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Discrete Bethe–Sommerfeld Conjecture

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Abstract: In this paper, we prove a discrete version of the Bethe–Sommerfeld conjecture. Namely, we show that the spectra of multi-dimensional discrete periodic Schrödinger operators on $\mathbb{Z}^d$ lattice with sufficiently small potentials contain at most two intervals. Moreover, the spectrum is a single interval, provided at least one of the periods is odd, and can have a gap whenever all periods are even.

1. Introduction

Bethe–Sommerfeld conjecture states that for $d \geq 2$ and any periodic function $V : \mathbb{R}^d \to \mathbb{R}$, the spectrum of the continuous Schrödinger operator:

$$-\Delta + V$$

contains only finitely many gaps, so no gaps for large energies. This conjecture has been studied extensively with many important advances [1,3,4,7–12]. Finally, Parnovski [6], proved it in any dimension $d \geq 2$, under smoothness conditions on the potential V (see [13] for an alternative approach).

In this paper, we consider a discrete version of this conjecture. A discrete multi-dimensional periodic Schrödinger operator on $l^2(\mathbb{Z}^d)$ is given by:

$$(H_V u)(\boldsymbol{n}) = \sum_{|\boldsymbol{m}-\boldsymbol{n}|=1} u(\boldsymbol{m}) + V(\boldsymbol{n})u(\boldsymbol{n}), \tag{1.1}$$

where $|\boldsymbol{m}-\boldsymbol{n}| = \sum_{j=1}^{d} |m_j - n_j|$. We assume $V(\cdot)$ is a bounded real-valued periodic function on $\mathbb{Z}^d$ with period $\boldsymbol{q} = (q_1, q_2, \ldots, q_d)$, namely, $V(\boldsymbol{n} + q_j \boldsymbol{b}_j) = V(\boldsymbol{n})$, with $\{\boldsymbol{b}_j\}_{j=1}^{d}$ being the standard basis for $\mathbb{R}^d$.

Remark 1.1. The most general periodic case may seem to be $V(\boldsymbol{n}+\boldsymbol{w}_j) = V(\boldsymbol{n})$, where $\boldsymbol{w}_j \in \mathbb{Z}^d$, $j = 1, \ldots, d$, are linearly independent vectors. This however reduces to our assumption because such operators are periodic with period $\boldsymbol{q} = (\det W, \ldots, \det W)$, where W is the matrix with $\boldsymbol{w}_j$ as columns.

In the high energy regime continuous Schrödinger operators can be viewed as small perturbations of the free Laplacian. In this sense, the proper discrete analogy of the Bethe–Sommerfeld conjecture is absence of gaps for small coupling discrete periodic operators.

The discrete Bethe–Sommerfeld conjecture has been proved for $d = 2$ by Embree-Fillman [2], with a partial result (for coprime periods) earlier by Krüger [5]. The approach of [2] runs into combinatorial/algebraic difficulties for $d > 2$. Here we prove this conjecture for arbitrary dimension:

Theorem 1.1. *Let $d \geq 2$ and a period $\boldsymbol{q} = (q_1, q_2, \ldots, q_d)$ be given. There exists a constant $c_{\boldsymbol{q}} > 0$ such that the following statements hold:*

(1) *If $\|V\|_\infty \leq c_{\boldsymbol{q}}$, then the spectrum of H_V contains at most two intervals.*
(2) *If at least one of q_i is odd, and $\|V\|_\infty \leq c_{\boldsymbol{q}}$, then the spectrum of H_V is a single interval.*

Remark 1.2. We should point out that our proof does not currently allow us to give a quantitative estimate on $c_{\boldsymbol{q}}$.

Our result is sharp in the sense that if all the q_j's are even, then there exists V (see example in Sect. 6) with *minimal period* $\boldsymbol{q}$, and arbitrarily small $\|V\|_\infty$ such that $\Sigma(H_V)$ contains *exactly* two intervals. The example we give is a modification of Krüger's example [5], in which $V(\boldsymbol{n}) = \delta(-1)^{|\boldsymbol{n}|}$ has minimal period $(2, 2, \ldots, 2)$.

It is well-known that in the one dimensional case, a generic q-periodic operator has spectrum with q connected components. Also for $d \geq 2$, for any positive integer q, one can construct a periodic V with large $\|V\|_\infty$ such that the spectrum of H_V consists of q intervals. A simple example is to take $V(\boldsymbol{n}) = V(n_1)$ with $V(n_1) = \lambda n_1 (\mathrm{mod}\ q)$ for $\lambda > 4(d-1)$. Hence both $d \geq 2$ and the smallness of $\|V\|_\infty$ are needed. It is an interesting question to find the sharp threshold on $\|V\|_\infty$ for existence of periodic potentials with q gaps (a slight modification of the above example gives $4(d-1)\frac{q-1}{2}$ as an upper bound for odd q).

