



Manifestations of Dynamical Localization in the Disordered XXZ Spin Chain

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Abstract: We study disordered XXZ spin chains in the Ising phase exhibiting droplet localization, a single cluster localization property we previously proved for random XXZ spin chains. It holds in an energy interval I near the bottom of the spectrum, known as the droplet spectrum. We establish dynamical manifestations of localization in the energy window I , including non-spreading of information, zero-velocity Lieb–Robinson bounds, and general dynamical clustering. Our results do not rely on knowledge of the dynamical characteristics of the model outside the droplet spectrum. A byproduct of our analysis is that for random XXZ spin chains this droplet localization can happen only inside the droplet spectrum.

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1. Introduction

We study disordered XXZ spin chains in the Ising phase exhibiting droplet localization. This is a single cluster localization property we previously proved for random XXZ spin chains inside the droplet spectrum [16].

The basic phenomenon of Anderson localization in the single particle framework is that disorder can cause localization of electron states and thereby manifest itself in properties such as non-spreading of wave packets under time evolution and absence of dc transport. The mechanism behind this behavior is well understood by now, both physically and mathematically (e.g., [4, 6, 15, 18, 22, 28]). Many manifestations of single-particle Anderson localization remain valid if one considers a fixed number of interacting particles, e.g., [3, 13, 29].

The situation is radically different in the many-body setting. Little is known about the thermodynamic limit of an interacting electron gas in a random environment, i.e., an infinite volume limit in which the number of electrons grows proportionally to the volume. Even simplest models where the individual particle Hilbert space is finite dimensional (spin systems) pose considerable analytical and numerical challenges, due to the fact that the number of degrees of freedom involved grows exponentially fast with the size of the system.

The limited evidence from perturbative [5, 9, 20, 24, 27, 42] and numerical [12, 33, 37, 38] approaches supports the persistence of a many-body localized (MBL) phase for one-dimensional spin systems in the presence of weak interactions. The numerics also suggests the existence of transition from a many-body localized (MBL) phase to delocalized phases as the strength of interactions increases, [8, 10, 37, 38, 40].

Mathematically rigorous results on localization in a true many-body system have been until very recently confined to investigations of exactly solvable (quasi-free) models (see [1, 30, 41]). More recent progress has been achieved primarily in the study of the XXZ spin chain, a system that is not integrable but yet amenable to rigorous analysis. The first results in this direction established the exponential clustering property for zero temperature correlations of the André-Aubry quasi-periodic model [31, 32]. The authors recently proved localization results for the random XXZ spin chain in the droplet spectrum [16]. Related results are given in [11].

In [16, Theorem 2.1], the authors obtained a localization result for the droplet spectrum eigenstates of the random XXZ spin chain in the Ising phase. This result can be interpreted as the statement that a typical eigenstate in this part of the spectrum behaves as an effective quasi-particle, localized, in the appropriate sense, in the presence of a random field.

In this paper we study disordered XXZ spin chains exhibiting the same droplet localization property we proved in [16, Theorem 2.1], which we call Property DL. We draw conclusions concerning the dynamics of the XXZ spin chain based exclusively on Property DL.

For completely localized many-body systems, the dynamical manifestation of localization is often expressed in terms of the non-spreading of information under the time evolution. An alternative (and equivalent) description is the zero-velocity Lieb–Robinson bound. (See, e.g., [21].)

There is, however, some difficulty in even formulating our results for disordered XXZ spin chains. Property DL has been established in [16, Theorem 2.1] only inside the droplet spectrum, an interval at the bottom of the spectrum. In fact, Theorem 1 below shows that Property DL as stated can only hold inside the droplet spectrum for random XXZ spin chains, showing the near optimality of the energy window where

it was proven in [16, Theorem 2.1]. To resolve this issue, we recast non-spreading of information and the zero-velocity Lieb–Robinson bound as a problem on the subspace of the Hilbert space associated with the given energy window in which Property DL holds. This leads to a number of interesting findings, formulated below in Theorem 2 (non-spreading of information), Theorem 3 (zero-velocity Lieb–Robinson bounds), and Theorem 4 (general dynamical clustering).

As we mentioned earlier, our methodology in [16] is limited to the states near the bottom of the spectrum and sheds light only on what physicists call zero temperature localization. It is unrealistic to expect that this approach can yield insight about extensive energies of magnitude comparable to the system size which is the essence of MBL. In fact, numerical studies suggest the presence of a many-body mobility edge in the random XXZ chain, at least for sufficiently small disorder [8, 10, 37, 38]. Nonetheless, we believe that the ideas presented here will be useful in understanding the transport properties of interacting systems that have a mobility edge, such as the Quantum Hall Effect [14, 23, 39].

Some of the results in this paper were announced in [17].

This paper is organized as follows: The model, Property DL, and the main theorems are stated in Sect. 2. We introduce some useful tools and collect needed technical results in Sect. 3, and a lemma about spin chains is presented in Appendix A. Section 4 is devoted to the proof that Property DL only holds inside the droplet spectrum for random XXZ spin chains (Theorem 1). Non-spreading of information (Theorem 2) is proven in Sect. 5. Zero-velocity Lieb–Robinson bounds (Theorem 3) are proven in Sect. 6. Finally, the proof of general dynamical clustering (Theorem 4) is given in Sect. 7.

2. Model and Results

The infinite disordered XXZ spin chain (in the Ising phase) is given by the (formal) Hamiltonian

$$H = H_\omega = H_0 + \lambda B_\omega, \quad H_0 = \sum_{i \in \mathbb{Z}} h_{i,i+1}, \quad B_\omega = \sum_{i \in \mathbb{Z}} \omega_i \mathcal{N}_i, \quad (1)$$

acting on $\bigotimes_{i \in \mathbb{Z}} \mathbb{C}_i^2$, with $\mathbb{C}_i^2 = \mathbb{C}^2$ for all $i \in \mathbb{Z}$, the quantum spin configurations on the one-dimensional lattice \mathbb{Z} , where

- (i) $h_{i,i+1}$, the local next-neighbor Hamiltonian, is given by

$$h_{i,i+1} = \frac{1}{4} (I - \sigma_i^z \sigma_{i+1}^z) - \frac{1}{4\Delta} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y), \quad (2)$$

where $\sigma^x, \sigma^y, \sigma^z$ are the standard Pauli matrices ($\sigma_i^x, \sigma_i^y, \sigma_i^z$ act on \mathbb{C}_i^2) and $\Delta > 1$ is a parameter;

- (ii) $\mathcal{N}_i = \frac{1}{2}(1 - \sigma_i^z)$ is the local number operator at site i (the projection onto the down-spin state at site i);
- (iii) $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ are identically distributed random variables whose joint probability distribution is ergodic with respect to shifts in \mathbb{Z} , and the single-site probability distribution μ satisfies

$$\{0, 1\} \subset \text{supp } \mu \subset [0, 1] \quad \text{and} \quad \mu(\{0\}) = 0; \quad (3)$$

- (iv) $\lambda > 0$ is the disorder parameter.

If in addition $\{\omega_i\}_{i \in \mathbb{Z}}$ are independent random variables we call H_ω a *random XXZ spin chain*.

The choice $\Delta > 1$ specifies the Ising phase. The Heisenberg chain corresponds to $\Delta = 1$, and the Ising chain is obtained in the limit $\Delta \rightarrow \infty$.

We set $e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, spin up and spin down, respectively. Recall $\sigma^z e_\pm = \pm e_\pm$. Thus, if $\mathcal{N} = \frac{1}{2}(1 - \sigma^z)$, we have $\mathcal{N}e_+ = 0$ and $\mathcal{N}e_- = e_-$.

The operator H_ω as in (1) with $B_\omega \geq 0$ can be defined as an unbounded nonnegative self-adjoint operator as follows: Let \mathcal{H}_0 be the vector subspace of $\otimes_{i \in \mathbb{Z}} \mathbb{C}_i^2$ spanned by tensor products of the form $\otimes_{i \in \mathbb{Z}} e_i$, $e_i \in \{e_+, e_-\}$, with a finite number of spin downs, equipped with the tensor product inner product, and let \mathcal{H} be its Hilbert space completion. H_ω , defined in \mathcal{H}_0 by (1), is an essentially self-adjoint operator on \mathcal{H} . Moreover, the ground state energy of H_ω is 0, with the unique ground state (or *vacuum*) given by the all-spins up configuration $\psi_0 = \otimes_{i \in \mathbb{Z}} e_+$. Note that $\mathcal{N}_i \psi_0 = 0$ for all $i \in \mathbb{Z}$ and $\|\psi_0\| = 1$.

The spectrum of H_0 is known to be of the form [19,36] (recall $\Delta > 1$):

$$\sigma(H_0) = \{0\} \cup \left[1 - \frac{1}{\Delta}, 1 + \frac{1}{\Delta}\right] \cup \left\{ \left[2\left(1 - \frac{1}{\Delta}\right), \infty\right) \cap \sigma(H_0) \right\}. \tag{4}$$

We will call $I_1 = \left[1 - \frac{1}{\Delta}, 2\left(1 - \frac{1}{\Delta}\right)\right)$ the *droplet spectrum*. (Droplet states in the Ising phase of the XXZ chain were first described in [36] (see also [19,35]); they have energies in the interval $\left[1 - \frac{1}{\Delta}, 1 + \frac{1}{\Delta}\right]$. The pure droplet spectrum is actually $I_1 \cap \sigma(H_0)$; we call I_1 the droplet spectrum for convenience.)

Since the disordered XXZ spin chain Hamiltonian H_ω is ergodic with respect to translation in \mathbb{Z} , $B_\omega \geq 0$, and $B_\omega \psi_0 = 0$, standard considerations imply that H_ω has nonrandom spectrum Σ , and

$$\sigma(H_\omega) = \Sigma = \{0\} \cup \left\{ \left[1 - \frac{1}{\Delta}, \infty\right) \cap \Sigma \right\} \quad \text{almost surely.} \tag{5}$$

(In the case of a random XXZ spin chain Hamiltonian H_ω with a continuous single-site probability distribution standard arguments yield $\Sigma = \{0\} \cup \left[1 - \frac{1}{\Delta}, \infty\right)$.)

We consider the restrictions of H_ω to finite intervals $[-L, L]$, $L \in \mathbb{N}$ (We will write $[-L, L]$ for $[-L, L] \cap \mathbb{Z}$, etc., when it is clear from the context.) We let $\mathcal{H}^{(L)} = \mathcal{H}_{[-L, L]}$, where $\mathcal{H}_S = \otimes_{i \in S} \mathbb{C}_i^2$ for $S \subset \mathbb{Z}$ finite, and define the self-adjoint operator

$$H^{(L)} = H_\omega^{(L)} = \sum_{i=-L}^{L-1} h_{i,i+1} + \lambda \sum_{i=-L}^L \omega_i \mathcal{N}_i + \beta(\mathcal{N}_{-L} + \mathcal{N}_L) \quad \text{on } \mathcal{H}^{(L)}. \tag{6}$$

We take (and fix) $\beta \geq \frac{1}{2}\left(1 - \frac{1}{\Delta}\right)$ in the boundary term, which guarantees that the random spectrum of $H_\omega^{(L)}$ preserves the spectral gap of size $1 - \frac{1}{\Delta}$ above the ground state energy:

$$\sigma(H_\omega^{(L)}) = \{0\} \cup \left\{ \left[1 - \frac{1}{\Delta}, \infty\right) \cap \sigma(H_\omega^{(L)}) \right\}. \tag{7}$$

The ground state energy of $H_\omega^{(L)}$ is 0, with the all-spins up configuration state $\psi_0^{(L)} = \otimes_{i \in [-L, L]} e_+ \in \mathcal{H}^{(L)}$ being a ground state, which is unique almost surely since $\sum_{i=-L}^L \omega_i \mathcal{N}_i \neq 0$ almost surely (which rules out the all-spins down configuration in $H_\omega^{(L)}$ as a ground state).

Given an interval I , we set $\sigma_I(H_\omega^{(L)}) = \sigma(H_\omega^{(L)}) \cap I$, and let

$$G_I = \{g : \mathbb{R} \rightarrow \mathbb{C} \text{ Borel measurable, } |g| \leq \chi_I\}. \tag{8}$$

In this article we consider a disordered XXZ spin chain as in (1) for which we have localization in an interval $[1 - \frac{1}{\Delta}, \Theta_1]$ in the following form, where $\| \cdot \|_1$ is the trace norm.

Property DL. *Let $H = H_\omega$ be a disordered XXZ spin chain. There exist $\Theta_1 > \Theta_0 = 1 - \frac{1}{\Delta}$ and constants $C < \infty$ and $m > 0$, such that, setting $I = [\Theta_0, \Theta_1]$, we have, uniformly in L ,*

$$\mathbb{E} \left(\sup_{g \in G_I} \left\| \mathcal{N}_i g(H^{(L)}) \mathcal{N}_j \right\|_1 \right) \leq C e^{-m|i-j|} \text{ for all } i, j \in [-L, L]. \tag{9}$$

We proved in [16] that Property DL holds in the droplet spectrum for random XXZ spin chains, and show in Theorem 1 below that it can only hold inside the droplet spectrum, suggesting the name Property DL (for Droplet Localization). Property DL is relevant only for the eigenstates that populate the droplet spectrum, an interval at the bottom of the spectrum. These states are localized near energetically preferable droplets (single cluster) configurations in the underlying Hilbert space, since non-droplet (i.e., many-clusters) configurations always have energy above the droplet spectrum. In addition to non-droplet configurations, large droplets will pick up enough field energy to have total energy above the droplet spectrum. We make no assertions about eigenstates with energies above the droplet spectrum.

