb{C}:\Im(h(\tau))=0\}\cap\{\tau\in\mathbb{C}:0\le\Re(\tau)\le\Re(\tau^*)\}\cap\{\tau\in\mathbb{C}:|\tau-i|\ge c\,\varepsilon\}\\
\Gamma_3&=\{\tau\in\mathbb{C}:\Im(h(\tau))\le 0\}\cap\{\tau\in\mathbb{C}:|\tau-i|=c\,\varepsilon\}.
\end{aligned} \tag{90}$$

By the Cauchy-Goursat Theorem plus a limit argument, the integral $I(q,m,n)$, defined in (88) over the real axis, is equal to the one taken over the path Γ thinking of τ as a complex number (see [LS80]). In fact, by the same argument, its value depends neither on ε nor on τ^*.

The positive branch of the hyperbola defined by $\Im(h(\tau))=0$ intersects the circumference of radius ε in two points C_ε and $-\overline{C_\varepsilon}$ given by

$$C_\varepsilon=\Gamma_3\cap\Gamma_4 \qquad -\overline{C}_\varepsilon=\Gamma_3\cap\Gamma_2. \tag{91}$$

Since the integral over Γ does not depend on ε, we will choose a particular value of ε to bound $I(q,m,n)$ and consequently $N(q,m,n)$ defined in (73). Later on, in Proposition 22, we will just compute the ε-independent terms of this integral.

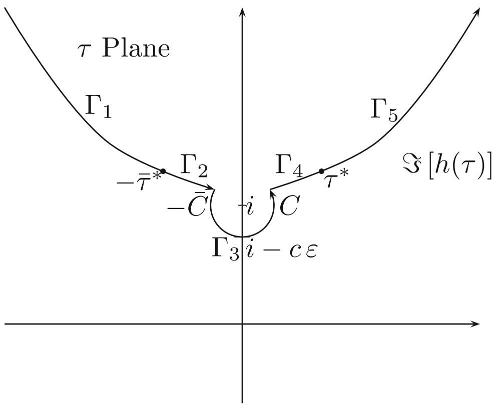

Fig. 4. The complex path Γ

It is not difficult to see that, if we define the function

$$u(\tau) = h(i) - h(\tau) = -\frac{2}{3} - i\left(\frac{\tau^3}{3} + \tau\right) = (\tau - i)^2 - \frac{i}{3}(\tau - i)^3, \tag{92}$$

then

$$u(\Gamma_1 \cup \Gamma_2),\ u(\Gamma_4 \cup \Gamma_5) \subset \mathbb{R}_0^+.$$

Moreover, if $\tau^- \in \Gamma_1 \cup \Gamma_2$ then $\tau^+ = -\bar{\tau}^- \in \Gamma_4 \cup \Gamma_5$ and

$$u(\tau^-) = u(\tau^+).$$

On the other hand, one can see that u is an increasing function while moving along $\Gamma_1 \cup \Gamma_2$ or $\Gamma_4 \cup \Gamma_5$ in the direction of increasing imaginary part. Therefore, u has two inverses in $\mathbb{R}_0^+$: τ^+ and τ_-. Before writing them down let us notice that the point C_ε defined in (91) can be written as

$$C_\varepsilon = i + \varepsilon\, e^{i\theta_\varepsilon} \quad \text{with} \quad \theta_\varepsilon \in (0, \pi/2) \tag{93}$$

and has the following expression in the coordinates u defined in (92)

$$u(C_\varepsilon) = |u(C_\varepsilon)| = \varepsilon^2 \left|1 - \frac{\varepsilon}{3} i e^{i\theta_\varepsilon}\right| = \varepsilon^2\ k_\varepsilon \tag{94}$$

with

$$k_\varepsilon = \left|1 - \frac{\varepsilon}{3} i e^{i\theta_\varepsilon}\right| = \sqrt{\left(1 + \frac{\varepsilon}{3}\sin\theta_\varepsilon\right)^2 + \left(\frac{\varepsilon}{3}\cos\theta_\varepsilon\right)^2} \geq 1,$$

since by construction, $\theta_\varepsilon \in (0, \pi/2)$ and then $0 < \sin\theta_\varepsilon$.

Now, we can write the inverses of the function u

$$\begin{aligned} \tau^+ : [u(C_\varepsilon), +\infty) &\longrightarrow \Gamma_4 \cup \Gamma_5 \\ u &\longmapsto \xi(u) + i\eta(u), \end{aligned} \qquad \begin{aligned} \tau^- : [u(C_\varepsilon), +\infty) &\longrightarrow \Gamma_1 \cup \Gamma_2 \\ u &\longmapsto -\xi(u) + i\eta(u). \end{aligned}$$

The change (92) is useful over $\Gamma_1 \cup \Gamma_2$ and $\Gamma_4 \cup \Gamma_5$, thus performing this change in (73), we have that for any $\varepsilon > 0$

$$N(q,m,n) = \frac{d_{m,n} e^{-q\frac{G^3}{3}}}{G^{2m+2n-1}} \left[\int_{u(C_\varepsilon)}^{\infty} [F^+_{m,n}(u) - F^-_{m,n}(u)] e^{-qG^3u/2} du \right. \\ \left. +(-i) e^{q\frac{G^3}{3}} \int_{\Gamma_3} f^q_{m,n}(\tau) d\tau \right] \tag{95}$$

where

$$d_{m,n} = i\, 2^{m+n} \binom{-1/2}{n} \binom{-1/2}{m} \tag{96}$$

$$F^{\pm}_{m,n}(u) = \frac{1}{(\tau^{\pm}(u) - i)^{2m+1} (\tau^{\pm}(u) + i)^{2n+1}} \tag{97}$$

$$f^q_{m,n}(\tau) = \frac{e^{q\frac{G^3}{2} h(\tau)}}{(\tau - i)^{2m} (\tau + i)^{2n}}, \tag{98}$$

and $h(\tau)$ is given in (87). To give a bound for $N(q,m,n)$ given by (95), some estimates for $d_{m,n}$ and $F_{m,n}$ are needed. We begin with the constants $d_{m,n}$.

Lemma 16. *Let $m, n \in \mathbb{Z}$, $m, n \geq 0$ and $d_{m,n}$ be defined by Eq. (96). Then*

$$|d_{m,n}| \leq e^{-1/2} 2^{m+n} \qquad \text{if } m + n > 0\,.$$

Proof. Let $s \in \mathbb{N}$, then

$$\left| \binom{-1/2}{s} \right| = \left| \frac{(-1)^s}{s!} \Big(\frac{1}{2}\Big)\Big(\frac{1}{2}+1\Big) \cdots \Big(\frac{1}{2}+s-1\Big) \right| = \frac{1}{2^s} \Big[1 \cdot \frac{3}{2} \cdots \frac{2s-1}{s}] \Big] \\ \leq \frac{1}{2^s} \Big(2 - \frac{1}{s}\Big)^s = \Big(1 - \frac{1}{2s}\Big)^s \leq \lim_{s\to\infty} \Big(1 - \frac{1}{2s}\Big)^s = e^{-1/2}.$$

□

The next Lemma gives information about the functions $F^{\pm}_{m,n}(u)$.

Lemma 17. *The function $F^{\pm}_{m,n}(u)$ defined in (97) has the expansion*

$$F^{\pm}_{m,n}(u) = (\pm\sqrt{u})^{-2m-1} \sum_{j=0}^{\infty} d^{m,n}_j (\pm\sqrt{u})^j \tag{99}$$

where the coefficients $d^{m,n}_j$ satisfy

$$d^{m,n}_0 = 1/(2i)^{2n+1}, \quad |d^{m,n}_j| \leq \left(\frac{4}{3}\right)^m \left(\frac{3}{2}\right)^{\frac{j+3}{2}}. \tag{100}$$

Consequently, the series (99) is convergent for $|\sqrt{u}| < \sqrt{2/3}$.

Proof. Let us introduce the function

$$T^{\pm}_{m,n}(x) := (\pm)x^{2m+1}F^{\pm}_{m,n}(x^2) = \sum_{j=0}^{\infty} d_j^{m,n}(\pm x)^j,$$

which is well defined since $u = x^2$ is a good change of variables in $\mathbb{R}^+$ and has the two inverses $x = \pm\sqrt{u}$. To bound the coefficients $d_j^{m,n}$ we use Cauchy formula:

$$(\pm 1)^j d_j^{m,n} = \frac{1}{2\pi i}\int_{|x|=\varepsilon} \frac{T^{\pm}_{m,n}(x)}{x^{j+1}}dx = \frac{-1}{2\pi i}\int_{|x|=\varepsilon} \frac{F^{\pm}_{m,n}(x^2)}{x^{j-2m}}dx.$$

Applying the change of variables

$$x = \pm\sqrt{(\tau-i)^2 - \frac{i}{3}(\tau-i)^3} = \pm(\tau-i)\sqrt{(1-\frac{i}{3}(\tau-i))} = \pm\frac{\tau-i}{\sqrt{3}}(\sqrt{2-i\tau}), \tag{101}$$

we obtain

$$\begin{aligned}(\pm 1)^j d_j^{m,n} &= \mp\frac{1}{2\pi i}\int_{|\tau-i|=\rho} \frac{(\tau-i)^{2m-j}}{3^{\frac{2m-j}{2}}}(2-i\tau)^{\frac{2m-j}{2}}\frac{1}{(\tau-i)^{2m+1}(\tau+i)^{2n+1}} \\ &\quad \frac{3(1-i\tau)}{2\sqrt{3}(2-i\tau)^{\frac{1}{2}}}d\tau \\ &= \mp\frac{1}{2}\frac{i^{\frac{j+1}{2}-m}}{2\pi 3^{m-\frac{j+1}{2}}}\int_{|\tau-i|=\rho}\frac{d\tau}{(\tau-i)^{j+1}(\tau+i)^{2n}(\tau+2i)^{\frac{j+1-2m}{2}}}.\end{aligned}$$

Now, taking $\rho = 1$ and using that $|\tau+i| \geq 1$ and that $2 \leq |\tau+2i| \leq 4$ we have

$$|d_j^{m,n}| \leq \left(\frac{4}{3}\right)^m\left(\frac{3}{2}\right)^{\frac{j+3}{2}},$$

which is the desired bound. From this bound it is clear that the series defining $T^{\pm}_{m,n}(x)$ is convergent for $|x| < \sqrt{2/3}$ and therefore the one for $F^{\pm}_{m,n}(u)$ is convergent for $\sqrt{u} < \sqrt{2/3}$. □

From Eq. (99) we have

$$F^{\pm}_{m,n}(u) = (\pm\sqrt{u})^{-2m-1}\sum_{j=0}^{2m} d_j^{m,n}(\pm\sqrt{u})^j + g^{\pm}_{m,n}(\pm\sqrt{u}), \tag{102}$$

where the regular part of the function $F^{\pm}_{m,n}(u)$ is given by

$$g^{\pm}_{m,n}(\pm\sqrt{u}) = (\pm\sqrt{u})^{-2m-1}\sum_{j=2m+1}^{\infty} d_j^{m,n}(\pm\sqrt{u})^j \tag{103}$$

and $d_j^{m,n}$ are defined by Eq. (99) and satisfy bounds (100). The next Lemma bounds $g^{\pm}_{m,n}$ inside its domain of convergence.

