Commun. Math. Phys. 358, 767–790 (2018)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-018-3107-y

Communications in Mathematical Physics

Optimal Hardy inequalities for Schrödinger operators on graphs

Matthias Keller[1], **Yehuda Pinchover**[2], **Felix Pogorzelski**[3]

[1] Institut für Mathematik, Universität Potsdam, 14476 Potsdam, Germany.
E-mail: mkeller@math.uni-potsdam.de
[2] Department of Mathematics, Technion-Israel Institute of Technology, 32000 Haifa, Israel.
E-mail: pincho@technion.ac.il
[3] Institut für Mathematik, Universität Leipzig 04109 Leipzig, Germany.
E-mail: felix.pogorzelski@math.uni-leipzig.de

Received: 15 December 2016 / Accepted: 23 December 2017
Published online: 26 February 2018 –

Abstract: For a given subcritical discrete Schrödinger operator H on a weighted infinite graph X, we construct a Hardy-weight w which is *optimal* in the following sense. The operator $H - \lambda w$ is subcritical in X for all $\lambda < 1$, null-critical in X for $\lambda = 1$, and supercritical near any neighborhood of infinity in X for any $\lambda > 1$. Our results rely on a criticality theory for Schrödinger operators on general weighted graphs.

0. Introduction

In 1921, Landau wrote a letter to Hardy which includes a proof of the following inequality

$$\sum_{n=0}^{\infty} |\varphi(n) - \varphi(n+1)|^p \geq C_p \sum_{n=1}^{\infty} w(n)|\varphi(n)|^p \tag{0.1}$$

for all finitely supported $\varphi : \mathbb{N}_0 \to \mathbb{R}$ such that $\varphi(0) = 0$, where

$$w(n) := \frac{1}{n^p}, \quad \text{and } C_p := \left(\frac{p-1}{p}\right)^p,$$

with $n \geq 1$, $1 < p < \infty$ and $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$. This inequality was stated before by Hardy, and therefore, it is called a *Hardy inequality* (see [19] for a marvelous description on the prehistory of the celebrated Hardy inequality). Since then, Hardy-type inequalities have received an enormous amount of attention.

By a Hardy-type inequality for a *nonnegative* operator P we roughly mean that the inequality $P \geq Cw$ holds for some "large" weight function w and optimal constant $C > 0$. One particular focus in the literature lies on finding the sharp constant C to a prescribed Hardy-weight w which is typically an inverse square weight. For the classical literature we refer here to the monographs [2,22] and to references therein. Recent developments include relationships with other functional inequalities, Hardy-type inequalities related to different boundary conditions [1,8,18], Hardy-type inequalities for fractional Schrödinger operators [4,10,11,14,20,24], and Hardy inequalities for the Laplacian on metric trees [9,21].

A conceptually different approach was taken by [6] in the context of general elliptic Schrödinger operators P in the case $p = 2$. There, the weight function $w \geq 0$ is intrinsically derived from P in terms of superharmonic functions such as the Green function. This weight w is shown to be optimal in three aspects:

- For every $\tilde{w} \gneqq w$, the Hardy inequality fails ("Criticality").
- The ground state is not an eigenfunction ("Null-criticality").
- For any $\lambda > 0$, the Hardy inequality outside of any compact set fails for the weight $(1 + \lambda)w$ ("Optimality near infinity").

Such a weight w is called an *optimal Hardy-weight* (for a rigorous definition see Sect. 1.2). Using a different approach, the main results of [6] were later generalized to the p-Laplacian for $1 < p < \infty$ [7].

In the present paper we follow the approaches in [6,7] in the context of Schrödinger operators on weighted graphs (with $p = 2$). Similarly to [6,7], we prove a Hardy inequality with an optimal Hardy weight under the assumption of the existence of certain positive (super)harmonic functions. There are two versions of the result: one for bounded positive (super)harmonic functions and one for unbounded positive (super)harmonic functions.

Let us present two special cases of our results in the context of graphs with standard weights and bounded vertex degree. Let X be a countably infinite vertex set. We denote $x \sim y$ whenever two vertices x, y are connected by an edge in which case we call x and y *adjacent*. The *degree* $\deg(x)$ of a vertex $x \in X$ is the number of vertices adjacent to x. A function $u : X \to \mathbb{R}$ is said to be *harmonic* on $W \subseteq X$ if

$$\Delta u(x) := \sum_{y \sim x} (u(x) - u(y)) = 0 \qquad x \in W.$$

Correspondingly, a function u is called *superharmonic* on W if $\Delta u(x) \geq 0$ for all $x \in W$.

Recall that a function $u : X \to \mathbb{R}$ is called *proper* on $W \subseteq X$ if $u^{-1}(I)$ is compact, (i.e., finite) for all compact $I \subseteq u(W) := \{u(x) \mid x \in W\}$.

Theorem 0.1. *Let a connected graph X with bounded vertex degree and a finite set $K \subseteq X$ be given, and let $u : X \to (0, \infty)$ be an positive function which is harmonic and proper on $X \setminus K$ and such that $u = 0$ on K. Then the following Hardy-type inequality holds true*

$$\frac{1}{2} \sum_{x,y \in X, x \sim y} (\varphi(x) - \varphi(y))^2 \geq \sum_{x \in X \setminus K} w(x)\varphi(x)^2$$

for all finitely supported functions φ with support in $X \setminus K$, where the weight function w is given by

$$w(x) := \frac{1}{2u(x)} \sum_{y \sim x} \left(u(x)^{1/2} - u(y)^{1/2} \right)^2 \qquad x \in X \setminus K.$$

Moreover, w is an optimal Hardy weight in $X \setminus K$.

In various cases it is hard to find explicit non-trivial harmonic functions. However, for transient graphs there are plenty of positive superharmonic functions in terms of the positive minimal Green function

$$G_x(y) := \sum_{n=0}^{\infty} p_n(x, y)$$

where $p_n(x, y)$ are the matrix elements of the n-th power of the transition matrix given by the matrix elements $p_1(x, y) = 1/\deg(x)$, $x, y \in X$. The case when the sum converges for all $x, y \in X$ is called transient. In this case, $u = G_o$ is known to be a strictly positive superharmonic and harmonic outside of $o \in X$ (cf e.g. [25]). Furthermore, G_o is bounded, see [15, Theorem B.1] By the minimality of G it follows that $\inf G_o = 0$. We prove the following Hardy inequality.

Theorem 0.2. *Let a transient connected graph with bounded vertex degree be given, and let $o \in X$ be fixed. Let $G_o : X \to (0, \infty)$ be the positive minimal Green function, and assume that $G = G_o$ is proper. Then the following Hardy-type inequality holds true*

$$\frac{1}{2} \sum_{x,y \in X,\, x \sim y} (\varphi(x) - \varphi(y))^2 \geq \sum_{x \in X} w(x)\varphi(x)^2$$

for all finitely supported functions φ on X, where

$$w(x) := \frac{\Delta\Big(G(x)^{1/2}\Big)}{G(x)^{1/2}}$$

is an optimal Hardy weight in X, and for all $x \neq o$

$$w(x) = \frac{1}{2G(x)} \sum_{y \sim x} \Big(G(x)^{1/2} - G(y)^{1/2}\Big)^2 .$$

These theorems are special cases of Theorem 1.1 and Corollary 1.4 which are the main results of the paper. In Sect. 6 we show how they can be derived from the main results.

As for the proofs of our theorems, we are faced with the challenge that the two approaches of [6,7] both rely heavily on the chain rule. However, it is well known that, in general, a chain rule is not valid in non-local settings such as graphs. The remedy is twofold. Firstly it was observed in [3] that an analogue of the chain rule holds for the square root. This is crucial since the square root of a positive superharmonic function is superharmonic. Secondly, we rely on a coarea formula that is inspired by the treatment in [7] (see also [6]). This is in line with the meta-strategy that local estimates should be replaced by integrated estimates in the case of graphs.

The paper is structured as follows. In the following section, the basic setting is introduced and the main results are stated. Then, in Sect. 2, we provide the major tools for the proof which consists of a discrete chain rule, the ground state transform and a coarea formula which are backbone of the proof of our main result. In Sects. 3, 4 and 5 the proofs of the criticality, null-criticality and optimality near infinity, of our Hardy weights are respectively presented for the case of Laplace-type operators. In Sect. 6, we deduce these results for Schrödinger operators from the previously proven results using the ground state transform. Finally, in Sect. 7 we present some first basic examples, where our results can be applied to concrete graphs.

1. Set Up and Main Result

1.1. Graphs, formal Schrödinger operators and forms. A graph over an infinite discrete set X is a symmetric function $b : X \times X \to [0, \infty)$ with zero diagonal such that

$$\sum_{y \in X} b(x, y) < \infty \quad \text{for all } x \in X.$$

We call the elements of X *vertices*. We say that $x, y \in X$ are *adjacent* or *neighbors* or *connected by an edge* if $b(x, y) > 0$ in which case we write $x \sim y$.