If $H = H_\omega$ is a random XXZ spin chain, then $H^{(L)}$ almost surely has simple spectrum. A simple analyticity based argument for this can be found in [2, Appendix A]. (The argument is presented there for the XY chain, but it holds for every random spin chain of the form $H_0 + \sum_{k=-L}^L \omega_k \mathcal{N}_k$ in $\bigotimes_{i \in [-L, L]} \mathbb{C}_i^2$.) Thus, almost surely, all its normalized eigenstates can be labeled as ψ_E where E is the corresponding eigenvalue. In particular,

$$\left\| \mathcal{N}_i P_E^{(L)} \mathcal{N}_j \right\|_1 = \|\mathcal{N}_i \psi_E\| \|\mathcal{N}_j \psi_E\|, \tag{10}$$

where $P_E^{(L)} = \chi_{\{E\}}(H^{(L)})$ and $\| \cdot \|_1$ is the trace norm.

Given $0 \leq \delta < 1$, we set

$$I_{1,\delta} = \left[1 - \frac{1}{\Delta}, (2 - \delta) \left(1 - \frac{1}{\Delta} \right) \right]; \tag{11}$$

note that $I_{1,\delta} \subsetneq I_1$ if $0 < \delta < 1$. The following result is proved in [16].

Droplet localization ([16, Theorem 2.1]). *Let $H = H_\omega$ be a random XXZ spin chain whose single-site probability distribution is absolutely continuous with a bounded density. There exists a constant $K > 0$ with the following property: If $\Delta > 1$, $\lambda > 0$, and $0 < \delta < 1$ satisfy*

$$\lambda (\delta(\Delta - 1))^{\frac{1}{2}} \min \{1, (\delta(\Delta - 1))\} \geq K, \tag{12}$$

then there exist constants $C < \infty$ and $m > 0$ such that we have, uniformly in L ,

$$\mathbb{E} \left(\sum_{E \in \sigma_{I_{1,\delta}}(H^{(L)})} \|\mathcal{N}_i \psi_E\| \|\mathcal{N}_j \psi_E\| \right) \leq C e^{-m|i-j|} \text{ for all } i, j \in [-L, L], \tag{13}$$

and, as a consequence,

$$\mathbb{E} \left(\sup_{g \in G_{I_1, \delta}} \left\| \mathcal{N}_i g(H^{(L)}) \mathcal{N}_j \right\|_1 \right) \leq C e^{-m|i-j|} \text{ for all } i, j \in [-L, L]. \tag{14}$$

In particular, Property DL holds in the interval $I_{1, \delta}$.

The interval $I_{1, \delta}$ in [16, Theorem 1.1] is close to optimal, as the following theorem shows that for a random XXZ spin chain localization in the form of Property DL (as in (9)) is only allowed in the droplet spectrum.

Theorem 1 (Optimality of the droplet spectrum). *Suppose Property DL is valid for a random XXZ spin chain H . Then $\Theta_1 \leq 2\Theta_0$, that is, if I is the interval in Property DL, then we must have $I = I_{1, \delta}$ for some $0 \leq \delta < 1$.*

Let H_ω be a disordered XXZ spin chain satisfying Property DL. We consider the intervals $I = [\Theta_0, \Theta_1]$ and $I_0 = [0, \Theta_1]$, where Θ_0, Θ_1 are given in Property DL. We mostly omit ω from the notation. We write $P_B^{(L)} = \chi_B(H^{(L)})$ for a Borel set $B \subset \mathbb{R}$, and let $P_E^{(L)} = P_{\{E\}}^{(L)}$ for $E \in \mathbb{R}$. It follows from (7) that $P_{I_0}^{(L)} = P_0^{(L)} + P_I^{(L)}$. Since $\mathcal{N}_i P_0^{(L)} = P_0^{(L)} \mathcal{N}_i = 0$ for all $i \in [-L, L]$, G_I may be replaced by G_{I_0} in (9). By $m > 0$ we will always denote the constant in (9). C will always denote a constant, independent of the relevant parameters, which may vary from equation to equation, and even inside the same equation.

Given an interval $J \subset [-L, L]$, a local observable X with support J is an operator on $\otimes_{j \in J} \mathbb{C}_j^2$, considered as an operator on $\mathcal{H}^{(L)}$ by acting as the identity on spins not in J . (We defined supports as intervals for convenience. Note that we do not ask J to be the smallest interval with this property, supports of observables are not uniquely defined.)

Given a local observable X , we will generally specify a support for X , denoted by $\mathcal{S}_X = [s_X, r_X]$. We always assume $\emptyset \neq \mathcal{S}_X \subset [-L, L]$. Given two local observables X, Y we set $\text{dist}(X, Y) = \text{dist}(\mathcal{S}_X, \mathcal{S}_Y)$.

Given $\ell \geq 1$ and $B \subset [-L, L]$, we set $B_\ell = \{j \in [-L, L]; \text{dist}(j, B)\} \leq \ell$. In particular, given a local observable X we let

$$\mathcal{S}_{X, \ell} = (\mathcal{S}_X)_\ell = [s_X - \ell, r_X + \ell] \cap [-L, L]. \tag{15}$$

In this paper we derive several manifestations of dynamical localization for H from Property DL. The time evolution of a local observable under $H^{(L)}$ is given by

$$\tau_t^{(L)}(X) = e^{itH^{(L)}} X e^{-itH^{(L)}} \text{ for } t \in \mathbb{R}. \tag{16}$$

(We also mostly omit L from the notation, and write τ_t for $\tau_t^{(L)}$.)

For a completely localized many-body system (i.e., localized at all energies), dynamical localization is often expressed as the *non-spreading of information under the time evolution*: Given a local observable X , for all $\ell \geq 1$ and $t \in \mathbb{R}$ there is a local observable $X_\ell(t)$ with support $\mathcal{S}_{X, \ell}$, such that $\|X_\ell(t) - \tau_t(X)\| \leq C \|X\| e^{-c\ell}$, with the constants C and $c > 0$ independent of X, t , and L . Since we only have localization in the energy interval I , and hence also in I_0 , we should only expect non-spreading of information in these energy intervals.

Thus, given an energy interval J , we consider the sub-Hilbert space $\mathcal{H}_J^{(L)} = \text{Ran } P_J^{(L)}$, spanned by the eigenstates of $H^{(L)}$ with energies in J , and localize an observable

X in the energy interval J by considering its restriction to $\mathcal{H}_J^{(L)}$, $X_J = P_J^{(L)} X P_J^{(L)}$. Clearly $\tau_t \left(X_J^{(L)} \right) = \left(\tau_t \left(X^{(L)} \right) \right)_J$.

Property DL implies non-spreading of information in the energy interval I_0 .

Theorem 2 (Non-spreading of information). *Let $H = H_\omega$ be a disordered XXZ spin chain satisfying Property DL. There exists $C < \infty$, independent of L , such that for all local observables X , $t \in \mathbb{R}$ and $\ell > 0$ there is a local observable $X_\ell(t) = (X_\ell(t))_\omega$ with support $\mathcal{S}_{X,\ell}$ satisfying*

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| (X_\ell(t) - \tau_t(X))_{I_0} \right\|_1 \right) \leq C \|X\| e^{-\frac{1}{16} m \ell}. \tag{17}$$

We give an explicit expression for $X_\ell(t)$ in (114). Note that $X_I = (X_{I_0})_I$, and hence (17) implies the same statement with I substituted for I_0 .

Another manifestation of dynamical localization is the existence of zero-velocity Lieb–Robinson (LR) bounds in the interval of localization. The following theorem states a zero-velocity Lieb–Robinson bound in the energy interval I . If we include the ground state, i.e., if we look for Lieb–Robinson type bounds in the energy interval I_0 , the situation is more complicated, and the zero-velocity Lieb–Robinson bound holds for the double commutator; the commutator requires counterterms. Note that $[\tau_t(X_I), Y_I] \neq ([\tau_t(X), Y])_I$. (We mostly omit ω and L from the notation.)

Theorem 3 (Zero velocity LR bounds). *Let $H = H_\omega$ be a disordered XXZ spin chain satisfying Property DL. Let X, Y and Z be local observables. The following holds uniformly in L :*

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| [\tau_t(X_I), Y_I] \right\|_1 \right) \leq C \|X\| \|Y\| e^{-\frac{1}{8} m \text{dist}(X,Y)}, \tag{18}$$

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| [\tau_t(X_{I_0}), Y_{I_0}] - (\tau_t(X) P_0 Y - Y P_0 \tau_t(X))_I \right\|_1 \right) \\ \leq C \|X\| \|Y\| e^{-\frac{1}{8} m \text{dist}(X,Y)}, \end{aligned} \tag{19}$$

$$\begin{aligned} \mathbb{E} \left(\sup_{t,s \in \mathbb{R}} \left\| [[\tau_t(X_{I_0}), \tau_s(Y_{I_0})], Z_{I_0}] \right\|_1 \right) \\ \leq C \|X\| \|Y\| \|Z\| e^{-\frac{1}{8} m \min\{\text{dist}(X,Y), \text{dist}(X,Z), \text{dist}(Y,Z)\}}. \end{aligned} \tag{20}$$

Moreover, for the random XXZ spin chain the estimate (19) is not true without the counterterms.

The proofs of these zero-velocity Lieb–Robinson bounds do not involve tricky cancellations from the commutators. Cancellations from the commutator are only used to justify that it suffices to prove the zero-velocity Lieb–Robinson bounds for local observables Z satisfying $\bigotimes_{j \in S_Z} (1 - \mathcal{N}_j) Z \bigotimes_{j \in S_Z} (1 - \mathcal{N}_j) = 0$ (see Sect. 3.1 for details). Under this assumption each term in the expansion of the commutators is shown to satisfy the desired bound.

The counterterms in (19) are generated by the interaction between the ground state and states corresponding to the energy interval I under the dynamics. Here, and also in

Theorem 4 below, they are linear combinations of terms of the form $(\tau_t(X) P_0 Y)_I$ and $(Y P_0 \tau_t(X))_I$. Note that

$$\begin{aligned} \|(\tau_t(X) P_0 Y)_I\|_1 &= \|(\tau_t(X) P_0 Y)_I\| = \|P_I Y^* \psi_0\| \|P_I X \psi_0\|, \\ \|(Y P_0 \tau_t(X))_I\|_1 &= \|(Y P_0 \tau_t(X))_I\| = \|P_I X^* \psi_0\| \|P_I Y \psi_0\|, \end{aligned} \tag{21}$$

which do not depend on either t or $\text{dist}(X, Y)$.

Another manifestation of localization is the dynamical exponential clustering property. Let $B \subset \mathbb{R}$ be a Borel set. We define the truncated time evolution of an observable X by $(H = H_\omega^{(L)})$,

$$\tau_t^B(X) = e^{itH_B} X e^{-itH_B}, \quad \text{where } H_B = P_B H. \tag{22}$$

Note that $(\tau_t^B(X))_B = (\tau_t(X))_B = \tau_t(X_B)$.

The correlator operator of two observables X and Y in the energy window B is given by $(\bar{P}_B = 1 - P_B)$

$$R_B(X, Y) = P_B X \bar{P}_B Y P_B = (X \bar{P}_B Y)_B. \tag{23}$$

If E is a simple eigenvalue with normalized eigenvector ψ_E , we have, with $R_E(X, Y) = R_{\{E\}}(X, Y)$,

$$\text{tr}(R_E(X, Y)) = \langle \psi_E, X Y \psi_E \rangle - \langle \psi_E, X \psi_E \rangle \langle \psi_E, Y \psi_E \rangle. \tag{24}$$

The following result is proved in [16].

Dynamical exponential clustering ([16, Theorem 1.1]). *Let $H = H_\omega$ be a random XXZ spin chain, and assume (13) holds in an interval I . Then, for all local observables X and Y we have, uniformly in L ,*

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \sum_{E \in \sigma_I(H^{(L)})} \left| \text{tr} \left(R_E(\tau_t^I(X), Y) \right) \right| \right) \leq C \|X\| \|Y\| e^{-m \text{dist}(X, Y)}, \tag{25}$$

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \sum_{E \in \sigma_I(H^{(L)})} \left| \text{tr} \left(R_E(\tau_t(X_I), Y_I) \right) \right| \right) \leq C \|X\| \|Y\| e^{-m \text{dist}(X, Y)}, \tag{26}$$

and

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \left| \text{tr} \left(R_I(\tau_t^I(X), Y) \right) \right| \right) \leq C \|X\| \|Y\| e^{-m \text{dist}(X, Y)}. \tag{27}$$

The estimate (26) is not the same as (25), but it can be proven the same way; the proof of [16, Lemma 3.1] is actually simpler in this case.

Since

$$\text{tr} \left(R_I(\tau_t^I(X), Y) \right) = \sum_{E \in \sigma_I(H^{(L)})} \left\langle \psi_E, \tau_t^I(X) \bar{P}_I Y \psi_E \right\rangle, \tag{28}$$

(27) is a statement about the diagonal elements of the correlator operator $R_I(\tau_t^I(X), Y)$. We will now state a more general dynamical clustering result that is not restricted to diagonal elements. The result, which holds in an interval of localization satisfying the conclusions of Theorem 1, requires counterterms.

