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The Set of Smooth Quasi-periodic Schrödinger Cocycles with Positive Lyapunov Exponent is Not Open

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Abstract: One knows that the set of quasi-periodic Schrödinger cocycles with positive Lyapunov exponent is open and dense in analytic topology. In this paper, we construct cocycles with positive Lyapunov exponent which can be arbitrarily approximated by ones with zero Lyapunov exponent in the space of $\mathcal{C}^l$ ($1 \le l \le \infty$) smooth quasi-periodic cocycles, which shows that the set of quasi-periodic Schrödinger cocycles with positive Lyapunov exponent is not open in smooth topology.

1. Introduction and Results

Given an $\mathcal{C}^r$ compact manifold X, let $A(x)$ be a $SL(2,\mathbb{R})$-valued function on X and (X, T, μ) be ergodic with μ a normalized T-invariant measure. The dynamical system in $X \times \mathbb{R}^2$: $(x, w) \to (T(x), A(x)w)$ is called a $SL(2,\mathbb{R})$ cocycle (or cocycle for simplicity). We will simply denoted it as (T, A). If the base system is a rotation on torus, i.e., $X = \mathbb{T}^n = \mathbb{R}^n \backslash \mathbb{Z}^n$, $T = T_\omega : x \to x + \omega$ with rational independent ω, we call (T_ω, A) a quasi-periodic cocycle, which is simply denoted by (ω, A). If furthermore $A(x) = S_v(x)$ is of the form $S_v(x) = \begin{pmatrix} v(x) & -1 \\ 1 & 0 \end{pmatrix}$ with $v(x+1) = v(x)$, we call $(\omega, S_v(x))$ a quasi-periodic Schrödinger cocycle.

For any $n \in \mathbb{N}$ and $x \in X$, we denote

$$A^n(x) = A(T^{n-1}x) \dots A(Tx)A(x)$$

and

$$A^{-n}(x) = A^{-1}(T^{-n}x) \dots A^{-1}(T^{-1}x) = (A^n(T^{-n}x))^{-1}.$$

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For fixed ergodic base system (X, T, μ), the (maximum) Lyapunov exponent of (T, A) is defined as

$$L(A) = \lim_{n\to\infty} \frac{1}{n} \int \log \|A^n(x)\| d\mu := \lim_{n\to\infty} \int L_n(A(x)) d\mu \in [0, \infty),$$

which measures the average growth rate of $\|A^n(x)\|$.

Regularity and positivity of Lyapunov exponent (LE) are the central subjects in smooth dynamical systems, both subtly depend on the base dynamics T and the smoothness of the matrix A. For nonlinear systems, they also depend on the geometry of the manifold.

The chaoticity in the base favors the positivity and regularity of LE. The classical Furstenberg theory [25] showed that for random product of matrices or cocyles over full shifts, the largest LE is positive under very general conditions. Furstenberg–Kifer [26] and Hennion [28] proved continuity of the largest LE of i.i.d random matrices under a condition of almost irreducibility. For Schrödinger cocycles, Kotani [40] proved that LE is positive for almost every energy for some class of non-deterministic potential. Viana [48] proved that for any $s > 0$, the set of C^s linear cocycles over any hyperbolic ergodic transformation contains an open and dense subset of cocycles with nonzero LE; and LE is continuous for $SL(2, \mathbb{R})$-cocycles over Markov shifts [43]. For other related results, see [6,10] and [49].

However, when the base dynamics is quasiperiodic, the positivity and continuity of LE seem to be more sensitive to the smoothness of the matrix-valued function $A(x)$. LE was proved to be discontinuous at any non-uniformly hyperbolic cocycles in C^0 topology by Furman [24] (Continuity at uniformly hyperbolic cocycles is trivial). Motivated by Mañé [41,42], Bochi [11] further proved that any non-uniformly hyperbolic $SL(2, \mathbb{R})$-cocycle over a fixed ergodic system on a compact space, can be arbitrarily approximated by cocycles with zero LE in the C^0 topology, which shows that any non-uniformly hyperbolic cocycle can not be an inner point of cocycles with positive LE in C^0 topology. For further related results, we refer to [8,12,13,28,35,36,39,47].

On the other hand, there are many tremendously positive results on both the positivity and continuity of LE in analytic topology.

For the positivity of LE, Herman proved that, by the subharmonicity method, LE is uniformly positive for Schrödinger cocycles with the potential $2\lambda \cos x$ if $|\lambda| > 1$. The result remains true for trigonometric polynomials $\lambda v(x)$ with large λ [29]. The generalization to arbitrary one-frequency nonconstant real analytic potentials was given by Sorets and Spencer [46]. The same result for Diophantine multi-frequency was established by Bourgain and Schlag [17] and Goldstein and Schlag [27]. Zhang [54] gave a different proof of the results in [46], and applied it to a certain class of analytic Szegö cocycle. For more references, we refer to [16,22,38].

For the continuity of LE, Large Deviation Theorems (LDT), established by Bourgain and Goldstein in [17] for real analytic potentials with Diophantine frequency, is an important tool. In [27], Goldstein and Schlag gave some sharp version of LDT and developed the Avalanche Principle(AP), and proved that if ω is a Diophantine irrational number and $v(x)$ is analytic, then the Lyapunov exponent $L(E)$ is Hölder continuous provided $L(E) > 0$. Later, Jitomirskaya, Koslover and Schulteis [31] proved the continuity of the LE for a class of analytic one-frequency quasiperiodic $M(2, \mathbb{C})$-cocycles with singularities. The continuity of LE implies that the set of the cocycles with positive LE is open in analytic topology. Together with the denseness result by Avila [1], one knows that the set of quasi-periodic cocycles with positive LE is open and dense in analytic topology. More related references can be found at the end of this section.

So, the behavior of LE are totally different in C^0 and analytic topology. We are curious about its behavior in smooth case. A natural question is whether the set of quasi-periodic cocycles with positive LE is open and dense in C^∞ topology, same as in analytic topology. The problem turns out to be very subtle as Avila [1] already proved, among many other results, that cocycles with positive LE is dense in smooth quasi-periodic cocycles. In this paper, we will prove that, different from analytic case, the set of smooth quasi-periodic cocycles with positive exponent is not open in smooth topology. More precisely, we will construct smooth non-uniformly hyperbolic Schrödinger cocycles which are accumulated by ones with zero LE in $\mathcal{C}^l$ topology ($l = 1, 2, \ldots, \infty$).[1] The following is the main result of this paper.

Theorem 1. *Consider quasi-periodic Schrödinger cocycles over $\mathbb{S}^1$ with ω being a fixed irrational number of bounded-type.*[2] *For any $0 \leq l \leq \infty$, there exists a Schrödinger cocycle S_v with a positive Lyapunov exponent and a sequence of Schrödinger cocycles S_{v_n} with zero Lyapunov exponent such that $v_n(x) \to v(x)$ in C^l topology. As a consequence, the set of quasi-periodic cocycles with positive LE is not open in $C^l, l = 1, 2, \ldots, \infty$.*

Theorem 1 can be obtained from Theorem 2 in the same way as in [50] to obtain examples in Schrödinger cocycles from examples in $SL(2, \mathbb{R})$ cocycles. Thus we only need to prove Theorem 2.

Theorem 2. *Consider quasi-periodic $SL(2, \mathbb{R})$ cocycles over $\mathbb{S}^1$ with ω being a fixed irrational number of bounded-type. For any $0 \leq l \leq \infty$, there exists a cocycle $D_l \in \mathcal{C}^l(\mathbb{S}^1, SL(2, \mathbb{R}))$ with positive Lyapunov exponent and a sequence of cocycles $C_k \in \mathcal{C}^l(\mathbb{S}^1, SL(2, \mathbb{R}))$ with zero Lyapunov exponent such that $C_k \to D_l$ in C^l topology. As a consequence, the set of cocycles with positive Lyapunov exponent is not open in $C^l, l = 1, 2, \ldots, \infty$.*

Remark 1.1. Avila and Krikorian [5] proved that the LE is smooth in the space of smooth monotonic quasi-periodic cocycles not homotopic to the identity. Our result thus shows that LE has a totally different behavior in the space of smooth quasi-periodic Schrödinger cocycles homotopic to the identity in comparison with the space of ones not homotopic to the identity.

This paper is a continuation of [50], where the authors constructed smooth quasi-periodic Schrödinger cocycles, at which the LE is not continuous in smooth topology. Here we further prove that LE can jump down to zero. There are also some results in the other side. For some type of $\mathcal{C}^2$ potentials, Anderson Localization and positivity of LE has been established by Sinai [45] and Fröhlich–Spencer–Wittwer [23], also see Bjerklöv [9]. For the model in [45], Wang and Zhang [51] showed the continuity of the LE, which implies some non-uniform hyperbolic quasi-periodic cocycles can be inner points of smooth quasi-periodic cocycles with positive exponents. Those results together show that the topological structure of the set of quasi-periodic cocycles with positive LE is more complicated in smooth topology comparing with C^0 topology and analytic topology.

The LE of the Schrödingier cocycles encodes enormous information on the spectrum of the corresponding quasi-periodic Schödinger operators. It is known from Kotani theory that the positive LE implies singular spectrum, and typically Anderson localization, see [30,37,44]; while zero Lyapunov spectrum usually implies continuous, typically absolutely continuous spectrum. Positivity of the LE is also a starting point for many

[1] The authors would like to thank Jitomirskaya for drawing their attention to the problem.

[2] Bounded type means $\frac{p_k}{q_k}$, the best approximation of ω, satisfies $q_{k+1} \leq Mq_k$ for some $M > 0$.

other problems in dynamical systems and spectral theory, such as Hölder continuity of LE, topological structure of spectrum, continuity of spectrum. Also the recent developed methods, such as Avalanche Principle and Green's function estimates, etc.(see [15]), depend crucially on the positivity of LE.

A related more interesting question is the robustness of Anderson localization. i.e., if the perturbations of a Schrödinger operator exhibiting Anderson localization still have Anderson localization (assuming that the base dynamics is a rigid Diophantine rotation)? The answer is affirmative in analytic category since the LE is continuous and thus keeps positive under perturbations. However, the problem is completely open in the smooth topology. The result of this paper implies that the positivity of LE is not a robust property in smooth topology, so it is reasonable to guess that Anderson localization is not a robust property in smooth topology. However this problem is widely open.

The proof of Theorem 2 is constructive. Recall in [50], we have constructed a smooth cocycles A with positive LE and a smooth cocycle A_1 in $\frac{1}{2}\delta$-neighborhood of A in C^l topology for any given δ such that the finite LE of A_1, defined by $L_{n_1}(A_1) = \frac{1}{n_1}\int_{\mathbb{S}^1} \log \|A_1^{n_1}(x)\| dx$, is smaller that $(1-\delta_1)L(A)$ for a fixed number $\delta_1 > 0$ independent of δ. As a consequence of subadditivity of finite LE, $L(A_1) < (1-\delta_1)L(A)$. It follows that the LE is discontinuous at A. However, the construction in [50] did not tell us how small $L(A_1)$ can be. In this paper we will further locally modify A_1 such that the modified cocycle, say A_2, satisfies $\|A_2 - A_1\|_{C^l} < \frac{1}{4}\delta$ and $L_{n_2}(A_2) < (1-\delta_1)L_{n_1}(A_1)$. It follows that A_2 is in the δ-neighborhood of A and $L(A_2) < (1-\delta_1)^2 L(A)$. Inductively, we locally modify A_k such that the modified cocycle, say A_{k+1}, satisfies $\|A_{k+1} - A_k\|_{C^l} < \frac{1}{2^k}\delta$ and $L_{n_{k+1}}(A_{k+1}) < (1-\delta_1)L_{n_k}(A_k)$, where $n_k \to \infty$ will be specified later. It follows that all A_k are in the δ-neighborhood of A and $L(A_{k+1}) \leq (1-\delta_1)^k L(A)$. It is easy to see that A_k has a limit, say $\bar{A}$, with $L(\bar{A}) = 0$.

A and $\bar{A}$ we constructed are of the form $\Lambda R_{\phi(x)}$ and $\Lambda R_{\bar{\phi}(x)}$ where $\Lambda = diag\{\lambda, -\lambda\}$, $\lambda \gg 1$ with $L(A) \sim \ln \lambda$ and $L(\bar{A}) = 0$. Moreover, $\bar{\phi}(x)$ is an arbitrarily small modification of $\phi(x)$ in an arbitrarily small neighborhood of two points. So a small change makes a big difference! For Schrödinger cocycles, we actually construct implicitly, for arbitrarily large λ, smooth $v(x)$ and $\bar{v}(x)$ which are slightly different from each other at the neighborhood of two critical points such that $L(S_{\lambda\phi(x)})$ is very big while $L(S_{\lambda\bar{\phi}(x)}) = 0$. The result is surprising as we have even not seen any smooth example of the form $S_{\lambda\bar{\phi}(x)}$ with $\lambda \gg 1$ such that $L(S_{\lambda\bar{\phi}(x)}) = 0$.

From our construction, one can see how and where to modify a cocycle so as to control the LE. This might be useful for other problems.

More historical remarks on the continuity of LE in analytic topology: When the underlying dynamics is a shift or skew-shift of a higher dimensional torus, the log-continuity of LE was proved in [18] by Bourgain, Goldstein and Schlag. Recently, the result of [31] was generalized by Jitomirskaya and Marx [32] for all non-trivial singular analytic quasiperiodic cocycles with one-frequency. With this result, Jitomirskaya and Marx [33] can determine the LE of extended Harper's model.