Throughout the paper we assume that X is a countably infinite set equipped with the discrete topology and b is a graph over X. Furthermore we assume that b is *connected*, that is, for every x and y in X there are $x_0, \dots, x_n$ in X such that $x_0 = x$, $x_n = y$ and $x_i \sim x_{i+1}$ for $i = 0, \dots, n-1$.

For $W \subseteq X$, we denote by $C(W)$ (resp., $C_c(W)$) the space of real valued functions on W (resp., with compact support in W). By extending functions by zero on $X \setminus W$ the space $C(W)$ will be considered as a subspace of $C(X)$.

We say that $f \in C(W)$ is *positive* in W if $f \geq 0$ and $f \neq 0$ in W, in this case, we also use the notation $f \gneq 0$.

Given a graph b over X, we introduce the associated *formal Laplacian* $L = L_b$ acting on the space

$$F(X) := \{f \in C(X) \mid \sum_{y \in X} b(x, y)|f(y)| < \infty \text{ for all } x \in X\},$$

by

$$Lf(x) := \sum_{y \in X} b(x, y)(f(x) - f(y)).$$

By the summability assumption on b we have $\ell^\infty(X) \subset F(X)$.

For a potential $q : X \to \mathbb{R}$, we define the *formal Schrödinger operator* H on $F(X)$ by

$$H := L + q.$$

The associated *bilinear form* h of H on $C_c(X) \times C_c(X)$ is given by

$$h(\varphi, \psi) := \frac{1}{2} \sum_{x, y \in X} b(x, y)(\varphi(x) - \varphi(y))(\psi(x) - \psi(y)) + \sum_{x \in X} q(x)\varphi(x)\psi(x).$$

We denote by $h(\varphi) := h(\varphi, \varphi)$ the induced quadratic form on $C_c(X)$, and write $h \geq 0$ on $C_c(X)$ (or in short $h \geq 0$) if $h(\varphi) \geq 0$ for all $\varphi \in C_c(X)$.

Any function $w : X \to \mathbb{R}$ gives rise to a canonical quadratic form on $C_c(X)$ which we denote (with a slight abuse of notation) by w. It acts as

$$w(\varphi) := \sum_{x \in X} w(x)\varphi(x)^2.$$

We denote by $\ell^2(X)$ the Hilbert space of square summable functions equipped with the scalar product

$$\langle f, g \rangle := \sum_X fg = \sum_{x \in X} f(x)g(x) \qquad f, g \in \ell^2(X).$$

Finally, we introduce the notion of (super)harmonic functions on a graph.

Definition. We say that a function u is *H-(super)harmonic* on $W \subseteq X$ if $u \in F(X)$ and $Hu = 0$, ($Hu \geq 0$) on W. We write

$$H \geq 0 \quad \text{on } W$$

if there exists a positive H-superharmonic function u on W.

1.2. Critical Hardy-weights. In this subsection we define the notions of criticality, subcriticality and null-criticality that are fundamental for the present paper. These notions are discussed in detail in [17] to which we also refer for references.

Definition (*Critical/subcritical*). Let h be a quadratic form associated with the formal Schrödinger operator H, such that $h \geq 0$ on $C_c(X)$. The form h is called *subcritical* in X if there is a positive $w \in C(X)$ such that $h - w \geq 0$ on $C_c(X)$. A positive form h that is not subcritical is called *critical* in X.

In [17, Theorem 5.3], a characterization of criticality is presented. There it is shown, that for a form h being critical is equivalent to the existence of a unique positive H-superharmonic function v (which is in fact H-harmonic), and which is called the *(Agmon) ground state* of h. Furthermore, criticality is equivalent to existence of a *null-sequence*, i.e. a sequence (e_n) in $C_c(X)$ such that $0 \leq e_n \leq v$, $e_n \to v$ pointwise and $h(e_n) \to 0$ for $n \to \infty$. (Here v is H-superharmonic which is in fact, the ground state).

Definition (*Null-critical/positive-critical*). Let h be a quadratic form associated with the formal Schrödinger operator H, such that $h \geq 0$ on $C_c(X)$. The form h is called *null-critical* (resp., *positive-critical*) in X with respect to a positive potential w if h is critical in X and $\sum_X \psi^2 w = \infty$ (resp., $\sum_X \psi^2 w < \infty$), where ψ is the ground state of h in X.

Note that the null/positive-criticality of a critical form depends also on the weight w. By [17, Theorem 6.2], the form $h - w$ is null-critical with respect to w if and only if the ground state ψ of the critical form $h - w$ cannot be approximated by compactly supported functions, i.e., by convergence with respect to $h - w$ and pointwise convergence.

Finally, we define the optimality criterion for Hardy weights we are interested in this paper.

Definition *(Optimal Hardy-weight)*. We say that a positive function $w : X \to [0, \infty)$ is an *optimal Hardy-weight* for h in X if

- $h - w$ is critical in X,
- $h - w$ is null-critical with respect to w in X,
- $h - w \geq \lambda w$ fails to hold on $C_c(X \setminus W)$ for all $\lambda > 0$ and all finite $W \subseteq X$. In this case, we say that w is *optimal near infinity for h*.

1.3. The main results. The following theorem is the main result of our paper.

Theorem 1.1. *Let b be a connected graph over X, and let q be a given potential. Let u and v be positive H-superharmonic functions that are H-harmonic outside of a finite set. Let $u_0 := u/v$, and assume that*

(a) $u_0 : X \to (0, \infty)$ *is proper.*
(b) $\displaystyle\sup_{\substack{x,y\in X\\ x\sim y}} \frac{u_0(x)}{u_0(y)} < \infty.$

Then the function

$$w := \frac{H\left[(uv)^{1/2}\right]}{(uv)^{1/2}}$$

is an optimal Hardy-weight in X, and

$$w(x) = \frac{1}{2}\sum_{y\in X} b(x,y)\left[\left(\frac{u(y)}{u(x)}\right)^{1/2} - \left(\frac{v(y)}{v(x)}\right)^{1/2}\right]^2$$

for all $x \in X$ satisfying $Hu(x) = Hv(x) = 0$.

Remark 1.2. Let us discuss assumptions (a) and (b) on the function u_0:

The properness assumption (a) says that $u_0^{-1}(I)$ is a finite set for any compact $I \subseteq u_0(X)$. Since $u_0(X) \subseteq (0, \infty)$ for positive superharmonic functions by the Harnack inequality, [17, Lemma 4.5], the assumption implies that 0 and ∞ are the only (possible) accumulation points of the range of u_0 (and without loss of generality we may assume $\sup u_0 = \infty$, since otherwise, we can only replace u_0 with $\widetilde{u}_0 := 1/u_0$). For example this can be achieved if the limit of u_0 in the one-point compactification $X \cup \{\infty\}$ of X towards ∞ is $\sup u_0 = \infty$.

The second assumption bounds the quotient of the function u_0 on neighboring vertices. It can be understood as an anti-oscillation assumption guaranteeing that the values of u_0 at neighbors do not oscillate over the whole range of u_0. As above, it is also satisfied if the limit of u_0 towards ∞ (in the one-point compactification of X) exists and is either 0 or ∞. Another case when this is automatically satisfied is if the weighted degree function $x \mapsto \sum_y b(x, y)$ is bounded on X and $\inf_{x\sim y} b(x, y) > 0$.

A downside of the assumptions (a) and (b) is that together they exclude locally infinite graphs, i.e. graphs that have vertices with infinitely many neighbors.

Remark 1.3. Via complexification of quadratic forms, the optimality of the Hardy inequality described in the theorem above remains valid on the space of complex valued functions. However, it is more natural to study criticality for real valued functions (as is recurrence). Indeed, considering complex valued functions adds only an additional step in the proofs by decomposing functions into their real and imaginary part while it does not add to the phenomena.

Theorem 1.1 uses the so called *supersolution construction* (see [6]) with the function $(uv)^{1/2}$ in the case where u_0, the quotient of the supersolution u and v, is a proper map. The corollary below applies for the case when the quotient is bounded. It uses the a similar supersolution construction that was developed in [7] for the p-Laplacian.

Corollary 1.4. *Let b be a connected graph over X, and let q be a given potential. Let u and v be positive functions such that u and $v-u$ are positive H-superharmonic functions on X that are H-harmonic outside of a finite set. Let $u_0 := u/v$ and assume*

(a) *The map* $u_0 : X \to (0,1)$ *is proper and satisfies* $\sup u_0 = 1$.