Theorem 4 (General dynamical clustering). *Let $H = H_\omega$ be a disordered XXZ spin chain satisfying Property DL. Fix an interval $K = [\Theta_0, \Theta_2]$, where $\Theta_0 < \Theta_2 < \min \{2\Theta_0, \Theta_1\}$, and $\alpha \in (0, 1)$. There exists $\tilde{m} > 0$, such that for all local observables X and Y we have, uniformly in L ,*

$$\begin{aligned} & \mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| R_K \left(\tau_t^K(X), Y \right) - \left(\tau_t^K(X) P_0 Y + \tau_t^K(Y) P_0 X \right)_K \right\| \right) \\ & \leq C (1 + \ln (\min \{ |S_X|, |S_Y| \})) \|X\| \|Y\| e^{-\tilde{m}(\text{dist}(X,Y))^\alpha}, \end{aligned} \tag{29}$$

and

$$\begin{aligned} & \mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| \left[\tau_t^K(X), Y \right] \right\| \right) \\ & \leq C (1 + \ln (\min \{ |S_X|, |S_Y| \})) \|X\| \|Y\| e^{-\tilde{m}(\text{dist}(X,Y))^\alpha}, \end{aligned} \tag{30}$$

where

$$\begin{aligned} & \left[\tau_t^K(X), Y \right] = \left[\tau_t^K(X), Y \right] \\ & - \left(\tau_t^K(X) P_0 Y + \tau_t^K(Y) P_0 X \right) + \left(Y P_0 \tau_t^K(X) + X P_0 \tau_t^K(Y) \right). \end{aligned} \tag{31}$$

Moreover, for the random XXZ spin chain the estimates (29) and (30) are not true without the counterterms.

While it is obvious where the counterterms in (19) come from, the same is not true in (29), where the time evolution in the second term seems to sit in the *wrong* place: it is $\tau_t^K(Y)$ and not $\tau_t^K(X)$. It turns out this term encodes information about the states above the energy window K , and the appearance of $\tau_t^K(Y)$ is related to the reduction of this data to P_0 , as can be seen in the proof.

Remark 1. One may wonder why the counterterms in (29) do not appear in (27). The reason is that their traces obey decay estimates similar to (27) with $\alpha = 1$, see Lemma 4.

3. Preliminaries

We make extensive use of the decomposition of a local observable with respect to the projection on the ground state on its support, introduced in Sect. 3.1. This decomposition will play a prominent role in Sect. 3.2, where it is used to establish several localization properties of operators $g(H)$ for g supported in the interval I where Property DL is assumed. These technical lemmas are extensively used in the proofs of non-spreading of information and zero-velocity LR bounds in Sects. 5 and 6 below.

The proofs of Theorems 1 and 4 will in addition require bounds on operators of the form $Xf(H)Y$, which will be established in Sect. 3.3 by using an argument of Hastings, combining LR-bounds with the Fourier transform.

A final technical lemma in Sect. 3.4 provides several lower bounds which show that σ^x correlations in the local ground states do not decay. This will be the key ingredient for the proofs of the non-triviality of the counterterms in Theorems 3 and 4.

3.1. *Decomposition of local observables.* Given $S \subset [-L, L] \subset \mathbb{Z}$, $S \neq \emptyset$, we define projections $P_{\pm}^{(S)}$ by

$$P_+^{(S)} = \bigotimes_{j \in S} \frac{1}{2}(1 + \sigma_j^z) = \bigotimes_{j \in S} (1 - \mathcal{N}_j) \quad \text{and} \quad P_-^{(S)} = 1 - P_+^{(S)}. \tag{32}$$

Note that

$$P_-^{(S)} \leq \sum_{i \in S} \mathcal{N}_i. \tag{33}$$

In particular,

$$P_-^{(S)} P_0 = P_0 P_-^{(S)} = 0. \tag{34}$$

We also set $S^c = [-L, L] \setminus S$, and note that

$$P_+^{(S)} P_+^{(S^c)} = P_+^{(S^c)} P_+^{(S)} = P_+^{[-L, L]} = P_0. \tag{35}$$

Given an observable X , we set $P_{\pm}^{(X)} = P_{\pm}^{(S_X)}$, obtaining the decomposition

$$X = \sum_{a,b \in \{+, -\}} X^{a,b}, \quad \text{where} \quad X^{a,b} = P_a^{(X)} X P_b^{(X)}. \tag{36}$$

Moreover, since $P_+^{(X)}$ is a rank one projection on \mathcal{H}_{S_X} , we must have

$$X^{+,+} = \zeta_X P_+^{(X)}, \quad \text{where} \quad \zeta_X \in \mathbb{C}, \quad |\zeta_X| \leq \|X\|. \tag{37}$$

In particular,

$$(X - \zeta_X)^{+,+} = 0 \quad \text{and} \quad \|X - \zeta_X\| \leq 2 \|X\|. \tag{38}$$

We will use (38) to replace general local observables by local observables satisfying $X^{+,+} = 0$ in the proofs of Theorems 2, 3, and 4.

3.2. *Consequences of Property DL.* Property DL is defined in terms of one type of local observable, the local number operators \mathcal{N}_i . For proving the theorems we need to apply it to general local observables. Using the decomposition (36) it suffices to consider local observables of the type $P_{\pm}^{(S)}$ as in (32). The following lemmas give the consequences of Property DL we need for the proofs of the theorems.

Let H_{ω} be a disordered XXZ spin chain satisfying Property DL. We write $H = H_{\omega}^{(L)}$, and generally omit ω and L from the notation. The following results hold uniformly on L .

Lemma 1. *Let X, Y be local observables. Then*

$$\mathbb{E} \left(\sup_{g \in G_{I_0}} \left\| P_-^{(X)} g(H) P_-^{(Y)} \right\|_1 \right) \leq C e^{-m \text{dist}(X, Y)}, \tag{39}$$

$$\mathbb{E} \left(\left\| P_-^{(Y)} P_-^{(X)} P_{I_0} \right\|_1 \right) \leq C e^{-\frac{1}{2} m \text{dist}(X, Y)}. \tag{40}$$

Proof. It follows from (33) that setting $Z = (\sum_{i \in \mathcal{S}} \mathcal{N}_i)^{-1} P_-^{(\mathcal{S})}$, we have $\|Z\| \leq 1$ and $P_-^{(\mathcal{S})} = (\sum_{i \in \mathcal{S}} \mathcal{N}_i) Z = Z (\sum_{i \in \mathcal{S}} \mathcal{N}_i)$, and hence we have

$$\left\| P_-^{(X)} g(H) P_-^{(Y)} \right\|_1 \leq \sum_{i \in \mathcal{S}_X, j \in \mathcal{S}_Y} \left\| \mathcal{N}_i g(H) \mathcal{N}_j \right\|_1. \tag{41}$$

The estimate (39) then follows immediately from from (9) using [16, Eq. (3.25)]
 Similarly,

$$\left\| P_-^{(Y)} P_-^{(X)} P_{I_0} \right\|_1 = \left\| P_-^{(Y)} P_-^{(X)} P_I \right\|_1 \leq \sum_{k=-L}^L \left\| P_-^{(Y)} P_-^{(X)} P_I \mathcal{N}_k \right\|_1. \tag{42}$$

Since $[P_-^{(Y)}, P_-^{(X)}] = 0$,

$$\left\| P_-^{(Y)} P_-^{(X)} P_I \mathcal{N}_k \right\|_1 \leq \min \left\{ \left\| P_-^{(X)} P_I \mathcal{N}_k \right\|_1, \left\| P_-^{(Y)} P_I \mathcal{N}_k \right\|_1 \right\}, \tag{43}$$

so it follows from (39) that

$$\mathbb{E} \left(\left\| P_-^{(Y)} P_-^{(X)} P_I \mathcal{N}_k \right\|_1 \right) \leq C e^{-m \max\{\text{dist}(k, \mathcal{S}_X), \text{dist}(k, \mathcal{S}_Y)\}}. \tag{44}$$

Suppose, say, $\max \mathcal{S}_X < \min \mathcal{S}_Y$, and let $K = \frac{1}{2} (\max \mathcal{S}_X + \min \mathcal{S}_Y)$. Then,

$$\begin{aligned} \mathbb{E} \left(\left\| P_-^{(Y)} P_-^{(X)} P_I \right\|_1 \right) &\leq \sum_{k \leq K} e^{-m \text{dist}(k, \mathcal{S}_Y)} + \sum_{k \geq K} e^{-m \text{dist}(k, \mathcal{S}_X)} \\ &\leq C e^{-\frac{1}{2} m \text{dist}(X, Y)}, \end{aligned} \tag{45}$$

where the last calculation is done as in [16, Eq. (3.25)], yielding (40). \square

Lemma 2. *Let X and Y be local observables and $\ell \geq 1$.*

(i) *We have*

$$\mathbb{E} \left(\sup_{I \in G_I} \left\| P_-^{(X)} g(H) P_+^{(\mathcal{S}_{X,\ell})} \right\|_1 \right) \leq C e^{-m\ell}. \tag{46}$$

(ii) *If $\ell \leq \frac{1}{2} \text{dist}(X, Y)$, we have*

$$\mathbb{E} \left(\sup_{g \in G_I} \left\| P_+^{(\mathcal{S}_{Y,\ell}^c)} g(H) P_+^{(\mathcal{S}_{X,\ell}^c)} \right\|_1 \right) \leq C e^{-m(\text{dist}(X, Y) - 2\ell)}. \tag{47}$$

Proof. Let $\ell \geq 1$ and $g \in G_I$. If $\mathcal{S}_{X,\ell}^c = \emptyset$, (46) is obvious since $P_+^{(\mathcal{S}_{X,\ell})} = P_0$. If $\mathcal{S}_{X,\ell}^c \neq \emptyset$, using (35) we get

$$\left\| P_-^{(X)} g(H) P_+^{(\mathcal{S}_{X,\ell})} \right\|_1 = \left\| P_-^{(X)} g(H) P_-^{(\mathcal{S}_{X,\ell}^c)} P_+^{(\mathcal{S}_{X,\ell})} \right\|_1 \leq \left\| P_-^{(X)} g(H) P_-^{(\mathcal{S}_{X,\ell}^c)} \right\|_1, \tag{48}$$

and (46) follows from (48) and (39).

Similarly, using (35) twice, we get

$$\begin{aligned} \left\| P_+^{(\mathcal{S}_{Y,\ell}^c)} g(H) P_+^{(\mathcal{S}_{X,\ell}^c)} \right\|_1 &= \left\| P_+^{(\mathcal{S}_{Y,\ell}^c)} P_-^{(\mathcal{S}_{Y,\ell})} g(H) P_-^{(\mathcal{S}_{X,\ell})} P_+^{(\mathcal{S}_{X,\ell}^c)} \right\|_1 \\ &\leq \left\| P_-^{(\mathcal{S}_{Y,\ell})} g(H) P_-^{(\mathcal{S}_{X,\ell})} \right\|_1. \end{aligned} \tag{49}$$

If $\ell \leq \frac{1}{2} \text{dist}(X, Y)$, then $\text{dist}(\mathcal{S}_{X,\ell}, \mathcal{S}_{Y,\ell}) \geq \text{dist}(X, Y) - 2\ell$. In this case (47) follows from (49) and (39). \square

Lemma 3. *Let X, Y be local observables with $X^{+,+} = Y^{+,+} = 0$. Then*

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \sup_{g \in G_I} \left\| (\tau_t(X) g(H) Y)_I \right\|_1 \right) \leq C \|X\| \|Y\| e^{-\frac{1}{8}m \text{dist}(X,Y)}. \tag{50}$$

Proof. Since

$$\left\| (\tau_t(X) g(H) Y)_I \right\|_1 = \left\| (X e^{-itH} g(H) Y)_I \right\|_1, \tag{51}$$

it suffices to prove

$$\mathbb{E} \left(\sup_{g \in G_I} \left\| (X g(H) Y)_I \right\|_1 \right) \leq C \|X\| \|Y\| e^{-\frac{1}{8}m \text{dist}(X,Y)}. \tag{52}$$

Let X, Y be local observables with $X^{+,+} = Y^{+,+} = 0$, and let $0 < 2\ell = \text{dist}(X, Y)$. Set $\mathcal{S}_1 = \mathcal{S}_{X,\frac{\ell}{2}}^c, \mathcal{S}_2 = \mathcal{S}_{Y,\frac{\ell}{2}}^c$. Given $g \in G_I$, and inserting $1 = P_-^{(\mathcal{S}_j)} + P_+^{(\mathcal{S}_j)}, j = 1, 2$, we get

$$X g(H) Y = \sum_{a=\pm; b=\pm} X P_a^{(\mathcal{S}_1)} g(H) P_b^{(\mathcal{S}_2)} Y. \tag{53}$$

We estimate the norms of the terms on the right hand side separately. If one of the indices a, b , say $a = -$, we get

$$\begin{aligned} \left\| (X P_-^{(\mathcal{S}_1)} g(H) P_b^{(\mathcal{S}_2)} Y)_I \right\|_1 &\leq \|Y\| \left\| P_I X P_-^{(\mathcal{S}_1)} e^{-itH} g(H) \right\|_1 \\ &\leq \|Y\| \left\| P_I X P_-^{(\mathcal{S}_1)} P_I \right\|_1 = \|Y\| \left\| P_I P_-^{(\mathcal{S}_1)} X P_-^{(\mathcal{S}_1)} P_I \right\|_1 \\ &\leq \|X\| \|Y\| \left(\left\| P_I P_-^{(\mathcal{S}_1)} P_-^{(X)} \right\|_1 + \left\| P_-^{(X)} P_-^{(\mathcal{S}_1)} P_I \right\|_1 \right), \end{aligned} \tag{54}$$

where we have used the fact that $[P_-^{(\mathcal{S}_1)}, X] = 0, X^{+,+} = 0$, and $g \in G_I$. If $a = b = +$, we bound the corresponding contribution as

$$\left\| (X P_+^{(\mathcal{S}_1)} g(H) P_+^{(\mathcal{S}_2)} Y)_I \right\|_1 \leq \|X\| \|Y\| \left\| P_+^{(\mathcal{S}_1)} g(H) P_+^{(\mathcal{S}_2)} \right\|_1. \tag{55}$$

Using (40) and (47) we get

$$\begin{aligned} E \left(\sup_{g \in G_I} \left\| (X g(H) Y)_I \right\|_1 \right) &\leq C \|X\| \|Y\| \left(2e^{-\frac{m}{4}\ell} + e^{-m(\text{dist}(X,Y)-\ell)} \right) \\ &\leq C \|X\| \|Y\| e^{-\frac{1}{8}m \text{dist}(X,Y)}. \end{aligned} \tag{56}$$

\square

The following lemma justifies Remark 1.