An arithmetic version of large deviations and inductive scheme were developed by Bourgain and Jitomirskaya in [19] allowing to obtain joint continuity of LE for $\mathrm{SL}(2,\mathbb{C})$ cocycles, in frequency and cocycle map, at any irrational frequencies. This result has been crucial in many important developments, such as the proof of the Ten Martini problem [3], Avila's global theory of one-frequency cocycles [2]. It was extended to multi-frequency case by Bourgain [16] and to general $\mathrm{M}(2,\mathbb{C})$ case by Jitomirskaya and Marx [33]. More recently, a completely different method without LDT or AP was developed by Avila, Jitomirskaya and Sadel [4] and used to prove the continuity of LE for the general case $M(d, \mathbb{C})$, $d \geq 2$. For further works, see [20,21,34,38,52].

In the following, we will use c, C, $C(l)$, etc, to denote some universal positive constants independent of iterative steps.

2. The Construction of D_l

We will not distinguish A and its lift in $\mathbb{R}^1$. In this paper, we consider the case $n = 1$. Clearly, the norm of an $SL(2, R)$-matrix is not less than 1 and the equality holds if and only if it is a rotation matrix. Thus we call an $SL(2, R)$-matrix is hyperbolic if its norm is strictly larger than 1. A quasi-periodic cocycle $(\omega, A(x))$ of degree d is defined by a matrix function $A(x) = R_{\psi(x)} \cdot \Lambda(x) \cdot R_{\phi(x)}$ on $\mathbb{R}^1$, with $\Lambda(x + 1) = \Lambda(x) = diag\{\|A\|, \frac{1}{\|A\|}\}$, $\psi(x + 1) = 2\pi d + \psi(x), \phi(x + 1) = 2\pi d + \phi(x)$ where $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. It is easy to see that $(\phi(x) + \psi(x - \omega))$ is uniquely determined by $A(x)$ up to $2\pi\mathbb{Z}$ and $L(A) = L(\Lambda(x) \cdot R_{\phi(x)+\psi(x-\omega)})$ as A is conjugated to $\Lambda(x) \cdot R_{\phi(x)+\psi(x-\omega)}$. We will construct examples in smooth topology for all degrees.

Let $\Lambda = diag\{\lambda, \frac{1}{\lambda}\}$ with $\lambda \gg 1$. In this section, we will construct a sequence of smooth cocycles B_k of the form $\Lambda \cdot R_{\xi_k(x)}$, converging in C^l such that $L(\lim B_k) > 0$. Moreover $\xi_k(x)$ will be specially designed so that, in the next section, we can further constructed cocycles C_k with zero Lyapunov exponent in any small neighborhood of B_k. When λ is big, we will see that the Lyapunov exponent of B_k crucially depends on the local behavior, more precisely the degeneracy, of $\xi_k(x)$ at the critical points $\{c : \xi_k(c) = \frac{\pi}{2} \pmod{\pi}\}$ due to the cancellation.

Let ω be a fixed irrational number and $\frac{p_k}{q_k}$ be its best approximation. Throughout the paper, we assume that ω is of the bounded type, i.e., $q_{k+1} \leq Mq_k$; $\epsilon > 0$ is small. l is a fixed positive integer reflecting the smoothness of cocycles. Let λ and N are large enough so that

$$\lambda^{-1} \ll q_N^{-2}, \quad 10l \sum_{k=N}^{\infty} \frac{\log q_{k+1}}{q_k} \leq \epsilon. \tag{2.1}$$

We define the decaying sequence $\{\lambda_k\}$ inductively by $\log \lambda_k = \log \lambda_{k-1} - \frac{10l \log q_k}{q_{k-1}}$ where $\lambda_N = \lambda \gg 1$. It is easy to see that λ_k converges to λ_∞ with $\lambda_\infty > \lambda^{1-\epsilon}$.

For $k \geq N$, we define $\mathcal{C}_0 = \{0, \frac{1}{2}\}$, $I_{k,1} = [-\frac{1}{q_k^2}, \frac{1}{q_k^2}]$, $I_{k,2} = [\frac{1}{2} - \frac{1}{q_k^2}, \frac{1}{2} + \frac{1}{q_k^2}]$ and $I_k = I_{k,1} \bigcup I_{k,2}$. For $C \geq 1$, we denote by $\frac{I_{k,1}}{C} = [-\frac{1}{Cq_k^2}, \frac{1}{Cq_k^2}]$, $\frac{I_{k,2}}{C} = [\frac{1}{2} - \frac{1}{Cq_k^2}, \frac{1}{2} + \frac{1}{Cq_k^2}]$, and by $\frac{I_k}{C}$ the set $\frac{I_{k,1}}{C} \cup \frac{I_{k,2}}{C}$. Denote Lebesgue measure of I_k by $|I_k|$. For each $x \in I_k$, let $r_k^+(x)$ (respectively $r_k^-(x)$) be the smallest positive integer such that $T^{r_k^+(x)}(x) \in I_k$ (respectively $T^{-r_k^-(x)}(x) \in I_k$). Let $r_k^\pm = \min_{x \in I_k} r_k^\pm(x)$ and $r_k = \min\{r_k^+, r_k^-\}$. Obviously, $r_k \geq q_k$. Moreover, from the symmetry between $I_{k,1}$ and $I_{k,2}$, we have $r_k = r_k^+ = r_k^-$.

We define ξ_0 on $I = I_1 \bigcup I_2 = \{x : |x| \leq \frac{1}{2q_N^2}\} \bigcup \{x : |x - \frac{1}{2}| \leq \frac{1}{2q_N^2}\}$ by

$$\xi_0(x) = \begin{cases} \xi_{01}(x), & |x| \leq \frac{1}{2q_N^2}; \\ -\xi_{02}(x) \text{ (or } \xi_{02}(x)), & |x - \frac{1}{2}| \leq \frac{1}{2q_N^2} \end{cases} \tag{2.2}$$

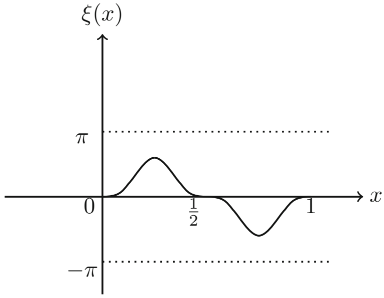

Fig. 1. Homotopic to identity

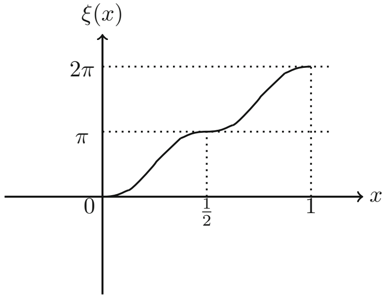

Fig. 2. Nonhomotopic to identity

where

$$\xi_{01}(x) = \mathrm{sgn}(x)|x|^{l+1}, \quad \xi_{02}(x) = \mathrm{sgn}(x - \frac{1}{2})|x - \frac{1}{2}|^{l+1}. \tag{2.3}$$

$\xi(x)$ is a lift of a 1-periodic C^l function satisfying

$$\xi(x) = \begin{cases} \xi_{01}(x), & |x| \le \frac{1}{2q_N^2}; \\ -\xi_{02}(x) \text{ (or } \pi + \xi_{02}(x)), & |x - \frac{1}{2}| \le \frac{1}{2q_N^2}, \end{cases} \tag{2.4}$$

and $|\xi(x)(\mathrm{mod}\ \pi)| > \frac{1}{2q_N^2}$ for any $x(\mathrm{mod}\ 1) \notin I$. The picture of the function ξ is as in Figs. 1 and 2.

Let $\xi_N(x) = \xi(x)$ defined above. Define $B_N(x) = \Lambda R_{\frac{\pi}{2} - \xi_N(x)}$. The following result on concatenation of non-rotating blocks is a generalization of Proposition 3.1 in [50].

Proposition 2.1. *There are C^l functions $\xi_k(x)$ $(k = N + 1, N + 2, \ldots)$ constructed inductively such that*

1. $|\xi_k(x) - \xi_{k-1}(x)|_{C^l} \le C(l) \cdot \lambda_k^{-2r_k} \cdot |I_k|^{-l^2}, \quad \text{if } k > N.$ (2.5)

2. *Let $B_k = \Lambda R_{\frac{\pi}{2} - \xi_k(x)}$. For each $x \in I_k$, we have*

$$\|B_k^{r_k^{\pm}(x)}(x)\| \ge \lambda_k^{r_k^{\pm}(x)}. \tag{2.6}$$

3. Let

$$B_k^{r_k^+}(x) = R_{\psi_{B_k,r_k^+}(x)} \cdot \Lambda_{B_k,r_k^+}(x) \cdot R_{\phi_{B_k,r_k^+}(x)},$$
$$B_k^{r_k^-}(T^{-r_k^-}x) = R_{\psi_{B_k,-r_k^-}(x)} \cdot \Lambda_{B_k,-r_k^-}(x) \cdot R_{\phi_{B_k,-r_k^-}(x)}.$$

Then for $x \in I_k$, we have

$$(1)_{\mathrm{k}} \quad \psi_{B_k,-r_k^-}(x) + \phi_{B_k,r_k^+}(x) - \frac{\pi}{2} = \xi_0(x) \quad \text{on } \frac{I_k}{10};$$

$$(2)_{\mathrm{k}} \quad |\psi_{B_k,-r_k^-}(x) + \phi_{B_k,r_k^+}(x) - \frac{\pi}{2}| \geq \frac{1}{(20q_k^2)^{l+1}}, \quad x \in I_k \backslash \frac{I_k}{10},$$

where $\xi_0(x)$ is defined in (2.2) *and* (2.3).

Remark 2.1. It is easy to see from (2.5) that B_k converges to a limit D_l in $\mathcal{C}^l$-topology. Moreover, from (2.5) and (2.6) as well as Theorem 3 in [50], we can show $L(D_l) \geq (1-\epsilon)\ln\lambda$.

We first describe the idea for the proof of Proposition 2.1, which is similar to the one for Proposition 3.1 in [50]. Under a non-degenerate condition, Young [53] obtained a positive lower bound for the Lyapunov exponent. In contrast, To find a series of cocycles with zero Lyapunov exponent which converges to a cocycle, some degeneracy on the limit cocycle is necessary. Thus in order to use Young's method to obtain the positivity of the Lyapunov exponent of the limit cocycle, we need a higher-order non-degenerate condition (see $(1)_{\mathrm{k}}$, $(2)_{\mathrm{k}}$ and the definition of the function ξ_0) on the limit cocycle as well as some modification of Young's method as follows. After each iteration step, the original higher-order non-degenerate condition is destroyed due to small perturbation. Thus we need to modify the definition of the cocycle a little such that the non-degenerate condition in $(1)_{\mathrm{k}}$ is recovered (see the definition of $\hat{f}_k$ and the construction of f_k in the proof of Proposition 2.1). Moreover the modification should be small enough to ensure the convergence of the series of cocycles, which is estimated by (2.5). Then similar to [53], we obtain (2.6).

For any cocycle $A(x)$, $n \in Z^+$ and $x \in I$, we decompose $A^n(x)$ as $R_{\psi_{A,n}(x)} \cdot \Lambda_{A,n}(x) \cdot R_{\phi_{A,n}(x)}$ when $A^n(x)$ is non-rotating in I, where $\Lambda_{A,n}(x) \in SL(2,R)$ is a diagonal matrix satisfying $\|A^n(x)\| = \|\Lambda_{A,n}(x)\|$ and $R_{\psi_{A,n}(x)}$ and $R_{\phi_{A,n}(x)}$ are two rotation matrix with $\psi_{A,n}$ and $\phi_{A,n}$ two angle functions. We make a similar decomposition for $A^n(T^{-n}x)$ as $R_{\psi_{A,-n}(x)} \cdot \Lambda_{A,-n}(x) \cdot R_{\phi_{A,-n}(x)}$ when $A^n(T^{-n}x)$ is non-rotating in I.

To prove Proposition 2.1, we first give the following Lemma 2.1 to estimate derivatives of angles and norms for the product of non-rotating blocks. It shows that the quotient of derivative of the norm by the norm is much smaller than the norm, while the quotient of derivative of angle by the norm is very small.

