



Stability Results, Almost Global Generalized Beltrami Fields and Applications to Vortex Structures in the Euler Equations

Alberto Enciso¹, David Poyato², Juan Soler²

¹ Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Científicas, 28049 Madrid, Spain.
E-mail: aenciso@icmat.es

² Departamento de Matemática Aplicada and Excellence Research Unit “Modeling Nature” (MNat), Universidad de Granada, 18071 Granada, Spain.
E-mail: davidpoyato@ugr.es; jsoler@ugr.es

Received: 9 May 2017 / Accepted: 25 October 2017

Published online: 18 December 2017 – © Springer-Verlag GmbH Germany, part of Springer Nature 2017

Abstract: Strong Beltrami fields, that is, vector fields in three dimensions whose curl is the product of the field itself by a constant factor, have long played a key role in fluid mechanics and magnetohydrodynamics. In particular, they are the kind of stationary solutions of the Euler equations where one has been able to show the existence of vortex structures (vortex tubes and vortex lines) of arbitrarily complicated topology. On the contrary, there are very few results about the existence of generalized Beltrami fields, that is, divergence-free fields whose curl is the field times a non-constant function. In fact, generalized Beltrami fields (which are also stationary solutions to the Euler equations) have been recently shown to be rare, in the sense that for “most” proportionality factors there are no nontrivial Beltrami fields of high enough regularity (e.g., of class $C^{6,\alpha}$), not even locally. Our objective in this work is to show that, nevertheless, there are “many” Beltrami fields with non-constant factor, even realizing arbitrarily complicated vortex structures. This fact is relevant in the study of turbulent configurations. The core results are an “almost global” stability theorem for strong Beltrami fields, which ensures that a global strong Beltrami field with suitable decay at infinity can be perturbed to get “many” Beltrami fields with non-constant factor of arbitrarily high regularity and defined in the exterior of an arbitrarily small ball, and a “local” stability theorem for generalized Beltrami fields, which is an analogous perturbative result which is valid for any kind of Beltrami field (not just with a constant factor) but only applies to small enough domains. The proof relies on an iterative scheme of Grad–Rubin type. For this purpose, we study the Neumann problem for the inhomogeneous Beltrami equation in exterior domains via a boundary integral equation method and we obtain Hölder estimates, a sharp decay at infinity and some compactness properties for these sequences of approximate solutions. Some of the parts of the proof are of independent interest.

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1. Introduction

Beltrami fields, that is, three dimensional vector fields whose curl is proportional to the field, are a particularly important class of smooth stationary solutions of the three-dimensional incompressible Euler equations:

$$\partial_t u + (u \cdot \nabla)u = -\nabla p, \quad \operatorname{div} u = 0.$$

In a way, what makes them so special is the celebrated structure theorem of Arnold [3], which asserts that, under suitable technical hypotheses, the velocity field of a smooth stationary solution to the Euler equations is either a Beltrami field or “laminar”, in the sense that it admits a regular first integral whose smooth level sets provide “layers” to which the fluid flow is tangent. In fluid mechanics, a Beltrami field is interpreted as a fluid whose velocity is parallel to its vorticity.

Understanding the knot and link type of stream lines and tubes in stationary fluids has also attracted the attention of many researchers, both from the theoretical and the experimental points of view [18, 19, 28, 44], because knotted stationary vortex structures turned out to play a key role in the so called Lagrangian theory of turbulence. From a numerical point of view, the description of the flows in the literature that allow for arbitrary vortex structures is mainly based on an active vector formulation of Euler’s equations (see [11] and the references therein). The existence of knotted and linked vortex lines and tubes in stationary solutions to the Euler equations has been recently established in [18, 19] using *strong Beltrami fields*, that is, Beltrami fields with a constant proportionality factor:

$$\operatorname{curl} u = \lambda u, \quad \lambda \in \mathbb{R} \setminus \{0\}. \quad (1)$$

Notice that the Beltrami fields in [18, 19] can be assumed to fall off as $1/|x|$ at infinity, and that this decay rate is optimal (see the global obstructions in the form of a Liouville type theorem in [7, 36]). Concrete examples of Beltrami fields with constant proportionality factor are the ABC flows, whose analysis has yielded considerable insight into the aforementioned phenomenon of Lagrangian turbulence [17].

The main objective of this paper is to study the existence, regularity and stability results of *generalized Beltrami fields* (i.e., Beltrami fields with nonconstant proportionality factor). These vector fields play a fundamental role in the understanding of turbulence. The idea that turbulent flows can be understood as a superposition of Beltrami flows has already been proposed in [12, 39]. They are also relevant in magnetohydrodynamics in the context of vanishing Lorentz force (*force-free fields*) and they can be used to model magnetic relaxation, which is relevant in some astrophysical applications [27, 29, 34, 35]. Indeed, to the best of our knowledge there are just a handful of explicit examples, all of which have Euclidean symmetries, and the analysis of Beltrami fields with nonconstant factor has proved to be extremely hard. The heart of the matter is that, as it was recently proved in [20], the equation for a generalized Beltrami field,

$$\operatorname{curl} u = f u, \quad \operatorname{div} u = 0, \quad (2)$$

does not admit any nontrivial solution, even locally, for a “generic” nonconstant function f . In a very precise sense, it shows that Beltrami fields with a nonconstant factor are rare and such obstruction is of a purely local nature. These results have been carefully stated in “Appendix B” for the reader’s convenience.

One of the aims of this paper is to show that, although generalized Beltrami fields are indeed rare, one can still prove some kind of partial stability result. Specifically, we will show that for each nontrivial Beltrami field, there are “many” close enough nonconstant proportionality factors that enjoy close nontrivial generalized Beltrami fields. The stability result is “partial” in the sense that a “full” stability result cannot be expected since the space of factors that enjoy nontrivial generalized Beltrami fields does not contain any ball in the $C^{k,\alpha}$ norm by the above-mentioned obstructions. The analysis of stability can be crucial to shed some light on the interactions between the different scales in the study of relevant configurations in a fully turbulent state.

More concretely, we will prove two stability results for generalized Beltrami fields. The first one (Theorem 3.7) is an “almost global” perturbation result for strong Beltrami fields defined on \mathbb{R}^3 . Roughly speaking, it asserts that given any nontrivial solution of (1) on \mathbb{R}^3 with optimal fall-off at infinity (i.e., $1/|x|$) and any arbitrarily small ball G , there are infinitely many nonconstant factors f , as close to the constant λ as one wishes in $C^{k,\alpha}(\mathbb{R}^3)$, such that the corresponding Eq. (2) admits nontrivial solutions on the complement $\mathbb{R}^3 \setminus \bar{G}$. This can be combined with the results in [18, 19] to construct almost global Beltrami fields with a nonconstant factor that feature vortex lines and vortex tubes of arbitrarily complicated topology (Theorem 4.1). The second stability result (Theorem 5.3) states an analogue for perturbations of nontrivial Beltrami fields with constant or nonconstant factor defined in a small enough open set where the field does not vanish. The point of these stability results is that the perturbation of the initial proportionality factor is defined by recursively propagating a two-variable function along the integral curves of a velocity vector field, so that is the flexibility in choosing the proportionality factor that is granted by the method of proof. Notice that the idea of constructing the proportionality factor by dragging along the integral curves of a field is somehow inherent to the problem, as the incompressibility condition $\operatorname{div} u = 0$ implies

that, if it is nonconstant, the factor must be a first integral of the generalized Beltrami field, i.e.,

$$u \cdot \nabla f = 0.$$

Let us outline the key aspects of the proofs. For concreteness, since all the ideas involved in the proof of the local partial stability result are essentially present in that of the almost global theorem, we shall only discuss the latter result in this Introduction. As we have already mentioned, the point of the partial stability result is to develop a perturbation technique allowing us to deform the initial factor f , which for the purpose of this discussion can be taken to be a nonzero constant λ . This requires analyzing a related boundary value problem, namely, the *Neumann boundary value problem for the inhomogeneous Beltrami equation* with constant proportionality factor λ in exterior domains. To our best knowledge, this problem has not been directly studied in the literature. Our analysis is based on a boundary integral equation method for complex-valued solutions which requires some potential theory estimates for generalized volume and single layer potentials and an analysis of the decay properties and radiation conditions of the solutions. They will be determined through the natural connections between the complex-valued solutions of the Beltrami, Helmholtz and Maxwell systems.

In [27], the authors show that one can perturb a harmonic field (i.e., a Beltrami field with $\lambda = 0$) defined in an exterior domain to construct a generalized Beltrami field with a nonconstant factor. However, the perturbed fields and factors are of low regularity (of class $C^{1,\alpha}$ and $C^{0,\alpha}$, respectively). In view of the relevance and important applications of Beltrami fields with nonzero λ , we have striven to extend the result for harmonic fields to general Beltrami fields, and also to show the existence of perturbations of arbitrarily high regularity (the field will be in $C^{k+1,\alpha}$ and the factor in $C^{k,\alpha}$ for any fixed integer k). It should be stressed that the passing from $\lambda = 0$ to nonzero λ is not a trivial matter, since the behavior of the equations at infinity is completely different (oversimplifying a little, for $\lambda = 0$ the behavior of the fields at infinity is that of a harmonic function, so one gets uniqueness simply from a decay condition, while for nonzero λ , Beltrami fields solve Helmholtz’s equation, so radiation conditions must be specified to obtain uniqueness.) We will present a detailed treatment of these topics (Sect. 2 and “Appendix A”), since we consider that they are of independent interest.