Lemma 18. *Let* $g^{\pm}_{m,n}(\pm\sqrt{u})$ *as in Eq.* (103), $0 < \beta < 1$ *and* $0 < \sqrt{u} < \beta\sqrt{2/3}$. *Then*

$$\left|g^{\pm}_{m,n}(\pm\sqrt{u})\right| < \frac{9}{1-\beta}2^{m-2}.$$

Proof. It is clear from Eq. (103) that

$$g^{\pm}_{m,n}(\pm\sqrt{u}) = \sum_{s=0}^{\infty} d^{m,n}_{s+2m+1}(\pm\sqrt{u})^s.$$

Since by hypothesis $0 < \sqrt{u} < \beta\sqrt{2/3}$ with $\beta < 1$, we can apply Lemma 17 to get

$$\begin{aligned} |g^{\pm}_{m,n}(\pm\sqrt{u})| &\le \left(\frac{4}{3}\right)^m \left(\frac{3}{2}\right)^{\frac{2m+4}{2}} \sum_{s=0}^{\infty} \left(\frac{3}{2}\right)^{\frac{s}{2}} (\sqrt{u})^s \\ &\le \left(\frac{4}{3}\right)^m \left(\frac{3}{2}\right)^{\frac{2m+4}{2}} \sum_{s=0}^{\infty} \left(\frac{3}{2}\right)^{\frac{s}{2}} \left(\beta\sqrt{2/3}\right)^s = \frac{9}{1-\beta}2^{m-2} \end{aligned}$$

which proves the Lemma. □

We are now in conditions to give a general bound for $N(q,m,n)$ for $q \ge 1$.

Proposition 19. *Let* $N(q,m,n)$ *as defined in* (95) *for* $q \ge 1$, $m,n \ge 0$, $m+n > 0$, $G > 1$. *Then*

$$|N(q,m,n)| \le 2^{n+m+3}\, e^q\, G^{m-2n-1/2}\, e^{-qG^3/3}.$$

Proof. We will bound the integrals of $N(q,m,n)$ in (95) choosing

$$\varepsilon = G^{-3/2}, \quad G > 1.$$

We write down then, using (93), (94), (97), and that $k_\varepsilon > 1$,

$$\begin{aligned} &\left|\int_{u(C_\varepsilon)}^{\infty} F^{\pm}_{m,n}(u)\mathrm{e}^{-qG^3u/2}\,du\right| \le \int_{G^{-3}k_\epsilon}^{\infty} |F^{\pm}_{m,n}(u)|\mathrm{e}^{-qG^3u/2}\,du \\ &\le \int_{G^{-3}}^{\infty} |F^{\pm}_{m,n}(u)|\mathrm{e}^{-qG^3u/2}\,du \\ &\le \left|F^{\pm}_{m,n}(u(C_\varepsilon))\right| \int_{G^{-3}}^{\infty} \mathrm{e}^{-qG^3u/2}\,du \le \frac{G^{3m+\frac{3}{2}}}{\left(2-(G^{-3/2})\right)^{2n+1}} \frac{2\mathrm{e}^{-q/2}}{qG^3} \le 2G^{3m-3/2}. \end{aligned} \tag{104}$$

It only remains to bound the last integral of (95) where the integrand is given in (98) and the domain Γ_3 in (90). The path Γ_3 can be parameterized by

$$\tau = i + G^{-3/2}\mathrm{e}^{\mathrm{i}\theta} \quad \text{with } \theta \in [\theta_1,\theta_2] = [\pi-\theta_\varepsilon,\theta_\varepsilon], \tag{105}$$

with θ_ε given in (93). If we define

$$\tilde{h}(\theta) = h(\tau(\theta)) = i\left(\frac{\tau(\theta)^3}{3} + \tau(\theta)\right),$$

a straightforward computation using (92) shows that

$$\tilde{h}(\theta) = -\frac{2}{3} - G^{-3}\left(\mathrm{e}^{2i\theta} + \frac{1}{3i} G^{-\frac{3}{2}} \mathrm{e}^{3i\theta}\right)$$

and then, as $G > 1$,

$$\left|\mathrm{e}^{qG^3\tilde{h}(\theta)/2}\right| = \mathrm{e}^{-qG^3/3}\mathrm{e}^{-q\left(\cos 2\theta + G^{-3/2}\sin 3\theta/3\right)/2} \leq \mathrm{e}^{-qG^3/3}\mathrm{e}^{\frac{q}{2}(1+\frac{1}{3}G^{-3/2})} \leq \mathrm{e}^{-qG^3/3}\mathrm{e}^{q}. \tag{106}$$

Note that, by (105), over Γ_3 we have that $|\tau - i| = G^{-3/2} < 1$ and therefore $|\tau + i| > 1$, and we can bound the last integral of (95) using (106):

$$\begin{aligned}\left|\int_{\Gamma_3} \frac{\mathrm{e}^{qG^3 h(\tau)/2}}{(\tau - i)^{2m}(\tau + i)^{2n}}\, d\tau\right| &= \left|\int_{\theta_1}^{\theta_2} \frac{\mathrm{e}^{qG^3\tilde{h}(\theta)/2}}{(\tau(\theta) - i)^{2m}(\tau(\theta) + i)^{2n}} i\, G^{-3/2}\mathrm{e}^{i\theta}\, d\theta\right| \\ &\leq \int_{\theta_1}^{\theta_2} \frac{\left|\mathrm{e}^{qG^3\tilde{h}(\theta)}/2\right|}{(G^{-3/2})^{2m}} G^{-3/2}\, d\theta \leq \int_{\theta_1}^{\theta_2} \frac{\mathrm{e}^{-qG^3/3}\mathrm{e}^{q}}{G^{-3m}} G^{-3/2}\, d\theta \leq \pi G^{3m-3/2}\mathrm{e}^{-qG^3/3}\mathrm{e}^{q}.\end{aligned} \tag{107}$$

From Lemma 16 and the bounds (104) and (107), we can finally bound $N(q, m, n)$ given by equation (95) as follows

$$|N(q, m, n)| \leq \mathrm{e}^{-1/2} 2^{m+n} \mathrm{e}^{-qG^3/3} G^{m-2n-1/2}\left(4 + \pi \mathrm{e}^{q}\right) \leq 2^{m+n+3}\mathrm{e}^{q}\mathrm{e}^{-qG^3/3}G^{m-2n-1/2}.$$

□

From this Proposition and the one estimating the constants $c_q^{n,m}$, we can provide general estimates for the Fourier coefficients $L_{q,k}$ of the Melnikov potential for $q \geq 1$.

Proposition 20. *Assume $G \geq 32$. Then for $q \geq 1$, $k \geq 2$, the Fourier coefficients of the Melnikov potential (38) verify the following bounds:*

$$\begin{aligned}|L_{q,0}| &\leq 2^9 \left(2e^2\right)^q \epsilon_J^q\, G^{-3/2}\, e^{-qG^3/3} \\ |L_{q,1}| &\leq 2^{11} e^{2q} \frac{\epsilon_J^{q+1}}{\sqrt{1 - \epsilon_J^2}} G^{-7/2} e^{-qG^3/3} \\ |L_{q,-1}| &\leq 2^9 \left(2e^2\right)^q \epsilon_J^{q-1}\, G^{-1/2} e^{-qG^3/3} \\ |L_{q,k}| &\leq 2^{2k+5} e^{2q} \frac{\epsilon_J^{q+k}}{(\sqrt{1 - \epsilon_J^2})^k} G^{-2k-1/2} e^{-qG^3/3} \\ |L_{q,-k}| &\leq 2^5 2^{2k} \left(2e^2\right)^q \epsilon_J^{|k-q|} G^{k-1/2} e^{-qG^3/3}.\end{aligned}$$

Proof. From Eq. (72) and Propositions 15 and 19 we have

$$|L_{q,0}| \leq 2^4 \mathrm{e}^{q}\mathrm{e}^{-qG^3/3}(2\epsilon_{\mathrm{J}}\mathrm{e}^{\sqrt{1-\epsilon_{\mathrm{J}}^2}})^q G^{-1/2} \sum_{l \geq 1} (2^4 G^{-1})^l$$

$$|L_{q,1}| \le 2^2 \mathrm{e}^q \mathrm{e}^{q\sqrt{1-\epsilon_\mathrm{J}^2}} \frac{\epsilon_\mathrm{J}^{q+1}}{\sqrt{1-\epsilon_\mathrm{J}^2}} \mathrm{e}^{-qG^3/3} G^{-3/2} \sum_{l\ge 2} (2^4 G^{-1})^l$$

$$|L_{q,-1}| \le \mathrm{e}^q \mathrm{e}^{-qG^3/3} \mathrm{e}^{q\sqrt{1-\epsilon_\mathrm{J}^2}} 2^q \epsilon_\mathrm{J}^{|1-q|} G^{3/2} \sum_{l\ge 2} (2^4 G^{-1})^l$$

$$|L_{q,k}| \le 2^{4-k} \mathrm{e}^q \mathrm{e}^{q\sqrt{1-\epsilon_\mathrm{J}^2}} \frac{\epsilon_\mathrm{J}^{q+k}}{(\sqrt{1-\epsilon_\mathrm{J}^2})^k} \mathrm{e}^{-qG^3/3} G^{-k-1/2} \sum_{l\ge k} (2^4 G^{-1})^l$$

$$|L_{q,-k}| \le 2^4 2^{-2k} \mathrm{e}^q \mathrm{e}^{-qG^3/3} \mathrm{e}^{q\sqrt{1-\epsilon_\mathrm{J}^2}} 2^q \epsilon_\mathrm{J}^{|k-q|} G^{2k-1/2} \sum_{l\ge k} (2^4 G^{-1})^l.$$

Since by hypothesis $2^4/G \le 1/2$, all these series converge and the Proposition is proven using that $0 \le \epsilon_\mathrm{J} \le 1$ and that $\mathrm{e}^{q\sqrt{1-\epsilon_\mathrm{J}^2}} \le \mathrm{e}^q$. □

The Melnikov potential $\mathcal{L}$ (37) has a Fourier Cosine series (38) which can be split with respect to the variable s as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \cdots$, like in (40–41), as well as a complex Fourier series (71) $\mathcal{L} = \sum_{q\in\mathbb{Z}} L_q \mathrm{e}^{iqs}$. Both series are related through $\mathcal{L}_0 = L_0$ and $\mathcal{L}_q = 2\Re\left\{\mathrm{e}^{iqs} L_q\right\}$ for $q \ge 1$. In the next Lemma we see that the terms

$$\mathcal{L}_{\ge 2}(\alpha, G, s; \epsilon_\mathrm{J}) = 2 \sum_{q\ge 2} \sum_{k\in\mathbb{Z}} L_{q,k} \cos(qs + k\alpha) = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \cdots$$

of second order with respect to s satisfy a very exponentially small bound for large G.