Lemma 2.1. *For any function $\sigma(x)$ defined on S^1, denote $d_k(\sigma) = \min_{x \notin I_k}\{|\sigma(x)|\}$. Assume that for any $x \in I_k$,*

$$\log\|A^{r_k}(x)\| \gg -\log d_{k+1}, \tag{2.7}$$

where $d_{k+1} = d_{k+1}(\phi_{A,r_k^+}(x) + \psi_{A,-r_k^-}(x) - \frac{\pi}{2})$*. Furthermore assume that, for* $i \le l$ *and* $m^{\pm} = r_k^{\pm}(x)$*,*

$$\begin{cases} \left|\frac{d^i}{dx^i}\phi_{A,m^+}(x)\right|, \quad \left|\frac{d^i}{dx^i}\psi_{A,-m^-}(x)\right| \le C(i)\cdot d_{k+1}^{-i} & (1)_k \\ \left|\frac{d^i\|A^{\pm m}(x)\|}{dx^i}\right|\cdot\|A^{\pm m}(x)\|^{-1} \le C(i)\cdot d_{k+1}^{-i}. & (2)_k \end{cases}$$

Then for $i \le l$*,* $x \in I_{k+1}$ *and* $\hat{m}^{\pm} = r_{k+1}^{\pm}(x)$ *it holds that*

$$\begin{cases} \left|\frac{d^i}{dx^i}\phi_{A,\hat{m}^+}(x)\right|, \quad \left|\frac{d^i}{dx^i}\psi_{A,-\hat{m}^-}(x)\right| \le C(i)\cdot d_{k+1}^{-i}, & (1)_{k+1} \\ \left|\frac{d^i\|A^{\pm \hat{m}}(x)\|}{dx^i}\right|\cdot\|A^{\pm \hat{m}}(x)\|^{-1} \le C(i)\cdot d_{k+1}^{-i}. & (2)_{k+1} \end{cases}$$

Moreover, for any $i \ge 0$*,* $x \in I_{k+1}$*, it holds that*

$$\begin{aligned} &\left|\frac{d^i}{dx^i}(\phi_{A,r_{k+1}^+}(x) - \phi_{A,r_k^+}(x))\right| \le C(i)\cdot L_{A,r_k^+}^{-2}\cdot d_k^{-i}, \\ &\left|\frac{d^i}{dx^i}(\psi_{A,-r_{k+1}^-}(x) - \psi_{A,-r_k^-}(x))\right| \le C(i)\cdot L_{A,r_k^-}^{-2}\cdot d_k^{-i}. \end{aligned} \tag{2.8}$$

The proof of Lemma 2.1 will be given in the Appendix.

Proof of Proposition 2.1:
For each $k \ge N$ and $x \in I_k$, let $\phi_{B_{k-1},r_k}(x)$ and $\psi_{B_{k-1},r_k}(x)$ correspond to $B_{k-1}^{r_k}(x)$. Since usually $\hat{f}_k(x) := (\psi_{B_{k-1},-r_k^-}(x) + \phi_{B_{k-1},r_k^+}(x)) - (\psi_{B_{k-1},-r_{k-1}^-}(x) + \phi_{B_{k-1},r_{k-1}^+}(x)) \ne 0$ and $\psi_{B_{k-1},-r_{k-1}^-}(x) + \phi_{B_{k-1},r_{k-1}^+}(x) - \frac{\pi}{2} = \xi_0(x)$ on I_{k-1}, we have that usually $\psi_{B_{k-1},-r_k^-}(x) + \phi_{B_{k-1},r_k^+}(x) - \frac{\pi}{2} \ne \xi_0(x)$ on I_k. To satisfy $(1)_k$ in Proposition 2.1, we modify $\xi_{k-1}(x)$ into a new $\xi_k(x)$ on I_k by defining $\xi_k(x) = \xi_{k-1}(x) + f_k(x)$, where $f_k(x) \in \mathcal{C}^l$ is the following 1-periodic function:

$$f_k(x) = \begin{cases} \hat{f}_k(x) & x \in \frac{I_k}{10} \\ h_k^{\pm}(x), & x \in I_k \backslash \frac{I_k}{10} \\ 0, & x \in \mathbb{S}^1 \backslash I_k \end{cases}$$

where $h_k^{\pm}(x)$ is a polynomial of degree $2l+1$ restricted in each interval of $I_k \backslash \frac{I_k}{10}$ satisfying

$$\begin{aligned} \frac{d^j h_k^{\pm}}{dx^j}(\pm\frac{1}{10q_k^2}) &= \frac{d^j \hat{f}_k}{dx^j}(\pm\frac{1}{10q_k^2}) \\ \frac{d^j h_k^{\pm}}{dx^j}(\pm\frac{1}{q_k^2}) &= 0, \quad i = 1, 2, \quad 0 \le j \le l. \end{aligned}$$

From (2.8) in Lemma 2.1, we have that

$$|(\psi_{B_{k-1},-r_k^-}(x) + \phi_{B_{k-1},r_k^+}(x)) - (\psi_{B_{k-1},-r_{k-1}^-}(x) + \phi_{B_{k-1},r_{k-1}^+}(x))|_{C^l} \le C(l)\cdot\lambda_k^{-2r_k}\cdot|I_k|^{-l^2}, \tag{2.9}$$

where (2.7) is fulfilled by conclusion 2 and 3 of the induction assumption for the case $k-1$.

Thus from the definition of $f_k(x)$ we obtain

$$|f_k|_{C^l} \le C(l) \cdot \lambda_k^{-2r_k} \cdot |I_k|^{-l^2}. \tag{2.10}$$

Let $B_k(x) = \Lambda \cdot R_{\frac{\pi}{2}-\xi_k(x)}$, then we have

Lemma 2.2. *For $x \in I_k$, it holds that*

$$B_k^{r_k^+(x)}(x) = B_{k-1}^{r_k^+(x)}(x) \cdot R_{-f_k(x)}$$

and

$$B_k^{r_k^-(x)}(T^{-r_k^-(x)}x) = B_{k-1}^{r_k^-(x)}(T^{-r_k^-(x)}x).$$

Proof. Obviously $T^i x \in \mathbb{S}^1 \backslash I_k$ for $x \in I_k$ and $1 \le i \le r_k^+(x) - 1$. Since $B_k(x) = B_{k-1}(x)$ for $x \in \mathbb{S}^1 \backslash I_k$, we have that

$$B_k^{r_k^+(x)}(x) = B_{k-1}^{r_k^+(x)}(x) \cdot (B_{k-1}^{-1}(x) B_k(x)), \quad x \in I_k.$$

From the definition, we have $B_k(x) = B_{k-1}(x) \cdot R_{\xi_{k-1}(x)-\xi_k(x)}$, which implies $B_{k-1}^{-1}(x) B_k(x) = R_{\xi_{k-1}(x)-\xi_k(x)}$. Thus we obtain the first equation in Lemma 2.2. Similarly we can prove the second. □

Lemma 2.3. *It holds that*

$$f_k(x) = (\psi_{B_{k-1},-r_k^-}(x) + \phi_{B_{k-1},r_k^+}(x)) - (\psi_{B_k,-r_k^-}(x) + \phi_{B_k,r_k^+}(x)), \quad x \in I_k.$$

Proof. Since a rotation does not change the norm of a vector, for a non-rotating matrix A and a rotation matrix R_θ, we have

$$\phi_{A \cdot R_\theta} = \phi_A + \theta. \tag{2.11}$$

From Lemma 2.2, we have

$$\phi_{B_k,r_k^+}(x) = \phi_{B_{k-1},r_k^+}(x) - f_k(x), \quad \psi_{B_k,-r_k^-}(x) = \psi_{B_{k-1},-r_k^-}(x).$$

Thus

$$f_k(x) = (\psi_{B_{k-1},-r_k^-}(x) + \phi_{B_{k-1},r_k^+}(x)) - (\psi_{B_k,-r_k^-}(x) + \phi_{B_k,r_k^+}(x)), \quad x \in I_k,$$

which concludes the proof. □

Proof of $(1)_k$ *and* $(2)_k$. From the definition of $f_k(x)$, we have $f_k(x) = (\psi_{B_{k-1},-r_k^-}(x) + \phi_{B_{k-1},r_k^+}(x)) - (\psi_{B_{k-1},-r_{k-1}^-}(x) + \phi_{B_{k-1},r_{k-1}^+}(x))$ on $\frac{I_k}{10}$, which together with Lemma 2.3 implies that for each $x \in \frac{I_k}{10}$, $\psi_{B_k,-r_k^-}(x) + \phi_{B_k,r_k^+}(x) = (\psi_{B_{k-1},-r_k^-}(x) + \phi_{B_{k-1},r_k^+}(x)) - f_k(x) = \psi_{B_{k-1},-r_{k-1}^-}(x) + \phi_{B_{k-1},r_{k-1}^+}(x)$. Since $\psi_{B_{k-1},-r_{k-1}^-}(x) + \phi_{B_{k-1},r_{k-1}^+}(x) = \xi_0(x)$ on $\frac{I_{k-1}}{10}$ by induction assumption $(1)_{k-1}$, we obtain $(1)_k$ in proposition 2.1.

Obviously $\lambda_k^{q_{k-1}} \gg q_k^{2l}$. Hence $(2)_\mathrm{k}$ in Proposition 2.1 can be obtained from the induction assumption $(2)_{k-1}$ for $|\psi_{B_{k-1},-r_{k-1}^-}(x) + \phi_{B_{k-1},r_{k-1}^+}(x) - \frac{\pi}{2}|$ on I_{k-1} and (2.10).

Proof of conclusion 1. Conclusion 1 can be obtained from (2.9).

Proof of conclusion 2. For $x \in I_k$, let $i_1(x) < i_2(x) < \ldots < i_{j(x)}(x) \le r_k$ be the returning times of I_{k-1} not larger than r_k. Since $|I_k| \le \frac{1}{4}|I_{k-1}|$ (we can make a slight modification of the definition of I_k if necessary), from the symmetry between $I_{k,1}$ and $I_{k,2}$, we have that for any $x \in I_k$, we have $T^{r_k}x \in I_{k-1}$. Then we have that $i_{j(x)}(x) = r_k$. Since $T^{i_s(x)}x \notin I_k$ for $s < j(x)$, $|\theta_s - \frac{\pi}{2}| \ge \frac{1}{q_k^{2l}}$, where $\theta_s = \phi_{B_k, i_{s+1}(x)-i_s(x)}(T^{i_s(x)}x) + \psi_{B_k, i_s(x)-i_{s-1}(x)}(T^{i_{s-1}(x)}x)$. Together with conclusion 3 of the induction assumption for $(k-1)$-th step we have that $|\tilde{\theta}_s - \frac{\pi}{2}| \ge \frac{1}{2q_k^{2l}}$, where $\tilde{\theta}_s = \phi_{B_k, i_{s+1}(x)-i_s(x)}(T^{i_s(x)}x) + \psi_{B_k, i_s(x)}(x)$. Thus from the definition of λ_k, we obtain conclusion 2 for k-th step by repeated applications of Lemma A.1.

3. The Construction of $C_k(x)$

Now we start to construct a C_k in any small $\mathcal{C}^l$-neighborhood of B_k such that $L(C_k) = 0$. It is obvious that $C_k \to D_l$ in $\mathcal{C}^l$ topology. C_k will be constructed as limit of a sequence of converging cocycles, say $A_{k,i}$, in any small neighborhood of B_k such that $L(A_{k,i}) \to 0$ as $i \to \infty$. By the construction, we can show that $L(C_k) \le \lim_{i\to\infty} L(A_{k,i}) = 0$, see Corollary 3.1. In the following, we shall simply denote $A_{k,i}$ by A_i.

The following lemma is of key importance for the construction:

Iterative Lemma: *Let* $A_0(x) = \Lambda \cdot R_{\frac{\pi}{2}-\theta_0(x)}$ *satisfy that* $\|A_0^{r_{n_0}(x)}(x)\| \ge \mu^{r_{n_0}(x)}$ *with* $\lambda \ge \mu \gg 1$ *and* $\psi_{A_0,-r_{n_0}}(x) + \phi_{A_0,r_{n_0}}(x) - \frac{\pi}{2} = \xi_0(x)$, $x \in I_{n_0}$. *Then we can find two small positive numbers* $\delta_1 > \delta_2$ *such that for any* $i \ge 0$, *there exist* $A_i(x) = \Lambda \cdot R_{\frac{\pi}{2}-\theta_i(x)}$ *and an unbounded sequence of integers* $\{r_{n_i}\}_{i=1}^{\infty}$ *with* $r_{n_i} \ge q_{n_i}$, *such that the following properties hold true:*

$$
(\mathrm{P_i}) : \begin{cases}
(1).\ |\theta_{i+1} - \theta_i|_{\mathcal{C}^l} \le q_{n_i}^{4Ml^2} \cdot \mu^{-\frac{1}{2}(1-\delta_1)^i \cdot q_{n_i}} + q_{n_i}^{-2}; \\
(2).\ \|A_i^{r_{n_j}(x)}(x)\| \le \lambda^{(1-\delta_2)^j \cdot r_{n_j}(x)} \text{ for } x \in I_{n_j} \text{ and } j \le i; \\
(3).\ \psi_{A_i,-r_{n_i}}(x) + \phi_{A_i,r_{n_i}}(x) - \frac{\pi}{2} = \xi_0(x) \text{ on } I_{n_i}; \\
(4).\ \|A_i^{r_{n_i}(x)}(x)\| \ge \mu^{(1-\delta_1)^i \cdot r_{n_i}(x)} \text{ on } I_{n_i} \text{ and } \mu^{(1-\delta_1)^i \cdot q_{n_i}} \gg \frac{1}{|I_{n_i}|}; \\
(5).\ \bar{\mu}_{n_i} \le \underline{\mu}_{n_i}^{1+w_i} \text{ with } 0 \le w_i \le \frac{C}{\log \lambda} \text{ and } w_i \to 0 \text{ as } i \to \infty.
\end{cases}
$$

In the above, $\bar{\mu}_{n_i} = \max_{x\in I_{n_i}} \|A_i^{r_{n_i}(x)}(x)\|^{\frac{1}{r_{n_i}(x)}}$ *and* $\underline{\mu}_{n_i} = \min_{x\in I_{n_i}} \|A_i^{r_{n_i}(x)}(x)\|^{\frac{1}{r_{n_i}(x)}}$. *Therefore,* $\underline{\mu}_{n_i} \ge \mu^{(1-\delta_1)^i}$ *and* $\bar{\mu}_{n_i} \le \lambda^{(1-\delta_2)^i}$.

