



A Conformally Invariant Gap Theorem in Yang–Mills Theory

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Abstract: We show a sharp conformally invariant gap theorem for Yang–Mills connections in dimension 4 by exploiting an associated Yamabe-type problem.

1. Introduction

Let (X^n, g) be a smooth Riemannian manifold, and suppose ∇ is a connection on a smooth vector bundle over X . The Yang–Mills energy associated to ∇ is

$$\int_X |F_\nabla|_g^2 dV_g.$$

Critical points of this functional are known as Yang–Mills connections, and satisfy $D_\nabla^* F_\nabla = 0$. In the case $n = 4$, the Yang–Mills energy admits a special class of critical points, namely those with (anti)self-dual curvature, i.e. $\star F_\nabla = \pm F_\nabla$, known as instantons. Instantons are the key ingredient in developing applications of Yang–Mills theory to four-dimensional topology through Donaldson invariants (cf. [10–12]). When they exist, instantons always have the minimum possible Yang–Mills energy. However, on many interesting bundles where the instantons are understood, there exist non-minimizing Yang–Mills connections (e.g. [5, 23, 27, 31]). Moreover, many basic questions about the structure of Yang–Mills connections beyond instantons remain unanswered, for instance on the allowable energy levels. We note that Bourguignon–Lawson ([7] Theorems C, D) have shown some gap results assuming pointwise smallness of some pieces of the curvature, exploiting primarily the Bochner formula. This was later improved by Min-Oo ([22] Theorems 2, 4) to an L^2 gap theorem, exploiting an ϵ -regularity argument which requires positivity of a certain curvature quantity associated to the background metric. A

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closely related gap result in the presence of a curvature positivity condition was shown by Parker ([24] Proposition 2.2). Later Feehan ([13]) showed a more general L^2 gap theorem, which removes this positivity hypothesis, and again ultimately relies on ϵ -regularity style estimates, where the smallness of Yang–Mills energy is balanced against the Sobolev constant to obtain a key estimate. The main result of this paper improves upon all of these prior gap theorems. In particular, we show a sharp, conformally invariant improvement of these gap theorems that is nontrivial when the Yamabe invariant $Y([g])$ of (X^4, g) (see [21]) is positive.

Theorem 1.1. *Let (X^4, g) be a closed, oriented four-manifold. Suppose ∇ is a Yang–Mills connection on a vector bundle E over X^4 with structure group $G \subset \text{SO}(E)$, and curvature F_∇ . Then one of the following must hold:*

- (1) $F_\nabla^+ \equiv 0$; or
- (2) F_∇^+ satisfies

$$Y([g]) \leq 3\gamma_1 \|F_\nabla^+\|_{L^2} + 2\sqrt{6} \|W^+\|_{L^2}, \tag{1.1}$$

where $\gamma_1 = \gamma_1(E) \leq \frac{4}{\sqrt{6}}$ is a constant which depends on the structure group of the bundle (see Definition 2.3 below), and W^+ is the self-dual Weyl tensor. Moreover, if equality holds in (1.1) then $[g]$ admits a Yamabe metric \bar{g} with respect to which W^+ has constant norm, $\nabla_{\bar{g}} F_{\nabla_{\bar{g}}}^+ \equiv 0$, and

$$R_{\bar{g}} - 2\sqrt{6} |W^+|_{\bar{g}} = 3\gamma_1 |F_\nabla^+|_{\bar{g}}, \tag{1.2}$$

where $R_{\bar{g}}$ is the scalar curvature of \bar{g} . Furthermore, if $\gamma_1 > 0$ we must have $b_2^+(X^4) = 0$, whereas if $\gamma_1 = 0$ then all harmonic self-dual forms are parallel.

The key idea to prove the inequality (1.1) is to interpret a certain Böchner estimate for F_∇^+ in light of the modified Yamabe problem introduced in [16]. This technique has been used in various contexts to prove sharp L^2 -curvature estimates under topological and geometric assumptions (see [17, 18, 20]). In each of these applications one studies a generalization of the Yamabe problem in which the scalar curvature is modified by adding a conformal density of the correct weight. Carrying this method out requires a number of sharp linear algebraic estimates carried out in Sect. 2.1, as well as a sharp improved Kato inequality for bundle-valued differential forms shown in Sect. 2.2.

The estimate (1.1) is sharp, as illustrated in a key geometric situation. First, note that as follows from Lemmas 2.4 and 2.6, one has the universal bound $\gamma_1 \leq \frac{4}{\sqrt{6}}$. Using this together with explicit calculations for the Yamabe constant and the Yang–Mills energy as spelled out in Sect. 3.2, one sees that the classic example of $\text{SU}(2)$ ADHM/BPST instantons on $(\mathbb{S}^4, g_{\mathbb{S}^4})$ ([1, 3]) yields equality in (1.1). Moreover, the existence of a conformally related metric for which the curvature is parallel reflects the fact that all these instantons are determined by the action of the conformal group on the unique $\text{SO}(4)$ -invariant ADHM/BPST instanton (the “standard” ADHM/BPST instanton) which has parallel curvature.

