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Lax Integrability and the Peakon Problem for the Modified Camassa–Holm Equation

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Abstract: *Peakons* are special weak solutions of a class of nonlinear partial differential equations modelling non-linear phenomena such as the breakdown of regularity and the onset of shocks. We show that the natural concept of weak solutions in the case of the modified Camassa–Holm equation studied in this paper is dictated by the distributional compatibility of its Lax pair and, as a result, it differs from the one proposed and used in the literature based on the concept of weak solutions used for equations of the Burgers type. Subsequently, we give a complete construction of peakon solutions satisfying the modified Camassa–Holm equation in the sense of distributions; our approach is based on solving certain inverse boundary value problem, the solution of which hinges on a combination of classical techniques of analysis involving Stieltjes' continued fractions and multi-point Padé approximations. We propose sufficient conditions needed to ensure the global existence of peakon solutions and analyze the large time asymptotic behaviour whose special features include a formation of pairs of peakons that share asymptotic speeds, as well as *Toda-like* sorting property.

Contents

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1. Introduction

The nonlinear partial differential equation

$$m_t + \Big((u^2 - u_x^2)m\Big)_x = 0, \qquad m = u - u_{xx}, \tag{1.1}$$

is an intriguing modification of the Camassa–Holm equation (CH) [8]:

$$m_t + um_x + 2u_x m = 0, \qquad m = u - u_{xx}, \tag{1.2}$$

for the shallow water waves. Originally, Eq. (1.1) appeared in the papers of Fokas [23], Fuchssteiner [24], Olver and Rosenau [55] and was, later, rediscovered by Qiao [56,57]. It is worth mentioning at this point that the same equation was also studied, including its Lax formulation, in a somewhat unappreciated paper by Schiff [60].

We note that the derivation of this equation in [55] followed from the general method of tri-Hamiltonian duality applied to the bi-Hamiltonian representation of the modified Korteweg–de Vries equation (see also [37] for a recent generalization of this idea). Since the CH equation can be obtained from the Korteweg–de Vries equation by the same tri-Hamiltonian duality, it is therefore natural to refer to Eq. (1.1) as the modified CH equation (mCH), in full agreement with other authors [30,42], even though the name FORQ to denote (1.1) is sometimes used as well (e.g. [31,32]).

We are interested in the class of non-smooth solutions of (1.1) given by the *peakon ansatz* [8,30,58], that is, we assume

$$u = \sum_{j=1}^{n} m_j(t) e^{-|x - x_j(t)|}, \tag{1.3}$$

where all coefficients $m_j(t)$ are taken to be positive, and hence

$$m = u - u_{xx} = 2\sum_{j=1}^{n} m_j \delta_{x_j}$$

is a positive discrete measure. The relevance of this ansatz proved to be supported by the fact that these special solutions seem to capture main attributes of solutions of this class of equations: the breakdown of regularity, which can be interpreted as collisions of peakons, and the nature of long time asymptotics, which can be loosely described as peakons becoming free particles in the asymptotic region [3]. For the CH equation peakons do not exhibit any asymptotic cooperative behaviour, while for other equations, for example, for the Geng–Xue equation [27] or the Novikov equation [54], one observes pairing or even more elaborate patterns of clustering of peakons in the asymptotic region [38,39]. At the same time, for still not entirely clear reasons, the peakon dynamics has an ever growing number of connections with classical analysis. This was observed for the first time in the CH case [2] where the dynamics of peakons was shown to be related, in fact, solved, in terms of the classical theory of Stieltjes continued fractions—the connection that goes through the fundamental theory of the inhomogeneous string of M. G. Krein [18]—eventually leading to sharp estimates on the patterns of the breakdown of regularity [3,49,51].

Although it does not seem possible in a short introduction to do justice to the enormous literature on the CH equation we would like to mention a few works related to issues raised in the present paper. Thus [13] discusses the concept of weak solutions for the CH equation that set the stage for numerous studies of related equations as well as gave the first general results regarding the wave breaking for CH. In [16] the authors discuss the issue of stability of CH peakons, while the stability of multipeakons is discussed in [19,20].

One of the most interesting issues, both physically and mathematically, is the question of the continuation of solutions past the breakdown of regularity in the CH equation. This question was addressed from many different perspectives by several authors, the most comprehensive being the work of Bressan and Constantin [6,7], in which the authors attach semigroups of global solutions both in the energy conserving sector as well as for dissipative solutions. Through a judicious change of variables which resolves singularities due to wave braking one can naturally continue past the time of the breakdown with, in the dissipative scenario, the only loss of energy occurring at the time of the breakdown. Another related direction, although focused primarily on the development of suitable numerical techniques is the work of Holden and Raynaud [33,34]. This line of research is closer in spirit to the present work since it puts the multipeakon solutions front and center, indirectly building on the result of [3] that the breakdown of regularity can be in a precise way traced back to the collisions of peakons.

Meanwhile the literature on the peakon ansatz has grown considerably since its discovery in [8]. In the following years the peakon ansatz was successfully applied to another, well studied by now, equation, namely the Degasperis-Procesi equation [17]

$$m_t + um_x + 3u_x m = 0, \qquad m = u - u_{xx}, \tag{1.4}$$

which despite its superficial similarity to the CH equation (1.2) has in addition shock solutions [11,12,44], while its peakon sector leads to new questions regarding Nikishin systems [53] studied in approximation theory [4,45]. For potential applicability to water wave theory the reader is invited to consult [14,36]; for a discussion of weak solutions see [21]; [41,43] present important results regarding stability, and finally [63,64] deal with collisions of peakons and the onset of shocks in the form of shockpeakons [44].

Another feature of peakon sectors of Lax integrable peakon equations is the *omnipresence* of *total positivity* [25,40]. In its simplest form, namely speaking of matrices, a totally positive matrix is a matrix whose minors, of all sizes, are positive. This concept is then generalized to kernels of linear integral equations. Total positivity appears in all peakon problems known to us, although admittedly we cannot yet explain from first principles the underlying reasons for the presence of such a strong form of positivity; however, we remark that peakons are in a nutshell disguised *oscillatory systems* in the sense of Gantmacher and Krein [25].

What is germane to this paper is that the peakon problem at hand is coming from studying a distibutional Lax pair which forces us to view (1.1) as a distribution equation, requiring in particular that we define the product $u_x^2 m$. With this in mind we show in Appendix A that the choice consistent with Lax integrability is to take $u_x^2 m$ to mean $\langle u_x^2 \rangle m$, where $\langle f \rangle$ denotes the average function (the arithmetic average of the right hand and left hand limits). Subsequently, Eq. (1.1) reduces to the system of ODEs:

$$\dot{m}_j = 0, \qquad \dot{x}_j = u(x_j)^2 - \langle u_x^2 \rangle (x_j), \tag{1.5}$$

or, more explicitly, assuming the ordering condition $x_1 < x_2 < \cdots < x_n$,

$$\dot{m}_j = 0, \qquad \dot{x}_j = 2 \sum_{\substack{1 \le k \le n, \\ k \neq j}} m_j m_k e^{-|x_j - x_k|} + 4 \sum_{1 \le i < j < k \le n} m_i m_k e^{-|x_i - x_k|}. \tag{1.6}$$

In broad terms we can say that our general interest in (1.1) is to understand how integrability manifests itself in the non-smooth sector of solutions, in particular how it determines the properties of, initially, ill-defined operations, which acquire well-defined meaning thanks to the condition of Lax integrability.

We note that the system given by (1.6) is not the same as the one proposed in [30]; the difference being precisely in the definition of the singular product $u_x^2 m$. We clarify the details of the difference in the remark below.

Remark 1.1. In [30], Gui, Liu, Olver and Qu showed that the mCH equation admits weak n-peakon solutions with x_j, m_j satisfying

$$\dot{m}_j = 0, \qquad \dot{x}_j = \frac{2}{3} m_j^2 + 2 \sum_{\substack{1 \le k \le n, \\ k \neq j}} m_j m_k e^{-|x_j - x_k|} + 4 \sum_{1 \le i < j < k \le n} m_i m_k e^{-|x_i - x_k|}, \tag{1.7}$$

(these equations also appear as a special case in [58]); we note that these equations differ from (1.6) by the constant term $\frac{2}{3} m_j^2$. For identical m_j this term can be absorbed by redefining x_j but in general this cannot be done without violating the invariance of $|x_i - x_k|$. It is not difficult to verify that, following the definition of weak solutions adopted in [30], the singular product $u_x^2 m$ appearing in (1.1) equals to

$$\left(\frac{\langle u_x^2 \rangle + 2 \langle u_x \rangle^2}{3} \right) m,$$

which is an abbreviated way of saying that the value of the multiplier of δ_{x_j} equals $\frac{\langle u_x^2 \rangle + 2 \langle u_x \rangle^2}{3} (x_j)$. This is markedly different than what the Lax integrability implies for the multiplier, namely $\langle u_x^2 \rangle (x_j)$. Indeed, in our case, as we shall prove in Appendix A, (1.6) can be derived from the compatibility condition of a distribution Lax pair, which in turn leads to explicit solutions of these equations by the inverse spectral method, following a successful solution to the appropriate inverse problem. Finally, in view of the comments above our solution is also a solution to the special case of the peakon problem in [30] for which all the masses m_j are assumed equal.

For other work related to (1.1) done recently the reader is invited to consult [5,42,59].

Remark 1.2. Another important feature that sets apart our definition of peakons is that the Sobolev H^1 norm of u defined by (1.3) for peakons satisfying Eqs. (1.6) is preserved. In other words

$$\frac{d}{dt}||u||_{H^1} = 0.$$

Even though this point is fully explained in the followup shorter paper [9], we nevertheless compute $||u||^2_{H^1}$ in Corollary 6.11 in terms of spectral variables, obtaining a trace-like identity akin to the one known from the CH theory [15,50]. We stress that the time preservation of the H^1 norm is of considerable importance if one recalls that one of the Hamiltonians defining the theory of (1.1) is $\mathscr{H}_1 = ||u||^2_{H^1}$ (see, for example, [37]).

In the present paper, we shall formulate and apply an inverse spectral method to solve the peakon ODEs (1.6) and hence (1.1) under the following assumptions:

(1) all m_k are positive,
(2) the initial positions are assumed to be ordered as $x_1(0) < x_2(0) < \cdots < x_n(0)$.

Remark 1.3. If m_k are negative, the corresponding problem may be solved by the transformation $m_k \to -m_k$. This results in pure antipeakon solutions.

In the remainder of this introduction we outline the content of individual sections, highlighting the main results. Thus in Sect. 2 we reformulate the Lax pair in a way suitable for further analysis; in particular, for the peakon ansatz we obtain a difference equation and we solve explicitly the affiliated initial value problem. This section uses in an essential way the result from Appendix A about the admissible ways of defining the distributional Lax pair.

In Sect. 3 we give a full characterization of the spectrum of the boundary value problem from Sect. 2. We prove that the spectrum is positive and simple, and in this sense the mCH peakons confirm the "experimental" fact that all known integrable peakon equations have a substantial amount of positivity built in. The spectral data, which involves not only the eigenvalues but also some positive constants known in scattering theory as the *norming constants*, are elegantly encoded in the Weyl function $W(z)$ of the boundary value problem and the main theorem, namely Theorem 3.1, which states that $W(z)$ is a shifted Stieltjes transform is proven in its entirety therein.

In Sect. 4 we solve the inverse boundary value problem which in a nutshell amounts to reconstructing the measures g, h appearing in the original formulation of the boundary value problem (2.3) from the spectral data encoded in the Weyl function $W(z)$. We subsequently give two constructions of the inverse map: one is based on recurrence relations and Stieltjes' method of continued fractions, the other method is explicit and it involves certain Cauchy-Jacobi interpolation problem which is shown in Theorem 4.20 to admit an explicit solution in terms of Cauchy-Stieltjes-Vandermonde matrices introduced in Definition 4.16.

In Sects. 5 and 6 we analyze the actual peakon solutions u constructed out of the peakon ansatz (1.3) and the determintal solution of the inverse problem studied in Sect. 4. This material is covered in two sections because there are some subtle differences in the character of solutions depending on whether the total number of peakons, n, is even or odd. In either case we present and prove sufficient conditions for the global existence of peakon solutions. This is done in Theorems 5.6 and 6.7. We also give large time asymptotic formulas for peakons, showing that in both cases the peakons form asymptotic pairs.

2. The Lax Formalism: The Boundary Value Problem

The Lax pair for (1.1) reads [56,60]:

$$\Psi_x = \frac{1}{2}U\Psi, \quad \Psi_t = \frac{1}{2}V\Psi, \quad \Psi = \begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix} \tag{2.1}$$

with

$$U = \begin{bmatrix}-1 & \lambda m\\ -\lambda m & 1\end{bmatrix}, \quad V = \begin{bmatrix}4\lambda^{-2}+Q & -2\lambda^{-1}(u-u_x)-\lambda m Q\\ 2\lambda^{-1}(u+u_x)+\lambda m Q & -Q\end{bmatrix}, \quad Q = u^2 - u_x^2, \quad \lambda \in \mathbf{C}.$$

Performing the gauge transformation $\Phi = \operatorname{diag}(\frac{e^{\frac{x}{2}}}{\lambda}, e^{-\frac{x}{2}})\Psi$ results in a simpler x-equation

$$\Phi_x = \begin{bmatrix}0 & h\\ -zg & 0\end{bmatrix}\Phi, \qquad g = \sum_{j=1}^{n} g_j\delta_{x_j}, \qquad h = \sum_{j=1}^{n} h_j\delta_{x_j}, \tag{2.2}$$

where $g_j = m_j e^{-x_j}$, $h_j = m_j e^{x_j}$, $z = \lambda^2$. For future use note that $g_j h_j = m_j^2$.

We will be interested in solving (2.2) subject to boundary conditions $\Phi_1(-\infty) = 0$, $\Phi_2(+\infty) = 0$. To make the boundary value problem

$$\Phi_x = \begin{bmatrix}0 & h\\ -zg & 0\end{bmatrix}\Phi, \qquad \Phi_1(-\infty) = \Phi_2(+\infty) = 0, \tag{2.3}$$

well posed we need to define the multiplication of the measures h and g by Φ. Guided by the results of Appendix A we require that Φ be left continuous and we define the terms $\Phi_a\delta_{x_j} = \Phi_a(x_j)\delta_{x_j}$, $a = 1, 2$. This choice makes the Lax pair well defined as a distributional Lax pair and, as it is shown in the Appendix A, the compatibility condition of the x and t components of the Lax pair indeed implies (1.5).

The solution Φ is a piecewise constant function which, for convenience, we can normalize by setting $\Phi_2(-\infty) = 1$. The distributional boundary value problem (2.3) is in our special case of the discrete measure m equivalent to a finite difference equation.

Lemma 2.1. *Let $q_k = \Phi_1(x_k+)$, $p_k = \Phi_2(x_k+)$, then the difference form of the boundary value problem reads:*

$$\begin{aligned} q_k - q_{k-1} &= h_k p_{k-1}, \qquad 1 \le k \le n,\\ p_k - p_{k-1} &= -zg_k q_{k-1}, \quad 1 \le k \le n,\\ q_0 = 0, \quad p_0 &= 1, \quad p_n = 0. \end{aligned} \tag{2.4}$$

An easy proof by induction leads to the following corollary.

Corollary 2.2. *$q_k(z)$ is a polynomial of degree $\lfloor\frac{k-1}{2}\rfloor$ in z, and $p_k(z)$ is a polynomial of degree $\lfloor\frac{k}{2}\rfloor$, respectively.*

Remark 2.3. Note that the difference form of the boundary value problem admits a simple matrix presentation, namely a 2×2 matrix encoding of (2.4)

$$\begin{bmatrix}q_k\\ p_k\end{bmatrix} = T_k\begin{bmatrix}q_{k-1}\\ p_{k-1}\end{bmatrix}, \qquad T_k = \begin{bmatrix}1 & h_k\\ -zg_k & 1\end{bmatrix}. \tag{2.5}$$

We point out that the transition matrix T_k is different from the difference equation for the inhomogeneous string boundary value problem $D^2 v = -zgv$, $v(0) = v(1) = 0$ discussed in [45] (Appendix A) for which $T_k = \begin{bmatrix} 1 & l_{k-1} \\ -zg_k & 1 - zg_k l_{k-1} \end{bmatrix}$, thus an element of the group $SL_2(\mathbf{C})$.

We can obtain more precise information about polynomials p_k, q_k by studying directly the solutions to the initial value problem

$$\Phi_x = \begin{bmatrix} 0 & h \\ -zg & 0 \end{bmatrix} \Phi, \qquad \Phi_1(-\infty) = 0, \quad \Phi_2(-\infty) = 1, \tag{2.6}$$

with the same rule regarding the multiplication of discrete measures g, h by piecewise smooth, left-continuous, functions f as specified above. With this proviso expressions like

$$\int_{-\infty}^{x} f(\xi)\, g(\xi) d\xi \overset{def}{=} \int_{\xi < x} f(\xi)\, g(\xi) d\xi$$

uniquely define piecewise constant functions which we choose to be left continuous. The same applies to iterated integrals over the regions $\{\xi_1 < \xi_2 < \cdots \xi_k < x\}$. For example

$$\int_{\xi_1 < \xi_2 < x} f(\xi_1) h(\xi_1) d\xi_1\, g(\xi_2) d\xi_2$$

is well defined. With this notation in place we obtain the following characterization of $\Phi_1(x)$ and $\Phi_2(x)$.

Lemma 2.4. *Let us set*

$$\Phi_1(x) = \sum_{0 \le k} \Phi_1^{(k)}(x) z^k, \qquad \Phi_2(x) = \sum_{0 \le k} \Phi_2^{(k)}(x) z^k.$$

Then

$$\Phi_1^{(0)}(x) = \int_{\eta_0 < x} h(\eta_0) d\eta_0, \qquad \Phi_2^{(0)}(x) = 1$$

for $k = 0$, otherwise

$$\Phi_1^{(k)}(x) = (-1)^k \int_{\eta_0 < \xi_1 < \eta_1 < \cdots < \xi_k < \eta_k < x} \Big[\prod_{p=1}^{k} h(\eta_p) g(\xi_p)\Big] h(\eta_0)\; d\eta_0 d\xi_1 \dots d\eta_k, \tag{2.7a}$$

$$\Phi_2^{(k)}(x) = (-1)^k \int_{\xi_1 < \eta_1 < \cdots < \xi_k < \eta_k < x} \Big[\prod_{p=1}^{k} g(\eta_p) h(\xi_p)\Big]\; d\xi_1 \dots d\eta_k. \tag{2.7b}$$

If the points of the support of the discrete measure g (and h) are ordered $x_1 < x_2 < \cdots < x_n$ then

$$\Phi_1^{(k)}(x) = (-1)^k \sum_{\substack{j_0<i_1<j_1<\cdots<i_k<j_k\\ x_{j_k}<x}} \Big[\prod_{p=1}^{k} h_{j_p} g_{i_p}\Big]\, h_{j_0}, \tag{2.8a}$$

$$\Phi_2^{(k)}(x) = (-1)^k \sum_{\substack{i_1<j_1<\cdots<i_k<j_k\\ x_{j_k}<x}} \Big[\prod_{p=1}^{k} g_{j_p} h_{i_p}\Big]. \tag{2.8b}$$

Proof. First we observe that solving Eq. (2.6) is equivalent to solving the system of integral equations:

$$\Phi_1(x) = \int_{\xi<x} \Phi_2(\xi)h(\xi)d\xi, \qquad \Phi_2(x) = 1 - z\int_{\xi<x} \Phi_1(\xi)g(\xi)d\xi,$$

with in turn implies

$$\Phi_1(x) = \int_{\eta_0<x} h(\eta_0)d\eta_0 - z\int_{\xi_1<\eta_1<x} h(\eta_1)g(\xi_1)\Phi_1(\xi_1)d\xi_1 d\eta_1,$$

$$\Phi_2(x) = 1 - z\int_{\xi_1<\eta_1<x} g(\eta_1)h(\xi_1)\Phi_2(\xi_1)d\xi_1 d\eta_1.$$

Elementary iterations yield the final result in the integral form. Finally, once the ordering conditions is in place, the evaluation of integrals as sums follows. □

We will now introduce a multi-index notation, later used to facilitate writing explicit formulas for peakon solutions, but also helpful in capturing the properties of solutions to (2.4). A similar notation turned out to be very helpful in stating and proving the Canada Day Theorem in [28] (see also [35]).

The formulas in Lemma 2.4 involve a choice of j-element index sets I and J from the set $[k] = \{1, 2, \dots, k\}$. We will use the notation $\binom{[k]}{j}$ for the set of all j-element subsets of $[k]$, listed in increasing order; for example $I \in \binom{[k]}{j}$ means that $I = \{i_1, i_2, \dots, i_j\}$ for some increasing sequence $i_1 < i_2 < \cdots < i_j \le k$. Furthermore, given the multi-index I we will abbreviate $g_I = g_{i_1} g_{i_2} \dots g_{i_j}$ etc.

Definition 2.5. Let $I, J \in \binom{[k]}{j}$, or $I \in \binom{[k]}{j+1}$, $J \in \binom{[k]}{j}$.