Lemma 21. *Assume $G \ge 32$, $\epsilon_J G \le 1/8$. Then for $q \ge 2$*

$$|L_q| \le \sum_{k\in\mathbb{Z}} |L_{q,k}| \le 2^{13} e^{-qG^3/3} (e^2 2^3 G)^q G^{-1/2}$$

$$\left|\mathcal{L}_{\ge 2}(\alpha, G, s; \epsilon_J)\right| \le 2^{28} G^{3/2} e^{-2G^3/3}.$$

Proof. From Proposition 20 we have, using that $\frac{\epsilon_\mathrm{J}}{\sqrt{1-\epsilon_\mathrm{J}^2}} \le 1$:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |L_{q,k}| &\le |L_{q,0}| + |L_{q,1}| + |L_{q,-1}| + \sum_{k\ge 2} \big(|L_{q,k}| + |L_{q,-k}|\big) \\
&\le \mathrm{e}^{-qG^3/3} \mathrm{e}^{2q} \Big[2^9 2^q \epsilon_\mathrm{J}^q G^{-3/2} + 2^{11} \epsilon_\mathrm{J}^q G^{-7/2} + 2^9 2^q \epsilon_\mathrm{J}^{q-1} G^{-1/2} \\
&\quad + 2^5 \sum_{k\ge 2} \Big(2^k \epsilon_\mathrm{J}^q G^{-2k-1/2} + 2^{2k+q} \epsilon_\mathrm{J}^{|k-q|} G^{k-1/2} \Big) \Big] \\
&\le \mathrm{e}^{-qG^3/3} \mathrm{e}^{2q} \Big[2^{10} 2^q \epsilon_\mathrm{J}^{q-1} G^{-1/2} + 2^4 \epsilon_\mathrm{J}^q G^{-7/2} + 2^5 G^{-1/2} \epsilon_\mathrm{J}^q \sum_{k=2}^{\infty} (2G^{-2})^k \\
&\quad + 2^5 G^{-1/2} 2^q \epsilon_\mathrm{J}^q \sum_{k=2}^{q-1} (4G\epsilon_\mathrm{J}^{-1})^k + 2^5 G^{-1/2} \epsilon_\mathrm{J}^{-q} 2^q \sum_{k=q}^{\infty} (4\epsilon_\mathrm{J} G)^k \Big].
\end{aligned}$$

Using now that $\epsilon_J G \le 1/8$, we obtain the required bound for $\sum_{k\in\mathbb{Z}} |L_{q,k}|$:

$$\sum_{k\in\mathbb{Z}} |L_{q,k}| \le \mathrm{e}^{-qG^3/3}\mathrm{e}^{2q}\Big[2^{10}2^q\epsilon_J^{q-1}G^{-1/2} + 2^{11}\epsilon_J^q G^{-7/2} + 2^8\epsilon_J^q G^{-9/2}$$
$$+ 2^4 2^{3q}\epsilon_J G^{q-3/2} + 2^6 G^{q-1/2}2^{3q}\Big]$$
$$\le 2^{10}\mathrm{e}^{-qG^3/3}\mathrm{e}^{2q}2^{3q}G^{q-1/2}\Big[2^{-2q}\epsilon_J^{q-1}G^{-q} + 2^{-3q}\epsilon_J^q G^{-3-q}$$
$$+ 2^{-3q-4}\epsilon_J^q G^{-4-q} + \frac{1}{2^6}\epsilon_J G^{-1} + \frac{1}{4}\Big] \le 2^{13}\mathrm{e}^{-qG^3/3}(\mathrm{e}^2 2^3 G)^q G^{-1/2}.$$

To get the bound for $\left|\mathcal{L}_{\ge 2}\right|$, we sum for $q \ge 2$,

$$\left|\mathcal{L}_{\ge 2}\right| \le 2^{13}G^{-1/2}\sum_{q\ge 2}\Big[\mathrm{e}^{-G^3/3}\mathrm{e}^2 2^3 G\Big]^q \le 2^{20}\mathrm{e}^4 G^{3/2}\mathrm{e}^{-2G^3/3}$$

where the last bound holds as long as

$$\mathrm{e}^{-2G^3/3}\mathrm{e}^2 2^3 G \le 1/2$$

which is true for every $G \ge 32$. Now, using that $\mathrm{e} < 4$ we get the result. □

6.3. Aymptotic estimate for $N(q,m,n)$. To estimate the term $\mathcal{L}_1$ we will need an asymptotic expression for $N(q,m,n)$, which is given in the next Proposition.

Proposition 22. *For $n+m > 0$ let $d_j^{m,n}$ the constants $d_j^{m,n}$ defined by Eq. (99) and $d_{n,m}$ given by Eq. (96). Then for $q \ge 1$ and $G > 1$ we have*

$$N(q,m,n) = \frac{d_{m,n}e^{-qG^3/3}}{G^{2m+2n-1}}\left[\sum_{s=0}^{m}(-1)^s\sqrt{\pi}\frac{2^{3/2}q^{s-1/2}}{(2s-1)!!}d_{2m-2s}^{m,n}G^{3s-3/2} + T_{m,n}^q + R_{m,n}^q\right]$$

where

$$|T_{m,n}^q| \le 45\, 2^{2m+2}\cdot G^{-3} \qquad |R_{m,n}^q| \le 18\, q^{m-1}G^{3m-3}.$$

When $s = 0$ the factor $1/(2s-1)\ldots$ in the formula above should be replaced by 1.

To prove this Proposition we will proceed as in the proof of Proposition 19 changing the path of integration to the path Γ defined in (89) leading to the integral (95). The important fact is that the integral (95) does not depend on ε. So, we will compute only the ε-independent terms of that integral. The rest of this subsection is dedicated to the proof of Proposition 22.

Lemma 23. *For $0 < \varepsilon < 1$ let $u(C_\varepsilon)$ be as in Eq. (94) and $F_{m,n}^{\pm}$ as defined by (97). For any $\varepsilon > 0$ small enough we have, if $G > 1$*

$$\int_{u(C_\varepsilon)}^{\infty} F_{m,n}^{\pm}(u)e^{-qG^3u/2}du = \sum_{j=0}^{2m}\int_{u(C_\varepsilon)}^{\infty} e^{-qG^3u/2}d_j^{m,n}(\pm\sqrt{u})^{-2m-1+j}du + \widehat{E}$$

where the constants $d_j^{m,n}$ are defined by Eq. (99) and $\widehat{E}$ satisfies

$$|\widehat{E}| \le 45\, 2^{2m+2}\, G^{-3}.$$

Proof. Let us take $\sqrt{u_*} = \beta\sqrt{2/3}$ with $\beta = -1 + \frac{\sqrt{11}}{4}\sqrt{3+\sqrt{11}/2} \simeq 0.79$. A simple calculation using (92) shows that $|\tau^{\pm}(u^*) - i| = 1/2$. By definition, for $\varepsilon > 0$ small enough we have that $0 < u(C_\varepsilon) < u_* < \sqrt{u_*} < \sqrt{2/3}$, so

$$\int_{u(C_\varepsilon)}^{\infty} F^{\pm}_{m,n}(u)\mathrm{e}^{-qG^3u/2}du = \int_{u(C_\varepsilon)}^{u_*} F^{\pm}_{m,n}(u)\mathrm{e}^{-qG^3u/2}du + \widehat{E}_1$$

with

$$\widehat{E}_1 = \int_{u_*}^{\infty} F^{\pm}_{m,n}(u)\mathrm{e}^{-qG^3u/2}du,$$

which can be bounded as

$$\begin{aligned}|\widehat{E}_1| &= \left|\int_{u_*}^{\infty} F^{\pm}_{m,n}(u)\mathrm{e}^{-qG^3u/2}\,du\right| \le \int_{u_*}^{\infty}\frac{\mathrm{e}^{-qG^3u/2}}{|(\tau^{\pm}(u)-i)^{2m+1}(\tau^{\pm}(u)+i)^{2n+1}|}\,du\\ &\le \frac{2\mathrm{e}^{-q\frac{G^3}{2}u_*}}{qG^3}\frac{1}{|\tau^{\pm}(u_*)-i|^{2m+1}}\frac{1}{|\tau^{\pm}(u_*)+i|^{2n+1}}\\ &\le 2^{2m+2}G^{-3}\mathrm{e}^{-q\frac{G^3}{2}u_*} \le 2^{2m+2}G^{-3}.\end{aligned}$$

By Lemma 17 and Eq. (102) we have

$$\int_{u(C_\varepsilon)}^{u_*} F^{\pm}_{m,n}(u)\mathrm{e}^{-qG^3u/2}du = \sum_{j=0}^{2m}\int_{u(C_\varepsilon)}^{u_*} d_j^{m,n}\mathrm{e}^{-qG^3u/2}(\pm\sqrt{u})^{-2m-1+j}du + \widehat{E}_2$$

where

$$\widehat{E}_2 = \int_{u(C_\varepsilon)}^{u_*} g^{\pm}_{m,n}(\pm\sqrt{u})\mathrm{e}^{-qG^3u/2}du.$$

Using that $\sqrt{u_*} = \beta\sqrt{2/3}$, by Lemma 18 we have that, for any $\varepsilon > 0$ small enough,

$$\begin{aligned}|\widehat{E}_2| &\le \int_{u(C_\varepsilon)}^{u_*} |g^{\pm}_{m,n}(\pm\sqrt{u})|\mathrm{e}^{-qG^3u/2}du \le 9\,\frac{2^{m-2}}{1-\beta}\int_0^{\infty}\mathrm{e}^{-qG^3u/2}du\\ &\le 9\,\frac{2^{m-1}}{q(1-\beta)}G^{-3} \le 9\,\frac{2^{m-1}}{1-\beta}G^{-3}.\end{aligned}$$

Finally,

$$\int_{u(C_\varepsilon)}^{u_*} d_j^{m,n}\mathrm{e}^{-qG^3u/2}(\pm\sqrt{u})^{-2m-1+j}du = \int_{u(C_\varepsilon)}^{\infty} d_j^{m,n}\mathrm{e}^{-qG^3u/2}(\pm\sqrt{u})^{-2m-1+j}du + \widehat{E}_3(j),$$

where

$$\widehat{E}_3(j) = -\int_{u_*}^{\infty} d_j^{m,n}\mathrm{e}^{-qG^3u/2}(\pm\sqrt{u})^{-2m-1+j}du.$$

We can bound $\widehat{E}_3(j)$ thanks to the inequalities of Lemma 17:

$$|\widehat{E}_3(j)| \le |d_j^{m,n}|(\sqrt{u_*})^{-2m-1+j}\int_{u_*}^{\infty}\mathrm{e}^{-qG^3u/2}du$$

$$\leq |d_j^{m,n}|(\sqrt{u_*})^{-2m-1+j} 2\mathrm{e}^{-q\frac{G^3}{2}u_*}\frac{G^{-3}}{q}$$
$$\leq 2|d_j^{m,n}|\left(\beta\sqrt{2/3}\right)^{-2m-1+j} G^{-3} \leq 2\left(\frac{4}{3}\right)^m\left(\frac{3}{2}\right)^{\frac{j+3}{2}}\left(\beta\sqrt{2/3}\right)^{-2m-1+j} G^{-3}$$
$$= 9\, 2^{m-1}\beta^{-2m-1+j}G^{-3}.$$

Denoting now $\widehat{E}_3 = \sum_{j=1}^{2m} \widehat{E}_3(j)$, we have

$$|\widehat{E}_3| \leq 9\, 2^{m-1}G^{-3}\sum_{j=0}^{2m}\beta^{-2m-1+j} \leq 9\, 2^{m-1}G^{-3}\frac{\beta^{-2m-1}}{1-\beta}.$$

Now the Lemma is proven using that $1/\beta < \sqrt{2}$ and

$$|\widehat{E}| = |\widehat{E}_1 + \widehat{E}_2 + \widehat{E}_3|.$$

□

The next Lemma is a straightforward application of the last one.

