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All-Order Volume Conjecture for Closed 3-Manifolds from Complex Chern–Simons Theory

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Abstract: We propose an extension of the recently-proposed volume conjecture for closed hyperbolic 3-manifolds, to all orders in perturbative expansion. We first derive formulas for the perturbative expansion of the partition function of complex Chern–Simons theory around a hyperbolic flat connection, which produces infinitely-many perturbative invariants of the closed oriented 3-manifold. The conjecture is that this expansion coincides with the perturbative expansion of the Witten–Reshetikhin–Turaev invariants at roots of unity $q = e^{2\pi i/r}$ with r odd, in the limit $r \to \infty$. We provide numerical evidence for our conjecture.

Contents

1. Introduction and Summary

The goal of this paper is twofold:

A. We derive a perturbative expansion (4), around a hyperbolic flat connection, for the partition function of $SL(2, \mathbb{C})$ Chern–Simons theory [1,2] on a general *closed* hyperbolic oriented 3-manifold. Our starting point is the finite-dimensional integral expression (17) for the partition function.

B. Based on the perturbative expansion mentioned above, we present a new all-order perturbative extension (5) of the recently-proposed volume conjecture [3] for Witten–Reshetikhin–Turaev (WRT) invariants [4,5] for closed oriented 3-manifolds.

In the rest of this introduction, let us explain these two points in more detail.

1.1. Volume conjecture for knot complements. The celebrated volume conjecture [6–8] (see [9] for review) states a surprising relation between two a-priori very different objects.

The first is the colored Jones polynomial [10] of a link (or a knot)[1] L in the three-sphere S^3. We denote this by $J_N(L; q)$, where N denotes the coloring (an integer specifying a representation of $SU(2)$), and q is a formal parameter.[2]

The second is the complex hyperbolic volume[3] of the knot complement $S^3 \backslash L$, which is a combination of the hyperbolic volume together with the Chern–Simons invariant: $\mathrm{Vol}(L) + i\mathrm{CS}(L)$.

The volume conjecture [6–8] states that the asymptotic behavior of the root-of-unity value of the former gives the latter:

$$\lim_{N \to \infty} J_N(L; q = e^{\frac{\pi i}{N}}) = \exp\left[\frac{N}{2\pi}\left(\mathrm{Vol}(L) + i\mathrm{CS}(L)\right)\right]. \qquad (1)$$

1.2. Volume conjecture for WRT invariants. A natural extension of the volume conjecture is to consider a closed hyperbolic oriented 3-manifold M, where the relevant quantity replacing the Jones polynomial would be the WRT invariant, which was formulated mathematically by Reshetikhin and Turaev [5] based on the physics idea of Witten [4]. This invariant is defined from a modular Hopf algebra, which can be obtained from a quantum group $U_q(\mathfrak{sl}_2)$ when q is a primitive root of unity.[4] For $q = \exp(\pi i/r)$ we denote the associated invariant by $\tau_r^{SU(2)}(M)$, where r is an integer $r \geq 3$.[5] It is then

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[1] A link in general has several components. A link is called a knot if it has only one component.

[2] The Jones polynomial referred here is the normalized Jones polynomial, namely an unknot has a trivial Jones polynomial. Also, the colored Jones polynomial in our convention is a Laurent polynomial in q, and not in $q^{1/2}$. In the literature q here is sometimes denoted by $q^{1/2}$.

[3] More generally this is a simplicial volume (Gromov norm) [11]. When L is a hyperbolic link, namely when $S^3 \backslash L$ is hyperbolic, then the simplicial volume coincides with the hyperbolic volume. Since we are mostly interested in the hyperbolic cases, we will hereafter mostly refer to this quantity as "hyperbolic volume".

[4] Contrary to the case of the colored Jones polynomial where q is a formal parameter, WRT invariant is defined only for q being a primitive root of unity. If we try to construct and expression in q whose values at root of unity reproduce the WRT invariants, we need to consider a certain cyclotomic completion of the polynomial ring in q [12].

[5] In the language of $SU(2)$ Chern–Simons theory this integer r is the 1-loop corrected level $r = k + 2$, where k is an integer known as the level and is a parameter in front of the classical Chern–Simons action.

natural to consider the limit $r \to \infty$, and expect that we again reproduce the complex simplicial volume of the closed 3-manifold:

$$\lim_{r \to \infty} \tau_r^{SU(2)}(M) \overset{??}{=} \exp\left[\frac{r}{2\pi}\left(\mathrm{Vol}(M) + i\mathrm{CS}(M)\right)\right]. \tag{2}$$

It turns out, however, this naive conjecture does not work; WRT partition function grows in a power-law in r, and not exponentially.[6]

A new insight was brought by the recent work of Chen and Yang [3], who considered the root-of-unity value $q = e^{2\pi i/r}$ with odd r. Let us denote the associated WRT invariant by $\tau_r^{SO(3)}(M)$, since this invariant is known to be related to the group $SO(3)$, rather than $SU(2)$. We will call this invariant the $SO(3)$-WRT invariant. This invariant is discussed for example by Kirby and Melvin [16], and studied further by Blanchet et al. [17] and Lickorish [18] in the context of skein-theory reformulation of WRT invariants by Lickorish [19–21].[7]

Now the conjecture due to [3] states the following asymptotics:

$$\lim_{r \in 2\mathbb{Z}+1; r \to \infty} \tau_r^{SO(3)}(M) = \exp\left[\frac{r}{2\pi}\left(\mathrm{Vol}(M) + i\mathrm{CS}(M)\right)\right]. \tag{3}$$

1.3. All-order generalization. We are now ready to state the main results of this paper.

First, we introduce a perturbative expansion for the state-integral model for complex Chern–Simons theory on a closed hyperbolic oriented 3-manifold M, whose partition function (which we denote by $Z_\hbar(M)$) is written as a finite-dimensional integral. We then consider the $\hbar$-expansion of this quantity around the complete hyperbolic flat connection:

$$Z_\hbar(M) \xrightarrow{\hbar \to 0} \exp\left[\frac{1}{\hbar}S_0^{\mathrm{hyp}}(M) + S_1^{\mathrm{hyp}}(M) + \hbar S_2^{\mathrm{hyp}}(M) + \cdots\right]. \tag{4}$$

Here each of the expansion coefficient $S_n^{\mathrm{hyp}}(M)$ is an interesting perturbative invariant of the closed 3-manifold M. We will give a concrete prescription for computing these invariants.[8]

We then claim that the expansion (4) coincides with the $r \to \infty$ of the WRT invariant $\tau_r^{SO(3)}(M)$:[9]

$$\tau_r^{SO(3)}(M) \xrightarrow{r \in 2\mathbb{Z}+1; r \to \infty} \exp\left[\frac{1}{\hbar}S_0^{\mathrm{hyp}}(M) + S_1^{\mathrm{hyp}}(M) + \hbar S_2^{\mathrm{hyp}}(M) + \cdots\right]\Big|_{\hbar = \frac{2\pi i}{r}}. \tag{5}$$

Since the first coefficient $S_0(M)$ is shown to coincide with the complex volume $i\mathrm{Vol}(M) - \mathrm{CS}(M)$, (5) contains and generalizes the conjecture (3).

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[6] See nevertheless [13–15] for related discussion, which applies a formal saddle point analysis to the WRT invariant $\tau_r^{SU(2)}$ and obtained the complex volume.

[7] $\tau_r^{SO(3)}(M)$ (r odd) was denoted by τ' in [16], and θ_r in [17]; they all coincide, up to overall normalization factors [22]. Our normalization follows [18].

[8] We can also consider non-perturbative evaluation of the state-integral along a proper converging integration cycle. The non-perturbative partition function can be identified as a Borel resummation of the perturbative expansion [23]. Thanks to the so-called 3d–3d correspondence [24,25], the partition function can be interpreted as a partition function of a three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory on a curved background.

[9] More precisely, the perturbative invariants $S_{n=0,1,2}$ has ambiguities as stated in (24), and the match is meant to be modulo this ambiguity.

The rest of the paper is organized as follows. In Sect. 2, we explain how to obtain the integral formula for the partition function for the closed 3-manifold, as motivated from complex Chern–Simons theory. In Sect. 3, we discuss the perturbative expansion of the partition function around a hyperbolic flat connection. In Sect. 4, we check for some examples the generalized conjecture (5) numerically. We also include appendices for review materials.

Note added: After submission of the manuscript to arXiv, we have been noticed of the preprint by Tomotada Ohtsuki [26], which also discusses asymptotic expansion of WRT invariants.

2. State-Integral Model for Closed 3-Manifolds

In this section, we introduce a partition function $Z_\hbar(M)$ for a closed oriented hyperbolic 3-manifold M. We then go on to discuss its perturbative expansion around a hyperbolic flat connection.

2.1. Derivation. Our discussion of closed 3-manifolds relies on the famous mathematical theorem by Lickorish and Wallace [27,28], which states that any closed 3-manifold can be obtained by a Dehn filling of the complement $S^3 \backslash L$ of a link L inside S^3.

The task of deriving a partition function is therefore divided into two. The first is to derive a formula for the partition function for a link complement $S^3 \backslash L$. The second is to study the effect of the Dehn filling on the partition function.