(b) $\sup\limits_{\substack{x,y\in X\\ x\sim y}} \dfrac{u_0(x)(1-u_0(y))}{u_0(y)(1-u_0(x))} < \infty.$

Then the function

$$w := \frac{H\left[u^{1/2}(v-u)^{1/2}\right]}{u^{1/2}(v-u)^{1/2}}$$

is an optimal Hardy-weight in X, and

$$w(x) = \frac{1}{2}\sum_{y\in X} b(x,y)\left[\left(\frac{u(y)}{u(x)}\right)^{1/2} - \left(\frac{(v-u)(y)}{(v-u)(x)}\right)^{1/2}\right]^2$$

for all $x \in X$ *satisfying* $Hu(x) = Hv(x) = 0$.

2. The Toolbox

In this section we discuss the three major tools needed for the proof of the main theorem. The first is a discrete chain rule for the square root which is the basis for the supersolution construction of the Hardy weight. Second, we briefly recall the ground state transform which allows us to deal with Schrödinger operators instead of Laplacians only. Finally, we prove a coarea formula for Laplace type operators.

Throughout the section we are given a graph b over a discrete set X and a potential q such that the associated form satisfies $h \geq 0$ on $C_c(X)$. Let $H = L + q$ be the corresponding Schrödinger operator on $F(X)$ with the graph Laplacian $L = L_b$.

2.1. Product and chain rules. We present the product rule for the discrete Laplacian and a discrete version of the chain rule for the square root.

Lemma 2.1. (Product rule) *Let* $f, g \in F(X)$. *Then for all* $x \in X$,

$$H(fg)(x) = (fHg)(x) + (gLf)(x) - \sum_{y\in X} b(x,y)(f(x)-f(y))(g(x)-g(y)).$$

Proof. This is a straightforward calculation. □

In general, there is no chain rule for the discrete setting. However, for the square root it was noticed by [3] that a chain rule holds. This observation is a crucial point for the analysis in this paper. Specifically, we use it to show that the square root of a product of positive superharmonic functions is superharmonic.

Lemma 2.2. (Chain rule for the square root). *Let* $f, g \in F(X)$ *be positive functions. Then for all* $x \in X$,

$$\begin{aligned}2(fg)^{\frac12} H\left[(fg)^{\frac12}\right](x) &= (fHg)(x) + (gHf)(x)\\ &\quad + \sum_{y\in X} b(x,y)\left[g(x)^{\frac12}\left(f(x)^{\frac12} - f(y)^{\frac12}\right) - f(x)^{\frac12}\left(g(x)^{\frac12} - g(y)^{\frac12}\right)\right]^2.\end{aligned}$$

Proof. To shorten notation we write $\nabla f = (f(x) - f(y))$. Then,

$$2(f^{1/2}Lf^{1/2})(x) = \sum_{y \in X} b(x,y)\Big(f(x) - f(y) + f(x) - 2(f(x)f(y))^{1/2} + f(y)\Big)$$
$$= Lf(x) + \sum_{y \in X} b(x,y)\left(\nabla f^{1/2}\right)^2.$$

We use the product rule

$$2(fg)^{\frac{1}{2}} H(fg)^{\frac{1}{2}}(x) = 2(fg)^{\frac{1}{2}}\left(L(fg)^{\frac{1}{2}}(x) + q(fg)^{\frac{1}{2}}(x)\right)$$
$$= 2(fg^{\frac{1}{2}}Lg^{\frac{1}{2}})(x) + 2(gf^{\frac{1}{2}}Lf^{\frac{1}{2}})(x) + 2(qfg)(x)$$
$$-2(fg)^{\frac{1}{2}} \sum_{x,y \in X} b(x,y)\nabla f^{\frac{1}{2}} \nabla g^{\frac{1}{2}}.$$

By the equality above we obtain

$$\ldots = (fHg)(x) + (gHf)(x)$$
$$+ \sum_{y \in X} b(x,y)\left(f(\nabla g^{\frac{1}{2}})^2 + g(\nabla f^{\frac{1}{2}})^2 - 2(fg)^{\frac{1}{2}} \nabla f^{\frac{1}{2}} \nabla g^{\frac{1}{2}}\right).$$

□

Corollary 2.3. *If $f, g \in F(X)$ are positive H-superharmonic (resp., H-harmonic) functions, then the function $(fg)^{1/2}$ is H-superharmonic (resp., H-harmonic), and*

$$\frac{H[(fg)^{1/2}]}{(fg)^{1/2}}(x) = \frac{1}{2}\sum_{y \in X} b(x,y)\left[\left(\frac{f(y)}{f(x)}\right)^{1/2} - \left(\frac{g(y)}{g(x)}\right)^{1/2}\right]^2$$

for all x satisfying $Hf(x) = Hg(x) = 0$.

In particular, if f is positive L-superharmonic (resp., L-harmonic), then $f^{1/2}$ is L-superharmonic (resp., L-harmonic), and

$$\frac{L[f^{1/2}]}{f^{1/2}}(x) = \frac{1}{2f(x)}\sum_{y \in X} b(x,y)\left(f(x)^{1/2} - f(y)^{1/2}\right)^2$$

for all x satisfying $Lf(x) = 0$.

Proof. We calculate $H[(fg)^{1/2}]$ using the chain rule for the square root (Lemma 2.2) to obtain the super-harmonicity and the explicit expression for w at points where f and g are H-harmonic. Indeed, $\frac{H[(fg)^{1/2}]}{(fg)^{1/2}}(x)$

$$= \sum_{y \in X} \frac{b(x,y)}{2(fg)(x)}\left[g(x)^{\frac{1}{2}}\left(f(x)^{\frac{1}{2}} - f(y)^{\frac{1}{2}}\right) - f(x)^{\frac{1}{2}}\left(g(x)^{\frac{1}{2}} - g(y)^{\frac{1}{2}}\right)\right]^2$$
$$= \frac{1}{2}\sum_{y \in X} b(x,y)\left[f(x)^{-\frac{1}{2}}\left(f(x)^{\frac{1}{2}} - f(y)^{\frac{1}{2}}\right) - g(x)^{-\frac{1}{2}}\left(g(x)^{\frac{1}{2}} - g(y)^{\frac{1}{2}}\right)\right]^2$$
$$= \frac{1}{2}\sum_{y \in X} b(x,y)\left[\left(\frac{f(y)}{f(x)}\right)^{\frac{1}{2}} - \left(\frac{g(y)}{g(x)}\right)^{\frac{1}{2}}\right]^2.$$

The statement for L follows from the calculation above with $g = 1$. □

2.2. The ground state transform. The ground state transform is vital to deal with Schrödinger operators $H = L + q$ specifically under the presences of potentials q with non-vanishing negative part $q_- \neq 0$. Namely, given a strictly positive (super)harmonic function one can reduce the analysis to the one of Laplace type operators (resp. Schrödinger operators with positive potentials).

This transform was first used to show Hardy inequalities by [12] and later by [5]. We present here the notation that is needed for the paper and refer the reader for a more detailed discussion to [17, Section 4.2].

For any function $v \in C(X)$, we denote the operator of multiplication by v by T_v and note that whenever v does not vanish its inverse is given by $T_{v^{-1}}$. For strictly positive v, we define the ground state transform of a Schrödinger operator H on $T_v^{-1}F(X)$

$$H_v = T_v^{-1} H T_v$$

and if additionally $v \in F(X)$, we define the bilinear form $h_v : C_c(X) \times C_c(X) \to \mathbb{R}$ via

$$h_v(\varphi, \psi) := \frac{1}{2} \sum_{x,y \in X} b(x, y) v(x) v(y) (\varphi(x) - \varphi(y))(\psi(x) - \psi(y)).$$

These two notions are related by the following formula. See [17, Proposition 4.8] for a proof and references therein for a discussion of the history in the discrete setting.

Proposition 2.4. (Ground state transform) *Let $v \in F(X)$ be strictly positive, $f \in C(X)$ such that $Hv = fv$. Then*

$$h(\varphi, \psi) = h_v\left(\frac{\varphi}{v}, \frac{\psi}{v}\right) + \langle f\varphi, \psi \rangle, \qquad \varphi, \psi \in C_c(X).$$

Clearly, whenever there exists a strictly positive H-harmonic function v, criticality can be carried over directly from h to h_v and vice versa. In this paper we are concerned with criticality of a form $h - w$ for some positive function $w \geq 0$. However, criticality, null-criticality and even optimality near infinity can be carried over from $h_v - v^2 w$ to $h - w$ as well.