We note here that it may be possible to extend Theorem 1.1 to the case when the underlying Riemannian manifold is complete. Previous results in this direction have been shown ([9, 30, 35]), which rely on a curvature positivity condition (see also [15]). As our proof relies on solving a kind of modified Yamabe problem, extending it to the complete setting requires knowledge of the asymptotics of the underlying metric and

the given Yang–Mills connection. This makes the sharp statement in this direction not completely clear, and we do not pursue this further here.

Theorem 1.1 has some immediate corollaries, which we collect below. First, using the characteristic class formula from Chern–Weil theory we derive a lower bound which we make explicit in the case of conformally flat metrics, which includes of course the round sphere. Here we take the convention following ([12] (2.1.40)) for the meaning of the characteristic number $\kappa(E)$. Note in particular that $\kappa(E) = c_2(E)$ for $SU(r)$ bundles. However, our metric convention differs from ([12]), see (Sect. 3.2, (3.17)). To make things concrete, estimates (1.4) and (1.5) below are saying for instance on a $|\kappa(E)| = 1$ bundle, any Yang–Mills connection which is not an instanton must have at least *three* units of charge, where intuitively one unit each of SD/ASD charge cancel out to preserve the characteristic class condition.

Corollary 1.2. *Let (X^4, g) be a closed, oriented, conformally flat four-manifold with $Y([g]) > 0$. Suppose ∇ is a Yang–Mills connection on a vector bundle E over X^4 with structure group $G \subset SO(E)$, and curvature F_∇ . Then ∇ is either an instanton, or satisfies*

$$\int_X |F_\nabla|_g^2 dV_g \geq 16\pi^2 |\kappa(E)| + \frac{2Y([g])^2}{9\gamma_1^2} \geq 16\pi^2 |\kappa(E)| + \frac{Y([g])^2}{12}. \tag{1.3}$$

In particular,

1. For $E \rightarrow (\mathbb{S}^4, g_{\mathbb{S}^4})$ an $SU(2)$ bundle, a Yang–Mills connection ∇ is either an instanton, or satisfies

$$\int_{\mathbb{S}^4} |F_\nabla|_g^2 dV_g \geq 16\pi^2 |\kappa(E)| + 32\pi^2. \tag{1.4}$$

2. For $E \rightarrow (\mathbb{S}^4, g_{\mathbb{S}^4})$ an $SO(3)$ bundle, a Yang–Mills connection ∇ is either an instanton, or satisfies

$$\int_{\mathbb{S}^4} |F_\nabla|_g^2 dV_g \geq 16\pi^2 |\kappa(E)| + 64\pi^2. \tag{1.5}$$

Another application of Theorem 1.1 is to the Yang–Mills flow. Fundamental work of Chen–Shen [8], Struwe, [34], Schlatter [29], Kozono–Maeda–Naito [19] shows that finite time singularities of the Yang–Mills flow occur via energy concentration, and moreover bubbling limits can be constructed which are Yang–Mills connections over $(\mathbb{S}^4, g_{\mathbb{S}^4})$. Although not explicitly stated, an immediate corollary of those works is a global existence and convergence statement for connections over the trivial bundle with sufficiently small energy, where this constant depends on a gap theorem for connections on \mathbb{S}^4 . Thus Theorem 1.1 gives a statement about Yang–Mills flow that is much stronger than that previously attainable, yielding a computable universal threshold below which the flow must exist globally and converge. In the case of $(\mathbb{S}^4, g_{\mathbb{S}^4})$ the limit is flat, and the corollary gives the sharp energy inequality, which guarantees convergence to a flat connection. Note that the hypothesis that the bundle is trivial is a necessary consequence of the energy upper bound hypothesis, but we state it for clarity.

Corollary 1.3. *Let (X^4, g) be a closed, oriented, four-manifold. Suppose $E \rightarrow X^4$ is a trivial bundle with structure group $G \subset \text{SO}(E)$. Suppose ∇ is a connection on E with curvature F_∇ satisfying*

$$\int_X |F_\nabla|^2 dV_g < 16\pi^2. \tag{1.6}$$

Then the solution to Yang–Mills flow with initial condition ∇ exists on $[0, \infty)$, and converges as $t \rightarrow \infty$ to a Yang–Mills connection. If $(X^4, g) \cong (\mathbb{S}^4, g_{\mathbb{S}^4})$ then the limit is flat.