Then I, J are said to be *interlacing* if

$$i_1 < j_1 < i_2 < j_2 < \cdots < i_j < j_j$$

or,

$$i_1 < j_1 < i_2 < j_2 < \cdots < i_j < j_j < i_{j+1},$$

in the latter case. We abbreviate this condition as $I < J$ in either case, and, furthermore, use the same notation, that is $I < J$, for $I \in \binom{[k]}{1}$, $J \in \binom{[k]}{0}$.

Remark 2.6. We point out that the multi-indices I, J satisfying $I < J$ are called in [28] *strictly interlacing*.

Corollary 2.7. *Let* $q_k = \Phi_1(x_k+)$, $p_k = \Phi_2(x_k+)$, *then the difference form of the initial value problem* (2.6) *reads:*

$$\begin{aligned} q_k - q_{k-1} &= h_k p_{k-1}, \qquad 1 \le k \le n,\\ p_k - p_{k-1} &= -z g_k q_{k-1}, \quad 1 \le k \le n,\\ q_0 = 0, &\quad p_0 = 1 \end{aligned} \tag{2.9}$$

whose unique solution (restoring the dependence on z *in* q_k *and* p_k*) is given by:*

$$q_k(z) = \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \Big(\sum_{\substack{I \in \binom{[k]}{j+1}, J \in \binom{[k]}{j} \\ I < J}} h_I g_J \Big) (-z)^j, \tag{2.10a}$$

$$p_k(z) = 1 + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \Big(\sum_{\substack{I, J \in \binom{[k]}{j} \\ I < J}} h_I g_J \Big) (-z)^j. \tag{2.10b}$$

Our next goal is to study the spectrum of the boundary value problem (2.3).

Definition 2.8. A complex number z is an *eigenvalue* of the boundary value problem (2.3) if there exists a solution $\{q_k, p_k\}$ to (2.9) for which $p_n(z) = 0$. The set of all eigenvalues is the *spectrum* of the boundary value problem (2.3).

The relevance of the spectrum of (2.3) is captured in the following lemma which follows from examining the t part of the Lax pair (2.1) in the region $x > x_n$.

Lemma 2.9. *Let* $\{q_k, p_k\}$ *satisfy the system of difference equations* (2.9). *Then*

$$\dot{q}_n = \frac{2}{z} q_n - \frac{2L}{z} p_n, \qquad \dot{p}_n = 0, \tag{2.11}$$

where $L = \sum_{j=1}^n h_j$. *Thus* $p_n(z)$ *is independent of time and, in particular, its zeros, i.e. the spectrum, are time invariant.*

Since Corollary 2.7 gives an explicit form of $p_n(z)$ we can easily identify the constants of motion implied by isospectrality of the boundary value problem (2.3).

Lemma 2.10. *The quantities*

$$M_j = \sum_{\substack{I, J \in \binom{[n]}{j} \\ I < J}} h_I g_J, \qquad 1 \le j \le \lfloor \frac{n}{2} \rfloor$$

form a set of $\lfloor \frac{n}{2} \rfloor$ *constants of motion for the system* (1.6).

Example 2.11. Let us consider the case $n = 4$. Then the constants of motion, written in the original variables (m_j, x_j), with positions x_j satisfying $x_1 < x_2 < x_3 < x_4$, are

$$\begin{aligned} M_1 &= m_1 m_2 e^{x_1 - x_2} + m_1 m_3 e^{x_1 - x_3} + m_1 m_4 e^{x_1 - x_4} + m_2 m_3 e^{x_2 - x_3}\\ &\quad + m_2 m_4 e^{x_2 - x_4} + m_3 m_4 e^{x_3 - x_4},\\ M_2 &= m_1 m_2 m_3 m_4 e^{x_1 - x_2 + x_3 - x_4}. \end{aligned}$$

3. Forward Map: Spectrum and Spectral Data

We will characterize the spectrum of the boundary value problem (2.3), or equivalently, (2.4) by associating it with the *Weyl function*

$$W(z) = \frac{q_n(z)}{p_n(z)}. \tag{3.1}$$

The remainder of this section is devoted to the proof of the following theorem characterizing our boundary value problem in terms of $W(z)$.

Theorem 3.1. *$W(z)$ is a (shifted) Stieltjes transform of a positive, discrete measure $d\mu$ with support inside $\mathbf{R}_+$. More precisely:*

$$W(z) = c + \int \frac{d\mu(x)}{x-z}, \qquad d\mu = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} b_j \delta_{\zeta_j}, \qquad 0 < \zeta_1 < \cdots < \zeta_{\lfloor \frac{n}{2} \rfloor},$$
$$0 < b_j, \quad 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor, \tag{3.2}$$

where $c > 0$ when n is odd and $c = 0$ when n even.

The next corollary describes the properties of the spectrum.

Corollary 3.2.

(1) The spectrum of the boundary value problem (2.2) is positive and simple.
(2) $W(z) = c + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{b_j}{\zeta_j - z}$, where all residues satisfy $b_j > 0$ and $c \ge 0$.

The strategy of the proof of Theorem 3.1 is to show that W has a continued fraction expansion of Stieltjes's type, the term explained below. We start by reformulating the recurrence relation (2.9).

Lemma 3.3. *Let $\{q_k, p_k\}$ be the solution to (2.9) and let $w_{2k} = \frac{q_k}{p_k}$, $w_{2k-1} = \frac{q_{k-1}}{p_k}$. Then*

$$w_1 = 0, \qquad w_{2k} = (1 + z m_k^2) w_{2k-1} + h_k, \qquad 1 \le k \le n \tag{3.3a}$$
$$\frac{1}{w_{2k}} = \frac{1}{w_{2k+1}} + z g_{k+1}, \qquad 1 \le k \le n-1 \tag{3.3b}$$

Proof. The first line follows readily by rewriting the first line of (2.9) as

$$\frac{q_k}{p_k} - \frac{q_{k-1}}{p_k} = h_k \frac{(p_{k-1} - p_k)}{p_k} + h_k,$$

then using the second equation of (2.9) to eliminate $p_{k-1} - p_k$, on the way employing the relation $m_k^2 = g_k h_k$, and finally rewriting the result using the definition of w_{2k} and w_{2k-1}. The condition $w_1 = 0$ corresponds to the boundary condition at index $k = 1$, recalling that $w_1 = \frac{q_0}{p_1} = 0$ because $q_0 = 0$, $p_1 = 1$. The second line follows from the second formula in (2.9). □

Remark 3.4. The recurrence in Lemma 3.3 can be viewed as the recurrence on the Weyl functions corresponding to shorter strings obtained by truncating at the index k. Then W_{2k} is precisely the Weyl function corresponding to the measures $\sum_{j=1}^{k} h_j \delta_{x_j}$ and $\sum_{j=1}^{k} g_j \delta_{x_j}$, while W_{2k-1} corresponds to the measures $\sum_{j=1}^{k-1} h_j \delta_{x_j}$ and $\sum_{j=1}^{k} g_j \delta_{x_j}$ respectively.

Before we state the second lemma we briefly review some old results of T. Stieltjes, appropriately adapted to our setup. More specifically, the following description of rational functions follows from general results proved by T. Stieltjes in his famous memoir [62].

Theorem 3.5 (T. Stieltjes). *Any rational function $F(z)$ admitting the integral representation*

$$F(z) = c + \int \frac{d\nu(x)}{x-z}, \tag{3.4}$$

where $d\nu(x)$ is the (Stieltjes) measure corresponding to the piecewise constant non-decreasing function $\nu(x)$ with finitely many jumps in $\mathbf{R}_+$ has a finite (terminating) continued fraction expansion

$$F(z) = c + \cfrac{1}{a_1(-z) + \cfrac{1}{a_2 + \cfrac{1}{a_3(-z) + \cfrac{1}{\ddots}}}}, \tag{3.5}$$

where all $a_j > 0$ and, conversely, any rational function with this type of continued fraction expansion has the integral representation (3.4).

We will refer to the integral representation (3.4) as the *shifted Stieltjes transform* of a measure $d\nu(x)$. Now we are ready to state the second lemma.

Lemma 3.6. *Given $h_j > 0$, $h_j g_j = m_j^2 > 0$, $1 \le j \le n$, let w_js satisfy the recurrence relations of Lemma 3.3. Then w_js are shifted Stieltjes transforms of finite, discrete Stieltjes measures supported on $\mathbf{R}_+$, with nonnegative shifts. More precisely:*

$$w_{2k-1} = \int \frac{d\mu^{(2k-1)}(x)}{x-z},$$
$$w_{2k} = c_{2k} + \int \frac{d\mu^{(2k)}(x)}{x-z},$$

where $c_{2k} > 0$ when k is odd, otherwise, $c_{2k} = 0$. Furthermore, the number of points in the support $d\mu^{(2k)}(x)$ and $d\mu^{(2k-1)}$ is $\lfloor \frac{k}{2} \rfloor$.

Proof. The proof proceeds by induction on k. The base case $k = 1$ is trivial since $w_1 = 0$ while $w_2 = h_1$ by the first equation in Lemma 3.3 confirming that $c_2 > 0$. Suppose now the claim is valid for the index k. Thus w_{2k-1} and w_{2k} are shifted Stieltjes transforms of some measures $d\mu^{(2k-1)}$ and $d\mu^{(2k)}$, both being finite, discrete and supported on $\mathbf{R}_+$. We now solve (3.3b) for w_{2k+1} obtaining:

$$w_{2k+1} = \frac{1}{-zg_{k+1} + \frac{1}{w_{2k}}},$$

then use the induction hypothesis, which implies that w_{2k} has the form (3.5), resulting in the continued fraction expansion:

$$w_{2k+1} = \cfrac{1}{-zg_{k+1} + \cfrac{1}{c + \cfrac{1}{a_1(-z) + \cfrac{1}{a_2 + \cfrac{1}{a_3(-z) + \cfrac{1}{\ddots}}}}}}.$$

If $c > 0$ then w_{2k+1} has already the required form. If, on the other hand, $c = 0$ the first term in the continued fraction expansion is $-(g_{k+1}+a_1)z$ and since $a_1 > 0$ the coefficient is positive. Thus w_{2k+1} satisfies the conditions of Stieltjes's Theorem 3.5 and, as a result, w_{2k+1} is the Stieltjes transform with zero shift of a finite, discrete measure, say $d\mu^{2k+1}(x)$ supported on $\mathbf{R}_+$. In either case

$$w_{2k+1} = \int d\mu^{2k+1}(x)(-\frac{1}{z}) + \mathcal{O}(\frac{1}{z^2}), \qquad z \to \infty,$$

where $\int d\mu^{2k+1}(x) = \frac{1}{g_{k+1}}$ if $c > 0$ and $\int d\mu^{2k+1}(x) = \frac{1}{g_{k+1}+a_1}$ if $c = 0$. Let us now examine Eq. (3.3a), shifting $k \to k+1$. First, we have

$$w_{2k+2} = (1 + zm_{k+1}^2)w_{2k+1} + h_{k+1},$$

and, upon using the integral representation for w_{2k+1} we obtain:

$$\begin{aligned} w_{2k+2} &= \int \frac{(1 + xm_{k+1}^2)d\mu^{(2k+1)}(x)}{x - z} + h_{k+1} - m_{k+1}^2 \int d\mu^{(2k+1)}(x) \\ &= \int \frac{(1 + xm_{k+1}^2)d\mu^{(2k+1)}(x)}{x - z} + h_{k+1}\left(1 - g_{k+1}\int d\mu^{(2k+1)}(x)\right). \end{aligned}$$

If the shift c in the formula for w_{2k} is positive then $\int d\mu^{(2k+1)}(x) = \frac{1}{g_{k+1}}$, as remarked earlier, and the shift in the formula for w_{2k+2} is 0. When $c = 0$, $\int d\mu^{(2k+1)}(x) = \frac{1}{g_{k+1}+a_1} < \frac{1}{g_{k+1}}$ and then the shift is positive since $\left(1 - g_{k+1}\int d\mu^{(2k+1)}(x)\right) > 0$. This proves the integral representation for w_{2k+2} and shows that the shift alternates between 0 and positive numbers, depending on whether k is even or odd as claimed since $c_2 > 0$. Finally, the number of the points in the support of $d\mu^{(2k)}(x)$ and $d\mu^{(2k-1)}(x)$ follows from Corollary 2.2. □

Now, with all the preparation, the proof of Theorem 3.1 follows readily from Lemma 3.6 by observing that

$$W(z) = c_{2n} + \int \frac{d\mu^{(2n)}(x)}{x - z}, \qquad d\mu^{(2n)} = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} b_j^{(2n)}\delta_{\zeta_j}. \tag{3.6}$$

This concludes the spectral characterization of the boundary value problem (2.3), or equivalently (2.4).

4. Inverse Problem

4.1. A solution by recursion. The inverse problem associated with the boundary value problem (2.3) can be stated: given positive constants m_j, $1 \le j \le n$, and a rational function $W(z)$ with integral representation (3.6), we seek to invert the map $S : \{x_1, x_2, \dots x_n\} \longrightarrow W$.

To solve the inverse problem we proceed in two stages: first we reconstruct the positive coefficients g_j, h_j such that $g_j h_j = m_j^2$ then we use the relation $\frac{h_j}{g_j} = e^{2x_j}$ to determine x_j. In this section we concentrate on the first stage.

The reconstruction of h_j, g_j amounts to solving recurrence relations (3.3a) and (3.3b) following the steps below:

(1) starting with $w_{2n} = W(z)$ define $h_n = w_{2n}(-\frac{1}{m_n^2})$, $g_n = \frac{m_n^2}{h_n}$ and solve

$$w_{2n} = (1 + zm_n^2) w_{2n-1} + h_n, \qquad \frac{1}{w_{2n-2}} = \frac{1}{w_{2n-1}} + zg_n,$$

for w_{2n-1} and w_{2n-2};

(2) restart the procedure from w_{2n-2} shifting $n \to n - 1$.

We remark that the procedure encodes solving (3.3a), (3.3b) backwards. However, for the procedure to make sense, w_{2n-2} needs to be of the form (3.6). Let us therefore turn to analyzing w_{2n-2}. First, from the recurrence relation we easily get

$$h_n = c_{2n} + \int \frac{d\mu^{(2n)}(x)}{x + \frac{1}{m_n^2}} = c_{2n} + m_n^2 \int d\mu^{(2n-1)}(x),$$

where $d\mu^{(2n-1)}(x) = \frac{d\mu^{(2n)}(x)}{1+m_n^2 x}$, while solving for w_{2n-1} yields

$$w_{2n-1}(z) = \int \frac{d\mu^{(2n-1)}(x)}{x - z}.$$

Thus by Stieltjes's theorem 3.5

$$w_{2n-1}(z) = \cfrac{1}{a_1(-z) + \cfrac{1}{a_2 + \cfrac{1}{a_3(-z) + \cfrac{1}{\ddots}}}}$$

for some $a_j > 0$. Next, we write

$$w_{2n-2} = \cfrac{1}{zg_n + \cfrac{1}{w_{2n-1}}} = \cfrac{1}{(g_n - a_1)z + \cfrac{1}{a_2 + \cfrac{1}{a_3(-z) + \cfrac{1}{\ddots}}}}$$

and observe that for w_{2n-2} to have the spectral representation (3.6) $g_n - a_1$ must be negative or 0. However,

$$\frac{1}{a_1} - \frac{1}{g_n} = \int d\mu^{(2n-1)}(x) - \frac{h_n}{m_n^2} = -\frac{c_{2n}}{m_n^2},$$

hence $g_n - a_1 \leq 0$, which proves the existence of the spectral representation (3.6) for w_{2n-2} for some measure $d\mu_{2n-2}$ supported on a finite number of points in $\mathbf{R}_+$.

Similar to the content of Lemma 3.6 we have the following dichotomy: if $c_{2n} = 0$, which by the same Lemma happens if n is even, the support of w_{2n-2} has one less point in the spectrum of the corresponding measure compensated by the appearance of non-zero c_{2n-2}. If, on the other hand, n is odd, in which case $c_{2n} > 0$, then $c_{2n-2} = 0$ and the number of points in the support of $d\mu^{(2n-2)}$ does not differ from that of $d\mu^{(2n)}$. In either case, by iterating, one reaches w_2 which is a positive constant equal by definition to h_1 and the iteration stops. We conclude the discussion of the solution to the inverse problem of recovering $\{g_j, h_j\}$ by recursion with the following theorem.

Theorem 4.1. *The inverse spectral problem is uniquely solvable for any positive masses m_j and the inverse map is continuous both with respect to the masses m_j as well as the spectral data $\{\zeta_1 < \zeta_2 < \cdots < \zeta_{\lfloor \frac{n}{2} \rfloor}; b_1, b_2, \ldots, b_{\lfloor \frac{n}{2} \rfloor}; c\}$.*

Proof. The uniqueness follows by construction of the inverse map. As discussed earlier there are no obstructions to invertibility present at each stage of the recursion and the updated spectral data is obtained by evaluation and algebraic inversions of continuous functions (Weyl functions) at points $(-\frac{1}{m_j^2})$ where those Weyl functions are strictly positive. □

4.2. A solution by interpolation; basic ideas. The iteration proposed above requires $2n - 2$ steps to reach w_2, each step leading to a new input rational function w_j. The formulas for h_j get increasingly more complicated and a natural question presents itself: can one compute h_j using directly the spectral data c_{2n} and $d\mu^{(2n)}$? The answer is affirmative and this section outlines the main steps of the construction leaving the detailed formulas for the following sections in which we present a complete solution to the peakon problem (1.6).

First we give a brief summary of main ideas behind the solution by interpolation. Let us rewrite (2.5) in terms of the Weyl function $W = w_{2n}$ as

$$\begin{bmatrix} W(z) \\ 1 \end{bmatrix} = T_n(z)T_{n-1}(z)\ldots T_{n-k+1}(z) \begin{bmatrix} \frac{q_{n-k}(z)}{p_n(z)} \\ \frac{p_{n-k}(z)}{p_n(z)} \end{bmatrix}. \tag{4.1}$$

Clearly, the transpose of the matrix of cofactors of each $T_j(z)$ is $\begin{bmatrix} 1 & -h_j \\ zg_j & 1 \end{bmatrix} \stackrel{\text{def}}{=} C_{n-j+1}(z)$, which allows one to express Eq. (4.1) as

$$C_k(z)\ldots C_1(z) \begin{bmatrix} W(z) \\ 1 \end{bmatrix} = \det(T_n(z))\det(T_{n-1}(z))\ldots\det(T_{n-k+1}(z)) \begin{bmatrix} \frac{q_{n-k}(z)}{p_n(z)} \\ \frac{p_{n-j}(z)}{p_n(z)} \end{bmatrix},$$

which, recalling that $\det T_j(z) = 1 + zm_j^2$ and that the roots of $p_n(z)$ are all positive, implies

$$\left(C_k(z)\ldots C_1(z)\begin{bmatrix}W(z)\\1\end{bmatrix}\right)\Big|_{z=-\frac{1}{m_{n-i+1}^2}} = 0, \qquad \text{for any } 1 \le i \le k. \tag{4.2}$$

Theorem 4.2. *Let the matrix of products of Cs in Eq.* (4.2) *be denoted by* $\begin{bmatrix}a_k(z) & b_k(z)\\ c_k(z) & d_k(z)\end{bmatrix} \overset{def}{=} \hat{S}_k(z)$. *Then the polynomials* $a_k(z), b_k(z), c_k(z), d_k(z)$ *solve the following interpolation problem:*

$$a_k(-\frac{1}{m_{n-i+1}^2})W(-\frac{1}{m_{n-i+1}^2}) + b_k(-\frac{1}{m_{n-i+1}^2}) = 0, \qquad 1 \le i \le k, \tag{4.3a}$$

$$\deg a_k = \lfloor\frac{k}{2}\rfloor, \qquad \deg b_k = \lfloor\frac{k-1}{2}\rfloor, \qquad a_k(0) = 1, \tag{4.3b}$$

$$c_k(-\frac{1}{m_{n-i+1}^2})W(-\frac{1}{m_{n-i+1}^2}) + d_k(-\frac{1}{m_{n-i+1}^2}) = 0, \qquad 1 \le i \le k, \tag{4.3c}$$

$$\deg c_k = \lfloor\frac{k+1}{2}\rfloor, \qquad \deg d_k = \lfloor\frac{k}{2}\rfloor, \qquad c_k(0) = 0, \quad d_k(0) = 1. \tag{4.3d}$$

Proof. The approximation statements follow directly from (4.2), while the degrees follow, by induction, from the definition of C_j and the formula for $\hat{S}_k$. □

Remark 4.3. The interpolation (4.3) is an example of a *Cauchy-Jacobi interpolation problem* [29,52,61], studied as part of a general multi-point Padé approximation theory [1].

Before we solve the interpolation problem it is helpful to understand how information about the measures g and h is encoded in the coefficients $a_j(z), b_j(z), c_j(z), d_j(z)$. To this end we define another initial value problem, following the general philosophy of scattering theory, this time specifying initial conditions at $x = +\infty$.

$$\widehat{\Phi}_x = \begin{bmatrix}0 & h\\ -zg & 0\end{bmatrix}\widehat{\Phi}, \qquad \widehat{\Phi}_1(+\infty) = 1, \quad \widehat{\Phi}_2(+\infty) = 0, \tag{4.4}$$

and seeking, in contrast to (2.6), the right-continuous solutions, interpreting the products $\widehat{\Phi}_a\delta_{x_j}$ as $\widehat{\Phi}_a\delta_{x_j} = \widehat{\Phi}_a(x_j)\delta_{x_j}$, $a = 1, 2$. Subsequently we define the (right) boundary value problem:

$$\widehat{\Phi}_x = \begin{bmatrix}0 & h\\ -zg & 0\end{bmatrix}\widehat{\Phi}, \qquad \widehat{\Phi}_1(-\infty) = 0, \quad \widehat{\Phi}_2(+\infty) = 0, \tag{4.5}$$

seeking right continuous solutions.