Lemma 4. *Let X, Y be local observables. Then for all intervals $K \subset I$ we have*

$$E \left(\sup_{t \in \mathbb{R}} \left| \text{tr} \left(\tau_t^K(X) P_0 Y \right)_K \right| \right) \leq C \|X\| \|Y\| e^{-m \text{dist}(X, Y)}. \tag{57}$$

Proof. Given $K \subset I$, we have

$$\begin{aligned} \text{tr} \left(\tau_t^K(X) P_0 Y \right)_K &= \text{tr} P_K \tau_t(X) P_0 Y P_K = \text{tr} P_0 Y P_K \tau_t(X) P_0 \\ &= \text{tr} P_0 Y P_-^{(Y)} P_K e^{itH} P_-^{(X)} X P_0, \end{aligned} \tag{58}$$

where we used (37), (34), and $P_K P_0 = 0$. It follows that

$$\left| \text{tr} \left(\tau_t^K(X) P_0 Y \right)_K \right| \leq \|X\| \|Y\| \left\| P_-^{(Y)} P_K e^{itH} P_-^{(X)} \right\|_1. \tag{59}$$

The estimate (57) now follows from (39). \square

3.3. *Estimates with Fourier transforms.* Lemmas 5 and 6 below are used to prove Theorems 1 and 4.

Let H_ω be a disordered XXZ spin chain. Given a function $f \in C_c^\infty(\mathbb{R})$, we write its Fourier transform as

$$\hat{f}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itx} f(x) dx, \quad \text{and recall} \quad f(x) = \int_{\mathbb{R}} e^{-itx} \hat{f}(t) dt. \tag{60}$$

The following lemma is an adaptation of an argument of Hastings [25, 26], which combines the Lieb–Robinson bound with estimates on Fourier transforms.

Lemma 5. *Let $\alpha \in (0, 1)$, and consider a function $f \in C_c^\infty(\mathbb{R})$ such that*

$$\left| \hat{f}(t) \right| \leq C_f e^{-m_f |t|^\alpha} \quad \text{for all} \quad |t| \geq 1, \tag{61}$$

where C_f and $m_f > 0$ are constants. Then for all local observables X and Y we have

$$\begin{aligned} \left\| Xf(H)Y - \int_{\mathbb{R}} e^{-irH} Y \tau_r(X) \hat{f}(r) dr \right\| \\ \leq C_1 \|X\| \|Y\| \left(1 + \|\hat{f}\|_1 \right) e^{-m_1 (\text{dist}(X, Y))^\alpha}, \end{aligned} \tag{62}$$

where C_1 and $m_1 > 0$ are suitable constants (depending on C_f , m_f , and α), uniformly in L .

Proof. We have

$$\begin{aligned} Xf(H)Y &= X \left(\int_{\mathbb{R}} e^{-irH} \hat{f}(r) dr \right) Y = \int_{\mathbb{R}} e^{-irH} \tau_r(X) Y \hat{f}(r) dr \\ &= \int_{\mathbb{R}} e^{-irH} [\tau_r(X), Y] \hat{f}(r) dr + \int_{\mathbb{R}} e^{-irH} Y \tau_r(X) \hat{f}(r) dr \end{aligned} \tag{63}$$

The commutator in the first term can be estimated by the Lieb–Robinson bound (e.g. [34]):

$$\|[\tau_r(X), Y]\| \leq C \|X\| \|Y\| \min \left\{ e^{-\mu_1(\text{dist}(X,Y)-v|r|)}, 1 \right\}, \tag{64}$$

where $C, \mu_1 > 0, v > 0$ are constants, independent of L and of the random parameter ω . We get

$$\begin{aligned} & \left\| \int_{\mathbb{R}} e^{-irH} [\tau_r(X), Y] \hat{f}(r) dr \right\| \\ & \leq C \|X\| \|Y\| \left(\int_{|r| \leq \frac{\text{dist}(X,Y)}{2v}} e^{-\mu_1(\text{dist}(X,Y)-v|r|)} |\hat{f}(r)| dr + \int_{|r| \geq \frac{\text{dist}(X,Y)}{2v}} |\hat{f}(r)| dr \right) \\ & \leq C \|X\| \|Y\| \left(\|\hat{f}\|_1 e^{-\frac{\mu_1}{2} \text{dist}(X,Y)} + \int_{|r| \geq \frac{\text{dist}(X,Y)}{2v}} |\hat{f}(r)| dr \right) \\ & \leq C \|X\| \|Y\| \left(\|\hat{f}\|_1 e^{-\frac{\mu_1}{2} \text{dist}(X,Y)} + C_f e^{-\frac{m_f}{2} \left(\frac{\text{dist}(X,Y)}{2v}\right)^\alpha} \int_{\mathbb{R}} e^{-\frac{m_f}{2}|r|^\alpha} dr \right), \end{aligned} \tag{65}$$

where we assumed $\text{dist}(X, Y) \geq 2v$. The estimate (62) follows. \square

Lemma 5 will be combined with the following lemma.

Lemma 6. *Let $K = [\Theta_0, \Theta_2]$ and $f \in C_c^\infty(\mathbb{R})$ with $\text{supp } f \subset [a_f, b_f]$. Then for all local observables X and Y we have*

$$\int_{\mathbb{R}} \left(e^{-irH} Y \tau_r(X) \right)_K \hat{f}(r) dr = \int_{\mathbb{R}} \left(e^{-irH} Y P_{K_f} \tau_r(X) \right)_K \hat{f}(r) dr, \tag{66}$$

where

$$K_f = K + K - \text{supp } f \subset [2\Theta_0 - b_f, 2\Theta_2 - a_f]. \tag{67}$$

Proof. Let $K = [\Theta_0, \Theta_2], f \in C_c(\mathbb{R})$ with $\text{supp } f \subset [a_f, b_f]$. Then for all $E, E' \in K$ we have

$$\begin{aligned} P_E \left(\int_{\mathbb{R}} e^{-irH} Y \tau_r(X) \hat{f}(r) dr \right) P_{E'} &= \int_{\mathbb{R}} P_E e^{-irH} Y e^{irH} X e^{-irH} P_{E'} \hat{f}(r) dr \\ &= P_E Y \left(\int_{\mathbb{R}} e^{ir(H-E-E')} \hat{f}(r) dr \right) X P_{E'} = P_E Y f(E + E' - H) X P_{E'} \\ &= P_E Y P_{K_f} f(E + E' - H) X P_{E'} \\ &= P_E \left(\int_{\mathbb{R}} e^{-irH} Y P_{K_f} \tau_r(X) \hat{f}(r) dr \right) P_{E'}, \end{aligned} \tag{68}$$

where K_f is given in (67). The equality (66) follows. \square

3.4. *Counterterms.* The following lemma is used to prove the need for counterterms in Theorems 3 and 4.

Given vectors $\psi_1, \psi_2 \in \mathcal{H}^{(L)}$, we denote by $T(\psi_1, \psi_2)$ the rank one operator $T(\psi_1, \psi_2) = \langle \psi_2, \cdot \rangle \psi_1$. Recall

$$\|T(\psi_1, \psi_2)\| = \|T(\psi_1, \psi_2)\|_1 = \|\psi_1\| \|\psi_2\|.$$

Note that for all observables X and Y we have

$$X P_0^{(L)} Y = T\left(Y^* \psi_0^{(L)}, X \psi_0^{(L)}\right). \tag{69}$$

Lemma 7. *Let H_ω be a random XXZ spin chain. Consider an interval $K \subset [1 - \frac{1}{\Delta}, 1 + \frac{1}{\Delta}]$. Then there exist constants $\gamma_K > 0$ and R_K such that for all $i, j \in \mathbb{Z}$ with $|i - j| \geq R_K$, we have*

$$\mathbb{E} \left(\liminf_{L \rightarrow \infty} \left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x \right)_K \right\| \right) \geq \gamma_K > 0, \tag{70}$$

$$\mathbb{E} \left(\liminf_{L \rightarrow \infty} \left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x \pm \sigma_j^x P_0^{(L)} \sigma_i^x \right)_K \right\|_2^2 \right) \geq \gamma_K, \tag{71}$$

and

$$\mathbb{E} \left(\liminf_{L \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left\| \left(A^{(L)}(t) - \left(A^{(L)}(t) \right)^* \right)_K \right\|_2^2 dt \right) \geq 2\gamma_K, \tag{72}$$

where

$$A^{(L)}(t) = \tau_t^{(L)}(\sigma_i^x) P_0^{(L)} \sigma_j^x + \tau_t^{(L)}(\sigma_j^x) P_0^{(L)} \sigma_i^x. \tag{73}$$

Proof. Let H be a random XXZ spin chain, and let $\mathcal{N} = \sum_{i \in \mathbb{Z}} \mathcal{N}_i$ denote the total (down) spin number operator on \mathcal{H} . The self-adjoint operator \mathcal{N} has pure point spectrum. Its eigenvalues are $N = 0, 1, 2, \dots$, and the corresponding eigenspaces \mathcal{H}_N are spanned by all the spin basis states with N down spins. Since $[H, \mathcal{N}] = 0$, the eigenspaces \mathcal{H}_N are left invariant by H . The restriction H_N of H to \mathcal{H}_N is unitarily equivalent to an N -body discrete Schrödinger operator restricted to the fermionic subspace (e.g., [16, 19]).

In particular, $H_1 = H_{\omega,1}$ is unitarily equivalent to an one-dimensional Anderson model:

$$H_{\omega,1} \cong -\frac{1}{2\Delta} \mathcal{L}_1 + \left(1 - \frac{1}{\Delta}\right) + \lambda V_\omega \text{ on } \ell^2(\mathbb{Z}), \tag{74}$$

where \mathcal{L}_1 is the graph Laplacian on $\ell^2(\mathbb{Z})$ and V_ω is the random potential given by $V_\omega(i) = \omega_i$ for $i \in \mathbb{Z}$.

The same is true for restrictions to finite intervals $[-L, L]$, where we have the unitary equivalence

$$H_{\omega,1}^{(L)} \cong -\frac{1}{2\Delta} \mathcal{L}_1^{(L)} + \left(1 - \frac{1}{\Delta}\right) + \lambda V_\omega + \left(\beta - \frac{1}{2}\left(1 - \frac{1}{\Delta}\right)\right) (\chi_{\{-L\}} + \chi_{\{L\}}), \tag{75}$$

acting on $\ell^2([-L, L])$, where now $\mathcal{L}_1^{(L)}$ is the graph Laplacian on $\ell^2([-L, L])$ (e.g., [16]). Note that $H_{\omega,1}^{(L)}$ is the restriction of $H_{\omega,1}$ to $\ell^2([-L, L])$, up to a boundary term.