The strategy of our proof relies on analysing the overlaps of adjacent bands of the spectrum. We refer the readers to [5] for a detailed background on discrete multi-dimensional Schrödinger operators. Here we only introduce some notations and known results. Let us denote the spectrum of H by $\Sigma(H)$. By the Floquet–Bloch decomposition, $\Sigma(H_V)$ can be decomposed into $\cup_{\boldsymbol{\theta}\in\Theta}\Sigma(H_V^{\boldsymbol{\theta}})$, where $\Theta = \{\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_d) : 0 \leq \theta_j < \frac{1}{q_j},\ 1 \leq j \leq d\}$ is a d-dimensional torus (by gluing 0 and $\frac{1}{q_j}$ together in the $\boldsymbol{b}_j$ direction), and $H_V^{\boldsymbol{\theta}}$ is the periodic Schrödinger operator with the following boundary condition:

$$u_{\boldsymbol{n}+q_j\boldsymbol{b}_j} = e^{2\pi i q_j \theta_j} u_{\boldsymbol{n}}.$$

Each operator $H_V^{\boldsymbol{\theta}}$ clearly has $Q = \prod_{j=1}^d q_j$ eigenvalues, which we will arrange in the increasing order and denote them by $E_V^1(\boldsymbol{\theta}) \leq E_V^2(\boldsymbol{\theta}) \leq \cdots \leq E_V^Q(\boldsymbol{\theta})$. Let $F_V^k = \cup_{\boldsymbol{\theta}\in\Theta} E_V^k(\boldsymbol{\theta})$ be the k-th band of the spectrum. Theorem 1.1 is thus reduced to proving non-empty overlaps of arbitrary two adjacent bands, with only possible exception around the point 0. Employing a standard perturbation argument (see Theorem 3.1), this is made possible via proving non-empty overlaps of the *interiors* of adjacent bands of the free Laplacian H_0. Two of our key lemmas are as follows:

Lemma 1.2. *If* $E \in (-2d, 2d) \setminus \{0\}$, *then* $E \in \mathrm{int}(F_0^k)$ *for some* $1 \le k \le Q$.

Lemma 1.3. *If at least one of* q_j*'s is odd, then* $0 \in \mathrm{int}(F_0^k)$ *for some* $1 \le k \le Q$.

We will prove Lemma 1.2 in Sect. 4 and Lemma 1.3 in Sect. 5. Different from the existing two-dimensional proofs in [2,5], our argument proceeds by contradiction. Namely, we assume $E_0^{k_0}(\tilde{\theta}) = \max F_0^{k_0} = \min F_0^{k_0+1}$ for certain k_0, and then apply a novel *perturb-and-count* technique. We perturb the phase $\tilde{\theta}$ and count the number of eigenvalues that move up and down. It is then argued that different chosen directions lead to different numbers of up/down eigenvalues, hence a contradiction. The difficulty of counting up/down eigenvalues along a given direction $\boldsymbol{\beta}$ arises in the situation when it is perpendicular to the gradients of certain eigenvalues. We resolve this issue by finding a particular $\boldsymbol{\beta}$ such that even in this case, the shifts of eigenvalues are still predictable.

2. Preliminaries

For $\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}} \in \Theta$, let $\|\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}\|_\Theta$ be the torus distance between them, defined by

$$\|\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}\|_\Theta^2 = \sum_{j=1}^{d} \|\theta_j - \tilde{\theta}_j\|_{\mathbb{T}_j}^2,$$

where $\|\theta\|_{\mathbb{T}_j} := \mathrm{dist}(\theta, \frac{1}{q_j}\mathbb{Z})$.

2.1. Spectrum of the free Laplacian. It is a well-known result that the spectrum of the free Laplacian H_0 is the whole interval:

$$\Sigma(H_0) = [-2d, 2d]. \tag{2.1}$$

By the Floquet–Bloch decomposition,

$$\Sigma(H_0) = [-2d, 2d] = \cup_{\boldsymbol{\theta} \in \Theta} \Sigma(H_0^{\boldsymbol{\theta}}). \tag{2.2}$$

Furthermore, each $\Sigma(H_0^{\boldsymbol{\theta}})$ can be written down explicitly,

$$\Sigma(H_0^{\boldsymbol{\theta}}) = \left\{ e_0^{\boldsymbol{l}}(\boldsymbol{\theta}) := 2\sum_{j=1}^{d} \cos 2\pi \left(\theta_j + \frac{l_j}{q_j}\right) \right\}_{\boldsymbol{l} \in \Lambda}, \tag{2.3}$$

where $\Lambda = \{\boldsymbol{l} = (l_1, l_2, \ldots, l_d) \in \mathbb{Z}^d : 0 \le l_j \le q_j - 1,\ 1 \le j \le d\}$.

3. Proof of Theorem 1.1

We say the bands $\{F_k\}_{k=1}^{Q}$ of H are δ-overlapping if $\max F^k - \min F^{k+1} \geq \delta$ for any $1 \leq k \leq Q-1$. Theorem 1.1 follows from a quick combination of Lemmas 1.2, 1.3 with Hausdorff continuity of the spectrum. The form of continuity convenient to us is presented in:

Theorem 3.1 [5, Theorem 3.8]. *Let the bands of H be δ-overlapping. Then the bands of $H+V$ are $\delta - 2\|V\|_\infty$-overlapping.* □

4. Proof of Lemma 1.2

Our strategy is to prove by contradiction, namely we assume $\max F_0^{k_0} = \min F_0^{k_0+1} \neq 0$ for some $1 \leq k_0 \leq Q$ and try to get a contradiction. Without loss of generality, we assume $\max F_0^{k_0} = \min F_0^{k_0+1} > 0$.

We will use the following elementary lemma, whose proof is included in the appendix.