Remark 4.4. We refer to (4.5) as the (right) boundary value problem, even though it is formally the same boundary value problem as (2.3) but we stress that the rules of defining the singular operation of multiplication of a measure by piecewise-smooth functions has changed, hence, we don't know *a priori* if the boundary value problems are indeed the same. We will establish below that they are.

Lemma 4.5. *Let $\hat{q}_j = \widehat{\Phi}_1(x_{j'}-)$, $\hat{p}_j = \widehat{\Phi}_2(x_{j'}-)$, where $j' = n+1-j$. Then the difference form of the (right) boundary value problem* (4.5) *reads:*

$$\begin{aligned} \hat{q}_j - \hat{q}_{j-1} &= -h_{j'}\hat{p}_{j-1}, \qquad 1 \le j \le n, \\ \hat{p}_j - \hat{p}_{j-1} &= zg_{j'}\hat{q}_{j-1}, \qquad 1 \le j \le n, \\ \hat{p}_0 = 0, &\quad \hat{q}_n = 0. \end{aligned} \tag{4.6}$$

The accompanying initial value problem is chosen for the remainder of the discussion to have initial conditions $\hat{q}_0 = 1$, $\hat{p}_0 = 0$. Furthermore, the notation $j' = n+1-j$ (reflection of the interval $[1,n]$, or counting from the right end n) is in force from this point onward.

Lemma 4.6. *The difference form of the (right) boundary value problem* (4.6) *can be written in matrix form*

$$\begin{bmatrix} \hat{q}_j \\ \hat{p}_j \end{bmatrix} = \widehat{T}_j \begin{bmatrix} \hat{q}_{j-1} \\ \hat{p}_{j-1} \end{bmatrix}, \qquad \widehat{T}_j = \begin{bmatrix} 1 & -h_{j'} \\ zg_{j'} & 1 \end{bmatrix}, \tag{4.7}$$

and $C_{j'}(z)$, the transpose of the cofactor matrix of $T_j(z)$ appearing in (4.2)*, satisfies*

$$C_j(z) = \hat{T}_j(z), \qquad 1 \le j \le n,$$

and its product $\hat{S}_k(z)$, also defined in Theorem 4.2, is the transition matrix for the right boundary value problem, namely,

$$\hat{S}_k(z) = \hat{T}_k(z) \cdots \hat{T}_1(z).$$

Lemma 4.7. *Consider the initial value problem given by Eq.* (4.4) *and let us set*

$$\widehat{\Phi}_1(x) = \sum_{0 \le k} \widehat{\Phi}_1^{(k)}(x) z^k, \qquad \widehat{\Phi}_2(x) = \sum_{0 \le k} \widehat{\Phi}_2^{(k)}(x) z^k.$$

Then

$$\widehat{\Phi}_1^{(0)}(x) = 1, \qquad \widehat{\Phi}_2^{(0)}(x) = 0, \tag{4.8a}$$

$$\widehat{\Phi}_1^{(k)}(x) = (-1)^k \int\limits_{x<\xi_1<\eta_1<\cdots<\xi_k<\eta_k} \Big[\prod_{j=1}^{k} h(\xi_j)g(\eta_j)\Big]\, d\xi_1 \ldots d\eta_k, \qquad 1 \le k, \tag{4.8b}$$

$$\widehat{\Phi}_2^{(k)}(x) = (-1)^{k-1} \int\limits_{x<\eta_0<\xi_1<\eta_1<\cdots<\xi_{k-1}<\eta_{k-1}} g(\eta_0)\Big[\prod_{j=1}^{k-1} h(\xi_j)g(\eta_j)\Big]\, d\eta_0 d\xi_1 \ldots d\eta_{k-1}, \qquad 1 \le k, \tag{4.8c}$$

where, for $k = 1$, $\prod_{j=1}^{0}$ is defined to be 1 and the integration is carried out with respect to η_0 only.

Furthermore, if the points of the support of g (and h) are ordered $x_1 < x_2 < \cdots < x_n$ then

$$\widehat{\Phi}_1^{(k)}(x) = (-1)^k \sum_{\substack{i_1<j_1<\cdots<i_k<j_k\\ x<x_{i_1}}} \Big[\prod_{l=1}^{k} h_{i_l} g_{j_l}\Big], \tag{4.9a}$$

$$\widehat{\Phi}_2^{(k)}(x) = (-1)^{k-1} \sum_{\substack{j_0<i_1<j_1<\cdots<i_k<j_k\\ x<x_{j_0}}} g_{j_0}\Big[\prod_{l=1}^{k-1} g_{i_l} h_{j_l}\Big]. \tag{4.9b}$$

Clearly, by setting $\hat{q}_k = \widehat{\Phi}_1(x_{k'}-)$, $\hat{p}_k = \widehat{\Phi}_2(x_{k'}-)$, with the help of (4.9), we obtain the solution to difference equations (4.6) to initial conditions $\hat{q}_0 = 1$, $\hat{p}_0 = 0$. The procedure can be repeated for the case of initial conditions $\hat{Q}_0 = 0$, $\hat{P}_0 = 1$, yielding a complementary solution to (4.6). We will skip the intermediate steps since they are very similar to the computations leading up to Lemma 4.7. To state the final result we remark that the map $i \to i' = n+1-i$ is a bijection between $[1,k]$ and $[n+1-k,n]$. This map can be lifted to multi-indices $I \in \binom{[1,k]}{j}$ introduced earlier, in particular given $I = i_1 < i_2 < \cdots < i_j \in \binom{[1,k]}{j}$ let us denote by I' its image $\{i'_1 > i'_2 > \cdots > i'_j\} \in \binom{[n-k+1,n]}{j}$.

Theorem 4.8. *Consider the right boundary value problem* (4.6) *with its transition matrix*

$$\hat{S}_k = \hat{T}_k \cdots \hat{T}_1.$$

Then

$$\hat{S}_k = \begin{bmatrix} \hat{q}_k & \hat{Q}_k \\ \hat{p}_k & \hat{P}_k \end{bmatrix}, \quad 1 \le k \le n,$$

where

$$\hat{q}_k(z) = 1 + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \Big(\sum_{\substack{I,J \in \binom{[k]}{j}\\ I<J}} g_{I'} h_{J'} \Big)(-z)^j, \qquad \hat{Q}_k(z) = -\sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \Big(\sum_{\substack{I \in \binom{[k]}{j+1}, J \in \binom{[k]}{j}\\ I<J}} h_{I'} g_{J'} \Big)(-z)^j, \tag{4.10a}$$

$$\hat{p}_k(z) = -\sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} \Big(\sum_{\substack{I \in \binom{[k]}{j}, J \in \binom{[k]}{j-1}\\ I<J}} g_{I'} h_{J'} \Big)(-z)^j, \quad \hat{P}_k(z) = 1 + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \Big(\sum_{\substack{I,J \in \binom{[k]}{j}\\ I<J}} h_{I'} g_{J'} \Big)(-z)^j. \tag{4.10b}$$

Corollary 4.9. *Let $\hat{S}_k$ be the transition matrix for the right boundary value problem as specified above.*

(1) The entries of $\hat{S}_k$ solve the interpolation problems (4.3)*, that is: $\hat{q}_k, \hat{p}_k, \hat{Q}_k, \hat{P}_k$ satisfy*

$$\hat{q}_k(-\frac{1}{m_{i'}^2})W(-\frac{1}{m_{i'}^2}) + \hat{Q}_k(-\frac{1}{m_{i'}^2}) = 0, \qquad 1 \le i \le k, \tag{4.11a}$$

$$\deg \hat{q}_k = \lfloor \frac{k}{2} \rfloor, \qquad \deg \hat{Q}_k = \lfloor \frac{k-1}{2} \rfloor, \qquad \hat{q}_k(0) = 1, \tag{4.11b}$$

$$\hat{p}_k(-\frac{1}{m_{i'}^2})W(-\frac{1}{m_{i'}^2}) + \hat{P}_k(-\frac{1}{m_{i'}^2}) = 0, \qquad 1 \le i \le k, \tag{4.11c}$$

$$\deg \hat{p}_k = \lfloor \frac{k+1}{2} \rfloor, \qquad \deg \hat{P}_k = \lfloor \frac{k}{2} \rfloor, \qquad \hat{p}_k(0) = 0, \quad \hat{P}_k(0) = 1. \tag{4.11d}$$

(2) Given $f(z) \in \mathbf{C}[z]$, let f^+ denote the coefficient of the term of the highest degree. Then

$$g_{k'} = \frac{\hat{p}_k^+}{\hat{q}_{k-1}^+}, \qquad \textit{if } k \textit{ is odd}, \tag{4.12a}$$

$$g_{k'} = \frac{\hat{P}_k^+}{\hat{Q}_{k-1}^+}, \qquad \textit{if } k \textit{ is even}. \tag{4.12b}$$

(3) The right boundary value problem (4.5) *has the same spectrum as the left boundary value problem* (2.3).

Proof. The interpolation problem was stated in Theorem 4.2 for the matrix elements of $C_k C_{(k-1)} \cdots C_1$ (and that's where $\hat{S}_k$ was introduced). However, by Theorem 4.8, $\hat{S}_k$ is the same as $\begin{bmatrix} \hat{q}_k & \hat{Q}_k \\ \hat{p}_k & \hat{P}_k \end{bmatrix}$, hence the first claim.

To prove the second claim with consider first the case of odd k. Then, by the formulas in Theorem 4.8 we get:

$$\hat{q}_{k-1}^+ = (-1)^{\frac{k-1}{2}} \sum_{\substack{I,J \in \binom{[k-1]}{\frac{k-1}{2}} \\ I<J}} g_{I'} h_{J'} = (-1)^{\frac{k-1}{2}} g_{1'} h_{2'} g_{3'} \cdots g_{(k-2)'} h_{(k-1)'},$$

$$\hat{p}_k^+ = -(-1)^{\frac{k+1}{2}} \sum_{\substack{I \in \binom{[k]}{\frac{k+1}{2}}, J \in \binom{[k]}{\frac{k-1}{2}} \\ I<J}} g_{I'} h_{J'} = (-1)^{\frac{k-1}{2}} g_{1'} h_{2'} g_{3'} \cdots g_{(k-2)'} h_{(k-1)'} g_{k'},$$

whose ratio gives the desired formula for $g_{k'}$, recalling that $\hat{q}_0 = 1$ to cover the case of $k = 1$. The argument for even k is similar except that one uses the formulas for the second column of $\hat{S}_k$.

Finally, to prove the last claim, we observe that the map $i \to n + 1 - i$ is a bijection of the set $[n]$. Upon comparing Corollary 2.7 with the formula for $\hat{q}_n$ given above we see that $\hat{q}_n(z) = p_n(z)$, hence the two boundary value problems are equivalent. □

4.3. Solving the inverse problem by interpolation. The inverse problem we are interested in solving explicitly can be stated as follows:

Definition 4.10. Given a rational function (see Theorem 3.1)

$$W(z) = c + \int \frac{d\mu(x)}{x - z}, \qquad d\mu = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} b_j \delta_{\zeta_j}, \qquad 0 < \zeta_1 < \cdots < \zeta_{\lfloor \frac{n}{2} \rfloor}, \qquad 0 < b_j, \quad 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor, \tag{4.13}$$

where $c > 0$ when n is odd and $c = 0$ when n even, as well as positive, distinct, constants $m_1, m_2, \ldots, m_n$, find positive constants g_j, h_j, $1 \le j \le n$, such that $g_j h_j = m_j^2$ and the unique solution of the initial value problem:

$$\begin{aligned} q_k - q_{k-1} &= h_k p_{k-1}, \qquad 1 \le k \le n, \\ p_k - p_{k-1} &= -z g_k q_{k-1}, \quad 1 \le k \le n, \\ q_0 = 0, &\quad p_0 = 1, \end{aligned}$$

satisfies

$$W(z) = \frac{q_n(z)}{p_n(z)}.$$

Remark 4.11. The restriction that the constants m_j be distinct has been made to facilitate the argument and will be eventually relaxed by taking appropriate limits of the generic case (see Theorem 4.22).

The key observation leading to the solution of this inverse problem is the realization that the interpolation problem (4.11) (the same as (4.3)) has a unique solution.

Theorem 4.12. *Given a rational function $W(z)$ as above, and positive, distinct constants $m_1, m_2, \ldots, m_n$, there exist unique solutions $\hat{q}_k, \hat{p}_k, \hat{Q}_k, \hat{P}_k$, $1 \le k \le n$ to the interpolations problems* (4.11).

Let $z_i = -\frac{1}{m_{i'}^2}$, $1 \le i \le k$, then the solution to the first interpolation problem (4.11a), (4.11b) *is*

$$\hat{q}_k(z) + z^{\lfloor \frac{k}{2} \rfloor + 1} \hat{Q}_k(z) = \frac{1}{D_k} \det \begin{bmatrix} 1 & z & \ldots & z^{\lfloor \frac{k}{2} \rfloor} & z^{\lfloor \frac{k}{2} \rfloor+1} & z^{\lfloor \frac{k}{2} \rfloor+2} & \ldots & z^k \\ W(z_1) & z_1 W(z_1) & \ldots & z_1^{\lfloor \frac{k}{2} \rfloor} W(z_1) & 1 & z_1 & \cdots & z_1^{\lfloor \frac{k-1}{2} \rfloor} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ W(z_k) & z_k W(z_k) & \ldots & z_k^{\lfloor \frac{k}{2} \rfloor} W(z_k) & 1 & z_k & \cdots & z_k^{\lfloor \frac{k-1}{2} \rfloor} \end{bmatrix}, \tag{4.14}$$

where

$$D_k = \det \begin{bmatrix} z_1 W(z_1) & \ldots & z_1^{\lfloor \frac{k}{2} \rfloor} W(z_1) & 1 & z_1 & \cdots & z_1^{\lfloor \frac{k-1}{2} \rfloor} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ z_k W(z_k) & \ldots & z_k^{\lfloor \frac{k}{2} \rfloor} W(z_k) & 1 & z_k & \cdots & z_k^{\lfloor \frac{k-1}{2} \rfloor} \end{bmatrix}. \tag{4.15}$$

Likewise, the solution to the second interpolation problem (4.11c), (4.11d) *is*

$$\hat{P}_k(z) + z^{\lfloor \frac{k}{2} \rfloor} \hat{p}_k(z) = \frac{1}{E_k} \det \begin{bmatrix} 1 & z & \ldots & z^{\lfloor \frac{k}{2} \rfloor} & z^{\lfloor \frac{k}{2} \rfloor+1} & z^{\lfloor \frac{k}{2} \rfloor+2} & \ldots & z^k \\ 1 & z_1 & \ldots & z_1^{\lfloor \frac{k}{2} \rfloor} & z_1 W(z_1) & z_1^2 W(z_1) & \cdots & z_1^{\lfloor \frac{k+1}{2} \rfloor} W(z_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_k & \ldots & z_k^{\lfloor \frac{k}{2} \rfloor} & z_k W(z_k) & z_k^2 W(z_k) & \cdots & z_k^{\lfloor \frac{k+1}{2} \rfloor} W(z_k) \end{bmatrix}, \tag{4.16}$$

where

$$E_k = \det \begin{bmatrix} z_1 & \dots & z_1^{\lfloor \frac{k}{2} \rfloor} & z_1 W(z_1) & z_1^2 W(z_1) & \cdots & z_1^{\lfloor \frac{k+1}{2} \rfloor} W(z_1) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ z_k & \dots & z_k^{\lfloor \frac{k}{2} \rfloor} & z_k W(z_k) & z_k^2 W(z_k) & \cdots & z_k^{\lfloor \frac{k+1}{2} \rfloor} W(z_k) \end{bmatrix}. \tag{4.17}$$

Proof. Set

$$\hat{q}_k(z) = 1 + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} a_j z^j, \qquad \hat{Q}_k(z) = \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} A_j z^j.$$

Then the first interpolation problem reads:

$$\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (z_i)^j W(z_j) a_j + \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (z_i)^j A_j = -W(z_i), \qquad 1 \le i \le k,$$

whose solution, by virtue of Cramer's rule, can be written in the form of Eq. (4.14), provided that $D_k \neq 0$. Likewise, the solution to the second interpolation problem can be easily deduced by writing

$$\hat{P}_k(z) = 1 + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} B_j z^j, \qquad \hat{p}_k(z) = \sum_{j=1}^{\lfloor \frac{k+1}{2} \rfloor} b_j z^j,$$

substituting into the interpolation problem (4.11c) and, again, using Cramer's rule, with the same proviso that $E_k \neq 0$. Thus it remains to prove that D_k and E_k are not 0 under our assumption of distinct masses m_j. To this end we derive below explicit formulas for the determinants D_k and E_k from which we conclude that none of the determinants can be 0 in view of the non-degeneracy assumption on the masses m_j (see Corollary 4.19). □

4.4. Evaluation of determinants. In this subsection, we will derive explicit formulas for determinants appearing in the solution to the interpolation problems (see Theorem 4.12). We begin by introducing some additional notation to facilitate the presentation of formulas, reminding the reader that the multi-index notation was introduced earlier in the part leading up to the Definition 2.5. The following notation is in place: we denote $[i, j] = \{i, i+1, \dots, j\}$, $\binom{[1,K]}{k} = \{J = \{j_1, j_2, \dots, j_k\} | j_1 < \cdots < j_k, j_i \in [1, K]\}$. Then for two ordered multi-index sets I, J we define

$$\mathbf{x}_J = \prod_{j \in J} x_j, \qquad \Delta_J(\mathbf{x}) = \prod_{i<j \in J} (x_j - x_i),$$

$$\Delta_{I,J}(\mathbf{x}; \mathbf{y}) = \prod_{i \in I} \prod_{j \in J} (x_i - y_j), \qquad \Gamma_{I,J}(\mathbf{x}; \mathbf{y}) = \prod_{i \in I} \prod_{j \in J} (x_i + y_j),$$

along with the convention

$$\Delta_\emptyset(\mathbf{x}) = \Delta_{\{i\}}(\mathbf{x}) = \Delta_{\emptyset,J}(\mathbf{x}; \mathbf{y}) = \Delta_{I,\emptyset}(\mathbf{x}; \mathbf{y}) = \Gamma_{\emptyset,J}(\mathbf{x}; \mathbf{y}) = \Gamma_{I,\emptyset}(\mathbf{x}; \mathbf{y}) = 1,$$

$$\binom{[1, K]}{0} = 1; \qquad \binom{[1, K]}{k} = 0, \quad k > K.$$

Definition 4.13. Given two vectors $\mathbf{e} \in \mathbf{R}^k, \mathbf{d} \in \mathbf{R}^l$, $0 \leq l \leq k$ such that $e_i + d_j \neq 0$ for any pair of indices, a *Cauchy–Vandermonde matrix* [22,26,47,48] is a matrix of the form

$$CV_k^{(l)}(\mathbf{e}, \mathbf{d}) = \begin{pmatrix} \frac{1}{e_1+d_1} & \frac{1}{e_1+d_2} & \cdots & \frac{1}{e_1+d_l} & 1 & e_1 & \cdots & e_1^{k-l-1} \\ \frac{1}{e_2+d_1} & \frac{1}{e_2+d_2} & \cdots & \frac{1}{e_2+d_l} & 1 & e_2 & \cdots & e_2^{k-l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{e_k+d_1} & \frac{1}{e_k+d_2} & \cdots & \frac{1}{e_k+d_l} & 1 & e_k & \cdots & e_k^{k-l-1} \end{pmatrix}. \tag{4.18}$$

Two special cases are: for $l = 0$ the matrix defined by (4.18) is a classical Vandermonde matrix and for $l = k$ it is a classical Cauchy matrix, and for both these special cases there exist classical formulas expressing their determinants. Luckily, there exists also a compact formula for the determinant of the (generic) Cauchy–Vandermonde matrix [26,47,48]:

$$\det(CV_k^{(l)}(\mathbf{e}, \mathbf{d})) = \frac{\Delta_{[1,k]}(\mathbf{e})\Delta_{[1,l]}(\mathbf{d})}{\Gamma_{[1,k],[1,l]}(\mathbf{e}; \mathbf{d})}. \tag{4.19}$$

As a side note we would like to mention that the Cauchy–Vandermonde matrix (4.18) appears naturally as the coefficient matrix of a rational interpolation problem of Lagrange type: given k pairs of interpolation data $(e_1, t_1), \ldots, (e_k, t_k)$, where $e_1, \ldots, e_k$ are different real numbers, find a function

$$f(x) = \sum_{j=1}^{l} s_j \frac{1}{x + d_j} + \sum_{j=l+1}^{k} s_j x^{j-l-1},$$

with s_i to be determined, such that $f(e_i) = t_i$, $i = 1, \cdots, k$.

The interpolation problems (4.3) can be viewed as slight variations on the theme of rational interpolations problem of Largrange type and to effect the explicit solution of these problems one is led to a generalization of the Cauchy–Vandermonde matrix.