In what follows we will consider these unitary equivalences as equalities. In this case, if $i \in [-L, L]$ we have $\sigma_i^x \psi_0^{(L)} = \delta_i \in \ell^2([-L, L])$, Note that for the infinite volume Anderson model in (74) we have

$$\sigma(H_1) \supset \Sigma_1 := [1 - \frac{1}{\Delta}, 1 + \frac{1}{\Delta}] \text{ almost surely.} \tag{76}$$

The following holds for all $\omega \in [0, 1]^{\mathbb{Z}}$: We have $\lim_{L \rightarrow \infty} H_1^{(L)} = H_1$ in the strong resolvent sense, and hence $\lim_{L \rightarrow \infty} f(H_1^{(L)}) = f(H_1)$ strongly for all bounded continuous functions f on \mathbb{R} . (For an interval $J \subset \mathbb{Z}$, we consider $\ell^2(J)$ as a subspace of $\ell^2(\mathbb{Z})$ in the obvious way: $\ell^2(\mathbb{Z}) = \ell^2(J) \oplus \ell^2(\mathbb{Z} \setminus J)$.) In particular, for f real valued with $\|f\|_{\infty} \leq 1$,

$$\sup_L \|f(H_1^{(L)})\delta_u\| \leq 1 \text{ and } \lim_{L \rightarrow \infty} f(H_1^{(L)})\delta_u = f(H_1)\delta_u \text{ for all } u \in \mathbb{Z}. \tag{77}$$

Moreover,

$$\begin{aligned} \lim_{L \rightarrow \infty} \mathbb{E} \left(\|f(H_1^{(L)})\delta_u\|^2 \right) &= \mathbb{E} \left(\|f(H_1)\delta_u\|^2 \right) = \mathbb{E} \left(\langle \delta_u, (f(H_1))^2 \delta_u \rangle \right) \\ &= \mathbb{E} \left(\langle \delta_0, (f(H_1))^2 \delta_0 \rangle \right) = \int f^2(t) d\eta(t), \end{aligned} \tag{78}$$

where η is the density of states measure for the Anderson model H_1 . It also follows from (77) by bounded convergence that

$$\lim_{L \rightarrow \infty} \mathbb{E} \left(\|f(H_1^{(L)})\delta_j\| \|f(H_1^{(L)})\delta_i\| \right) = \mathbb{E} \left(\|f(H_1)\delta_j\| \|f(H_1)\delta_i\| \right). \tag{79}$$

We now fix a function $f \in C_c(\mathbb{R})$ such that $\text{supp } f \subset K \cap \Sigma_1$ and $\chi_{K'} \leq f \leq \chi_{K \cap \Sigma_1}$ for some nonempty interval $K' \subset K \cap \Sigma_1$. Note that

$$D := \int f^2(t) d\eta(t) > 0, \tag{80}$$

Given $i, j \in \mathbb{Z}$, if $i, j \in [-L, L]$, we have

$$\begin{aligned} \left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x \right)_K \right\| &= \left\| P_K^{(L)} \sigma_j^x \psi_0^{(L)} \right\| \left\| P_K^{(L)} \sigma_i^x \psi_0^{(L)} \right\| \\ &= \left\| P_K^{(L)} \delta_j \right\| \left\| P_K^{(L)} \delta_i \right\| \geq \left\| f(H_1^{(L)})\delta_j \right\| \left\| f(H_1^{(L)})\delta_i \right\|, \end{aligned} \tag{81}$$

and hence it follows from (77) that

$$\liminf_{L \rightarrow \infty} \left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x \right)_K \right\| \geq \|f(H_1)\delta_j\| \|f(H_1)\delta_i\|, \tag{82}$$

Given $u \in \mathbb{Z}$, let $H_1^{(u,L)}$ denote the restriction of H_1 to the interval $[u - L, u + L] = u + [-L, L]$, and note that (77) and (78) hold with $H_1^{(u,L)}$ substituted for $H_1^{(L)} = H_1^{(0,L)}$. In particular,

$$\lim_{L \rightarrow \infty} \varepsilon^{(u,L)} = 0, \text{ where } \varepsilon^{(u,L)} = \mathbb{E} \left(\left\| \left(f(H_1^{(u,L)}) - f(H_1) \right) \delta_u \right\|^2 \right), \tag{83}$$

and note that $\varepsilon_L = \varepsilon^{(u,L)} \leq 2$ is independent of $u \in \mathbb{Z}$. Moreover,

$$\mathbb{E} (\|f(H_1)\delta_u\|) \geq \mathbb{E} \left(\|f(H_1)\delta_u\|^2 \right). \tag{84}$$

It follows that for all $i, j \in \mathbb{Z}$ and $L \in \mathbb{N}$, with $|i - j| \geq 3L$ we have (recall $\|f\|_\infty \leq 1$)

$$\begin{aligned} \mathbb{E} (\|f(H_1)\delta_j\| \|f(H_1)\delta_i\|) &\geq \mathbb{E} \left(\left\| f(H_1^{(j,L)})\delta_j \right\| \left\| f(H_1^{(i,L)})\delta_i \right\| \right) - 2\varepsilon_L \\ &= \mathbb{E} \left(\left\| f(H_1^{(j,L)})\delta_j \right\| \right) \mathbb{E} \left(\left\| f(H_1^{(i,L)})\delta_i \right\| \right) - 2\varepsilon_L \\ &\geq \mathbb{E} (\|f(H_1)\delta_j\|) \mathbb{E} (\|f(H_1)\delta_i\|) - 4\varepsilon_L \\ &\geq \mathbb{E} \left(\|f(H_1)\delta_j\|^2 \right) \mathbb{E} \left(\|f(H_1)\delta_i\|^2 \right) - 4\varepsilon_L \\ &= \mathbb{E} \left(\|f(H_1)\delta_0\|^2 \right)^2 - 4\varepsilon_L \geq D^2 - 4\varepsilon_L \geq \frac{1}{2}D^2, \end{aligned} \tag{85}$$

where we used (83), the fact that the collections of random variables $\{\omega_k\}_{k \in [j-L, j+L]}$ and $\{\omega_s\}_{s \in [i-L, i+L]}$ are independent, used (83) again, used (84), and the last inequality follows from (78), (80), and (83), taking L sufficiently large. In particular, there exists \tilde{R} such that (85) holds if $|i - j| \geq \tilde{R}$.

It follows from (82) and (85) that for $|i - j| \geq \tilde{R}$ we have

$$\mathbb{E} \left(\liminf_{L \rightarrow \infty} \left(\left\| P_K^{(L)} \sigma_j^x \psi_0^{(L)} \right\| \left\| P_K^{(L)} \sigma_i^x \psi_0^{(L)} \right\| \right) \right) \geq \frac{1}{2}D^2, \tag{86}$$

which is (70).

Note that $\sqrt{f} \in C_c(\mathbb{R})$ and $\chi_{K'} \leq f \leq \sqrt{f} \leq \chi_{K \cap \Sigma_1}$. Given an observable X we have

$$\begin{aligned} \|X_K\|_2^2 &= \left\| P_K^{(L)} X P_K^{(L)} \right\|_2^2 = \text{tr} \left(P_K^{(L)} X^* P_K^{(L)} X P_K^{(L)} \right) \\ &\geq \text{tr} \left(P_K^{(L)} X^* f(H^{(L)}) X P_K^{(L)} \right) = \text{tr} \left(\sqrt{f}(H^{(L)}) X P_K^{(L)} X^* \sqrt{f}(H^{(L)}) \right) \\ &\geq \text{tr} \left(\sqrt{f}(H^{(L)}) X f(H^{(L)}) X^* \sqrt{f}(H^{(L)}) \right) = \left\| \sqrt{f}(H^{(L)}) X \sqrt{f}(H^{(L)}) \right\|_2^2. \end{aligned} \tag{87}$$

Thus, we can estimate

$$\begin{aligned} &\left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x \pm \sigma_j^x P_0^{(L)} \sigma_i^x \right)_K \right\|_2^2 \\ &\geq \left\| \sqrt{f}(H_1^{(L)}) \left(\sigma_i^x P_0^{(L)} \sigma_j^x \pm \sigma_j^x P_0^{(L)} \sigma_i^x \right) \sqrt{f}(H_1^{(L)}) \right\|_2^2 \\ &= \left\| T \left(\sqrt{f}(H_1^{(L)})\delta_i, \sqrt{f}(H_1^{(L)})\delta_j \right) \pm T \left(\sqrt{f}(H_1^{(L)})\delta_j, \sqrt{f}(H_1^{(L)})\delta_i \right) \right\|_2^2 \\ &= 2 \left(\left\| \sqrt{f}(H_1^{(L)})\delta_i \right\|^2 \left\| \sqrt{f}(H_1^{(L)})\delta_j \right\|^2 \pm \text{Re} \left(\left\langle \delta_j, f(H_1^{(L)})\delta_i \right\rangle \right)^2 \right) \\ &\geq 2 \left(\left\| f(H_1^{(L)})\delta_i \right\|^2 \left\| f(H_1^{(L)})\delta_j \right\|^2 - \left| \left\langle \delta_j, f(H_1^{(L)})\delta_i \right\rangle \right| \right). \end{aligned} \tag{88}$$

It follows from (88) and (77) that

$$\begin{aligned} \liminf_{L \rightarrow \infty} \left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x \pm \sigma_j^x P_0^{(L)} \sigma_i^x \right)_K \right\|_2^2 \\ \geq 2 \left(\|f(H_1)\delta_i\|^2 \|f(H_1)\delta_j\|^2 - |\langle \delta_j, f(H_1)\delta_i \rangle| \right). \end{aligned} \tag{89}$$

Given a scale ℓ and $|i - j| \geq 3\ell$, we have

$$|\langle \delta_j, f(H_1)\delta_i \rangle| = \left| \langle \delta_j, \left(f(H_1) - f(H_1^{(i,\ell)}) \right) \delta_i \rangle \right| \leq \left\| \left(f(H_1) - f(H_1^{(i,\ell)}) \right) \delta_i \right\| \tag{90}$$

Since $\mathbb{E} \left(\|f(H_1)\delta_i\| \|f(H_1)\delta_j\| \right) \leq \left(\mathbb{E} \left(\|f(H_1)\delta_i\|^2 \|f(H_1)\delta_j\|^2 \right) \right)^{\frac{1}{2}}$, it follows from (89), (90), (85) and (83), that there exists ℓ_1 , such that for $|i - j| \geq 3\ell_1$ we have

$$\begin{aligned} \mathbb{E} \left(\liminf_{L \rightarrow \infty} \left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x \pm \sigma_j^x P_0^{(L)} \sigma_i^x \right)_K \right\|_2^2 \right) \\ \geq 2 \left(\mathbb{E} \left(\|f(H_1)\delta_i\| \|f(H_1)\delta_j\| \right) \right)^2 - \mathbb{E} \left(\left\| \left(f(H_1) - f(H_1^{(i,\ell_1)}) \right) \delta_i \right\|^2 \right) \\ \geq 2 \left(\frac{1}{4} D^4 - \varepsilon_{\ell_1} \right) \geq \frac{1}{4} D^4. \end{aligned} \tag{91}$$

The estimate (71) is proven.

Now let $A^{(L)}(t)$ be as in (73) (we mostly omit L from the notation), and let

$$\begin{aligned} Z^{(L)}(t) &= \left(A^{(L)}(t) - \left(A^{(L)}(t) \right)^* \right)_K \\ &= e^{itH} \left(\sigma_i^x P_0 \sigma_j^x + \sigma_j^x P_0 \sigma_i^x \right)_K - \left(\sigma_i^x P_0 \sigma_j^x + \sigma_j^x P_0 \sigma_i^x \right)_K e^{-itH} \\ &= e^{itH} A_K - A_K e^{-itH} = B_t - B_t^*, \end{aligned} \tag{92}$$

where

$$A = A^{(L)}(0) = \sigma_i^x P_0 \sigma_j^x + \sigma_j^x P_0 \sigma_i^x = A^* \quad \text{and} \quad B_t = e^{itH} A_K. \tag{93}$$

We have

$$\begin{aligned} \left\| Z^{(L)}(t) \right\|_2^2 &= \|B_t - B_t^*\|_2^2 \\ &= \text{tr} \left(B_t B_t^* \right) + \text{tr} \left(B_t^* B_t \right) - \text{tr} \left(B_t B_t \right) - \text{tr} \left(B_t^* B_t^* \right) \\ &= 2 \|A_K\|_2^2 - 2 \text{Re} \text{tr} \left(P_K e^{itH} A P_K e^{itH} A P_K \right). \end{aligned} \tag{94}$$

Since

$$\text{tr} \left(P_K e^{itH} A P_K e^{itH} A P_K \right) = \sum_{E, E' \in \sigma_K} e^{it(E+E')} \text{tr} \left(P_E A P_{E'} A P_E \right), \tag{95}$$

$K \subset (0, \infty)$, and $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{its} dt = 0$ if $s \neq 0$, we conclude that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left\| Z^{(L)}(t) \right\|_2^2 dt = 2 \|A_K\|_2^2 = 2 \left\| \left(\sigma_i^x P_0 \sigma_j^x + \sigma_j^x P_0 \sigma_i^x \right)_K \right\|_2^2. \tag{96}$$

The estimate (72) now follows from (71) \square

4. Optimality of the Droplet Spectrum

We are ready to prove Theorem 1.

Proof. (Theorem 1) Suppose Property DL is valid for a disordered XXZ spin chain H with $\Theta_1 > 2\Theta_0$. Let $K = [\Theta_0, \Theta_2]$, where $\Theta_0 < \Theta_2 < \Theta_1$, and $\varepsilon = \min \{\Theta_1 - 2\Theta_2, \Theta_0\} > 0$. We pick and fix a Gevrey class function h such that

$$0 \leq h \leq 1, \text{ supp } h \subset (-\varepsilon, \varepsilon), h(0) = 1, \text{ and } \left| \hat{h}(t) \right| \leq C e^{-c|t|^{\frac{1}{2}}} \text{ for all } t \in \mathbb{R},$$

in particular, $\left\| \hat{h} \right\|_1 < \infty$. Note that $P_0 = h(H)$.

Let X, Y be local observables with $X^{+,+} = Y^{+,+} = 0$. It follows from Lemmas 5 and 6 that

$$\begin{aligned} \|(X P_0 Y)_K\| &= \|(X h(H) Y)_K\| \\ &\leq C \|X\| \|Y\| e^{-m_1(\text{dist}(X,Y))^{\frac{1}{2}}} + C' \sup_{r \in \mathbb{R}} \|(Y P_{K_h} \tau_r(X))_K\|, \end{aligned} \tag{97}$$

where

$$K_h \subset [2\Theta_0 - \varepsilon, 2\Theta_2 + \varepsilon] \subset [\Theta_0, \Theta_1] = I. \tag{98}$$

It follows from Lemma 3 that

$$\begin{aligned} \mathbb{E} \left(\sup_{r \in \mathbb{R}} \|(Y P_{K_h} \tau_r(X))_K\| \right) &\leq \mathbb{E} \left(\sup_{r \in \mathbb{R}} \|(Y P_{K_h} \tau_r(X))_I\| \right) \\ &\leq C \|X\| \|Y\| e^{-\frac{1}{8}m \text{dist}(X,Y)}, \end{aligned} \tag{99}$$

so we conclude that

$$\mathbb{E} (\|(X P_0 Y)_K\|) \leq C \|X\| \|Y\| e^{-m_2(\text{dist}(X,Y))^{\frac{1}{2}}}, \tag{100}$$

where $m_2 = \min \{m_1, \frac{1}{8}m\} > 0$.