Lemma 4.1. *Let $d \geq 2$. For any $E \in (-2d, 2d)$, there exist $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_d)$ with $\theta_j \in [0, 1)$ such that*

$$\begin{cases} \sum_{j=1}^{d} 2\cos 2\pi\theta_j = E, \\ \sum_{j=1}^{d} \sin 2\pi\theta_j = 0, \\ \sum_{j=1}^{d} \sin^2 2\pi\theta_j \neq 0. \end{cases}$$

Now let us prove Lemma 1.2.

First, by Lemma 4.1, there exist $\tilde{\boldsymbol{\theta}} = (\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_d) \in \Theta$ and $\boldsymbol{l}^{(1)} = (l_1^{(1)}, l_2^{(1)}, \ldots, l_d^{(1)}) \in \Lambda$ such that

$$\begin{cases} \max F_0^{k_0} = \sum_{j=1}^{d} 2\cos 2\pi\left(\tilde{\theta}_j + \frac{l_j^{(1)}}{q_j}\right) = e_0^{\boldsymbol{l}_j^{(1)}}(\tilde{\boldsymbol{\theta}}), \\ 0 = \sum_{j=1}^{d} \sin 2\pi\left(\tilde{\theta}_j + \frac{l_j^{(1)}}{q_j}\right), \\ 0 \neq \sum_{j=1}^{d} \sin^2 2\pi\left(\tilde{\theta}_j + \frac{l_j^{(1)}}{q_j}\right). \end{cases} \tag{4.1}$$

Next, let us choose $\boldsymbol{l}^{(2)}, \boldsymbol{l}^{(3)}, \ldots, \boldsymbol{l}^{(r)} \in \Lambda$ (if any) be *all* the vectors in Λ such that

$$e_0^{\boldsymbol{l}^{(1)}}(\tilde{\boldsymbol{\theta}}) = e_0^{\boldsymbol{l}^{(2)}}(\tilde{\boldsymbol{\theta}}) = \cdots = e_0^{\boldsymbol{l}^{(r)}}(\tilde{\boldsymbol{\theta}}).$$

Then clearly they are $E_0^{k_0-s}(\tilde{\boldsymbol{\theta}}) = \cdots = E_0^{k_0}(\tilde{\boldsymbol{\theta}}) = \cdots = E_0^{k_0+r-s-1}(\tilde{\boldsymbol{\theta}})$, for some $0 \leq s \leq r-1$. And also we have $E_0^{k_0-s-1}(\tilde{\boldsymbol{\theta}}) < E_0^{k_0-s}(\tilde{\boldsymbol{\theta}})$, $E_0^{k_0+r-s-1}(\tilde{\boldsymbol{\theta}}) < E_0^{k_0+r-s}(\tilde{\boldsymbol{\theta}})$. By the continuity of each eigenvalue, we could choose $\epsilon > 0$ small enough, such that for any $\|\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}\|_\Theta < \epsilon$, we always have

$$E_0^{k_0-s-1}(\boldsymbol{\theta}) < E_0^{k_0-s}(\boldsymbol{\theta}) \text{ and } E_0^{k_0+r-s-1}(\boldsymbol{\theta}) < E_0^{k_0+r-s}(\boldsymbol{\theta}). \tag{4.2}$$

Let $J_0 \geq 0$ be the number of j's such that $\nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \mathbf{0}$. For $\boldsymbol{\beta} \in \mathbb{R}^d$, we also introduce $J_{\boldsymbol{\beta}}$ and $J_{\boldsymbol{\beta}}^0$: let $J_{\boldsymbol{\beta}}$ be the number of j's such that $\boldsymbol{\beta} \cdot \nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) > 0$, and $J_{\boldsymbol{\beta}}^0$ be the number of j's such that $\nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) \neq \mathbf{0}$ and $\boldsymbol{\beta} \cdot \nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = 0$.
Perturbing $e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}})$ along the direction of $\boldsymbol{\beta}$ we get:

$$e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}) = e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) + t\boldsymbol{\beta} \cdot \nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) + O(t^2) \tag{4.3}$$

$$= e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) + t\boldsymbol{\beta} \cdot \nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) + \frac{t^2}{2}\left(-4\pi^2 \sum_{m=1}^{d} 2\beta_m^2 \cos 2\pi \left(\tilde{\theta}_m + \frac{l_m^{(j)}}{q_m}\right)\right) + O(t^3). \tag{4.4}$$

Step 1 Let $\tilde{\boldsymbol{\beta}} = \frac{1}{\sqrt{d}}(1, 1, \ldots, 1)$. By (4.1), we have

$$\tilde{\boldsymbol{\beta}} \cdot \nabla e_0^{\boldsymbol{l}^{(1)}}(\tilde{\boldsymbol{\theta}}) = 0 \quad \text{and} \quad \nabla e_0^{\boldsymbol{l}^{(1)}}(\tilde{\boldsymbol{\theta}}) \neq \mathbf{0}, \tag{4.5}$$

which implies $J_{\tilde{\boldsymbol{\beta}}}^0 \geq 1$.