Definition 4.14. Given three vectors $\mathbf{e} \in \mathbf{R}^k, \mathbf{d}, \mathbf{a} \in \mathbf{R}^l$, $0 \leq l \leq k$ such that $e_i + d_j \neq 0$ for any pair of indices, a *modified Cauchy–Vandermonde matrix* is that of the form

$$CV_k^{(l,p)}(\mathbf{e}, \mathbf{d}, \mathbf{a}) = \begin{pmatrix} \frac{a_1 e_1^p}{e_1+d_1} & \frac{a_2 e_1^{p+1}}{e_1+d_2} & \cdots & \frac{a_l e_1^{p+l-1}}{e_1+d_l} & 1 & e_1 & \cdots & e_1^{k-l-1} \\ \frac{a_1 e_2^p}{e_2+d_1} & \frac{a_2 e_2^{p+1}}{e_2+d_2} & \cdots & \frac{a_l e_2^{p+l-1}}{e_2+d_l} & 1 & e_2 & \cdots & e_2^{k-l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{a_1 e_k^p}{e_k+d_1} & \frac{a_2 e_k^{p+1}}{e_k+d_2} & \cdots & \frac{a_l e_k^{p+l-1}}{e_k+d_l} & 1 & e_k & \cdots & e_k^{k-l-1} \end{pmatrix}, \tag{4.20}$$

with $p \geq 0, 0 \leq l \leq k, p + l - 1 \leq k - l$.

Theorem 4.15. *Let* $0 \leq p$, $0 \leq l \leq k$ *and* $p + l - 1 \leq k - l$, *then*

$$\det(CV_k^{(l,p)}(\mathbf{e}, \mathbf{d}, \mathbf{a})) = C_{l,p} \frac{\Delta_{[1,k]}(\mathbf{e})\Delta_{[1,l]}(\mathbf{d})}{\Gamma_{[1,k],[1,l]}(\mathbf{e}; \mathbf{d})}, \tag{4.21}$$

where $C_{l,p} = (-1)^{lp+\frac{l(l-1)}{2}} \mathbf{a}_{[1,l]} \cdot \mathbf{d}_{[1,l]}^p \cdot d_1^0 d_2^1 \ldots d_l^{l-1}$.

Proof. By multilinearity of the determinant we can factor all coefficients a_j from the first l rows. Hence it is sufficient to work with $\mathbf{a} = [1, 1, \ldots, 1]$. Let us drop the reference to $\mathbf{a}$ for the remainder of the proof and simply write

$$CV_k^{(l,p)}(\mathbf{e}, \mathbf{d}) = \begin{pmatrix} \frac{e_1^p}{e_1+d_1} & \frac{e_1^{p+1}}{e_1+d_2} & \cdots & \frac{e_1^{p+l-1}}{e_1+d_l} & 1 & e_1 & \cdots & e_1^{k-l-1} \\ \frac{e_2^p}{e_2+d_1} & \frac{e_2^{p+1}}{e_2+d_2} & \cdots & \frac{e_2^{p+l-1}}{e_2+d_l} & 1 & e_2 & \cdots & e_2^{k-l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{e_k^p}{e_k+d_1} & \frac{e_k^{p+1}}{e_k+d_2} & \cdots & \frac{e_k^{p+l-1}}{e_k+d_l} & 1 & e_k & \cdots & e_k^{k-l-1} \end{pmatrix}, \tag{4.22}$$

maintaining the assumptions $0 \le p,\ 0 \le l \le k$ and $p + l - 1 \le k - l$.

Let us now consider the l-th column of the matrix; we may write it as

$$\begin{pmatrix} \frac{e_1^{p+l-1}-(-d_l)^{p+l-1}+(-d_l)^{p+l-1}}{e_1+d_l} \\ \frac{e_2^{p+l-1}-(-d_l)^{p+l-1}+(-d_l)^{p+l-1}}{e_2+d_l} \\ \vdots \\ \frac{e_k^{p+l-1}-(-d_l)^{p+l-1}+(-d_l)^{p+l-1}}{e_k+d_l} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{p+l-1} e_1^{p+l-1-j}(-d_l)^{j-1} + \frac{(-d_l)^{p+l-1}}{e_1+d_l} \\ \sum_{j=1}^{p+l-1} e_2^{p+l-1-j}(-d_l)^{j-1} + \frac{(-d_l)^{p+l-1}}{e_2+d_l} \\ \vdots \\ \sum_{j=1}^{p+l-1} e_k^{p+l-1-j}(-d_l)^{j-1} + \frac{(-d_l)^{p+l-1}}{e_k+d_l} \end{pmatrix}.$$

Since, thanks to the assumption $p + l - 1 \le k - l$, the first terms above are linear combinations of columns $l + 1$ through k we obtain

$$\det(CV_k^{(l,p)}(\mathbf{e}, \mathbf{d})) = \det \begin{pmatrix} \frac{e_1^p}{e_1+d_1} & \frac{e_1^{p+1}}{e_1+d_2} & \cdots & \frac{(-d_l)^{p+l-1}}{e_1+d_l} & 1 & e_1 & \cdots & e_1^{k-l-1} \\ \frac{e_2^p}{e_2+d_1} & \frac{e_2^{p+1}}{e_2+d_2} & \cdots & \frac{(-d_l)^{p+l-1}}{e_2+d_l} & 1 & e_2 & \cdots & e_2^{k-l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{e_k^p}{e_k+d_1} & \frac{e_k^{p+1}}{e_k+d_2} & \cdots & \frac{(-d_l)^{p+l-1}}{e_k+d_l} & 1 & e_k & \cdots & e_k^{k-l-1} \end{pmatrix},$$

which after implementing similar operations for the remaining first $l - 1$ columns leads to:

$$\det(CV_k^{(l,p)}(\mathbf{e}, \mathbf{d})) = \det \begin{pmatrix} \frac{(-d_1)^p}{e_1+d_1} & \frac{(-d_2)^{p+1}}{e_1+d_2} & \cdots & \frac{(-d_l)^{p+l-1}}{e_1+d_l} & 1 & e_1 & \cdots & e_1^{k-l-1} \\ \frac{(-d_1)^p}{e_2+d_1} & \frac{(-d_2)^{p+1}}{e_2+d_2} & \cdots & \frac{(-d_l)^{p+l-1}}{e_2+d_l} & 1 & e_2 & \cdots & e_2^{k-l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-d_1)^p}{e_k+d_1} & \frac{(-d_2)^{p+1}}{e_k+d_2} & \cdots & \frac{(-d_l)^{p+l-1}}{e_k+d_l} & 1 & e_k & \cdots & e_k^{k-l-1} \end{pmatrix}.$$

Now it suffices to factor $(-d_1)^p, \ldots, (-d_l)^{p+l-1}$ in order to obtain a straightforward relation between the determinants of the modified Cauchy–Vandermonde matrix and the Cauchy–Vandermonde matrix (4.18), (4.19)

$$\det(CV_k^{(l,p)}(\mathbf{e},\mathbf{d})) = (-d_1)^p(-d_2)^{p+1}\cdots(-d_l)^{p+l-1}\cdot\det(CV_k^l(\mathbf{e},\mathbf{d})),$$

from which, after restoring a general $\mathbf{a}$ which contributes the factor $\mathbf{a}_{[1,l]}$, the result follows. □

In the final step of generalizing Cauchy–Vandermonde matrices we introduce a family of matrices of this type attached to a Stieltjes transform of a positive measure.

Definition 4.16. Given a (strictly) positive vector $\mathbf{e} \in \mathbf{R}^k$, a non-negative number c, an index l such that $0 \le l \le k$, another index p such that $0 \le p,\ p+l-1 \le k-l$, and a positive measure ν with support in $\mathbf{R}_+$, a *Cauchy-Stieltjes-Vandermonde (CSV) matrix* is that of the form

$$CSV_k^{(l,p)}(\mathbf{e},\nu,c) = \begin{pmatrix} e_1^p\hat{\nu}_c(e_1) & e_1^{p+1}\hat{\nu}_c(e_1) & \cdots & e_1^{p+l-1}\hat{\nu}_c(e_1) & 1 & e_1 & \cdots & e_1^{k-l-1} \\ e_2^p\hat{\nu}_c(e_2) & e_2^{p+1}\hat{\nu}_c(e_2) & \cdots & e_2^{p+l-1}\hat{\nu}_c(e_2) & 1 & e_2 & \cdots & e_2^{k-l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ e_k^p\hat{\nu}_c(e_k) & e_k^{p+1}\hat{\nu}_c(e_k) & \cdots & e_k^{p+l-1}\hat{\nu}_c(e_k) & 1 & e_k & \cdots & e_k^{k-l-1} \end{pmatrix}, \tag{4.23}$$

where $\hat{\nu}_c$ is the (shifted) Stieltjes transform of the measure ν and is given by $\hat{\nu}_c(y) = c + \int \frac{d\nu(x)}{y+x}$.

In the next theorem we establish explicit formulas for the determinant of the CSV matrix. This theorem is essential for our solution of the interpolation problems (4.3).

Theorem 4.17. *Let ν be a positive measure with support in $\mathbf{R}_+$ and let $\mathbf{x}$ denote the vector $[x_1, x_2, \ldots, x_l] \in \mathbf{R}^l$ and $d\nu^p(y) = y^p d\nu(y)$, respectively. Then*

(1) if either $c = 0$ or $p+l-1 < k-l$ then

$$\det CSV_k^{(l,p)}(\mathbf{e},\nu,c) = (-1)^{lp+\frac{l(l-1)}{2}}\Delta_{[1,k]}(\mathbf{e})\int_{0<x_1<x_2<\cdots<x_l}\frac{\Delta_{[1,l]}(\mathbf{x})^2}{\Gamma_{[1,k],[1,l]}(\mathbf{e};\mathbf{x})}d\nu^p(x_1)d\nu^p(x_2)\ldots d\nu^p(x_l); \tag{4.24}$$

(2) if $c > 0$ and $p+l-1 = k-l$ then

$$\begin{aligned}\det CSV_k^{(l,p)}(\mathbf{e},\nu,c) &= (-1)^{lp+\frac{l(l-1)}{2}}\Delta_{[1,k]}(\mathbf{e}) \\ &\cdot\Big(\int_{0<x_1<x_2<\cdots<x_l}\frac{\Delta_{[1,l]}(\mathbf{x})^2}{\Gamma_{[1,k],[1,l]}(\mathbf{e};\mathbf{x})}d\nu^p(x_1)d\nu^p(x_2)\ldots d\nu^p(x_l) \\ &+ c\int_{0<y_1<y_2<\cdots<y_{l-1}}\frac{\Delta_{[1,l-1]}(\mathbf{y})^2}{\Gamma_{[1,k],[1,l-1]}(\mathbf{e};\mathbf{y})}d\nu^p(y_1)d\nu^p(y_2)\ldots d\nu^p(y_{l-1})\Big).\end{aligned} \tag{4.25}$$

Proof. Let us first consider the case $c = 0$. Using multilinearity of the determinant we obtain

$$\det CSV_k^{(l,p)}(\mathbf{e}, \nu, 0) = \int \det \begin{pmatrix} \frac{e_1^p}{e_1+x_1} & \frac{e_1^{p+1}}{e_1+x_2} & \cdots & \frac{e_1^{p+l-1}}{e_1+x_l} & 1 & e_1 & \dots & e_1^{k-l+1} \\ \frac{e_2^p}{e_2+x_1} & \frac{e_2^{p+1}}{e_2+x_2} & \cdots & \frac{e_2^{p+l-1}}{e_2+x_l} & 1 & e_2 & \dots & e_2^{k-l+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{e_k^p}{e_k+x_1} & \frac{e_k^{p+1}}{e_k+x_2} & \cdots & \frac{e_k^{p+l-1}}{e_k+x_l} & 1 & e_k & \dots & e_k^{k-l+1} \end{pmatrix} d\nu(x_1)d\nu(x_2)\cdots d\nu(x_l).$$

Then by Theorem 4.15

$$\det CSV_k^{(l,p)}(\mathbf{e}, \nu, 0) = (-1)^{lp+\frac{l(l-1)}{2}} \int (x_1x_2\cdots x_l)^p x_1^0 x_2^1 \cdots x_l^{l-1} \frac{\Delta_{[1,k]}(\mathbf{e})\Delta_{[1,l]}(\mathbf{x})}{\Gamma_{[1,k],[1,l]}(\mathbf{e};\mathbf{x})} d\nu(x_1)d\nu(x_2)\cdots d\nu(x_l).$$

Let us now consider the action of the group of permutations on l letters, denoted S_l, on individual terms of the integrand. The product measure is invariant under the action and so are $(x_1x_2\cdots x_l)^p$ and $\Gamma_{[1,n],[1,l]}(\mathbf{e};\mathbf{x})$. Let $\sigma.\mathbf{x} = [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(l)}]$ then

$$\begin{aligned}\det CSV_k^{(l,p)}(\mathbf{e}, \nu, 0) &= \frac{(-1)^{lp+\frac{l(l-1)}{2}}\Delta_{[1,k]}(\mathbf{e})}{l!} \int \frac{(x_1x_2\cdots x_l)^p}{\Gamma_{[1,k],[1,l]}(\mathbf{e};\mathbf{x})} \\ &\times \Big(\sum_{\sigma\in S_l} x_{\sigma(1)}^0 x_{\sigma(2)}^1 \cdots x_{\sigma(l)}^{l-1}\Delta_{[1,l]}(\sigma.\mathbf{x})\Big)\, d\nu(x_1)d\nu(x_2)\cdots d\nu(x_l) \\ &= \frac{(-1)^{lp+\frac{l(l-1)}{2}}\Delta_{[1,k]}(\mathbf{e})}{l!} \int \frac{(x_1x_2\cdots x_l)^p\Delta^2_{[1,l]}(\mathbf{x})}{\Gamma_{[1,k],[1,l]}(\mathbf{e};\mathbf{x})}\, d\nu(x_1)d\nu(x_2)\cdots d\nu(x_l),\end{aligned}$$

where in the last step we used $\Delta_{[1,l]}(\sigma.\mathbf{x}) = \operatorname{sgn}(\sigma)\Delta_{[1,l]}(\mathbf{x})$. Since now the integrand is invariant under the action of S_l we integrate over $x_1 < x_2 < \cdots < x_l$, multiply by $l!$, and restrict integration to R_+ in view of the condition on the support of ν, obtaining the final formula for this case.

The next case is $0 \le c$ but $p + l - 1 < k - l$. In this case every column j, $1 \le j \le l$ is a sum:

$$c\begin{pmatrix} e_1^{p+j-1} \\ e_2^{p+j-1} \\ \vdots \\ e_k^{p+j-1}\end{pmatrix} + \begin{pmatrix} e_1^{p+j-1}\nu_0(e_1) \\ e_2^{p+j-1}\nu_0(e_2) \\ \vdots \\ e_k^{p+j-1}\nu_0(e_k)\end{pmatrix},$$

and the condition $p + l - 1 < k - l$ ensures that the first vector, namely the one multiplied by c, appears in the Vandermonde part of the matrix, hence by antisymmetry of the determinant implying that this case reduces to the case $c = 0$.

Finally, the last case $p + l - 1 = k - l$ can handled in a similar fashion, except that now in the lth column we have a term which does not appear in the original Cauchy part:

$$\det CSV_k^{(l,p)}(\mathbf{e}, \nu, c)$$
$$= \det \begin{pmatrix} e_1^p \hat{\nu}_0(e_1) & e_1^{p+1}\hat{\nu}_0(e_1) & \cdots & ce_1^{p+l-1} + e_1^{p+l-1}\hat{\nu}_0(e_1) & 1 & e_1 & \cdots & e_1^{k-l-1} \\ e_2^p \hat{\nu}_0(e_2) & e_2^{p+1}\hat{\nu}_0(e_2) & \cdots & ce_2^{p+l-1} + e_2^{p+l-1}\hat{\nu}_0(e_2) & 1 & e_2 & \cdots & e_2^{k-l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ e_k^p \hat{\nu}_0(e_k) & e_k^{p+1}\hat{\nu}_0(e_k) & \cdots & ce_k^{p+l-1} + e_k^{p+l-1}\hat{\nu}_0(e_k) & 1 & e_k & \cdots & e_k^{k-l-1} \end{pmatrix}.$$

It suffices now to split the determinant into two, then move the term involving c to the Vandermonde part, effectively lowering l to $l-1$ for this term and then apply the same method as in the proof of the $c = 0$ case. □

Our goal in the remainder of this section is to connect the determinantal formalism we have developed to the interpolation problem (4.3) (see Theorem 4.12). To this end we set (see (4.13) and Theorem 4.12):

$$e_j = -z_j = \frac{1}{m_{j'}^2}, \qquad \nu = \mu, \quad 1 \le j \le n,$$

and observe that $W(z_j) = \hat{\mu}_c(e_j)$ by (4.13).

Theorem 4.18. *Let D_k, E_k be the determinants defined in Theorem 4.12. Then*

$$D_k = (-1)^{\lfloor \frac{k}{2} \rfloor \lfloor \frac{k+1}{2} \rfloor} \det CSV_k^{(\lfloor \frac{k}{2} \rfloor, 1)}(\mathbf{e}, \mu, c),$$
$$E_k = (-1)^{k + \lfloor \frac{k}{2} \rfloor \lfloor \frac{k+1}{2} \rfloor} \mathbf{e}_{[1,k]} \det CSV_k^{(\lfloor \frac{k+1}{2} \rfloor, 0)}(\mathbf{e}, \mu, c).$$

Proof. This is a straightforward computation requiring only to factor $(-1)^j$ from any column containing $z_i^j = (-e_i)^j$ and, in the case of E_k, we also need to factor $(z_1 z_2 \cdots z_k)$ and reshuffle the columns to bring the matrix to the CSV form. □

Now, it suffices to use theorem 4.17 to conclude that both D_k and E_k are nonzero, provided that all e_j are distinct (which is the same as our nondegeracy conditions on the masses m_j).

Corollary 4.19. *The determinants D_k and E_k appearing in the solution to the interpolation problem stated in Theorem 4.12 are non zero for any k, $1 \le k \le n$.*

We finish this subsection by giving a complete solution to the inverse problem in terms of determinants of CSV matrices. To lessen the burden of keeping track of signs resulting from manipulations of matrix columns needed to bring matrices to the CSV form we opt for the display using the absolute value of determinants to quickly and compactly present the formulas. Thus, till further notice, we will denote

$$\mathscr{D}_k^{(l,p)} = \left| \det CSV_k^{(l,p)}(\mathbf{e}, \mu, c) \right|, \tag{4.26}$$

with the proviso that the arguments $\mathbf{e}, \mu, c$ are fixed. With this notation in place the formulas of Theorem 4.18 take the form:

$$|D_k| = \mathscr{D}_k^{(\lfloor \frac{k}{2} \rfloor, 1)}, \quad |E_k| = \mathbf{e}_{[1,k]} \mathscr{D}_k^{(\lfloor \frac{k+1}{2} \rfloor, 0)}, \tag{4.27}$$

from which one obtains determinantal formulas for the coefficients of the polynomials $\hat{q}_k, \hat{p}_k, \hat{Q}_k, \hat{P}_k$ leading via Eqs. (4.12a), (4.12b) to a complete solution of the inverse problem 4.10.

Theorem 4.20. *Suppose the Weyl function $W(z)$ is given by (4.13) along with positive distinct constants (masses) $m_1, m_2, \dots, m_n$. Then there exists a unique solution to the inverse problem specified in Definition 4.10:*

$$g_{k'} = \frac{\mathscr{D}_k^{(\frac{k-1}{2},1)} \mathscr{D}_{k-1}^{(\frac{k-1}{2},1)}}{\mathbf{e}_{[1,k]} \mathscr{D}_k^{(\frac{k+1}{2},0)} \mathscr{D}_{k-1}^{(\frac{k-1}{2},0)}}, \qquad \textit{if } k \textit{ is odd}, \tag{4.28a}$$

$$g_{k'} = \frac{\mathscr{D}_k^{(\frac{k}{2},1)} \mathscr{D}_{k-1}^{(\frac{k}{2}-1,1)}}{\mathbf{e}_{[1,k]} \mathscr{D}_k^{(\frac{k}{2},0)} \mathscr{D}_{k-1}^{(\frac{k}{2},0)}}, \qquad \textit{if } k \textit{ is even}. \tag{4.28b}$$

Likewise,

$$h_{k'} = \frac{\mathbf{e}_{[1,k-1]} \mathscr{D}_k^{(\frac{k+1}{2},0)} \mathscr{D}_{k-1}^{(\frac{k-1}{2},0)}}{\mathscr{D}_k^{(\frac{k-1}{2},1)} \mathscr{D}_{k-1}^{(\frac{k-1}{2},1)}}, \qquad \textit{if } k \textit{ is odd}, \tag{4.29a}$$

$$h_{k'} = \frac{\mathbf{e}_{[1,k-1]} \mathscr{D}_k^{(\frac{k}{2},0)} \mathscr{D}_{k-1}^{(\frac{k}{2},0)}}{\mathscr{D}_k^{(\frac{k}{2},1)} \mathscr{D}_{k-1}^{(\frac{k}{2}-1,1)}}, \qquad \textit{if } k \textit{ is even}. \tag{4.29b}$$

Proof. The formulas follow from Eqs. (4.12a) and (4.12b), as well as Theorem 4.12. The question of signs involved in the identification of the CSV determinants is addressed by taking the absolute values in all formulas needed to produce positive outcomes $g_{k'}$. The formulas for $h_{k'}$ follow from the relation $g_{k'} h_{k'} = m_{k'}^2$. □

Finally, recalling that the original peakon problem (1.6) was formulated in the x space, using the relation $h_j = m_j e^{x_j}$ (see Eq. (2.2)), we arrive at the inverse formulae relating the spectral data and the positions of peakons given by x_j.