Remark 3.1. In the above lemma, (1) shows the convergence of cocycles and upper bound estimate (2) can imply that the Lyapunov of the limit cocycle is zero. (3) played a similar role as the one played by $(1)_\mathrm{k}$ in Proposition 2.1, i.e., it ensures some higher-order non-degenerate condition. The realization of (1) and (2) is based on the condition that $\|A_i^{r_{n_i}(x)}(x)\|$ is large enough. Thus we need a lower bound estimates as in (4). To prove (1)–(4), we need (5) to show that the growth of $\|A_i^{r_{n_i}(x)}(x)\|$ is uniformly for all $x \in I_{n_i}$.

The main result Theorem 2 is an easy consequence of the following corollary.

Corollary 3.1. *There exists a $SL(2,\mathbb{R})$-sequence $\{C_k\}_{k=N}^{\infty}$ such that C_k has the limit D_l in $\mathcal{C}^l$-topology with $L(D_l) \geq (1-\epsilon)\ln\lambda$ and $L(C_k) = 0$ for each k.*

Proof. For any $k \in \mathbb{N}$, we apply Iterative Lemma by setting $A_0 = B_k$, $n_0 = q_k$ and $\mu = \lambda^{1-\epsilon}$ where B_k is defined in Proposition 2.1. Hence for each i we obtain A_i such that ($\mathrm{P_i}$) holds true. By (1) of ($\mathrm{P_i}$), A_i has a limit, say C_k, in $\mathcal{C}^l$-topology. From the second inequality in (2) of ($\mathrm{P_i}$), as $i \to \infty$, we obtain $\|C_k^{r_{n_j}(x)}(x)\|^{\frac{1}{r_{n_j}(x)}} \leq \lambda^{(1-\delta_2)^j}$ for any $j \leq i$ and $x \in I_{n_j}$. By the subadditivity of Lyapunov exponent and the definition of $r_i(x)$, it implies that for every x in the base space, it holds that $\liminf_{n\to\infty} \frac{1}{n}\log\|C_k^n(x)\| \leq (1-\delta_2)^j \log\lambda$, see also the argument in [50]. Hence we have $L(C_k) \leq (1-\delta_2)^j \log\lambda$ for any j. Let $j \to \infty$, we obtain $L(C_k) = 0$. Moreover, from (1) of ($\mathrm{P_i}$) it holds that

$$\begin{aligned}\|C_k - D_l\|_{\mathcal{C}^l} &\leq \|C_k - B_k\|_{\mathcal{C}^l} + \|B_k - D_l\|_{\mathcal{C}^l}\\ &= \|C_k - A_0\|_{\mathcal{C}^l} + \|B_k - D_l\|_{\mathcal{C}^l}\\ &\leq 2q_k^{4M^2 l}\cdot\lambda^{-(1-\epsilon)(1-\delta_1)\cdot q_k} + q_k^{-2} + \|B_k - D_l\|_{\mathcal{C}^l},\end{aligned}$$

which, with the help of the inequality $\mu^{(1-\delta_1)^i\cdot q_{n_i}} \gg \frac{1}{|I_{n_i}|}$ in (4) of ($\mathrm{P_i}$), implies $C_k \to D_l$ in $\mathcal{C}^l$-topology as $k \to \infty$. On the other hand, Proposition 2.1 says that $L(D_l) \geq (1-\epsilon)\ln\lambda$. □

Roughly speaking, the idea for the proof of Iterative Lemma can be described as follows. Consider two non-rotating blocks $A_i(x) = R_{\psi_i(x)}\Lambda_i(x)R_{\phi_i(x)}$ with $\|A_i(x)\| \gg 1$, $i = 1, 2$ for all x in some interval. Then the difference between $\|A_2(x)A_1(x)\|$ and $\|A_2(x)\|\cdot\|A_1(x)\|$ is determined by $\phi_2(x)+\psi_1(x)-\frac{\pi}{2}$. Now we add some small perturbation on $\phi_2(x)+\psi_1(x)-\frac{\pi}{2}$ to reduce $\|A_2(x)\cdot A_1(x)\|$. Obviously, the fastest way to reduce it is to change $\phi_2(x)+\psi_1(x)-\frac{\pi}{2}$ into zero for all x, since then $\|A_2(x)\cdot A_1(x)\|$ will be much smaller than $\|A_2(x)\|\cdot\|A_1(x)\|$. With such a modification, however, we lose the control on the lower bound of $\|A_2(x)\cdot A_1(x)\|$, while a large norm $\|A_2(x)A_1(x)\|$ is necessary for us to continue the reduction of the Lyapunov exponent in later iterations. Thus instead, we will modify $\phi_2(x)+\psi_1(x)-\frac{\pi}{2}$ to equal some small $\epsilon > 0$, see Fig. 3. With a suitable ϵ, on one hand we can reduce the Lyapunov exponent $\frac{1}{n_1+n_2}\log\|A_2(x)A_1(x)\|$ remarkably, where n_i is the length of the block A_i, $i = 1, 2$; on the other hand a large lower bound for the norm $\|A_2(x)A_1(x)\|$ is still available.

Proof of Iterative Lemma. Let $0 < \delta_0 \ll \frac{1}{l^2}$ be a fixed number and $\delta_1 = 8\delta_0\cdot l$, $\delta_2 = M^{-k_1}\cdot\delta_0 l$ with k_1 defined in Proposition 3.1. Obviously, $M^{k_1} > 8$.

When $i = N$, ($\mathrm{P_i}$) obviously holds true for A_N with $\lambda \gg N$. Assuming that $A_N, \ldots, A_{i-1}$ have been constructed with ($\mathrm{P_N}$), …, ($\mathrm{P_{i-1}}$), we will construct A_i such that ($\mathrm{P_i}$) holds. From (5) of (P_{i-1}), we have $\|A_{i-1}^{r_{n_{i-1}}(x)}(x)\| \leq \|A_{i-1}^{r_{n_{i-1}}(y)}(y)\|^{1+w_{i-1}}$ for $x, y \in I_{n_{i-1}}$ provided that n_{i-1} is sufficiently large such that $\frac{\log q_{n_{i-1}+1}}{q_{n_{i-1}}} \ll 1$.

Let $n_i \gg n_{i-1}$ such that $\mu^{(1-\delta_1)^i\cdot q_{n_i}} \gg q_{n_i}^2 \gg \lambda^{2\delta_0\cdot r_{n_{i-1}}}$, hence the second part of (4) of (P_i) holds true. It is worthy to note that one purpose to do so is to ensure that the norm of non-rotating block $\|A_i^{r_{n_i}(x)}(x)\| \gg 1$ although the corresponding finite Lyapunov exponent maybe very small. Then the Diophantine condition implies that $r_{n_i} \geq q_{n_i}$. Thus I_{n_i} can be defined as before. We will determine n_i later, see step 3 as below.

Next we construct a modification of A_{i-1}. For our purpose, we first make a local modification for A_{i-1} on $I_{n_{i-1}}$ such that there is a low platform in the image of $\phi_{\tilde{A}_{i-1},r_{n_{i-1}}}(x)+\psi_{\tilde{A}_{i-1},-r_{n_{i-1}}}(x)-\frac{\pi}{2}$ (see Fig. 3) for the new cocycle $\tilde{A}_{i-1}$, which is critical to reduce the Lyapunov exponent when keeping the norm large.

We denote the sub-interval $[0,\frac{1}{q^2_{n_{i-1}}}]$ of $I_{n_{i-1}}$ by $[a,b]$. Define $a<c<\tilde{c}<d<b$ such that $|ac|=\underline{\mu}_{i,0}^{-2\delta_0\cdot r_{n_{i-1}}}$, $|a\tilde{c}|=M^2\cdot|ac|$, $d=\frac{a+b}{2}$, see Fig. 3. From the definition of n_i, we have $|I_{n_i}|<|ac|$.

Define

$$e_i(x)=\begin{cases}2|ac|^{l+1}-(\phi_{A_{i-1},r_{n_{i-1}}}(x)+\psi_{A_{i-1},-r_{n_{i-1}}}(x)), & x\in[\tilde{c},d];\\ 0, & x\in[a,c]\text{ or equals } b;\\ \tilde{h}_i(x) & x\in[c,\tilde{c}]\text{ or }[d,b],\end{cases}$$

where $\tilde{h}_i(x)$ are polynomials of degree $2l+1$ restricted on each interval and for $0\le j\le l$ satisfies

$$\frac{d^j\tilde{h}_i}{dx^j}(b)=0,\quad \frac{d^j\tilde{h}_i}{dx^j}(d)=\frac{d^j(2|ac|^{l+1}-(\phi_{A_{i-1},r_{n_{i-1}}}+\psi_{A_{i-1},-r_{n_{i-1}}}))}{dx^j}(d),$$

$$\frac{d^j\tilde{h}_i}{dx^j}(c)=0,\quad \frac{d^j\tilde{h}_i}{dx^j}(\tilde{c})=\frac{d^j(2|ac|^{l+1}-(\phi_{A_{i-1},r_{n_{i-1}}}+\psi_{A_{i-1},-r_{n_{i-1}}}))}{dx^j}(\tilde{c}).$$

We have the following estimates:

Lemma 3.1. *It holds that* $|e_i(x)|_{C^l}\le C\cdot q^{-2}_{n_{i-1}}$.

Proof. From (3) in ($\mathrm{P_{i-1}}$) and $(1)_{n_i-1}$ in Proposition 2.1, it holds for $0\le j\le l$ that $|(2|ac|^{l+1}-(\phi_{A_{i-1},r_{n_{i-1}}}+\psi_{A_{i-1},-r_{n_{i-1}}}))(x)|_{C^j}\le C\cdot q_{n_{i-1}}^{-2(l+1-j)}$. Hence from Cramer's rule we have that $|\tilde{h}_i(x)|_{C^l}\le C\cdot q^{-2}_{n_{i-1}}$. Consequently, $|e_i(x)|_{C^l}\le C\cdot q^{-2}_{n_{i-1}}$. □

We can make the definition of $e_i(x)$ on other subintervals of $I_{n_{i-1}}$ in a similar way. Let $\tilde{\theta}_i=\theta_{i-1}+e_i(x)$. Thus for $\tilde{A}_{i-1}=\Lambda\cdot R_{\frac{\pi}{2}-\tilde{\theta}_i}$, $\psi_{\tilde{A}_{i-1},-r_{n_{i-1}}}(x)+\phi_{\tilde{A}_{i-1},r_{n_{i-1}}}(x)-\frac{\pi}{2}$ on $I_{n_{i-1}}$ is of the shape as in the Fig. 3.

In the following, we will do some routine modifications on $\tilde{A}_{i-1}$ as in the proof of Proposition 2.1 such that the process of iterations can go forward until at the step n_i we obtain a cocycle A_i satisfying (1)–(5) of ($\mathrm{P_i}$). Note that the modification defined by $e_i(x)$ is the only one which is essential in the proof of Iteration Lemma.

Step 1. The construction of A_i and proof of (1), (3) and (4) of ($\mathrm{P_i}$).

Lemma 3.2. *Let $A_{i,0}=\Lambda\cdot R_{\tilde{\theta}_i}:=\Lambda\cdot R_{\theta_{i,0}}$ satisfy $\|A_{i,0}^{r_{n_{i-1}}(x)}(x)\|\ge\nu_0^{r_{n_{i-1}}(x)}$ for $x\in I_{n_i}$ with $\nu_0=\mu^{(1-\delta_1)^{i-1}}$. Then for any $n_i-n_{i-1}\ge j\ge1$, there exist $\theta_{i,j}$ and $A_{i,j}=\Lambda\cdot R_{\frac{\pi}{2}-\theta_{i,j}}$ such that the following properties hold true:*

$$(\tilde{\mathrm{P}}_{i,j}):\begin{cases}(\tilde{1}).\ \|A_{i,j}^{r_{n_{i-1}+j}(x)}(x)\|\ge\nu_j^{r_{n_{i-1}+j}(x)}\text{ on } I_{n_i};\\ (\tilde{2}).\ \phi_{A_{i,j},r_{n_{i-1}+j}}(x)+\psi_{A_{i,j},-r_{n_{i-1}+j}}(x)-\frac{\pi}{2}=\theta_{i,0}(x)\text{ on } I_{n_i};\\ (\tilde{3}).\ |\theta_{i,j}-\theta_{i,j-1}|_{C^l}\le r_{i,j}\cdot\nu_{j-1}^{-q_{n_{i-1}+j-1}},\ r_{i,j}\approx\max\{\nu_{j-1}^{2l^2\delta_0q_{n_{i-1}}},q^{2l^2}_{n_{i-1}+j}\},\end{cases}$$

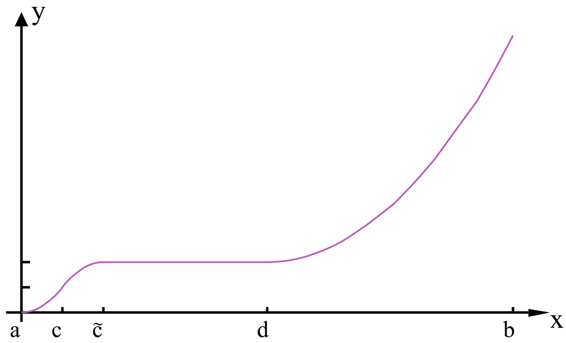

Fig. 3. The image of $\phi_{\tilde{A}_{i-1}, r_{n_{i-1}}}(x) + \psi_{\tilde{A}_{i-1}, -r_{n_{i-1}}}(x) - \frac{\pi}{2}$

where ν_j are iteratively defined by

$$\nu_j = \nu_0 \cdot \nu_0^{-\delta_0(\sqrt{2}^{-(j-1)}+\ldots+\sqrt{2}^{-1}+1)\cdot 2(l+1)} \geq \nu_0^{(1-8\delta_0\cdot(l+1))} = \mu^{(1-\delta_1)^i}.$$

Let $\underline{\mu}_{i,j} = \min_{x\in I_{n_i}} \|(A_{i,j}^{r_{n_{i-1}+j}}(x))\|^{\frac{1}{r_{n_{i-1}+j}}}$ for any $j \leq n_i - n_{i-1}$. Thus we have $\underline{\mu}_{i,j} \geq \nu_j$ and $\underline{\mu}_{n_i} = \underline{\mu}_{i,n_i-n_{i-1}} \geq \nu_{n_i-n_{i-1}}$.