By (4.4) for j such that $\tilde{\boldsymbol{\beta}} \cdot \nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = 0$ (in total $J_0 + J_{\tilde{\boldsymbol{\beta}}}^0$ many such j's), we have

$$e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\tilde{\boldsymbol{\beta}}) = \left(1 - \frac{2\pi^2}{d}t^2\right) e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) + O(t^3) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}), \tag{4.6}$$

for $|t|$ small enough. Let us mention that in (4.6), we used the fact that $e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \max F_0^{k_0} > 0$.

Now combine (4.3) with (4.6). On one hand, we have, for $t \to 0_+$ (meaning $t > 0$ small enough),

- there are $J_{\tilde{\boldsymbol{\beta}}}$ many j's such that $E^{k_0+r-s}(\tilde{\boldsymbol{\theta}} + t\tilde{\boldsymbol{\beta}}) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\tilde{\boldsymbol{\beta}}) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \max F_0^{k_0}$, thus these $J_{\tilde{\boldsymbol{\beta}}}$ many eigenvalues $e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\tilde{\boldsymbol{\beta}})$ must belong to the bands $F_0^{k_0+r-s-1}$, $F_0^{k_0+r-s-2}$,..., $F_0^{k_0+1}$. Hence by counting the numbers, we get $J_{\tilde{\boldsymbol{\beta}}} \leq (k_0 + r - s - 1) - k_0 = r - s - 1$.
- for the other $r - J_{\tilde{\boldsymbol{\beta}}}$ many j's, we have $E^{k_0-s-1}(\tilde{\boldsymbol{\theta}} + t\tilde{\boldsymbol{\beta}}) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\tilde{\boldsymbol{\beta}}) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \min F_0^{k_0+1}$, so these $r - J_{\tilde{\boldsymbol{\beta}}}$ eigenvalues must belong to the bands $F_0^{k_0}$, $F_0^{k_0-1}$,..., $F_0^{k_0-s}$. Hence we have $r - J_{\tilde{\boldsymbol{\beta}}} \leq k_0 - (k_0 - s) + 1 = s + 1$.

Thus

$$J_{\tilde{\boldsymbol{\beta}}} = r - s - 1. \tag{4.7}$$

On the other hand, for for $t \to 0_-$ (meaning $t < 0$ with $|t|$ small enough), we have,

- there are $r - J_{\tilde{\boldsymbol{\beta}}} - J_{\tilde{\boldsymbol{\beta}}}^0 - J_0$ many j's such that $E^{k_0+r-s}(\tilde{\boldsymbol{\theta}} + t\tilde{\boldsymbol{\beta}}) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\tilde{\boldsymbol{\beta}}) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \max F_0^{k_0}$,
- for the other $J_{\tilde{\boldsymbol{\beta}}} + J_{\tilde{\boldsymbol{\beta}}}^0 + J_0$ many j's, we have $E^{k_0-s-1}(\tilde{\boldsymbol{\theta}} + t\tilde{\boldsymbol{\beta}}) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\tilde{\boldsymbol{\beta}}) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \min F_0^{k_0+1}$.

Thus

$$J_{\tilde{\boldsymbol{\beta}}} + J^0_{\tilde{\boldsymbol{\beta}}} + J_0 = s + 1. \tag{4.8}$$

Combining this with (4.7), we have,

$$2s - r = J^0_{\tilde{\boldsymbol{\beta}}} + J_0 - 2. \tag{4.9}$$

Step 2 We choose $\boldsymbol{\beta} \in \mathbb{R}^d$, $\|\boldsymbol{\beta}\|_{\mathbb{R}^d} = 1$, such that $\boldsymbol{\beta} \cdot \nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) \neq 0$ for any $1 \leq j \leq r$ with $\nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) \neq \mathbf{0}$, and satisfies the following:

$$\sum_{m=1}^{d} 2|\beta_m^2 - \frac{1}{d}| < \frac{1}{2d} \min F_0^{k_0+1}. \tag{4.10}$$

Inequality (4.10) means $\boldsymbol{\beta}$ is a small perturbation of $\tilde{\boldsymbol{\beta}}$.

For j such that $\nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \mathbf{0}$, we have, by (4.4), (4.10)

$$\begin{aligned} e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}) &= e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) \\ &\quad + \frac{t^2}{2}\left(-\frac{4\pi^2}{d} e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) + 4\pi^2 \sum_{m=1}^{d} 2\left(\frac{1}{d} - \beta_m^2\right) \cos 2\pi \left(\tilde{\theta}_m + \frac{l_m^{(j)}}{q_m}\right)\right) + O(t^3) \\ &\leq \left(1 - \frac{\pi^2}{d} t^2\right) e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) + O(t^3) \qquad (4.11) \\ &< e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}). \qquad (4.12) \end{aligned}$$

Combining (4.3) with (4.12), on one hand, we have that for $t \to 0_+$,

- there are $J_{\boldsymbol{\beta}}$ many j's such that $E^{k_0+r-s}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \max F_0^{k_0}$,
- for the other $r - J_{\boldsymbol{\beta}}$ many j's, we have $E^{k_0-s-1}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \min F_0^{k_0+1}$.

Thus

$$J_{\boldsymbol{\beta}} = r - s - 1. \tag{4.13}$$

On the other hand, we have that for $t \to 0_-$,

- there are $r - J_{\boldsymbol{\beta}} - J_0$ many j's such that $E^{k_0+r-s}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \max F_0^{k_0}$,
- for the other $J_{\boldsymbol{\beta}} + J_0$ many j's, we have $E^{k_0-s-1}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \min F_0^{k_0+1}$.