Theorem 4.21. *Given positive and distinct constants m_j, let Φ be the solution to the boundary value problem 2.3 with associated spectral data $\{d\mu, c\}$. Then the positions x_j (of peakons) in the discrete measure $m = 2\sum_{j=1}^n m_j \delta_{x_j}$ can be expressed in terms of the spectral data as:*

$$x_{k'} = \ln \frac{\mathbf{e}_{[1,k-1]} \mathscr{D}_k^{(\frac{k+1}{2},0)} \mathscr{D}_{k-1}^{(\frac{k-1}{2},0)}}{m_{k'} \mathscr{D}_k^{(\frac{k-1}{2},1)} \mathscr{D}_{k-1}^{(\frac{k-1}{2},1)}}, \qquad \textit{if } k \textit{ is odd}, \tag{4.30a}$$

$$x_{k'} = \ln \frac{\mathbf{e}_{[1,k-1]} \mathscr{D}_k^{(\frac{k}{2},0)} \mathscr{D}_{k-1}^{(\frac{k}{2},0)}}{m_{k'} \mathscr{D}_k^{(\frac{k}{2},1)} \mathscr{D}_{k-1}^{(\frac{k}{2}-1,1)}}, \qquad \textit{if } k \textit{ is even}, \tag{4.30b}$$

with $\mathscr{D}_k^{(l,p)}$ *defined in* (4.26), $k' = n - k + 1$, $1 \le k \le n$ *and the convention that* $\mathscr{D}_0^{l,p} = 1$.

Finally, we can relax the condition that the masses be distinct, observing that the Vandermonde determinants $\Delta_{[1,r]}(\mathbf{e})$, $r = k, k-1$, cancel out in all expressions of the type

$$\frac{\mathscr{D}_k^{(l_1,p_1)} \mathscr{D}_{k-1}^{(l_2,p_2)}}{\mathscr{D}_k^{(l_3,p_3)} \mathscr{D}_{k-1}^{(l_4,p_4)}}$$

as can be seen from the determinantal expressions given in Theorem 4.17 (see (4.24) and (4.25)).

Theorem 4.22. *Given* **positive** *constants* m_j*, let* Φ *be the solution to the boundary value problem 2.3 with associated spectral data* $\{d\mu, c\}$*. Then the positions* x_j *(of peakons) in the discrete measure* $m = 2\sum_{j=1}^n m_j \delta_{x_j}$ *can be expressed in terms of the spectral data as given by the continuous extension of the formulas* (4.30a) *and* (4.30b) *to all, including coinciding, masses.*

Proof. We give a short proof of this statement. The forward map $\mathscr{S} : (m_1, m_2, \ldots, m_n; x_1, x_2, \ldots, x_n) \to (m_1, m_2, \ldots, \zeta, \mathbf{b}, c)$ and its inverse $\mathscr{S}^{-1}$, which exists by Theorem 4.1, are continuous. The formulas (4.30a) and (4.30b) were originally defined for distinct masses but, after cancellation of the Vandermonde determinants mentioned above, have continuous extensions to all positive masses, distinct or not. By uniqueness and continuity of $\mathscr{S}$ the extended formulas are then the formulas valid for all positive masses. □

5. Multipeakons for *n* = 2*K*

Even though the only difference between even and odd n is that $c = 0$ when n is even (see Theorem 3.1), and $c > 0$ otherwise, we nevertheless present these two cases separately in an attempt to underscore subtle differences in the asymptotic behaviour of peakons for these two cases.

5.1. Closed formulae for $n = 2K$. If we assume that $x_1(0) < x_2(0) < \cdots < x_{2K}(0)$ then this condition will hold at least in a small interval containing $t = 0$. Thus by Theorem 4.21 we have the following result.

Theorem 5.1. *Assuming the notation of Theorem 4.21, the mCH Eq.* (1.1) *with the regularization of the singular term* $u_x^2 m$ *given by* $\langle u_x^2 \rangle m$ *admits the multipeakon solution*

$$u(x,t) = \sum_{k=1}^{2K} m_{k'}(t) \exp(-|x - x_{k'}(t)|), \tag{5.1}$$

where $x_{k'}$ *are given by Eqs.* (4.30a) *and* (4.30b)*, with the peakon spectral measure*

$$d\mu = \sum_{j=1}^{K} b_j(t) \delta_{\zeta_j}, \tag{5.2}$$

$b_j(t) = b_j(0) e^{\frac{2t}{\zeta_j}}$, $0 < b_j(0)$*, ordered eigenvalues* $0 < \zeta_1 < \cdots < \zeta_K$ *and* $c = 0$ *in* (4.26).

Proof. We only need to discuss the time evolution of the spectral measure. To this end we recall that the Weyl function $W(z)$ defined in (3.1) undergoes the time evolution dictated by (2.11); a simple computation gives

$$\dot{W} = \frac{2}{z}W - \frac{2L}{z},$$

which, in turn, implies $\dot{b}_j = \frac{2}{\zeta_j} b_j$, $1 \leq j \leq K$ by virtue of Corollary 3.2. The remaining statement regarding the multipeakon solutions follows from our solution of the inverse problem and the fact that to formulate the time evolution we used the distributionally compatible Lax pair (see Apendix A). □

Even though one could easily give examples of peakon solutions based directly on Theorem 5.1 it is helpful to examine the explicit formulas for the evaluation of CSV determinants presented in Theorem 4.17 (see Eq. (4.26) for notation), adjusted to the case of even n.

Theorem 5.2. *Let $n = 2K$, $0 \leq l \leq K$, $0 \leq p$, $p + l - 1 \leq k - l$, $1 \leq k \leq 2K$ and let the peakon spectral measure be given by (5.2). Then*

(1)

$$\mathscr{D}_k^{(l,p)} = \left|\Delta_{[1,k]}(\mathbf{e})\right| \sum_{I \in \binom{[1,K]}{l}} \frac{\Delta_I^2(\zeta)\mathbf{b}_I \zeta_I^p}{\Gamma_{[1,k],I}(\mathbf{e};\zeta)}; \tag{5.3}$$

(2) in the asymptotic region $t \to +\infty$

$$\mathscr{D}_k^{(l,p)} = \left|\Delta_{[1,k]}(\mathbf{e})\right| \frac{\Delta_{[1,l]}^2(\zeta)\mathbf{b}_{[1,l]}\zeta_{[1,l]}^p}{\Gamma_{[1,k],[1,l]}(\mathbf{e};\zeta)}\Big[1 + \mathscr{O}(e^{-\alpha t})\Big], \quad 0 < \alpha; \tag{5.4}$$

(3) in the asymptotic region $t \to -\infty$

$$\mathscr{D}_k^{(l,p)} = \left|\Delta_{[1,k]}(\mathbf{e})\right| \frac{\Delta_{[1,l]^*}^2(\zeta)\mathbf{b}_{[1,l]^*}\zeta_{[1,l]^*}^p}{\Gamma_{[1,k],[1,l]^*}(\mathbf{e};\zeta)}\Big[1 + \mathscr{O}(e^{\beta t})\Big], \quad 0 < \beta, \tag{5.5}$$

where $[1,l]^* = [l^* = K - l + 1, 1^* = K]$ (reflection of the interval $[1,K]$).

Proof. Equation (5.3) follows from Theorem 4.17, in particular Eq. (4.24), by taking $d\nu = d\mu$ there, and carrying out integration. The formula (5.4) can be obtained from (5.3) by observing that the time dependence is confined to terms $\mathbf{b}_I = b_{j_1}(0)e^{\frac{2t}{\zeta_{j_1}}} b_{j_2}(0)e^{\frac{2t}{\zeta_{j_2}}} \cdots b_{j_l}(0)e^{\frac{2t}{\zeta_{j_l}}}$ of which the term with the smallest l-tuple of eigenvalues is dominant; the rest then follows from our ordering of the eigenvalues. Finally, formula (5.5) follows from a similar argument, except that for $t \to -\infty$ the dominant term corresponds to the largest l-tuple of eigenvalues. □

Before we display examples of formulas for multipeakons in the case of $n = 2K$ we remind the reader that $e_j = \frac{1}{m_{j'}^2}$, $j' = 2K - j + 1$. All examples below are derived following the same pattern: one takes formulas (4.30a), (4.30b) and uses (5.3) to derive explicit expressions for positions $x_1, \ldots, x_{2K}$.

Example 5.3. (2-peakon solution; K = 1)

$$x_1 = \ln\left(\frac{b_1}{\zeta_1 m_1(1+\zeta_1 m_2^2)}\right), \qquad x_2 = \ln\left(\frac{b_1 m_2}{1+\zeta_1 m_2^2}\right).$$

Example 5.4. (4-peakon solution; K = 2)

$$x_1 = \ln\left(\frac{1}{m_1}\cdot\frac{b_1 b_2(\zeta_2-\zeta_1)^2}{\zeta_1\zeta_2\left(b_1\zeta_1(1+\zeta_2 m_2^2)(1+\zeta_2 m_3^2)(1+\zeta_2 m_4^2)+b_2\zeta_2(1+\zeta_1 m_2^2)(1+\zeta_1 m_3^2)(1+\zeta_1 m_4^2)\right)}\right),$$

$$x_2 = \ln\left(m_2\cdot\frac{b_1 b_2(\zeta_2-\zeta_1)^2\left(b_1(1+\zeta_2 m_3^2)(1+\zeta_2 m_4^2)+b_2(1+\zeta_1 m_3^2)(1+\zeta_1 m_4^2)\right)}{\left(b_1\zeta_1(1+\zeta_2 m_3^2)(1+\zeta_2 m_4^2)+b_2\zeta_2(1+\zeta_1 m_3^2)(1+\zeta_1 m_4^2)\right)}\right.$$
$$\left.\cdot\frac{1}{\left(b_1\zeta_1(1+\zeta_2 m_2^2)(1+\zeta_2 m_3^2)(1+\zeta_2 m_4^2)+b_2\zeta_2(1+\zeta_1 m_2^2)(1+\zeta_1 m_3^2)(1+\zeta_1 m_4^2)\right)}\right),$$

$$x_3 = \ln\left(\frac{1}{m_3}\cdot\frac{\left(b_1(1+\zeta_2 m_4^2)+b_2(1+\zeta_1 m_4^2)\right)\left(b_1(1+\zeta_2 m_3^2)(1+\zeta_2 m_4^2)+b_2(1+\zeta_1 m_3^2)(1+\zeta_1 m_4^2)\right)}{(1+\zeta_1 m_4^2)(1+\zeta_2 m_4^2)\left(b_1\zeta_1(1+\zeta_2 m_3^2)(1+\zeta_2 m_4^2)+b_2\zeta_2(1+\zeta_1 m_3^2)(1+\zeta_1 m_4^2)\right)}\right),$$

$$x_4 = \ln\left(m_4\cdot\frac{b_1(1+\zeta_2 m_4^2)+b_2(1+\zeta_1 m_4^2)}{(1+\zeta_1 m_4^2)(1+\zeta_2 m_4^2)}\right).$$

Example 5.5. (a general formula for the last position $x_{2K} = x_{1'}$)

$$x_{2K} = x_{1'} = \ln\frac{\mathscr{D}_1^{(1,0)}}{m_{1'}} = \ln\frac{\hat{\mu}_0(e_1)}{m_{2K}} = \ln\frac{\sum_{i=1}^K \frac{b_i}{\frac{1}{m_{2K}^2}+\zeta_i}}{m_{2K}} = \ln m_{2K}\sum_{i=1}^K \frac{b_i}{1+m_{2K}^2\zeta_i}.$$

5.2. Global existence for $n = 2K$. As time varies, the initial order $x_1(0) < x_2(0) < \cdots < x_{2K}(0)$ might cease to hold. In this subsection, we formulate a sufficient condition needed to ensure that the peakon flow exists globally in time.

Theorem 5.6. *Given arbitrary spectral data*

$$\{b_j > 0,\ 0 < \zeta_1 < \zeta_2 < \cdots < \zeta_K : 1 \le j \le K\},$$

suppose the masses m_k satisfy

$$\frac{\zeta_K^{\frac{k-1}{2}}}{\zeta_1^{\frac{k+1}{2}}} < m_{(k+1)'}m_{k'}, \qquad \textit{for all odd } k, \quad 1 \le k \le 2K-1, \tag{5.6a}$$

$$\frac{m_{(k+2)'}m_{(k+1)'}}{(1+m_{(k+1)'}^2\zeta_1)(1+m_{(k+2)'}^2\zeta_1)} < \frac{\zeta_1^{\frac{k+1}{2}}}{\zeta_K^{\frac{k-1}{2}}}\frac{2\min_j(\zeta_{j+1}-\zeta_j)^{k-1}}{(k+1)(\zeta_K-\zeta_1)^{k+1}}, \qquad \textit{for all odd } k, \quad 1 \le k \le 2K-3. \tag{5.6b}$$

Then the positions obtained from inverse formulas (4.30a), (4.30b) *are ordered $x_1 < x_2 < \cdots < x_{2K}$ and the multipeakon solutions* (5.1) *exist for arbitrary $t \in R$.*

Proof. The solutions described in Theorem 4.21 are valid peakon solutions as long as $x_1 < x_2 < \cdots < x_{2K}$ holds. We write these conditions as:

$$x_{(k+1)'} < x_{k'}, \qquad \text{for all odd } k, \qquad 1 \le k \le 2K-1, \tag{5.7a}$$
$$x_{(k+2)'} < x_{(k+1)'}, \qquad \text{for all odd } k, \qquad 1 \le k \le 2K-3, \tag{5.7b}$$

and use Eqs. (4.30a), (4.30b) to obtain equivalent conditions

$$\frac{1}{m_{(k+1)'}m_{k'}} < \frac{\mathcal{D}_{k+1}^{(\frac{k+1}{2},1)}\mathcal{D}_{k-1}^{(\frac{k-1}{2},0)}}{\mathcal{D}_{k+1}^{(\frac{k+1}{2},0)}\mathcal{D}_{k-1}^{(\frac{k-1}{2},1)}}, \qquad \text{for all odd } k, \qquad 1 \le k \le 2K-1, \tag{5.8a}$$
$$\frac{1}{m_{(k+2)'}m_{(k+1)'}} < \frac{\mathcal{D}_{k+2}^{(\frac{k+1}{2},1)}\mathcal{D}_{k}^{(\frac{k+1}{2},0)}}{\mathcal{D}_{k+2}^{(\frac{k+3}{2},0)}\mathcal{D}_{k}^{(\frac{k-1}{2},1)}}, \qquad \text{for all odd } k, \qquad 1 \le k \le 2K-3. \tag{5.8b}$$

Note that Eq. (5.3) implies easily that the inequality

$$\frac{\mathcal{D}_{k+1}^{(\frac{k+1}{2},1)}\mathcal{D}_{k-1}^{(\frac{k-1}{2},0)}}{\mathcal{D}_{k+1}^{(\frac{k+1}{2},0)}\mathcal{D}_{k-1}^{(\frac{k-1}{2},1)}} > \frac{\zeta_1^{\frac{k+1}{2}}}{\zeta_K^{\frac{k-1}{2}}} \tag{5.9}$$

holds uniformly in t (we recall that the coefficients b_j depend on t). Thus if we impose

$$\frac{1}{m_{(k+1)'}m_{k'}} < \frac{\zeta_1^{\frac{k+1}{2}}}{\zeta_K^{\frac{k-1}{2}}} \text{ for all odd } k, \qquad k \le 2K-1,$$

then Eq. (5.7a) hold automatically.

Now we turn to the second inequality, namely (5.8b), which is needed whenever $K \ge 2$. It is convenient to consider a slightly more general expression, namely,

$$\frac{\mathcal{D}_{k+2}^{(l,1)}\mathcal{D}_{k}^{(l,0)}}{\mathcal{D}_{k+2}^{(l+1,0)}\mathcal{D}_{k}^{(l-1,1)}}, \qquad 1 \le l \le K-1,$$

for which after using (5.3) we obtain the inequality

$$\frac{\mathcal{D}_{k+2}^{(l,1)}\mathcal{D}_{k}^{(l,0)}}{\mathcal{D}_{k+2}^{(l+1,0)}\mathcal{D}_{k}^{(l-1,1)}} > \frac{\zeta_1^{l}}{\zeta_K^{l-1}} \frac{\displaystyle\sum_{A,B\in\binom{[1,K]}{l}} \frac{\Delta_A^2\Delta_B^2\mathbf{b}_A\mathbf{b}_B}{\Gamma_{[1,k+2],A}\Gamma_{[1,k],B}}}{\displaystyle\sum_{I\in\binom{[1,K]}{l-1},J\in\binom{[1,K]}{l+1}} \frac{\Delta_I^2\Delta_J^2\mathbf{b}_I\mathbf{b}_J}{\Gamma_{[1,k],I}\Gamma_{[1,k+2],J}}}, \tag{5.10}$$

where, to ease off notation, we temporarily suspended displaying the dependence on $\zeta, \mathbf{e}$.

We focus now on rewriting the denominator of the above expression. First, we note that since the cardinality of J exceeds that of I it is always possible to find a unique smallest index i in J which is not in I. This leads to the map:

$$\Phi : \binom{[1,K]}{l-1} \times \binom{[1,K]}{l+1} \longrightarrow \binom{[1,K]}{l} \times \binom{[1,K]}{l},$$
$$(I,J) : \longmapsto (A = I \cup \{i\},\ B = J \backslash \{i\}), \quad \text{for all } l, \quad l \leq K-1. \tag{5.11}$$

For $A = I \cup \{i\},\ B = J\backslash\{i\}$ we clearly have $A, B \in \binom{[1,K]}{l}$, $\mathbf{b}_I \mathbf{b}_J = \mathbf{b}_A \mathbf{b}_B$, as well as

$$\begin{aligned}
&\frac{\Delta_I^2 \Delta_J^2}{\Gamma_{[1,k],I}\Gamma_{[1,k+2],J}} \\
&= \frac{\Delta_{\{i\},B}^2}{\Delta_{\{i\},I}^2} \cdot \frac{1}{(e_{k+1}+\zeta_i)(e_{k+2}+\zeta_i)} \cdot \frac{\Delta_A^2 \Delta_B^2}{\Gamma_{[1,k],A}\Gamma_{[1,k+2],B}} \\
&\leq \frac{(\zeta_K - \zeta_1)^{2l}}{\min_j (\zeta_{j+1}-\zeta_j)^{2(l-1)}} \cdot \frac{1}{(e_{k+1}+\zeta_1)(e_{k+2}+\zeta_1)} \frac{\Delta_A^2 \Delta_B^2}{\Gamma_{[1,k],A}\Gamma_{[1,k+2],B}}
\end{aligned}$$

which implies

$$\begin{aligned}
&\sum_{I \in \binom{[1,K]}{l-1}, J \in \binom{[1,K]}{l+1}} \frac{\Delta_I^2 \Delta_J^2 \mathbf{b}_I \mathbf{b}_J}{\Gamma_{[1,k],I}\Gamma_{[1,k+2],J}} \\
&\leq \frac{(\zeta_K - \zeta_1)^{2l}}{\min_j (\zeta_{j+1}-\zeta_j)^{2(l-1)}} \cdot \frac{1}{(e_{k+1}+\zeta_1)(e_{k+2}+\zeta_1)} \\
&\quad \cdot \sum_{\substack{A,B \in \binom{[1,K]}{l} \\ (A,B) \in \mathrm{Image}(\Phi)}} \#[\Phi^{-1}(A,B)] \frac{\Delta_A^2 \Delta_B^2 \mathbf{b}_A \mathbf{b}_B}{\Gamma_{[1,k],A}\Gamma_{[1,k+2],B}},
\end{aligned}$$

where $\#[\Phi^{-1}(A, B)]$ counts the number of pairs (I, J) which are mapped by Φ into the same (A, B). However, by construction, $\#[\Phi^{-1}(A, B)] \leq 1$ for $l = 1$, while for $1 < l$ two distinct pairs $(I_1, J_1) \neq (I_2, J_2)$ are mapped to the same (A, B) if, for the smallest $i_1 \in J_1\backslash I_1$ and the smallest $i_2 \in J_2\backslash I_2$, there exists $L \in \binom{[1,K]}{l-2}$, $M \in \binom{[1,K]}{l}$ such that

$$\begin{aligned}
I_1 &= L \cup \{i_2\}, & J_1 &= M \cup \{i_1\}, \\
I_2 &= L \cup \{i_1\}, & J_2 &= M \cup \{i_2\},
\end{aligned}$$

in which case $A = L \cup \{i_1\} \cup \{i_2\},\ B = M$. Thus $\#[\Phi^{-1}(A, B)]$ is bounded from above by the number of ways we can select an individual entry from A, since once i_1 is selected so is i_2 and M, hence $\#[\Phi^{-1}(A, B)] \leq l$, and

$$\sum_{I\in\binom{[1,K]}{l-1},\,J\in\binom{[1,K]}{l+1}} \frac{\Delta_I^2\Delta_J^2\mathbf{b}_I\mathbf{b}_J}{\Gamma_{[1,k],I}\Gamma_{[1,k+2],J}} \tag{5.12}$$

$$\leq l\,\frac{(\zeta_K-\zeta_1)^{2l}}{\min_j(\zeta_{j+1}-\zeta_j)^{2(l-1)}}\cdot\frac{1}{(e_{k+1}+\zeta_1)(e_{k+2}+\zeta_1)}\cdot\sum_{\substack{A,B\in\binom{[1,K]}{l}\\(A,B)\in\mathrm{Image}(\Phi)}}\frac{\Delta_A^2\Delta_B^2\mathbf{b}_A\mathbf{b}_B}{\Gamma_{[1,k],A}\Gamma_{[1,k+2],B}}$$

$$\leq l\,\frac{(\zeta_K-\zeta_1)^{2l}}{\min_j(\zeta_{j+1}-\zeta_j)^{2(l-1)}}\cdot\frac{1}{(e_{k+1}+\zeta_1)(e_{k+2}+\zeta_1)}\cdot\sum_{A,B\in\binom{[1,K]}{l}}\frac{\Delta_A^2\Delta_B^2\mathbf{b}_A\mathbf{b}_B}{\Gamma_{[1,k],A}\Gamma_{[1,k+2],B}}, \tag{5.13}$$

which, upon substituting into (5.10), proves the bound

$$\begin{aligned}\frac{\mathscr{D}_{k+2}^{(l,1)}\mathscr{D}_k^{(l,0)}}{\mathscr{D}_{k+2}^{(l+1,0)}\mathscr{D}_k^{(l-1,1)}} &> \frac{\zeta_1^l}{\zeta_K^{l-1}}\frac{\min_j(\zeta_{j+1}-\zeta_j)^{2(l-1)}}{l(\zeta_K-\zeta_1)^{2l}}\cdot(e_{k+1}+\zeta_1)(e_{k+2}+\zeta_1)\\ &= \frac{\zeta_1^l}{\zeta_K^{l-1}}\frac{\min_j(\zeta_{j+1}-\zeta_j)^{2(l-1)}}{l(\zeta_K-\zeta_1)^{2l}}\cdot\frac{(1+m_{(k+1)'}^2\zeta_1)(1+m_{(k+2)'}^2\zeta_1)}{m_{(k+1)'}^2m_{(k+2)'}^2}.\end{aligned} \tag{5.14}$$