2.1.1. State-integral model for link complements. For a knot/link complement, there are several developed state-integral models [29–31]. For our purpose, in particular, we will use the state-integral model developed in [30,32] (see also [33–36] for discussion of higher order terms for knot complements). This result was motivated from complex Chern–Simons theory [1,2]; in our context this is natural since Jones polynomial is nothing but the vacuum expectation value of the Wilson line in Chern–Simons theory [4] and an interpretation for the volume conjecture (for a link complement) is provided in [37].

Given a hyperbolic knot/link complement, we can consider its regular ideal triangulation. Let us denote the number of ideal tetrahedra by k. The gluing rules of the ideal tetrahedra are specified by the gluing datum $\{A, B, C, D, \vec{v}, \vec{v}'\}$. Here A, B, C, D are $k \times k$ matrices, and $\vec{v}, \vec{v}'$ are k-vectors. For details, see "Appendix 4". Then the state-integral expression for the link complement is [30,32]

$$Z_\hbar^{(\mathbb{X}_\alpha, \mathbb{P}_\alpha)}(\hat{M}\backslash L; X_\alpha)$$
$$= \frac{2}{\sqrt{\det B}} \int \prod_{i=1}^{k} \frac{dZ_i}{\sqrt{2\pi\hbar}} \exp\left(\frac{1}{\hbar} Q(\vec{Z}, \vec{X}; \{A, B, C, D, \vec{v}, \vec{v}'\})\right) \prod_{i=1}^{k} \Psi_\hbar(Z_i). \quad (6)$$

This finite dimensional integration can be interpreted as a $SL(2, \mathbb{C})$ Chern–Simons partition function on the link complement $\hat{M}\backslash L$ with analytically-continued Chern–Simons level $\frac{2\pi i}{\hbar}$.[10] Let us explain this formula in detail. In order to compute the partition

[10] It is known that this state-integral model does not capture reducible flat connections. We will, however, be interested in the perturbative expansion of our partition function around a hyperbolic flat connection, which is irreducible, and hence this subtlety is not important for the considerations of this paper.

function we need to specify the choice of polarization on the boundary of $\hat{M}\setminus L$. In our case, this is to choose a basis of $H_1(\partial(\hat{M}\setminus L), \mathbb{Z})$, and if the link L has S components we need to pick up S pairs of generators $(\mathbb{X}_\alpha, \mathbb{P}_\alpha)_{\alpha=1}^S$. For our later purposes it is actually sufficient to restrict to the case $\hat{M} = S^3$. Then we have a canonical choice

$$\mathbb{X}_\alpha = \mathfrak{m}_\alpha \text{ (merdian)}, \quad \mathbb{P}_\alpha = \mathfrak{l}_\alpha \text{ (longitude)}. \tag{7}$$

for each boundary torus labeled by α. Once we fix a polarization $(\mathbb{X}_\alpha, \mathbb{P}_\alpha)_{\alpha=1}^S$, the partition function depends non-trivially on the deformation parameters (boundary conditions) $(X_\alpha)_{\alpha=1}^S$; their effect is to modify the holonomy along the meridian cycles of the knot complements (see (78) and (79) in "Appendix"). The integral (6) is over k parameters $\{Z_i\}_{i=1}^k$, one for each tetrahedron. For the integrand, the expression Q on the exponent is a quadratic expression in $\vec{Z}$ and $\vec{X}$:[11]

$$Q(\vec{Z}, \vec{X}; \{A, B, C, D, \vec{v}, \vec{v}'\}) := \frac{1}{2}\vec{Z}B^{-1}A\vec{Z} + 2\vec{X}DB^{-1}\vec{X} + (2\pi i + \hbar)\vec{f}B^{-1}\vec{X}$$

$$+ \frac{1}{2}\left(i\pi + \frac{\hbar}{2}\right)^2 \vec{f}B^{-1}\vec{v} - \vec{Z}B^{-1}\left(\left(i\pi + \frac{\hbar}{2}\right)\vec{v} + 2\vec{X}\right). \tag{8}$$

The rest of the integrand is a product of the quantum dilogarithm function $\Psi_\hbar(Z)$, which is defined as (for $\text{Re}(\hbar) < 0$) [38][12]

$$\Psi_\hbar(Z) := \prod_{r=1}^\infty \frac{1 - (e^\hbar)^r e^{-Z}}{1 - \left(e^{-\frac{(2\pi i)^2}{\hbar}}\right)^{r-1} e^{-2\pi i \frac{Z}{\hbar}}}. \tag{9}$$

The parameter $\hbar$ is the expansion parameter of the partition function, which is to be identified with the parameter of the same name in (5).

Given a knot complement, the choice of an ideal triangulation, as well as its gluing datum, is far from unique. It can be shown, however, that the partition function (6) is independent of such choices up to the following ambiguity [32]:[13]

$$\exp\left(\frac{\pi^2}{6\hbar}\mathbb{Z} + \frac{i\pi}{4}\mathbb{Z} + \frac{\hbar}{24}\mathbb{Z}\right). \tag{10}$$

In the following $\hbar$ will be taken to be pure imaginary (see (20)), in which case the ambiguity is only a phase factor.

2.1.2. Dehn-filling formula. The second ingredient is the Dehn-filling formula, which specifies the change of the partition function under the Dehn filling (a similar formula for compact-group Chern–Simons theory is well-known, see [4]).

Consider a 3-manifold obtained by performing (p_α, q_α)-Dehn filling ($\alpha = 1, \ldots, S'$) for the knot complement $\hat{M}\setminus L$ along $S'(\leq S)$ components of a link L out of S components. We denote this manifold as

$$(\hat{M}\setminus L)_{\{p_\alpha \mathbb{X}_\alpha + q_\alpha \mathbb{P}_\alpha\}_{\alpha=1}^{S'}}. \tag{11}$$

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[11] We did not include $\vec{f}, \vec{f}''$ in the arguments of Q since they are determined by other arguments (84).

[12] This function has a symmetry under $b \to 1/b$, where b is a parameter related to $\hbar$ by $\hbar = 2\pi i b^2$. This symmetry, however, is not important for a perturbative consideration of this paper.

[13] The $\frac{\pi^2}{6}$ ambiguity at $\hbar^{-1}$ can actually be lifted to $2\pi^2$ [39].

where in the notation $p_\alpha \mathbb{X}_\alpha + q_\alpha \mathbb{P}_\alpha$ denotes the cycle of the boundary torus which becomes contractible after Dehn filling (see (72) in "Appendix"). Let us first assume that $q_\alpha \neq 0$ for all α. Then our Dehn-filling formula is given by [23][14]

$$Z_\hbar^{(\mathbb{X}_\alpha, \mathbb{P}_\alpha)_{\alpha=S'+1}^S} \left((\hat{M} \backslash L)_{\{p_\alpha \mathbb{X}_\alpha + q_\alpha \mathbb{P}_\alpha\}_{\alpha=1}^{S'}} ; \{X_\alpha\}_{\alpha=S'+1}^S \right)$$

$$= \int \left(\prod_{\alpha=1}^{S'} \frac{dX_\alpha}{\sqrt{2\pi \hbar q_\alpha}} \mathcal{K}_{p_\alpha, q_\alpha}(X_\alpha) \right) Z_\hbar^{(\mathbb{X}_\alpha, \mathbb{P}_\alpha)}(\hat{M} \backslash L; X_\alpha), \qquad (12)$$

where s_α (and r_α) is defined by the condition[15]

$$\begin{pmatrix} r_\alpha & s_\alpha \\ p_\alpha & q_\alpha \end{pmatrix} \in PSL(2, \mathbb{Z}), \qquad (13)$$

and the integral kernel for the Dehn filling is given by

$$\mathcal{K}_{p,q}(X) := \exp\left[\frac{s}{q} \left(\frac{\pi^2}{\hbar} - \frac{\hbar}{4} \right) + \frac{pX^2}{q\hbar} \right]$$

$$\times \left(e^{\frac{2\pi i X}{\hbar q}} \sinh\left(\frac{X - i\pi s}{q} \right) - e^{-\frac{2\pi i X}{\hbar q}} \sinh\left(\frac{X + i\pi s}{q} \right) \right). \qquad (14)$$

When $q_\alpha = 0$, the formula (12) should be modified as

$$\int \frac{dX_\alpha}{\sqrt{2\pi \hbar q_\alpha}} \mathcal{K}_{p_\alpha, q_\alpha}(X_\alpha) \xrightarrow{q_\alpha=0} \int \frac{dY_\alpha dX_\alpha}{\sqrt{2\pi \hbar}} \sinh(Y_\alpha) \sinh\left(\frac{2\pi i Y_\alpha}{\hbar} \right) e^{-\frac{2X_\alpha Y_\alpha}{\hbar}}. \qquad (15)$$

The resulting 3-manifold has $(S - S')$ cusp boundaries and its partition function $Z_\hbar$ depends on the same number of variables $\{X_\alpha\}_{\alpha=S+1}^S$. When $S = S'$, the 3-manifold $(\hat{M} \backslash L)_{\{p_\alpha \mathbb{X}_\alpha + q_\alpha \mathbb{P}_\alpha\}_{\alpha=1}^S}$ is a closed 3-manifold.