Thus

$$J_{\boldsymbol{\beta}} + J_0 = s + 1. \tag{4.14}$$

Combining this with (4.13), we have,

$$2s - r = J_0 - 2. \tag{4.15}$$

However, this contradicts (4.9), since $J^0_{\tilde{\boldsymbol{\beta}}} \geq 1$. □

5. Proof of Lemma 1.3

The spirit of this proof is similar to that of Lemma 1.2, but requires different choices of $\tilde{\boldsymbol{\theta}}, \boldsymbol{l}^{(1)}$ and $\boldsymbol{\beta}, \tilde{\boldsymbol{\beta}}$.

Without loss of generality, we assume q_1 is odd. We assume q_m's, $m \geq 2$, are even, since otherwise, we could simply replace q_m with $2q_m, m \geq 2$. Throughout this section, we will consider the case when $\max F_0^{k_0} = \min F^{k_0+1} = 0$.

5.1. $d = 2$. This result has already been proved in [2]. Here we give an alternative self-contained proof.

We let $\tilde{\boldsymbol{\theta}} = (\frac{1}{2q_1}, 0)$, $\boldsymbol{l}^{(1)} = (\frac{q_1-1}{2}, 0)$, and observe that

$$\begin{cases} 0 = 2\cos\pi + 2\cos 0 = e_0^{\boldsymbol{l}^{(1)}}(\tilde{\boldsymbol{\theta}}), \\ \mathbf{0} = \nabla e_0^{\boldsymbol{l}^{(1)}}(\tilde{\boldsymbol{\theta}}). \end{cases} \tag{5.1}$$

Again, we let $\boldsymbol{l}^{(2)}, \ldots, \boldsymbol{l}^{(r)} \in \Lambda$ (if any) to be *all* the vectors in Λ such that $e_0^{\boldsymbol{l}^{(1)}}(\tilde{\boldsymbol{\theta}}) = e_0^{\boldsymbol{l}^{(2)}}(\tilde{\boldsymbol{\theta}}) = \cdots = e_0^{\boldsymbol{l}^{(r)}}(\tilde{\boldsymbol{\theta}}) = 0$. Let $0 \leq s \leq r-1$ be such that $E_0^{k_0-s-1}(\boldsymbol{\theta}) < E_0^{k_0-s}(\boldsymbol{\theta}) = \cdots = E_0^{k_0}(\boldsymbol{\theta}) = \cdots = E_0^{k_0+r-s-1}(\boldsymbol{\theta}) < E_0^{k_0+r-s}(\boldsymbol{\theta})$ for any $\|\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}\|_\Theta < \epsilon$.

Let $\boldsymbol{l}^{(j)}$, $1 \leq j \leq r$, be such that $\nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \mathbf{0}$. Then $\sin 2\pi(\tilde{\theta}_1 + \frac{l_1^{(j)}}{q_1}) = \sin 2\pi(\tilde{\theta}_2 + \frac{l_2^{(j)}}{q_2}) = 0$. Taking into account that $e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = 0$, this implies $j = 1$. Hence the number of j's such that $\nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \mathbf{0}$ is equal to 1.

Now let $\boldsymbol{\beta}^+ = (1, 0)$ and $\boldsymbol{\beta}^- = (0, 1)$. Let $J_{\boldsymbol{\beta}^\pm}$, $J^0_{\boldsymbol{\beta}^\pm}$ be as in the proof of Lemma 1.2.

First, it is easy to see that $J^0_{\boldsymbol{\beta}^+} = J^0_{\boldsymbol{\beta}^-} = 0$. Indeed, if there is j such that $\nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) \neq \mathbf{0}$ and $\boldsymbol{\beta}^+ \cdot \nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = 0$, then $\sin 2\pi(\tilde{\theta}_1 + \frac{l_1^{(j)}}{q_1}) = 0$, which implies $\cos 2\pi(\tilde{\theta}_1 + \frac{l_1^{(j)}}{q_1}) = \pm 1$. This in turn implies $\cos 2\pi(\tilde{\theta}_2 + \frac{l_2^{(j)}}{q_2}) = \mp 1$, and hence $\nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \mathbf{0}$, contradiction. The case $J^0_{\boldsymbol{\beta}^-} = 0$ can be argued in the same way.

Secondly, by (4.4), we have that for $|t| < \epsilon$ small enough,

$$e_0^{\boldsymbol{l}^{(1)}}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}^\pm) = \pm 4\pi^2 t^2 + O(t^3), \tag{5.2}$$

so $e_0^{\boldsymbol{l}^{(1)}}$ increases in the direction of $\boldsymbol{\beta}^+$ and decreases in the direction of $\boldsymbol{\beta}^-$.

Combining (4.3) with (5.2) for $\boldsymbol{\beta}^+$, on one hand, we have, for $t \to 0_+$,

- there are $J_{\boldsymbol{\beta}^+} + 1$ many j's such that $E^{k_0+r-s}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}^+) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}^+) > 0 = \max F_0^{k_0}$,
- for the other $r - J_{\boldsymbol{\beta}^+} - 1$ many j's, we have $E^{k_0-s-1}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}^+) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}^+) < 0 = \min F_0^{k_0+1}$.