The gist of the proof of the almost global partial stability result for strong Beltrami fields is to study the convergence in $C^{k,\alpha}$ of an iterative scheme that takes the form

$$\begin{cases} \nabla \varphi_n \cdot u_n = 0, & x \in \Omega, \\ \varphi_n = \varphi^0, & x \in \Sigma, \end{cases} \quad \begin{cases} \operatorname{curl} u_{n+1} - \lambda u_{n+1} = \varphi_n u_n, & x \in \Omega, \\ u_{n+1} \cdot \eta = u_0 \cdot \eta, & x \in S. \end{cases}$$

Here, Ω stands for an exterior domain with smooth boundary S , η is its outward unit normal vector field and Σ is some open subset of the boundary. This is a modified Grad–Rubin method (see [1,5] for the original Grad–Rubin method in the setting of force-free fields perturbations of harmonic fields), which we will start up with a strong Beltrami field u_0 of constant proportionality factor λ (which can be assumed to exhibit knotted and linked vortex structures) and prescribes the value φ^0 of the perturbation of the proportionality factor λ over Σ . Notice that $\{\varphi_n\}_{n \in \mathbb{N}}$ and $\{u_n\}_{n \in \mathbb{N}}$ are taken in a consistent way so that whenever they have limits φ and u in some sense, then φ is a global first integral of u and such vector field verifies the Beltrami equation (2) with $f = \lambda + \varphi$.

Our approach will be based again on the analysis of *stationary transport equations* along stream tubes and a sequence of inhomogeneous problems of div-curl type that

we will call *inhomogeneous Beltrami equations* and which are intimately linked to the Helmholtz equation. In fact, we will start with the complex-valued fundamental solution of the Helmholtz equation in \mathbb{R}^3

$$\Gamma_\lambda(x) = \frac{e^{i\lambda|x|}}{4\pi|x|}, \quad x \in \mathbb{R}^3 \setminus \{0\},$$

and will arrive at a representation formula of Helmholtz–Hodge type for its complex-valued solutions. Then, it is necessary to specify the optimal decay and radiation conditions that allow dealing with generalized volume and single layer potentials, namely,

$$\int_{\partial B_R(0)} |u(x)| \, d_x S = o(R^2), \quad R \rightarrow +\infty, \tag{3}$$

$$\int_{\partial B_R(0)} \left| i \frac{x}{R} \times u(x) - u(x) \right| \, d_x S = o(R), \quad R \rightarrow +\infty. \tag{4}$$

Here, (3) is nothing but a weak decay condition of the velocity field u in L^1 and (4) will be called the L^1 *Silver–Müller–Beltrami radiation condition* (L^1 SMB) and will be deduced from both the classical Sommerfeld and Silver–Müller radiation conditions, whose connections with the Helmholtz equation and the Maxwell system are classical.

Summing up, we will be interested in analyzing the existence and uniqueness of complex-valued smooth solution with high order Hölder-type regularity of the general *Neumann boundary value problem for the inhomogeneous Beltrami equation* (NIB)

$$\begin{cases} \operatorname{curl} u - \lambda u = w, & x \in \Omega, \\ u \cdot \eta = g, & x \in \Omega, \\ + L^1 \text{ decay property (3)}, \\ + L^1 \text{ SMB radiation condition (4)}. \end{cases} \tag{5}$$

Notice that although we were originally interested in real-valued Beltrami fields, we will be concerned with complex-valued solutions to (5) and we will then take real parts to obtain the real-valued ones. The reason to do it is twofold. Firstly, this will allow us to employ a representation formula for complex-valued *radiating* fields. Secondly, this presents no problems related to the application to knotted structures as one can realize the fields in [18, 19] as the real parts of complex-valued radiating Beltrami fields. Problem (5) was previously studied in [29], who proved C^1 regularity results in bounded domains. We introduce some potential theory estimates of high order for generalized potentials associated with inhomogeneous kernels in exterior domains and adapt the boundary integral method to the unbounded setting. We will also improve regularity from C^1 to $C^{k+1,\alpha}$.

Consequently, we will rely on the complex-valued counterpart of the modified Grad–Rubin method:

$$\begin{cases} \nabla \varphi_n \cdot u_n = 0, & x \in \Omega, \\ \varphi_n = \varphi^0, & x \in \Sigma, \end{cases} \quad \begin{cases} \operatorname{curl} v_{n+1} - \lambda v_{n+1} = \varphi_n u_n, & x \in \Omega, \\ v_{n+1} \cdot \eta = u_0 \cdot \eta, & x \in S, \\ + L^1 \text{ Decay property (3)}, \\ + L^1 \text{ SBM radiation condition (4)}, \end{cases} \tag{6}$$

where $u_n = \Re v_n$ are the real parts of the complex-valued solutions v_n . The compactness of $\{u_n\}_{n \in \mathbb{N}}$ in $C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ follows from some Schauder estimates of Eq. (5) in Hölder spaces. Similarly, $\{\varphi_n\}_{n \in \mathbb{N}}$ will be shown to be compact in $C^{k,\alpha}(\overline{\Omega})$ too. Concerning

the application to solutions u_0 with knotted vortex structures of the kind constructed in [18, 19], we will see that the solution u inherits the knotted vortex structures from u_0 (up to a small deformation) by virtue of structural stability. This is a straightforward consequence of the fact that u can be chosen close to u_0 as long as the prescribed value φ^0 is small enough.

The paper is organized as follows. Section 3 is devoted to studying the iterative scheme (6). First, we analyze the linear transport equations in the right hand side and the convergence of the iterative scheme will then follow from the analysis of (5). Such problem will be studied in Sect. 2 by extending the results in [29, 38, 43]. By comparison with the *vector-valued divergence-free Helmholtz equation*, the *reduced Maxwell system* and the Beltrami equation, we will deduce the appropriate radiation and decay conditions. The SMB radiation condition (4) will then be connected with the classical Silver–Müller and Sommerfeld radiation conditions and we will then present a representation formula of Helmholtz–Hodge type which involves these radiation conditions and that will be extremely useful to obtain our existence, uniqueness and regularity results. In Sect. 4, we combine the above results to construct small perturbations of the constant proportionality factor λ leading to nontrivial generalized Beltrami fields that exhibit the same kind of knots and links and so to construct stationary solutions to the Euler equations. Finally, in order to support the above regularity results, “Appendix A” will focus on obtaining Hölder estimates of high order for volume and single layer potentials associated with the inhomogeneous kernel $\Gamma_\lambda(x)$. The underlying ideas can be adapted to many other general inhomogeneous kernels with a controlled decay at infinity. The local partial stability result for generalized Beltrami fields will be discussed in Sect. 5. “Appendix B” recalls, for the benefit of the reader, the results on the generic non-existence of generalized Beltrami fields proved in [20].

Notation Let us conclude this Introduction by summing up some notation that will be used throughout the paper without further notice. The notation regarding the domains can be stated as follows:

- $$\left\{ \begin{array}{l} \bullet \ G \text{ is a } C^{k+5} \text{ bounded domain homeomorphic to an Euclidean ball and} \\ \quad \text{containing the origin, i.e., } 0 \in G. \\ \bullet \ \Omega := \mathbb{R}^3 \setminus \overline{G} \text{ is its exterior domain and } S := \partial\Omega = \partial G \text{ is the boundary surface.} \\ \bullet \ \eta \text{ denotes the outward unit normal vector field of } S. \end{array} \right. \quad (7)$$

Although most of our results hold under weaker assumption on the boundary regularity (specifically $C^{k+1,\alpha}$ boundaries), there are certain results concerning a singular boundary integral equation which need S to be at least C^{k+5} because higher order derivatives of the normal vector field η are involved (see for instance Theorem A.10).

Concerning functional spaces, we will essentially use the same notation as in [23]. Let us agree to say that $C^k(\Omega)$ is the space of functions of class C^k on Ω with finite C^k norm (meaning that all their derivatives up to order k are bounded). We will replace Ω by $\overline{\Omega}$ when the function and all its derivatives up to order k can be continuously extended to the closure of Ω . The space $C^{k,\alpha}(\Omega)$ is the inhomogeneous Hölder space with exponent $\alpha \in (0, 1)$ and k -th order regularity. We will use similar notation $C^k(S)$, $C^{k,\alpha}(S)$ for functions defined on S . Vector-valued analogs of these spaces are denoted in the usual fashion, e.g. $C^{k,\alpha}(\Omega, \mathbb{R}^3)$.

2. Neumann Problem for the Inhomogeneous Beltrami Equation and Radiation Conditions

In this section we analyze the existence and uniqueness of solutions in $C^{k+1,\alpha}$ of the NIB problems (5) arising in the modified Grad–Rubin iterative method (6). The key tool is a representation formula of Helmholtz–Hodge type for its solutions, which we will combine with the well-posedness of the underlying boundary integral equation for the tangential components in the space of $C^{k+1,\alpha}$ tangent vector fields to the boundary. For this we will need to deal with some regularity results for high order derivatives of generalized volume and single layer potentials arising in the classical potential theory, which will require some potential-theoretic estimates for inhomogeneous singular integral kernels that are relegated to “Appendix A” for simplicity of exposition. Regarding the representation formula, we will introduce and discuss in detail the weakest decay and radiation conditions under which this formula holds (namely, (3) and (4)), as this topic is of independent interest. Notice that many other radiation conditions have been used in the literature for related models: the natural one for the scalar complex-valued Helmholtz equation is the *Sommerfeld radiation condition* and those of the reduced Maxwell system are called the *Silver–Müller radiation conditions* (SM) (see e.g. [9, 10, 37, 45]).