Proof. For $j = 1$, from (4) of (P_{i-1}) and the definition of $\tilde{\theta}_i$ and μ_0, we have

$$\frac{1}{r_{n_{i-1}+1}(x)} \log \|A_{i,1}^{r_{n_{i-1}+1}(x)}(x)\| \geq \log \underline{\mu}_{i,n_{i-1}} + \frac{1}{q_{n_{i-1}+1}} \log \underline{\mu}_{i,n_{i-1}}^{-2(l+1)\delta_0 q_{n_{i-1}}\cdot\frac{q_{n_{i-1}+1}}{q_{n_{i-1}}}}$$
$$= (1-2(l+1)\delta_0)\log \underline{\mu}_{i,n_{i-1}} \geq \nu_1.$$

Thus we obtain $(\tilde{1})$. Moreover $(\tilde{2})$ and $(\tilde{3})$ can be proved by Proposition 2.1 and Lemma 2.1 with $d_{n_{i-1}+1} \geq \frac{1}{q_{n_{i-1}+1}^{2(l+1)}}$.

Assume $(\tilde{P}_{i,j})$ hold true. We will prove $(\tilde{P}_{i,j+1})$ holds true. We define $\theta_{i,j+1}(x)$ starting from $\theta_{i,j}(x)$ in the same way as we define ξ_{k+1} starting from ξ_k in Proposition 2.1. Applying Proposition 2.1 and Lemma 2.1 with $d_{n_{i-1}+j} \geq \min\{\nu_{j-1}^{-2(l+1)\delta_0 q_{n_{i-1}}}, q_{n_{i-1}+j}^{-2(l+1)}\}$ and $i = l$, we have that $(\tilde{2})$ and $(\tilde{3})$ hold true.

For $(\tilde{1})$, one can sees that if $|a\tilde{c}| < |I_{n_{i-1}+j}|$, then from $q_{m+1} \geq \sqrt{2}\cdot q_m$ for each m,

$$\frac{1}{r_{n_{i-1}+j}(x)} \log \|A_{i,j+1}^{r_{n_{i-1}+j}(x)}(x)\| \geq \log \nu_j + \frac{1}{q_{n_{i-1}+j+1}} \cdot \log \underline{\mu}_{i,n_{i-1}}^{-\delta_0\cdot q_{n_{i-1}}\cdot\frac{q_{n_{i-1}+j+1}}{q_{n_{i-1}+j}}\cdot 2(l+1)}$$

$$\geq (1-\delta_0(\sqrt{2}^{-(j-1)}+\ldots+\sqrt{2}^{-1}+1)\cdot 2(l+1))\log \nu_0 - \delta_0\cdot\sqrt{2}^{-j}\cdot 2(l+1)\log \nu_0$$

$$= \log \nu_{j+1}.$$

Now we consider the case $|a\tilde{c}| \geq |I_{n_{i-1}+j}|$. Let j^* be the smallest integer such that $|I_{q_{n_{i-1}+j^*}}| \leq |a\tilde{c}|$ (Obviously, j^* depends on n_{i-1} and we can choose n_i large enough such that $j^* \ll n_i$). Since for any s, it holds that $M^2 \cdot |I_{s+1}| \geq |I_s|$. Thus from the definition of $|ac|$ and $|a\tilde{c}|$, we have $|I_{q_{n_{i-1}+j^*}}| \geq |ac|$ since $|I_{q_{n_{i-1}+j^*-1}}| \geq |a\tilde{c}|$. Then since $\nu_{j^*}^{q_{n_{i-1}+j^*}} \gg q^{2l}_{n_{i-1}+j^*+1}$, we follow Proposition 2.1 to construct ψ_{i,j^*+m} and $A_{i,j^*+m} = \Lambda \cdot R_{\psi_{i,j^*+m}}$ such that if $m \geq 1$, then

$$\underline{\mu}_{i,n_{i-1}+j^*+m} \geq \nu_{j^*+m}.$$

Thus $(\tilde{1})$ holds true. □

Define $\theta_i(x) = \theta_{i,j^*+m^*}(x)$ and $A_i(x) = \Lambda \cdot R_{\frac{\pi}{2}-\theta_i(x)}$, where $m^* = n_i - n_{i-1} - j^*$. Then (4) of (P$_\mathrm{i}$) can be proved by $(\tilde{1})$ in $(\tilde{P}_{i,j})$. From the inequality $0 < \delta_0 \ll \frac{1}{l^2}$, (1) of (P$_\mathrm{i}$) can be proved by $(\tilde{3})$ in $(\tilde{P}_{i,j})$ and Lemma 3.1. (3) of (P$_\mathrm{i}$) is obvious by the method of constructing $A_{i,j}$.

Step 2. The proof of (2) of (P$_\mathrm{i}$).
Now we will give an upper bound estimate for the Lyapunov exponent. For this purpose, we need the following proposition in [50]:

Proposition 3.1. *Let I_1, I_2 be two intervals in S^1 satisfying $I_2 = I_1 + 1/2$. Define $I = I_1 \bigcup I_2$.* $\min r(x) = \min_{x\in I} \min\{i > 0 | T^i x \ (mod\ 2\pi) \in I\}$ *and* $\max\ r(x) = \max_{x\in \frac{1}{10}I_1} \min\{i > 0 | T^i x\ (mod\ 2\pi) \in \frac{1}{10} I_1\}$. *Then there exists $k_1 \in \mathbb{N}$ such that* $M^{-k_1} \leq \frac{\min r(x)}{\max r(x)} \leq 1$.

From the definition of $[a, d]$, we have $|\phi_{A_i, r_{n_{i-1}}}(x) + \psi_{A_i, -r_{n_{i-1}}}(x) - \frac{\pi}{2}| \leq 2|ac|^l = 2\underline{\mu}_{n_{i-1}}^{-2l\delta_0 q_{n_{i-1}}}$ for each $x \in [a, d]$. Apply Proposition 3.1 with $I_1 = [a, d]$. Similar to the definition of $\underline{\mu}_{i,j}$ in Lemma 3.2, let $\overline{\mu}_{i,j} = \max_{x\in I_{n_i}} \|(A_{i,j}^{r_{n_{i-1}+j}}(x))\|^{\frac{1}{r_{n_{i-1}+j}}}$. Then with $\log \lambda \gg 1$ it follows from (5) of (P_{i-1}) and Lemma A.1 that

$$\bar{\mu}_{n_i} = \bar{\mu}_{i,n_i} \leq \bar{\mu}_{i,n_{i-1}}^{1-lM^{-k_1}\cdot\delta_0} \leq \bar{\mu}_{n_{i-1}}^{1-lM^{-k_1}\cdot\delta_0} \leq \bar{\mu}_{n_{i-1}}^{1-\delta_2} \leq \lambda^{(1-\delta_2)^i}.$$

Step 3. The proof of (5) of (P$_\mathrm{i}$) For $i = N$, (5) of (P_i) can be achieved by choosing $\lambda \gg N \gg 1$. For $i > N$, recall that $n_i = m^* + n_{i-1} + j^*$ in the proof of Lemma 3.2. Let $n_i \gg \hat{n}_i \gg j^* + n_{i-1}$. Then $|I_{n_i}| \ll |I_{\hat{n}_i}| \ll |I_{n_{i-1}}|$, which implies that

$$\|A_i^{r_{\hat{n}_i}(x)}(x)\|^{\frac{1}{r_{\hat{n}_i}(x)}} \approx \|A_i^{r_{\hat{n}_i}(y)}(y)\|^{\frac{1}{r_{\hat{n}_i}(y)}} \tag{3.1}$$

for any $x, y \in I_{n_i}$ (in fact both sides of (3.1) tend to each other as $n_i \to \infty$). From the definition of j^* we know that in the iterations from step $n_{i-1} + j^*$ to step n_i for $x \in I_{n_i}$, we need not to consider the existence of the platform, see the definition of $\tilde{\theta}_i$ as above. Hence based on the estimate on concatenation of non-rotating blocks in [53] or Proposition 3.1 in [50] (see also the proof of (2.6) in Proposition 2.1) and by the fact $n_i \gg \hat{n}_i \gg n_{i-1} + j^*$, for any $x \in I_{n_i}$ we obtain that

$$\|A_i^{r_{\hat{n}_i}(x)}(x)\|^{\frac{1}{r_{\hat{n}_i}(x)}} \geq \frac{1}{2} \cdot \|A_i^{r_{j^*+n_{i-1}}(x)}(x)\|^{\frac{1}{r_{j^*+n_{i-1}}(x)}} \tag{3.2}$$

and

$$\left| \|A_i^{r_{n_i}(x)}(x)\|^{\frac{1}{r_{n_i}(x)}} - \|A_i^{r_{\hat{n}_i}(x)}(x)\|^{\frac{1}{r_{\hat{n}_i}(x)}} \right| < C(l) \cdot \frac{\log q_{\hat{n}_i}}{q_{\hat{n}_i - 1}} \ll \|A_i^{r_{j^*+n_{i-1}}(x)}(x)\|^{\frac{1}{r_{j^*+n_{i-1}}(x)}}. \tag{3.3}$$

In the last inequality, we use the condition that $\hat{n}_i \gg j^* + n_{i-1}$. Combining this together with (3.1), (3.2) and (3.3), we obtain (5) of (P_i).
This ends the proof of Iterative Lemma. □

4. The Proof for the $\mathcal{C}^\infty$ Case

In this section, we will prove Theorems 1 and 2 for the $\mathcal{C}^\infty$ case. The basic idea is the same as the one in finitely differential case. Essentially, we only need to modify cocycles in C^∞ category. We will focus on the difference between the two cases. First we follow the steps in section 3 to construct a sequence of C^∞ cocycle which is $\mathcal{C}^1$-convergent. Then we will prove that it actually converges in $\mathcal{C}^\infty$ topology.

Assume $\lambda \gg e^{q_N^{a+1}} \gg 1$ with $0 < a < \frac{1}{10}$. For $n > N$, let $\lambda_{n+1}^{q_{n+1}} = \lambda_n^{q_{n+1}} \cdot e^{-(10q_{n+1}^2)^a}$ with $\lambda_N = \lambda$. From the definition of λ_n, we have $\lambda_n^{q_n} \geq \lambda_{n-1}^{q_n} \cdot e^{-q_n^{2a}} \geq \lambda_{n-2}^{q_n} \cdot e^{-q_n \cdot q_{n-1}^{2a-1}} \geq \cdots \geq \lambda^{q_n} \cdot \lambda_N^{-C \cdot q_n^{2a}} \geq \lambda^{(1-\epsilon)q_n}$ for some small positive ϵ if $\lambda \gg 1$ and $N \gg 1$. It implies that λ_n decrease to $\lambda_\infty > \lambda^{1-\epsilon}$.