Hence

$$J_{\boldsymbol{\beta}^+} + 1 = r - s - 1. \tag{5.3}$$

On the other hand, for $t \to 0_-$, we have,

- there are $r - J_{\boldsymbol{\beta}^+}$ many j's such that $E^{k_0+r-s}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}^+) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}^+) > 0 = \max F_0^{k_0}$,
- for the other $J_{\boldsymbol{\beta}^+}$ many j's, we have $E^{k_0-s-1}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}^+) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}^+) < 0 = \min F_0^{k_0+1}$.

Hence

$$J_{\boldsymbol{\beta}^+} = s + 1. \tag{5.4}$$

Thus combining (5.3) with (5.4), we have

$$r = 2s + 3. \tag{5.5}$$

Similarly, combining (4.3) with (5.2) for $\boldsymbol{\beta}^-$, on one hand, we have, for $t \to 0_+$,

- there are $J_{\boldsymbol{\beta}^-}$ many j's such that $E^{k_0+r-s}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}^-) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}^-) > 0 = \max F_0^{k_0}$,
- for the other $r - J_{\boldsymbol{\beta}^-}$ many j's, we have $E^{k_0-s-1}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}^-) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}^-) < 0 = \min F_0^{k_0+1}$.

Hence

$$J_{\boldsymbol{\beta}^-} = r - s - 1. \tag{5.6}$$

On the other hand, for $t \to 0_-$, we have,

- there are $r - J_{\boldsymbol{\beta}^-} - 1$ many j's such that $E^{k_0+r-s}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}^-) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}^-) > 0 = \max F_0^{k_0}$,
- for the other $J_{\boldsymbol{\beta}^-} + 1$ many j's, we have $E^{k_0-s-1}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}^-) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}^-) < 0 = \min F_0^{k_0+1}$.

Hence

$$J_{\boldsymbol{\beta}^-} + 1 = s + 1. \tag{5.7}$$

Thus combining (5.6) with (5.7), we have

$$r = 2s + 1. \tag{5.8}$$

This contradicts (5.5). □

5.2. $d \geq 3$. Let us choose $\tilde{\boldsymbol{\theta}}, \boldsymbol{l}^{(1)}$ with $\tilde{\theta}_1 = \frac{1}{2q_1}$, $l_1^{(1)} = \frac{q_1-1}{2}$ and $\tilde{\theta}_m, l_m^{(1)}$, $2 \leq m \leq d$, be such that $\cos 2\pi(\tilde{\theta}_m + \frac{l_m^{(1)}}{q_m}) = \frac{1}{d-1} < 1$ and $\sin 2\pi(\tilde{\theta}_m + \frac{l_m^{(1)}}{q_m}) > 0$. Let $\boldsymbol{\beta} = (1, 0, 0, \ldots, 0)$, then clearly we have,

$$\nabla e_0^{\boldsymbol{l}^{(1)}}(\tilde{\boldsymbol{\theta}}) \neq \mathbf{0} \text{ and } \boldsymbol{\beta} \cdot \nabla e_0^{\boldsymbol{l}^{(1)}}(\tilde{\boldsymbol{\theta}}) = 0. \tag{5.9}$$

Let $\boldsymbol{l}^{(2)}, \ldots, \boldsymbol{l}^{(r)} \in \Lambda$ (if any) be *all* the vectors in Λ such that $e_0^{\boldsymbol{l}^{(1)}}(\tilde{\boldsymbol{\theta}}) = e_0^{\boldsymbol{l}^{(2)}}(\tilde{\boldsymbol{\theta}}) = \cdots = e_0^{\boldsymbol{l}^{(r)}}(\tilde{\boldsymbol{\theta}})$. Let $0 \leq s \leq r - 1$ be such that $E_0^{k_0-s-1}(\boldsymbol{\theta}) < E_0^{k_0-s}(\boldsymbol{\theta}) = \cdots = E_0^{k_0}(\boldsymbol{\theta}) = \cdots = E_0^{k_0+r-s-1}(\boldsymbol{\theta}) < E_0^{k_0+r-s}(\boldsymbol{\theta})$ for any $\|\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}\|_\Theta < \epsilon$.

Let J_0, $J_{\boldsymbol{\beta}}$, $J^0_{\boldsymbol{\beta}}$ be as in the proof of Lemma 1.2. Then by (5.9), $J^0_{\boldsymbol{\beta}} \geq 1$.

Clearly, for $J_0+J^0_{\boldsymbol{\beta}}$ many j's, we have $\boldsymbol{\beta}\cdot\nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = 0$, which means $\sin 2\pi(\tilde{\theta}_1 + \frac{l_1^{(j)}}{q_1}) = 0$. Since our $\tilde{\theta}_1$ equals $\frac{1}{2q_1}$, we must have

$$\cos 2\pi\left(\tilde{\theta}_1 + \frac{l_1^{(j)}}{q_1}\right) = -1. \tag{5.10}$$

Thus, by (4.4) and (5.10), we have that for j (in total $J_0 + J^0_{\boldsymbol{\beta}}$ many) such that $\boldsymbol{\beta}\cdot\nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = 0$, for $|t| < \epsilon$ small enough,

$$\begin{aligned} e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}) &= e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) + \frac{t^2}{2}\left(-8\pi^2\cos 2\pi\left(\tilde{\theta}_1 + \frac{l_1^{(j)}}{q_1}\right)\right) + O(t^3) \\ &= 4\pi^2 t^2 + O(t^3) \\ &> 0. \end{aligned} \tag{5.11}$$

Hence, combining (4.3) with (5.11), on one hand, we have, for $t \to 0_+$,

- there are $J_{\boldsymbol{\beta}} + J_0 + J^0_{\boldsymbol{\beta}}$ many j's such that $E^{k_0+r-s}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\boldsymbol{\beta}) > 0 = \max F_0^{k_0}$,
- for the other $r - J_{\boldsymbol{\beta}} - J_0 - J^0_{\boldsymbol{\beta}}$ many j's, we have $E^{k_0-s-1}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}) < 0 = \min F_0^{k_0+1}$.