For all $k \in \mathbb{Z}$ we have $\sigma_k^x = (\sigma_k^x)^*$, $(\sigma_k^x)^{+,+} = 0$, and $\|\sigma_k^x\| = 1$. Thus it follows from (100) that for all $i, j \in [-L, L]$ we have (we put L back in the notation)

$$\mathbb{E} \left(\left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x \right)_K \right\| \right) \leq C e^{-m_2(|i-j|)^{\frac{1}{2}}}, \tag{101}$$

uniformly in L .

If H is a random XXZ spin chain, (101) contradicts (70) in Lemma 7 if $|i - j|$ is sufficiently large. Thus we conclude that we cannot have $\Theta_1 > 2\Theta_0$, that is, we must have $\Theta_1 \leq 2\Theta_0$. \square

5. Non-spreading of Information

In this section we prove Theorem 2.

Proof (Theorem 2). Let H_ω be a disordered XXZ spin chain satisfying Property DL. Let X be a local observable with support $\mathcal{S} = \mathcal{S}_X = [s_X, r_X]$. In view of (38) we can assume $X^{+,+} = 0$.

We take $\ell \geq 1$, and set (recall (15))

$$\begin{aligned} \mathcal{O} &= [-L, L] \setminus \mathcal{S}_{\frac{\ell}{2}} = [-L, s_X - \frac{\ell}{2}) \cup (r_X + \frac{\ell}{2}, L] \\ \mathcal{T} &= \mathcal{S}_\ell \cap \mathcal{O} = [s_X - \ell, s_X - \frac{\ell}{2}) \cup (r_X + \frac{\ell}{2}, r_X + \ell] \end{aligned} \tag{102}$$

We start by proving that

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| \left(P_+^{(\mathcal{O})} \tau_t(X_{I_0}) P_+^{(\mathcal{O})} - \tau_t(X) \right)_{I_0} \right\|_1 \right) \leq C \|X\| e^{-\frac{1}{16} m \ell}. \tag{103}$$

Given an observable Z , we write $Z_{I_0} = Z_1 + Z_2 + Z_3 + Z_4$, where

$$Z_1 = P_0 Z P_0; \quad Z_2 = P_I Z P_I = Z_I; \quad Z_3 = P_0 Z P_I; \quad Z_4 = P_I Z P_0. \tag{104}$$

Since $(X_i)_{I_0} = (X_{I_0})_i = X_i$ and $\tau_t(X_i) = (\tau_t(X))_i$ for $i = 1, 2, 3, 4$, $X_1 = X_1^{+,+} = 0$, and $(X_4)^* = (X^*)_3$, to prove (103) it suffices to prove

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| \left(P_+^{(\mathcal{O})} \tau_t(X_{I_0}) P_+^{(\mathcal{O})} \right)_i - \tau_t(X_i) \right\|_1 \right) \leq C \|X\| e^{-\frac{1}{16} m \ell} \tag{105}$$

in the cases $i = 2, 3$.

If $i = 3$, we have

$$\begin{aligned} &\left\| \tau_t(X_3) - \left(P_+^{(\mathcal{O})} \tau_t(X_{I_0}) P_+^{(\mathcal{O})} \right)_3 \right\|_1 = \left\| \left(\tau_t(X_{I_0}) - P_+^{(\mathcal{O})} \tau_t(X_{I_0}) P_+^{(\mathcal{O})} \right)_3 \right\|_1 \\ &= \left\| \left(\tau_t(X_{I_0}) P_-^{(\mathcal{O})} \right)_3 \right\|_1 = \left\| P_0 X P_-^{(X)} e^{-itH} P_I P_-^{(\mathcal{O})} P_I \right\|_1 \\ &\leq \|X\| \left\| P_-^{(X)} e^{-itH} P_I P_-^{(\mathcal{O})} \right\|_1, \end{aligned} \tag{106}$$

where we used $P_0 X = P_0 X P_-^{(X)}$ since $X^{+,+} = 0$. Thus it follows from (39) that

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| \tau_t(X_3) - \left(P_+^{(\mathcal{O})} \tau_t(X_{I_0}) P_+^{(\mathcal{O})} \right)_3 \right\|_1 \right) \leq C \|X\| e^{-\frac{1}{2} m \ell}. \tag{107}$$

If $i = 2$, recall that $Z_2 = Z_I$. Since $P_I P_+^{(\mathcal{O})} P_0 = P_I P_0 = 0$, we have

$$\left(P_+^{(\mathcal{O})} \tau_t(X_{I_0}) P_+^{(\mathcal{O})} \right)_I = \left(P_+^{(\mathcal{O})} \tau_t(X_I) P_+^{(\mathcal{O})} \right)_I. \tag{108}$$

Thus

$$\begin{aligned} &\left\| \tau_t(X_I) - \left(P_+^{(\mathcal{O})} \tau_t(X_{I_0}) P_+^{(\mathcal{O})} \right)_I \right\|_1 \\ &= \left\| \left(\tau_t(X_I) P_-^{(\mathcal{O})} \right)_I + \left(P_-^{(\mathcal{O})} \tau_t(X_I) P_+^{(\mathcal{O})} \right)_I \right\|_1 \\ &\leq \left\| \left(\tau_t(X_I) P_-^{(\mathcal{O})} \right)_I \right\|_1 + \left\| \left(P_-^{(\mathcal{O})} \tau_t(X_I) \right)_I \right\|_1 \\ &= \left\| \left(\tau_t(X_I) P_-^{(\mathcal{O})} \right)_I \right\|_1 + \left\| \left(\tau_t(X_I^*) P_-^{(\mathcal{O})} \right)_I \right\|_1. \end{aligned} \tag{109}$$

Since

$$\left\| \left(\tau_t(X_I) P_-^{(\mathcal{O})} \right)_I \right\|_1 = \left\| \left(\tau_t(X) P_I P_-^{(\mathcal{O})} \right)_I \right\|_1, \quad (110)$$

it follows from Lemma 3 that

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| \tau_t(X_I) - \left(P_+^{(\mathcal{O})} \tau_t(X_{I_0}) P_+^{(\mathcal{O})} \right)_I \right\|_1 \right) \leq C \|X\| e^{-\frac{1}{16} m \ell}. \quad (111)$$

This finishes the proof of (105), and hence of (103).

We now observe that for all observables Z we have

$$P_+^{(\mathcal{O})} Z P_+^{(\mathcal{O})} = \tilde{Z} P_+^{(\mathcal{O})} = P_+^{(\mathcal{O})} \tilde{Z}, \quad (112)$$

where \tilde{Z} is an observable with $\mathcal{S}_{\tilde{Z}} = \mathcal{S}_{\frac{\ell}{2}}$ and $\|\tilde{Z}\| \leq \|Z\|$. To see this, we write the Hilbert space as $\mathcal{H}^{(L)} = \mathcal{H}_{\mathcal{O}} \otimes \mathcal{H}_{\mathcal{S}_{\frac{\ell}{2}}}$, and let $\psi_{\mathcal{O}} = \otimes_{i \in \mathcal{O}} e_+$ be the all spins up vector in $\mathcal{H}_{\mathcal{O}}$. We define $T : \mathcal{H}_{\mathcal{S}_{\frac{\ell}{2}}} \rightarrow \mathcal{H}^{(L)}$ by $T\eta = \psi_{\mathcal{O}} \otimes \eta$ and $R : \mathcal{H}^{(L)} \rightarrow \mathcal{H}_{\mathcal{S}_{\frac{\ell}{2}}}$ by $P_+^{(\mathcal{O})} \varphi = \psi_{\mathcal{O}} \otimes R\varphi$. i.e., $P_+^{(\mathcal{O})} = TR$. Note $\|T\|, \|R\| \leq 1$. Given an observable Z , we define $\hat{Z} : \mathcal{H}_{\mathcal{S}_{\frac{\ell}{2}}} \rightarrow \mathcal{H}_{\mathcal{S}_{\frac{\ell}{2}}}$ by $\hat{Z} = RZT$. Then $\tilde{Z} = I_{\mathcal{H}_{\mathcal{O}}} \otimes \hat{Z}$ satisfies (112).

It follows from (103) and (112) that

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| \left(P_+^{(\mathcal{O})} \widetilde{\tau_t(X_{I_0})} - \tau_t(X) \right)_{I_0} \right\|_1 \right) \leq C \|X\| e^{-\frac{1}{16} m \ell}. \quad (113)$$

Since $P_+^{(\mathcal{O})} \widetilde{\tau_t(X_{I_0})}$ does not have support in \mathcal{S}_{ℓ} , we now define

$$X_{\ell}(t) = P_+^{(\mathcal{T})} \widetilde{\tau_t(X_{I_0})} = \widetilde{\tau_t(X_{I_0})} P_+^{(\mathcal{T})} \quad \text{for } t \in \mathbb{R}, \quad (114)$$

an observable with support in $\mathcal{S}_{\frac{\ell}{2}} \cup \mathcal{T} = \mathcal{S}_{\ell}$. and claim that $X_{\ell}(t)$ satisfies (17).

To show that (17) follows from (113), we consider an observable Y with $\mathcal{S}_Y = \mathcal{O}^c = \mathcal{S}_{\frac{\ell}{2}}$, and note that

$$\left(P_+^{(\mathcal{T})} - P_+^{(\mathcal{O})} \right) Y = P_-^{(\mathcal{O} \setminus \mathcal{T})} P_+^{(\mathcal{T})} Y. \quad (115)$$

Since $P_0 P_-^{(\mathcal{O} \setminus \mathcal{T})} = P_-^{(\mathcal{O} \setminus \mathcal{T})} P_0 = 0$, we have

$$\left(P_-^{(\mathcal{O} \setminus \mathcal{T})} P_+^{(\mathcal{T})} Y \right)_{I_0} = \left(P_-^{(\mathcal{O} \setminus \mathcal{T})} P_+^{(\mathcal{T})} Y \right)_I. \quad (116)$$

We now apply (115) and (116) with $Y = \widetilde{\tau_t(X_{I_0})}$. We have

$$\begin{aligned} P_+^{(\mathcal{O})} \left(\widetilde{\tau_t(X_{I_0})} \right)^{+,+} P_+^{(\mathcal{O})} &= P_+^{(\mathcal{O})} P_+^{(\mathcal{O}^c)} \widetilde{\tau_t(X_{I_0})} P_+^{(\mathcal{O}^c)} P_+^{(\mathcal{O})} \\ &= P_+^{(\mathcal{O}^c)} P_+^{(\mathcal{O})} \widetilde{\tau_t(X_{I_0})} P_+^{(\mathcal{O})} P_+^{(\mathcal{O}^c)} = P_+^{(\mathcal{O}^c)} P_+^{(\mathcal{O})} \tau_t(X_{I_0}) P_+^{(\mathcal{O})} P_+^{(\mathcal{O}^c)} \\ &= P_0 \tau_t(X_{I_0}) P_0 = P_0 X P_0 = P_0 X^{+,+} P_0 = 0, \end{aligned} \quad (117)$$

where we used (112), $P_+^{(\mathcal{O})} P_+^{(\mathcal{O}^c)} = P_0$ and $X^{+,+} = 0$. Since $\widetilde{\tau_t(X_{I_0})}$ is supported on \mathcal{O}^c , we conclude that $\left(\widetilde{\tau_t(X_{I_0})}\right)^{+,+} = 0$. Thus we only need to estimate $\left(P_-^{(\mathcal{O}\setminus T)} P_+^{(T)} Y^{a,b}\right)_I$, where $Y = \widetilde{\tau_t(X_{I_0})}$ and $a, b = \pm$, but either $a = -$ or $b = -$. If $a = -$, we have

$$P_-^{(\mathcal{O}\setminus T)} P_+^{(T)} Y^{-,b} = P_-^{(\mathcal{O}\setminus T)} P_+^{(T)} P_-^{(\mathcal{O}^c)} Y^{-,b} = P_-^{(\mathcal{O}\setminus T)} P_-^{(\mathcal{O}^c)} P_+^{(T)} Y^{-,b}, \tag{118}$$

and hence

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| \left(P_-^{(\mathcal{O}\setminus T)} P_+^{(T)} Y^{-,b} \right)_I \right\|_1 \right) &\leq \|Y\| \mathbb{E} \left(\left\| P_I P_-^{(\mathcal{O}\setminus T)} P_-^{(\mathcal{O}^c)} \right\|_1 \right) \\ &\leq C \|X\| e^{-\frac{1}{4}m\ell}, \end{aligned} \tag{119}$$

using (40). Since the $b = -$ case is similar we conclude from (115), (116), and (114) that

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| \left(P_+^{(\mathcal{O})} \widetilde{\tau_t(X_{I_0})} - X_\ell(t) \right)_{I_0} \right\|_1 \right) \leq C \|X\| e^{-\frac{1}{4}m\ell} \tag{120}$$

Combining (113) and (120) we get (17). \square

6. Zero-Velocity Lieb–Robinson Bounds

In this section we prove Theorem 3.

Proof (Theorem 3). In view of (38), we can assume $X^{+,+} = Y^{+,+} = 0$, and prove the theorem in this case. This is the only step where we use cancellations from the commutator. The estimate (18) then follows immediately from Lemma 3.