2. Preliminary Inequalities

In this section we establish preliminary inequalities necessary for the proof of Theorem 1.1. In particular, in Sect. 2.1 we show sharp matrix inequalities needed to estimate non-linear terms which arise. We also establish an improved Kato inequality for Lie algebra valued 2-forms in Sect. 2.2 which yields favorable inequalities for small powers of $|F_\nabla^+|$. To set the stage, we restate the fundamental Böchner formula here for convenience. First let us fix certain conventions. For sections of g_E we define the canonical inner product

$$\langle A, B \rangle := -\frac{1}{2} \text{tr}(AB). \tag{2.1}$$

The factor of $\frac{1}{2}$ is in keeping with the convention of Bourguignon-Lawson ([7] (2.14)). This inner product is positive definite since the only Lie algebras we consider satisfy $\mathfrak{g}_E \subset \mathfrak{so}_E$, the space of skew symmetric endomorphisms of E . In the Bochner formula below, the inner products use the given Riemannian metric in conjunction with (2.1).

Lemma 2.1 ([6,7] Theorem 3.10). *Let (X^4, g) be a Riemannian manifold, and suppose $E \rightarrow X$ is a smooth vector bundle with connection ∇ . If $\omega \in \Lambda_+^2(\mathfrak{g}_E)$ is a harmonic two-form, one has*

$$\frac{1}{2} \Delta |\omega|^2 = |\nabla \omega|^2 - \langle \omega, [F^+, \omega] \rangle - 2\langle \omega, W^+ \star \omega \rangle + \frac{1}{3} R |\omega|^2, \tag{2.2}$$

where with respect to local bases one has, for $P, Q \in \Lambda_+^2(\mathfrak{g}_E)$, the tensor $[P, Q] \in \Lambda_+^2(\mathfrak{g}_E)$ defined via

$$[P, Q]_{ij\alpha}^\beta := g^{kl} \left(P_{ik\delta}^\beta Q_{jl\alpha}^\delta - P_{ik\alpha}^\delta Q_{jl\delta}^\beta - P_{jk\delta}^\beta Q_{il\alpha}^\delta + P_{jk\alpha}^\delta Q_{il\delta}^\beta \right), \tag{2.3}$$

and

$$(W^+ \star \omega)_{ij\alpha}^\beta := g^{kp} g^{lq} W_{ijkl}^+ \omega_{pq\alpha}^\beta.$$

2.1. Sharp matrix inequalities. In this subsection we establish estimates for the curvature and covariant derivative terms in (2.2). First we estimate the Weyl curvature term. As the action is induced from the natural action on real valued self-dual two-forms, the proof is a straightforward modification of that case.

Lemma 2.2. *Let (X^4, g) be a Riemannian manifold, and suppose $E \rightarrow X$ is a smooth vector bundle with connection ∇ . If $\omega \in \Lambda_+^2(\mathfrak{g}_E)$ is a two-form, one has*

$$|\langle \omega, W^+ \star \omega \rangle| \leq \frac{2}{\sqrt{6}} |W^+| |\omega|^2. \tag{2.4}$$

Proof. This follows directly from the fact that W^+ is a trace-free endomorphism of Λ_+^2 , a rank 3 vector bundle (cf. [33] p 234). \square

The estimate of the bracket term in (2.2) will depend on the Lie algebra. We define two constants relevant to understanding the bracket term in the Böchner formula.

Definition 2.3. Given $\mathfrak{g} \subset \mathfrak{so}_E$, let

$$\begin{aligned} \gamma_0 &:= \sup_{A, B \in \mathfrak{g} \setminus \{0\}} \frac{|[A, B]|}{|A| |B|}, \\ \gamma_1 &:= \sup_{\omega \in \Lambda_+^2(\mathfrak{g}) \setminus \{0\}} \frac{\langle \omega, [\omega, \omega] \rangle}{|\omega|^3}. \end{aligned}$$

These suprema are certainly attained as the quantities are scale invariant and defined on finite dimensional vector spaces. A crucial point however is that these quantities depend on the choice of metric on \mathfrak{g}_E . Thus the choice (2.1) is important to what follows, and fixes the “scale” of various terms involving Lie algebra values. A fundamental lemma of Bourguignon-Lawson gives a universal upper bound for γ_0 , which is further improvable for some special Lie algebras.

Lemma 2.4 ([7] Lemma 2.30). *Given $\mathfrak{g} \subset \mathfrak{so}_E$, one has $\gamma_0 \leq \sqrt{2}$, with the supremum defining γ_0 attained by pairs A, B which are simultaneously equivalent to a pair of Pauli matrices, i.e.*

$$A = \left(\begin{array}{cc|c|c} 0 & t & & 0 \\ -t & 0 & & 0 \\ \hline 0 & & 0 & t \\ & & -t & 0 \\ \hline & & 0 & 0 \end{array} \right), \quad B = \left(\begin{array}{cc|c|c} & & -s & 0 \\ & & 0 & s \\ \hline s & 0 & & 0 \\ 0 & -s & & 0 \\ \hline & & 0 & 0 \end{array} \right). \tag{2.5}$$

Furthermore, in the case $\mathfrak{g} = \mathfrak{so}_3$ one has $\gamma_0 = 1$.