Hence

$$J_{\boldsymbol{\beta}} + J_0 + J^0_{\boldsymbol{\beta}} = r - s - 1. \tag{5.12}$$

On the other hand, for $t \to 0_-$, we have,

- there are $r - J_{\boldsymbol{\beta}}$ many j's such that $E^{k_0+r-s}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}) > 0 = \max F_0^{k_0}$,
- for the other $J_{\boldsymbol{\beta}}$ many j's, we have $E^{k_0-s-1}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}+t\boldsymbol{\beta}) < 0 = \min F_0^{k_0+1}$.

Hence

$$J_{\boldsymbol{\beta}} = s + 1. \tag{5.13}$$

Thus combining (5.12) with (5.13), we have

$$r - 2s = J_0 + J^0_{\boldsymbol{\beta}} + 2. \tag{5.14}$$

Now we choose $\tilde{\boldsymbol{\beta}} \in \mathbb{R}^d$, $\|\tilde{\boldsymbol{\beta}}\|_{\mathbb{R}^d} = 1$, such that $\tilde{\boldsymbol{\beta}}\cdot\nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) \neq 0$ for any $1 \leq j \leq r$ with $\nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) \neq \mathbf{0}$, and satisfies the following:

$$1 - \tilde{\beta}_1^2 + \sum_{m=2}^{d}\tilde{\beta}_m^2 < \frac{1}{2}. \tag{5.15}$$

This inequality essentially says $\tilde{\boldsymbol{\beta}}$ is a small perturbation of $\boldsymbol{\beta}$.

With $J_{\tilde{\boldsymbol{\beta}}}$ defined as before, by (4.4), (5.10) and (5.15), we have that for j (in total J_0 many) such that $\nabla e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}) = \mathbf{0}$, for $|t| < \epsilon$ small enough,

$$\begin{aligned} e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\tilde{\boldsymbol{\beta}}) &= \frac{t^2}{2}\left(8\pi^2 - 8\pi^2(1-\tilde{\beta}_1^2) - 8\pi^2\sum_{m=2}^{d}\tilde{\beta}_m^2\cos 2\pi\left(\tilde{\theta}_m + \frac{l_m^{(j)}}{q_m}\right)\right) + O(t^3) \\ &> 2\pi^2 t^2 + O(t^3) > 0. \end{aligned} \tag{5.16}$$

As before, combining (4.3) with (5.16), on one hand, we have, for $t \to 0_+$,

- there are $J_0 + J_{\tilde{\boldsymbol{\beta}}}$ many j's such that $E^{k_0+r-s}(\tilde{\boldsymbol{\theta}}+t\tilde{\boldsymbol{\beta}}) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}+t\tilde{\boldsymbol{\beta}}) > 0 = \max F_0^{k_0}$,
- for the other $r - J_0 - J_{\tilde{\boldsymbol{\beta}}}$ many j's, we have $E^{k_0-s-1}(\tilde{\boldsymbol{\theta}} + t\tilde{\boldsymbol{\beta}}) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}} + t\tilde{\boldsymbol{\beta}}) < 0 = \min F_0^{k_0+1}$.

Hence

$$J_0 + J_{\tilde{\boldsymbol{\beta}}} = r - s - 1. \tag{5.17}$$

On the other hand, for $t \to 0_-$, we have,

- there are $r - J_{\tilde{\boldsymbol{\beta}}}$ many j's such that $E^{k_0+r-s}(\tilde{\boldsymbol{\theta}}+t\tilde{\boldsymbol{\beta}}) > e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}+t\tilde{\boldsymbol{\beta}}) > 0 = \max F_0^{k_0}$,
- for the other $J_{\tilde{\boldsymbol{\beta}}}$ many j's, we have $E^{k_0-s-1}(\tilde{\boldsymbol{\theta}}+t\tilde{\boldsymbol{\beta}}) < e_0^{\boldsymbol{l}^{(j)}}(\tilde{\boldsymbol{\theta}}+t\tilde{\boldsymbol{\beta}}) < 0 = \min F_0^{k_0+1}$.

Hence

$$J_{\tilde{\boldsymbol{\beta}}} = s + 1. \tag{5.18}$$

Thus combining (5.17) with (5.18), we have

$$r - 2s = J_0 + 2. \tag{5.19}$$

This contradicts (5.14) since $J_{\boldsymbol{\beta}}^0 \geq 1$. □

6. Example with Exactly Two Intervals

Let all the q_m's be even and $\delta > 0$ be any small positive number. We are going to construct V with minimal period $\boldsymbol{q}$, such that $\|V\|_\infty = \delta$ and the spectrum of H_V does not contain the point 0. This example is a modification of Krüger's example (see Theorem 6.3 in [5]), where V is $(2, 2, \ldots, 2)$-periodic.