Corollary 2.5. *Let $v \in F(X)$ be a strictly positive H-(super)harmonic function and $w \geq 0$.*

(a) *$h - w$ is critical if and only if $h_v - v^2 w$ is critical.*
(b) *$h - w$ is null-critical with respect to w if and only if $h_v - v^2 w$ is null-critical with respect to $v^2 w$.*
(c) *w is optimal near infinity for h if and only if $v^2 w$ is optimal near infinity for h_v.*

Proof. By the ground state, transform the quadratic form associated to the operator $H_v - w$ is $h_v - v^2 w$ (where Proposition 2.4 is applied for the operator $H - w$ and $f = w$). Furthermore, for any $\psi \in F(X)$ we see that

$$(H - w)\psi = (T_v H_v T_{v^{-1}} - w)\psi = T_v (H_v - w) T_{v^{-1}} \psi = T_v (H_v - w) \frac{\psi}{v}.$$

Therefore, every positive $(H - w)$-(super)harmonic function ψ yields a positive $(H_v - w)$-(super)harmonic function ψ / v. Thus, $h - w$ is critical if and only if $h_v - v^2 w$ is critical by [17, Theorem 5.3]. This proves (a). For the very same reason, statement (b) follows immediately by the definition of null-criticality.

Finally, w not being optimal near infinity for h means there is a finite $W \subseteq X$ and $\lambda > 0$ such that $h - w \geq \lambda w$ on $C_c(X \setminus W)$. Hence, $h - (1+\lambda/2)w$ is subcritical in $X \setminus W$ which according to (a) yields that $h_v - (1+\lambda/2)v^2 w$ is subcritical there. Consequently, $h_v - v^2 w \geq (\lambda/2)v^2 w$ on $C_c(X \setminus W)$ and thus, $v^2 w$ is not optimal near infinity (for h_v). By the same argument, w is not optimal near infinity for h when $v^2 w$ is not optimal near infinity for h_v which shows (c). □

2.3. Coarea formula. In this section we establish the pivotal tool for the proof of the main theorems. It allows us to translate calculations and estimates of infinite sums over graphs to one dimensional integrals.

Theorem 2.6. *Let b be a connected graph over X, and let $u \in C(X)$ be positive. Let $f : (\inf u, \sup u) \to [0, \infty)$ be a Riemann integrable function. Then*

$$\frac{1}{2} \sum_{x,y \in X \times X} b(x, y)(u(x) - u(y)) \int_{u(y)}^{u(x)} f(t)\, \mathrm{d}t = \int_{\inf u}^{\sup u} f(t) g(t)\, \mathrm{d}t, \tag{2.1}$$

where both sides can take the value $+\infty$, and $g : (\inf u, \sup u) \to [0, \infty]$ is given by

$$g(t) := \sum_{\substack{x,y \in X \\ u(y) < t \leq u(x)}} b(x, y)(u(x) - u(y)).$$

Assume further that $u \in F(X)$ is L-harmonic outside of a finite set, and

(a) *$u^{-1}(I)$ is finite for any compact $I \subseteq (\inf u, \sup u)$,*

(b) $\displaystyle\sup_{\substack{x,y \in X \\ x \sim y}} \frac{u(x)}{u(y)} < \sup_{x,y \in X} \frac{u(x)}{u(y)}$.

Then there are positive constants c and C such that

$$c \leq g \leq C,$$

and if in addition u is L-harmonic in X, then g is constant.

Remark 2.7. (a) Note that by $f \geq 0$, the terms in the sum on both sides of the equality (2.1) above are always greater than or equal to zero.

(b) Let u and f be as in Theorem 2.6 with f being continuous. Then by the Lagrange's mean value theorem, if $u(x) \neq u(y)$ there is $\theta_{x,y} \in (u(x) \wedge u(y), u(x) \vee u(y))$ such that

$$f(\theta_{x,y}) = \frac{\int_{u(y)}^{u(x)} f(t)\, \mathrm{d}t}{u(x) - u(y)}.$$

Consequently, the coarea formula reads as

$$\frac{1}{2} \sum_{x,y \in X \times X} b(x, y)(u(x) - u(y))^2 f(\theta_{x,y}) = \int_{\inf u}^{\sup u} f(t) g(t)\, \mathrm{d}t.$$

The following lemma can be interpreted as a Stokes type theorem. Specifically, the function g can be viewed as the integral of the normal derivative over the boundary of the level set $\{x \mid u(x) > t\}$ of u for some t. Formula (2.2) of the Lemma 2.8 then shows that the function g at t_1 and t_2 differs only by the nonharmonic contribution of u on the set $A = \{x \mid t_1 < u(x) \leq t_2\}$. Provided (2.2), the proof of the coarea-formula reduces to algebraic manipulations and the application of Tonelli's theorem.

Lemma 2.8 (Stokes-type formula). *Let $u \in F(X)$ be a positive nonconstant function such that $u^{-1}(I)$ is finite for any compact $I \subseteq (\inf u, \sup u)$. Let*

$$g : (\inf u, \sup u) \to [0, \infty], \qquad g(t) := \sum_{\substack{x,y\in X \\ u(y)<t\leq u(x)}} b(x, y)(u(x) - u(y)).$$

Then for any $t_1, t_2 \in (\inf u, \sup u)$ such that $t_1 \leq t_2$, the set

$$A := \{x \in X \mid t_1 < u(x) \leq t_2\}$$

is finite, and

$$g(t_2) = g(t_1) - \sum_{x\in A} Lu(x), \tag{2.2}$$

where both sides may take the value $+\infty$. In particular, g is monotone decreasing whenever u is L-superharmonic.

Moreover, if

$$\sup_{x,y\in X\, x\sim y} \frac{u(x)}{u(y)} < \sup_{x,y\in X} \frac{u(x)}{u(y)},$$

and u is L-harmonic outside of a finite set, then g is piecewise constant with finitely many jumps, and for some positive constants c, C

$$0 < c \leq g \leq C < \infty.$$

Furthermore, if u is L-harmonic in X, then g is constant.

Proof. For this proof we denote $\nabla f := f(x) - f(y)$ for functions f whenever summing over x and y such that $x \sim y$.

For $t > 0$, define

$$\Omega_t := \{x \in X \mid u(x) > t\}.$$

Let $t_1, t_2 \in (\inf u, \sup u)$ with $t_1 \leq t_2$, and define A to be

$$A := \Omega_{t_1} \setminus \Omega_{t_2} = \{x \in X \mid t_1 < u(x) \leq t_2\}.$$

By the assumption that the pre-images of u of compact sets in the interval $(\inf u, \sup u)$ are finite, the set A is finite. Therefore, the characteristic function 1_A of A is in $C_c(X)$. For $B \subset X$ we denote

$$\partial B := \{(x, y) \in X \times X \mid x \in B,\ y \notin B\}.$$

Since $u \in F(X)$, we can apply the Green formula, [13, Lemma 4.7], for u paired with 1_A to see

$$\sum_A Lu = \sum_X 1_A Lu = \frac{1}{2}\sum_{X\times X} b\nabla u \nabla 1_A = \sum_{\partial A} b\nabla u,$$

where the right hand side also converges absolutely.

In the next step we show that the sum on the left hand side can be split into a difference of a sum over the boundary of Ω_{t_1} and Ω_{t_2}. To this end, we observe that for any $B \subseteq X$, we have $(x, y) \in \partial B$ if and only if $(y, x) \in \partial(X\setminus B)$. Moreover, since $\Omega_{t_2} \subseteq \Omega_{t_1}$, we conclude

$$\partial A = \left(\partial\Omega_{t_1} \cap \partial A\right) \cup \left(\partial(X\setminus\Omega_{t_2}) \cap \partial A\right),$$

and

$$\partial\Omega_{t_1}\setminus\partial A = \partial\Omega_{t_2}\setminus\partial(X\setminus A).$$

For the sake of illustration, see Fig. 1.

Thus, since $\sum_{(x,y)\in\partial A} b(x, y)(u(x) - u(y))$ converges absolutely, we obtain by the considerations above

$$\sum_{\partial A} b\nabla u = \sum_{\partial\Omega_{t_1}\cap\partial A} b\nabla u + \sum_{\partial(X\setminus\Omega_{t_2})\cap\partial A} b\nabla u = \sum_{\partial\Omega_{t_1}\cap\partial A} b\nabla u - \sum_{\partial\Omega_{t_2}\cap\partial(X\setminus A)} b\nabla u.$$

We employ the equalities above to see

$$\begin{aligned}
g(t_2) = \sum_{\partial\Omega_{t_2}} b\nabla u &= \sum_{\partial\Omega_{t_2}\setminus\partial(X\setminus A)} b\nabla u + \sum_{\partial\Omega_{t_2}\cap\partial(X\setminus A)} b\nabla u \\
&= \sum_{\partial\Omega_{t_1}\setminus\partial A} b\nabla u + \sum_{\partial\Omega_{t_2}\cap\partial(X\setminus A)} b\nabla u \\
&= g(t_1) - \sum_{\partial\Omega_{t_1}\cap\partial A} b\nabla u + \sum_{\partial\Omega_{t_2}\cap\partial(X\setminus A)} b\nabla u \\
&= g(t_1) - \sum_{\partial\Omega_{t_1}\cap\partial A} b\nabla u - \sum_{\partial(X\setminus\Omega_{t_2})\cap\partial A} b\nabla u \\
&= g(t_1) - \sum_{\partial A} b\nabla u = g(t_1) - \sum_A Lu.
\end{aligned}$$

As $\sum_A Lu < \infty$, this shows for all $t_1, t_2 \in (\inf u, \sup u)$ we have $g(t_1) < \infty$ if and only if $g(t_2) < \infty$.