It is worth pointing out that a version of the Dehn-filling formula was proposed in [29, Section 3]. The integral kernel there coincides with the leading semiclassical piece of our integral kernel $\mathcal{K}_{p,q}$,

$$\mathcal{K}_{p,q} \xrightarrow{\hbar \to 0} \frac{1}{2} \exp\left[\frac{1}{\hbar} \left(\frac{p}{q} X^2 + \frac{2\pi i}{q} X + \frac{\pi^2 s}{q} \right) + \mathcal{O}(\hbar^0) \right], \qquad (16)$$

where we assumed $\mathrm{Re}(X/q) > 0$. This means that the two proposals give the same result as far as the leading classical results are concerned. Leading classical part of the state-integral model in [29] is shown to give the complex hyperbolic volume of M, so does our state-integral model. However, the two proposals give different answers for higher orders in $\hbar$, and the difference will crucially affect the discussion of the volume conjecture below.

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[14] See also [40] for recent discussion on Dehn fillings.

[15] As discussed before s_α is defined up to $q_\alpha \mathbb{Z}$. This ambiguity does not change the formula (12) modulo (10). The formula is also invariant under the sign flip of (p, q), which can be seen explicitly by noting that $Z_\hbar^{(\mathbb{X}_\alpha, \mathbb{P}_\alpha)}(\hat{M} \backslash L; X_\alpha) = Z_\hbar^{(\mathbb{X}_\alpha, \mathbb{P}_\alpha)}(\hat{M} \backslash L; -X_\alpha)$ by Weyl invariance and $\mathcal{K}_{p_\alpha, q_\alpha}(X_\alpha) = \mathcal{K}_{-p_\alpha, -q_\alpha}(-X_\alpha)$.

2.2. Main formula. We are now ready to give the final expression for the state-integral model. Suppose that a closed 3-manifold M is obtained from the knot complement $\hat{M}\setminus L$ by (p_α, q_α)-Dehn surgeries on the α-th component: in our previous notation we have $M = (\hat{M}\setminus L)_{\{p_\alpha \mathbb{X}_\alpha + q_\alpha \mathbb{P}_\alpha\}_{\alpha=1}^S}$. For this 3-manifold, our formula is given by a concrete finite-dimensional integral expression,[16] which is obtained by (6) and (12):[17]

$$Z_\hbar(M) = \frac{2}{\sqrt{\det B}} \int \prod_{\alpha=1}^{S} \frac{\mathcal{K}_{p_\alpha, q_\alpha}(X)dX_\alpha}{\sqrt{2\pi\hbar q_\alpha}} \prod_{i=1}^{k} \frac{\Psi_\hbar(Z_i)dZ_i}{\sqrt{2\pi\hbar}}$$
$$\times \exp\left(\frac{1}{\hbar} Q(\vec{Z}, \vec{X}; \{A, B, C, D, \vec{v}, \vec{v}'\})\right). \tag{17}$$

The formula is only valid for $q_\alpha \neq 0$ and it should be modified as (15) when $q_\alpha = 0$. The expression of state-integral depends both on basis choice $(\mathbb{X}_\alpha, \mathbb{P}_\alpha)_{\alpha=1}^S$ of $H_1(\partial(\hat{M}\setminus L), \mathbb{Z})$ and filling slopes $(p_\alpha, q_\alpha)_{\alpha=1}^S$. The final state-integral, however, turns out to depend only on the combination $\{p_\alpha \mathbb{X}_\alpha + q_\alpha \mathbb{P}_\alpha\}_{\alpha=1}^S$ and invariant under following transformation:

$$\begin{pmatrix} \tilde{\mathbb{X}}_\alpha \\ \tilde{\mathbb{P}}_\alpha \end{pmatrix} = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix}\begin{pmatrix} \mathbb{X}_\alpha \\ \mathbb{P}_\alpha \end{pmatrix}, \quad \begin{pmatrix} \tilde{p}_\alpha \\ \tilde{q}_\alpha \end{pmatrix} = \begin{pmatrix} d_\alpha & -c_\alpha \\ -b_\alpha & a_\alpha \end{pmatrix}\begin{pmatrix} p_\alpha \\ q_\alpha \end{pmatrix},$$
$$\text{with } \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{18}$$

This is consistent with the obvious fact that the closed 3-manifold M is invariant under the transformation. The gluing datum $(A, B, \vec{v})$ and $(C, D, \vec{v}')$ depends on the choice of $\mathbb{X}_\alpha$ and $\mathbb{P}_\alpha$ respectively and let denote them as $(A_\mathbb{X}, B_\mathbb{X}, \vec{v}_\mathbb{X})$ and $(C_\mathbb{P}, D_\mathbb{P}, \vec{v}'_\mathbb{P})$. Under the transformation of $(\mathbb{X}_\alpha, \mathbb{P}_\alpha)$, the matrices transforms as

$$\begin{pmatrix} (A_{\tilde{\mathbb{X}}})_{\alpha i} & (B_{\tilde{\mathbb{X}}})_{\alpha i} & (v_{\tilde{\mathbb{X}}})_\alpha \\ (C_{\tilde{\mathbb{P}}})_{\alpha i} & (D_{\tilde{\mathbb{P}}})_{\alpha i} & (v'_{\tilde{\mathbb{P}}})_\alpha \end{pmatrix} = \begin{pmatrix} a_\alpha & 2b_\alpha \\ c_\alpha/2 & d_\alpha \end{pmatrix}\begin{pmatrix} (A_\mathbb{X})_{\alpha i} & (B_\mathbb{X})_{\alpha i} & (v_\mathbb{X})_\alpha \\ (C_\mathbb{P})_{\alpha i} & (D_\mathbb{P})_{\alpha i} & (v'_\mathbb{P})_\alpha \end{pmatrix} \tag{19}$$

for $\alpha = 1, \ldots, S$ and other components do not transform.

Notice that the Dehn-filling representation of a closed 3-manifold is far from unique, and there are ambiguities associated with the Kirby moves [44]. While the connection with complex Chern–Simons theory suggest that our partition function is invariant under such moves (up to possibly overall ambiguities discussed in (10)), it would be interesting to prove this more directly from the formulas above. We leave the detailed proof for future work.

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[16] Our partition function (17) can be thought of as finite-dimensional counterparts of the infinite-dimensional path integral. Since the complex Chern–Simons theory has a complex gauge field, its action is also complex and the precise definition of the path integral requires a subtle choice of the integration contour [2]. This is reflected in the choice of integration contour in (17). These subtleties, however, are irrelevant for perturbative expansions discussed in this paper.

[17] We can also apply the Dehn filling prescription (12) to the cluster partition of [41–43]. It would be interesting to study the resulting partition function.

3. Perturbative Expansion

In this section, we work out the perturbative expansion of the partition function of the state-integral model (17) for a closed hyperbolic 3-manifold M. For the expansion, we always assume that

$$\hbar = 2\pi i b^2 \in i\mathbb{R}_+. \tag{20}$$

Using the symmetry in (18) we can always make

$$q_\alpha \neq 0 \tag{21}$$

and we assume it in the most discussions of this section. For the expansion, let us first expand the integrand of the state-integral model in powers of $\hbar$:

$$\int \prod_{\alpha=1}^{S} dX_\alpha \prod_{i=1}^{k} dZ_i \exp\left(\sum_{n=0}^{\infty} \mathcal{W}_n(\vec{Z}, X)\hbar^{n-1}\right). \tag{22}$$

For this expansion we can use the $\hbar$-expansion of the quantum dilogarithm function $\Psi_\hbar$ (9) [45]:

$$\log \Psi_\hbar(Z) \sim \sum_{n=0}^{\infty} \frac{B_n \hbar^{n-1}}{n!} \mathrm{Li}_{2-n}(e^{-Z}), \quad \text{for } 0 < \mathrm{Im}[Z] < \pi. \tag{23}$$

where B_n is the n-th Bernoulli–Seki number with $B_1 = 1/2$.

By evaluating (22) in the saddle point approximation (around a saddle point, whose choice we will discuss momentarily), we will obtain an expansion of the form (4). Each of the $S_n(M)$ is a well-defined perturbative invariant of 3-manifolds, with curious number-theoretic properties. For $SU(2)$ Chern–Simons theory, such an expansion is considered for example in [46,47]. For $SL(2, \mathbb{C})$ Chern–Simons theory, see [32,45] for similar perturbative invariants for knot complements.

Due the ambiguity in (10), the perturbative series $\{S_n(M)\}$ is defined up to

$$S_0 \sim S_0 + \frac{\pi^2}{6}\mathbb{Z}, \quad S_1 \sim S_1 + i\frac{\pi}{4}\mathbb{Z}, \quad S_2 \sim S_2 + \frac{\mathbb{Z}}{24}. \tag{24}$$

Under the change of orientation $M \to \overline{M}$,

$$Z_\hbar(\overline{M}) = (Z_\hbar(M))^* \Rightarrow S_n(\overline{M}) = (-1)^{n+1}(S_n(M))^*. \tag{25}$$

In the rest of this section, we assume for simplicity of notation that $S = S' = 1$; namely, the closed 3-manifold M is obtained by a Dehn filling along a one-component link (knot) L, which we also denote by K, to match standard notation: $M = (\hat{M} \backslash K)_{p\mathbb{X}+q\mathbb{P}}$. It is straightforward to repeat the discussion for general values of S.