Let us define

$$V_{\boldsymbol{q}}(\boldsymbol{n}) = \begin{cases} (1-\delta^2/d)\delta & \text{if } \boldsymbol{n} \equiv \mathbf{0} \pmod{\boldsymbol{q}} \\ \delta(-1)^{|\boldsymbol{n}|} & \text{otherwise} \end{cases} \tag{6.1}$$

It can be easily checked that $V_{\boldsymbol{q}}$ has minimal period $\boldsymbol{q}$ and $\|V\|_\infty = \delta$. The fact that the spectrum of H_V does not contain 0 will follow from the following lemma.

Lemma 6.1. *There exists constant $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$, we have*

$$\|(H_0 + V_{\boldsymbol{q}})u\| > \frac{1}{2}\delta$$

holds for any unit vector $u \in l^2(\mathbb{Z}^d)$.

Proof of Lemma 6.1. Let us consider

$$\|(H_0 + V_{\boldsymbol{q}})u\|^2 = \|H_0 u\|^2 + \|V_{\boldsymbol{q}} u\|^2 + 2(H_0 u, V_{\boldsymbol{q}} u) \geq \|V_{\boldsymbol{q}} u\|^2 + 2(H_0 u, V_{\boldsymbol{q}} u), \tag{6.2}$$

in which the first term obviously satisfies

$$\|V_{\boldsymbol{q}} u\|^2 = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} |V_{\boldsymbol{q}}(\boldsymbol{n})|^2 |u(\boldsymbol{n})|^2 \geq (1-\delta^2/d)^2\delta^2 \geq (1-\delta^2)^2\delta^2. \tag{6.3}$$

Let $\{\boldsymbol{b}_i\}$ be the standard basis for $\mathbb{R}^d$. The second term in (6.2) could be estimated in the following way:

$$(H_0 u, V_{\boldsymbol{q}} u) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} \left(\sum_{m=1}^{d} u(\boldsymbol{n} \pm \boldsymbol{b}_m)\right) V_{\boldsymbol{q}}(\boldsymbol{n}) u(\boldsymbol{n})$$
$$= \sum_{m=1}^{d} \sum_{\boldsymbol{n}\in\mathbb{Z}^d} u(\boldsymbol{n}+\boldsymbol{b}_m) u(\boldsymbol{n}) (V_{\boldsymbol{q}}(\boldsymbol{n}) + V_{\boldsymbol{q}}(\boldsymbol{n}+\boldsymbol{b}_m)). \tag{6.4}$$

Note that by our construction and the fact that q_i's are even,

$$V_{\boldsymbol{q}}(\boldsymbol{n}) + V_{\boldsymbol{q}}(\boldsymbol{n}+\boldsymbol{b}_m) = \begin{cases} -\delta^3/d & \text{if } \boldsymbol{n} \equiv -\boldsymbol{b}_m \text{ or } \boldsymbol{0} \pmod{\boldsymbol{q}} \\ 0 & \text{otherwise} \end{cases} \tag{6.5}$$

Combining (6.4) with (6.5), we get

$$|(H_0 u, V_{\boldsymbol{q}} u)| \leq \frac{\delta^3}{d} \sum_{m=1}^{d} \sum_{\boldsymbol{n}\in\mathbb{Z}^d} |u(\boldsymbol{n}+\boldsymbol{b}_m)||u(\boldsymbol{n})| \leq \delta^3. \tag{6.6}$$

Now combining (6.2), (6.3) with (6.6), we get

$$\|(H_0 + V_{\boldsymbol{q}})u\|^2 \geq (1-\delta^2)^2\delta^2 - 2\delta^3 > \frac{1}{4}\delta^2,$$

provided δ small. □

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Appendix A.

Proof of Lemma 4.1. Without loss of generality we could assume $E \geq 0$.
If $d = 2\tilde{d}$ is an even number, then we could take $(0, 1/2) \ni \theta_1 = \cdots = \theta_{\tilde{d}} = 1-\theta_{\tilde{d}+1} = \cdots = 1-\theta_{2\tilde{d}}$ be such that $\cos 2\pi\theta_1 = \frac{E}{4\tilde{d}} \neq \pm 1$.
If $d = 2\tilde{d}+1$ is an odd number and $E \in [2, 4\tilde{d}+2)$, then we could take $\theta_{2\tilde{d}+1} = 0$ and $(0, 1/2) \ni \theta_1 = \cdots = \theta_{\tilde{d}} = 1-\theta_{\tilde{d}+1} = \cdots = 1-\theta_{2\tilde{d}}$ be such that $\cos 2\pi\theta_1 = \frac{E-2}{4\tilde{d}} \neq \pm 1$.
If $d = 2\tilde{d}+1$ is an odd number and $E \in [0, 2)$, then we could take $\theta_{2\tilde{d}+1} = \frac{1}{2}$ and $(0, 1/2) \ni \theta_1 = \cdots = \theta_{\tilde{d}} = 1-\theta_{\tilde{d}+1} = \cdots = 1-\theta_{2\tilde{d}}$ be such that $\cos 2\pi\theta_1 = \frac{E+2}{4\tilde{d}} \neq \pm 1$. □

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