Let us first recall some previous results in the literature about the exterior NIB boundary value problem (5). Although the same problem is studied in [29] for bounded domains and C^1 vector fields, the technique that we present in this section has not been studied in the case of exterior domains and $C^{k,\alpha}$ -regularity. We recall that in [29] it was essential to assume that λ is “regular” with respect to the interior problem. This is the case when λ is not a Dirichlet eigenvalue of the Laplacian in the interior domain, or if it is a simple eigenvalue whose eigenfunction has non-zero mean, so this condition holds generically (as it can be seen e.g. by considering arbitrarily small rescalings of the domain). Related results for exterior domains are proved in [38]. Indeed, the technique used in bounded domain by [43] and [29] (for $\lambda = 0$ and $\lambda \neq 0$, respectively) goes through to the case of $\lambda = 0$ and exterior domains via sharp estimates of harmonic volume and single layer potentials in $C^{1,\alpha}$. In our case λ is a nonzero constant, which leads to inhomogeneous kernels where the estimates in unbounded domains are much harder to obtain.

There is some literature regarding Laplace’s equation in less regular settings (e.g. L^p data and Lipschitz domains). For C^1 domains, [14, 15] solved it via the analysis of harmonic measures and [22] introduced a method of layer potentials. The latter looks like the method that we propose and is supported by Fredholm’s theory: some boundary singular integral operator is shown to be compact and one to one in the C^1 setting, leading to bijectivity and an useful lower estimate that entails the well posedness. For purely Lipschitz domains, compactness does no longer hold [21] whilst bijectivity is preserved [16]. Regarding non-symmetric elliptic operators $L = -\operatorname{div} A(x)\nabla$ in the half-space $(x, t) \in \mathbb{R}^n \times \mathbb{R}^+$, the well posedness of the Dirichlet problem with L^p data [26] follows from the method of “ ε -approximability” and the absolute continuity of the L -harmonic measure with respect to the surface measure.

Let us now analyze the representation formula, the radiation conditions and some existence and uniqueness results for the scalar complex-valued Helmholtz equation. We will introduce some classical notation and powerful tools like the *far field pattern* of a *radiating* solution not only in the homogeneous setting but also in the inhomogeneous one. All these results will be later used and extended to the NIB problem (5) in the subsequent parts of this section.

2.1. Inhomogeneous Helmholtz equation in the exterior domain. The Helmholtz equation with wave number $\lambda \in \mathbb{R}$ in the exterior domain Ω stands for the elliptic PDE

$$\Delta a + \lambda^2 a = 0, \quad x \in \Omega,$$

where the unknown is a possibly complex-valued scalar function $a \in C^2(\Omega, \mathbb{C})$. This equation arises in acoustic and electromagnetic mathematics [10, 37] and in the study of Beltrami fields arising either from the incompressible Euler equation or from the force-free field system of magnetohydrodynamics. Indeed, it can be derived from (1) by taking curl and noting that Beltrami fields are divergence-free when $\lambda \neq 0$.

This relation with the Beltrami equation suggests studying the representation formulas, radiation conditions and uniqueness result for the Helmholtz equation.

Definition 2.1. We will say that a complex-valued scalar function $a \in C^1(\Omega, \mathbb{C})$ verifies

- the L^1 Sommerfeld radiation condition if

$$\int_{\partial B_R(0)} \left| \nabla a(y) \cdot \frac{y}{R} - i\lambda a(y) \right| d_y S = o(R), \quad R \rightarrow +\infty. \tag{8}$$

- the L^1 decay property at infinity if

$$\int_{\partial B_R(0)} |a(y)| d_y S = o(R^2), \quad \text{when } R \rightarrow +\infty. \tag{9}$$

Other stronger radiation conditions may be assumed to obtain representation formulas and certain uniqueness results [10, 37]. For instance, the L^2 Sommerfeld radiation condition

$$\int_{\partial B_R(0)} \left| \nabla a(y) \cdot \frac{y}{R} - i\lambda a(y) \right|^2 d_y S = o(1), \quad R \rightarrow +\infty, \tag{10}$$

implies (8) and, in turns, the classical (L^∞) Sommerfeld radiation condition

$$\sup_{y \in \partial B_R(0)} \left| \nabla a(y) \cdot \frac{y}{R} - i\lambda a(y) \right| = o\left(\frac{1}{R}\right), \quad R \rightarrow +\infty, \tag{11}$$

implies (10). There is another stronger link between the L^2 and L^1 conditions that will be exhibited in the next results. The proof follows from a simple expansion of the square in the L^2 condition (10) and an integration by parts argument in the Helmholtz equation multiplied by the solution itself.

Remark 2.2. Let $a \in C^2(\Omega, \mathbb{C}) \cap C^1(\overline{\Omega}, \mathbb{C})$ be any complex-valued solution to the Helmholtz equation such that (10) holds. Then

$$\lim_{R \rightarrow +\infty} \int_{\partial B_R(0)} \left(\left| \frac{\partial a}{\partial \eta} \right|^2 + \lambda^2 |a|^2 \right) d_x S = -2\lambda \Im \left(\int_S a \frac{\partial \bar{a}}{\partial \eta} d_x S \right).$$

In particular, (10) \Rightarrow (8) + (9) for each complex-valued solution of the Helmholtz equation.

Before showing that this radiation condition leads to aforementioned formula, let us analyze it in the case of the fundamental solution to the 3-D Helmholtz equation,

$$\Gamma_\lambda(x) = \frac{e^{i\lambda|x|}}{4\pi|x|} = \left(\frac{\cos(\lambda|x|)}{4\pi|x|} + i \frac{\sin(\lambda|x|)}{4\pi|x|} \right). \tag{12}$$

Since

$$\nabla\Gamma_\lambda(x) = \left(i\lambda - \frac{1}{|x|} \right) \Gamma_\lambda(x) \frac{x}{|x|}, \tag{13}$$

a straightforward inductive argument shows that all the partial derivatives of $\Gamma_\lambda(x)$ up to second order verify an even stronger version of the Sommerfeld radiation condition (11). Hence we easily infer:

Proposition 2.3. *The fundamental solution of the Helmholtz equation, together with its partial derivatives up to order 2 satisfy the identities*

$$\begin{aligned} \nabla\Gamma_\lambda(x) \cdot \frac{x}{|x|} - i\lambda\Gamma_\lambda(x) &= -\frac{\Gamma_\lambda(x)}{|x|}, \\ \nabla\left(\frac{\partial\Gamma_\lambda}{\partial x_i}\right)(x) \cdot \frac{x}{|x|} - i\lambda\frac{\partial\Gamma_\lambda}{\partial x_i}(x) &= \left(\frac{2}{|x|} - i\lambda\right)\Gamma_\lambda(x)\frac{x_i}{|x|^2}, \\ \nabla\left(\frac{\partial^2\Gamma_\lambda}{\partial x_i\partial x_j}\right)(x) \cdot \frac{x}{|x|} - i\lambda\frac{\partial^2\Gamma_\lambda}{\partial x_i\partial x_j}(x) &= -\nabla\left(\frac{\partial\Gamma_\lambda}{\partial x_i}\right)(x) \cdot \frac{\partial}{\partial x_j}\left(\frac{x}{|x|}\right) \\ &\quad + \frac{\partial}{\partial x_j}\left(\left(\frac{2}{|x|} - i\lambda\right)\Gamma_\lambda(x)\frac{x_i}{|x|^2}\right), \end{aligned}$$

for every $i, j \in \{1, 2, 3\}$. Consequently,

$$\sup_{x \in \partial B_R(0)} \left| \nabla(D^\gamma\Gamma_\lambda)(x) \cdot \frac{x}{R} - i\lambda D^\gamma\Gamma_\lambda(x) \right| = O\left(\frac{1}{R^2}\right), \text{ for } R \rightarrow +\infty,$$

for every multi-index with $|\gamma| \leq 2$.

In particular, $\Gamma_\lambda(x)$ together with its partial derivatives up to order two verify the Sommerfeld radiation condition (11). It is then an easy task to obtain new complex-valued solutions to the homogeneous Helmholtz equation enjoying such radiation condition through the definition of the generalized single layer potentials associated with the kernel $\Gamma_\lambda(x)$.

Proposition 2.4. *Let a be the generalized single layer potential with density $\zeta \in C(S)$ associated with the Helmholtz equation, i.e., $a(x) := (\mathcal{S}_\lambda\zeta)(x) = \int_S \Gamma_\lambda(x-y)\zeta(y) d_yS$, for every $x \in \Omega$. Then, a solves the homogeneous Helmholtz equation $\Delta a + \lambda^2 a = 0$ in the exterior domain Ω . Moreover, a and all its partial derivatives up to second order verify the Sommerfeld radiation condition (11).*

The same result remains true for generalized volume potential with compactly supported densities. In this case, radiating solutions for the inhomogeneous complex-valued Helmholtz equation can be obtained.

Proposition 2.5. *Let a be the generalized volume potential with density $\zeta \in C_c(\overline{\Omega})$ associated with the Helmholtz equation, i.e., $a(x) := (\mathcal{N}_\lambda \zeta)(x) = \int_\Omega \Gamma_\lambda(x-y)\zeta(y) d_y S$, for every $x \in \Omega$. Then, a solves the inhomogeneous Helmholtz equation $-(\Delta a + \lambda^2 a) = \zeta$ in the exterior domain Ω . Moreover, a and all its partial derivatives up to second order verify the Sommerfeld radiation condition (11).*

To establish the representation formula for the inhomogeneous Helmholtz equation, we study the radiation conditions for the volume and single layer potentials, as well as its decay properties at infinity. We will need the *Hardly–Littlewood–Sobolev estimates of fractional integrals* [42, Theorem 1.2.1], which we state not in terms of L^p integrability conditions but in terms of pointwise decay at infinity. The proof follows from similar arguments.