3.1. Classical part: complex hyperbolic volume. As already mentioned above, to define the perturbative expansion $S_n^{(c)}$, we need to specify a saddle point $(X^{(c)}, Z^{(c)})$. For the formulation of the generalized volume conjecture for WRT invariants (5), we are particularly interested in perturbative expansion S_n^{hyp} around saddle points corresponding to the hyperbolic structure of M.

To discuss this more explicitly, let us start wth the leading term of the integrand (22). For which is given by ($\varepsilon := \mathrm{Sign}(\mathrm{Re}[X/q])$)

$$
\mathcal{W}_0(X, \vec{Z}) = \mathcal{W}_0(\vec{Z}) + \frac{p}{q}X^2 + \varepsilon \frac{2\pi i}{q}X + \frac{\delta}{q}\pi^2
$$
$$
+ \left(2\vec{X}DB^{-1}\vec{X} + (2\pi i)fB^{-1}\vec{X} - \frac{1}{2}\pi^2 \vec{f}B^{-1}\vec{v} - \vec{Z}B^{-1}\left(i\pi\vec{v} + 2\vec{X}\right) \right),
$$
$$
\text{with } \mathcal{W}_0(\vec{Z}) = \frac{1}{2}\vec{Z}B^{-1}A\vec{Z} + \sum_{i=1}^{k}\mathrm{Li}_2(e^{-Z_i}). \tag{26}
$$

Here the vector $\vec{X}$ is $(X, 0, \ldots, 0)$ as defined in Eq. (79). The formula (26) coincides with the so-called Neumann–Zagier potential for a knot complement [48], and the formula (26) recovers its transformation rule under the Dehn filling. Indeed, saddle point equations with respect to Z_i are

$$
\frac{\partial \mathcal{W}_0}{\partial Z_i} = 0 \Rightarrow A\vec{Z} + B\vec{Z}'' = i\pi\vec{v} + 2\vec{X}, \tag{27}
$$

where we defined $Z_i'' := \log(1 - \frac{1}{z_i})$. This equation coincides with the gluing Eq. (77) upon exponentiation and under the identification $e^{Z_i} = z_i$, $e^{Z_i''} = z_i'' = 1 - z_i^{-1}$ (see also (76)). We also need to take into account the saddle point equation for X:

$$
\frac{\partial \mathcal{W}_0}{\partial X} = 0 \Rightarrow pX + qP = -\pi i \varepsilon, \tag{28}
$$

where

$$
P := (C\vec{Z} + D\vec{Z}'' - i\pi\vec{v}')_1. \tag{29}
$$

In the above expressions, we simplify the equations using the equation of motion (27) and the symplectic property of gluing matrices (82). These saddle point Eqs. (27), (28), (29) are equivalent to the gluing equations for closed 3-manifold studied by Neumann and Zagier [48].

Generically a solution to these equations gives a $PSL(2, \mathbb{C})$ flat connection on M. We can explicitly construct $\mathrm{Hom}(\pi_1(M) \to PSL(2, \mathbb{C}))$ from the solution [43,49]. But there are two subtle points as emphasized in [50]. First, some flatconnections on a closed 3-manifold M cannot be constructed from solutions of the gluing Eqs. (27) and (28). This is because we are using a state-integral model based on ideal triangulations which do not capture reducible flat connections on a knot complement, $\hat{M} \backslash K$. This means flat connections on M that originated from a reducible flat connection on the knot complement cannot be found as a saddle point in the state-integral model. Second, some solutions of the gluing equations might not give a flat connection on M. The solutions of the gluing Eqs. (27) and (28) are guaranteed to correspond to a flat connection on $\hat{M} \backslash K$ whose holonomy along $p\mathbb{X} + q\mathbb{P}$ has eigenvalue ± 1. If the holonomy is $\pm(\text{identity})$,

then it gives a $PSL(2, \mathbb{C})$ flat-connection on M. But if the holonomy is conjugate to $\pm(\text{parabolic}) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, it does not give a flat connection on M. In the latter case, the perturbative expansion around the parabolic solution cannot be interpreted as a perturbative expansion of $PSL(2, \mathbb{C})$ Chern–Simons theory on M. One simple example is $S^3 = (S^3 \backslash K)_{\mathfrak{m}}$ with K a hyperbolic knot. In this case, there are at least three flat-connections on $S^3 \backslash K$ whose meridian ($\mathfrak{m}$) holonomy has eigenvalue ± 1. Two of them can be constructed from the unique complete hyperbolic structure on the knot complement, say $\mathcal{A}^{\text{hyp}}$ and its conjugate $\mathcal{A}^{\overline{\text{hyp}}}$.[18] The third flat connection is an Abelian flat connection. General Abelian flat connection has trivial longitude holonomy and its meridian holonomy can be an arbitrary element of $PSL(2, \mathbb{C})$. The unique trivial $PSL(2, \mathbb{C})$ flat connection on S^3 comes from this Abelian flat connection on the knot complement which cannot be captured by an ideal triangulation. Therefore, for models based on ideal triangulations, the solutions to the gluing equations correspond to the flat-connections $\mathcal{A}^{\text{hyp}}$ or $\mathcal{A}^{\overline{\text{hyp}}}$ which have $\pm(\text{parabolic})$ as meridian holonomy and thus do not give a flat connection on S^3.

In the case that the resulting manifold M, after Dehn filling, is hyperbolic, the gluing Eqs. (27) and (28) have a solution $\mathcal{A}^{\overline{\text{hyp}}}$ corresponding to the conjugate of the complete hyperbolic metric on M, satisfying

$$0 < \text{Im}[Z_i] < \pi \quad \text{for all} \quad i = 1, \dots k. \tag{30}$$

This flat connection $\mathcal{A}^{\overline{\text{hyp}}}$ has the maximal value $(2\text{Vol}(M))$ of $\text{Im}[\text{CS}[\mathcal{A}]]$ among all $PSL(2, \mathbb{C})$ flat connections $\mathcal{A}$ on M, where $\text{CS}[\mathcal{A}]$ is the holomorphic Chern–Simons functional defined by

$$\text{CS}[\mathcal{A}] := \int_M \text{Tr}\left[\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right]. \tag{31}$$

We can also consider perturbative series $\{S_n^{\text{hyp}}\}$ around a flat connection $\mathcal{A}^{\text{hyp}}$ which has lowest value $(-2\text{Vol}(M))$ of $\text{Im}[\text{CS}[\mathcal{A}]]$. The two expansions are related by complex conjugation,

$$S_n^{\text{hyp}} = (S_n^{\overline{\text{hyp}}})^* \quad \text{for all} \quad n \geq 0. \tag{32}$$

Coming back to the flat connection $\mathcal{A}^{\overline{\text{hyp}}}$, there is actually a twofold degeneracy as originated from the $\mathbb{Z}_2$ Weyl-symmetry of $SL(2)$; the two saddle points are $(X^{\overline{\text{hyp}}, \varepsilon=1}, Z^{\overline{\text{hyp}}, \varepsilon=1})$ and $(X^{\overline{\text{hyp}}, \varepsilon=-1}, Z^{\overline{\text{hyp}}, \varepsilon=-1})$, with $X^{\overline{\text{hyp}}, \varepsilon=1} = -X^{\overline{\text{hyp}}, \varepsilon=-1}$. Perturbative expansions around two saddle points are expected to be equal to all orders

$$S_n^{\overline{\text{hyp}}, \varepsilon=1} = S_n^{\overline{\text{hyp}}, \varepsilon=-1} \quad \text{for all} \quad n \geq 0. \tag{33}$$

---

[18] A complete hyperbolic 3-manifold M (knot complement or closed) with finite volume can be realized as a quotient 3-dimensional hyperbolic upper half-plane, $\mathbb{H}^3 / \Gamma$ with a discrete, torsion-free action Γ. The action Γ gives a representation $\text{Hom}(\pi_1(M) \rightarrow PSL(2, \mathbb{C}) = \text{Isom}^+(\mathbb{H}^3))$ which defines a $PSL(2, \mathbb{C})$ flat-connection $\mathcal{A}^{\text{hyp}}$. Its conjugate representation defines $\mathcal{A}^{\overline{\text{hyp}}}$. In the language of three-dimensional gravity, the flat connection $\mathcal{A}^{\text{hyp}}$ on hyperbolic 3-manifold can be written as $\mathcal{A}^{\text{hyp}} = \omega + ie$ where ω and e are spin connection and dreibein on M constructed using the unique complete hyperbolic structure. The conjugate flat connection is $\mathcal{A}^{\overline{\text{hyp}}} = \omega - ie$.

due to the Weyl-symmetry. The leading classical contributions from the two saddle points $S_n^{\overline{\text{hyp}},\varepsilon}$ thus sum up to

$$\exp\left(\sum_{n=0}^{\infty} S_n^{\overline{\text{hyp}}}\hbar^{n-1}\right) = \exp\left(\sum_{n=0}^{\infty} S_n^{\overline{\text{hyp}},\varepsilon=1}\hbar^{n-1}\right) + \exp\left(\sum_{n=0}^{\infty} S_n^{\overline{\text{hyp}},\varepsilon=-1}\hbar^{n-1}\right)$$

$$\implies S_n^{\overline{\text{hyp}}} = S_n^{\overline{\text{hyp}},\varepsilon=1} + \delta_{n,1}\log 2. \tag{34}$$