If u is L-harmonic outside of a finite set, then the mapping $\{B \subseteq X\} \to \mathbb{R}$, $B \mapsto \sum_B Lu$ takes only finitely many values. Therefore, by (2.2), g takes only finitely many values. Moreover, since $\sum_A Lu$ as a function of t_2 changes only at finitely many t_2 and vice versa for t_1, the function g is piecewise constant with finitely many jumps.

Hence, $g \geq c > 0$ can fail only if $g(t) = 0$ for some $t \in (\inf u, \sup u)$. However, this is impossible as $g(t) = 0$ implies $\partial\Omega_t = \emptyset$ which implies either $t < \inf u$ or $t \geq \sup u$ by the connectedness of the graph.

To see the upper bound for g, we employ the assumption

$$\sup_{x,y\in X\, x\sim y} \frac{u(x)}{u(y)} < \sup_{x,y\in X} \frac{u(x)}{u(y)}.$$

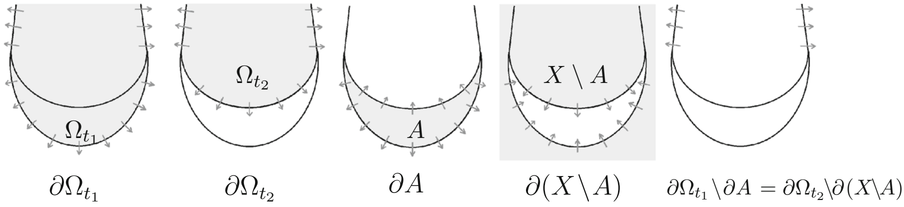

Fig. 1. Illustrations of the sets and their boundaries

This assumption is equivalent to

$$c := \sup_{x\sim y}(u(x) - u(y)) < \sup_{x\in X} u(x) - \inf_{x\in X} u(x).$$

Hence, there are $t_1, t_2 \in (\inf u, \sup u)$ such that $t_2 - t_1 > c$. For the choice of these t_1, t_2, there is no vertex in Ω_{t_2} that is connected to a vertex outside of Ω_{t_1}. Hence, $\partial\Omega_{t_2} = \partial\Omega_{t_2} \cap \partial(X\setminus A)$ and we have by the considerations above

$$g(t_2) = \sum_{\partial\Omega_{t_2}\cap\partial(X\setminus A)} b\nabla u \le \sum_{\partial A} b|\nabla u| < \infty.$$

Thus, g stays finite on $(\inf u, \sup u)$ and since g is piecewise constant with finitely many jumps, there is C such that $g \le C$. This finishes the proof. □

Proof of Theorem 2.6. Let $t > 0$, and recall that

$$\Omega_t = \{x \in X \mid u(x) > t\}.$$

Let $1_{x,y}$ be the characteristic function of the interval

$$I_{x,y} = [u(x) \wedge u(y), u(x) \vee u(y)].$$

Observe that (x, y) or (y, x) are in $\partial\Omega_t := \Omega_t \times X\setminus\Omega_t$ if and only if $t \in I_{x,y}$. With this observation in mind, we calculate

$$\sum_{x,y\in X\times X} b(x, y)(u(x) - u(y)) \int_{u(y)}^{u(x)} f(t)\,\mathrm{d}t$$
$$= \sum_{x,y\in X\times X} b(x, y)|u(x) - u(y)| \int_{\inf u}^{\sup u} f(t)1_{x,y}(t)\,\mathrm{d}t.$$

Now, by Tonelli's theorem we obtain

$$\begin{aligned}
\ldots &= \int_{\inf u}^{\sup u} f(t) \sum_{x,y\in X\times X} b(x, y)|u(x) - u(y)|1_{x,y}(t)\,\mathrm{d}t \\
&= 2\int_{\inf u}^{\sup u} f(t) \sum_{(x,y)\in\partial\Omega_t} b(x, y)|u(x) - u(y)|\,\mathrm{d}t \\
&= 2\int_{\inf u}^{\sup u} f(t) \sum_{(x,y)\in\partial\Omega_t} b(x, y)(u(x) - u(y))\,\mathrm{d}t
\end{aligned}$$

since $u(x) \ge u(y)$ for $(x, y) \in \partial\Omega_t$. This shows the first part of the theorem. The second part follows from Lemma 2.8. □

3. Critical Hardy-Weights

In the following three sections we will prove Theorem 1.1 for operators $H = L + q$ with finitely supported $q \geq 0$. This will be achieved by the virtue of the coarea formula (Theorem 2.6). The general case will be deduced in Sect. 6 using the ground state transform. We start by proving criticality in this section, and show null-criticality and optimality at infinity in the two succeeding sections.

Theorem 3.1. *Let b be a connected graph and $q \geq 0$ be a finitely supported potential. Suppose that u is a positive H-superharmonic function that is H-harmonic outside of a finite set, and satisfies*

(a) $u : X \to (0, \infty)$ *is proper,*
(b) $\displaystyle\sup_{\substack{x,y\in X\\ x\sim y}} \frac{u(x)}{u(y)} < \infty.$

Let h be the quadratic form associated to H. Then $h - w$ with $w := \frac{H\left[u^{1/2}\right]}{u^{1/2}}$, is critical in X.

Proof. We set $v := u^{1/2}$. Then v is positive and H-superharmonic by the chain rule for the square root (Lemma 2.2) since $q \geq 0$. Furthermore, v is obviously a positive $(H - w)$-harmonic function in X.

The strategy of the proof is to construct a null-sequence (e_n) in $C_c(X)$ with respect to $(h - w)_v$, i.e. $(h - w)_v(e_n) \to 0$ and $e_n \to 1$ pointwise. By [17, Theorem 5.3 (iv)], this then implies that $(h - w)_v$ is critical, and hence the criticality of $h - w$.

Set $\varphi_n : \mathbb{R} \to \mathbb{R}$

$$\varphi_n(t) := \left(2 + \frac{1}{\log n}\log(t)\right) 1_{[\frac{1}{n^2},+\frac{1}{n}]}(t) + 1_{[\frac{1}{n},n]}(t) + \left(2 - \frac{1}{\log n}\log(t)\right) 1_{[n,n^2]}(t),$$

and let $e_n := \varphi_n \circ u$. Since $\operatorname{supp}\varphi_n \subseteq (0, \infty)$, and $\sup u = \infty$ or $\inf u = 0$, we have $e_n \in C_c(X)$ by assumption (a). Obviously, $e_n \to 1$ pointwise as $n \to \infty$. So, we are left to show $(h - w)_v(e_n) \to 0$ as $n \to \infty$. We compute

$$\begin{aligned}(h - w)_v(e_n) &= \frac{1}{2}\sum_{x,y\in X} b(x, y)(u(x)u(y))^{1/2}(\varphi_n(u(x)) - \varphi_n(u(y)))^2 \\ &= \frac{1}{2}\sum_{x,y\in X} b(x, y)\,(u(x) - u(y))\,c(x, y)\left(\int_{u(y)}^{u(x)} t\varphi_n'(t)^2\,\mathrm{d}t\right),\end{aligned}$$

where

$$c(x, y) := \frac{(u(x)u(y))^{1/2}(\varphi_n(u(x)) - \varphi_n(u(y)))^2}{(u(x) - u(y))\int_{u(y)}^{u(x)} t\varphi_n'(t)^2\,\mathrm{d}t}$$

whenever the denominator is nonzero and $c(x, y) = 0$ otherwise.

Since $c(x, y)$ always appears in a product with $b(x, y)$ it suffices to consider x, y with $x \sim y$. By the anti-oscillation assumption (b) there is a constant $C_0 := \sup_{z\sim w} u(z)/u(w)$ such that for $n > \sqrt{C_0}$ we have $u(x), u(y) \in (0, n]$ or $u(x), u(y) \in [1/n, \infty)$ for $x \sim y$. We now use the definition of φ_n and the elementary inequalities

$$|a \wedge c - b \wedge c| \leq |a - b|, \qquad a, b, c \in \mathbb{R},$$
$$\frac{\log b - \log a}{b - a} \leq \log'(a) = \frac{1}{a}, \qquad 0 < a \leq b < \infty,$$

to estimate

$$c(x, y) \leq \frac{(u(x)u(y))^{1/2}(\log u(x) - \log u(y))}{(u(x) - u(y))}$$
$$\leq \sup_{z,w\in X, z\sim w} \left(\frac{u(z)}{u(w)}\right)^{1/2} = C_0$$

for all $x \sim y$ and $n > \sqrt{C_0}$. Notice that $C_0 < \infty$ by our assumption (b).