To prove (19), recall $P_{I_0} = P_I + P_0$, and note that since $X^{+,+} = Y^{+,+} = 0$ we have $P_0 X P_0 = P_0 Y P_0 = 0$, so

$$\begin{aligned} [\tau_t(X_{I_0}), Y_{I_0}] &= [\tau_t(X_I), Y_I] + P_I (\tau_t(X) P_0 Y - Y P_0 \tau_t(X)) P_I \\ &\quad + P_0 (\tau_t(X) P_I Y - Y P_I \tau_t(X)) P_0 + (\tau_t(X_I) Y P_0 - P_0 Y \tau_t(X_I)) \\ &\quad + (P_0 X e^{-itH} Y_I - Y_I e^{itH} X P_0). \end{aligned} \tag{121}$$

Note that $[\tau_t(X_I), Y_I]$ can be estimated by (18). We have

$$\begin{aligned} \|P_0 \tau_t(X) P_I Y P_0\|_1 &= \|P_0 \tau_t(X^{+,-}) P_I Y^{-,+} P_0\|_1 \\ &\leq \|X\| \|Y\| \left\| P_-^{(X)} e^{-itH} P_I P_-^{(Y)} \right\|_1, \end{aligned} \tag{122}$$

so it can be estimated by (39), with a similar estimate for $\|P_0 Y P_I \tau_t(X) P_0\|_1$. Moreover,

$$\begin{aligned} \|\tau_t(X_I) Y P_0\|_1 &= \|\tau_t(X_I) Y^{-,+} P_0\|_1 \\ &\leq \|X\| \|Y\| \left\| P_-^{(X)} e^{-itH} P_I P_-^{(Y)} \right\|_1 + \|Y\| \left\| P_I X^{-,+} e^{-itH} P_I P_-^{(Y)} \right\|_1. \end{aligned} \tag{123}$$

The first term can be estimated by (40). To estimate the second term, let $\ell = \text{dist}(X, Y) \geq 1$. Then

$$\begin{aligned} & \left\| P_I X^{-,+} e^{-itH} P_I P_-^{(Y)} \right\|_1 \\ & \leq \left\| P_I X^{-,+} P_+^{(S_{Y, \frac{\ell}{2}})} e^{-itH} P_I P_-^{(Y)} \right\|_1 + \left\| P_I X^{-,+} P_-^{(S_{Y, \frac{\ell}{2}})} e^{-itH} P_I P_-^{(Y)} \right\|_1 \\ & \leq \|X\| \left(\left\| P_+^{(S_{Y, \frac{\ell}{2}})} e^{-itH} P_I P_-^{(Y)} \right\|_1 + \left\| P_I P_-^{(S_{Y, \frac{\ell}{2}})} P_-^{(X)} \right\|_1 \right), \end{aligned} \tag{124}$$

where we used $[X^{-,+}, P_-^{(S_{Y, \frac{\ell}{2}})}] = 0$. Thus the second term in last line of (123) can be estimated by (46) and (40).

The remaining three terms in (121) can be similarly estimated. (Although (121) is stated for the commutator, it could have been stated separately for each term of the commutator. The above argument does not use cancellations from the commutator.) Combining all these estimates we get (19).

It remains to prove (20). Let X, Y and Z be local observables. In view of (19), we only need to estimate the commutator of the counterterms in (19) with Z_I , that is,

$$\mathbb{E} \left(\sup_{t,s \in \mathbb{R}} \|[(\tau_t(X) P_0 \tau_s(Y) - \tau_s(Y) P_0 \tau_t(X))_I, Z_I]\|_1 \right). \tag{125}$$

If we expand the commutator, we get to estimate several terms, the first one being

$$\mathbb{E} \left(\sup_{t,s \in \mathbb{R}} \|P_I \tau_t(X) P_0 \tau_s(Y) P_I Z P_I\|_1 \right) \leq \mathbb{E} \left(\sup_{s \in \mathbb{R}} \|P_0 \tau_s(Y) P_I Z P_I\|_1 \right) \tag{126}$$

This can be estimated as in (122) and (123), and the other terms can be similarly estimated, yielding (20). In particular, there are no counterterms in (20) since the quantity in (125) decays.

We will now show that for the random XXZ spin chain the estimate (19) is not true without the counterterms. In fact, a stronger statement holds. Let now H be a random XXZ spin chain, and assume that for all local observables X and Y we have

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \|[\tau_t(X_{I_0}), Y_{I_0}]\|_1 \right) \leq C \|X\| \|Y\| \Upsilon(\text{dist}(X, Y)), \tag{127}$$

uniformly in L , where the function $\Upsilon : \mathbb{N} \rightarrow [0, \infty)$ satisfies $\lim_{r \rightarrow \infty} \Upsilon(r) = 0$. Assume (19) holds with the same right hand side as (127).

It follows from (19) and (127) that

$$\begin{aligned} \mathbb{E} (\|(X P_0 Y - Y P_0 X)_I\|_1) & \leq \mathbb{E} \left(\sup_{t \in \mathbb{R}} \|(\tau_t(X) P_0 Y - Y P_0 \tau_t(X))_I\|_1 \right) \\ & \leq \mathbb{E} \left(\sup_{t \in \mathbb{R}} \|[\tau_t(X_{I_0}), Y_{I_0}] - (\tau_t(X) P_0 Y - Y P_0 \tau_t(X))_I\|_1 \right) \\ & \quad + \mathbb{E} \left(\sup_{t \in \mathbb{R}} \|[\tau_t(X_{I_0}), Y_{I_0}]\|_1 \right) \\ & \leq 2C \|X\| \|Y\| \Upsilon(\text{dist}(X, Y)) \end{aligned} \tag{128}$$

In particular, taking $X = \sigma_i^x$ and $Y = \sigma_j^x$ we get (putting L back in the notation)

$$\mathbb{E} \left(\left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x - \sigma_j^x P_0^{(L)} \sigma_i^x \right)_I \right\|_1 \right) \leq 2C \Upsilon (|i - j|). \tag{129}$$

Thus, using $\|A\|_2^2 \leq \|A\| \|A\|_1$ and $\left\| \sigma_i^x P_0^{(L)} \sigma_j^x - \sigma_j^x P_0^{(L)} \sigma_i^x \right\| \leq 2$, we get

$$\begin{aligned} \mathbb{E} \left(\left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x - \sigma_j^x P_0^{(L)} \sigma_i^x \right)_I \right\|_2^2 \right) &\leq 2\mathbb{E} \left(\left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x - \sigma_j^x P_0^{(L)} \sigma_i^x \right)_I \right\|_1 \right) \\ &\leq 4C \Upsilon (|i - j|). \end{aligned} \tag{130}$$

Since (130) is not compatible with (71), we have a contradiction, so (127) cannot hold. \square

7. General Dynamical Clustering

We now turn to the proof of Theorem 4. We will use the following lemma.

Lemma 8. *Let $\Theta_2 < \Theta_1$. Given $\alpha \in (0, 1)$, there exist constants $m_\alpha > 0$ and $C_\alpha < \infty$, such that, given $\Theta_3 \geq \Theta_1$, there exists a function $f \in C_c^\infty(\mathbb{R})$, such that*

- (i) $0 \leq f \leq 1$;
- (ii) $\text{supp } f \subset [\Theta_2, \Theta_3 + \Theta_1 - \Theta_2]$;
- (iii) $f(x) = 1$ for $x \in [\Theta_1, \Theta_3]$;
- (iv) $\left| \hat{f}(t) \right| \leq C_\alpha e^{-m_\alpha |t|^\alpha}$ for $|t| \geq 1$;
- (v) $\left\| \hat{f} \right\|_1 \leq C_\alpha \max \{1, \ln (\Theta_3 - \Theta_2)\}$.

Proof. Let $\theta = \Theta_1 - \Theta_2$. Pick a Gevrey class function $h \geq 0$ such that

$$\text{supp } h \subset [0, \theta]; \quad \int_{\mathbb{R}} h(x) dx = 1; \quad \text{and} \quad \left| \hat{h}(t) \right| \leq C_h e^{-m_h |t|^\alpha} \text{ for all } t \in \mathbb{R},$$

where C_h and $m_h > 0$ are constants. Let

$$k(x) = \int_{-\infty}^x h(y) dy \quad \text{for } x \in \mathbb{R},$$

then $k \in C^\infty(\mathbb{R})$ is non-decreasing and satisfies

$$0 \leq k \leq 1, \quad \text{supp } k \subset [0, \infty), \quad \text{and} \quad k(x) = 1 \text{ for } x \geq \theta.$$

Given $\Theta_3 \geq \Theta_1$, we claim that the function

$$f(x) = k(x - \Theta_2) - k(x - \Theta_3) \tag{131}$$

has all the required properties. Indeed, properties (i)–(iii) are obvious. To finish, we compute

$$\hat{f}(t) = \int_{\mathbb{R}} e^{-itx} \left(\int_{x-\Theta_3}^{x-\Theta_2} h(y) dy \right) dx. \tag{132}$$

Integrating by parts and noticing that the boundary terms vanish, we get

$$\begin{aligned} \hat{f}(t) &= \frac{-i}{t} \int_{\mathbb{R}} e^{-itx} (h(x - \Theta_2) - h(x - \Theta_3)) dx = \frac{-i}{t} \left(e^{-i\Theta_2 t} - e^{-i\Theta_3 t} \right) \hat{h}(t) \\ &= \frac{-i}{t} e^{-i\Theta_2 t} \left(1 - e^{-i(\Theta_3 - \Theta_2)t} \right) \hat{h}(t). \end{aligned} \tag{133}$$

Thus

$$\left| \hat{f}(t) \right| \leq 2C_h \left| \frac{\sin\left(\frac{1}{2}(\Theta_3 - \Theta_2)t\right)}{t} \right| e^{-m_h|t|^\alpha} \quad \text{for all } t \in \mathbb{R}. \tag{134}$$

Parts (iv) and (v) follow. \square

We are ready to prove Theorem 4.

Proof (Theorem 4). Let H_ω be a disordered XXZ spin chain satisfying Property DL. Let $K = [\Theta_0, \Theta_2]$, where $\Theta_0 < \Theta_2 < \min\{2\Theta_0, \Theta_1\}$. Since (9) holds for the interval $[\Theta_0, \min\{2\Theta_0, \Theta_1\}]$, we assume $\Theta_1 \leq 2\Theta_0$ without loss of generality. We set $K' = (\Theta_2, \infty)$.

Let X and Y be local observables. In view of (38), we can assume $X^{+,+} = Y^{+,+} = 0$, and prove the theorem in this case. For a fixed L (we omit L from the notation), we have

$$R_K(\tau_t^K(X), Y) = \left(\tau_t^K(X) \bar{P}_K Y \right)_K = \left(\tau_t^K(X) P_{K'} Y \right)_K + \left(\tau_t^K(X) P_0 Y \right)_K. \tag{135}$$

Fix $\alpha \in (0, 1)$, let $\Theta_3 \geq 2\Theta_2$, to be chosen later, and let f be the function given in Lemma 8. We have

$$\left(\tau_t^K(X) P_{K'} Y \right)_K = \left(\tau_t^K(X) (P_{K'} - f(H)) Y \right)_K + \left(\tau_t^K(X) f(H) Y \right)_K. \tag{136}$$

To estimate the first term, note that $P_{K'} - f(H) = g(H)$, where $|g| \leq 1$ and $g(H) = g(H)P_I + g(H)\bar{P}^{(\Theta_3)}$, where $P^{(\Theta_3)} = P_{(-\infty, \Theta_3]}$ and $\bar{P}^{(\Theta_3)} = 1 - P^{(\Theta_3)}$. The term with $g(H)P_I$ can be estimated by Lemma 3,

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| \left(\tau_t^K(X) g(H) P_I Y \right)_K \right\| \right) &\leq \mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| \left(\tau_t(X) g(H) P_I Y \right)_I \right\| \right) \\ &\leq C \|X\| \|Y\| e^{-\frac{1}{8}m \text{dist}(X, Y)}. \end{aligned} \tag{137}$$

The contribution of $g(H)\bar{P}^{(\Theta_3)}$ is estimated by Lemma 9,

$$\begin{aligned} \left\| \left(\tau_t^K(X) g(H) \bar{P}^{(\Theta_3)} Y \right)_K \right\| &\leq \|Y\| \left\| P_K X g(H) \bar{P}^{(\Theta_3)} \right\| \\ &\leq \|Y\| \left\| P^{(\Theta_2)} X g(H) \bar{P}^{(\Theta_3)} \right\| \leq \|Y\| \left\| P^{(\Theta_2)} X \bar{P}^{(\Theta_3)} \right\| \\ &\leq C_F \|X\| \|Y\| e^{-\frac{mF}{|\mathcal{S}_X|}(\Theta_3 - \Theta_2)}. \end{aligned} \tag{138}$$

To estimate the second term on the right hand side of (136), we recall that $H_K = 0$ on $\text{supp } f$, so

$$\left(\tau_t^K(X) f(H) Y \right)_K = e^{itH_K} (X f(H) Y)_K. \tag{139}$$

it follows from Lemmas 5, 6 and 8 that

$$\left(\tau_t^K(X) f(H)Y\right)_K = A + T(K_f), \tag{140}$$

where

$$\|A\| \leq 2C_1 C_\alpha \|X\| \|Y\| \max\{1, \ln(\Theta_3 - \Theta_2)\} e^{-m_1(\text{dist}(X,Y))^\alpha}, \tag{141}$$

$$T(J) = e^{iH_K} \left(\int_{\mathbb{R}} e^{-irH} Y P_J \tau_r(X) \hat{f}(r) dr\right)_K \text{ for } J \subset \mathbb{R}, \tag{142}$$

and

$$[0, 2\Theta_2 - \Theta_1] \subset K_f \subset [2\Theta_0 - \Theta_3 - (\Theta_1 - \Theta_2), 2\Theta_2 - \Theta_2] \subset (-\infty, \Theta_2]. \tag{143}$$