Theorem 2.6. *Consider any dimension N and $0 < \alpha < N$. Define the associated Riesz potential by*

$$R_\alpha(x) := |x|^{-\alpha}, \quad x \in \mathbb{R}^N.$$

For any measurable function $f : \mathbb{R}^N \rightarrow \mathbb{R}$, we have

- (1) *If $f = O(|x|^{-\rho})$ for $|x| \rightarrow +\infty$ and ρ is any nonnegative exponent such that $N - \alpha < \rho < N$ then,*

$$|(R_\alpha * f)(x)| \leq C \| |x|^\rho f \|_{L^\infty(\mathbb{R}^N)} |x|^{(N-\rho)-\alpha},$$

holds for every $x \in \mathbb{R}^N$. Here, C stands for a positive constant that depends on N , α and ρ but do not depend on f .

- (2) *The optimal decay $|x|^{-\alpha}$ is obtained in the compactly supported case, i.e.,*

$$\|(R_\alpha * f)(x)\| \leq C \|f\|_{L^\infty(\mathbb{R}^N)} |x|^{-\alpha},$$

for every $x \in \mathbb{R}^N$, as long as $f \in L^\infty(\mathbb{R}^N)$ has compact support inside some ball $B_{R_0}(0)$. Now, not only does C depend on N and α but also on the size $R_0 > 0$ of the support.

The above results permit obtaining a Stokes-type formula to represent the solutions to the inhomogeneous Helmholtz equation. Now, we deal with the weakest radiation condition, namely, the L^1 Sommerfeld radiation condition and some property of weak decay at infinity in L^1 . We will skip the proof, since it is completely analogous to the more important result for complex-valued solutions of the inhomogeneous Beltrami equation that we present in the next Sect. (Theorem 2.12). See also [10, Theorem 2.4] and [37, Theorem 3.1.1] for a proof with more restrictive radiation conditions that can be recovered from the next stronger version via Remark 2.2.

Theorem 2.7. *Let $a \in C^2(\Omega, \mathbb{C}) \cap C^1(\overline{\Omega}, \mathbb{C})$ be any function which verifies the L^1 Sommerferld radiation condition (8) and the L^1 decay property at infinity (9). Assume that $\Delta a + \lambda^2 a = O(|x|^{-\rho})$ when $|x| \rightarrow +\infty$, for some exponent $2 < \rho < 3$. Then,*

$$\begin{aligned} a(x) = & - \int_\Omega \Gamma_\lambda(x-y)(\Delta a(y) + \lambda^2 a(y)) dy + \int_S \frac{\partial \Gamma_\lambda(x-y)}{\partial \eta(y)} a(y) d_y S \\ & - \int_S \Gamma_\lambda(x-y) \frac{\partial a}{\partial \eta}(y) d_y S, \end{aligned} \tag{14}$$

for every $x \in \Omega$ and, as a consequence, $a = O(|x|^{-(\rho-2)})$, when $|x| \rightarrow +\infty$. Indeed, when $\Delta a + \lambda^2 a$ has compact support, one obtains the optimal decay at infinity, namely, $a = O(|x|^{-1})$, when $|x| \rightarrow +\infty$.

The properties follow from Theorem 2.6 and they may also be found in [10,37]. Notice that the decay rates $|x|^{-(\rho-2)}$ (for the inhomogeneous equation) and $|x|^{-1}$ (for the homogeneous one) are straightforward consequences of the representation formula.

An immediate consequence of the representation formulas in Theorem 2.7 is that a *far field pattern* at infinity exists for each solution to the Helmholtz equation (see [10] for details). It is a very powerful tool since it provides a description of the asymptotic behavior at infinity and easy uniqueness criteria for radiating solutions.

Although most of the literature is only devoted to far field patterns of complex-valued radiating solutions to the homogeneous Helmholtz equation, our problem concerns the inhomogeneous setting. For this, consider any solution $a \in C^2(\Omega, \mathbb{C}) \cap C^1(\overline{\Omega}, \mathbb{C})$ to the inhomogeneous Helmholtz equation

$$-(\Delta a + \lambda^2 a) = f, \quad x \in \Omega,$$

where f is compactly supported in $\overline{\Omega}$ and a verifies both the decay condition (9) and the L^1 Sommerfeld radiation condition (8). Then, Theorem 2.7 leads to

$$a(x) = \int_{\Omega} \Gamma_{\lambda}(x - y) f(y) dy + \int_S \frac{\partial \Gamma_{\lambda}(x - y)}{\partial \eta(y)} a(y) d_y S - \int_S \Gamma_{\lambda}(x - y) \frac{\partial a}{\partial \eta}(y) d_y S.$$

Consider the compact subset $K := \text{supp } f$ and notice the asymptotic behavior

$$\begin{aligned} \Gamma_{\lambda}(x - y) &= \Gamma_{\lambda}(x) \left\{ e^{-i\lambda \frac{x}{|x|} \cdot y} + O\left(\frac{1}{|x|}\right) \right\}, & \text{when } |x| \rightarrow +\infty, \\ \frac{\partial \Gamma_{\lambda}(x - y)}{\partial \eta(y)} &= \Gamma_{\lambda}(x) \left\{ \frac{\partial e^{-i\lambda \frac{x}{|x|} \cdot y}}{\partial \eta(y)} + O\left(\frac{1}{|x|}\right) \right\}, & \text{when } |x| \rightarrow +\infty, \end{aligned}$$

where $O(|x|^{-1})$ is uniform in $y \in K \cup S$ in the first formula and uniform in $y \in S$ in the second one. From here we deduce the asymptotic behavior

$$a(x) = \Gamma_{\lambda}(x) \left\{ a_{\infty} \left(\frac{x}{|x|} \right) + O\left(\frac{1}{|x|}\right) \right\}, \quad \text{when } |x| \rightarrow +\infty, \tag{15}$$

where a_{∞} is called the *far field pattern* of a , and reads as

$$a_{\infty}(\sigma) = \int_{\Omega} e^{-i\lambda \sigma \cdot y} f(y) dy + \int_S \frac{\partial e^{-i\lambda \sigma \cdot y}}{\partial \eta(y)} a(y) d_y S - \int_S e^{-i\lambda \sigma \cdot y} \frac{\partial a}{\partial \eta}(y) d_y S,$$

for each point $\sigma \in \partial B_1(0)$.

It is apparent that a_{∞} is uniquely determined from formula (15). Hence, we can define the following well-defined linear and one to one map

$$\begin{aligned} \mathcal{D}_{\infty} &\longrightarrow C^{\infty}(\partial B_1(0)) \\ a &\longmapsto a_{\infty}, \end{aligned} \tag{16}$$

where the domain of the *far field pattern mapping* is

$$\begin{aligned} \mathcal{D}_{\infty} &:= \{a \in C^2(\Omega, \mathbb{C}) \cap C^1(\overline{\Omega}, \mathbb{C}) : \Delta a + \lambda^2 a \\ &\quad \text{has compact support and (8) and (9) hold}\}. \end{aligned}$$

A similar reasoning leads to an explicit formula for the far field pattern of the derivatives of a , namely,

$$(\nabla a)_\infty(\sigma) = i\lambda a_\infty(\sigma)\sigma, \quad \forall \sigma \in \partial B_1(0). \tag{17}$$

The splitting in (15) ensures that

$$\lim_{R \rightarrow +\infty} \int_{\partial B_R(0)} |a(x)|^2 dx = \frac{1}{4\pi} \int_{\partial B_1(0)} |a_\infty(\sigma)|^2 d_\sigma S. \tag{18}$$

The celebrated *Rellich Lemma* [10, Lemma 2.11] states that the only complex-valued solution $a \in C^2(\Omega, \mathbb{C})$ to the exterior homogeneous Helmholtz equation such that the limit in the left hand side of the preceding formula becomes zero is the zero function identically. Therefore, whenever a solution to the homogeneous Helmholtz equation has a well-defined far field pattern and it vanishes, then a vanishes everywhere.

Combining Rellich Lemma with Remark 2.2, the following uniqueness result follows. It is of great interest to deal with Dirichlet and Neumann boundary value problems in the exterior domain, see [10, Theorem 2.12].

Lemma 2.8. *Consider any solution $a \in C^2(\Omega, \mathbb{C}) \cap C^1(\overline{\Omega}, \mathbb{C})$ to the complex-valued homogeneous Helmholtz equation in the exterior domain Ω fulfilling the L^2 Sommerfeld radiation condition (10). Then, a verifies*

$$\lambda \Im \left(\int_S a(x) \frac{\partial \bar{a}}{\partial \eta}(x) d_x S \right) \leq 0.$$

If the equality holds, then a vanishes everywhere in Ω .