Next, using Lemma 2.4, we establish an estimate for the wedge product commutator of Lie algebra valued 2-forms. We first record an elementary linear algebra fact. Note here that we take the metric induced on two-forms as

$$\langle \eta, \mu \rangle := g^{ik} g^{jl} \eta_{ij} \mu_{kl}.$$

Lemma 2.5. *Given (V^4, g) an inner product space, for $\eta, \mu \in \Lambda_+^2(V^*)$ define $(\eta \circ \mu) \in \Lambda_+^2(V^*)$ via*

$$(\eta \circ \mu)_{ij} = g^{kl} (\eta_{ik} \mu_{jl} - \eta_{jk} \mu_{il}).$$

Then given $\{e_1, e_2, e_3\}$ an orthonormal basis for $\Lambda_+^2(V^)$, one has that $\{(e_1 \circ e_2), (e_1 \circ e_3), (e_2 \circ e_3)\}$ is another orthonormal basis.*

Lemma 2.6. *Let $E \rightarrow (M^4, g)$ be a smooth vector bundle. Given $P \in \Lambda_+^2(\mathfrak{g}_E)$, one has*

$$|[P, P]| \leq \frac{2}{\sqrt{3}} \gamma_0 |P|^2.$$

Proof. Fix a point $p \in M$ and let $\{e_i\}, i = 1, 2, 3$ be an orthonormal basis for Λ_+^2 at p . Now let us express $P = P^i e_i$, where $P^i \in \mathfrak{g}_E$. Due to the skew commutativity of the bracket structure we may estimate

$$\begin{aligned}
 |[P, P]|^2 &= |[P^i e_i, P^j e_j]|^2 \\
 &= |2(e_1 \circ e_2) \otimes [P^1, P^2] + 2(e_1 \circ e_3) \otimes [P^1, P^3] + 2(e_2 \circ e_3) \otimes [P^2, P^3]|^2 \\
 &\leq 4 \left(|[P^1, P^2]|^2 + |[P^1, P^3]|^2 + |[P^2, P^3]|^2 \right) \\
 &\leq 4\gamma_0^2 \left(|P^1|^2 |P^2|^2 + |P^1|^2 |P^3|^2 + |P^2|^2 |P^3|^2 \right) \\
 &\leq \frac{4\gamma_0^2}{3} \left(|P^1|^2 + |P^2|^2 + |P^3|^2 \right)^2 = \frac{4\gamma_0^2}{3} |P|^4.
 \end{aligned}
 \tag{2.6}$$

Taking the square root yields the claim. \square

Remark 2.7. We note that equality in Lemma 2.6 is achieved by the curvature of the standard ADHM/BPST instanton. At a fixed point this tensor takes the form (cf. (3.16))

$$F_\nabla = \lambda \left\{ (dx^{12} + dx^{34}) \otimes \mathbf{i} + (dx^{13} - dx^{24}) \otimes \mathbf{j} + (dx^{14} + dx^{23}) \otimes \mathbf{k} \right\}.$$

Note that, in the notation of Lemma 2.6, this curvature is expressed as $e_i P^i = e_1 \mathbf{i} + e_2 \mathbf{j} + e_3 \mathbf{k}$ for $\{e_1, e_2, e_3\}$ the standard basis for Λ_+^2 . Using the quaternion relations, it is clear that the pairwise commutators between the P^i are thus orthogonal, making the third line of (2.6) an equality. These matrices P^i are Pauli matrices, so by Lemma 2.4 the fourth line is also an equality. Lastly, as each P^i has the same norm, the fifth line of (2.6) is an equality.

2.2. Improved Kato inequality. In this subsection we prove a sharp Kato inequality for Lie algebra valued harmonic two-forms on four-manifolds. This was proved for Yang–Mills connections on \mathbb{R}^4 in [25]. Our proof is an elementary modification of the method of Seaman [28], who showed a sharp Kato inequality for harmonic real valued two-forms on four-manifolds, and exploited it to derive vanishing results for positively curved four-manifolds. Seaman’s method exploits the conformal invariance of harmonic two-forms in four dimensions, together with a delicate comparison of the Böchner formula for two choices of conformal factor. The Yang–Mills equation is also conformally invariant in four-dimensions, and the relevant Bochner formula only differs by a conformally invariant term, the proof is adapted in a straightforward manner.