We use this estimate and we apply now the coarea formula with $f(t) = t\varphi_n'(t)^2$. To this end, we note that the assumptions of Theorem 2.6 are fulfilled: The function u is L-harmonic outside of the finite set (including the finite set where q is supported), Theorem 2.6 (a) is fulfilled by assumption (a), and Theorem 2.6 (b) is fulfilled by assumption (b) and $\sup u = \infty$ or $\inf u = 0$. Moreover, by the L-harmonicity of u outside of a finite set, the function g in the coarea formula (2.1) is piecewise constant. Therefore, there exists a constant C_1 such that

$$(h - w)_v(e_n) \leq C_0 \sum_{x,y\in X} b(x, y)\,(u(x) - u(y)) \left(\int_{u(y)}^{u(x)} t\varphi_n'(t)^2 \,\mathrm{d}t\right)$$
$$\leq C_1 \int_{\inf u}^{\sup u} t\varphi_n'(t)^2 \,\mathrm{d}t$$
$$\leq C_2 \left(\frac{1}{\log n}\right)^2 \left(\int_{\frac{1}{n^2}}^{\frac{1}{n}} \frac{\mathrm{d}t}{t} + \int_{n}^{n^2} \frac{\mathrm{d}t}{t}\right) = \frac{2C_2}{\log n} \xrightarrow[n\to\infty]{} 0.$$

Thus, (e_n) is a null-sequence which implies that $h - w$ is critical by the discussion in the beginning of the proof. □

4. Null-Criticality

In the present section we prove the null-criticality assertion of Theorem 1.1 under the additional assumption that q is a positive finitely supported potential. For general q the statement is then deduced in Sect. 6 using the ground state transform.

In the case $q \geq 0$, it is convenient to extend the quadratic form h for a graph b which is defined on $C_c(X)$ to a map $C(X) \to [0, \infty]$ via

$$f \mapsto \frac{1}{2} \sum_{x,y\in X} b(x, y)(f(x) - f(y))^2 + \sum_{x\in X} q(x) f(x)^2.$$

It is easily seen that this defines a quadratic form and with a slight abuse of notation we denote this form also by h. Moreover, whenever there is a positive harmonic function

u for the operator H associated to a form h with general potential q, such an extension can be employed for the ground state transform h_u of h.

Theorem 4.1. *Let b be a graph, $q \geq 0$ be finitely supported and $H = L + q$. Let u be a positive H-superharmonic function that is H-harmonic outside of a finite set. In addition, assume that*

(a) $u : X \to (0, \infty)$ *is proper,*
(b) $\sup_{\substack{x,y\in X\\ x\sim y}} \dfrac{u(x)}{u(y)} < \infty$.

Let h be the quadratic form associated to H. Then

$$h(u^{1/2}) = \infty,$$

and the form $h - w$ with $w := \frac{H[u^{1/2}]}{u^{1/2}}$ is null-critical in X with respect to w.

Proof. By the elementary inequality

$$\frac{(b^{1/2} - a^{1/2})^2}{(b-a)\int_a^b \frac{\mathrm{d}t}{t}} \geq \frac{a}{4b}, \qquad 0 < a < b < \infty,$$

we have for $x, y \in X$ with $x \sim y$

$$c(x, y) := \frac{(u^{1/2}(x) - u^{1/2}(y))^2}{(u(x) - u(y)) \int_{u(y)}^{u(x)} \frac{\mathrm{d}t}{t}} \geq \frac{1}{4} \inf_{x'\sim y'} \frac{u(x')}{u(y')} =: C_0,$$

where $C_0 > 0$ by assumption (b) whenever $u(x) \neq u(y)$. We apply the coarea formula (Theorem 2.6) with $f(t) = 1/t$. Since q is finitely supported, u is L-harmonic outside of a finite set. Thus, by Theorem 2.6, the function g in the coarea formula (2.1) is bounded away from zero, and we get

$$\begin{aligned}\frac{1}{2}\sum_{x,y\in X} b(x, y)(u^{1/2}(x) - u^{1/2}(y))^2 &= \frac{1}{2}\sum_{x,y\in X} b(x, y)c(x, y)(u(x) - u(y))\int_{u(y)}^{u(x)} \frac{\mathrm{d}t}{t} \\ &\geq C_0 \sum_{x,y\in X} b(x, y)(u(x) - u(y)) \int_{u(y)}^{u(x)} \frac{\mathrm{d}t}{t} \geq C_1 \int_{\inf u}^{\sup u} \frac{1}{t}\,\mathrm{d}t = \infty,\end{aligned}$$

where C_1 is a positive constant. Consequently, $h(u^{1/2}) = \infty$.

It remains to show that this implies the null-criticality of $h - w$ with respect to w in the case $\sup u = \infty$ or $\inf u = 0$. Note that the function $u^{1/2}$ is $(H - w)$-harmonic with $w = \frac{H[u^{1/2}]}{u^{1/2}}$. Since $\sup u = \infty$ or $\inf u = 0$, it follows from Theorem 3.1 that, under our assumptions, $h - w$ is critical.

In the critical case, there is a unique positive harmonic function (up to linear dependence), see [17, Theorem 5.3 (iii)]. Hence, this ground state is $u^{1/2}$. By $h(u^{1/2}) = \infty$ it follows, in particular, that $u^{1/2}$ can not be approximated by compactly supported functions. By a characterization of null-criticality [17, Theorem 6.2], this implies that the form $h - w$ is null-critical with respect to w. □

5. Optimality Near Infinity

In this section we give a criterion for optimality near infinity. Recall that if h, u and $q \geq 0$ satisfy the assumptions of Theorem 4.1, then

$$h(u^{1/2}) = \frac{1}{2} \sum_{x,y \in X} b(x,y) \left(u(x)^{1/2} - u(y)^{1/2} \right)^2 + \sum_{x \in X} q(x) u(x)^{1/2} = \infty,$$

where h denotes again the extension of the form on $C_c(X)$ to $C(X)$. We will deduce optimality at infinity for $q \geq 0$ directly from the divergence of the sum above. The case of general q is then covered in Sect. 6.

Theorem 5.1. *Consider a graph b and $q \geq 0$ with a finite support. Let $H := L + q$. Let u be a positive nonconstant H-superharmonic function in X that is L-harmonic outside of a finite set. Furthermore, let $w = L(u^{1/2})/u^{1/2}$, and assume $h - w$ is critical and*

$$h(u^{1/2}) = \infty.$$

Then for all finite $W \subseteq X$ and all $\lambda > 0$, the inequality

$$h - (1 + \lambda) w \geq 0$$

fails to hold on $C_c(X \backslash W)$.

We prove this theorem by assuming that the inequality in the theorem holds for some $\lambda > 0$ and show that contradicts the null-criticality. To this end, we need to show that we can extend the inequalities in question to a larger class of functions. This is achieved by the following lemma

Lemma 5.2. *Let $b, \widetilde{b}$ be graphs, q, q' potentials, and let h, h' be the corresponding forms with the associated operators H and H'. Let v be a positive H-superharmonic function and suppose that $v^{1/2}$ is a positive H'-harmonic function. Assume h' is critical on X and that there is $C \geq 0$ such that*

$$h(\varphi) \leq C h'(\varphi), \quad \varphi \in C_c(X \backslash W),$$

for $W \subseteq X$. Then

$$h_v(v^{-1} f) \leq C h'_{v^{1/2}}(v^{-1/2} f), \qquad f \in C(X \backslash W).$$

Proof. Since h' is critical on X, it follows that $h'_{v^{1/2}}$ is critical as well by Corollary 2.5. Then by [17, Theorem 5.3 (iv)], there exists a the null-sequence (e_n) for $h'_{v^{1/2}}$ such that $e_n \in C_c(X)$, $0 \leq e_n \leq 1$, $e_n \to 1$ and $h'_{v^{1/2}}(e_n) \to 0$.