Lemma 24. *For $0 < \varepsilon < 1$ let $u(C_\varepsilon)$ be as in Eq. (94) and $F^{\pm}_{m,n}$ as in (97). Then for any $\varepsilon > 0$ small enough we have, if $G > 1$*

$$\int_{u(C_\varepsilon)}^{\infty}\left[F^+_{m,n}(u)-F^-_{m,n}(u)\right]e^{-qG^3u/2}du=2\sum_{s=0}^{m}\int_{u(C_\varepsilon)}^{\infty} e^{-qG^3u/2}d^{m,n}_{2m-2s}(\sqrt{u})^{-2s-1}du+2\widehat{E}$$

where $\widehat{E}$ is the same as in Lemma 23.

Proof. By Lemma 23 we have

$$\int_{u(C_\varepsilon)}^{\infty}\left[F^+_{m,n}(u) - F^-_{m,n}(u)\right]\mathrm{e}^{-qG^3u/2}du$$
$$= \sum_{j=0}^{2m}\int_{u(C_\varepsilon)}^{\infty}\mathrm{e}^{-qG^3u/2}d_j^{m,n}[1-(-1)^{-2m-1+j}](\sqrt{u})^{-2m-1+j}du + 2\widehat{E}$$

and the terms in the sum are not zero only for $-2m-1+j = -2s-1$ with $s = 0, \ldots, m$. This observation proves the Lemma. □

Lemma 25. *Let $0 < \varepsilon < 1$ and $u(C_\varepsilon)$ be as in Eq. (95). Then the ε-independent term of*

$$\int_{u(C_\varepsilon)}^{\infty} e^{-qG^3u/2}d^{m,n}_{2m-2s}(\sqrt{u})^{-2s-1}du$$

is

$$(-1)^s 2^{s+3/2}(2s+1)\frac{(s+1)!}{(2s+2)!}d^{m,n}_{2m-2s}q^{s-1/2}G^{3s-3/2}\,\Gamma(1/2).$$

Proof. By Eq. (94) we know that $u(C_\varepsilon) = O(\varepsilon^2)$ and then the following definitions make sense, calling $\delta = G^3/2$:

$$I_{p,s}(\varepsilon) = \int_{u(C_\varepsilon)}^{\infty} \mathrm{e}^{-q\delta u}\, u^{p-(2s+1)/2}\, du$$
$$f_{p,s}(\varepsilon) = u(C_\varepsilon)^{p-(2s+1)/2}\, \mathrm{e}^{-q\delta u(C_\varepsilon)}.$$

Using this notation and integrating by parts we have

$$\begin{aligned} I_{p-1,s}(\varepsilon) &= \frac{q\delta}{p-s-1/2} \int_{u(C_\varepsilon)}^{\infty} \mathrm{e}^{-q\delta u} u^{p-(2s+1)/2}\, du - \frac{u(C_\varepsilon)^{p-(2s+1)/2}\, \mathrm{e}^{-q\delta u(C_\varepsilon)}}{p-s-1/2} \\ &= \frac{1}{p-s-1/2} \left(q\delta I_{p,s}(\varepsilon) - f_{p,s}(\varepsilon) \right) \end{aligned} \tag{108}$$

and also

$$\int_{u(C_\varepsilon)}^{\infty} \mathrm{e}^{-qG^3u/2} d_{2m-2s}^{m,n} (\sqrt{u})^{-2s-1} du = d_{2m-2s}^{m,n} I_{0,s}(\varepsilon). \tag{109}$$

Now, for $s > 0$ we use recursively Eq. (108) s times to get

$$\begin{aligned} I_{0,s}(\varepsilon) =& \frac{(q\delta)^s}{(-s-1/2+1)(-s-1/2+2)\cdots(-1/2)} I_{s,s}(\varepsilon) \\ & - \sum_{p=1}^{s} \frac{(q\delta)^{p-1} f_{p,s}(\varepsilon)}{(-s-1/2+1)\cdots(-s-1/2+p)}. \end{aligned}$$

The ε-independent term of $I_{0,s}(\varepsilon)$ is given by

$$\begin{aligned} & \frac{(q\delta)^s}{(-s-1/2+1)(-s-1/2+2)\cdots(-1/2)} \lim_{\varepsilon \to 0} I_{s,s}(\varepsilon) \\ &= \frac{(q\delta)^s}{(-s-1/2+1)(-s-1/2+2)\cdots(-1/2)} \frac{1}{\sqrt{q\delta}}\, \Gamma(1/2) \\ &= \frac{(\sqrt{q\delta})^{2s-1}}{(-s-1/2+1)(-s-1/2+2)\cdots(-1/2)}\, \Gamma(1/2). \end{aligned}$$

Then the ε-independent term of the integral in Eq. (109) is

$$\frac{d_{2m-2s}^{m,n} (\sqrt{q\delta})^{2s-1}}{(-s-1/2+1)(-s-1/2+2)\cdots(-1/2)}\, \Gamma(1/2)$$

when $s > 0$.

In the same way, we have that the ε-independent term of

$$I_{0,0}(\varepsilon) = \int_{u(C_\varepsilon)}^{\infty} \mathrm{e}^{-qG^3u/2} d_{2m}^{m,n} (\sqrt{u})^{-1} du$$

is $d_{2m}^{m,n}(\sqrt{q\delta})^{-1}\,\Gamma(1/2)$. Therefore the Lemma is proved if we notice that

$$(-s-1/2+1)(-s-1/2+2)\cdots(-1/2)=\frac{(-1)^s}{2^s}(2s-1)(2s-3)\cdots 1$$
$$=\frac{(-1)^s}{2^s}\frac{(2s+1)!!}{2s+1}=\frac{(-1)^s}{2^{2s+1}(2s+1)}\frac{(2s+2)!}{(s+1)!}$$

where we have used that

$$(2s+1)!!=\frac{(2s+2)!}{2^s(s+1)!}\,.$$

This expression allow us to write the cases $s>0$ and $s=0$ in one equation which completes the proof. □

Next Lemma is a straightforward application of Lemmas 24 and 25.

Lemma 26. *Let $u(C_\varepsilon)$ given in Eq. (94) and $F^{\pm}_{m,n}$ defined by (97), then the ε-independent terms of*

$$\int_{u(C_\varepsilon)}^{\infty}\left[F^{+}_{m,n}(u)-F^{-}_{m,n}(u)\right]e^{-qG^3u/2}du$$

are given by

$$\sum_{s=0}^{m}(-1)^s 2^{s+5/2}(2s+1)\frac{(s+1)!}{(2s+2)!}d^{m,n}_{2m-2s}q^{s-1/2}G^{3s-3/2}\,\Gamma(1/2)+2\widehat{E}$$

where $\widehat{E}$ is the same as in Lemma 23.

Lemma 27. *Let $f^{q}_{m,n}$ be defined in Eq. (98), then*

$$\mathrm{Res}(f^{q}_{m,n}(\tau),i)=2i\,e^{-qG^3/3}\sum_{l=0}^{m-1}\frac{1}{l!}\left(\frac{-qG^3}{2}\right)^l d^{m,n}_{2m-1-2l}. \tag{110}$$

Proof. We use the definition of $f^{q}_{m,n}$ given in (98), with $h(\tau)$ given in (87), or equivalently by (92).

$$h(\tau)=-2/3-(\tau-i)^2+i(\tau-i)^3/3.$$

Taking any $\delta>0$ small enough, we have

$$\mathrm{Res}(f^{q}_{m,n}(\tau),i)=\frac{1}{2\pi i}\int_{|\tau-i|=\delta}f^{q}_{m,n}(\tau)d\tau=\frac{1}{2\pi i}\int_{|\tau-i|=\delta}\frac{e^{q\frac{G^3}{2}h(\tau)}}{(\tau-i)^{2m}(\tau+i)^{2n}}\,d\tau.$$

We use again one of the changes (101), for instance

$$x=\sqrt{h(i)-h(\tau)}=\frac{\tau-i}{\sqrt{3}}(\sqrt{2-i\tau}),$$

to obtain

$$\mathrm{Res}\left(f_{m,n}^{q}(\tau), i\right) = \frac{\mathrm{e}^{-qG^3/3}}{\pi}\int_{|x|=\bar{\delta}} \frac{\mathrm{e}^{-qG^3x^2/2}}{(\tau_+(x)-i)^{2m+1}(\tau_+(x)+i)^{2n+1}}\, x\, d\tau$$
$$= 2i\,\mathrm{e}^{-qG^3/3}\,\mathrm{Res}\left(xF_+^{n,m}(x^2)\mathrm{e}^{-q\frac{G^3}{2}x^2}, 0\right).$$

We can now use the Taylor expansion of the function $F_+^{n,m}(x^2) = \sum_{j\geq 0} d_j^{m,n} x^{j-2m-1}$ and the expansion of $\mathrm{e}^{-qG^3x^2/2} = \sum_{l\geq 0}\left(-qG^3x^2/2\right)^l/l!$ to obtain the desired formula (110). □

From this Lemma one and the bounds for $d_j^{m,n}$ given in (100), we have

$$\left|\mathrm{Res}\left(f_{m,n}^{q}(\tau), i\right)\right| \leq 3\,2^m \mathrm{e}^{-qG^3/3}\sum_{l=0}^{m-1}\frac{1}{l!}\left(\frac{qG^3}{3}\right)^l$$
$$\leq 3\,2^{m+1}\mathrm{e}^{-qG^3/3}\left(\frac{qG^3}{3}\right)^{m-1} = \frac{2^{m+1}q^{m-1}G^{3m-3}}{3^{m-2}}\mathrm{e}^{-qG^3/3}. \quad (111)$$

We are finally in conditions to prove Proposition 22. $N(q, m, n)$ is given in (95), and since it does not depend on ε we can apply Lemmas 26 and 27 and the bound above (111) to obtain

$$N(q, m, n) = \frac{d_{m,n}\mathrm{e}^{-q\frac{G^3}{3}}}{G^{2m+2n-1}}\left[\sum_{s=0}^{m}(-1)^s 2^{s+\frac{5}{2}}(2s+1)\frac{(s+1)!}{(2s+2)!}d_{2m-2s}^{m,n}q^{s-1/2}G^{3s-\frac{3}{2}}\,\Gamma(1/2)\right.$$
$$\left. + T_{m,n}^{q} + R_{m,n}^{q}\right]$$

where by Lemma 26

$$|T_{m,n}^{q}| = 2\widehat{E} \leq 45\,2^{2m+2}\cdot G^{-3}$$

and

$$R_{m,n}^{q} = (-i)\mathrm{e}^{q\frac{G^3}{3}}\int_{\Gamma_3} f_{m,n}^{q}(\tau)d\tau$$

is bounded by Lemma 27

$$|R_{m,n}^{q}| \leq \frac{2^{m+1}q^{m-1}}{3^{m-2}}G^{3m-3} < 18\,q^{m-1}G^{3m-3}.$$

Using that $2^{s+1}(s+1)!(2s+1)!! = (2s+2)!$ to show that

$$\frac{(2s+1)(s+1)!}{(2s+2)!} = \frac{1}{2^{s+1}(2s-1)!!}$$

the formula for $N(q, m, n)$ of Proposition 22 follows. Due to the fact that the right hand side of this last expression is not defined for $s = 0$ but the left hand side is and is equal to one, we need to point out that for $s = 0$, the term $1/(2s-1)!!$ in the final formula should be replaced by 1.