Proposition 2.8. *Let $E \rightarrow (X^4, g)$ be a vector bundle over a smooth Riemannian four-manifold. Given ∇ a connection on E , and $\omega \in \Lambda^2(\mathfrak{g}_E)$ a harmonic two-form, one has the pointwise inequality*

$$|\nabla\omega|^2 \geq \frac{3}{2} |d|\omega||^2.
 \tag{2.7}$$

Proof. First, by applying the Bochner formula (Lemma 2.1) to ω (summing over self-dual and antiself-dual parts) we obtain

$$\frac{1}{2} \Delta |\omega|^2 = |\nabla \omega|^2 + \langle [F, \omega] - 2W \star \omega + \frac{R}{3} \omega, \omega \rangle. \tag{2.8}$$

We now make a conformal modification of the metric and derive a second Böchner identity. In particular, let $\hat{g} = |\omega| g$, which defines a smooth Riemannian metric away from the zero locus of $|\omega|$. Note by construction that $|\omega|_{\hat{g}}^2 \equiv 1$. Furthermore, as the condition that ω is harmonic is conformally invariant, ω is harmonic with respect to \hat{g} , and thus we apply the Bochner formula again to conclude

$$0 = \frac{1}{2} \hat{\Delta} |\omega|_{\hat{g}}^2 = \left| \hat{\nabla} \omega \right|_{\hat{g}}^2 + \left\langle [\hat{F}, \omega] - 2 \hat{W} \star \omega + \frac{\hat{R}}{3} \omega, \omega \right\rangle_{\hat{g}}. \tag{2.9}$$

The bundle curvature F and the Weyl tensor W are conformally covariant, yielding

$$\left\langle [\hat{F}, \omega] - 2 \hat{W} \star \omega, \omega \right\rangle_{\hat{g}} = |\omega|^{-3} \langle [F, \omega] - 2W \star \omega, \omega \rangle. \tag{2.10}$$

On the other hand, using the transformation formula for the scalar curvature under conformal change one obtains

$$\left\langle \frac{\hat{R}}{3} \omega, \omega \right\rangle_{\hat{g}} = |\omega|^{-3} \left(\left\langle \frac{R}{3} \omega, \omega \right\rangle - \frac{1}{2} \Delta |\omega|^2 + \frac{3}{2} |d|\omega|^2 \right). \tag{2.11}$$

Plugging (2.10) and (2.11) into (2.9), and incorporating (2.8) we conclude

$$\begin{aligned} \left| \hat{\nabla} \omega \right|_{\hat{g}}^2 &= -|\omega|^{-3} \left(\langle [F, \omega] + W \star \omega + \frac{R}{6} \omega, \omega \rangle - \frac{1}{2} \Delta |\omega|^2 + \frac{3}{2} |d|\omega|^2 \right) \\ &= |\omega|^{-3} \left(|\nabla \omega|^2 - \frac{3}{2} |d|\omega|^2 \right). \end{aligned}$$

This implies the desired inequality away from the vanishing locus of $|\omega|$, which in turn implies the inequality at all points.

3. Main Proofs

In this section we give the proof of Theorem 1.1. As discussed in the introduction the proof involves a delicate application of ideas from conformal geometry to the Böchner formula for F_{∇}^+ . The proof of Theorem 1.1 appears in Sect. 3.1. Then in Sect. 3.2 we give an example illustrating the sharpness of the estimate. We conclude in Sect. 3.3 with the proofs of Corollaries 1.2 and 1.3.

3.1. Proof of Theorem 1.1.

Proof. Let ∇ denote a Yang–Mills connection, and let us set $F = F_{\nabla}$ for convenience. Then $DF = 0$, $D^*F = 0$, and it follows easily that $\star F$ is also closed and co-closed, hence $DF^+ = 0$, $D^*F^+ = 0$. Thus we may apply Lemma 2.1 to F^+ to yield

$$\frac{1}{2} \Delta |F^+|^2 = |\nabla F^+|^2 - \langle F^+, [F^+, F^+] \rangle - 2 \langle F^+, W^+ \star F^+ \rangle + \frac{1}{3} R |F^+|^2. \tag{3.1}$$

By the result of Lemma 2.2 and the definition of γ_1 , we immediately obtain

$$\frac{1}{2}\Delta|F^+|^2 \geq |\nabla F^+|^2 + \frac{1}{3}(R - 2\sqrt{6}|W^+| - 3\gamma_1|F^+|)|F^+|^2. \tag{3.2}$$

By the Leibniz rule, away from the zero locus of F^+ we have

$$\frac{1}{2}\Delta|F^+|^2 = |F^+|\Delta|F^+| + |\nabla|F^+||^2,$$

hence by (2.7)

$$\Delta|F^+| \geq \frac{1}{2}\frac{|\nabla|F^+||^2}{|F^+|} + \frac{1}{3}(R - 2\sqrt{6}|W^+| - 3\gamma_1|F^+|)|F^+|. \tag{3.3}$$

It follows that

$$\Delta|F^+|^{1/2} \geq \frac{1}{6}(R - 2\sqrt{6}|W^+| - 3\gamma_1|F^+|)|F^+|^{1/2} \tag{3.4}$$

off the zero locus of F^+ , or in the sense of distributions.