As stated above around (16), the classical part $S_0^{\overline{\text{hyp}}}$ coincides with the complex hyperbolic volume of M:

$$S_0^{\overline{\text{hyp}}}(M) = -i\,\text{Vol}(M) - \text{CS}(M) := -\frac{1}{2}\text{CS}[\mathcal{A}^{\overline{\text{hyp}}}]. \tag{35}$$

3.2. One-loop part: reidemeister torsion. Having specified the saddle point we can now discuss the perturbative expansion (4). We define from the next $\mathcal{O}(\hbar^0)$ term in the expansion a quantity

$$\tau(M) := e^{-2S_1^{\overline{\text{hyp}}}(M)}. \tag{36}$$

Then, as will be derived in Sect. 3.3

$$\tau(M) = \pm\frac{1}{8}\det\left(A\Delta_{z''} + B\Delta_z^{-1} - R\cdot\Delta_{z''}\right)$$

$$\times \frac{p + 2q(DB^{-1})_{1,1}}{(\sinh[\frac{X-\varepsilon i\pi s}{q}])^2}\prod_{i=1}^{k} z_i^{f_i''}(z_i'')^{-f_i}\Bigg|_{(z,X)\to(e^Z,X)^{\overline{\text{hyp}},\varepsilon}}, \tag{37}$$

where $R_{ij} := \frac{2q\,\delta_{i1}(B^{-1})_{j,1}}{p+2q(DB^{-1})_{1,1}}, \quad \Delta_t := \text{diag}(t_1,\ldots,t_k) \quad t \in \{z, z', z''\}.$

Using the equations of motion, the factor $\sinh[\frac{X-\varepsilon i\pi s}{q}]^2$ can be written in terms of edge variables

$$\sinh\left[\frac{(A\vec{Z} + B\vec{Z}'' - i\pi v)_1 - 2\varepsilon i\pi s}{2q}\right]^2 \quad \text{or equivalently,}$$

$$\sinh\left[\frac{(C\vec{Z} + D\vec{Z}'' - i\pi v')_1 - i\varepsilon\pi r}{p}\right]^2. \tag{38}$$

In the derivation we have assumed q is non-zero. Nevertheless, the formula gives the correct answer even for $q = 0$, if we use the second expression above.

There is a useful trick for the evaluation of $\tau(M)$. Using the transformation (18) and (19), we can map $[(\mathbb{X}, \mathbb{P}); (p, q)]$ to $[(p\mathbb{X}+q\mathbb{P}, -r\mathbb{X}-s\mathbb{P}); (1, 0)]$. After transformation, the 1-loop is expressed as

$$\tau(M) = \pm\frac{1}{8}\det(A_{p\mathbb{X}+q\mathbb{P}}\Delta_{z''} + B_{p\mathbb{X}+q\mathbb{P}}\Delta_{z^{-1}})$$

$$\frac{1}{(\sinh[rX + sP])^2}\prod_{i=1}^{k} z_i^{f_i''}(z_i'')^{-f_i}\Bigg|_{(z,X)\to(e^Z,X)^{\overline{\text{hyp}},\varepsilon}}. \tag{39}$$

Where $A_{p\mathbb{X}+q\mathbb{P}}$ and $B_{p\mathbb{X}+q\mathbb{P}}$ are the A, B matrices corresponding to the polarization $p\mathbb{X} + q\mathbb{P}$. As checked in [32], by overwhelming experimental evidence, is expected that

$$\text{Tor}(\hat{M}\backslash L; p\mathbb{X} + q\mathbb{P})$$
$$= \pm\frac{1}{2}\det(A_{p\mathbb{X}+q\mathbb{P}}\Delta_{z''} + B_{p\mathbb{X}+q\mathbb{P}}\Delta_{z^{-1}})\prod_{i=1}^{k} z_i^{f_i''}(z_i'')^{-f_i}\Big|_{(z,X)\to(e^Z,X)^{\overline{\text{hyp}},\varepsilon}}, \qquad (40)$$

where $\text{Tor}(\hat{M}\backslash L; p\mathbb{X} + q\mathbb{P})$ denotes the Reidemeister torsion of adjoint representation twisted by the flat connection $\mathcal{A}^{\overline{\text{hyp}}}$ on $\hat{M}\backslash L$ associated to the boundary one-cycle $p\mathbb{X} + q\mathbb{P}$. This means that under Dehn filling

$$\tau(M) = \text{Tor}(\hat{M}\backslash L; p\mathbb{X} + q\mathbb{P})\frac{1}{4\sinh[rX + sP]^2}. \qquad (41)$$

This is exactly the same as the change of torsion under the Dehn filling (see, for example, [51])

$$\text{Tor}\big((\hat{M}\backslash L)_{p\mathbb{X}+q\mathbb{P}}\big) = \text{Tor}(\hat{M}\backslash L; p\mathbb{X} + q\mathbb{P})\frac{1}{4\sinh[rX + sP]^2}. \qquad (42)$$

We have therefore proven (modulo the assumption of (40)) that our 1-loop invariant coincides with the Reidemeister torsion:

$$\tau(M) = \text{Tor}(M). \qquad (43)$$

This is an expected result since the 1-loop part of Chern–Simons partition function is given by the Reidemeister torsion [1].

3.3. Higher order results from Feynman diagrams.
Let us now we will derive the Feynman rules, which are useful for systematic computation of higher-order perturbative invariants.

Our starting point is the formula (12). We are interested in the limit $\hbar \to 0$. As we already discussed in Sect. 3.1, depending on the sign of $\text{Re}(X/q)$, the dominant term in the exponential of the integral kernel $\mathcal{K}_{p,q}(X)$ will be either $\frac{p}{q}X^2 \pm \frac{X}{q}$. Under these assumptions, the leading pice of (12) in the limit $\hbar \to 0$ yields to the potential $\mathcal{W}_0(\vec{X}, \vec{Z})$ (26). We have also seen in Sect. 3.2 that there is a twofold degeneracy in the saddle point corresponding to the sign $\varepsilon = +1$ and $\varepsilon = -1$, with the same perturbative expansion to all orders (12). This means we only need to focus on one choice of ε. We are then left with the following integral:

$$\varepsilon e^{\frac{s}{q}\left(\frac{\pi^2}{\hbar} - \frac{\hbar}{4}\right)}\int \frac{dX}{\sqrt{2\pi\hbar q}}e^{\varepsilon\frac{2\pi iX}{q\hbar} + \frac{pX^2}{q\hbar}}\sinh\left(\frac{X - i\pi s\varepsilon}{q}\right)Z_\hbar^{(\mathbb{X},\mathbb{P})}(\hat{M}\backslash L; X). \qquad (44)$$

Let us combine all the integration variables X and $(Z_i)_{i=1,\ldots k}$ together into a $(k+1)$-component vector $y := (y_a)_{a=1,\ldots,k+1} = (X, \vec{Z})$. Let us choose a critical point $y^{(c)} = (X^c, \vec{Z}^c)$ of $\mathcal{W}_0$. The integral we want to perform perturbatively is given by:

$$Z_{\text{pert}}^{(c),\varepsilon}(M) = \frac{2\varepsilon}{\sqrt{\det B}} e^{\Gamma^{(0)}} \int [dy] \, e^{\frac{1}{2\hbar} H^{ab} y_a y_b} \sinh\left(\frac{X + X^c - i\pi s\varepsilon}{q}\right) e^{(fB^{-1})_1 X}$$

$$\times \prod_{i=1}^{k} e^{\sum_{s\geq 1} \Gamma_i^{(s)}/s! Z_i^s}$$

$$= \frac{2\varepsilon}{\sqrt{\det B}} e^{\Gamma^{(0)}} \int [dy] \, e^{\frac{1}{2\hbar} H^{ab} y_a y_b} \prod_{a=1}^{k+1} e^{\sum_{s\geq 1} \Gamma_a^{(s)}/s! y_a^s}. \tag{45}$$

Here the measure is

$$[dy] := \frac{dX}{\sqrt{2\pi \hbar q}} \prod_{j=1}^{k} \frac{dZ_j}{\sqrt{2\pi \hbar}}, \tag{46}$$

In the first line (45) we have defined some symbols, including the classical action:

$$\Gamma^{(0)} = \frac{1}{\hbar} \mathcal{W}(y^{(c)}) + \sum_{i} \sum_{n=1}^{\infty} B_n \frac{\hbar^{n-1}}{n!} \text{Li}_{2-n}(e^{-Z_i^c}) + (fB^{-1})_1 X^c$$

$$+ \frac{i}{2}(\pi + \frac{\hbar}{4}) fB^{-1} v - \frac{1}{2} Z^c B^{-1} v + \frac{s}{q}\left(\frac{\pi^2}{\hbar} - \frac{\hbar}{4}\right), \tag{47}$$

the linear vertex:

$$\Gamma_i^{(1)} = -\frac{1}{2}(B^{-1}v)_i - \sum_{n=1}^{\infty} B_n \frac{\hbar^{n-1}}{n!} \text{Li}_{1-n}(e^{-Z_i^c}), \tag{48}$$

the quadratic vertex:

$$\Gamma_i^{(2)} = \sum_{n=1}^{\infty} B_n \frac{\hbar^{n-1}}{n!} \text{Li}_{-n}(e^{-Z_i^c}), \tag{49}$$

the Hessian matrix:

$$H^{ab} := \frac{\partial \mathcal{W}(y^{(c)})}{\partial y_a \partial y_b},$$

$$H^{ij} = (B^{-1}A)_{i,j} + (z_j z_j'')^{-1} \delta_{i,j}, \quad H^{ix} = -2(B^{-1})_{i,1}, \tag{50}$$

$$H^{xx} = 2\frac{p}{q} + 4(DB^{-1})_{1,1},$$

and finally the higher order vertices:

$$\Gamma_i^{(s)} = (-1)^s \sum_{n=0}^{\infty} B_n \frac{\hbar^{n-1}}{n!} \text{Li}_{2-n-s}(e^{-Z_i^c}) \qquad s \geq 3. \tag{51}$$