6.4. Asymptotic estimate of $\mathcal{L}_1$. Let us first compute the coefficients $c_1^{n,m}$ which enter in the dominant terms of $\mathcal{L}_1$, more precisely $c_1^{3,1}$, $c_1^{2,2}$ and $c_1^{3,3}$. In passing, we will also compute $c_0^{2,0}$ and $c_0^{3,1}$, which will enter in the dominant terms of $\mathcal{L}_0$.

Lemma 28. *Let* $c_q^{n,m}$ *be defined by* (70). *Then*

$$c_1^{3,1} = 1 + Q_1, \quad c_1^{2,2} = -3\epsilon_J + Q_2, \quad c_0^{2,0} = 1 + Q_3, \quad c_0^{3,1} = -\frac{5}{2}\epsilon_J + Q_4,$$
$$c_1^{3,3} = \frac{57}{8}\epsilon_J^2 + Q_5,$$

with

$$|Q_i| \le 98\epsilon_J^2, \quad i = 1, 2, 3, 4, \quad |Q_5| \le 98\epsilon_J^3.$$

Proof. From the definition given in (70) plus the change of variable $t = E - \epsilon_{\mathrm{J}} \sin E$ we have

$$c_1^{3,1} = \frac{1}{2\pi}\int_0^{2\pi} \left(r e^{if(E)}\right) r^3 e^{-it} dE, \quad c_1^{j,j} = \frac{1}{2\pi}\int_0^{2\pi} \left(r e^{if(E)}\right)^j r e^{-it} dE, \quad j = 2, 3$$
$$c_0^{2,0} = \frac{1}{2\pi}\int_0^{2\pi} r^3 dE, \qquad c_0^{3,1} = \frac{1}{2\pi}\int_0^{2\pi} \left(r e^{if(E)}\right) r^3 dE.$$

From Eq. (83) we have

$$c_1^{3,1} = \frac{1}{2\pi}\int_0^{2\pi} [a^2 e^{iE} - \epsilon_{\mathrm{J}} + \frac{\epsilon_{\mathrm{J}}^2}{4a^2} e^{-iE}](1 - \epsilon_{\mathrm{J}} \cos E)^3 e^{-iE} e^{i\epsilon_{\mathrm{J}} \sin E} dE \tag{112}$$

$$c_1^{j,j} = \frac{1}{2\pi}\int_0^{2\pi} [a^2 e^{iE} - \epsilon_{\mathrm{J}} + \frac{\epsilon_{\mathrm{J}}^2}{4a^2} e^{-iE}]^j (1 - \epsilon_{\mathrm{J}} \cos E) e^{-iE} e^{i\epsilon_{\mathrm{J}} \sin E} dE, \quad j = 2, 3 \tag{113}$$

$$c_0^{2,0} = \frac{1}{2\pi}\int_0^{2\pi} (1 - \epsilon_{\mathrm{J}} \cos E)^3 dE \tag{114}$$

$$c_0^{3,1} = \frac{1}{2\pi}\int_0^{2\pi} [a^2 e^{iE} - \epsilon_{\mathrm{J}} + \frac{\epsilon_{\mathrm{J}}^2}{4a^2} e^{-iE}](1 - \epsilon_{\mathrm{J}} \cos E)^3 dE\,. \tag{115}$$

In what follows we will use (see (83b)) that

$$0 \le \epsilon_{\mathrm{J}} \le 1, \qquad \frac{1}{2} \le a^2 \le 1, \qquad |a^2 - 1| \le \frac{\epsilon_{\mathrm{J}}^2}{2}, \qquad a^2 + \frac{\epsilon_{\mathrm{J}}^2}{4a^2} = 1. \tag{116}$$

To compute $c_1^{3,1}$ we use Eq. (112). It is easy to see that

$$a^2 e^{iE} - \epsilon_{\mathrm{J}} + \tfrac{\epsilon_{\mathrm{J}}^2}{4a^2} e^{-iE} = e^{iE} - \epsilon_{\mathrm{J}} + \bar{E}_1,$$
$$(1 - \epsilon_{\mathrm{J}} \cos E)^3 = 1 - 3\epsilon_{\mathrm{J}} \cos E + \bar{E}_2, \quad e^{i\epsilon_{\mathrm{J}} \sin E} = 1 + i\epsilon_{\mathrm{J}} \sin E + \bar{E}_3, \tag{117}$$

where

$$\bar{E}_1 = (a^2-1)\mathrm{e}^{iE} + \frac{\epsilon_\mathrm{J}^2}{4a^2}\mathrm{e}^{-iE}, \qquad \bar{E}_2 = 3\epsilon_\mathrm{J}^2\cos^2 E - \epsilon_\mathrm{J}^3\cos^3 E,$$
$$\bar{E}_3 = \frac{1}{2}\sum_{j=0}^{\infty} 2\frac{(i\epsilon_\mathrm{J}\sin E)^{j+2}}{(j+2)!},$$

satisfy

$$\left|\bar{E}_1\right| \le \frac{\epsilon_\mathrm{J}^2}{2} + \frac{\epsilon_\mathrm{J}^2}{2} = \epsilon_\mathrm{J}^2, \qquad \left|\bar{E}_2\right| \le 4\epsilon_\mathrm{J}^2, \qquad \left|\bar{E}_3\right| \le \frac{\epsilon_\mathrm{J}^2}{2}\mathrm{e}^{\epsilon_\mathrm{J}} \le \epsilon_\mathrm{J}^2\frac{\mathrm{e}}{2} \le 2\epsilon_\mathrm{J}^2.$$

From the Eq. (112) defining $c_1^{3,1}$ plus equations (117), $c_1^{3,1}$ is the Fourier coefficient of order 1 of the function

$$\begin{aligned}&(\mathrm{e}^{iE} - \epsilon_\mathrm{J} + \bar{E}_1)(1 - 3\epsilon_\mathrm{J}\cos E + \bar{E}_2)(1 + i\epsilon_\mathrm{J}\sin E + \bar{E}_3)\\ &\quad \mathrm{e}^{iE} - \epsilon_\mathrm{J} - 3\epsilon_\mathrm{J}\cos E\mathrm{e}^{iE} + i\epsilon_\mathrm{J}\sin E\mathrm{e}^{iE} + \tilde{Q}_1(E)\end{aligned}$$

where

$$\begin{aligned}\tilde{Q}_1(E) = {} & \bar{E}_1 - 3\epsilon_\mathrm{J}^2\cos E - 3\epsilon_\mathrm{J}\bar{E}_1\cos E + \bar{E}_2(\mathrm{e}^{iE} - \epsilon_\mathrm{J} + \bar{E}_1) - i\epsilon_\mathrm{J}^2\sin E\\ & - 3i\epsilon_\mathrm{J}^2\cos E\sin E\mathrm{e}^{iE}\\ & - 3i\epsilon_\mathrm{J}^3\cos E\sin E - 3i\epsilon_\mathrm{J}^2\sin E\cos E\bar{E}_2 + i\epsilon_\mathrm{J}\sin E\bar{E}_2(\mathrm{e}^{iE} - \epsilon_\mathrm{J} + \bar{E}_1)\\ & + \bar{E}_3(\mathrm{e}^{iE} - \epsilon_\mathrm{J} + \bar{E}_1 - 3\epsilon_\mathrm{J}\cos E\mathrm{e}^{iE} - 3\epsilon_\mathrm{J}^2\cos E - 3\epsilon_\mathrm{J}\bar{E}_1\cos E\\ & + \bar{E}_2(\mathrm{e}^{iE} - \epsilon_\mathrm{J} + \bar{E}_1)),\end{aligned}$$

which implies that, up to order one in ϵ_J, the Fourier coefficient $c_1^{3,1}$ is exactly 1. From the bounds for $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ we find $|\tilde{Q}_1(E)| \le 98\epsilon_\mathrm{J}^2$, which implies the Lemma for $c_1^{3,1}$.

From Eq. (117), it is easy to see that

$$\left(a^2\mathrm{e}^{iE} - \epsilon_\mathrm{J} + \frac{\epsilon_\mathrm{J}^2}{4a^2}\mathrm{e}^{-iE}\right)^2 = \left(\mathrm{e}^{iE} - \epsilon_\mathrm{J} + \bar{E}_1\right)^2 = \mathrm{e}^{2iE} - 2\epsilon_\mathrm{J}\mathrm{e}^{iE} + \bar{E}_4$$

where

$$\bar{E}_4 = \epsilon_\mathrm{J}^2 + 2\bar{E}_1(\mathrm{e}^{iE} - \epsilon_\mathrm{J}) + \bar{E}_1^2$$

can be bounded, in regard of Eq. (116) and the bound for $\tilde{E}_1$, as

$$|\bar{E}_4| \le \epsilon_\mathrm{J}^2 + 2\epsilon_\mathrm{J}^2(1+\epsilon_\mathrm{J}) + \epsilon_\mathrm{J}^4 \le 6\epsilon_\mathrm{J}^2.$$

Using Eq. (117), we see from Eq. (113) that $c_1^{2,2}$ is the Fourier coefficient of order 1 of the function

$$\begin{aligned}&(\mathrm{e}^{2iE} - 2\epsilon_\mathrm{J}\mathrm{e}^{iE} + \bar{E}_4)(1 - \epsilon_\mathrm{J}\cos E)(1 + i\epsilon_\mathrm{J}\sin E + \bar{E}_3)\\ &\quad = \mathrm{e}^{2iE} - \epsilon_\mathrm{J}\cos E\mathrm{e}^{2iE} - 2\epsilon_\mathrm{J}\mathrm{e}^{iE} + i\epsilon_\mathrm{J}\sin E\mathrm{e}^{2iE} + \tilde{Q}_2(E)\end{aligned}$$

where

$$\tilde{Q}_2(E) = 2\epsilon_\mathrm{J}^2\cos E\mathrm{e}^{iE} + \bar{E}_4 - \epsilon_\mathrm{J}\bar{E}_4\cos E$$

$$= i\epsilon_J \sin E(-\epsilon_J \cos E e^{2iE} - 2\epsilon_J e^{iE} + 2\epsilon_J^2 \cos E e^{iE} + \bar{E}_4 - \epsilon_J \bar{E}_4 \cos E)$$
$$= \bar{E}_3(e^{2iE} - \epsilon_J \cos E e^{2iE} - 2\epsilon_J e^{iE} + 2\epsilon_J^2 \cos E e^{iE} + \bar{E}_4 - \epsilon_J \bar{E}_4 \cos E).$$

From the above expressions we conclude that, up to order one in ϵ_J, the Fourier coefficient $c_1^{2,2}$ is exactly $-3\epsilon_J$, and from the bounds for $\bar{E}_4$ and $\bar{E}_3$ we find that $|\tilde{Q}_2(E)| \leq 50\epsilon_J^2$ which implies the Lemma for $c_1^{2,2}$.