We next exploit (3.4) in conjunction with as a modified Yamabe problem introduced in [16]. In particular, given a metric $\hat{g} \in [g]$ we define

$$\Phi_{\hat{g}} = R_{\hat{g}} - 2\sqrt{6}|W^+|_{\hat{g}} - 3\gamma_1|F^+|_{\hat{g}}. \tag{3.5}$$

If we drop the Weyl and F^+ -terms then Φ is just the scalar curvature. Moreover, because these terms transform by scalings of the same weight under conformal changes of metric, the transformation law for Φ is essentially the same as the scalar curvature. More precisely, if we define the natural generalization of the conformal Laplacian by

$$L = -6\Delta + \Phi,$$

then given $\hat{g} = u^2g$ it follows that

$$\Phi_{\hat{g}} = u^{-3}L_g u. \tag{3.6}$$

Moreover, L is conformally covariant:

$$L_{\hat{g}}\phi = u^{-3}L_g(u\phi).$$

Consequently, if $\lambda_1(L)$ denotes the first eigenvalue of L ,

$$\lambda_1(L_g) = \inf_{\phi \in C^\infty, \phi \neq 0} \frac{\int_X \phi L_g \phi \, dV_g}{\int_X \phi^2 \, dV_g}, \tag{3.7}$$

then the sign of $\lambda_1(g)$ is a conformal invariant. In particular, by using an eigenfunction associated with $\lambda_1(L)$ as a conformal factor, it follows that $[g]$ admits a metric \hat{g} with $\Phi_{\hat{g}} > 0$ (*resp.*, $= 0$, < 0) if and only if $\lambda_1(L_g) > 0$ (*resp.*, $= 0$, < 0).

One departure from the classical Yamabe problem is that the modified scalar curvature may only be Lipschitz continuous. Therefore, the Schauder estimates imply that the first eigenfunction is in $C^{2,\alpha}$ and hence defines a conformal metric which is only $C^{2,\alpha}$. One can smooth $|F^+|$ and approximate (see Section 3 of [16] for details), but in our setting this will not be necessary.

Returning to the inequality (3.4), we can now express this as

$$0 \geq L_g(|F^+|^{1/2}). \tag{3.8}$$

Multiplying by $|F^+|^{1/2}$ and integrating over X^4 gives

$$0 \geq \int_X |F^+|^{1/2} L_g(|F^+|^{1/2}) dV_g.$$

It thus follows that either $F^+ \equiv 0$ or $\lambda_1(L_g) \leq 0$. The case $F^+ \equiv 0$ is case (1) of the statement, thus we proceed to analyze the case $\lambda_1(L_g) \leq 0$. Let $\phi_1 > 0$ denote an eigenfunction associated to $\lambda_1(L)$, and define the metric $\bar{g} = \phi_1^2 g$. By (3.6),

$$\Phi_{\bar{g}} = \phi_1^{-3} L_g \phi_1 = \lambda_1 \phi_1^{-2} \leq 0.$$

Therefore,

$$0 \geq \int_X \Phi_{\bar{g}} dV_{\bar{g}} = \int_X (R_{\bar{g}} - 2\sqrt{6}|W^+| - 3\gamma_1|F^+|_{\bar{g}}) dV_{\bar{g}},$$

or

$$\int_X R_{\bar{g}} dV_{\bar{g}} \leq 2\sqrt{6} \int_X |W^+|_{\bar{g}} dV_{\bar{g}} + 3\gamma_1 \int_X |F^+|_{\bar{g}} dV_{\bar{g}}. \tag{3.9}$$

We can estimate the integral on the left-hand side in terms of the Yamabe invariant of $[g]$:

$$\int_X R_{\bar{g}} dV_{\bar{g}} \geq Y([g]) \text{Vol}(\bar{g})^{1/2}. \tag{3.10}$$

For the terms on the right-hand side of (3.9) we use Cauchy-Schwartz:

$$\begin{aligned} 2\sqrt{6} \int_X |W^+|_{\bar{g}} dV_{\bar{g}} + 3\gamma_1 \int_X |F^+|_{\bar{g}} dV_{\bar{g}} &\leq 2\sqrt{6} \left(\int_X |W^+|_{\bar{g}}^2 dV_{\bar{g}} \right)^{1/2} \text{Vol}(\bar{g})^{1/2} + 3\gamma_1 \left(\int_X |F^+|_{\bar{g}}^2 dV_{\bar{g}} \right)^{1/2} \text{Vol}(\bar{g})^{1/2} \\ &= 2\sqrt{6} \left(\int_X |W^+|_{\bar{g}}^2 dV_{\bar{g}} \right)^{1/2} \text{Vol}(\bar{g})^{1/2} + 3\gamma_1 \left(\int_X |F^+|_{\bar{g}}^2 dV_{\bar{g}} \right)^{1/2} \text{Vol}(\bar{g})^{1/2}, \end{aligned} \tag{3.11}$$

where the second line follows from conformal invariance of the integrals. Combining (3.9)–(3.11) and dividing by the square root of the volume we arrive at (1.1).