In the second line (45), the sine-hyperbolic piece has been expanded as

$$\sinh\left(\frac{X + X^c - i\pi s\varepsilon}{q}\right) = \sinh\left(\frac{X^c - i\pi s\varepsilon}{q}\right) \exp\left(\frac{X}{q} - \sum_{s=1}^{\infty} \frac{X^s}{s!} C_s(X^c)\right), \tag{52}$$

where

$$C_s(X^c) := \left(-\frac{2}{q}\right)^s \mathrm{Li}_{1-s}\left(e^{-2\left(\frac{X^c - i\pi s\varepsilon}{q}\right)}\right).$$ (53)

We can then define the combined s-vertex $\Gamma_a^{(s)}$ for $y = (X, \vec{Z})$ by combining (51), (52) and the exponential linear term in X in (45):

$$\boxed{\text{Vertices: } \Gamma_a^{(s)} := \left((fB^{-1})_1 + q^{-1})\delta_{s,1} - C_s(X^c), \Gamma_i^{(s)}\right)}.$$ (54)

With this information, we can obtain the perturbative expansion of (45):

$$Z_{\mathrm{pert}}^{(c),\varepsilon}(M) = \exp\left(\frac{1}{\hbar}S_0^{(c),\varepsilon} + \sum_{n=1}^{\infty} S_n^{(c),\varepsilon}\hbar^{n-1}\right),$$ (55)

where the index (c) in (55) is labelling the choice of critical point $y^{(c)}$. The first two terms in the $\hbar$-expansion (55) are given by

$$S_0^{(c),\varepsilon} = \mathrm{coeff}[\Gamma^{(0)}, \hbar^{-1}],$$
$$\exp(S_1^{(c),\varepsilon}) = 2\varepsilon \sinh\left(\frac{X^c - i\pi s\varepsilon}{q}\right)\frac{i^{k+1}}{\sqrt{q\det(B)\det(H)}}e^{\mathrm{coeff}[\Gamma^{(0)}, \hbar^0]}.$$ (56)

Here for a given a Laurent series $f(\hbar)$ on $\hbar$, $\mathrm{coeff}[f(\hbar), \hbar^a]$ denotes the coefficient of $\hbar^a$ in $f(\hbar)$. We can verify that the expression (56) reduces to

$$e^{S_1^{(c),\varepsilon}} = 2\varepsilon \sinh\left(\frac{X^c - i\pi s\varepsilon}{q}\right)\frac{i^{k+1}\prod_j z_j^{-\frac{f_j''}{2}}(z_j'')^{-\frac{f_j}{2}}}{\sqrt{(2p + 4q(DB^{-1})_{1,1})\det(A\Delta_{z''} + B\Delta_{z^{-1} - R\Delta_{z''}})}}.$$ (57)

After including a factor 2 from (34) we can verify Eq. (37).

The higher order terms in the $\hbar$ expansion can be computed by the Feynman diagram techniques. The situation is very analogous to [32], except here we have the sinh term as in (52) and the vertex (54) is more involved. The terms $S_{n>1}$ will be extracted from a sum of connected graphs. Consider a connected graph $\mathcal{G}_\Gamma$ with vertices of valences $k \geq 1$. Then, we associate a weight to $\mathcal{G}_\Gamma$: to each k-vertex we associate a factor $\Gamma_{t_i}^{(k)}$ and a label t_i and to each internal line connecting two vertices with labels t_i and t_j a factor Π_{t_i,t_j}, where we defined the propagator:

$$\boxed{\text{Propagator: } \Pi_{a,b} := -\hbar(H^{-1})_{a,b}},$$ (58)

Then the weight associated to the graph $\mathcal{G}_\Gamma$ is:

$$W_\Gamma(\mathcal{G}_\Gamma) := \frac{1}{|\mathrm{Aut}(\mathcal{G}_\Gamma)|}\sum_{\text{labels}}\prod_{v \in \text{vertices}}(-1)^{k_v}\Gamma_{t_v}^{(k_v)}\prod_{e \in \text{edges}}\Pi_e,$$ (59)

where $|\mathrm{Aut}(\mathcal{G}_\Gamma)|$ is the symmetry factor (the rank of the group of automorphisms of $\mathcal{G}_\Gamma$). Given a connected graph $\mathcal{G}_\Gamma$ then is easy to see that $W_\Gamma(\mathcal{G}_\Gamma)$ is of order $\hbar^{-V+E}$ or higher,

where E is the number of internal lines and V the number of vertices with valence $k \geq 3$ in $\mathcal{G}_\Gamma$. After some computation one can show that $E = V + \mathcal{L} + V_1 + V_2 - 1$ where $\mathcal{L}$ is the number of loops and V_1, V_2 are the number of 1 and 2-vertices respectively.

The Feynman rule for the perturbative invariant is then

$$S_n^{(c),\varepsilon} = \text{coeff} \left[\Gamma^{(0)} + \sum_{\mathcal{G}_\Gamma \in \mathcal{G}_n} W_\Gamma(\mathcal{G}_\Gamma), \hbar^{n-1} \right] \qquad n \geq 2, \tag{60}$$

where we defined

$$\mathcal{G}_n := \{ \text{ Connected graphs } \mathcal{G}_\Gamma \text{ such that } \mathcal{L} + V_1 + V_2 \leq n \}. \tag{61}$$

For example, $S_2^{(c),\varepsilon}$ is given by:

$$\frac{1}{4} \sum_i \text{Li}_0 \left(e^{-Z_i^c} \right) + \frac{i}{8} f B^{-1} v - \frac{s}{4q} + \text{coeff} \left[\frac{1}{8} \Gamma_a^{(4)} (\Pi_{aa})^2 + \frac{1}{8} \Pi_{aa} \Gamma_a^{(3)} \Pi_{ab} \Gamma_b^{(3)} \Pi_{bb} \right.$$
$$\left. + \frac{1}{12} \Gamma_a^{(3)} (\Pi_{ab})^3 \Gamma_b^{(3)} + \frac{1}{2} \Gamma_a^{(1)} \Pi_{ab} \Gamma_b^{(3)} \Pi_{bb} + \frac{1}{2} \Gamma_a^{(2)} \Pi_{aa} + \frac{1}{2} \Gamma_a^{(1)} \Pi_{ab} \Gamma_b^{(1)}, \hbar \right]. \tag{62}$$

3.4. Examples. Let us discuss an example of $(S^3 \backslash \mathbf{4}_1)_{p\mathfrak{m}+q\mathfrak{l}}$. Here $\mathbf{4}_1$ denote the figure-eight knot, the simplest hyperbolic knot. Since $\hat{M} = S^3$ in this case, we can use the canonical choices (7). In this choice the gluing datum $(A, B, C, D, \vec{v}, \vec{v}')$ for the figure-eight knot complement $(S^3 \backslash \mathbf{4}_1)$ are [52]

$$A = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \vec{v}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{63}$$

Using the gluing datum and the perturbative expansion developed in previous section, we can compute $S_n^{\overline{\text{hyp}}}((S^3 \backslash \mathbf{4}_1)_{p\mathfrak{m}+q\mathfrak{l}})$. The knot is amphichiral and thus topologically $(S^3 \backslash \mathbf{4}_1)_{p\mathfrak{m}+q\mathfrak{l}} = (S^3 \backslash \mathbf{4}_1)_{-p\mathfrak{m}+q\mathfrak{l}}$ for all (p, q)s. For $M = (S^3 \backslash \mathbf{4}_1)_{\pm 5\mathfrak{m}+\mathfrak{l}}$, which is called Thurston manifold, the saddle point (for $\varepsilon = +1$) is

$$(X, Z_1, Z_2)^{\overline{\text{hyp}}, \varepsilon=1}$$
$$= (0.360784 - 0.575606i, 1.59632 + 0.348931i, -0.929172 + 1.23658i),$$

and the perturbative invariants are

$$S_0^{\overline{\text{hyp}}}(M) = 1.52067 - 0.981369i, \quad S_1^{\overline{\text{hyp}}}(M) = -0.343697 + 3.78189i,$$
$$S_2^{\overline{\text{hyp}}}(M) = -0.512461 + 0.0155225i, \quad S_3^{\overline{\text{hyp}}}(M) = 0.00927226 + 0.00617571i,$$
$$S_4^{\overline{\text{hyp}}}(M) = 0.00312943 + 0.00434039i, \quad S_5^{\overline{\text{hyp}}}(M) = 0.00164586 + 0.00407186i. \tag{64}$$