In the case of vector-valued solutions, the decay property and radiation conditions can be considered componentwise. For instance, given any vector-valued solution $u \in C^2(\Omega, \mathbb{C}^3) \cap C^1(\overline{\Omega}, \mathbb{C}^3)$ to

$$-(\Delta u + \lambda^2 u) = F, \quad x \in \Omega,$$

where F is compactly supported, then the decay property and radiation condition read

$$\begin{aligned} \int_{\partial B_R(0)} |u(x)| dx &= o(R^2), & \text{when } R \rightarrow +\infty, \\ \int_{\partial B_R(0)} \left| \text{Jac } u(x) \frac{x}{R} - i\lambda u(x) \right| dx &= o(R), & \text{when } R \rightarrow +\infty. \end{aligned}$$

One can wonder whether there are more natural radiation conditions for vector-valued solutions to Helmholtz equation, see [9, Theorem 4.13] and [45, Section 5, Theorem 2]. Straightforward computations using $\text{curl}(\text{curl } u) - \nabla(\text{div } u) - \lambda^2 u = F$ in Ω , and $\text{curl}(\text{curl}(\text{curl } u)) - \lambda^2 \text{curl } u = \text{curl } F$ in Ω show that the terms associated with the far field patterns vanish and we obtain the radiation conditions

$$\begin{aligned} \sup_{x \in \partial B_R(0)} \left| \frac{x}{R} \times \text{curl } u(x) - \text{div } u(x) \frac{x}{R} + i\lambda u(x) \right| &= o\left(\frac{1}{R}\right), & \text{when } R \rightarrow +\infty, \\ \sup_{x \in \partial B_R(0)} \left| \lambda \frac{x}{R} \times u(x) + i \text{curl } u(x) \right| &= o\left(\frac{1}{R}\right), & \text{when } R \rightarrow +\infty. \end{aligned}$$

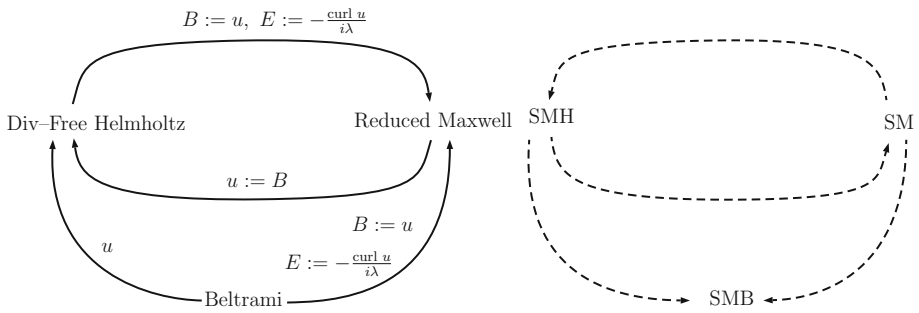


Fig. 1. Sketch of the connections between the three related models: *divergence-free Helmholtz equation, reduced Maxwell system* and *Beltrami equation*. The picture in the left shows the bonds between such models whilst the picture in the right exhibits the associated relations between its natural radiation conditions

When u is a divergence-free solution to the Helmholtz equation (as in our case), the radiation condition are simpler and read

$$\sup_{x \in \partial B_R(0)} \left| \frac{x}{R} \times \text{curl } u(x) + i\lambda u(x) \right| = o\left(\frac{1}{R}\right), \quad \text{when } R \rightarrow +\infty, \quad (19)$$

$$\sup_{x \in \partial B_R(0)} \left| \lambda \frac{x}{R} \times u(x) + i \text{curl } u(x) \right| = o\left(\frac{1}{R}\right), \quad \text{when } R \rightarrow +\infty. \quad (20)$$

2.2. Inhomogeneous Beltrami equation in the exterior domain. Now, we move to the complex-valued inhomogeneous Beltrami equation. In order to understand where the natural radiation condition (4) comes from, we will connect three different systems that will provide an appropriate terminology. The heuristic idea is summarized in Fig. 1. Through the relations between the vector fields u and B in the left hand side of such pictures, we find (see [10, Theorem 6.4] and [45]) that the divergence-free Helmholtz equation and the reduced Maxwell system [10, Definition 6.5] are completely equivalent, i.e.,

$$\begin{cases} \Delta u + \lambda^2 u = 0, & x \in \Omega, \\ \text{div } u = 0, & x \in \Omega. \end{cases} \iff \begin{cases} \text{curl } E - i\lambda B = 0, & x \in \Omega, \\ \text{curl } B + i\lambda E = 0, & x \in \Omega. \end{cases}$$

In order that the solutions to this system could be represented through the classical *Stratton–Chu formulas* [10, Theorem 6.6], the *Silver–Müller radiation conditions* (SM) have to be considered:

$$\begin{aligned} \sup_{x \in \partial B_R(0)} \left| B(x) \times \frac{x}{R} - E(x) \right| &= o\left(\frac{1}{R}\right), \\ \sup_{x \in \partial B_R(0)} \left| E(x) \times \frac{x}{R} + B(x) \right| &= o\left(\frac{1}{R}\right), \quad \text{when } R \rightarrow +\infty. \end{aligned}$$

Due to our choice of B and E , the SM radiation conditions leads to (19)–(20) again. Thus, the natural radiation conditions for the divergence-free vector-valued Helmholtz equation are actually a consequence of the SM radiation conditions for the reduced Maxwell system. Therefore, we will call them the *Silver–Müller–Helmholtz radiation conditions* (SMH).

Let us now consider the case of the Beltrami equation

$$\operatorname{curl} u - \lambda u = 0, \quad x \in \Omega.$$

When $\lambda \neq 0$, then u is a solution to the divergence-free Helmholtz equation, and consequently it also solves the reduced Maxwell system. Therefore, one may want to transfer the SMH or the original SM radiation condition to the Beltrami framework. An easy substitution in (19) and (20) leads to the *Silver–Müller–Beltrami radiation condition* (SMB):

$$\sup_{x \in \partial B_R(0)} \left| i \frac{x}{R} \times u(x) - u(x) \right| = o\left(\frac{1}{R}\right), \quad \text{when } R \rightarrow +\infty.$$

It might seem that the only connection between the Beltrami equation and the divergence-free vector-valued Helmholtz equation is the first implication sketched in Fig. 1, but the connection is actually much stronger. The reason is the following. Given any solution u to the Beltrami equation, it is obviously a solution to the divergence-free Helmholtz equation. The point is that, conversely, given any solution \widehat{u} to the divergence-free Helmholtz equation,

$$u := \frac{\operatorname{curl} \widehat{u} + \lambda \widehat{u}}{2\lambda}. \quad (21)$$

is a solution to the Beltrami equation, and all the solutions can be constructed this way.

In view of this converse relation, it is natural to wonder about the radiation conditions that one should assume on \widehat{u} in order for u to verify the SMB radiation condition. For this, notice that

$$i \frac{x}{R} \times u(x) - u(x) = \frac{i}{2\lambda} \left(\frac{x}{R} \times \operatorname{curl} \widehat{u}(x) + i\lambda \widehat{u}(x) \right) + \frac{i}{2\lambda} \left(\lambda \frac{x}{R} \times \widehat{u}(x) + i \operatorname{curl} \widehat{u}(x) \right),$$

for every $x \in \partial B_R(0)$. Therefore, the SMB radiation condition on u is recovered from the SMH radiation conditions on \widehat{u} , so all the possible links between the three models and its corresponding radiation conditions in Fig. 1 follow.

Remark 2.9. The complex-valued Beltrami fields u satisfying the SMB radiation condition take the form (21) for some solution \widehat{u} of the divergence-free Helmholtz equation satisfying the SMH radiation conditions.

Definition 2.10. We will say that u verify

- (1) the L^1 Silver–Müller–Beltrami condition if

$$\int_{\partial B_R(0)} \left| i \frac{x}{R} \times u(x) - u(x) \right| d_x S = o(R), \quad R \rightarrow +\infty; \quad (22)$$

- (2) the L^1 decay property at infinity if

$$\int_{\partial B_R(0)} |u(x)| d_x S = o(R^2), \quad \text{when } R \rightarrow +\infty. \quad (23)$$

Analogously to the case of the Helmholtz equation, one might consider the L^2 SMB radiation condition

$$\int_{\partial B_R(0)} \left| i \frac{x}{R} \times u(x) - u(x) \right|^2 d_x S = o(1), \quad R \rightarrow +\infty, \tag{24}$$

or the (L^∞) SMB radiation condition

$$\sup_{x \in \partial B_R(0)} \left| i \frac{x}{R} \times u(x) - u(x) \right| = o\left(\frac{1}{R}\right), \quad R \rightarrow +\infty. \tag{25}$$

As in the Helmholtz equation, similar reasonings yield the next remark that links (24) to (22) and (23).

Remark 2.11. Let $u \in C^1(\overline{\Omega}, \mathbb{C}^3)$ be any complex-valued solution to the Beltrami equation such that (24) holds. Then

$$\lim_{R \rightarrow +\infty} \int_{\partial B_R(0)} \left(\left| \frac{x}{R} \times u(x) \right|^2 + |u(x)|^2 \right) d_x S = 2\Im \left(\int_S \overline{u}(x) \cdot (\eta(x) \times u(x)) d_x S \right).$$

In particular, (24) \Rightarrow (22) + (23) for each complex-valued solution of the Beltrami equation.