If equality is achieved then all of the inequalities above become equalities. Equality in (3.10) implies that \bar{g} is a Yamabe metric (hence C^∞), and

$$R_{\bar{g}} - 2\sqrt{6}|W^+|_{\bar{g}} = 3\gamma_1|F^+|_{\bar{g}}, \tag{3.12}$$

which proves (1.2). Since equality is attained in (3.11), it follows that both $|W^+|_{\bar{g}}$ and $|F^+|_{\bar{g}}$ are constant. By conformal invariance of the Yang–Mills energy, we can write the Böchner formula (3.1) with respect to any metric in $[g]$. If we use \bar{g} in place of g , then (3.2) becomes

$$0 = \frac{1}{2} \Delta_{\bar{g}} |F^+|_{\bar{g}}^2 \geq |\nabla F^+|_{\bar{g}}^2 + \frac{1}{3} \Phi_{\bar{g}} |F^+|_{\bar{g}}^2 = |\nabla F^+|_{\bar{g}}^2, \tag{3.13}$$

hence F^+ is parallel.

Finally, we establish the statements concerning b_2^+ . By the Bochner formula for real-valued self-dual two-forms (a special case of (3.1)),

$$\frac{1}{2} \Delta_{\bar{g}} |\omega|^2 = |\nabla \omega|^2 - 2W_{\bar{g}}^+(\omega, \omega) + \frac{1}{3} R_{\bar{g}} |\omega|^2. \tag{3.14}$$

As a special case of Lemma 2.2 we have

$$|W_{\bar{g}}^+(\omega, \omega)| \leq \frac{2}{\sqrt{3}} |W^+| |\omega|^2.$$

Therefore,

$$\begin{aligned} \frac{1}{2} \Delta_{\bar{g}} |\omega|^2 &= |\nabla \omega|^2 - 2W_{\bar{g}}^+(\omega, \omega) + \frac{1}{3} R_{\bar{g}} |\omega|^2 \\ &\geq |\nabla \omega|^2 + \frac{1}{3} (R_{\bar{g}} - 2\sqrt{6} |W^+|_{\bar{g}}) |\omega|^2 \\ &= |\nabla \omega|^2 + \gamma_1 |F^+|_{\bar{g}} |\omega|^2, \end{aligned}$$

where the last line follows from (3.12). Since $|F^+|_{\bar{g}}$ is constant and non-zero by assumption, we see that ω must vanish if $\gamma_1 > 0$, implying $b_2^+ = 0$. If $\gamma_1 = 0$ we see that ω must be parallel, as claimed. \square

3.2. *Sharpness via SU(2) instantons.* It is possible to achieve equality in (1.1) via the classic construction of BPST/ADHM SU(2) instantons on \mathbb{S}^4 ([1,3]). We recall the most basic connection in this class, the standard connection, for convenience (and also as a way to fix conventions). The standard connection is expressed on \mathbb{R}^4 , thought of as $\mathbb{S}^4 \setminus \{N\}$ as the $\mathfrak{su}(2)$ -valued 1-form (cf. [12] (3.4.2))

$$\theta = \frac{1}{1+|x|^2} (\theta_1 \otimes \mathbf{i} + \theta_2 \otimes \mathbf{j} + \theta_3 \otimes \mathbf{k}),$$

where,

$$\begin{aligned} \theta_1 &= x_1 dx^2 - x_2 dx^1 + x_3 dx^4 - x_4 dx^3 \\ \theta_2 &= x_1 dx^3 - x_3 dx^1 + x_4 dx^2 - x_2 dx^4 \\ \theta_3 &= x_1 dx^4 - x_4 dx^1 + x_2 dx^3 - x_3 dx^2. \end{aligned}$$

Also, in keeping with our previous convention, we represent $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as real matrices

$$\mathbf{i} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array} \right), \quad \mathbf{j} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right), \quad \mathbf{k} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right). \tag{3.15}$$

A classic calculation yields the relevant curvature tensor

$$F_{\nabla} = \frac{2}{(1+|x|^2)^2} \left\{ (dx^{12} + dx^{34}) \otimes \mathbf{i} + (dx^{13} - dx^{24}) \otimes \mathbf{j} + (dx^{14} + dx^{23}) \otimes \mathbf{k} \right\}. \tag{3.16}$$

Using our conventions for the metric induced by the Euclidean metric on two-forms, $|dx^{12}|^2 = 2$, etc. Moreover, using (3.15) and (2.1), it follows that $|\mathbf{i}|^2 = |\mathbf{j}|^2 = |\mathbf{k}|^2 = 2$. Putting these together we can compute the pointwise norm of F_∇ in this context to yield

$$\begin{aligned} |F_\nabla|^2 &= \frac{4}{(1+|x|^2)^4} \left(2 |dx^{12} + dx^{34}|^2 + 2 |dx^{13} - dx^{24}|^2 + 2 |dx^{14} + dx^{23}|^2 \right) \\ &= \frac{96}{(1+|x|^2)^4}. \end{aligned}$$