We compute the perturbative invariants for $(p, q) = (5, 1)$ and the invariants for $(p, q) = (-5, 1)$ are simply related by the orientation change (25). As another example, for $M = (S^3 \backslash \mathbf{4}_1)_{-\mathrm{m}+2\mathfrak{l}}$, we have

$$S_0^{\overline{\mathrm{hyp}}}(M) = 4.86783 - 1.39851i, \quad S_1^{\overline{\mathrm{hyp}}}(M) = -0.340874 - 3.9077i,$$
$$S_2^{\overline{\mathrm{hyp}}}(M) = -0.610686 + 0.0259448i, \quad S_3^{\overline{\mathrm{hyp}}}(M) = 0.0130034 + 0.00708517i.$$
$$(65)$$

4. Numerical Evidence for All-Order Volume Conjecture

Finally, let us present a numerical evidence for our conjecture (5) based on the technical developments in the previous sections. The $SO(3)$-WRT invariant for the closed 3-manifold $(S^3 \backslash K)_{p\mathrm{m}+\mathfrak{l}}$ is given by [18][19]

$$\tau_r^{SO(3)}\big((S^3 \backslash K)_{p\mathrm{m}+\mathfrak{l}}\big)$$
$$= \frac{2}{r} e^{\pi i \left(\frac{3+r^2}{r} - \frac{3-r}{4}\right)} \left(\sum_{N=0}^{r-2} \left(\sin \frac{2(N+1)\pi}{r} \right)^2 (-e^{\frac{\pi i}{r}})^{-p(N^2+2N)} J_{N+1}(K; e^{\frac{2\pi i}{r}}) \right). \tag{66}$$

where $J_N(K; e^{\frac{2\pi i}{r}})$ is the value of N-th colored Jones polynomial of K at $q = e^{\frac{2\pi i}{r}}$ with a normalization $J_N(\text{unknot}) = 1$.

Let us take the example $M = (S^3 \backslash \mathbf{4}_1)_{-5\mathrm{m}+\mathfrak{l}} = $ (Thurston Manifold). The colored Jones polynomial for the knot $K = \mathbf{4}_1$ is (this is due to Habiro [53], see also [54])

$$J_N(\mathbf{4}_1; q) = \sum_{k=0}^{N-1} \prod_{i=1}^{k} (q^{N-i} - q^{-N+i})(q^{N+i} - q^{-N-i}). \tag{67}$$

Combining this expression with (66), we can compute $\tau_r(M)$ for any r. Numerical value for the perturbative expansions $S_n^{\mathrm{hyp}}(M)$ up to $n = 5$ are given in (64) (see also Eq. (32)). The numerical test for the generalized conjecture (5) is given in Fig. 1.

In these plots we defined

$$\delta_k(r; M) := r^{k-1} \Big[\log \tau_r^{SO(3)}(M) - \sum_{n=0}^{k} \hbar^{n-1} S_n^{\mathrm{hyp}}(M) \Big]_{\hbar = \frac{2\pi i}{r}}, \tag{68}$$

We see that both of $\mathrm{Re}\, \delta_k(r; M)$ and $\mathrm{Im}\, \delta_k(r; M)$ quickly decreases as we increase the value of r.[20] This is a highly non-trivial evidence for our conjecture.

---

[19] This should be compared with

$$\tau_r^{SU(2)}\big((S^3 \backslash K)_{p\mathrm{m}+\mathfrak{l}}\big) = \sqrt{\frac{2}{r}} \frac{1}{\sin(\frac{\pi}{r})} \exp(\frac{(3-2p)\pi i}{4}) \sum_{N=1}^{r-1} \sin^2(\frac{\pi N}{r}) e^{\frac{\pi i p N^2}{2}} J_N(K; e^{\frac{\pi i}{r}}).$$

[20] For the imaginary part, we numerically fixed the phase ambiguity (10); once we fix the integers in (10) the rest can be used to the numerical values for $\mathrm{Im}\, \delta_k(r; M)$.
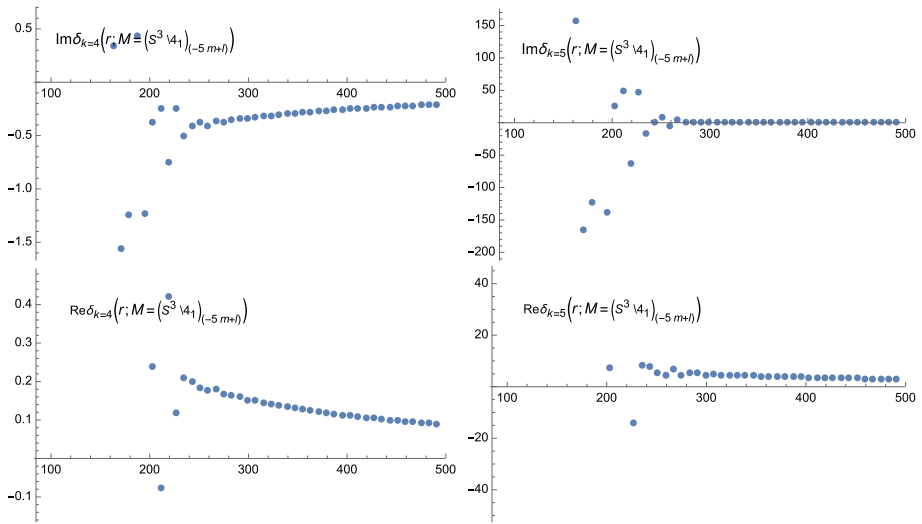
Fig. 1. Graphs of real and imaginary part of $\delta_k(r; M)$ for $k = 4, 5$ and $M = (S^3\backslash 4_1)_{-5m+l}$. The expression decreases quickly as r becomes large, especially for larger k
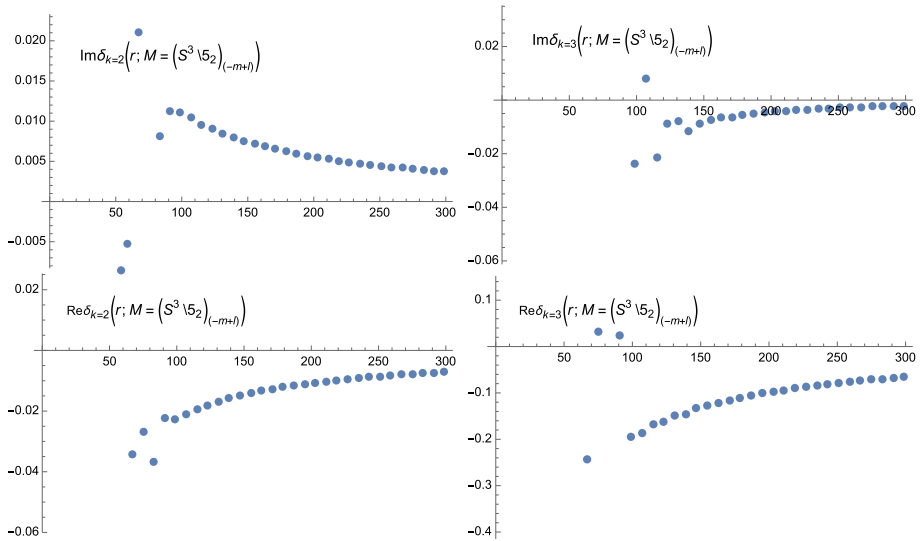
Fig. 2. Graphs of real and imaginary part of $\delta_k(r; M = (S^3\backslash 5_2)_{-m+l})$ for $k = 2, 3$

As another example, let us consider a Dehn filling of the knot $K = 5_2$. Its colored Jones polynomial is [54]

$$J_N(5_2; q) = \sum_{k=0}^{N-1} c_k \prod_{i=1}^{k} (q^{N-i} - q^{-N+i})(q^{N+i} - q^{-N-i}),$$

$$c_k := (-1)^k q^{3k^2+5k} \sum_{i=0}^{k} q^{i^2-2i-3ki} \frac{[k]!}{[i]![k-i]!}, \quad [n]! := \prod_{a=1}^{n} \frac{q^a - q^{-a}}{q - q^{-1}}. \quad (69)$$

For a manifold $M = (S^3\backslash\mathbf{5}_2)_{-\mathrm{m}+\mathfrak{l}} = (S^3\backslash\mathbf{4}_1)_{-\mathrm{m}+2\mathfrak{l}}$,[21] numerical value for the perturbative expansions $S_n^{\mathrm{hyp}}(M)$ up to $n = 3$ are given in (65) (see also Eq. (32)). The numerical test for the generalized conjecture (5) is given in Fig. 2.

Appendix A. Review: Surgery Construction of Closed 3-Manifolds

In this appendix we quickly summarize the concept of Dehn filling, for readers unfamiliar with the concept.