In the next result we show the desired decomposition theorem of Helmholtz–Hodge type is proved under the above L^1 decay and radiation hypotheses:

Theorem 2.12. *Let $u \in C^1(\overline{\Omega}, \mathbb{C}^3)$ be any vector field which verifies the L^1 SMB condition (22) and (23). Assume that $\operatorname{div} u, \operatorname{curl} u - \lambda u = O(|x|^{-\rho})$ when $|x| \rightarrow +\infty$ for $2 < \rho < 3$. Then, u can be decomposed as $u(x) = -\nabla\phi(x) + \operatorname{curl} A(x) + \lambda A(x)$, for every $x \in \Omega$, where ϕ and A are the scalar and vector fields*

$$\begin{aligned} \phi(x) &= \int_{\Omega} \Gamma_\lambda(x - y) \operatorname{div} u(y) dy + \int_S \Gamma_\lambda(x - y) \eta(y) \cdot u(y) d_y S, \\ A(x) &= \int_{\Omega} \Gamma_\lambda(x - y) (\operatorname{curl} u(y) - \lambda u(y)) dy + \int_S \Gamma_\lambda(x - y) \eta(y) \times u(y) d_y S. \end{aligned}$$

As a consequence, $u = O(|x|^{-(\rho-2)})$, when $|x| \rightarrow +\infty$. Indeed, when both $\operatorname{div} u$ and $\operatorname{curl} u - \lambda u$ are compactly supported, one obtains the optimal decay at infinity, namely, $u = O(|x|^{-1})$, when $|x| \rightarrow +\infty$, and u satisfies the Sommerfeld radiation condition (11) componentwise.

Proof. Consider any $x \in \Omega$ and fix any couple of radii $\varepsilon_0, R_0 > 0$ such that

$$\overline{B_{\varepsilon_0}}(x) \subseteq \Omega \quad \text{and} \quad \overline{B_{\varepsilon_0}}(x) \cup \overline{G} \subseteq B_{R_0}(0).$$

Define the subdomain $\Omega(x, \varepsilon, R) := \Omega \cap (B_R(0) \setminus \overline{B_\varepsilon}(x))$ for $R > R_0$ and $\varepsilon > \varepsilon_0$, as in Fig. 2.

Let $e \in \mathbb{C}^3$ be fixed. Since Γ_λ solves the scalar homogeneous Helmholtz equation outside the origin, then $\Gamma_\lambda e$ is a solution to the vector-valued Helmholtz equation too. Therefore, the following identity

$$0 = - \int_{\Omega(x, \varepsilon, R)} (\Delta(\Gamma_\lambda(x - y)e) + \lambda^2(\Gamma_\lambda(x - y)e)) \cdot u(y) dy$$

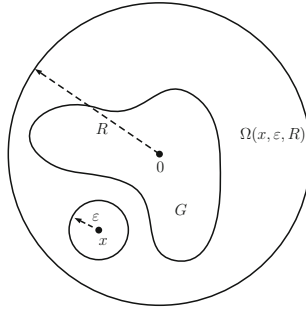


Fig. 2. Domain $\Omega(x, \varepsilon, R)$

holds. As in the classical Helmholtz–Hodge theorem, having in mind $\text{curl}(\text{curl}) = \nabla(\text{div}) - \Delta$, removing the dot product by e , subtracting and adding appropriate terms, we obtain the following formula

$$\begin{aligned}
 0 = & - \int_{\partial\Omega(x, \varepsilon, R)} \nabla_x \Gamma_\lambda(x - y) v(y) \cdot u(y) \, d_y S + \int_{\Omega(x, \varepsilon, R)} \nabla_x \Gamma_\lambda(x - y) \text{div} \, u(y) \, dy \\
 & + \int_{\partial\Omega(x, \varepsilon, R)} \nabla_x \Gamma_\lambda(x - y) \times (v(y) \times u(y)) \, d_y S \\
 & - \int_{\Omega(x, \varepsilon, R)} \nabla_x \Gamma_\lambda(x - y) \times (\text{curl} \, u(y) - \lambda u(y)) \, dy \\
 & + \lambda \left(- \int_{\Omega(x, \varepsilon, R)} \Gamma_\lambda(x - y) (\text{curl} \, u(y) - \lambda u(y)) \, dy \right. \\
 & \quad \left. + \int_{\partial\Omega(x, \varepsilon, R)} \Gamma_\lambda(x - y) v(y) \times u(y) \, d_y S \right). \tag{26}
 \end{aligned}$$

Taking limits when $\varepsilon \rightarrow 0$ and $R \rightarrow +\infty$ shows that the volume integrals converges to the integral over the whole exterior domain due to the dominated convergence theorem, the *Hardy–Littewood–Sobolev theorem of fractional integration* (Theorem 2.6) and the hypotheses on $\text{div} \, u$ and $\text{curl} \, u - \lambda u$:

$$\begin{aligned}
 \int_{\Omega(x, \varepsilon, R)} \nabla_x \Gamma_\lambda(x - y) \text{div} \, u(y) \, dy & \longrightarrow \int_{\Omega} \nabla_x \Gamma_\lambda(x - y) \text{div} \, u(y) \, dy, \\
 \int_{\Omega(x, \varepsilon, R)} \nabla_x \Gamma_\lambda(x - y) \times (\text{curl} \, u(y) - \lambda u(y)) \, dy & \longrightarrow \int_{\Omega} \nabla_x \Gamma_\lambda(x - y) \\
 & \quad \times (\text{curl} \, u(y) - \lambda u(y)) \, dy, \\
 \int_{\Omega(x, \varepsilon, R)} \Gamma_\lambda(x - y) (\text{curl} \, u(y) - \lambda u(y)) \, dy & \longrightarrow \int_{\Omega} \Gamma_\lambda(x - y) (\text{curl} \, u(y) \\
 & \quad - \lambda u(y)) \, dy,
 \end{aligned}$$

when $\varepsilon \rightarrow 0$ and $R \rightarrow +\infty$. Regarding the boundary integrals, it is worth splitting them into the three connected components of the boundary surface of $\Omega(x, \varepsilon, R)$, that is, $\partial\Omega(x, \varepsilon, R) = S \cup \partial B_\varepsilon(x) \cup \partial B_R(0)$. Since the integrals over S are not relevant in the limit $\varepsilon \rightarrow 0$ and $R \rightarrow +\infty$, we focus on the two remaining terms. On the one hand,

using (13) and Lagrange’s formula $v = (e \cdot v) e - e \times (e \times v)$, for any unit vector e and any general vector v , the boundary terms over the sphere $\partial B_\varepsilon(x)$ can be written as

$$\begin{aligned} I_\varepsilon &:= - \left(i\lambda - \frac{1}{\varepsilon} \right) \frac{e^{i\lambda\varepsilon}}{4\pi\varepsilon} \int_{\partial B_\varepsilon(x)} u(y) d_y S - \lambda \frac{e^{i\lambda\varepsilon}}{4\pi\varepsilon} \int_{\partial B_\varepsilon(x)} \frac{y-x}{\varepsilon} \times u(y) d_y S \\ &= i\lambda \frac{e^{i\lambda\varepsilon}}{4\pi\varepsilon} \int_{\partial B_\varepsilon(x)} \left(i \frac{y-x}{\varepsilon} \times u(y) - u(y) \right) d_y S + \frac{e^{i\lambda\varepsilon}}{4\pi\varepsilon^2} \int_{\partial B_\varepsilon(x)} u(y) d_y S. \end{aligned}$$

Consequently, the first term converges to zero as $\varepsilon \rightarrow 0$ while the second term converges to $u(x)$ due to the properties of the mean value over spheres of continuous functions.

In addition, the boundary terms over $\partial B_R(0)$ may also be written in a similar way

$$\begin{aligned} I_R &:= \int_{\partial B_R(0)} \left\{ -\nabla_x \Gamma_\lambda(x-y) \frac{y}{R} \cdot u(y) + \nabla_x \Gamma_\lambda(x-y) \right. \\ &\quad \left. \times \left(\frac{y}{R} \times u(y) \right) + \lambda \Gamma_\lambda(x-y) \frac{y}{R} \times u(y) \right\} d_y S \\ &= \int_{\partial B_R(0)} \left(i\lambda - \frac{1}{|x-y|} \right) \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} \frac{y-x}{|y-x|} \frac{y}{R} \cdot u(y) d_y S \\ &\quad - \int_{\partial B_R(0)} \left\{ \left(i\lambda - \frac{1}{|x-y|} \right) \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} \frac{y-x}{|y-x|} \right. \\ &\quad \left. \times \left(\frac{y}{R} \times u(y) \right) + \lambda \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} \frac{y}{R} \times u(y) \right\} d_y S. \end{aligned}$$

Lagrange’s formula for the triple vector product cannot be directly applied since $B_R(0)$ is not centered at x . See Remark 2.13 below for the behavior of this boundary integrals if we had defined $\Omega(x, \varepsilon, R) = \Omega \cap B_R(x) \cap (\mathbb{R}^3 \setminus \bar{B}_\varepsilon(x))$ instead of $\Omega(x, \varepsilon, R) = \Omega \cap B_R(0) \cap (\mathbb{R}^3 \setminus \bar{B}_\varepsilon(x))$. Adding and subtracting appropriate terms in order to apply Lagrange’s formula for the triple vector product

$$\begin{aligned} I_R &:= -i\lambda \int_{\partial B_R(0)} \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} \left(i \frac{y}{R} \times u(y) - u(y) \right) d_y S \\ &\quad - \int_{\partial B_R(0)} \frac{e^{i\lambda|x-y|}}{4\pi|x-y|^2} u(y) d_y S \\ &\quad + \int_{\partial B_R(0)} \left(i\lambda - \frac{1}{|x-y|} \right) \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} \left(\frac{y-x}{|y-x|} - \frac{y}{R} \right) \frac{y}{R} \cdot u(y) d_y S \\ &\quad - \int_{\partial B_R(0)} \left(i\lambda - \frac{1}{|x-y|} \right) \frac{e^{i\lambda|x-y|}}{4\pi|x-y|} \left(\frac{y-x}{|y-x|} - \frac{y}{R} \right) \times \left(\frac{y}{R} \times u(y) \right) d_y S. \end{aligned}$$

Then, a mean value argument leads to the following bound of the norm of I_R for $R > |x|$

$$\begin{aligned} |I_R| &\leq \frac{|\lambda|}{4\pi(R-|x|)} \int_{\partial B_R(0)} \left| i \frac{y}{R} \times u(y) - u(y) \right| d_y S \\ &\quad + \frac{1}{4\pi(R-|x|)^2} \int_{\partial B_R(0)} |u(y)| d_y S + \frac{2C|x|}{4\pi(R-|x|)^2} \int_{\partial B_R(0)} |u(y)| d_y S. \quad (27) \end{aligned}$$

Thereby, $I_R \rightarrow 0$ when $R \rightarrow +\infty$, thanks to the L^1 SMB radiation condition (22) and the decay property (23).