Using the conformal invariance of the Yang–Mills energy, we thus obtain

$$\|F_\nabla\|_{L^2(\mathbb{S}^4, g_{\mathbb{S}^4})}^2 = 96 \int_{\mathbb{R}^4} \frac{1}{(1+|x|^2)^4} dV_{Eucl} = 16\pi^2.$$

Having computed the Yang–Mills energy, we turn to the remaining quantities in (1.1). As follows from [2], the Yamabe invariant of the round 4-sphere is $4(3)(\omega_4)^{\frac{1}{2}} = 12 \left(\frac{8}{3}\pi^2\right)^{\frac{1}{2}} = 8\sqrt{6}\pi$. Thus, employing Lemmas 2.4 and 2.6 we observe that the right hand side of the right hand side of (1.1) can be estimated as

$$3\gamma_1 \|F_\nabla^+\|_{L^2} \leq 3 \left(\frac{2}{\sqrt{3}}\gamma_0\right) (4\pi) \leq 8\sqrt{6}\pi = Y([g_{\mathbb{S}^4}]).$$

Thus (1.1) is an equality, as are all the intermediate estimates. These equalities reflect many interesting geometric properties of this classic charge 1 $SU(2)$ instanton. First, as discussed in Remark 2.7, the curvature of this connection gives equality in the relevant algebraic inequalities we used. Furthermore, the theorem yields parallelism of F_∇^+ with respect to a particular representative of $[g]$. One can directly compute that the distinguished BPST/ADHM connection representing the center of the moduli space has parallel curvature with respect to the round metric. However, since all BPST/ADHM connections are given by pullback by an element of the conformal group, one immediately concludes again that any such connection has parallel curvature with respect to a particularly chosen element of the conformal class, which our method explicitly constructs in a more general fashion via the solution to the modified Yamabe problem.

3.3. Proofs of Corollaries.

Proof of Corollary 1.2. We recall the fundamental Chern-Weil formula

$$16\pi^2\kappa(E) = \int_X \text{tr}(F_\nabla \wedge F_\nabla) = \int_X \left(|F_\nabla^-|_g^2 - |F_\nabla^+|_g^2 \right) dV_g. \tag{3.17}$$

Let us first assume $\kappa(E) \geq 0$, and $F_\nabla^+ \neq 0$, with the case $\kappa(E) \leq 0$ directly analogous. Since we have assumed our metric is conformally flat and $Y([g]) > 0$, we may combine (1.1) and (3.17) to yield

$$\begin{aligned} \|F_\nabla\|_{L^2}^2 &= \|F_\nabla^-\|_{L^2}^2 + \|F_\nabla^+\|_{L^2}^2 \\ &= 16\pi^2 |\kappa(E)| + 2 \int_X |F_\nabla^+|_g^2 dV_g \geq 16\pi^2 |\kappa(E)| + \frac{2Y([g])^2}{9\gamma_1^2}, \end{aligned}$$

as claimed. For the special case of $(\mathbb{S}^4, g_{\mathbb{S}^4})$, as discussed in Sect. 3.2 know that the Yamabe invariant is $8\sqrt{6}\pi$. Moreover using Lemmas 2.4 and 2.6, we know that in the case of structure group $SU(2)$ we may choose $\gamma_1 = \frac{4}{\sqrt{6}}$, whereas for $SO(3)$ one has $\gamma_1 = \frac{2}{\sqrt{3}}$. This yields the remaining statements. \square

Proof of Corollary 1.3. We give a very brief sketch which assumes familiarity with the papers ([29,34]), and Yang–Mills flow in general. In particular, as discussed in ([34] Theorem 2.3, [29] Theorem 1.1), any smooth connection admits a unique solution to Yang–Mills flow with the prescribed initial data, which moreover encounters a singularity at either finite or infinite time via “concentration of energy.” As made precise in ([34] Theorem 2.4, [29] Theorem 1.2), at any singular point in spacetime one can construct at least a maximal bubble defined as a limit of blowup sequences, which converge in the Uhlenbeck sense to a nontrivial Yang–Mills connection over $(\mathbb{S}^4, g_{\mathbb{S}^4})$. Crucially, the energy of this limiting connection is no larger than the energy of the initial connection. Thus, comparing (1.6) against the results in Corollary 1.2, we see that the energy inequalities cannot hold, and therefore this limiting connection must be an instanton. However, comparing against (3.17), we see that any nonflat instanton must have energy at least $16\pi^2$, thus we have arrived at a contradiction. Thus the flow exists globally and the time slices converge subsequentially as time approaches infinity to a smooth limiting Yang–Mills connection. In the case of $(\mathbb{S}^4, g_{\mathbb{S}^4})$ it is clear by the argument above that the limiting connection is flat. By employing Łojasiewicz–Simon arguments (cf. [26] Proposition 7.2, [14,36] Theorem 7, all based on the classic [32] Theorem 2) one can improve this C^∞ Uhlenbeck subsequential convergence to convergence of the entire flow line. \square

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