In view of (7), $P_{K_f} = P_{K'_f} + P_0$, where $K'_f = K_f \cap K$, so $T(K_f) = T(K'_f) + T(\{0\})$. We have

$$\begin{aligned} \mathbb{E} \left(\sup_{r \in \mathbb{R}} \|T(K'_f)\|\right) &\leq \|\hat{f}\|_1 \mathbb{E} \left(\sup_{r \in \mathbb{R}} \left\| \left(Y P_{K'_f} \tau_r(X)\right)_K \right\|\right) \\ &\leq C \max\{1, \ln(\Theta_3 - \Theta_2)\} \|X\| \|Y\| e^{-\frac{1}{8}m \text{dist}(X,Y)}, \end{aligned} \tag{144}$$

where we used Lemmas 3 and 8. In addition,

$$T(\{0\}) = e^{iH_K} (Y P_0 X)_K = \left(\tau_t^K(Y) P_0 X\right)_K. \tag{145}$$

To see this, let $E, E' \in K$. Proceeding as in (68), we have

$$\begin{aligned} P_E \left(\int_{\mathbb{R}} e^{-irH} Y P_0 \tau_r(X) \hat{f}(r) dr\right) P_{E'} &= \int_{\mathbb{R}} P_E e^{-irH} Y P_0 X e^{-irH} P_{E'} \hat{f}(r) dr \\ &= \left(\int_{\mathbb{R}} e^{-ir(E+E')} \hat{f}(r) dr\right) P_E Y P_0 X P_{E'} = f(E + E') P_E Y P_0 X P_{E'} \\ &= P_E Y P_0 X P_{E'}, \end{aligned} \tag{146}$$

since $f(E + E') = 1$ as $E + E' \in [2\Theta_0, 2\Theta_2] \subset [\Theta_1, \Theta_3]$.

Combining (135), (136), (137), (138), (140), (141), (144), and (145), we obtain

$$\begin{aligned} &\left\| R_K(\tau_t^K(X), Y) - \left(\tau_t^K(X) P_0 Y\right)_K - \left(\tau_t^K(Y) P_0 X\right)_K \right\| \\ &\leq C \|X\| \|Y\| \left(\max\{1, \ln(\Theta_3 - \Theta_2)\} e^{-m_2(\text{dist}(X,Y))^\alpha} + e^{-\frac{m_F}{|S_X|}(\Theta_3 - \Theta_2)}\right), \end{aligned} \tag{147}$$

where $m_2 = \min\{m_1, \frac{1}{8}m\} > 0$.

We now choose $\Theta_3 = \Theta_2 + |S_X|(\text{dist}(X, Y))^\alpha$, note that $\Theta_3 \geq 2\Theta_2$ if $\text{dist}(X, Y) \geq \Theta_2^{\frac{1}{\alpha}}$, obtaining

$$\begin{aligned} &\left\| R_K(\tau_t^K(X), Y) - \left(\tau_t^K(X) P_0 Y\right)_K - \left(\tau_t^K(Y) P_0 X\right)_K \right\| \\ &\leq C \|X\| \|Y\| (1 + \ln |S_X|) e^{-m_3(\text{dist}(X,Y))^\alpha}, \end{aligned} \tag{148}$$

with $m_3 = \frac{1}{2} \min \{m_2, m_F\} > 0$, for $\text{dist}(X, Y)$ sufficiently large. Observing that the argument can be done with Y instead of X , we get (29).

Since

$$\left([\tau_t^K(X), Y] \right)_K = R_K \left(\tau_t^K(X), Y \right) - R_K \left(Y, \tau_t^K(X) \right) + [\tau_t(X_K), Y_K], \quad (149)$$

(30) follows immediately from (29) and (18).

To conclude the proof, we need to show that for a random XXZ spin chain H the estimates (29) and (30) are not true without the counterterms.

Suppose (29) holds without counterterms, even in a weaker form: for all local observables X and Y we have

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| R_K \left(\tau_t^K(X), Y \right) \right\| \right) \\ \leq C (\min \{|\mathcal{S}_X|, |\mathcal{S}_Y|\}) \|X\| \|Y\| \Upsilon (\text{dist}(X, Y)), \end{aligned} \quad (150)$$

uniformly in L , where the function $\Upsilon : \mathbb{N} \rightarrow [0, \infty)$ satisfies $\lim_{r \rightarrow \infty} \Upsilon(r) = 0$. Assume (29) holds with the same right hand side as (150). Taking $X = \sigma_i^x$ and $Y = \sigma_j^x$, and proceeding as in (128)-(129), we get (putting L back in the notation)

$$\mathbb{E} \left(\left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x + \sigma_j^x P_0^{(L)} \sigma_i^x \right)_K \right\| \right) \leq 4C \Upsilon (|i - j|). \quad (151)$$

Recall that (in the notation of the proof of Lemma 7, as in (69)),

$$Z := \left(\sigma_i^x P_0^{(L)} \sigma_j^x + \sigma_j^x P_0^{(L)} \sigma_i^x \right)_K = T \left(P_K^{(L)} \delta_i, P_K^{(L)} \delta_j \right) + T \left(P_K^{(L)} \delta_j, P_K^{(L)} \delta_i \right). \quad (152)$$

Let V be the two dimensional vector space spanned by the vectors $P_K^{(L)} \delta_i$ and $P_K^{(L)} \delta_j$, and let Q_V be the orthogonal projection onto V . We clearly have $Z = Q_V Z Q_V$ and $\|Z\| \leq 2$, and hence

$$\|Z\|_2^2 \leq 2 \|Z\|^2 \leq 4 \|Z\|, \quad (153)$$

so it follows from (71) that there exist constants $\gamma_K > 0$ and R_K such that

$$\begin{aligned} \mathbb{E} \left(\left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x + \sigma_j^x P_0^{(L)} \sigma_i^x \right)_K \right\| \right) &\geq \frac{1}{4} \mathbb{E} \left(\left\| \left(\sigma_i^x P_0^{(L)} \sigma_j^x + \sigma_j^x P_0^{(L)} \sigma_i^x \right)_K \right\|_2^2 \right) \\ &\geq \frac{1}{4} \gamma_K, \end{aligned} \quad (154)$$

for all $i, j \in \mathbb{Z}$ with $|i - j| \geq R_K$.

Since (151) and (154) establish a contradiction, we conclude that (150) cannot hold.

We show the necessity of the counterterms in (30) in a similar way. Note that the counterterm for $X = \sigma_i^x$ and $Y = \sigma_j^x$ is given by $Z^{(L)}(t)$ as in (92). If we assumed the validity of (30) without counterterms, we would have

$$\mathbb{E} \left(\sup_{t \in \mathbb{R}} \left\| Z^{(L)}(t) \right\| \right) \leq 4C \Upsilon (|i - j|), \quad (155)$$

where the function Υ is as in (151). Since $Z^{(L)}(t)$ is a rank 4 operator, we have

$$\left\| Z^{(L)}(t) \right\|_2^2 \leq 4 \|Z\|^2 \leq 16 \|Z\|, \quad (156)$$

and hence

$$\sup_{t \in \mathbb{R}} \|Z^{(L)}(t)\| \geq \frac{1}{16} \sup_{t \in \mathbb{R}} \|Z^{(L)}(t)\|_2^2 \geq \frac{1}{16} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|Z^{(L)}(t)\|_2^2 dt, \tag{157}$$

so (155) and (72) give a contradiction, an hence (155) cannot hold. \square

A. A Priori Transition Probabilities for Spin Chains

The following lemma is an adaptation of [7, Lemma 6.6(2)] to our needs. It holds for every spin chain with uniformly norm-bounded next-neighbor interactions (more generally, for uniformly norm-bounded interactions of fixed finite range).

Given a spin chain Hamiltonian $H^{(L)}$ and an energy $E \in \mathbb{R}$, we write $P^{(E,L)} = \chi_{(-\infty, E]}(H^{(L)})$ for the Fermi projection, and let $\bar{P}^{(E,L)} = 1 - P^{(E,L)}$.

Lemma 9. *Let $H^{(L)} = \sum_{i=-L}^{L-1} Y_{i,i+1}^{(L)}$ be a spin chain Hamiltonian on $\mathcal{H}^{(L)} = \otimes_{i \in [-L, L]} \mathbb{C}_i^2$, $\mathbb{C}_i^2 = \mathbb{C}^2$ for $i \in \mathbb{Z}$, where $Y_{i,i+1}^{(L)}$ is a local observable with support $\mathcal{S}_{Y_{i,i+1}^{(L)}} = [i, i + 1]$ for $i \in [-L, L - 1]$. Suppose*

$$\max_{i \in [-L, L-1]} \|Y_{i,i+1}^{(L)}\| \leq \theta < \infty. \tag{158}$$

Then there exists constants $m_F > 0$ and $C_F < \infty$, depending on θ , but and independent of L , such that for any local observable X and energies $E < E'$ we have

$$\|P^{(E,L)} X \bar{P}^{(E',L)}\| \leq C_F \|X\| e^{-\frac{m_F}{|S_X|}(E'-E)}. \tag{159}$$

Note that if $H = H_\omega$ is a disordered XXZ spin chain, we can write (cf. (6))

$$H_\omega^{(L)} = \sum_{i=-L}^{L-1} Y_{\omega; i, i+1}^{(L)}, \tag{160}$$

where $Y_{\omega; i, i+1}^{(L)}$ is a local observable with $\mathcal{S}_{Y_{\omega; i, i+1}^{(L)}} = [i, i + 1]$, and

$$\sup_{\omega \in [0, 1]^{\mathbb{Z}}, L \in \mathbb{N}} \max_{i \in [-L, L-1]} \|Y_{\omega; i, i+1}^{(L)}\| \leq \theta = \frac{1}{2} \left(1 + \frac{1}{\Delta}\right) + 2\lambda + \beta < \infty. \tag{161}$$

Proof (Lemma 9). Let $E < E'$ and let X be a local observable. Without loss of generality we take $\|X\| = 1$. We proceed as in [7, Proof of Lemma 6.6(2)]. For all $r > 0$ we have (we omit L from the notation)

$$\|\bar{P}^{(E')} X P^{(E)}\| \leq e^{-r(E'-E)} \|e^{rH} X e^{-rH}\| \tag{162}$$

Hadamard’s Lemma gives

$$e^{rH} X e^{-rH} = X + \sum_{n=1}^{\infty} \frac{r^n}{n!} ad_H^n(X), \quad \text{where } ad_H(\cdot) = [H, \cdot], \tag{163}$$

and hence

$$\left\| e^{rH} X e^{-rH} \right\| \leq 1 + \sum_{n=1}^{\infty} \frac{r^n}{n!} \left\| ad_H^n(X) \right\|. \tag{164}$$

Letting $\mathcal{S} = \mathcal{S}_X = [s_X, r_X]$, $\gamma = |\mathcal{S}_X| = r_X - s_X + 1$, and $\mathcal{S}_{j,k} = [s_X - j, r_X + k] \cap [-L, L]$ for $j, k = 0, 1, 2, \dots$, we can see that

$$ad_H(X) = [H, X] = \sum_{j=1}^{J_1} Z_j^{(1)}, \quad \text{with } J_1 \leq \gamma + 1, \tag{165}$$

where each $Z_j^{(1)} = [Y_{i,i+1}, X]$ for some $i \in \mathcal{S}_{1,0}$, so $\|Z_j^{(1)}\| \leq 2\theta$ and either $\mathcal{S}_{Z_j^{(1)}} \subset \mathcal{S}_{1,0}$ or $\mathcal{S}_{Z_j^{(1)}} \subset \mathcal{S}_{0,1}$, and we have

$$\|ad_H(X)\| \leq 2\theta(\gamma + 1). \tag{166}$$

Using induction, we can show that for $n = 1, 2, 3, \dots$ (with $J_0 = 1$)

$$ad_H^n(X) = \sum_{j=1}^{J_n} Z_j^{(n)}, \quad \text{with } J_n \leq (\gamma + n)J_{n-1}, \tag{167}$$

where each $Z_j^{(n)}$ is a local observable with $\|Z_j^{(n)}\| \leq (2\theta)^n$ and $\mathcal{S}_{Z_j^{(n)}} \subset \mathcal{S}_{k,k'}$ for some $k, k' \in \{0, 1, 2, \dots\}$ with $k + k' \leq n$, and we have

$$\begin{aligned} \|ad_H^n(X)\| &\leq (2\theta)^n J_n \leq (2\theta)^n \prod_{k=1}^n (\gamma + k) \leq (2\theta\gamma)^n \prod_{k=1}^n (1 + k) \\ &= (2\theta\gamma)^n (n + 1)! \end{aligned} \tag{168}$$

We conclude that

$$\left\| e^{rH} X e^{-rH} \right\| \leq C_r = 1 + \sum_{n=1}^{\infty} (2r\theta\gamma)^n (n + 1). \tag{169}$$

Choosing $r = (4\theta\gamma)^{-1}$, we get $\tilde{C} = C_{(4\theta\gamma)^{-1}} < \infty$, and it follows from (162) that

$$\left\| \bar{P}^{(E')} X P^{(E)} \right\| \leq \tilde{C} e^{-(4\theta\gamma)^{-1}(E' - E)}, \tag{170}$$

proving the lemma. \square

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