Now that we have the representation formula in the statement of the theorem, the asymptotic behavior at infinity follows from Theorem 2.6 and the componentwise Sommerfeld radiation condition in the compactly supported case is a direct consequence of Propositions 2.4 and 2.5. \square

Remark 2.13. Consider $\Omega(x, \varepsilon, R) = \Omega \cap B_R(x) \cap (\mathbb{R}^3 \setminus \overline{B_\varepsilon(x)})$ instead of $\Omega(x, \varepsilon, R) = \Omega \cap B_R(0) \cap (\mathbb{R}^3 \setminus \overline{B_\varepsilon(x)})$ in Eq. (26). We can argue in the same way both for the boundary terms over $\partial B_\varepsilon(x)$ and for those over $\partial B_R(x)$. Then, the former has already been studied in the above proof and the later reads

$$\begin{aligned} I_R &:= \left(i\lambda - \frac{1}{R} \right) \frac{e^{i\lambda R}}{4\pi R} \int_{\partial B_R(x)} u(y) d_y S + \lambda \frac{e^{i\lambda R}}{4\pi R} \int_{\partial B_R(x)} \frac{y-x}{R} \times u(y) d_y S \\ &= -i\lambda \frac{e^{i\lambda R}}{4\pi R} \int_{\partial B_R(x)} \left(i \frac{y-x}{\varepsilon} \times u(y) - u(y) \right) d_y S - \frac{e^{i\lambda R}}{4\pi R^2} \int_{\partial B_R(x)} u(y) d_y S. \end{aligned} \tag{28}$$

Therefore, the same representation theorem might have been obtained from the following radiation and decay conditions

$$\begin{aligned} \int_{\partial B_R(x)} \left(i \frac{y-x}{\varepsilon} \times u(y) - u(y) \right) d_y S &= o(R), \quad \text{when } R \rightarrow +\infty, \\ \int_{\partial B_R(x)} u(y) d_y S &= o(R^2), \quad \text{when } R \rightarrow +\infty, \end{aligned}$$

for every $x \in \Omega$. The hypotheses are stronger than (22) and (23) in the sense that they have to be assumed on every $x \in \Omega$. However, they are weaker in the sense that norms can be removed here. Therefore, one might take advantage of certain geometric cancellations of our vector fields to ensure these conditions.

An obvious but interesting feature of the above boundary terms is that in both cases, when $\Omega(x, \varepsilon, R) = \Omega \cap B_R(0) \cap (\mathbb{R}^3 \setminus \overline{B_\varepsilon(x)})$ (27) and $\Omega(x, \varepsilon, R) = \Omega \cap B_R(x) \cap (\mathbb{R}^3 \setminus \overline{B_\varepsilon(x)})$ (28), the harmonic case $\lambda = 0$ does not need to prescribe any radiation condition at infinity, as it is the case in the classical Helmholtz–Hodge theorem and in [38,43].

Again, Remark 2.11 and the Rellich lemma [10, Lemma 2.11] yields an uniqueness result, which is similar to that for the reduced Maxwell system in [10, Theorem 6.10]:

Lemma 2.14. *Consider any solution $u \in C^1(\overline{\Omega}, \mathbb{C}^3)$ to the complex-valued homogeneous Beltrami equation in the exterior domain satisfying the L^2 SMB radiation condition (24). Then, u verifies the inequality*

$$\Im \left(\int_S \overline{u}(x) \cdot (\eta(x) \times u(x)) d_x S \right) \geq 0.$$

If the equality holds, then u vanishes everywhere in Ω .

To conclude, let us state the existence result for the complex-valued inhomogeneous Beltrami equation that will be needed in the modified Grad–Rubin iterative scheme in Sect. 3. Since this iterative method only involves compactly supported inhomogeneities,

we will focus on this case although it is easy to extend it to general inhomogeneous terms with an appropriate fall off at infinity. Hereafter we will denote by $\mathfrak{X}^{k,\alpha}(S) \equiv \mathfrak{X}^{k,\alpha}(S, \mathbb{R}^3)$ the real vector space of all tangent vector fields on S of regularity $C^{k,\alpha}$, i.e.,

$$\mathfrak{X}^{k,\alpha}(S) := \{\xi \in C^{k,\alpha}(S, \mathbb{R}^3) : \xi \cdot \eta = 0 \text{ on } S\}.$$

Its complex counterpart will be denoted by $\mathfrak{X}^{k,\alpha}(S, \mathbb{C}^3)$.

Theorem 2.15. *Let $0 \neq \lambda \in \mathbb{R}$ be any constant that is not a Dirichlet eigenvalue of the Laplace operator in the interior domain, $w \in C_c^{k,\alpha}(\bar{\Omega}, \mathbb{C}^3)$ and $g \in C^{k+1,\alpha}(S, \mathbb{C})$ such that $\operatorname{div} w \in C^{k,\alpha}(\bar{\Omega}, \mathbb{C})$ and the following compatibility condition*

$$\int_S (\lambda g + w \cdot \eta) \, dS = 0 \tag{29}$$

is satisfied. Consider any solution $\xi \in \mathfrak{X}^{k+1,\alpha}(S, \mathbb{C}^3)$ to the boundary integral equation

$$\left(\frac{1}{2}I - T_\lambda\right)\xi = \mu, \quad x \in S, \tag{30}$$

where $T_\lambda \xi$ and μ are defined by

$$(T_\lambda \xi)(x) = \int_S \eta(x) \times (\nabla_x \Gamma_\lambda(x - y) \times \xi(y)) \, d_y S + \lambda \int_S \Gamma_\lambda(x - y) \eta(x) \times \xi(y) \, d_y S, \tag{31}$$

$$\begin{aligned} \mu(x) &= \frac{1}{\lambda} \int_\Omega \eta(x) \times \nabla_x \Gamma_\lambda(x - y) \operatorname{div} w(y) \, d_y \\ &\quad - \int_S \eta(x) \times \nabla_x \Gamma_\lambda(x - y) g(y) \, d_y S \\ &\quad + \int_\Omega \eta(x) \times (\nabla_x \Gamma_\lambda(x - y) \times w(y)) \, d_y \\ &\quad + \lambda \int_S \Gamma_\lambda(x - y) \eta(x) \times w(y) \, d_y S. \end{aligned} \tag{32}$$

Define the complex-valued vector field

$$u(x) := -\nabla \phi(x) + \operatorname{curl} A(x) + \lambda A(x), \quad x \in \Omega, \tag{33}$$

where ϕ and A stand for the scalar and vector fields

$$\phi(x) = -\frac{1}{\lambda} \int_\Omega \Gamma_\lambda(x - y) \operatorname{div} w(y) \, d_y + \int_S \Gamma_\lambda(x - y) g(y) \, d_y S, \tag{34}$$

$$A(x) = \int_\Omega \Gamma_\lambda(x - y) w(y) \, d_y + \int_S \Gamma_\lambda(x - y) \xi(y) \, d_y. \tag{35}$$

Then, u is a complex-valued solution to the exterior NIB problem

$$\begin{cases} \operatorname{curl} u - \lambda u = w, & x \in \Omega, \\ u \cdot \eta = g, & x \in \Omega, \\ + L^1 \text{ SMB radiation condition (22),} \\ + L^1 \text{ decay property (23).} \end{cases} \tag{36}$$

Furthermore, the decay and radiation conditions are stronger since u behaves as $O(|x|^{-1})$ at infinity and the Sommerfeld radiation condition (11) holds component-wise.