Applying the ground state transforms (Proposition 2.4), we see

$$h_v(v^{-1}\varphi) \leq h(\varphi) \leq C h'(\varphi) = C h'_{v^{1/2}}(v^{-1/2}\varphi)$$

for all $\varphi \in C_c(X \backslash W)$, and hence, multiplying all functions by $v^{1/2}$ yields

$$h_v(v^{-1/2}\varphi) \leq C h'_{v^{1/2}}(\varphi), \qquad \varphi \in C_c(X \backslash W).$$

We next employ [17, Lemma 5.11] which states that since $h'_{v^{1/2}}$ is critical and (e_n) is a null-sequence for $h'_{v^{1/2}}$ it follows that for every function $f \in C(X)$ we have

$\lim_{n\to\infty} h'(e_n f) = h'(f)$. Hence, for $f \in C(X)$ with support in $X\setminus W$, by Fatou's lemma and the inequality in the assumption (noting that $e_n f \in C_c(X\setminus W)$) we obtain

$$h_v(v^{-1/2} f) \leq \liminf_{n\to\infty} h_v(v^{-1/2} e_n f) \leq C \limsup_{n\to\infty} h'_{v^{1/2}}(e_n f) = C h'_{v^{1/2}}(f).$$

Replacing f by $v^{-1/2} f$ yields the claim of the lemma. □

We recall the following notation: Given a graph b, zero potential $q = 0$, and a strictly positive function g, the form h_g on $C(X)$ acts as

$$h_g(f) = \frac{1}{2} \sum_{x,y\in X} b(x, y)g(x)g(y)(f(x) - f(y))^2,$$

which happens to coincide with the extension of the ground state transform of h to $C(X)$ whenever g is H-harmonic.

Proof of Theorem 5.1. Throughout the proof c and C denote positive finite constants that may change from line to line.

We set $w = (Hu^{1/2})/u^{1/2}$. Assume there is $\lambda > 0$ such that $h - w \geq \lambda w$ on $C_c(X\setminus W)$ for finite W. So,

$$h \leq C(h - w)$$

on $C_c(X\setminus W)$ with $C = \frac{1+\lambda}{\lambda}$. We show that this leads to a contradiction.

By definition of w the function $u^{1/2}$ is $(H - w)$-harmonic and u is H-superharmonic by assumption. By Lemma 5.2, we have

$$h_u(u^{-1} f) \leq C(h - w)_{u^{1/2}}(u^{-1/2} f), \quad f \in C(X\setminus W), \tag{5.1}$$

with the ground state transforms h_u of h and $(h - w)_{u^{1/2}}$ of $h - w$.

Let us first estimate the left hand side of (5.1) from below. Let $f = u^{1/2} 1_{X\setminus W}$. By the equality $ab(a^{-1/2} - b^{-1/2})^2 = (a^{1/2} - b^{1/2})^2$ applied with $a = u(x)$, $b = u(y)$, we obtain

$$h_u(u^{-1} f) = h_u(u^{-1/2} 1_{X\setminus W}) = h_{1_{X\setminus W}}(u^{1/2}) + \sum_{x\in W, y\in X\setminus W} b(x, y)u(y),$$

where we observe that the second term on the right hand side is finite since $u \in F(X)$ and W is finite. Furthermore,

$$\begin{aligned} \ldots &= h_1(u^{1/2}) - h_{1_W}(u^{1/2}) - \sum_{x\in W, y\in X\setminus W} b(x, y)((u^{1/2}(x) - u^{1/2}(y))^2 - u(y)) \\ &\geq h_1(u^{1/2}) - h_{1_W}(u^{1/2}) - \sum_{x\in W} u(x) \sum_{y\in X\setminus W} b(x, y), \end{aligned}$$

where the second and the third term on the right hand side are finite since W is finite and $\sum_{y\in X} b(x, y) < \infty$ for all x. Since $q \geq 0$ is compactly supported and we assume that $h(u^{1/2}) = \infty$, it follows that $h_1(u^{1/2}) = \infty$ and, thus, we conclude that

$$h_u(u^{-1} f) = \infty.$$

For the right hand side of (5.1), we get with $f := u^{1/2} 1_{X \setminus W}$

$$
\begin{aligned}
(h-w)_{u^{1/2}}(u^{-1/2} f) &= (h-w)_{u^{1/2}}(1_{X\setminus W}) \\
&\le \sum_{x \in W, y \in X \setminus W} b(x,y)(u(x)u(y))^{1/2} \\
&\le \left(\sum_{x \in W, y \in X\setminus W} b(x,y)u(x) \right)^{1/2} \left(\sum_{x \in W, y \in X\setminus W} b(x,y)u(y) \right)^{1/2}.
\end{aligned}
$$

The right hand side is finite since $u \in F(X)$, the set W is finite, and $\sum_{y \in X} b(x, y) < \infty$ for all x.

Thus, $(h-w)_{u^{1/2}}(u^{-1/2} f) < \infty$ while $h_u(u^{-1} f) = \infty$ which is a contradiction to (5.1). This proves the theorem by the discussion in the beginning of the proof. □

6. Proof of the Main Theorem

Proof of Theorem 1.1. Let u, v be positive H-superharmonic functions on X, such that u, v are H-harmonic outside of a finite set. Let $u_0 := u/v$. Then the positive function u_0 is H_v-superharmonic and H_v-harmonic outside of the finite set. Let

$$
w := \frac{H_v \left[u_0^{1/2} \right]}{u_0^{1/2}} = \frac{H \left[(uv)^{1/2} \right]}{(uv)^{1/2}}.
$$

By Corollary 2.5, w is an optimal Hardy weight for h if and only if $w' = v^2 w$ is an optimal Hardy weight for $h' := h_v$. Note that by Green's formula the form $h' = h_v$ corresponds to the operator $H' = L' + q'$ where L' is the operator associated to the graph $b'(x, y) = b(x, y)v(x)v(y)$, $x, y \in X$, and $q' = v(Hv)$, i.e.,

$$
h'(\varphi, \psi) = \langle H' \varphi, \psi \rangle_1.
$$

Note that by assumption on v the potential q' is finitely supported and since $H' = T_{v^2} H_v$ the function u_0 is a positive H'-superharmonic function that satisfies the assumptions of Theorem 3.1, Theorem 4.1 and Theorem 5.1. Hence, $w' = \frac{H'(u_0^{1/2})}{u_0^{1/2}} = v^2 w$ is an optimal Hardy weight for $h' = h_v$ which finishes the proof. The explicit formula for w follows from the chain rule of the square root (see Lemma 2.2 and Corollary 2.3). □

Proof of Corollary 1.4. First of all, by the chain rule for the square root, the function $(u(v-u))^{1/2}$ is H-superharmonic, (see, Lemma 2.2 and Corollary 2.3, where also the explicit formula of w can be read from). Let us check that the assumptions of Corollary 1.4 imply the assumptions of Theorem 1.1. We let $u_0 = u/v$ and

$$
v_0 := \frac{u}{v-u} = \frac{u_0}{1-u_0} = \frac{1}{u_0^{-1} - 1}.
$$

Let us check the validity of the assumption (a) in Theorem 1.1. Let $[a, b] \subseteq (0, \infty)$ and $x \in X$ such that $v_0(x) \in [a, b]$. It follows that $u_0(x) \in [a/(a+1), b/(b+1)] \subset (0, 1)$. By assumption (a) of Corollary 1.4, there are only finitely many of these x and hence Theorem 1.1 (a) follows. Assumption (b) of Theorem 1.1 follows directly from assumption (b) of Corollary 1.4. □

Finally, we explain how the two special cases in the Introduction can be derived from the main theorems.

To this end we first prove the following lemma in the context of graphs with standard weights, i.e., $b(x, y) \in \{0, 1\}$, $x, y \in X$.

Lemma 6.1. *Assume that* $\deg(x) \leq C$ *for all* $x \in X$*, and let* u *be a positive* Δ*-superharmonic on* $W \subseteq X$*. Then*

$$\sup_{\substack{x \sim y \\ x \in W}} \frac{u(x)}{u(y)} \leq C.$$

Proof. Since $u > 0$ and $\Delta u(y) \geq 0$, we get for $x \sim y$

$$u(x) \leq \sum_{z \sim y} u(z) \leq \deg(y) u(y) \leq C u(y)$$

where $C > 0$ does not depend on $x \in W$. □

Proof of Theorem 0.1. Let $h_{X \setminus K}$ be the restriction of the form h to the space $C_c(X \setminus K)$. Then the operator $H_{X \setminus K}$ acts as

$$H_{X \setminus K} \varphi(x) = \sum_{y \in X \setminus K,\ y \sim x} (\varphi(x) - \varphi(y)) + q(x)\varphi(x),$$

with $q(x) := \#\{z \in K \mid z \sim x\}$. Hence, $v = 1$ is $H_{X \setminus K}$-superharmonic in $X \setminus K$ and $H_{X \setminus K}$-harmonic outside of the combinatorial neighborhood of K. Moreover, as $\Delta = H_{X \setminus K}$ for functions supported on $X \setminus K$, the restriction of u to $X \setminus K$ is $H_{X \setminus K}$-harmonic.