Hence, setting $l=\frac{k+1}{2}$, if one takes

$$\frac{1}{m_{(k+2)'}m_{(k+1)'}} < \frac{\zeta_1^{\frac{k+1}{2}}}{\zeta_K^{\frac{k-1}{2}}}\frac{2\min_j(\zeta_{j+1}-\zeta_j)^{k-1}}{(k+1)(\zeta_K-\zeta_1)^{k+1}}\cdot\frac{(1+m_{(k+1)'}^2\zeta_1)(1+m_{(k+2)'}^2\zeta_1)}{m_{(k+1)'}^2m_{(k+2)'}^2},$$

then (5.8b) and thus (5.7b) will hold. Finally, rewriting the last condition as:

$$\frac{m_{(k+2)'}m_{(k+1)'}}{(1+m_{(k+1)'}^2\zeta_1)(1+m_{(k+2)'}^2\zeta_1)} < \frac{\zeta_1^{\frac{k+1}{2}}}{\zeta_K^{\frac{k-1}{2}}}\frac{2\min_j(\zeta_{j+1}-\zeta_j)^{k-1}}{(k+1)(\zeta_K-\zeta_1)^{k+1}},\quad \text{for all odd } k,\quad k\leq 2K-3 \tag{5.15}$$

we obtain the second sufficient condition (5.6b). □

As an example illustrating the global existence of our multipeakon solutions let us consider the case $K=2$ (i.e. $n=4$).

Example 5.7. Let $K=2$, and $b_1(0)=10$, $b_2(0)=1$, $\zeta_1=0.3$, $\zeta_2=3$, $m_1=8$, $m_2=16$, $m_3=18$, $m_4=13$. It is easy to show that the condition in Theorem 5.6 is satisfied. Hence the order of $\{x_k, k=1,2,3,4\}$ will be preserved at all time and one can use the explicit formulae for the 4-peakon solution at all time, resulting in the following sequence of graphs (Fig. 1).

5.3. Large time peakon asymptotics for $n=2K$. In this short subsection we state the asymptotic behaviour of multipeakon solutions for large (positive and negative) time, thus implicitly assuming the global existence of solutions as guaranteed for example by imposing sufficient conditions of Theorem 5.6.

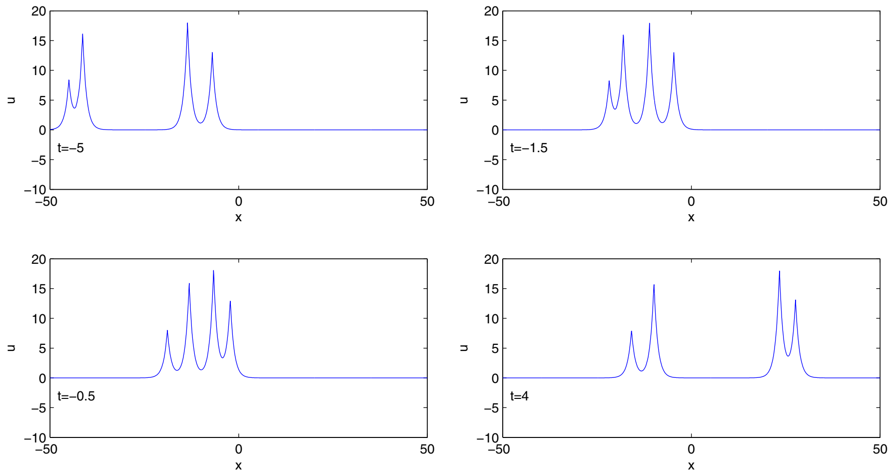

Fig. 1. Snapshots of $u(x,t)$ for $n = 4$ at times $t = -5,\ -1.5,\ -0.5,\ 4$ in the case of $b_1(0) = 10,\ b_2(0) = 1,\ \zeta_1 = 0.3,\ \zeta_2 = 3,\ m_1 = 8,\ m_2 = 16,\ m_3 = 18,\ m_4 = 13$

Theorem 5.8. *Suppose the masses m_j satisfy the conditions of Theorem 5.6. Then the asymptotic position of a k-th (counting from the right) peakon as $t \to +\infty$ is given by*

$$x_{k'} = \frac{2t}{\zeta_{\frac{k+1}{2}}} + \ln \frac{b_{\frac{k+1}{2}}(0)\mathbf{e}_{[1,k-1]}\Delta^2_{[1,\frac{k-1}{2}],\{\frac{k+1}{2}\}}(\zeta)}{m_{k'}\Gamma_{[1,k],\{\frac{k+1}{2}\}}(\mathbf{e};\zeta)\zeta^2_{[1,\frac{k-1}{2}]}} + \mathscr{O}(e^{-\alpha_k t}),$$

$$\text{for some positive } \alpha_k \text{ and odd } k, \tag{5.16a}$$

$$x_{k'} = \frac{2t}{\zeta_{\frac{k}{2}}} + \ln \frac{b_{\frac{k}{2}}(0)\mathbf{e}_{[1,k-1]}\Delta^2_{[1,\frac{k}{2}-1],\{\frac{k}{2}\}}(\zeta)}{m_{k'}\Gamma_{[1,k-1],\{\frac{k}{2}\}}(\mathbf{e};\zeta)\zeta^2_{[1,\frac{k}{2}-1]}\zeta_{\frac{k}{2}}} + \mathscr{O}(e^{-\alpha_k t}),$$

$$\text{for some positive } \alpha_k \text{ and even } k, \tag{5.16b}$$

$$x_{k'} - x_{(k+1)'} = \ln m_{(k+1)'}m_{k'}\zeta_{\frac{k+1}{2}} + \mathscr{O}(e^{-\alpha_k t}),$$

$$\text{for some positive } \alpha_k \text{ and odd } k. \tag{5.16c}$$

Likewise, as $t \to -\infty$, using the notation of Theorem 5.2, the asymptotic position of the k-th peakon is given by

$$x_{k'} = \frac{2t}{\zeta_{(\frac{k+1}{2})^*}} + \ln \frac{b_{(\frac{k+1}{2})^*}(0)\mathbf{e}_{[1,k-1]}\Delta^2_{([1,\frac{k-1}{2}])^*,\{(\frac{k+1}{2})^*\}}(\zeta)}{m_{k'}\Gamma_{[1,k],\{(\frac{k+1}{2})^*\}}(\mathbf{e};\zeta)\zeta^2_{[1,\frac{k-1}{2}]^*}} + \mathscr{O}(e^{\beta_k t}),$$

$$\text{for some positive } \beta_k \text{ and odd } k, \tag{5.17a}$$

$$x_{k'} = \frac{2t}{\zeta_{(\frac{k}{2})^*}} + \ln \frac{b_{(\frac{k}{2})^*}(0)\mathbf{e}_{[1,k-1]}\Delta^2_{[1,\frac{k}{2}-1]^*,\{(\frac{k}{2})^*\}}(\zeta)}{m_{k'}\Gamma_{[1,k-1],\{(\frac{k}{2})^*\}}(\mathbf{e};\zeta)\zeta^2_{[1,\frac{k}{2}-1]^*}\zeta_{(\frac{k}{2})^*}} + \mathscr{O}(e^{\beta_k t}),$$

$$\text{for some positive } \beta_k \text{ and even } k, \tag{5.17b}$$

$$x_{k'} - x_{(k+1)'} = \ln m_{(k+1)'} m_{k'} \zeta_{(\frac{k+1}{2})^*} + \mathcal{O}(e^{\beta_k t}),$$
$$\text{for some positive } \beta_k \text{ and odd } k. \tag{5.17c}$$

Proof. The proof is by a straightforward computation using the formulas for positions (4.30a), (4.30b), as well as asymptotic evaluations of determinants (5.4) and (5.5). □

Remark 5.9. In the closing remark for this section we note that multipeakons of the mCH exhibit *Toda-like sorting properties* of asymptotic speeds, a common occurrence among known to us peakon systems. However, it is apparent that the multipeakons of the mCH show also features known to occur in multi-component cases, for example in the Geng–Xue equation [46], a two-component modified Camassa–Holm equation [10], but also in the Novikov equation [38,39], for which one observes an *asymptotic pairing of peakons*, undoubtedly forced by a shortage of eigenvalues whose total number is K, versus $2K$ positions in need of asymptotic speeds. This feature does not show up in the CH equation.

6. Multipeakons for $n = 2K + 1$

The main source of difference with the even case is of course the presence of the positive shift c which impacts the evaluations of the CSV determinants as illustrated by Theorem 4.17, in particular formula (4.25). We will present the material in this section in a way parallel to the previous section on the even case.

6.1. Closed formulae for $n = 2K + 1$. Again, we assume that $x_1(0) < x_2(0) < \cdots < x_{2K+1}(0)$ then this condition will hold at least in a small interval containing $t = 0$. Thus Theorem 4.21 gives us the following *local existence* result.

Theorem 6.1. *Assuming the notation of Theorem 4.21, the mCH Eq. (1.1) with the regularization of the singular term $u_x^2 m$ given by $\langle u_x^2 \rangle m$ admits the multipeakon solution*

$$u(x,t) = \sum_{k=1}^{2K+1} m_{k'}(t) \exp(-|x - x_{k'}(t)|), \tag{6.1}$$

where $x_{k'}$ are given by Eqs. (4.30a) and (4.30b), with the peakon spectral measure

$$d\mu = \sum_{j=1}^{K} b_j(t) \delta_{\zeta_j}, \tag{6.2}$$

$b_j(t) = b_j(0) e^{\frac{2t}{\zeta_j}}$, $0 < b_j(0)$*, ordered eigenvalues* $0 < \zeta_1 < \cdots < \zeta_K$ *and* $c(t) = c(0) > 0$ *in* (4.26).

Proof. The time evolution of the spectral measure is the same as for the even case. To see this as well as that c is a constant we recall that the Weyl function $W(z)$ is defined in (3.1), regardless of whether n is even or odd, thus $W(z)$ undergoes the time evolution obtained earlier in the proof of Theorem 5.1, namely,

$$\dot{W} = \frac{2}{z} W - \frac{2L}{z},$$

which, in turn, implies $\dot{b}_j = \frac{2}{\zeta_j} b_j$, $1 \le j \le K$ as well as $\dot{c} = 0$ by virtue of Corollary 3.2. The rest of the proof is the same as for the even case. □

We examine now the explicit formulas for the evaluation of CSV determinants presented in Theorem 4.17 (see Eq. (4.26) for notation), with due care to two facts: $n = 2K+1$ and $c > 0$. The proof follows the same steps as in Theorem 5.2 and we omit it.

Theorem 6.2. *Let $n = 2K+1,\ 1 \le k \le 2K+1,\ 0 \le l \le K+1,\ 0 \le p,\ p+l-1 \le k-l$, and let the peakon spectral measure be given by (6.2) and a shift $c > 0$. Then*

(1)

$$\mathscr{D}_k^{(l,p)} = \left|\Delta_{[1,k]}(\mathbf{e})\right| \sum_{I\in\binom{[1,K]}{l}} \frac{\Delta_I^2(\zeta)\mathbf{b}_I\zeta_I^p}{\Gamma_{[1,k],I}(\mathbf{e};\zeta)} \quad \text{if} \quad p+l-1 < k-l, \quad k \le 2K+1; \tag{6.3a}$$

$$\mathscr{D}_k^{(l,p)} = \left|\Delta_{[1,k]}(\mathbf{e})\right| \Big(\sum_{I\in\binom{[1,K]}{l}} \frac{\Delta_I^2(\zeta)\mathbf{b}_I\zeta_I^p}{\Gamma_{[1,k],I}(\mathbf{e};\zeta)} + c \sum_{I\in\binom{[1,K]}{l-1}} \frac{\Delta_I^2(\zeta)\mathbf{b}_I\zeta_I^p}{\Gamma_{[1,k],I}(\mathbf{e};\zeta)} \Big) \quad \text{if} \quad p+l-1 = k-l, \quad k \le 2K+1; \tag{6.3b}$$

with the proviso that the first term inside the bracket is set to zero if $l = K+1$, which only happens when $k = 2K+1$, $p = 0$.

(2) in the asymptotic region $t \to +\infty$

$$\mathscr{D}_k^{(l,p)} = \left|\Delta_{[1,k]}(\mathbf{e})\right| \frac{\Delta_{[1,l]}^2(\zeta)\mathbf{b}_{[1,l]}\zeta_{[1,l]}^p}{\Gamma_{[1,k],[1,l]}(\mathbf{e};\zeta)} \Big[1 + \mathscr{O}(e^{-\alpha t})\Big], \quad 0 < \alpha, \quad \text{if} \quad 0 \le l \le K; \tag{6.4a}$$

$$\mathscr{D}_{2K+1}^{(K+1,0)} = c\left|\Delta_{[1,2K+1]}(\mathbf{e})\right| \frac{\Delta_{[1,K]}^2(\zeta)\mathbf{b}_{[1,K]}}{\Gamma_{[1,2K+1],[1,K]}(\mathbf{e};\zeta)}, \quad \text{if} \quad k = 2K+1, l = K+1, p = 0. \tag{6.4b}$$

(3) in the asymptotic region $t \to -\infty$

$$\mathscr{D}_k^{(l,p)} = \left|\Delta_{[1,k]}(\mathbf{e})\right| \frac{\Delta_{[1,l]^*}^2(\zeta)\mathbf{b}_{[1,l]^*}\zeta_{[1,l]^*}^p}{\Gamma_{[1,k],[1,l]^*}(\mathbf{e};\zeta)} \Big[1 + \mathscr{O}(e^{\beta t})\Big], \quad 0 < \beta, \quad \text{if} \quad p+l-1 < k-l, \quad k \le 2K+1; \tag{6.5a}$$

$$\mathscr{D}_k^{(l,p)} = c\left|\Delta_{[1,k]}(\mathbf{e})\right| \frac{\Delta_{[1,l-1]^*}^2(\zeta)\mathbf{b}_{[1,l-1]^*}\zeta_{[1,l-1]^*}^p}{\Gamma_{[1,k],[1,l-1]^*}(\mathbf{e};\zeta)} \Big[1 + \mathscr{O}(e^{\beta t})\Big], \quad 0 < \beta, \quad \text{if} \quad p+l-1 = k-l, \quad k < 2K+1; \tag{6.5b}$$

$$\mathscr{D}_{2K+1}^{(K+1,0)} = c\left|\Delta_{[1,2K+1]}(\mathbf{e})\right| \frac{\Delta_{[1,K]}^2(\zeta)\mathbf{b}_{[1,K]}}{\Gamma_{[1,2K+1],[1,K]}(\mathbf{e};\zeta)}, \quad \text{if} \quad k = 2K+1, l = K+1, p = 0, \tag{6.5c}$$

where, as before, $[1,l]^* = [l^* = K-l+1, 1^* = K]$.

For the future use, namely in the forthcoming proof of Theorem 6.7, we will formulate an elementary corollary aimed at comparing formulae with $c > 0$ and $c = 0$. For the duration of this corollary we explicitly display the dependence on c.

Corollary 6.3. *Let* $n = 2K+1$, $1 \leq k \leq 2K+1$, $0 \leq l \leq K+1$, $0 \leq p$, $p+l-1 \leq k-l$, *and let the peakon spectral measure be given by* (6.2) *and a shift* $c > 0$. *Then*

$$\mathscr{D}_k^{(l,p)}(c) = \mathscr{D}_k^{(l,p)}(0), \qquad \text{if} \quad p+l-1 < k-l, \quad k \leq 2K+1; \tag{6.6a}$$

$$\mathscr{D}_k^{(l,p)}(c) = \mathscr{D}_k^{(l,p)}(0) + c\mathscr{D}_k^{(l-1,p)}(0), \qquad \text{if} \quad p+l-1 = k-l, \quad k \leq 2K+1; \tag{6.6b}$$

with the convention that the first term in (6.6b) *is set to zero if* $l = K+1, k = 2K+1, p = 0$.

Proof. It suffices to compare formulas (6.3a) and (6.3b) with (5.3), for $k \leq 2K$, while the case $k = 2K+1$ can be directly obtained from Theorem 4.17 and the definition of $\mathscr{D}_k^{(l,p)}(c)$ (see Eq. (4.26)). □

Finally, by use of the formulas (4.30a), (4.30b), Theorems 6.1 and 6.2, in particular the formulas (6.3a) and (6.3b), we get exact formulae for (local) 1,3-peakon solutions with initial positions satisfying $x_1(0) < x_2(0) < \cdots < x_{2K+1}(0)$.

Example 6.4. (1-peakon solution; $K = 0$ (trivial, does not require inverse spectral machinery))

$$x_1 = \ln\left(\frac{c}{m_1}\right).$$

We note that by shifting this example covers the 1-peakon solution discussed in Theorem 6.1 in [30]. More generally, by shifting we cover all peakon solutions discussed therein for which masses are taken to be identical (see Remark 1.1 in the Introduction).

Example 6.5. (3-peakon solution; $K = 1$)

$$x_1 = \ln\left(\frac{b_1 c}{\zeta_1 m_1\left(b_1\zeta_1 m_2^2 m_3^2 + c(1+\zeta_1 m_2^2)(1+\zeta_1 m_3^2)\right)}\right),$$
$$x_2 = \ln\left(\frac{b_1 m_2}{b_1\zeta_1 m_2^2 m_3^2 + c(1+\zeta_1 m_2^2)(1+\zeta_1 m_3^2)}\left(\frac{b_1 m_3^2}{1+\zeta_1 m_3^2} + c\right)\right),$$
$$x_3 = \ln\left(\frac{1}{m_3}\left(\frac{b_1 m_3^2}{1+\zeta_1 m_3^2} + c\right)\right).$$

Example 6.6. (a general formula for the last position $x_{1'} = x_{2K+1}$) Recalling that $\hat{\mu}_c$ denotes the shifted Stieltjes transform of the spectral measure μ (introduced in Definition 4.16) and using (4.30a) we obtain:

$$x_{1'} = x_{2K+1} = \ln\frac{\hat{\mu}_c(e_1)}{m_{1'}} = \ln\frac{c + m_{2K+1}^2\sum_{i=1}^{K}\frac{b_i}{1+m_{2K+1}^2\zeta_i}}{m_{2K+1}}.$$

6.2. Global existence for $n = 2K + 1$. This section presents the main results regarding the global existence of peakon solutions when $n = 2K + 1$.