An analogous reasoning gives the value and the bounds for $c_1^{3,3}$ using

$$\left(a^2 e^{iE} - \epsilon_J + \frac{\epsilon_J^2}{4a^2} e^{-iE}\right)^3 = \frac{15}{4} a^2 \epsilon_J^2 e^{iE} - 3a^4 \epsilon_J e^{2iE} + a^6 e^{3iE} + \tilde{E}_{4,1}, \quad |\tilde{E}_{4,1}| \leq 8\epsilon_J^3$$

and

$$(1 - \epsilon_J \cos E) e^{i\epsilon_J \sin E} = 1 - \frac{\epsilon_J^2}{4} - \epsilon_J e^{-iE} - \frac{\epsilon_J^2}{8} e^{2iE} + \frac{3\epsilon_J^2}{8} e^{-2iE} + \tilde{E}_{4,2},$$
$$|\tilde{E}_{4,2}| \leq \frac{3}{2}\epsilon_J^3$$

which give

$$c_1^{3,3} = \frac{15}{4} a^2 \epsilon_J^2 (1 - \frac{\epsilon_J^2}{4}) + 3a^4 \epsilon_J^2 + \frac{3}{8} a^6 \epsilon_J^2 + \tilde{E}_{4,3}, \quad |\tilde{E}_{4,3}| \leq 56\epsilon_J^3.$$

Now, using (116) we obtain the value for $c_1^{3,3}$.

We compute $c_0^{2,0}$ using Eq. (114), as well as Eq. (117) to get

$$c_0^{2,0} = \frac{1}{2\pi} \int_0^{2\pi} (1 - 3\epsilon_J \cos E + \bar{E}_2) dE = 1 + Q_3$$

with

$$Q_3 = \frac{1}{2\pi} \int_0^{2\pi} \bar{E}_2 dE$$

and we have immediately, using the bound for $\bar{E}_2$, that $|Q_3| \leq 4\epsilon_J^2$, the desired result for $c_0^{2,0}$.

We finally compute $c_0^{3,1}$ using Eq. (115), as well as Eq. (117)

$$c_0^{3,1} = \frac{1}{2\pi} \int_0^{2\pi} (e^{iE} - \epsilon_J + \bar{E}_1)(1 - 3\epsilon_J \cos E + \bar{E}_2) dE.$$

We now want to find, up to order ϵ_J, the Fourier coefficient of order zero of the function

$$(e^{iE} - \epsilon_J + \bar{E}_1)(1 - 3\epsilon_J \cos E + \bar{E}_2) = e^{iE} - 3\epsilon_J e^{iE} \cos E - \epsilon_J + \bar{E}_5,$$

where

$$\bar{E}_5 = \bar{E}_2 e^{iE} + 3\epsilon_J^2 \cos E - \epsilon_J \bar{E}_2 + \bar{E}_1 - 3\epsilon_J \bar{E}_1 \cos E + \bar{E}_2 \bar{E}_1,$$

from where we find

$$c_0^{3,1} = -\frac{5}{2}\epsilon_{\mathrm{J}} + Q_4$$

with

$$Q_4 = \frac{1}{2\pi}\int_0^{2\pi} \bar{E}_5 dE,$$

and using the bounds for $\bar{E}_2$ and $\bar{E}_1$, we find $|Q_4| \leq 19\epsilon_{\mathrm{J}}^2$. □

The next step provides an asymptotic formula for $\mathcal{L}_1 = 2\Re\left\{\mathrm{e}^{is}L_1\right\}$.

Lemma 29. *For $G \geq 32$ and $\epsilon_J G \leq 1/8$ we have the following formula for L_1 given in (71)*

$$\Re\left\{e^{is}L_1\right\} = \Re\left\{e^{is}\left(L_{1,-1}\,e^{-i\alpha} + L_{1,-2}\,e^{-2i\alpha} + L_{1,-3}\,e^{-3i\alpha} + E_1\right)\right\} \tag{118}$$

where

$$\begin{aligned} L_{1,-1} &= c_1^{3,1}N(1,2,1) + E_3 \\ L_{1,-2} &= c_1^{2,2}N(1,2,0) + E_4 \\ L_{1,-3} &= c_1^{3,3}N(1,3,0) + \tilde{E}_4\,, \end{aligned} \tag{119}$$

$N(q,m,n)$ are defined by formula (73) and

$$\begin{aligned} |E_1(\alpha, G; \epsilon_J)| &\leq 2^{18}e^{-G^3/3}\epsilon_J\left[G^{-3/2} + \epsilon_J^2 G^{7/2}\right] \\ |E_3(\alpha, G; \epsilon_J)| &\leq 2^{20}e^{-G^3/3}G^{-3/2} \\ |E_4(\alpha, G; \epsilon_J)| &\leq 2^{18}e^{-G^3/3}\epsilon_J\,G^{1/2} \\ \left|\tilde{E}_4(\alpha, G; \epsilon_J)\right| &\leq 2^{20}e^{-G^3/3}\epsilon_J^2\,G^{3/2}. \end{aligned}$$

Proof. From Eq. (71), we have that

$$\begin{aligned} L_1 &= L_{1,0} + \sum_{k\geq 1}\left(L_{1,k}\mathrm{e}^{ik\alpha} + L_{1,-k}\mathrm{e}^{-ik\alpha}\right) \\ &= L_{1,-1}\mathrm{e}^{-i\alpha} + L_{1,-2}\mathrm{e}^{-2i\alpha} + L_{1,-3}\mathrm{e}^{-3i\alpha} + \sum_{k\geq 0}L_{1,k}\mathrm{e}^{ik\alpha} + \sum_{k\geq 4}L_{1,-k}\mathrm{e}^{-ik\alpha}. \end{aligned}$$

Now, setting

$$E_1 = \sum_{k\geq 0}L_{1,k}\mathrm{e}^{ik\alpha} + \sum_{k\geq 4}L_{1,-k}\mathrm{e}^{-ik\alpha} \tag{120}$$

we can write

$$\Re\left\{L_1\mathrm{e}^{is}\right\} = \Re\left\{\left(L_{1,-1}\mathrm{e}^{-i\alpha} + L_{1,-2}\mathrm{e}^{-2i\alpha} + L_{1,-3}\mathrm{e}^{-3i\alpha} + E_1\right)\mathrm{e}^{is}\right\}. \tag{121}$$

By definitions (72) we have

$$
\begin{aligned}
L_{1,-1} &= c_1^{3,1} N(1,2,1) + \sum_{l\geq 3} c_1^{2l-1,1} N(1,l,l-1) \\
L_{1,-2} &= c_1^{2,2} N(1,2,0) + \sum_{l\geq 3} c_1^{2l-2,2} N(1,l,l-2) \\
L_{1,-3} &= c_1^{3,3} N(1,3,0) + \sum_{l\geq 4} c_1^{2l-3,3} N(1,l,l-3).
\end{aligned} \tag{122}
$$

If we now set

$$
\begin{aligned}
E_3 &= \sum_{l\geq 3} c_1^{2l-1,1} N(1,l,l-1) \\
E_4 &= \sum_{l\geq 3} c_1^{2l-2,2} N(1,l,l-2) \\
\tilde{E}_4 &= \sum_{l\geq 4} c_1^{2l-3,3} N(1,l,l-3)
\end{aligned} \tag{123}
$$

we obtain just (118) from Eq. (122) and (121). Once we have obtained the formula (118), it only remains to bound properly the errors E_1, E_3, E_4 and $\tilde{E}_4$. From Eq. (120), the triangle inequality and Proposition 7 we have, using also that $\frac{\epsilon_J}{\sqrt{1-\epsilon_J^2}} \leq 1$:

$$
\begin{aligned}
|E_1| &\leq |L_{1,0}| + |L_{1,1}| + \sum_{k\geq 2} |L_{1,k}| + \sum_{k\geq 4} |L_{1,-k}| \\
&\leq e^2 e^{-G^3/3} \Bigg[2^{10}\epsilon_J G^{-3/2} + 2^{11}\epsilon_J G^{-7/2} + 2^5 \epsilon_J \sum_{k\geq 2} 2^{2k} G^{-2k-1/2} \\
&\quad + 2^6 \sum_{k\geq 4} 2^{2k} \epsilon_J^{k-1} G^{k-1/2} \Bigg] \\
&\leq e^2 e^{-G^3/3} \left[2^{10}\epsilon_J G^{-3/2} + 2^{11}\epsilon_J G^{-7/2} + 2^{10}\epsilon_J G^{-9/2} + 2^{14}\epsilon_J^3 G^{7/2} \right] \\
&\leq e^{-G^3/3} \left[2^{18}\epsilon_J G^{-3/2} + 2^{18}\epsilon_J^3 G^{7/2} \right] \\
&\leq 2^{18}\epsilon_J e^{-G^3/3} \left[G^{-3/2} + \epsilon_J^2 G^{7/2} \right].
\end{aligned} \tag{124}
$$

We now proceed with E_3, E_4 and $\tilde{E}_4$. By Propositions 15 and 19, from Eq. (123)

$$
\begin{aligned}
|E_3| &\leq \sum_{l\geq 3} |c_1^{2l-1,1} N(1,l,l-1)| \leq 2^3 e^{\sqrt{1-\epsilon_J^2}}\, e\, e^{-G^3/3} G^{3/2} \sum_{l\geq 3} (2^4 G^{-1})^l \\
&\leq 2^{16} e^2 e^{-G^3/3} G^{-3/2}, \\
|E_4| &\leq \sum_{l\geq 3} |c_1^{2l-2,2} N(1,l,l-2)| \leq 2\epsilon_J e^{\sqrt{1-\epsilon_J^2}}\, e\, e^{-G^3/3} G^{7/2} \sum_{l\geq 3} (2^4 G^{-1})^l
\end{aligned}
$$

$$\leq 2^{14}\mathrm{e}^2\mathrm{e}^{-G^3/3}\epsilon_\mathrm{J}G^{1/2}$$

$$|\tilde{E}_4| \leq \sum_{l\geq 4}|c_1^{2l-3,3}N(1,l,l-3)| \leq 2^{-1}\epsilon_\mathrm{J}^2\mathrm{e}^{\sqrt{1-\epsilon_\mathrm{J}^2}}\,\mathrm{e}\,\mathrm{e}^{-G^3/3}G^{11/2}\sum_{l\geq 4}(2^4G^{-1})^l$$
$$\leq 2^{16}\mathrm{e}^2\mathrm{e}^{-G^3/3}\epsilon_\mathrm{J}^2G^{3/2}.$$

The two estimates above, together with estimate (124) provide the desired bounds for the errors of Eq. (118). □

Putting together Lemmas 21 and 29 we already have

$$\mathcal{L} = L_0 + 2\Re\left\{\left[L_{1,-1}\mathrm{e}^{-i\alpha} + L_{1,-2}\mathrm{e}^{-2i\alpha} + L_{1,-3}\mathrm{e}^{-3i\alpha} + E_1\right]\mathrm{e}^{is}\right\} + \mathcal{L}_{\geq 2} \tag{125}$$

with $L_{1,-1}$, $L_{1,-2}$ and $L_{1,-3}$ as given in (119) and

$$\begin{aligned}
|E_1(\alpha, G; \epsilon_\mathrm{J})| &\leq 2^{18}\mathrm{e}^{-G^3/3}\epsilon_\mathrm{J}\left[G^{-3/2} + \epsilon_\mathrm{J}^2G^{7/2}\right]\\
|E_3(\alpha, G; \epsilon_\mathrm{J})| &\leq 2^{20}\mathrm{e}^{-G^3/3}G^{-3/2}\\
|E_4(\alpha, G; \epsilon_\mathrm{J})| &\leq 2^{18}\mathrm{e}^{-G^3/3}\epsilon_\mathrm{J}\,G^{1/2}\\
\left|\tilde{E}_4(\alpha, G; \epsilon_\mathrm{J})\right| &\leq 2^{20}\mathrm{e}^{-G^3/3}\epsilon_\mathrm{J}^2\,G^{3/2}\\
|\mathcal{L}_{\geq 2}(\alpha, G, s; \epsilon_\mathrm{J})| &\leq 2^{28}G^{3/2}\,\mathrm{e}^{-G^3\frac{4}{9}}.
\end{aligned} \tag{126}$$

We now compute $N(1,2,1)$, $N(1,2,0)$ and $N(1,3,0)$.