Assumption (a) of Theorem 1.1 is satisfied for $H_{X \setminus K}$ for $u_0 = u$. Furthermore, assumption (b) follows from Lemma 6.1. Hence, we obtain for $\varphi \in C_c(X \setminus K)$,

$$\frac{1}{2} \sum_{x \sim y} (\varphi(x) - \varphi(y))^2 = h(\varphi) \geq \sum_{x \in X \setminus K} w(x) \varphi^2(x)$$

with optimal w given by

$$w(x) = \frac{H_{X \setminus K} u^{1/2}}{u^{1/2}}(x) = \frac{1}{2u(x)} \sum_{y \sim x} \left(u(x)^{1/2} - u(y)^{1/2} \right)^2$$

for $x \in X \setminus K$. □

Proof of Theorem 0.2. We apply Theorem 1.1 with $v = G(o, \cdot)$ and $u = 1$. In particular, assumption (a) of Theorem 1.1 is satisfied for $u_0 = 1/G(o, \cdot)$. Furthermore, assumption (b) follows from the lemma above (Lemma 6.1). Hence, the statement follows. □

7. Examples

7.1. The $\mathbb{Z}^d$*-case.* It is a well-known that for $d \geq 3$, the Green function $G(x) := G(x, 0)$ associated to the Laplacian Δ on the standard $\mathbb{Z}^d$-lattice has the following asymptotic behaviour (see [23, Theorem 2], and the remark at the very end of Sect. 2 therein):

Theorem 7.1. *Let $d \geq 3$. Then as $|x| \to \infty$,*

$$G(x) = \frac{C_1(d)}{|x|^{d-2}} + C_2(d)\left(\left(\sum_{i=1}^{d}\left(\frac{x_i}{|x|}\right)^4\right) - \frac{3}{d+2}\right)\frac{1}{|x|^d} + \mathcal{O}\left(\frac{1}{|x|^{d+2}}\right),$$

where $C_1(d)$ and $C_2(d)$ are positive constants depending only on d.

It follows from Theorem 0.2 that

$$w(x) := \frac{\Delta[G^{1/2}(x)]}{G^{1/2}(x)}, \qquad x \in \mathbb{Z}^d \setminus \{0\},$$

is an optimal Hardy weight for Δ. We use Theorem 7.1 to derive the asymptotic behavior of w as $|x| \to \infty$.

Theorem 7.2. *Let $d \geq 3$. Then as $|x| \to \infty$ we have*

$$w(x) = \frac{(d-2)^2}{4}\frac{1}{|x|^2} + \mathcal{O}\left(\frac{1}{|x|^3}\right).$$

Proof. Using Theorem 7.1, one obtains for $|x| \to \infty$ and $y \sim x$ that

$$\frac{G(y)}{G(x)} = \frac{|x|^{d-2}}{|y|^{d-2}} + \mathcal{O}\left(\frac{1}{|x|^3}\right).$$

Keeping this observation in mind, it follows that

$$\sum_{y\sim x}\left(\frac{G(y)}{G(x)}\right)^{1/2} = \sum_{i=1}^{d}\sum_{\epsilon\in\{\pm 1\}}\left(\frac{|x|^{d-2}}{\left(|x|^2 + \epsilon 2 x_i + 1\right)^{(d-2)/2}}\right)^{1/2} + \mathcal{O}\left(\frac{1}{|x|^3}\right)$$

$$= \sum_{i=1}^{d}\sum_{\epsilon\in\{\pm 1\}}\left(1 + \frac{\epsilon 2 x_i}{|x|^2} + \frac{1}{|x|^2}\right)^{(2-d)/4} + \mathcal{O}\left(\frac{1}{|x|^3}\right).$$

We now use the Taylor series expansion

$$\left(1+z\right)^{\alpha} = 1 + \alpha z + \binom{\alpha}{2} z^2 + \mathcal{O}(|z|^3)$$

for $z \in \mathbb{R}$ with $|z| < 1$ and $\alpha \in \mathbb{R}$ (here the generalized binomial coefficient $\binom{\alpha}{2}$ is defined as $\binom{\alpha}{2} := \alpha\left(\alpha - 1\right)/2$). As $x \to \infty$, this yields

$$\sum_{y\sim x}\left(\frac{G(y)}{G(x)}\right)^{1/2} = \sum_{i=1}^{d}\left(2 + \frac{2(2-d)}{4}\frac{1}{|x|^2} + 2\binom{\frac{2-d}{4}}{2}\frac{4x_i^2}{|x|^4}\right) + \mathcal{O}\left(\frac{1}{|x|^3}\right)$$

$$= 2d + \frac{d(2-d)}{2}\frac{1}{|x|^2} + 8\,\frac{(2-d)(2-d-4)}{2\cdot 16}\frac{1}{|x|^2} + \mathcal{O}\left(\frac{1}{|x|^3}\right)$$

$$= 2d - \frac{(d-2)^2}{4}\frac{1}{|x|^2} + \mathcal{O}\left(\frac{1}{|x|^3}\right).$$

It follows that

$$w(x) = \frac{\Delta\left[G^{1/2}(x)\right]}{G^{1/2}(x)} = 2d - \sum_{y\sim x}\left(\frac{G(y)}{G(x)}\right)^{1/2} = \frac{(d-2)^2}{4}\frac{1}{|x|^2} + \mathcal{O}\left(\frac{1}{|x|^3}\right).$$

has the claimed asymptotics as $|x| \to \infty$. □

7.2. The half line. In this subsection we show that we not only recover the classical Hardy inequality (0.1), but that we can also improve it, in the sense that we can show lower order terms. In particular, in contrast to the continuous case, the operator associated with the classical Hardy inequality (0.1) is subcritical in $\mathbb{N}$.

Theorem 7.3 ([16]). *For all finitely supported functions* $\varphi : \mathbb{N}_0 \to \mathbb{R}$ *with* $\varphi(0) = 0$ *we have*

$$\sum_{n=0}^{\infty} |\varphi(n) - \varphi(n+1)|^2 \geq \sum_{n=1}^{\infty} w(n)|\varphi(n)|^2 \tag{7.1}$$

with an optimal Hardy-weight w *given by*

$$w(n) = \sum_{k=1}^{\infty} \binom{4k}{2k} \frac{1}{(4k-1)\,2^{(4k-1)}} \frac{1}{n^{2k}} = \frac{1}{4n^2} + \frac{5}{64n^4} + \cdots,$$

for $n \geq 2$ *and* $w(1) = 2 - \sqrt{2}$. *In particular,* $w(n) > \frac{1}{4n^2}$ *for any* $n \geq 1$.

Proof. Consider the Laplacian (with standard weights) acting on $\mathbb{N} \subset \mathbb{Z}$ as

$$\Delta u(n) := 2\varphi(n) - \varphi(n+1) - \varphi(n-1), \qquad n \in \mathbb{N}.$$

Then the identity function $u(n) := n$ on $\mathbb{N}_0$ is positive and harmonic on $\mathbb{N}$. Moreover, by choosing $v(n) := 1$ on $\mathbb{N}_0$ (which is harmonic on $\mathbb{N}$), it follows that the assumptions of Theorem 0.1 are satisfied. Hence, the corresponding optimal Hardy weight w is given by

$$\begin{aligned}
w(n) &= \frac{1}{2n}\left[\left(n^{1/2} - (n+1)^{1/2}\right)^2 + \left(n^{1/2} - (n-1)^{1/2}\right)^2\right] \\
&= \frac{1}{n}\left[2n - n^{1/2}\left((n+1)^{1/2} + (n-1)^{1/2}\right)\right] \\
&= 2 - \left(\left(1+\frac{1}{n}\right)^{1/2} + \left(1-\frac{1}{n}\right)^{1/2}\right).
\end{aligned}$$

Employing the Taylor expansion of the square root at 1, i.e.,

$$\left(1 \pm \frac{1}{n}\right)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k}\left(\frac{\pm 1}{n}\right)^k = 1 \pm \frac{1}{2n} - \frac{1}{8n^2} \pm \frac{1}{16n^3} - \frac{5}{128n^4} \pm \cdots$$

yields the result. □

Acknowledgements. The authors thank W. Cygan and W. Woess for alluding to the paper [23] and R. Frank for helpful comments on the literature. Furthermore, the authors thank R. Matveev for pointing out that the properties (a) and (b) in Theorem 1.1 together exclude graphs which are not locally finite. M. K. is grateful to the Department of Mathematics at the Technion for the hospitality during his visits and acknowledges the financial support of the German Science Foundation. Y. P. and F. P. acknowledge the support of the Israel Science Foundation (grants No. 970/15) founded by the Israel Academy of Sciences and Humanities. F. P. is grateful for support through a Technion Fine Fellowship.

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Communicated by R. Seiringer