Theorem 6.7. *Given arbitrary spectral data*

$$\{b_j > 0,\ 0 < \zeta_1 < \zeta_2 < \cdots < \zeta_K,\ c > 0 : 1 \le j \le K\},$$

suppose the masses m_k satisfy

$$\frac{1}{m_{(k+1)'}m_{k'}} < \frac{\zeta_1^{\frac{k+1}{2}}}{\zeta_K^{\frac{k-1}{2}}} min\{1, \hat{\beta}\}, \qquad \textit{for all odd } k, \quad 1 \le k \le 2K-1, \tag{6.7a}$$

$$\frac{1}{m_{(k+2)'}m_{(k+1)}} < \frac{\zeta_1^{\frac{k+1}{2}}}{\zeta_K^{\frac{k-1}{2}}} min\{1, \hat{\beta}_1\}, \qquad \textit{for all odd } k, \quad 1 \le k \le 2K-1, \tag{6.7b}$$

where

$$\hat{\beta} = \begin{cases} \frac{2\zeta_K\, min_j(\zeta_{j+1}-\zeta_j)^{k-3}}{\zeta_1(k-1)(\zeta_K-\zeta_1)^{k-1}} \frac{(1+m^2_{(k)'}\zeta_1)(1+m^2_{(k+1)'}\zeta_1)}{m^2_{(k)'}m^2_{(k+1)'}}, & \textit{for all odd } k, \quad 3 \le k \le 2K-1, \\ +\infty, & \textit{for } k = 1, \end{cases}$$

$$\hat{\beta}_1 = \frac{2\, min_j(\zeta_{j+1}-\zeta_j)^{k-1}}{(k+1)(\zeta_K-\zeta_1)^{k+1}} \frac{(1+m^2_{(k+1)'}\zeta_1)(1+m^2_{(k+2)'}\zeta_1)}{m^2_{(k+1)'}m^2_{(k+2)'}}.$$

Then the positions obtained from inverse formulas (4.30a)*,* (4.30b) *are ordered $x_1 < x_2 < \cdots < x_{2K+1}$ and the multipeakon solutions* (5.1) *exist for arbitrary $t \in \mathbf{R}$.*

Proof. The solutions described in Theorem 4.21 are valid peakon solutions as long as $x_1 < x_2 < \cdots < x_{2K+1}$ holds. We write these conditions as:

$$x_{(k+1)'} < x_{k'}, \qquad \text{for all odd } k, \quad 1 \le k \le 2K-1, \tag{6.8a}$$

$$x_{(k+2)'} < x_{(k+1)'}, \qquad \text{for all odd } k, \quad 1 \le k \le 2K-1, \tag{6.8b}$$

and use Eqs. (4.30a), (4.30b) to obtain equivalent conditions

$$\frac{1}{m_{(k+1)'}m_{k'}} < \frac{\mathscr{D}_{k+1}^{(\frac{k+1}{2},1)}(c)\mathscr{D}_{k-1}^{(\frac{k-1}{2},0)}(c)}{\mathscr{D}_{k+1}^{(\frac{k+1}{2},0)}(c)\mathscr{D}_{k-1}^{(\frac{k-1}{2},1)}(c)}, \qquad \text{for all odd } k, \quad 1 \le k \le 2K-1, \tag{6.9a}$$

$$\frac{1}{m_{(k+2)'}m_{(k+1)'}} < \frac{\mathscr{D}_{k+2}^{(\frac{k+1}{2},1)}(c)\mathscr{D}_{k}^{(\frac{k+1}{2},0)}(c)}{\mathscr{D}_{k+2}^{(\frac{k+3}{2},0)}(c)\mathscr{D}_{k}^{(\frac{k-1}{2},1)}(c)}, \qquad \text{for all odd } k, \quad 1 \le k \le 2K-1, \tag{6.9b}$$

displaying the dependence on c in anticipation of the use of Corollary 6.3. Note that Eqs. (6.6a) and (6.6b) suggest writing

$$\frac{\mathscr{D}_{k+1}^{(\frac{k+1}{2},1)}(c)\mathscr{D}_{k-1}^{(\frac{k-1}{2},0)}(c)}{\mathscr{D}_{k+1}^{(\frac{k+1}{2},0)}(c)\mathscr{D}_{k-1}^{(\frac{k-1}{2},1)}(c)} = \frac{\big(\mathscr{D}_{k+1}^{(\frac{k+1}{2},1)}(0)+c\mathscr{D}_{k+1}^{(\frac{k-1}{2},1)}(0)\big)\mathscr{D}_{k-1}^{(\frac{k-1}{2},0)}(0)}{\mathscr{D}_{k+1}^{(\frac{k+1}{2},0)}(0)\big(\mathscr{D}_{k-1}^{(\frac{k-1}{2},1)}(0)+c\mathscr{D}_{k-1}^{(\frac{k-3}{2},1)}(0)\big)}$$
$$= \frac{\mathscr{D}_{k+1}^{(\frac{k+1}{2},1)}(0)\mathscr{D}_{k-1}^{(\frac{k-1}{2},0)}(0)+c\mathscr{D}_{k+1}^{(\frac{k-1}{2},1)}(0)\mathscr{D}_{k-1}^{(\frac{k-1}{2},0)}(0)}{\mathscr{D}_{k+1}^{(\frac{k+1}{2},0)}(0)\mathscr{D}_{k-1}^{(\frac{k-1}{2},1)}(0)+c\mathscr{D}_{k+1}^{(\frac{k+1}{2},0)}(0)\mathscr{D}_{k-1}^{(\frac{k-3}{2},1)}(0)} \stackrel{def}{=} \frac{\mathscr{A}_1+\mathscr{B}_1}{\mathscr{A}_2+\mathscr{B}_2},$$

with the proviso that $\mathscr{B}_2 = 0$ for $k = 1$. Examining the ratios $\frac{\mathscr{A}_1}{\mathscr{A}_2}, \frac{\mathscr{B}_1}{\mathscr{B}_2}$ we observe that they satisfy (uniform in t) bounds

$$\frac{\mathscr{A}_1}{\mathscr{A}_2} > \frac{\zeta_1^{\frac{k+1}{2}}}{\zeta_K^{\frac{k-1}{2}}} \stackrel{def}{=} \alpha,$$
$$\frac{\mathscr{B}_1}{\mathscr{B}_2} > \frac{\zeta_1^{\frac{k-1}{2}}}{\zeta_K^{\frac{k-3}{2}}} \frac{2\min_j(\zeta_{j+1}-\zeta_j)^{k-3}}{(k-1)(\zeta_K-\zeta_1)^{k-1}} \frac{(1+m_{k'}^2\zeta_1)(1+m_{(k+1)'}^2\zeta_1)}{m_{k'}^2 m_{(k+1)'}^2} \stackrel{def}{=} \beta,$$

by Eqs. (5.9) and (5.14), respectively, with the convention that $\beta = \infty$ for the special case $k = 1$. Thus

$$\min\{\alpha,\beta\} < \frac{\mathscr{D}_{k+1}^{(\frac{k+1}{2},1)}(c)\mathscr{D}_{k-1}^{(\frac{k-1}{2},0)}(c)}{\mathscr{D}_{k+1}^{(\frac{k+1}{2},0)}(c)\mathscr{D}_{k-1}^{(\frac{k-1}{2},1)}(c)}$$

holds uniformly in t and if we impose

$$\frac{1}{m_{(k+1)'}m_{k'}} < \min\{\alpha,\beta\} \qquad \text{for all odd } k, \qquad 1 \le k \le 2K-1,$$

then Eq. (6.8a) will hold automatically.

Now we turn to the second inequality, namely (6.9b). Again, using Corollary 6.3 we obtain

$$\frac{\mathscr{D}_{k+2}^{(\frac{k+1}{2},1)}(c)\mathscr{D}_{k}^{(\frac{k+1}{2},0)}(c)}{\mathscr{D}_{k+2}^{(\frac{k+3}{2},0)}(c)\mathscr{D}_{k}^{(\frac{k-1}{2},1)}(c)} > \frac{\mathscr{A}_1+\mathscr{B}_1}{\mathscr{A}_2+\mathscr{B}_2}, \tag{6.10}$$

where, this time,

$$\frac{\mathscr{A}_1}{\mathscr{A}_2} > \frac{\zeta_1^{\frac{k+1}{2}}}{\zeta_K^{\frac{k-1}{2}}} = \alpha,$$
$$\frac{\mathscr{B}_1}{\mathscr{B}_2} > \frac{\zeta_1^{\frac{k+1}{2}}}{\zeta_K^{\frac{k-1}{2}}} \frac{2\min_j(\zeta_{j+1}-\zeta_j)^{k-1}}{(k+1)(\zeta_K-\zeta_1)^{k+1}} \frac{(1+m_{(k+1)'}^2\zeta_1)(1+m_{(k+2)'}^2\zeta_1)}{m_{(k+1)'}^2 m_{(k+2)'}^2} \stackrel{def}{=} \beta_1,$$

and,

$$\min\{\alpha,\beta_1\} < \frac{\mathscr{D}_{k+2}^{(\frac{k+1}{2},1)}(c)\mathscr{D}_{k}^{(\frac{k+1}{2},0)}(c)}{\mathscr{D}_{k+2}^{(\frac{k+3}{2},0)}(c)\mathscr{D}_{k}^{(\frac{k-1}{2},1)}(c)}$$

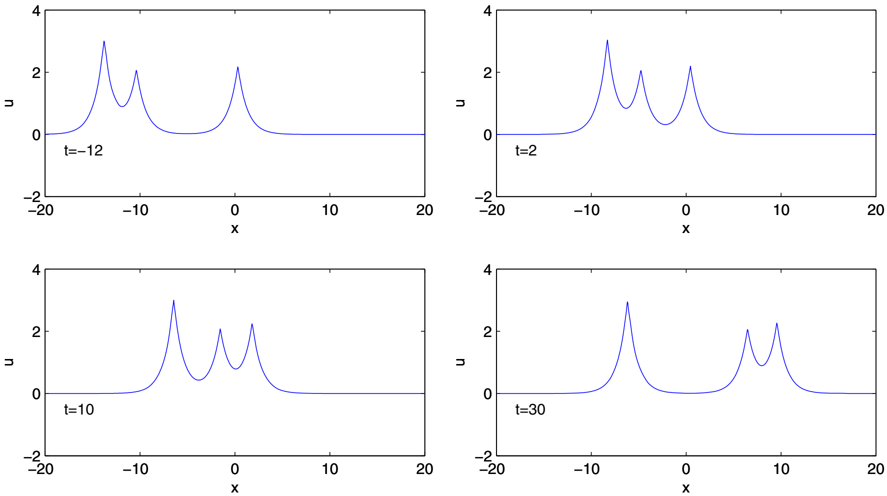

Fig. 2. Snapshots of $u(x,t)$ for $n=3$ at time $t=-12,\ 2,\ 10,\ 30$ in the case of $b_1(0)=1,\ c=3,\ \zeta_1=5,\ m_1=3,\ m_2=2,\ m_3=2.2$

is satisfied. Thus inequality

$$\frac{1}{m_{(k+2)'}m_{(k+1)'}} < \min\{\alpha, \beta_1\}, \qquad \text{for all odd } k, \qquad 1 \le k \le 2K-1,$$

implies (6.9b) and, consequently, (6.8b), thereby completing the proof. □

Example 6.8. Let $K=1$, and $b_1(0)=1,\ c=3,\ \zeta_1=5,\ m_1=3,\ m_2=2,\ m_3=2.2$. Then the sufficient conditions in Theorem 6.7 are satisfied. Hence the order of $\{x_k, k=1,2,3\}$ will be preserved at all time and one can use the explicit formulae for the 3-peakon solution at all time, resulting in the following sequence of graphs (Fig. 2).

6.3. Large time peakon asymptotics for $n=2K+1$. We will investigate in this section the long time asymptotics of global multipeakon solutions, guaranteed to exist by Theorem 6.7.

Theorem 6.9. *Suppose the masses m_j satisfy the conditions of Theorem 6.7. Then the asymptotic position of a k-th (counting from the right) peakon as $t \to +\infty$ is given by*

$$x_{k'} = \frac{2t}{\zeta_{\frac{k+1}{2}}} + \ln \frac{b_{\frac{k+1}{2}}(0)\mathbf{e}_{[1,k-1]}\Delta^2_{[1,\frac{k-1}{2}],\{\frac{k+1}{2}\}}(\zeta)}{m_{k'}\Gamma_{[1,k],\{\frac{k+1}{2}\}}(\mathbf{e};\zeta)\zeta^2_{[1,\frac{k-1}{2}]}} + \mathscr{O}(e^{-\alpha_k t}),$$

for some positive α_k and odd $k \le 2K-1$; (6.11a)

$$x_{(2K+1)'} = \ln \frac{c\mathbf{e}_{[1,2K]}}{m_{(2K+1)'}\zeta^2_{[1,K]}} + \mathscr{O}(e^{-\alpha t}), \text{ for some positive } \alpha; \qquad (6.11b)$$

$$x_{k'} = \frac{2t}{\zeta_{\frac{k}{2}}} + \ln \frac{b_{\frac{k}{2}}(0)\mathbf{e}_{[1,k-1]}\Delta^2_{[1,\frac{k}{2}-1],\{\frac{k}{2}\}}(\zeta)}{m_{k'}\Gamma_{[1,k-1],\{\frac{k}{2}\}}(\mathbf{e};\zeta)\zeta^2_{[1,\frac{k}{2}-1]}\zeta_{\frac{k}{2}}} + \mathscr{O}(e^{-\alpha_k t}),$$

$$\textit{for some positive } \alpha_k \textit{ and even } k \leq 2K; \tag{6.11c}$$

$$x_{k'} - x_{(k+1)'} = \ln m_{(k+1)'} m_{k'} \zeta_{\frac{k+1}{2}} + \mathcal{O}(e^{-\alpha_k t}),$$
$$\textit{for some positive } \alpha_k \textit{ and odd } k \leq 2K - 1. \tag{6.11d}$$

Likewise, as $t \to -\infty$, using the notation of Theorem 5.2, the asymptotic position of the k-th peakon is given by

$$x_{k'} = \frac{2t}{\zeta_{(\frac{k-1}{2})^*}} + \ln \frac{b_{(\frac{k-1}{2})^*}(0)\mathbf{e}_{[1,k-1]}\Delta^2_{[1,\frac{k-1}{2}-1]^*,\{(\frac{k-1}{2})^*\}}(\zeta)}{m_{k'}\Gamma_{[1,k-1],\{(\frac{k-1}{2})^*\}}(\mathbf{e};\zeta)\zeta^2_{[1,\frac{k-1}{2}-1]^*}\zeta_{(\frac{k-1}{2})^*}} + \mathcal{O}(e^{\beta_k t}),$$
$$\textit{for positive } \beta_k \textit{ and odd } 1 < k \leq 2K+1; \tag{6.12a}$$

$$x_{1'} = \ln \frac{c}{m_{1'}} + \mathcal{O}(e^{\beta_k t}), \textit{ for positive } \beta_k; \tag{6.12b}$$

$$x_{k'} = \frac{2t}{\zeta_{(\frac{k}{2})^*}} + \ln \frac{b_{(\frac{k}{2})^*}(0)\mathbf{e}_{[1,k-1]}\Delta^2_{([1,\frac{k}{2}-1])^*,\{(\frac{k}{2})^*\}}(\zeta)}{m_{k'}\Gamma_{[1,k],\{(\frac{k}{2})^*\}}(\mathbf{e};\zeta)\zeta^2_{[1,\frac{k}{2}-1]^*}} + \mathcal{O}(e^{\beta_k t}),$$
$$\textit{for positive } \beta_k \textit{ and even } k; \tag{6.12c}$$

$$x_{k'} - x_{(k+1)'} = \ln m_{(k+1)'} m_{k'} \zeta_{(\frac{k}{2})^*} + \mathcal{O}(e^{\beta_k t}), \textit{ for positive } \beta_k \textit{ and even } k. \tag{6.12d}$$

Proof. The proof is by a straightforward, but tedious, computation using the formulas for positions (4.30a), (4.30b), as well as asymptotic evaluations of determinants presented in Theorem 6.2. □

Remark 6.10. The Toda-like sorting property can also be observed in this case by examining more closely the asymptotic formulae but the pairing mechanism is subtly different. We point out that the constant c is a surrogate of an additional eigenvalue $\zeta_{K+1} = \infty$, which results in the formal asymptotic speed 0. Thus for large positive times the first particle counting from the left comes to a halt, while the remaining $2K$ peakons form pairs, sharing the remaining K eigenvalues. By contrast, for large, negative times, the first particle counting from the right comes to a halt, while the remaining peakons form pairs. This, somewhat intricate, breaking of symmetry is responsible for noticeable asymmetry in the indexing of positions seen when one compares asymptotic formulas for $n = 2K$ with $n = 2K + 1$.

We would like to conclude this section with an application of asymptotic formulas, valid for any n, to the computation of the Sobolev H^1 norm of u which, by a result of [9], is time invariant.

Corollary 6.11. *Suppose masses satisfy conditions guaranteeing the global existence of solutions. Then*

$$||u||^2_{H^1} = 2\sum_{j=1}^{n} m_j^2 + 4\sum_{j=1}^{K} \frac{1}{\zeta_j}. \tag{6.13}$$

Proof. First, as proven in [9], $||u||^2_{H^1} = \sum_{j=1}^n 2m_j u(x_j) = 2\sum_{j=1}^n m_j^2 + 4\sum_{i<j} m_i m_j e^{x_i - x_j}$, where we used the ordering condition $x_i < x_{i+1}$. Since $||u||^2_{H^1}$

is constant we can compute its value using asymptotic formulas. Thus from the asymptotic formulas in Theorem 6.9 (or 5.8 in the even case) we see that the only contribution to the last term above will come from pairs sharing the same asymptotic speeds. In other words,

$$||u||^2_{H^1} = \sum_{j=1}^{n} 2m_j u(x_j) = 2\sum_{j=1}^{n} m_j^2 + \lim_{t\to+\infty} 4\sum_{i=1}^{K} m_{2i}m_{2i+1}e^{x_{2i}-x_{2i+1}} = 2\sum_{j=1}^{n} m_j^2 + 4\sum_{i=1}^{K}\frac{1}{\zeta_i},$$

again, by asymptotic formulas of Theorem 6.9, 5.8, respectively. □

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Appendix A. Lax Pair for the mCH Peakon ODEs

Our technique of solving the peakon ODEs (1.6) hinges on the following steps:

(1) associate a Lax pair to the differential equation in question;
(2) formulate the boundary value problem compatible with the Lax pair;
(3) define the spectral data and its time evolution;
(4) solve the inverse problem of reconstructing the x component of the Lax pair;

One of the essential challenges of this program is to construct a well defined distribution Lax pair, i.e. a distribution version of (2.1), which is ordinarily given in the smooth sector of the equation. The transition from the smooth sector to the distribution sector is not canonical and this appendix addresses the main steps of our construction of the correct distribution Lax pair used in this paper.

Remark A.1. In fact, we started our search for a distribution Lax pair suitable for (1.7). We were, however, led to a different definition of distribution solutions to (1.1) than in [30] or [58]. Even though we do not have a result that would exclude (1.7) as coming from a suitably defined distribution Lax pair using some other way of defining the products of distributions appearing in the Lax pair we can state this: within the class of possible distribution Lax pairs which we will sharply define below no such a pair exists.

Notations:

- Ω_k: the region $x_k(t) < x < x_{k+1}(t)$, where x_k are smooth functions such that $-\infty = x_0(t) < x_1(t) < \cdots < x_n(t) < x_{n+1}(t) = +\infty$.
- PC^∞: the function space consisting of all the piecewise smooth functions $f(x,t)$ such that the restriction of f to each region Ω_k is a smooth function $f_k(x,t)$ defined on an open neighbourhood of Ω_k. Actually, for each fixed t, $f(x,t)$ defines a regular distribution $T_f(t)$ in the class of $\mathscr{D}'(R)$ (for simplicity we will write f instead of T_f). Note that the value of $f(x,t)$ on $x_k(t)$ does not need to be defined.
- $f_x(x_k-,t)$: the left limit of the function $f(x,t)$ at every point x_k, $f_x(x_k+,t)$: the right limit of the function $f(x,t)$ at every point x_k.
- $\big[f\big](x_k,t)$: the jump between $f_x(x_k-,t)$ and $f_x(x_k+,t)$, i.e.

$$\big[f\big](x_k,t) = f(x_k+,t) - f(x_k-,t).$$

- $\langle f\rangle(x_k,t)$: the arithmetic average of $f_x(x_k-,t)$ and $f_x(x_k+,t)$, i.e.

$$\langle f\rangle(x_k,t)=\frac{f(x_k+,t)+f(x_k-,t)}{2}.$$

- f_x, f_t: the ordinary (classical) partial derivative with respect to x, t respectively.
- $D_x f$: the distributional derivative with respect to x.
- $D_t f$: the distributional limit $D_t f(t)=\lim_{a\to 0}\frac{f(t+a)-f(t)}{a}$, .
- we will suppress the t-dependence throughout the remainder of this Appendix; thus $[f](x_k)$ will denote $[f](x_k,t)$ etc.

Then the following identities follow from elementary distributional calculus

$$D_x f=f_x+\sum_{k=1}^{n}[f](x_k)\delta_{x_k}.$$

$$D_t f=f_t-\sum_{k=1}^{n}\dot{x}_k[f](x_k)\delta_{x_k},$$

where $\dot{x}_k=\frac{dx_k}{dt}$.

Moreover, we also have:

$$\begin{aligned}&[fg]=\langle f\rangle[g]+[f]\langle g\rangle, \qquad \langle fg\rangle=\langle f\rangle\langle g\rangle+\frac{1}{4}[f][g],\\&\frac{d}{dt}[f](x_k)=[f_x](x_k)\dot{x}_k+[f_t](x_k),\\&\frac{d}{dt}\langle f\rangle(x_k)=\langle f_x\rangle(x_k)\dot{x}_k+\langle f_t\rangle(x_k),\end{aligned} \tag{A.1}$$

for any $f,g\in PC^\infty$.