Suppose that K is a knot inside a closed 3-manifold $\hat{M}$, namely has only one component, and consider a knot complement

$$\hat{M}\backslash K. \tag{70}$$

The boundary of this 3-manifold is a two-torus T^2, which has two non-trivial one-cycles, $\mathbb{X}$ and $\mathbb{P}$

$$\langle \mathbb{X}, \mathbb{P}\rangle = H_1\big(T^2 = \partial(\hat{M}\backslash K), \mathbb{Z}\big) = \mathbb{Z}\oplus\mathbb{Z}. \tag{71}$$

Now a (p, q)-Dehn filling of M, which we denote as $(\hat{M}\backslash K)_{p\mathbb{X}+q\mathbb{P}}$, is obtain by eliminating the boundary torus of the knot complement by filling in solid tori $D^2 \times S^1$, so that the combination $p\mathbb{X} + q\mathbb{P}$ is contractible inside $D^2 \times S^1$:

$$(\hat{M}\backslash K)_{p\mathbb{X}+q\mathbb{P}} := (D^2 \times S^1) \bigcup_{\varphi\in\mathrm{Aut}(T^2)} (\hat{M}\backslash K). \tag{72}$$

where an automorphisms φ on the two-torus is taken to be[22]

$$\varphi : H_1(\partial(\hat{M}\backslash K), \mathbb{Z}) \to H_1(\partial(D^2 \times S^1), \mathbb{Z}),$$
$$\begin{pmatrix}\mathbb{X}\\\mathbb{P}\end{pmatrix} \to g \cdot \begin{pmatrix}\alpha\\\beta\end{pmatrix}, \text{ with } g := \pm\begin{pmatrix}r & s\\p & q\end{pmatrix} \in PSL(2, \mathbb{Z}). \tag{73}$$

Here (α, β) is a basis of $H_1\big(\partial(D^2 \times S^1), \mathbb{Z}\big)$:

$$\alpha \sim (S^1 \text{ in } D^2 \times S^1), \quad \beta \sim (S^1 \text{ of } \partial D^2). \tag{74}$$

---

[21] The closed 3-manifold can be obtained by Dehn surgery on $\mathbf{5}_1^2$:=(Whitehead link), $M = (S^3\backslash\mathbf{5}_1^2)_{\mathrm{m}_1+\mathfrak{l}_1, -\mathrm{m}_2+2\mathfrak{l}_2}$. From surgery calculus [44,55], on the other hand, $(S^3\backslash\mathbf{5}_1^2)_{\mathrm{m}_1+\mathfrak{l}_1} = (S^3\backslash\mathbf{4}_1)$ and $(S^3\backslash\mathbf{5}_1^2)_{-\mathrm{m}_2+2\mathfrak{l}_2} = (S^3\backslash\overline{\mathbf{5}_2})$, where $\overline{K}$ means the mirror of a knot K. Thus $M = (S^3\backslash\mathbf{4}_1)_{-\mathrm{m}+2\mathfrak{l}} = (S^3\backslash\overline{\mathbf{5}_2})_{\mathrm{m}+\mathfrak{l}} = (S^3\backslash\mathbf{5}_2)_{-\mathrm{m}+\mathfrak{l}}$.

[22] The topological type of the resulting 3-manifold is invariant under continuous deformation of φ and hence we can regard φ as an element of the $PSL(2, \mathbb{Z})$.

The definition (72) apparently depends on the choice of integers (r, s) in (73), in addition to (p, q). Indeed, given (p, q) there is an ambiguity in the choice of (r, s) as $(r, s) \rightarrow (r, s) + n(p, q)$, with $n \in \mathbb{Z}$. However, this ambiguity is equivalent with the ambiguity of the longitude inside a general 3-manifold, and keeps the topology of the manifold after the Dehn filling (see e.g. discussion around Fig. 7 of [43]). Consequently this ambiguity preserves the partition function of the complex Chern–Simons theory, possible up to some overall pre-factors originating from framing anomaly. Note also that we also have an overall sign ambiguity $(p, q) \sim (-p, -q)$, which we can eliminate by considering a slope p/q.

In general, the link L has several (say S) connected components and we can choose (p_α, q_α) Dehn-surgeries for the α-th component, for some value of α $(1 \leq \alpha \leq S)$. The resulting 3-manifold is then a complement of a link with $S - 1$ components. When do the Dehn filing on all the link complements we obtain a closed 3-manifold $(\hat{M} \setminus L)_{p_1 \mathbb{X}_1 + q_1 \mathbb{P}, \dots, p_S \mathbb{X}_S + q_S \mathbb{P}_S}$.

Appendix B. Review: State-Integral Model for Link Complement

Let us first review the state-integral model for a knot/link complement $\hat{M} \setminus L$, following [30,32]. We consider a regular ideal triangulation of a link complement $\hat{M} \setminus L$:

$$\hat{M} \setminus L = \left(\bigcup_{i=1}^{k} \Delta_i \right) / \sim . \tag{75}$$

where each Δ_i is an ideal tetrahedron (ideal here means that all the vertices are located on the boundary). The symbol $\sim$ denotes the gluing of the k tetrahedra. The shape of an ideal tetrahedron can be parametrized by a shape parameter z_i. This is a complexification of the dihedral angles between two faces, and once we fix the choice of an edge the remaining dihedral angles are given by

$$z_i' = (1 - z_i)^{-1}, \quad z_i'' = 1 - z_i^{-1}, \quad z_i z_i' z_i'' = 1. \tag{76}$$

In this parametrization, we fixed a choice of which dihedral angle to call z_i (and not z_i' or z_i''). Such a choice is called the 'quad type'.

We next impose extra conditions originating from the gluing of tetrahedra. First we have gluing conditions at each internal edge. It follows from vanishing of the Euler number of the boundary tori that the number of edges is also given by k. We also need to impose conditions on the cups boundaries, and we therefore have extra S conditions. This naively means that we have $k + S$ conditions. However, it turns out that only $k - S$ out of the k conditions from internal edges are linearly independent, leading to total of $(k - S) + S = k$ constraints.

To describe this gluing, let us denote by z_i the shape parameter (modulus) of the i-th ideal tetrahedron $(i = 1, \dots, k)$. We can then express the constraint equation as [48]

$$\prod_{j=1}^{k} z_j^{A_{ij}} z_j''^{B_{ij}} = (-1)^{\nu_i}, \quad i = 1, \dots, k, \tag{77}$$

where the matrices $A = (A_{ij})$, $B = (B_{ij})$ are $k \times k$-matrices with integer entries, and $\vec{v}$ is an k-component integer vector. When we consider deformations of the boundary holonomy, the equations are modified to be

$$\prod_{j=1}^{k} z_j^{A_{ij}} z_j''^{B_{ij}} = e^{2X_i + i\pi v_i}, \quad i = 1, \ldots, k. \tag{78}$$

Here a length-k vector $\vec{X} = (X_i)$ is defined by from a set of S parameters $(X_\alpha)_{\alpha=1}^{S}$ to be

$$X_i := \begin{cases} X_i & \text{if } 1 \leq i \leq S \\ 0 & \text{if } S + 1 \leq i \leq k \end{cases}, \tag{79}$$

where we have chosen the indices I such that the first S conditions ($i = 1, \ldots, S$) come from cusp boundaries, and the remaining $k - S$ conditions ($i = S + 1 \ldots, k$) from internal edges. The parameters $(X_\alpha)_{\alpha=1}^{S}$ (or rather its exponential, to match with the standard definition in literature) parameterize the boundary $PSL(2, \mathbb{C})$-holonomies along S one-cycles $\mathbb{X}_\alpha$.

$$P \exp \left(\oint_{\mathbb{X}_\alpha} \mathcal{A} \right) \sim \pm \begin{pmatrix} e^{X_\alpha} & * \\ 0 & e^{-X_\alpha} \end{pmatrix} \tag{80}$$

with a conjugacy equivalence relation $\sim$. Similarly, we can introduce boundary holonomy variables P_α along S $\mathbb{P}_\alpha$ and these variables also can be written in terms of shape parameters

$$\prod_{j=1}^{k} z_j^{\hat{C}_{\alpha j}} z_j''^{\hat{D}_{\alpha j}} = e^{P_\alpha + i\pi \hat{v}'_\alpha}, \quad \alpha = 1, \ldots, S. \tag{81}$$

In general, the $\hat{C}$, $\hat{D}$ and $\hat{v}'$ are valued in half-integers. The partition function for the state-integral model is given by (6) in the main text. As discussed in Sect. 3.1, in the semiclassical limit, its saddle point value reproduces the shape modulus of the tetrahedron by the relation $e^{Z_i} = z_i$. The matrices $(\hat{C}, \hat{D})$ can be extended to $k \times k$ matrices (C, D) in a way that [48].

$$C_{ij} = \hat{C}_{ij}, \quad D_{ij} = \hat{D}_{ij} \quad \text{for } i = 1, \ldots, S,$$
$$\text{and} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2k, \mathbb{Q}). \tag{82}$$

Similarly, the vector $\hat{v}'$ is extended to k-component vector $\vec{v}'$:

$$\vec{v}' := (\hat{v}_1, \ldots, \hat{v}_S, 0, \ldots, 0). \tag{83}$$

The vectors $(\vec{f}, \vec{f}'') = (f_i, f_i'')_{i=1}^{k}$ are known as combinatorial flattening, and satisfy the constraints

$$\vec{v} = A \cdot \vec{f} + B \cdot \vec{f}'', \quad \vec{v}' = C \cdot \vec{f} + D \cdot \vec{f}''. \tag{84}$$

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