Construction of $B_N(x)$ Let
(a)

$$\xi_0(x) = \begin{cases} \xi_{01}(x) & \text{for } |x| \leq \delta, \\ \xi_{02}(x) (\text{or } -\xi_{02}(x)) & \text{for } |x - 1/2| \leq \delta, \end{cases}$$

where $\xi_{01}(x) = \text{sgn}(x) e^{-\frac{1}{|x|^a}}$ and $\xi_{02}(x) = \text{sgn}(x - 1/2) e^{-\frac{1}{|x-1/2|^a}}$, $\delta > 0$ is a small number. Let $\xi(x)$ be a lift of a $\mathcal{C}^\infty$ 1-periodic function satisfying

$$\xi(x) = \begin{cases} \xi_{01}(x), & |x| \leq \delta; \\ -\xi_{02}(x) \ (\text{ or } \pi + \xi_{02}(x)), & |x - \frac{1}{2}| \leq \delta. \end{cases} \tag{4.1}$$

(b) $\forall |x (\text{mod } 1)| > \delta$ and $|(x - 1/2)(\text{mod } 1)| > \delta$, $|\xi(x)(\text{mod } \pi)| > e^{-\frac{1}{\delta^a}}$.
Define $\xi_N(x) = \xi(x)$ and $B_N(x) = \Lambda \cdot R_{\frac{\pi}{2} - \xi_N(x)}$.
We restate Lemma 5.1 in [50] as follows:

Lemma 4.1. *For each $n \geq N$, there exist a $g_n(x) \in \mathcal{C}^\infty$ be a 1-periodic function such that*

$$g_n(x) : \begin{cases} = 1, & x \in \frac{I_n}{10}, \\ \in [0, 1], & x \in I_n \backslash \frac{I_n}{10} \\ = 0, & x \in \mathbb{S}^1 \backslash I_n \end{cases}$$

and

$$\left| \frac{d^r g_n(x)}{dx^r} \right| \leq q_n^{3r}, \quad 0 \leq r \leq [q_n^{\frac{1}{10}}]. \tag{4.2}$$

Using the same argument as that in finite smooth case, we have that for any $x \in I_N$, $\|B_N^{r_N^+(x)}(x)\| \geq \lambda_N^{r_N^+(x)}$ and

$$|\phi_{B_N,r_N}(x) + \psi_{B_N,-r_N}(x) - \frac{\pi}{2} - \xi_0(x)|_{\mathcal{C}^1} \leq \lambda_N^{-1} \tag{4.3}$$

for $x \in I_N$.

Define a 1-periodic function $e_N(x) \in \mathcal{C}^\infty$ such that $e_N(x) = -(\phi_{B_N,r_N}(x) + \psi_{B_N,-r_N}(x) - \frac{\pi}{2} - \xi_0(x))$ for $x \in I_N$.

Let $\hat{e}_N(x) = e_N(x) \cdot g_N(x)$ and $\xi_{N+1}(x) = \xi_N(x) + \hat{e}_N(x)$ for $x \in \mathbb{S}^1$. Define $B_{N+1}(x) = \Lambda \cdot R_{\frac{\pi}{2}-\xi_{N+1}(x)}$. Obviously, $B_{N+1}(x) = B_N(x) \cdot R_{-\hat{e}_N(x)}$. Then for any $x \in I_N$, $\|B_{N+1}^{r_N^+(x)}(x)\| \geq \lambda_N^{r_N^+(x)}$ and $\phi_{B_{N+1},r_N}(x)+\psi_{B_{N+1},-r_N}(x) = \phi_{B_N,r_N}(x)+\psi_{B_N,-r_N}(x)-\hat{e}_N(x)$, which implies $\phi_{B_{N+1},r_N}(x)+\psi_{B_{N+1},-r_N}(x)-\frac{\pi}{2} = \xi_0(x)$ on $\frac{I_N}{10}$. (4.3) implies that $|\hat{e}_N(x)|_{\mathcal{C}^1} \leq \lambda_N^{-1}$ in I_N. Thus we have $|\phi_{B_{N+1},r_N}(x)+\psi_{B_{N+1},-r_N}(x)-\frac{\pi}{2}| \geq \frac{1}{2}\cdot e^{-(10\cdot q_N^2)^a}$ on $I_N \backslash \frac{I_N}{10}$.

For any $n \geq N$, define a 1-periodic function $e_n(x) \in \mathcal{C}^\infty$ such that

$$e_n(x) = (\phi_{B_n,r_n}(x) + \psi_{B_n,-r_n}(x)) - (\phi_{B_n,r_{n+1}}(x) + \psi_{B_n,-r_{n+1}}(x)) \quad x \in I_n.$$

Define $\hat{e}_n(x) = e_n(x) \cdot g_n(x)$, $\xi_n(x) = \xi_{n-1}(x) + \hat{e}_n(x)$ and $B_n(x) = \Lambda \cdot R_{\frac{\pi}{2}-\xi_n(x)}$. Obviously, $B_n(x) = B_{n-1}(x) \cdot R_{-\hat{e}_n(x)}$. Then we obtain (2.5), (2.6) of Proposition 2.1 and

$$\begin{aligned} &|\phi_{B_n,r_n}(x) + \psi_{B_n,-r_n}(x) - \tfrac{\pi}{2}| = e^{-|x|^{-a}} (\text{ or } e^{-|x-1/2|^{-a}}), \quad x \in \tfrac{I_{n,i}}{10}, i = 1, 2 \\ &|\phi_{B_n,r_n}(x) + \psi_{B_n,-r_n}(x) - \tfrac{\pi}{2}| \geq \tfrac{1}{2} \cdot e^{-(10\cdot q_n^2)^a}, \quad x \in I_n \backslash \tfrac{I_n}{10}. \end{aligned}$$

From (2.5), one easily sees that $B_N(x), B_{N+1}(x), \ldots,$ is $\mathcal{C}^1$-convergent to some $D_\infty(x)$. Thus from (2.6), the Lyapunov exponent of $D_\infty(x)$ has a lower bound $\log \lambda_\infty > (1-\epsilon)\log \lambda$.

In the following, we will prove that $B_N(x), B_{N+1}(x), \ldots,$ is also convergent to $D_\infty(x)$ in $\mathcal{C}^\infty$-topology. To deal with $\mathcal{C}^\infty$ case, we need some refining estimates for finitely differentiable cases.

Corollary 4.1. *$B_N(x), B_{N+1}(x), \ldots,$ is also convergent to $D_\infty(x)$ in $\mathcal{C}^\infty$-topology.*

Proof. It is equivalent to prove that $\xi_n(x),\ n = N, N+1, \ldots$ is $\mathcal{C}^\infty$-convergent. From the definition of $\xi_n(x)$, we have $\xi_n(x) - \xi_{n-1}(x) = \hat{e}_n(x)$. From the definition of $\hat{e}_n(x)$, it is sufficient to estimate $e_n(x)$ and $g_n(x)$. Since $e_n(x)$ is determined by $\phi_{B_n,r_n}(x) - \phi_{B_n,r_{n+1}}(x)$ and $\psi_{B_n,-r_n}(x) - \psi_{B_n,-r_{n+1}}(x)$, with the help of Lemma 2.1, we have

$$\left|\frac{d^r e_n(x)}{dx^r}\right| \leq C(r) \cdot \lambda_n^{-q_{n-1}}, \quad 0 \leq r \leq [q_{n-1}^{\frac{1}{10}}].$$

Note that $C(r)$ is independent of n. Thus for any fixed $R \in \mathbb{N}$, we can choose n large enough such that $C(r) \leq \lambda_n^{\frac{1}{2}q_{n-1}}$ for any $r \leq R$. This together with (4.2) ends the proof. □

Construction of $C_n(x)$ *Next we will construct the sequence $C_n(x)$, $n = N, N+1, \ldots,$ which is also $\mathcal{C}^\infty$-convergent to D_∞, but the Lyapunov exponent of each $C_n(x)$ equals 0.*

We denote the sub-interval $[0, \frac{1}{k_{i-1}^2}]$ of $I_{k_{i-1}}$ by $[a, b]$. Define $a < c < \tilde{c} < d < b$ such that $|ac| = (2\delta_0 \cdot k_{i-1} \cdot \log \underline{\mu}_{i,0})^{-1/a}$, $|a\tilde{c}| = M^2 \cdot |ac|$, $d = \frac{a+b}{2}$. Let n_i be sufficiently large such that $I_{n_i} \not\supseteq [a, c]$.

Define

$$\bar{e}_i(x) = \begin{cases} e^{-|ac|^{-a}} - (\phi_{A_{i-1}, r_{n_{i-1}}}(x) + \psi_{A_{i-1}, -r_{n_{i-1}}}(x)), & x \in [\tilde{c}, d]; \\ 0, & x \in [a, c] \text{ or equals } b; \\ \bar{h}_i^{\pm}(x) & x \in [c, \tilde{c}] \text{ or } [d, b], \end{cases}$$

where $\bar{h}_i^{\pm}(x)$ is of a $\mathcal{C}^\infty$ connection between the parts in $[a, c]$ and $[\tilde{c}, d]$ as well as between the part in $[\tilde{c}, d]$ and the end point b of I_{n_i}. Then similar to Lemma 4.1, we have

$$\left| \frac{d^r \bar{h}_i(x)}{dx^r} \right| \leq C(r) \cdot q_{n_i}^{3r}, \quad 0 \leq r \leq [q_{n_i}^{\frac{1}{10}}].$$

Thus the $\mathcal{C}^\infty$-convergence of $C_n(x)$ is similar to the above argument.
The remain part of the proof is same as Section 4.

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Appendix A. Product of non-rotating matrices

Let A be a non-rotating $SL(2, R)$-matrix, i.e., $\|A\| > 1$. It is know that A can be written uniquely as $A = R_\psi \cdot \Lambda_A \cdot R_\phi$ with $\Lambda_A = diag(\|A\|, \|A\|^{-1})$. It is known that $-\phi$ is the most expanded direction of A and ψ is the most contracted direction of A^{-1}.

For two non-rotating matrices $A = R_{\psi_A} \cdot \Lambda_A \cdot R_{\phi_A}$, $B = R_{\psi_B} \cdot \Lambda_B \cdot R_{\phi_B}$ with big norms, let $BA = R_{\psi_{BA}} \cdot \Lambda_{BA} \cdot R_{\phi_{BA}}$. We firstly investigate how ϕ_{BA}, ψ_{BA} and $\|BA\|$ depend on A and B.

Lemma A.1. *Let A, B be non-rotating $SL(2, \mathbb{R})$ cocycles and $\theta = \phi_B + \psi_A$. Then it holds that $\frac{1}{4}N(\|A\|, \|B\|, \theta) \leq \|BA\|^2 \leq N(\|A\|, \|B\|, \theta)$, where $N(\|A\|, \|B\|, \theta) = (\|A\|^2\|B\|^2 + \|A\|^{-2}\|B\|^{-2}) \cdot \cos^2\theta + (\|A\|^2\|B\|^{-2} + \|A\|^{-2}\|B\|^2) \cdot \sin^2\theta$.*

Proof. For any $SL(2, \mathbb{R})$ matrix $A = (a_{ij})_{2\times 2}$, it is known that $\frac{1}{4}\sum_{i,j} a_{ij}^2 \leq \|A\|^2 \leq \sum_{i,j} a_{ij}^2$.

It is easy to see that

$$\|BA\| = \left\| \begin{pmatrix} \|B\| & 0 \\ 0 & \|B\|^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \|A\| & 0 \\ 0 & \|A\|^{-1} \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} \|A\|\|B\|\cos\theta & -\|A\|^{-1}\|B\|\sin\theta \\ \|A\|\|B\|^{-1}\sin\theta & \|A\|^{-1}\|B\|^{-1}\cos\theta \end{pmatrix} \right\|.$$

It thus implies the conclusion. $\square$

Lemma A.2. *Let $\phi = \phi_A - \phi_{BA}$, $\psi = \psi_{BA} - \psi_B$. Assume $\theta \in [0, \pi)$. Then ϕ can be chosen as the following continuous function*

$$\phi(\|A\|, \|B\|, \theta) = \begin{cases} 0, & \text{for } \theta = 0 \\ -\frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(a\cot\theta + b\tan\theta)\right), & \text{for } 0 < \theta < \frac{\pi}{2} \\ \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(a\cot\theta + b\tan\theta)\right), & \text{for } \frac{\pi}{2} < \theta < \pi \text{ if } b \geq 0 \\ -\frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(a\cot\theta + b\tan\theta)\right), & \text{for } \frac{\pi}{2} < \theta < \pi \text{ if } b < 0 \\ 0, & \text{for } \theta = \frac{\pi}{2} \text{ if } b \geq 0 \\ -\frac{\pi}{2}, & \text{for } \theta = \frac{\pi}{2} \text{ if } b < 0, \end{cases} \tag{A.1}$$

where

$$a = \frac{\|A\|^2\|B\|^2 - \|A\|^{-2}\|B\|^{-2}}{2(\|B\|^2 - \|B\|^{-2})}, \quad b = \frac{\|A\|^2\|B\|^{-2} - \|A\|^{-2}\|B\|^2}{2(\|B\|^2 - \|B\|^{-2})}.$$

Similarly, ψ can be chosen as the following continuous function

$$\psi(\|A\|, \|B\|, \theta)) = \begin{cases} 0, & \text{for } \theta = 0 \\ -\frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(a'\cot\theta - b'\tan\theta)\right), & \text{for } 0 < \theta < \frac{\pi}{2} \\ \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(a'\cot\theta - b'\tan\theta)\right), & \text{for } \frac{\pi}{2} < \theta < \pi \text{ if } b' \geq 0 \\ -\frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(a'\cot\theta - b'\tan\theta)\right), & \text{for } \frac{\pi}{2} < \theta < \pi \text{ if } b' < 0 \\ 0, & \text{for } \theta = \frac{\pi}{2} \text{ if } b' \geq 0 \\ -\frac{\pi}{2}, & \text{for } \theta = \frac{\pi}{2} \text{ if } b' < 0, \end{cases} \tag{A.2}$$

where

$$a' = \frac{\|A\|^2\|B\|^2 - \|A\|^{-2}\|B\|^{-2}}{2(\|A\|^2 - \|A\|^{-2})}, \quad b' = \frac{\|A\|^2\|B\|^{-2} - \|A\|^{-2}\|B\|^2}{2(\|A\|^2 - \|A\|^{-2})}.$$