Lemma 30. *Let $N(q, m, n)$ be defined by Eq. (73). Then*

$$\begin{aligned}
N(1,2,1) &= \frac{1}{4}\sqrt{\frac{\pi}{2}}G^{-1/2}e^{-G^3/3} + {}^1E_{TT}\\
N(1,2,0) &= \sqrt{\frac{\pi}{2}}G^{3/2}e^{-G^3/3} + {}^2E_{TT}\\
N(1,3,0) &= \frac{1}{3}\sqrt{\frac{\pi}{2}}G^{5/2}e^{-G^3/3} + {}^3E_{TT}
\end{aligned}$$

where

$$|{}^1E_{TT}| \leq 2^6\,9\,G^{-2}e^{-G^3/3}, \qquad |{}^2E_{TT}| \leq 2^5\,9\,e^{-G^3/3}, \qquad |{}^3E_{TT}| \leq 2^6\,9\,G\,e^{-G^3/3}.$$

Proof. From Proposition 22 we have

$$N(1,2,1) = \frac{d_{2,1}}{G^5}\mathrm{e}^{-G^3/3}\left[d_4^{2,1}\sqrt{\pi}\left(\frac{2}{G}\right)^{3/2} - 2^2d_2^{2,1}\sqrt{\pi}\sqrt{\frac{G^3}{2}} + \frac{2^3}{3}d_0^{2,1}\sqrt{\pi}\left(\sqrt{\frac{G^3}{2}}\right)^3 + T_{2,1}^1 + R_{2,1}^1\right] \tag{127}$$

where

$$|T_{2,1}^1| \leq 45\,2^6G^{-3}, \qquad |R_{2,1}^1| \leq 18\,G^3,$$

$$N(1,2,0)=\frac{d_{2,0}}{G^3}e^{-G^3/3}\Bigg[2d_4^{2,0}\sqrt{\pi}\left(\sqrt{\frac{G^3}{2}}\right)^{-1}$$
$$-2^2d_2^{2,0}\sqrt{\pi}\sqrt{\frac{G^3}{2}}+\frac{2^3}{3}d_0^{2,0}\sqrt{\pi}\left(\sqrt{\frac{G^3}{2}}\right)^3+T^1_{2,0}+R^1_{2,0}\Bigg] \quad (128)$$

where

$$|T^1_{2,0}|\le 45\,2^6G^{-3}\qquad |R^1_{2,0}|\le 18\,G^3,$$

and

$$N(1,3,0)=\frac{d_{3,0}}{G^5}e^{-G^3/3}\Bigg[2d_6^{3,0}\sqrt{2\pi}G^{-3/2}$$
$$-2d_4^{3,0}\sqrt{2\pi}G^{3/2}+\frac{2}{3}\sqrt{2\pi}d_2^{3,0}G^{9/2}-\frac{2}{15}d_0^{3,0}\sqrt{2\pi}G^{15/2}+T^1_{3,0}+R^1_{3,0}\Bigg] \quad (129)$$

where

$$|T^1_{3,0}|\le 45\,2^8G^{-3}\qquad |R^1_{3,0}|\le 18\,G^6.$$

Taking the dominant terms in (127), (128) and (129)we get:

$$N(1,2,1)=d_{2,1}d_0^{2,1}\frac{2\sqrt{2}}{3}\sqrt{\pi}G^{-1/2}e^{-G^3/3}+{}^1E+{}^1E_{TR} \quad (130)$$

where

$$^1E=2^{\frac{3}{2}}d_{2,1}\sqrt{\pi}\left(d_4^{2,1}G^{-13/2}-d_2^{2,1}G^{-7/2}\right)e^{-G^3/3},$$
$$^1E_{TR}=(T^1_{2,1}+R^1_{2,1})d_{2,1}G^{-5}e^{-G^3/3},$$
$$N(1,2,0)=d_{2,0}d_0^{2,0}\frac{2\sqrt{2}}{3}\sqrt{\pi}G^{3/2}e^{-G^3/3}+{}^2E+{}^2E_{TR} \quad (131)$$

where

$$^2E=2^{\frac{3}{2}}d_{2,0}\sqrt{\pi}\left(d_4^{2,0}G^{-9/2}-d_2^{2,0}G^{-3/2}\right)e^{-G^3/3}$$
$$^2E_{TR}=(T^1_{2,0}+R^1_{2,0})d_{2,0}G^{-3}e^{-G^3/3},$$

and

$$N(1,3,0)=-d_{3,0}d_0^{3,0}\frac{2}{15}\sqrt{2\pi}G^{5/2}e^{-G^3/3}+{}^3E+{}^3E_{TR} \quad (132)$$

where

$$^3E=2d_{3,0}\sqrt{2\pi}\left(d_6^{3,0}G^{-13/2}-d_4^{3,0}G^{-7/2}+\frac{d_2^{3,0}}{3}G^{-1/2}\right)e^{-G^3/3},$$
$$^3E_{TR}=(T^1_{3,0}+R^1_{3,0})d_{3,0}G^{-5}e^{-G^3/3}.$$

From the bounds given in Lemma 17 for $d_j^{m,n}$ and the bounds in Lemma 16 for $d_{m,n}$ we get:

$$
\begin{aligned}
|{}^1E| &\le 2^{\frac{3}{2}}|d_{2,1}|\sqrt{\pi}(|d_4^{1,2}|+|d_2^{2,1}|)G^{-7/2}\mathrm{e}^{-G^3/3} \le 2^7\,9\,G^{-7/2}\mathrm{e}^{-G^3/3}\\
|{}^1E_{TR}| &\le |d_{2,1}|\,36\,G^{-2}\mathrm{e}^{-G^3/3} \le 2^5 9\,G^{-2}\mathrm{e}^{-G^3/3},\\
|{}^2E| &\le 2^{\frac{3}{2}}|d_{2,0}|\sqrt{\pi}(|d_4^{2,0}|+|d_2^{2,0}|)G^{-3/2}\mathrm{e}^{-G^3/3} \le 2^6 9\,G^{-3/2}\mathrm{e}^{-G^3/3}\\
|{}^2E_{TR}| &\le |d_{2,0}|\,36\mathrm{e}^{-G^3/3} \le 2^4 9\,\mathrm{e}^{-G^3/3},
\end{aligned}
$$

and

$$
\begin{aligned}
|{}^3E| &\le 2|d_{3,0}|\sqrt{2\pi}(|d_6^{3,0}|+|d_4^{3,0}|+\tfrac{|d_2^{3,0}|}{3})G^{-1/2}\mathrm{e}^{-G^3/3} \le 2^8 9\,G^{-1/2}\mathrm{e}^{-G^3/3}\\
|{}^3E_{TR}| &\le |d_{3,0}|\,36\,G\mathrm{e}^{-G^3/3} \le 2^5 9\,G\mathrm{e}^{-G^3/3}.
\end{aligned}
$$

Using Lemma 17, $d_0^{m,n}=1/(2i)^{2n+1}$ and the definition (96) for $d_{m,n}$ we have that

$$
\begin{aligned}
d_{2,1}d_0^{2,1} &= -i2^3\binom{-1/2}{2}\binom{-1/2}{1}\left(\frac{i}{2^3}\right) = -\frac{3}{2^4}\\
d_{2,0}d_0^{2,0} &= i2^2\binom{-1/2}{2}\left(-\frac{i}{2}\right) = \frac{3}{2^2}\\
d_{3,0}d_0^{3,0} &= i2^3\binom{-1/2}{3}\left(-\frac{i}{2}\right) = -\frac{5}{2^2}.
\end{aligned}
$$

We can then write Eq. (130) as

$$N(1,2,1) = \frac{1}{4}\sqrt{\frac{\pi}{2}}G^{-1/2}\mathrm{e}^{-G^3/3} + {}^1E_{TT}$$

where

$${}^1E_{TT} = {}^1E + {}^1E_{TR}$$

satisfies

$$|{}^1E_{TT}| \le 2^7\,9\,G^{-\frac{7}{2}}\mathrm{e}^{-G^3/3} + 2^5 9\,G^{-2}\mathrm{e}^{-G^3/3} \le 2^6 9\,G^{-2}\mathrm{e}^{-G^3/3}.$$

In an analogous way, Eq. (131) can be written as

$$N(1,2,0) = \sqrt{\frac{\pi}{2}}G^{3/2}\mathrm{e}^{-G^3/3} + {}^2E_{TT}$$

where

$${}^2E_{TT} = {}^2E + {}^2E_{TR}$$

satisfies

$$|{}^2E_{TT}| \le 2^6 9\,G^{-\frac{3}{2}}\mathrm{e}^{-G^3/3} + 2^4 9\,\mathrm{e}^{-G^3/3} \le 2^5 9\,\mathrm{e}^{-G^3/3}.$$

Finally, Eq. (132) can be written as

$$N(1,3,0) = \frac{1}{3}\sqrt{\frac{\pi}{2}}G^{5/2}\mathrm{e}^{-G^3/3} + {}^3E_{TT}$$

where

$$^3E_{TT} = {}^3E + {}^3E_{TR}$$

satisfies

$$|^3E_{TT}| \leq 2^8 9\, G^{-\frac{1}{2}} \mathrm{e}^{-G^3/3} + 2^5 9\, G \mathrm{e}^{-G^3/3} \leq 2^6 9\, G \mathrm{e}^{-G^3/3},$$

and this proves the Lemma. □

Using the approximations given in Lemma 30 we have from Lemmas 21 and 29:

Lemma 31. *For $G \geq 32$ and $\epsilon_J G \leq 1/8$, the Melnikov potential $\mathcal{L}$ given in* (71) *satisfies*

$$\begin{aligned}\mathcal{L} = L_0 + 2L_{1,-1}\cos(s-\alpha) + 2L_{1,-2}\cos(s-2\alpha) + 2L_{1,-3}\cos(s-3\alpha) \\ +2\Re\{E_1 e^{is}\} + \mathcal{L}_{\geq 2}\end{aligned} \tag{133}$$

with

$$\begin{aligned}2\,L_{1,-1} &= c_1^{3,1}\sqrt{\tfrac{\pi}{8}}G^{-1/2}e^{-G^3/3} + E_3 + E_5\\ 2\,L_{1,-2} &= c_1^{2,2}\sqrt{2\pi}G^{3/2}e^{-G^3/3} + E_4 + E_6\\ 2\,L_{1,-3} &= c_1^{3,3}\tfrac{\sqrt{2\pi}}{3}G^{5/2}e^{-G^3/3} + \tilde{E}_4 + \tilde{E}_6\end{aligned}$$

and where $\mathcal{L}_{\geq 2}$ and E_k with $k = 1, 3, 4$ are given in Eq. (126) *and*

$$|E_5| \leq 2^{13}\, 9\, G^{-2}e^{-G^3/3}, \qquad |E_6| \leq 2^{11}\, 9\, \epsilon_J e^{-G^3/3}, \qquad |\tilde{E}_6| \leq 2^{13}\, 9\, G\epsilon_J^2 e^{-G^3/3}.$$