It is easy to see that the peakon solution $u(x,t)$ and the corresponding functions $\Psi_1,\ \Psi_2$ belong to the piecewise smooth class PC^∞. Indeed $u, u_x, \Psi_1,\ \Psi_2$ are smooth functions in $x_k<x<x_{k+1}$. However, u is continuous throughout $\mathbf{R}$; by contrast u_x, Ψ_1, Ψ_2 have a jump at each x_k.

Let us now set $\Psi=(\Psi_1,\Psi_2)^T$, and let us consider an overdetermined system

$$D_x\Psi=\frac{1}{2}\hat{L}\Psi, \qquad D_t\Psi=\frac{1}{2}\hat{A}\Psi, \tag{A.2}$$

where

$$\hat{L}=L+2\lambda\left(\sum_{k=1}^{n}m_k\delta_{x_k}\right)M, \tag{A.3}$$

$$\hat{A}=A-2\lambda\left(\sum_{k=1}^{n}m_kQ(x_k)\delta_{x_k}\right)M \tag{A.4}$$

with

$$L=\begin{pmatrix}-1&0\\0&1\end{pmatrix}, \qquad M=\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \qquad A=\begin{pmatrix}4\lambda^{-2}+Q & -2\lambda^{-1}(u-u_x)\\ 2\lambda^{-1}(u+u_x) & -Q\end{pmatrix}$$

and $Q = u^2 - u_x^2$. Note that in view of (A.3) the x-member of the Lax equation (A.2) involves multiplying $M\Psi = (\Psi_2, -\Psi_1)$ by δ_{x_k}. Thus we have to assign some values to Ψ_1, Ψ_2 at x_k. Likewise, for the t-Lax equation (A.4) to be defined as a distribution equation, $u_x^2 M\Psi = (u_x^2\Psi_2, -u_x^2\Psi_1)$ needs to be a multiplier of δ_{x_k}. Thus the values of $u_x^2(x_k)$ need to be assigned as well. Henceforth, we will refer to these assignments as *regularizations*. The compatibility condition $(D_x D_t - D_t D_x)\Psi = 0$ is a geometric condition (the zero curvature condition), and can be written as

$$\big(D_x(\hat{A}) - D_t(\hat{L}) + \frac{1}{2}[\hat{A}, \hat{L}]\big)\Psi = 0,$$

whose invariance includes the transformations $\Psi \to \Psi_R = R\Psi$, $R \in GL(2, \mathbf{R})$. These transformations leave the singular support of m invariant, and we require that the assignment of values to Ψ on the singular support respects that symmetry. Thus we postulate that for every x_k

$$\Psi_R(x_k) = R\Psi(x_k), \qquad R \in GL(2, \mathbf{R}).$$

Furthermore we consider local regularizations, depending only on the right and left hand limits at the points of singular support. In summary we consider regularizations of the form:

$$\Psi(x_k) = \alpha\big[\Psi\big](x_k) + \beta\big\langle\Psi\big\rangle(x_k), \qquad \alpha, \beta \in GL(2, \mathbf{R}), \tag{A.5}$$

which lead, under the invariance assumption, to the condition:

$$\Psi_R(x_k) = \alpha\big[\Psi_R\big](x_k) + \beta\big\langle\Psi_R\big\rangle(x_k) = R\big(\alpha\big[\Psi\big](x_k) + \beta\big\langle\Psi\big\rangle(x_k)\big)$$

valid for every $R \in GL(2, \mathbf{R})$ and resulting in the intertwining conditions

$$\alpha R = R\alpha, \quad \beta R = R\beta, \qquad \forall R \in GL(2, \mathbf{R}). \tag{A.6}$$

Consequently, by Schur's Lemma α and β are scalar matrices. This motivates the next definition.

Definition A.2. An invariant regularization of the Lax pair (A.2) is given by specifying the values of $\alpha, \beta \in \mathbf{R}$ and $Q(x_k) = (u^2 - u_x^2)(x_k)$ in the formulas below

$$\begin{aligned} \Psi(x)\delta_{x_k} &= \Psi(x_k)\delta_{x_k}, \\ \Psi(x_k) &= \alpha\big[\Psi\big](x_k) + \beta\big\langle\Psi\big\rangle(x_k), \\ Q(x)\delta_{x_k} &= Q(x_k)\delta_{x_k}. \end{aligned}$$

Theorem A.3. *Let m be the discrete measure associated to u defined by* (1.3). *Given an invariant regularization in the sense of* A.2 *the distributional Lax pair* (A.2) *is compatible, i.e. $D_t D_x\Psi = D_x D_t\Psi$, if and only if the following conditions hold:*

$$\beta^2 = 4\alpha^2, \tag{A.7a}$$

$$\beta = 1, \tag{A.7b}$$

$$Q(x_k) = \big\langle Q\big\rangle(x_k), \tag{A.7c}$$

$$\dot{m}_k = 0, \tag{A.7d}$$

$$\dot{x}_k = Q(x_k). \tag{A.7e}$$

Proof. The proof proceeds in a similar way to Theorem B.1 in [35] (also see [10]). We highlight the critical steps of the proof. First, we observe that since we are interested only in the behaviour of Lax pairs around the singular points x_k we can *localize* our computations to be carried out only locally on some open neighbourhoods of these points. Moreover, these computations look identical, regardless of the index k. In other words, without loss of generality we can assume $u(x) = m_1 e^{-|x-x_1|}$ for the sake of the computation, thus using $n = 1$, and then in the final step of the proof pass to a general n. With this simplification in mind, assuming invariant regularization A.2, we write Eq. (A.2) as

$$D_x\Psi = \frac{1}{2}L\Psi + \lambda m_1 M\Psi(x_1)\delta_{x_1},$$
$$D_t\Psi = \frac{1}{2}A\Psi - \lambda m_1 Q(x_1)M\Psi(x_1)\delta_{x_1}.$$

In particular, the first equation implies

$$\big[\Psi\big](x_1) = \lambda m_1 M\Psi(x_1). \tag{A.8}$$

The computation of the distribution compatibility condition $D_x D_t\Psi = D_t D_x\Psi$ produces a distribution condition which can be split into the regular and singular parts. The regular part is just the compatibility condition one gets in the smooth sector of the equation and we omit that. The singular part takes the form:

$$[\frac{1}{2}A\Psi](x_1)\delta_{x_1} - \lambda m_1 Q(x_1)M\Psi(x_1)\delta'_{x_1} =$$
$$-\frac{\lambda}{2}m_1 Q(x_1)\,LM\Psi(x_1)\delta_{x_1} + \lambda\big(\dot m_1 M\Psi(x_1) + m_1 M\dot\Psi(x_1)\big)\delta_{x_1} - \lambda m_1 M\Psi(x_1)\dot x_1\delta'_{x_1}.$$

The coefficients at δ'_{x_1} imply Eq. (A.7e), while the coefficients at δ_{x_1} give the condition:

$$[\frac{1}{2}A\Psi](x_1) = -\frac{\lambda}{2}m_1 Q(x_1)\,LM\Psi(x_1) + \lambda\big(\dot m_1 M\Psi(x_1) + m_1 M\dot\Psi(x_1)\big). \tag{A.9}$$

Since the value of $\Psi(x_1)$ is determined uniquely once the coefficients α and β are chosen and the values of $Q(x_k)$ are assigned (fixing a regularization) we can compute the term $\dot\Psi(x_1)$ appearing in (A.9) with the help of Eqs. (A.1), (A.8), the Definition A.2, and (A.7e). After several intermediate elementary steps we obtain:

$$\dot\Psi(x_1) = \{\langle\frac{A}{2}\rangle(x_1) + \frac{\alpha}{\beta}[\frac{A}{2}](x_1) + \lambda m_1(\frac{\beta}{4} - \frac{\alpha^2}{\beta})[\frac{A}{2}](x_1)M + \dot x_1\frac{L}{2}\}\Psi(x_1). \tag{A.10}$$

Likewise, we can express the right hand side of (A.9) by using (A.1), (A.8) and A.2. Again, after some straightforward computations we obtain:

$$[\frac{A}{2}\Psi](x_1) = \{\langle\frac{A}{2}\rangle(x_1)\lambda m_1 M + [\frac{A}{2}](x_1)\frac{1-\alpha\lambda m_1 M}{\beta}\}\Psi(x_1), \tag{A.11}$$

which finally gives us the compatibility condition we have set out to obtain:

$$\lambda m_1\langle\frac{A}{2}\rangle(x_1)M + [\frac{A}{2}](x_1)\frac{1-\alpha\lambda m_1 M}{\beta}$$
$$= -\lambda m_1 Q(x_1)\frac{1}{2}LM + \lambda\dot m_1 M + \lambda\, m_1 M\{\langle\frac{A}{2}\rangle(x_1)$$
$$+ \frac{\alpha}{\beta}[\frac{A}{2}](x_1) + \lambda m_1(\frac{\beta}{4} - \frac{\alpha^2}{\beta})[\frac{A}{2}](x_1)M + Q(x_1)\frac{L}{2}\}.$$

We now summarize the content of (A.12), broken down according to powers of λ, omitting conditions identically satisfied,

(1) λ^{-1}: $\frac{[u_x](x_1)}{\beta} = -2m_1$

(2) λ^1: $\dot{m}_1 = m_1\big(Q(x_1) - \langle Q\rangle(x_1)\big), \quad \dot{m}_1 = -m_1\big(Q(x_1) - \langle Q\rangle(x_1)\big),$

(3) λ^2: $\frac{\beta}{4} - \frac{\alpha^2}{\beta} = 0,$

which imply claims (A.7b), (A.7c), (A.7d), (A.7a) after restoring the number of singular points to n. □

Corollary A.4. *There are only two invariant regularizations of the Lax pair* (2.1) *for the peakon problem of the mCH equation* (1.1)*:*

$$\Psi(x_k) = \Psi(x_k+), \qquad or \qquad \Psi(x_k) = \Psi(x_k-). \tag{A.12}$$

For either of the two regularizations $u_x^2(x_k) = \langle u_x^2\rangle(x_k)$ and in both cases the equations of motion read:

$$\dot{m}_k = 0, \qquad \dot{x}_k = u^2(x_k) - \langle u_x^2\rangle(x_k). \tag{A.13}$$

Remark A.5. In the body of the paper we use both regularizations to define the right and the left boundary value problems.

Remark A.6. Observe that one does not need to specify the values of $u_x(x_k)$.

References

1. Baker, G., Graves-Morris, P.: Padé Approximants. Cambridge University Press, Cambridge (1996)
2. Beals, R., Sattinger, D.H., Szmigielski, J.: Multi-peakons and a theorem of Stieltjes. Inverse Probl. **15**(1), L1–L4 (1999)
3. Beals, R., Sattinger, D.H., Szmigielski, J.: Multipeakons and the classical moment problem. Adv. Math. **154**, 229–257 (2000)
4. Bertola, M., Gekhtman, M., Szmigielski, J.: Cauchy biorthogonal polynomials. J. Approx. Theory **162**(4), 832–867 (2010)
5. Bies, P.M., Górka, P., Reyes, E.G.: The dual modified Korteweg–de Vries–Fokas–Qiao equation: geometry and local analysis. J. Math. Phys. **53**(7), 073710 (2012)
6. Bressan, A., Constantin, A.: Global conservative solutions of the Camassa–Holm equation. Arch. Ration. Mech. Anal. **183**(2), 215–239 (2007)
7. Bressan, A., Constantin, A.: Global dissipative solutions of the Camassa–Holm equation. Anal. Appl. (Singap.) **5**(1), 1–27 (2007)
8. Camassa, R., Holm, D.D.: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71**(11), 1661–1664 (1993)
9. Chang, X., Szmigielski, J.: Lax integrability of the modified Camassa–Holm equation and the concept of peakons. J. Nonlinear Math. Phys. **23**(4), 563–572 (2016)
10. Chang, X.K., Hu, X.B., Szmigielski, J.: Multipeakons of a two-component modified Camassa–Holm equation and the relation with the finite Kac–van Moerbeke lattice. Adv. Math. **299**, 1–35 (2016)
11. Coclite, G.M., Karlsen, K.H.: On the well-posedness of the Degasperis–Procesi equation. J. Funct. Anal. **233**(1), 60–91 (2006)
12. Coclite, G.M., Karlsen, K.H.: On the uniqueness of discontinuous solutions to the Degasperis–Procesi equation. J. Differ. Equ. **234**(1), 142–160 (2007)
13. Constantin, A., Escher, J.: Wave breaking for nonlinear nonlocal shallow water equations. Acta Math. **181**(2), 229–243 (1998)
14. Constantin, A., Lannes, D.: The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations. Arch. Ration. Mech. Anal. **192**(1), 165–186 (2009)
15. Constantin, A., McKean, H.P.: A shallow water equation on the circle. Commun. Pure Appl. Math. **52**(8), 949–982 (1999)
16. Constantin, A., Strauss, W.A.: Stability of peakons. Commun. Pure Appl. Math. **53**(5), 603–610 (2000)

17. Degasperis, A., Procesi, M.: Asymptotic integrability. In: Degasperis, A., Gaeta, G. (eds.) Symmetry and Perturbation Theory (Rome, 1998), pp. 23–37. World Scientific Publishing, River Edge (1999)
18. Dym, H., McKean, H.P.: Gaussian processes, function theory, and the inverse spectral problem. Probability and Mathematical Statistics, vol. 31. Academic Press [Harcourt Brace Jovanovich Publishers], New York (1976)
19. El Dika, K., Molinet, L.: Stability of multi antipeakon–peakons profile. Discrete Contin. Dyn. Syst. Ser. B **12**(3), 561–577 (2009)
20. El Dika, K., Molinet, L.: Stability of multipeakons. Ann. Inst. H. Poincaré Anal. Non Linéaire **26**(4), 1517–1532 (2009)
21. Escher, J., Liu, Y., Yin, Z.: Global weak solutions and blow-up structure for the Degasperis–Procesi equation. J. Funct. Anal. **241**(2), 457–485 (2006)
22. Finck, T., Heinig, G., Rost, K.: An inversion formula and fast algorithms for Cauchy–Vandermonde matrices. Linear Algebra Appl. **183**, 179–191 (1993)
23. Fokas, A.: The Korteweg–de Vries equation and beyond. Acta Appl. Math. **39**(1–3), 295–305 (1995)
24. Fuchssteiner, B.: Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa–Holm equation. Phys. D Nonlinear Phenom. **95**(3), 229–243 (1996)
25. Gantmacher, F.R., Krein, M.G.: Oscillation matrices and kernels and small vibrations of mechanical systems. AMS Chelsea Publishing, Providence, RI, revised edition, 2002. Translation based on the 1941 Russian original, edited and with a preface by Alex Eremenko
26. Gasca, M., Martinez, J., Mühlbach, G.: Computation of rational interpolants with prescribed poles. J. Comput. Appl. Math. **26**(3), 297–309 (1989)
27. Geng, X., Xue, B.: An extension of integrable peakon equations with cubic nonlinearity. Nonlinearity **22**(8), 1847–1856 (2009)
28. Gomez, D., Lundmark, H., Szmigielski, J.: The Canada Day theorem. Electron. J. Combin. **20**(1), 20, 16 (2013)
29. Gončar, A.A., Lopes, G.L.: On Markov's theorem for multipoint Padé approximants. Math. USSR Sb. **34**, 449–459 (1978)
30. Gui, G., Liu, Y., Olver, P., Qu, C.: Wave-breaking and peakons for a modified Camassa–Holm equation. Commun. Math. Phys. **319**(3), 731–759 (2013)
31. Himonas, A., Mantzavinos, D.: The Cauchy problem for the Fokas–Olver–Rosenau–Qiao equation. Nonlinear Anal. Theory Methods Appl. **95**, 499–529 (2014)
32. Himonas, A., Mantzavinos, D.: The Cauchy problem for a 4-parameter family of equations with peakon traveling waves. Nonlinear Anal. **133**, 161–199 (2016)
33. Holden, H., Raynaud, X.: Global dissipative multipeakon solutions of the Camassa–Holm equation. Commun. Partial Differ. Equ. **33**(10–12), 2040–2063 (2008)
34. Holden, H., Raynaud, X.: A numerical scheme based on multipeakons for conservative solutions of the Camassa–Holm equation. In: Benzoni-Gavage, S., Serre, D. (eds.) Hyperbolic Problems: Theory, Numerics, Applications, pp. 873–881. Springer, Berlin (2008)
35. Hone, A.N.W., Lundmark, H., Szmigielski, J.: Explicit multipeakon solutions of Novikov's cubically nonlinear integrable Camassa–Holm type equation. Dyn. Partial Differ. Equ. **6**(3), 253–289 (2009)
36. Johnson, R.S.: Camassa–Holm, Korteweg–de Vries and related models for water waves. J. Fluid Mech. **455**, 63–82 (2002)
37. Kang, J., Liu, X., Olver, P.J., Qu, C.: Liouville correspondence between the modified KdV hierarchy and its dual integrable hierarchy. J. Nonlinear Sci. **26**(1), 141–170 (2016)
38. Kardell, M.: Contributions to the theory of peaked solitons. 2016. Ph.D. thesis, Linköping University (2016)
39. Kardell, M., Lundmark, H.: Peakon solutions of the Novikov equation. Preprint (2016)
40. Karlin, S.: Total Positivity. Vol. I. Stanford University Press, Stanford (1968)
41. Lin, Z., Liu, Y.: Stability of peakons for the Degasperis–Procesi equation. Commun. Pure Appl. Math. **62**(1), 125–146 (2009)
42. Liu, X., Liu, Y., Qu, C.: Orbital stability of the train of peakons for an integrable modified Camassa–Holm equation. Adv. Math. **255**, 1–37 (2014)
43. Liu, Y.: Wave breaking phenomena and stability of peakons for the Degasperis–Procesi equation. In: Trends in Partial Differential Equations, vol. 10 of Adv. Lect. Math. (ALM), pp. 265–293. Int. Press, Somerville (2010)
44. Lundmark, H.: Formation and dynamics of shock waves in the Degasperis–Procesi equation. J. Nonlinear Sci. **17**(3), 169–198 (2007)
45. Lundmark, H., Szmigielski, J.: Degasperis–Procesi peakons and the discrete cubic string. IMRP Int. Math. Res. Pap. **2**, 53–116 (2005)
46. Lundmark, H., Szmigielski, J.: An inverse spectral problem related to the Geng–Xue two-component peakon equation. Mem. Am. Math. Soc. **244**(1155) (2016)

47. Martínez, J., Peña, J.: Factorizations of Cauchy–Vandermonde matrices. Linear Algebra Appl. **284**(1), 229–237 (1998)
48. Martínez, J., Peña, J.: Fast algorithms of Björck–Pereyra type for solving Cauchy–Vandermonde linear systems. Appl. Numer. Math. **26**(3), 343–352 (1998)
49. McKean, H.P.: Breakdown of a shallow water equation. Asian J. Math. **2**(4):867–874, (1998) (Mikio Sato: a great Japanese mathematician of the twentieth century)
50. McKean, H.P.: Fredholm determinants and the Camassa–Holm hierarchy. Commun. Pure Appl. Math. **56**(5), 638–680 (2003)
51. McKean, H.P.: Breakdown of the Camassa–Holm equation. Commun. Pure Appl. Math. **57**(3), 416–418 (2004)
52. Meinguet, J.: On the solubility of the Cauchy interpolation problem. In: Talbot, A. (ed.) Approximation Theory, Academic Press, London (2007)
53. Nikishin, E.M., Sorokin, V.N.: Rational approximations and orthogonality, vol. 92 of Translations of Mathematical Monographs. American Mathematical Society, Providence. Translated from the Russian by Ralph P. Boas (1991)
54. Novikov, V.: Generalizations of the Camassa–Holm equation. J. Phys. A Math. Theor. **42**(34), 342002, 14 (2009)
55. Olver, P., Rosenau, P.: Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support. Phys. Rev. E **53**(2), 1900 (1996)
56. Qiao, Z.: A new integrable equation with cuspons and W/M-shape-peaks solitons. J. Math. Phys. **47**(11), 112701–112900 (2006)
57. Qiao, Z.: New integrable hierarchy, its parametric solutions, cuspons, one-peak solitons, and M/W-shape peak solitons. J. Math. Phys. **48**(8), 082701 (2007)
58. Qiao, Z., Xia, B., Li, J.: Integrable system with peakon, weak kink, and kink-peakon interactional solutions. arXiv:1205.2028 (2012)
59. Qu, C., Liu, X., Liu, Y.: Stability of peakons for an integrable modified Camassa–Holm equation with cubic nonlinearity. Commun. Math. Phys. **322**(3), 967–997 (2013)
60. Schiff, J.: Zero curvature formulations of dual hierarchies. J. Math. Phys. **37**(4), 1928–1938 (1996)
61. Stahl, H.: Existence and uniqueness of rational interpolants with free and prescribed poles. In: Saff, E.B. (ed.) Approximation Theory, Lecture Notes in Mathematics, vol. 1287, pp. 180–208. Springer, Berlin (1987)
62. Stieltjes, T. J.: Œuvres Complètes/Collected Papers. Vol. I, II. Springer, Berlin (1993)
63. Szmigielski, J., Zhou. L.: Colliding peakons and the formation of shocks in the Degasperis–Procesi equation. In: Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science, vol. 469(2158) (2013)
64. Szmigielski, J., Zhou, L.: Peakon–antipeakon interaction in the Degasperis–Procesi equation. Contemp. Math. **593**, 83–107 (2013)

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