Proof. Let

$$\begin{aligned} V(s) &= \begin{pmatrix} \|B\| & 0 \\ 0 & \|B\|^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \|A\| & 0 \\ 0 & \|A\|^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} \\ &= \begin{pmatrix} \|B\| & 0 \\ 0 & \|B\|^{-1} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \cdot \|A\| \cdot \cos s - \sin\theta \cdot \|A\|^{-1} \cdot \sin s \\ \sin\theta\|A\| \cdot \cos s + \cos\theta \cdot \|A\|^{-1} \cdot \sin s \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta \cdot \|A\|\|B\| \cdot \cos s - \sin\theta \cdot \|A\|^{-1} \cdot \|B\| \sin s \\ \sin\theta\|A\|\|B\|^{-1} \cdot \cos s + \cos\theta \cdot \|A\|^{-1}\|B\|^{-1} \cdot \sin s \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned}|V(s)|^2 = &(\cos\theta\|A\|\|B\|)^2 + (\sin^2\theta\|A\|^{-2}\|B\|^2 - \cos^2\theta\|A\|^2\|B\|^2)\sin^2 s\\ &+ \sin^2\theta\|A\|^2\|B\|^{-2}\\ &+ (\cos^2\theta\|A\|^{-2}\|B\|^{-2} - \sin^2\theta\|A\|^2\|B\|^{-2})\sin^2 s\\ &+ 2(\|B\|^{-2} - \|B\|^2)\sin\theta\cos\theta\sin s\cos s.\end{aligned}$$

Obviously $\frac{d}{ds}(|V(s)|^2) = 0$ at ϕ since $|V(s)|^2$ attains its extreme at ϕ, a simple computation leads to

$$\begin{aligned}&\Big((\|A\|^2\|B\|^2 - \|A\|^{-2}\|B\|^{-2})\cos^2\theta + (\|A\|^2\|B\|^{-2} - \|A\|^{-2}\|B\|^2)\sin^2\theta\Big)\sin 2\phi\\ &= -2(\|B\|^2 - \|B\|^{-2})\sin 2\theta\cos 2\phi.\end{aligned}$$

Thus

$$-\cot 2\phi = \frac{\|A\|^2\|B\|^2 - \|A\|^{-2}\|B\|^{-2}}{2(\|B\|^2 - \|B\|^{-2})}\cot\theta + \frac{\|A\|^2\|B\|^{-2} - \|A\|^{-2}\|B\|^2}{2(\|B\|^2 - \|B\|^{-2})}\tan\theta.$$

With the help of the equation $\frac{d^2}{ds^2}(|V(s)|^2) \leq 0$, we obtain the unique ϕ corresponding the maximum $\|BA\|^2$ of $|V(s)|^2$, which satisfies (A.1).

(A.2) is proved similarly. □

Later we will see that both $\|A\|$ and $\|B\|$ are very big. Thus

$$a = \tfrac{\|A\|^2\|B\|^2 - \|A\|^{-2}\|B\|^{-2}}{\|B\|^2 - \|B\|^{-2}} \sim \|A\|^2,$$

$$b = \tfrac{\|A\|^2\|B\|^{-2} - \|A\|^{-2}\|B\|^2}{\|B\|^2 - \|B\|^{-2}} \preceq \max\{\|A\|^{-2}, \tfrac{\|A\|^2}{\|B\|^4}\}.$$

If A, B are non-rotating, the functions $\phi(\|A\|, \|B\|, \theta)$, $\psi(\|A\|, \|B\|, \theta)$ defined above are continuous in all variables. In the following, we estimate the derivatives of ϕ and ψ with respect to θ, $\|A\|$ and $\|B\|$.

Lemma A.3. *It holds that*

$$|\phi(mod\ \pi)| \leq C(0)\cdot\|A\|^{-2}\cdot|\theta - \frac{\pi}{2}|^{-1} \tag{A.3}$$

and

$$|\psi(mod\ \pi)| \leq C(0)\cdot\|B\|^{-2}\cdot|\theta - \frac{\pi}{2}|^{-1}. \tag{A.4}$$

Suppose $|\theta - \frac{\pi}{2}|^{-1} \ll \|A\|^2$. *Then, for* $i \geq 1$, *we have that*

$$|\frac{\partial^i\phi}{\partial\theta^i}| \leq C(i)\cdot\|A\|^{-2}\cdot|\theta - \frac{\pi}{2}|^{-i-1}, \tag{A.5}$$

$$|\tfrac{\partial^i\phi}{\partial\|A\|^i}| \leq C(i)\cdot\|A\|^{-2}\cdot\|A\|^{-i}\cdot|\theta - \tfrac{\pi}{2}|^{-1}, \tag{A.6}$$

and

$$|\frac{\partial^i\phi}{\partial\|B\|^i}| \leq C(i)\cdot\|A\|^{-2}\cdot\|B\|^{-i}\cdot|\theta - \frac{\pi}{2}|^{-1}. \tag{A.7}$$

More generally, for $i + j + k \geq 1$, *we have*

$$|\frac{\partial^{i+j+k}\phi}{\partial\theta^i\partial\|A\|^j\partial\|B\|^k}| \leq C(i,j,k)\cdot|\theta-\frac{\pi}{2}|^{-i-1}\|A\|^{-2-j}\cdot\|B\|^{-k}; \tag{A.8}$$

Similarly, suppose $|\theta - \frac{\pi}{2}|^{-1} \ll \|B\|^2$. *Then we have*

$$|\frac{\partial^{i+j+k}\psi}{\partial\theta^i\partial\|A\|^j\partial\|B\|^k}| \leq C(i,j,k)\cdot|\theta-\frac{\pi}{2}|^{-i-1}\|A\|^{-j}\cdot\|B\|^{-2-k}. \tag{A.9}$$

Proof. To prove (A.3), it is sufficient to consider the situation $\theta \approx \frac{\pi}{2}$. We only consider the case $0 \leq \theta \leq \frac{\pi}{2}$ since the proof for the other cases is similar. From the fact $\lim_{x\to\infty}\frac{\frac{\pi}{2}-\arctan x}{x^{-1}} = 1$ and the definition of a, we have $|\phi| \leq C(0)\cdot a^{-1}\cdot|\theta-\frac{\pi}{2}|^{-1} \leq C(0)\cdot\|A\|^{-2}\cdot|\theta-\frac{\pi}{2}|^{-1}$. Thus we obtain (A.3). We can obtain (A.4) similarly.

for $i \geq 1$, from the definition of ϕ, we have

$$\frac{\partial^i\phi}{\partial\theta^i} = -\frac{1}{2}\sum_{l_1+\ldots+l_k=i}\frac{d^{k-1}(\frac{1}{1+f^2})}{df^{k-1}}\cdot\frac{\partial^{l_1}f}{\partial\theta^{l_1}}\cdots\frac{\partial^{l_k}f}{\partial\theta^{l_k}},$$

where $f(\|A\|,\|B\|,\theta) = a\cot\theta + b\tan\theta$. To estimate $\frac{\partial^i\phi}{\partial\theta^i}$, we have that

$$|\frac{\partial^{l_s}f}{\partial\theta^{l_s}}| = |\frac{\partial^{l_s}}{\partial\theta^{l_s}}(a\cot\theta+b\tan\theta)| \leq |a|\cdot|\cot^{(l_s)}(\theta)| + |b|\cdot|\tan^{(l_s)}(\theta)|.$$

By a direct computation, we have

$$|\tan^{(l_s)}\theta| = |(\cos^{-2}\theta)^{(l_s-1)}| \leq |\sum_{\kappa_1+\ldots+\kappa_t=l_s-1}\cos^{-(2+t)}\theta\cdot\cos^{(\kappa_1)}\theta\ldots\cos^{(\kappa_t)}\theta|$$

and

$$|\cot^{(l_s)}\theta| = |(\sin^{-2}\theta)^{(l_s-1)}| \leq |\sum_{\kappa_1+\ldots+\kappa_t=l_s-1}\sin^{-(2+t)}\theta\cdot\sin^{(\kappa_1)}\theta\ldots\sin^{(\kappa_t)}\theta|.$$

From the condition $|\theta-\frac{\pi}{2}|^{-1} \ll \|A\|^2$ and the fact that the signs of $\|B\|^2\cot\theta$ and $\|B\|^{-2}\tan\theta$ are the same, we have

$$|\frac{\partial^{l_s}f}{\partial\theta^{l_s}}| \leq C(l_s)\cdot(|a|\cdot|\theta-\frac{\pi}{2}|^{-(l_s-1)}+|b|\cdot|\theta-\frac{\pi}{2}|^{-(l_s+1)}) \leq C(l_s)\cdot|f|\cdot|\theta-\frac{\pi}{2}|^{-l_s}. \tag{A.10}$$

On the other hand, we have

$$\left|\frac{d^{k-1}(\frac{1}{1+f^2})}{df^{k-1}}\right| \preceq |f|^{-k-1} \quad \text{if } k \geq 1.$$

Thus from $|f| \succeq \|A\|^2\cdot|\cot\theta|$ we obtain

$$|\frac{\partial^i\phi}{\partial\theta^i}| \leq C(i)|\theta-\frac{\pi}{2}|^{-i}\cdot\frac{1}{|f|} \leq C(i)\|A\|^{-2}|\theta-\frac{\pi}{2}|^{-i-1}.$$

Next we estimate

$$\begin{aligned} |\tfrac{\partial^i \phi}{\partial \|A\|^i}| &\leq \textstyle\sum_{l_1+\cdots+l_k=i} |\frac{d^{k-1}(\frac{1}{1+f^2})}{df^{k-1}}| \cdot |\frac{\partial^{l_1} f}{\partial \|A\|^{l_1}}| \ldots |\frac{\partial^{l_k} f}{\partial \|A\|^{l_k}}| \quad l_j \geq 1,\ 1 \leq j \leq k \\ &\leq \textstyle\sum_{l_1+\cdots+l_k=i} |f|^{-k-1} \cdot |\frac{\partial^{l_1} f}{\partial \|A\|^{l_1}}| \cdots |\frac{\partial^{l_k} f}{\partial \|A\|^{l_k}}|. \end{aligned} \tag{A.11}$$

It is easy to see that $|f| \sim |a| \cdot |\cot\theta|$ with the condition $|\theta - \frac{\pi}{2}|^{-1} \ll \|A\|^2$. We also have

$$|\frac{\partial^{l_s} f}{\partial \|A\|^{l_s}}| \leq |\cot\theta| |\frac{\partial^{l_s} a}{\partial \|A\|^{l_s}}| + |\tan\theta| |\frac{\partial^{l_s} b}{\partial \|A\|^{l_s}}|.$$

By a direct computation, we obtain

$$|\frac{\partial^{l_s} a}{\partial \|A\|^{l_s}}| = \left| \frac{\partial^{l_s}}{\partial \|A\|^{l_s}} \left(\frac{\|A\|^2\|B\|^2 - \|A\|^{-2}\|B\|^{-2}}{\|B\|^2 - \|B\|^{-2}} \right) \right| \leq C(l_s) \cdot |a| \cdot \|A\|^{-l_s}$$

and

$$|\tfrac{\partial^{l_s} b}{\partial \|A\|^{l_s}}| = \left| \tfrac{\partial^{l_s}}{\partial \|A\|^{l_s}} \left(\tfrac{\|A\|^2\|B\|^{-2} - \|A\|^{-2}\|B\|^2}{\|B\|^2 - \|B\|^{-2}} \right) \right| \leq C(l_s) \cdot (\|A\|^{-2} + \tfrac{\|A\|^2}{\|B\|^4}) \cdot \|A\|^{-l_s}.$$

Thus we have

$$\begin{aligned} |\tfrac{\partial^{l_s} f}{\partial \|A\|^{l_s}}| &\leq C(l_s) \cdot \left\{ |\theta - \tfrac{\pi}{2}| \|A\|^{2-l_s} + |\theta - \tfrac{\pi}{2}|^{-1} \cdot \|A\|^{-l_s} \cdot (\|A\|^{-2} + \tfrac{\|A\|^2}{\|B\|^4}) \right\} \\ &\leq C(l_s) \cdot |f| \cdot \|A\|^{-l_s}. \end{aligned} \tag{A.12}$$

With the fact that $|f| \succeq \|A\|^2 \cdot |\cot\theta|$, it follows that

$$\begin{aligned} |f|^{-2} \cdot |\tfrac{\partial^{l_s} f}{\partial \|A\|^{l_s}}| &\leq C(l_s)|f|^{-1} \cdot \|A\|^{-l_s} \\ &\leq C(l_s) \cdot \|A\|^{-2-l_s} \cdot |\theta - \tfrac{\pi}{2}|^{-1}. \end{aligned} \tag{A.13}$$

Combining (A.11), (A.12) with (A.13), we obtain (A.6).
Similarly, we have (A.7) and (A.8).
The estimates for ψ can be proved similarly. □

Appendix B. Proof of Lemma 2.1.

In this section, we first give estimates on most contracted and expanded directions of the product of non-rotating blocks. Then we will give the proof of Lemma 2.1.

Let A, B, θ, ϕ and ψ be defined as in Lemmas A.1 and A.2.