Proof. By Lemma 30 we have that $N(1,2,1)$, $N(1,2,0)$ and $N(1,3,0)$ are real and then coincide with their real part. Equation (125) gives the correct estimation of $\mathcal{L}$. To complete the proof is enough to take

$$E_5 = c_1^{3,1}\cdot {}^1E_{TT}\,, \quad E_6 = c_1^{2,2}\cdot {}^2E_{TT} \quad \text{and} \quad \tilde{E}_6 = c_1^{3,3}\cdot {}^3E_{TT}$$

where $^1E_{TT}$, $^2E_{TT}$ and $^3E_{TT}$ are given in Lemma 30. Therefore by Proposition 15 we find directly the bounds of E_5, E_6 and $\tilde{E}_6$. □

The next Proposition contains the final asymptotic estimate for $\mathcal{L}_1$:

Proposition 32. *For $G \geq 32$ and $\epsilon_J G \leq 1/8$, the Melnikov potential $\mathcal{L}$* (71) *is given by* (133) *with:*

$$\begin{aligned}2\,L_{1,-1} &= \sqrt{\tfrac{\pi}{8}}G^{-1/2}e^{-G^3/3} + E_3 + E_5 + E_7\\ 2\,L_{1,-2} &= -3\sqrt{2\pi}\epsilon_J G^{3/2}e^{-G^3/3} + E_4 + E_6 + E_8\\ 2\,L_{1,-3} &= \tfrac{19}{8}\sqrt{2\pi}\epsilon_J^2 G^{5/2}e^{-G^3/3} + \tilde{E}_4 + \tilde{E}_6 + \tilde{E}_8\end{aligned}$$

and where $\mathcal{L}_{\geq 2}$ and E_k with $k = 1, 3, \ldots 6$ and $\tilde{E}_k$ with $k = 4, 5, 6$ are given in Eq. (126) *and*

$$|E_7| \leq 98\epsilon_J^2 G^{-1/2}e^{-G^3/3}\,, \quad |E_8| \leq 982^2\epsilon_J^2 G^{3/2}e^{-G^3/3}\,, \quad |\tilde{E}_8| \leq 982^2\epsilon_J^3 G^{5/2}e^{-G^3/3}.$$

Proof. From Lemma 28 we have

$$c_1^{3,1}\sqrt{\frac{\pi}{8}}G^{-1/2}\mathrm{e}^{-G^3/3} = \sqrt{\frac{\pi}{8}}G^{-1/2}\mathrm{e}^{-G^3/3} + E_7$$
$$c_1^{2,2}\sqrt{2\pi}G^{3/2}\mathrm{e}^{-G^3/3} = -3\sqrt{2\pi}\epsilon_{\mathrm{J}}G^{3/2}\mathrm{e}^{-G^3/3} + E_8$$
$$c_1^{3,3}\frac{\sqrt{2\pi}}{3}G^{5/2}\mathrm{e}^{-G^3/3} = \frac{19}{8}\sqrt{2\pi}\epsilon_{\mathrm{J}}^2G^{5/2}\mathrm{e}^{-G^3/3} + \tilde{E}_8$$

with

$$E_7 = Q_1\sqrt{\frac{\pi}{8}}G^{-1/2}\mathrm{e}^{-G^3/3}$$
$$E_8 = Q_2\sqrt{2\pi}G^{3/2}\mathrm{e}^{-G^3/3}$$
$$\tilde{E}_8 = Q_5\sqrt{2\pi}G^{5/2}\mathrm{e}^{-G^3/3}.$$

Therefore by Lemma 31 and the bounds of Q_1 and Q_2 given in Lemma 28 we conclude the proof. □

6.5. Asymptotic estimate of $\mathcal{L}_0$. It only remains to estimate the Fourier coefficient $L_0 = \mathcal{L}_0$ defined in (41) or (71).

Lemma 33. *Let $N(q, m, n)$ be defined by Eq.* (73)*. Then for $m, n \in \mathbb{N}$, $m + n > 0$,*

$$|N(0, m, n)| \leq 2^{m+n+2}G^{-2m-2n+1}.$$

Proof. Since $\tau \in \mathbb{R}$ in the integral (73), it is clear that

$$\frac{1}{|\tau + i|}, \frac{1}{|\tau - i|} \leq 1$$

and then

$$\frac{1}{|\tau + i|^{2n}}\frac{1}{|\tau - i|^{2m}} \leq \frac{1}{1 + \tau^2}.$$

For $n, m > 0$, using Eq. (73) and Lemma 16 to bound $d_{m,n}$, the Lemma follows:

$$|N(0, m, n)| \leq 2^{m+n}G^{-2m-2n+1}\mathrm{e}^{-1/2}\int_{-\infty}^{\infty}\frac{d\tau}{1 + \tau^2}$$
$$= 2^{m+n}G^{-2m-2n+1}\mathrm{e}^{-1/2}\pi \leq 2^{m+n+2}G^{-2m-2n+1}.$$

□

Lemma 34. *Let $k \in \mathbb{N}$ and $L_{0,k}$ defined by Eq.* (41)*. Then*

$$L_{0,k} = \sum_{l \geq k+1} c_0^{2l-k,-k}N(0, l - k, l).$$

Proof. From Eq. (72), we have just to prove $N(0,0,k) = N(0,k,0) = 0$ for $k \geq 2$. By Eq. (73) this reduces to show that

$$\int_{-\infty}^{\infty} \frac{d\tau}{(\tau \pm i)^{2k}} = 0$$

where the positive sign in the denominator correspond to $I(0,0,k)$ and the negative to $I(0,k,0)$. Since the variable $\tau \in \mathbb{R}$ this integral is trivially zero

$$\int_{-\infty}^{\infty} \frac{d\tau}{(\tau \pm i)^{2k}} = -\frac{1}{2k-1} \frac{1}{(\tau \pm i)^{2k-1}}\bigg|_{-\infty}^{\infty} = 0.$$

□

Lemma 35. *Let $L_{0,k}$ be defined by Eq. (72) for $k \geq 0$. If $G \geq 32$,*

$$|L_{0,k}| \leq 2^{2k+8} \epsilon_J^k G^{-2k-3}.$$

Proof. From Lemma 34 we have

$$|L_{0,k}| \leq \sum_{l \geq k+1} |c_0^{2l-k,-k}||N(0,l-k,l)|,$$

and by Propositions 15 and 19,

$$|L_{0,\pm k}| \leq 2^{-2k+3} \epsilon_{\mathrm{J}}^k G^{2k+1} \sum_{l \geq k+1} (2^4 G^{-4})^l \leq \epsilon_{\mathrm{J}}^k 2^{2k+8} G^{-2k-3}.$$

□

Lemma 36. *Let $L_0 = \mathcal{L}_0$ be defined by Eqs. (41) or (71). Then for $G \geq 32$*

$$L_0 = L_{0,0} + (c_0^{3,1} \frac{3}{4} \pi G^{-5} + F_2) \cos(\alpha) + F_3$$
$$L_{0,0} = c_0^{2,0} \frac{\pi}{2} G^{-3} + F_1$$

where

$$|F_1| \leq 2^{12} G^{-7}, \qquad |F_2| \leq 2^{15} \epsilon_J G^{-9}, \qquad |F_3| \leq 2^{14} \epsilon_J^2 G^{-7}.$$

Proof. From Proposition 14 we know that

$$L_0 = L_{0,0} + 2 \sum_{k \geq 1} L_{0,k} \cos k\alpha,$$

and from Lemma 34 we have that

$$L_{0,0} = c_0^{2,0} N(0,1,1) + \sum_{l \geq 2} c_0^{2l,0} N(0,l,l)$$
$$L_{0,1} = c_0^{3,-1} N(0,1,2) + \sum_{l \geq 3} c_0^{2l-1,-1} N(0,l-1,l)$$

$$L_{0,k} = \sum_{l \geq k+1} c_0^{2l-k,-k} N(0, l-k, l) \qquad \text{for } k \geq 2. \tag{134}$$

Introducing

$$F_1 = \sum_{l\geq 2} c_0^{2l,0} N(0,l,l), \quad F_2 = 2\sum_{l\geq 3} c_0^{2l-1,-1} N(0,l-1,l), \quad F_3 = 2\sum_{k\geq 2} \cos k\alpha L_{0,k},$$

and using $G \geq 32$ in Lemmas 33, 35 and Proposition 15, we have

$$|F_1| \leq 2^3 G \sum_{l\geq 2} (2^4 G^{-4})^l \leq 2^{12} G^{-7}$$

$$|F_2| \leq 2^2 \epsilon_J G^3 \sum_{l\geq 3} (2^4 G^{-4})^l \leq 2^{15} \epsilon_J G^{-9}$$

$$|F_3| \leq 2\sum_{k\geq 2} |L_{0,k}| \leq 2^{14} \epsilon_J^2 G^{-7}.$$

From definition (73) we have now that

$$N(0,1,1) = \frac{2^2}{G^3}\binom{-1/2}{1}\binom{-1/2}{1}\int_{-\infty}^{\infty}\frac{d\tau}{(\tau^2+1)^2} = 2^2\Big(-\frac{1}{2}\Big)\Big(-\frac{1}{2}\Big)G^{-3} = \frac{\pi}{2}G^{-3},$$

$$N(0,1,2) = \frac{2^3}{G^5}\binom{-1/2}{1}\binom{-1/2}{2}\int_{-\infty}^{\infty}\frac{d\tau}{(\tau-i)(\tau+i)^2} = 2^3\Big(-\frac{1}{2}\Big)\Big(\frac{3}{2^3}\Big)\Big(-\frac{\pi}{4}\Big)G^{-5}$$

$$= \frac{3}{8}\pi G^{-5}.$$

From these equations, substituting Eq. (134) in the definition of L_0 and the bounds given in equations (135) we have proven this Lemma. □

A refinement of this Lemma is

Lemma 37. *Let $L_0 = \mathcal{L}_0$ be defined by Eqs. (41) or (71). Then for $G \geq 2^{3/2}$*

$$L_0 = L_{0,0} + \Big(-\frac{15}{8}\pi\epsilon_J G^{-5} + F_2 + F_5\Big)\cos(\alpha) + F_3$$

$$L_{0,0} = \frac{\pi}{2}G^{-3} + F_1 + F_4$$

where F_1, F_2 and F_3 are given in Lemma 36 and

$$|F_4| \leq 2\,98\,G^{-3}\epsilon_J^2, \qquad |F_5| \leq 2^2\,98\,G^{-5}\epsilon_J^2.$$

Proof. In Lemma 28 we have computed the constants $c_0^{2,0}$ and $c_0^{3,1}$, then by setting

$$F_4 = \frac{\pi}{2}Q_3 G^{-3}, \qquad F_5 = \frac{3}{4}\pi Q_4 G^{-5}\cos\alpha,$$

and using the bounds for Q_3 and Q_4 we find the desired bound for F_4 and F_5. □

With this Lemma we can rewrite Proposition 32 exactly as Theorem 8, and so it is proven.

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