Lemma B.1. *Let $|\theta - \frac{\pi}{2}|^{-1} \ll \|A\|^2, \|B\|^2$. Suppose for any $i \geq 0$, it holds that*

$$\begin{aligned} \left|\frac{d^i \|A\|}{dx^i}\right| &\leq C(i) \cdot \|A\| \cdot |\theta - \frac{\pi}{2}|^{-i-1}, \\ \left|\frac{d^i \|B\|}{dx^i}\right| &\leq C(i) \cdot \|B\| \cdot |\theta - \frac{\pi}{2}|^{-i-1}, \\ |\frac{d^i \theta}{dx^i}| &\leq C(i) \cdot |\theta - \frac{\pi}{2}|^{-i-1}. \end{aligned} \tag{B.1}$$

Then we have

$$
\begin{aligned}
|\frac{d^i\phi}{dx^i}| &\le C(i)\cdot|\theta-\frac{\pi}{2}|^{-i-1}\cdot\|A\|^{-2},\\
|\frac{d^i\psi}{dx^i}| &\le C(i)\cdot|\theta-\frac{\pi}{2}|^{-i-1}\cdot\|B\|^{-2},\\
\left|\frac{d^i\|BA\|}{dx^i}\right| &\le C(i)\cdot\|BA\|\cdot|\theta-\frac{\pi}{2}|^{-i-1}.
\end{aligned}
\tag{B.2}
$$

Proof. For the first inequality of (B.2), we see that

$$
\begin{aligned}
\left|\frac{d^i\phi_1}{dx^i}\right| &= \sum\nolimits_{t_1+\cdots+t_{i_1}+s_1+\cdots+s_{i_2}+j_1+\cdots+j_{i_3}=i} \frac{\partial^{i_1+i_2+i_3}\phi_1}{\partial\|A\|^{i_1}\cdot\partial\|B\|^{i_2}\partial\theta^{i_3}}\cdot\frac{d^{t_1}\|A\|}{dx^{t_1}}\cdots\frac{d^{t_{i_1}}\|A\|}{dx^{t_{i_1}}}\\
&\cdot\frac{d^{s_1}\|B\|}{dx^{s_1}}\cdots\frac{d^{s_{i_2}}\|B\|}{dx^{s_{i_2}}}\cdot\left(\frac{d^{j_1}\theta}{dx^{j_1}}\right)\cdots\left(\frac{d^{j_{i_3}}\theta}{dx^{j_{i_3}}}\right),
\end{aligned}
\tag{B.3}
$$

where ϕ_1 satisfies $\tan 2\phi_1 := \cot 2\phi$.

From Lemma A.3, we have that

$$
\left|\frac{\partial^{i_1+i_2+i_3}\phi_1}{\partial\|A\|^{i_1}\cdot\partial\|B\|^{i_2}\cdot\partial\theta^{i_3}}\right| \le C(i_1,i_2,i_3)\cdot|\theta-\tfrac{\pi}{2}|^{-(1+i_3)}\cdot\|A\|^{-i_1-2}\cdot\|B\|^{-i_2}. \tag{B.4}
$$

Then from (B.1), (B.3), (B.4), we have that

$$
\left|\frac{d^i\phi_1}{dx^i}\right| \le C(i)\cdot\|A\|^{-2}\cdot|\theta-\frac{\pi}{2}|^{-(i+1)},
$$

thus the first inequality of the lemma is proved.

In the last inequality, we use the fact that

$$
\left|\frac{\partial f}{\partial\|A\|}\cdot\frac{\partial^j\|A\|}{\partial x^j}\right| \le |\theta-\frac{\pi}{2}|^{-(j+1)}\cdot|f|.
$$

We can prove the second inequality similarly. There remains the third inequality to be proved.

By a direct computation, we have

$$
\frac{\partial^i\|BA\|}{\partial\phi^i} = \frac{\partial^i(g^{\frac{1}{2}})}{\partial\phi^i} = \sum_{l_1+\cdots+l_k=i}(g^{\frac{1}{2}})^{(k)}\cdot\frac{\partial^{l_1}g}{\partial\phi^{l_1}}\cdots\frac{\partial^{l_k}g}{\partial\phi^{l_k}}, \tag{B.5}
$$

where

$$
\begin{aligned}
&g = g_1^2+g_2^2,\quad g_1 = \|A\|\cdot\|B\|\cdot\cos\theta\cos\phi - \|A\|^{-1}\cdot\|B\|^{-1}\sin\theta\sin\phi,\\
&g_2 = \|A\|\cdot\|B\|^{-1}\cdot\sin\theta\cdot\cos\phi + \|A\|^{-1}\cdot\|B\|^{-1}\cdot\cos\theta\cdot\sin\phi.
\end{aligned}
\tag{B.6}
$$

It is not difficult to see that $|(g^{\frac{1}{2}})^{(k)}| \le C(k)\cdot g^{\frac{1}{2}-k}$.

From the definition of g, we have

$$
\left|\frac{\partial^{l_s}g}{\partial\phi^{l_s}}\right| \le \left|\frac{\partial^{l_s}(g_1^2)}{\partial\phi^{l_s}}\right| + \left|\frac{\partial^{l_s}(g_2^2)}{\partial\phi^{l_s}}\right|,
$$

with

$$\left|\frac{\partial^{l_s}(g_1^2)}{\partial\phi^{l_s}}\right| \leq \sum_{l_{s,1}+l_{s,2}=l_s}\left|\frac{\partial^{l_{s,1}}g_1}{\partial\phi^{l_{s,1}}}\right|\cdot\left|\frac{\partial^{l_{s,2}}g_1}{\partial\phi^{l_{s,2}}}\right|.$$

It is easy to see that

$$\begin{aligned}\left|\frac{\partial^{l_{s,1}}g_1}{\partial\phi^{l_{s,1}}}\right| &\leq \|A\|\|B\||\cos\theta|\cdot|\cos(\phi+\tfrac{\pi}{2}\cdot l_{s,1})|+\|A\|^{-1}\|B\||\sin\theta|\cdot|\sin(\phi+\tfrac{\pi}{2}\cdot l_{s,1})|\\ &\leq \|A\|\cdot\|B\|\cdot|\cos\theta| \preceq \|BA\|.\end{aligned}$$

Then we obtain

$$\left|\frac{\partial^{l_s}(g_1^2)}{\partial\phi^{l_s}}\right| \preceq \|BA\|^2. \tag{B.7}$$

Similarly, we have

$$\left|\frac{\partial^{l_s}(g_2^2)}{\partial\phi^{l_s}}\right| \preceq \|BA\|^2. \tag{B.8}$$

Combining (B.5) with (B.7), (B.8), we then have

$$\left|\frac{\partial^i\|BA\|}{\partial\phi^i}\right| \leq C(i)\cdot\max_{k\leq i}(\|BA\|^{2k}\cdot g^{\frac{1}{2}-k}) = C(i)\cdot\|BA\|. \tag{B.9}$$

Similarly, it holds that

$$\left|\frac{\partial^i\|BA\|}{\partial\theta^i}\right| \leq C(i)\cdot\max_{k\leq i}\left(g^{\frac{1}{2}-k}\cdot(\|A\|\|B\|)^{2k}|\cos\theta|^k\right) \leq C(i)\cdot|\theta-\frac{\pi}{2}|^{1-i} \tag{B.10}$$

and

$$\left|\frac{\partial^i\|BA\|}{\partial\|A\|^i}\right| \leq C(i)\cdot\|BA\|\cdot\|A\|^{-i}, \quad \left|\frac{\partial^i\|BA\|}{\partial\|B\|^i}\right| \leq C(i)\cdot\|BA\|\cdot\|B\|^{-i}. \tag{B.11}$$

Similar to (B.9)–(B.11), we have that

$$\left|\frac{\partial^{i_1+\cdots+i_4}\|BA\|}{\partial\|A\|^{i_1}\cdot\partial\|B\|^{i_2}\cdot\partial^{i_3}\phi\cdot\partial^{i_4}\theta}\right| \leq C(i)\cdot\|BA\|\cdot\|A\|^{-i_1}\cdot\|B\|^{-i_2}\cdot|\theta-\tfrac{\pi}{2}|^{-i_4},$$

which, combining with (B.1), the first inequality in (B.2) and the fact

$$\begin{aligned}\frac{d^i\|BA\|}{dx^i} = \sum\frac{\partial^{i_1+\cdots+i_4}\|BA\|}{\partial\|A\|^{i_1}\cdot\partial\|B\|^{i_2}\cdot\partial^{i_3}\phi\cdot\partial^{i_4}\theta}\cdot\frac{\partial^{j_{1,1}}\|A\|}{\partial x^{j_{1,1}}}\cdots\frac{\partial^{j_{i_1,1}}\|A\|}{\partial x^{j_{i_1,1}}}\\ \cdots\frac{\partial^{j_{1,4}}\theta}{\partial x^{j_{1,4}}}\cdots\frac{\partial^{j_{i_4,4}}\theta}{\partial x^{j_{i_4,4}}},\end{aligned} \tag{B.12}$$

implies the third inequality. □

Proof of Lemma 2.1

For any $x \in I_{k+1}$, let $r_k(x) := r_{k,0}(x) < r_{k,1}(x) < \cdots < r_{k,s(x)}(x) := r_{k+1}(x)$ such that $T^{r_{k,j}(x)}x \in I_k$, $0 \le j \le s(x) \le C(M)$. Consider $A^{r_{k,0}(x)+r_{k,1}(x)}(x) = A^{r_{k,1}(x)}(T^{r_{k,0}(x)}(x)) \cdot A^{r_{k,0}(x)}(x)$.

Let $A^{r_{k,0}(x)}(x) := R_{\psi_k^-(x)} \cdot L_k^-(x) \cdot R_{\phi_k^-(x)}$ and $A^{r_{k,1}(x)}(T^{r_{k,0}(x)}x) := R_{\psi_k^+(x)} \cdot L_k^+(x) \cdot R_{\phi_k^+}(x)$. Then

$$\begin{aligned} A^{r_{k,0}(x)+r_{k,1}(x)}(x) &= R_{\psi_k^+(x)} \cdot L_k^+(x) \cdot R_{\phi_k^+ + \psi_k^-}(x) \cdot L_k^-(x) \cdot R_{\phi_k^-(x)} \\ &:= R_{\psi_{k+1,1}}(x) \cdot L_{k+1,1}(x) \cdot R_{\phi_{k+1,1}}(x). \end{aligned}$$

Since $T^{r_{k,j}(x)}x \in I_k \backslash I_{k+1}$ for $j < s(x)$, it holds that $|\phi_k^+ + \psi_k^- - \frac{\pi}{2}| \ge d_{k+1}$. From (2.7) and Lemma B.1, we have

$$\left| \frac{d^i(\phi_{k+1,1} - \phi_k^-)}{dx^i} \right| \le C(i) \cdot d_k^{-i} \cdot |L_k^+|^{-2}. \tag{B.13}$$

Similarly, it holds that

$$\left| \frac{d^i(\psi_{k+1,1} - \psi_k^+)}{dx^i} \right| \le C(i) \cdot d_k^{-i} \cdot |L_k^-|^{-2}. \tag{B.14}$$

In the above, we regard $\phi_{k+1,1} - \phi_k^-$ and $\phi_k^+ + \psi_k^-$ as ϕ and θ in Lemma B.1, respectively. Moreover, for each $x \in I_k$, $|\theta(x) - \frac{\pi}{2}| = |\phi_k^+(x) + \psi_k^-(x) - \frac{\pi}{2}| \ge d_{k+1}$ from the definition of d_{k+1}. It implies that

$$\left|\tfrac{d^i \phi_{k+1,1}}{dx^i}\right|, \quad \left|\tfrac{d^i \psi_{k+1,1}}{dx^i}\right| \le C(i) \cdot d_k^{-i} + \left|\tfrac{d^i \phi_k^-}{dx^i}\right| + \left|\tfrac{d^i \psi_k^+}{dx^i}\right|$$

$$\le C(i) \cdot (d_k^{-i} + d_{k-1}^{-i}) \le C(i) \cdot d_k^{-i}.$$

The last inequality is obtained from $(1)_k$.

Setting $L_{k+1,1} = \|BA\|$, we obtain $\left|\frac{d^i L_{k+1,1}}{dx^i}\right| \le C(i) \cdot \|L_{k+1,1}\| \cdot d_k^{-i}$.

Since $s(x) \le C(M)$, it follows from no more than $C(M)$-applications of the above argument that $(1)_{k+1}$ and $(2)_{k+1}$ hold true.

From no more than $C(M)$-applications of (B.13) and (B.14), we have

$$\left| \frac{d^i}{dx^i}(\phi_{A,r_{k+1}^+}(x) - \phi_{A,r_k^+}(x)) \right| \le C(i) \cdot |\theta_k - \frac{\pi}{2}|^{-i} \cdot |L_k^+|^{-2}$$

and

$$\left| \frac{d^i}{dx^i}(\psi_{A,-r_{k+1}^-}(x) - \psi_{A,-r_k^-}(x)) \right| \le C(i) \cdot |\theta_k - \frac{\pi}{2}|^{-i} \cdot |L_k^-|^{-2}.$$

Since $|\theta_k - \frac{\pi}{2}| \ge d_k$, we obtain (2.8). □

Remark B.1. In the proof of Lemma 2.1, it is not necessary that $s(x)$ is bounded by a constant. We make such an assumption only for the simplicity. And the condition that ω is of bound type is only used in constructing $C_k(x)$.

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