Proof. Since the divergence of any solution u can be recovered from the equation through the identity $\operatorname{div} u = -\frac{1}{\lambda} \operatorname{div} w$, then one arrives at the next expression for the candidate to be a solution to (36)

$$u(x) = -\nabla\phi(x) + \operatorname{curl} A(x) + \lambda A(x),$$

where ϕ and A are defined as follows

$$\begin{aligned} \phi(x) &= -\frac{1}{\lambda} \int_{\Omega} \Gamma_{\lambda}(x-y) \operatorname{div} w(y) dy + \int_S \Gamma_{\lambda}(x-y) g(y) d_y S, \\ A(x) &= \int_{\Omega} \Gamma_{\lambda}(x-y) w(y) dy + \int_S \Gamma_{\lambda}(x-y) \eta(y) \times u_+(y) d_y S. \end{aligned}$$

Consider $\xi := \eta \times u_+$, where u_{\pm} denotes the limits of u at S from Ω and G respectively. In order to obtain a more manageable formula for ξ , one can use the well known *jump relations* for the derivatives of a single layer potential associated with the fundamental solution to the Helmholtz equation, $\Gamma_{\lambda}(x)$ (see e.g. [9]). This formulas lead to the following identity

$$\begin{aligned} u_{\pm}(x) &= \frac{1}{\lambda} \int_{\Omega} \nabla_x \Gamma_{\lambda}(x-y) \operatorname{div} w(y) dy - \operatorname{PV} \int_S \nabla_x \Gamma_{\lambda}(x-y) g(y) d_y S \\ &\quad + \int_{\Omega} \nabla_x \Gamma_{\lambda}(x-y) \times w(y) dy + \operatorname{PV} \int_S \nabla_x \Gamma_{\lambda}(x-y) \times \xi(y) d_y S \\ &\quad + \lambda \int_{\Omega} \Gamma_{\lambda}(x-y) w(y) dy + \lambda \int_S \Gamma_{\lambda}(x-y) \xi(y) d_y S \\ &\quad \pm \frac{1}{2} \eta(x) g(x) \mp \frac{1}{2} \eta(x) \times \xi(x), \end{aligned} \tag{37}$$

where PV stands for the Cauchy principal value integral. It is clear that the terms in the last line are actually $\pm \frac{1}{2} u_{\pm}(x)$. Consequently, one can take cross products by $\eta(x)$ and arrive at the boundary integral equation in (30) for the tangential component ξ . There, we have intentionally avoided the PV signs because the $\eta(x)$ factor in such integrals provides certain geometrical cancellations (see ‘‘Appendix A’’) leading to absolutely convergent integrals.

Now, let us show that the field u thus defined is a solution to (36) as long as ξ solves the boundary integral equation (30). We will prove later that ξ is unique and, consequently, (36) is uniquely solvable. First, let us obtain some PDEs for the potentials ϕ and A both in the interior and the exterior domain. Since volume and single layer potentials are indeed complex-valued solutions to such PDEs, we have

$$\Delta\phi + \lambda^2\phi = \begin{cases} \frac{1}{\lambda} \operatorname{div} w, & x \in \Omega \\ 0, & x \in G \end{cases} \quad \Delta A + \lambda^2 A = \begin{cases} -w, & x \in \Omega \\ 0, & x \in G \end{cases} \tag{38}$$

Therefore,

$$\begin{aligned} \operatorname{curl} u - \lambda u &= \nabla(\operatorname{div} A) - \Delta A + \lambda \operatorname{curl} A + \lambda \nabla\phi - \lambda \operatorname{curl} A - \lambda^2 A \\ &= -(\Delta A + \lambda^2 A) + \nabla \underbrace{(\operatorname{div} A + \lambda\phi)}_a. \end{aligned}$$

A direct substitution of (38) leads to the following PDE for u at any side of the boundary surface S :

$$\operatorname{curl} u - \lambda u = \begin{cases} w + \nabla a, & x \in \Omega, \\ \nabla a, & x \in G. \end{cases} \quad (39)$$

In order to show that u solves (36), it remains to check that ∇a is zero in the exterior domain and u satisfies the boundary condition $u_+ \cdot \eta = g$ (the decay and radiation conditions will be studied later). To this end, it might be useful to find first a PDE for a . The same reasoning as above shows that a solves both in Ω and in G the homogeneous Helmholtz equation, specifically

$$\Delta a + \lambda^2 a = \operatorname{div}(\Delta A) + \lambda \Delta \phi + \lambda^2 \operatorname{div} A + \lambda^3 \phi = \operatorname{div}(\Delta A + \lambda^2 A) + \lambda(\Delta \phi + \lambda^2 \phi) = 0. \quad (40)$$

Let us show first the jump relations for the scalar potential a . Straightforward computations on the explicit formulas for ϕ and A involving the divergence theorem lead to

$$\begin{aligned} a(x) &= \operatorname{div} A(x) + \lambda \phi(x) \\ &= \int_{\Omega} \{\nabla_x \Gamma_{\lambda}(x-y) \cdot w(y) - \Gamma_{\lambda}(x-y) \operatorname{div} w(y)\} dy \\ &\quad + \int_S \{\nabla_x \Gamma_{\lambda}(x-y) \cdot \xi(y) + \lambda \Gamma_{\lambda}(x-y) g(y)\} d_y S \\ &= - \int_{\Omega} \operatorname{div}_y (\Gamma_{\lambda}(x-y) w(y)) dy + \int_S \nabla_x \Gamma_{\lambda}(x-y) \cdot \xi(y) d_y S \\ &\quad + \lambda \int_S \Gamma_{\lambda}(x-y) g(y) d_y S \\ &= \int_S \Gamma_{\lambda}(x-y) (\lambda g(y) + w(y) \cdot \eta(y)) d_y S + \int_S \nabla_x \Gamma_{\lambda}(x-y) \cdot \xi(y) d_y S. \end{aligned}$$

Finally, notice that $\nabla_x \Gamma_{\lambda}(x-y) \cdot \xi(y) = -(\nabla_S)_y [\Gamma_{\lambda}(x-y)] \cdot \xi(y)$ for every $y \in S$ because of ξ being a tangent vector field along S . Hence, the integration by parts formula over S yields the next simpler expression for a :

$$a(x) = \int_S \Gamma_{\lambda}(x-y) (\lambda g(y) + w(y) \cdot \eta(y) + \operatorname{div}_S \xi(y)) d_y S,$$

i.e., a is just a new single layer potential. As such, the first and second jumps relations read

$$a_+ - a_- \equiv 0, \quad \left(\frac{\partial a}{\partial \eta}\right)_+ - \left(\frac{\partial a}{\partial \eta}\right)_- \equiv -(\lambda g + w \cdot \eta + \operatorname{div}_S \xi), \quad (41)$$

on the surface S . In particular, a is continuous across S but its normal derivative exhibits a jump discontinuity with height $\lambda g + w \cdot \eta + \operatorname{div}_S \xi$. The same kind of reasoning yields the jump relation for u

$$u_+ - u_- = g \eta - \eta \times \xi, \quad x \in S. \quad (42)$$

Consequently, the boundary integral equation (30) along with the jump relation (42) ensure that

$$\eta \times u_+ = \xi, \quad \eta \times u_- = 0, \quad (43)$$

on S . Regarding a , let us show that it is indeed constant on S and to this end, define the next vector field in the interior domain G :

$$v := \lambda u + \nabla a, \quad x \in G.$$

Notice that v is a strong Beltrami field with factor λ by virtue of (39). Then, one can repeat the same kind of uniqueness criterion as in Lemma 2.14 in the simpler bounded setting, specifically

$$\lambda \int_G |v|^2 dx = \int_G \bar{v} \cdot \operatorname{curl} v dx = \int_G \operatorname{div}(v \times \bar{v}) dx = \int_S (\eta \times v) \cdot \bar{v} dS.$$

Now, notice that we can substitute both v and \bar{v} in the above formula with its tangential parts thanks to the presence of a cross product by the unit normal vector field η and

$$-\eta \times (\eta \times v) = -\lambda \eta \times (\eta \times u_-) + \nabla_S a = \nabla_S a,$$

by virtue of (43). Thereby, an integration by parts leads again to

$$\lambda \int_G |v|^2 dx = \int_S (\eta \times \nabla_S a) \cdot \nabla_S \bar{a} dS = - \int_S a \overline{\operatorname{curl}_S (\nabla_S a)} dS = 0,$$

where the well know formula $\operatorname{curl}_S \nabla_S = 0$ has been used in the last step. Consequently, v vanishes everywhere in G and, in particular, $\nabla_S a \equiv 0$, i.e., $a_{\pm} \equiv a_0 = \text{const}$ on S .

We will next prove that a vanishes everywhere in the exterior domain Ω using the uniqueness result in Lemma 2.8. Notice that since a can be written as a sum of volume and single layer potentials with compactly supported densities together with its first order partial derivatives, then a satisfies a stronger Sommerfeld radiation condition due to Propositions 2.4 and 2.5. Consequently, this lemma can be applied. We therefore want to show that

$$\Im \left(\int_S a_+ \left(\frac{\partial \bar{a}}{\partial \eta} \right)_+ dS \right) = 0. \tag{44}$$

To derive (44), we first pass from the exterior to the interior trace values using the jump relations (41)

$$\int_S a_+ \left(\frac{\partial \bar{a}}{\partial \eta} \right)_+ dS = a_0 \int_S (\lambda g + w \cdot \eta + \operatorname{div}_S \xi) dS + \int_S a_- \left(\frac{\partial \bar{a}}{\partial \eta} \right)_- dS = I + II.$$

On the one hand, I becomes zero because of the divergence theorem over surfaces and the compatibility condition (29) in the hypothesis. On the other hand, integrate by parts in II to arrive at

$$II := \int_S \operatorname{div} (a \nabla \bar{a}) dS = \int_G |\nabla a|^2 dx + \int a \Delta \bar{a} dx = \int_G |\nabla a|^2 dx - \lambda^2 \int_G |a|^2 dx,$$

where the Helmholtz equation (40) has being used. Therefore, one arrives at

$$\Im \left(\int_S a_+ \left(\frac{\partial \bar{a}}{\partial \eta} \right)_+ dS \right) = \Im \left(\int_G |\nabla a|^2 dx - \lambda^2 \int_G |a|^2 dx \right) = 0,$$

and consequently $a = 0$ in Ω and u solves the inhomogeneous Beltrami equation.

Before proving the boundary condition and the decay and radiation properties, let us show that a also vanishes in the interior domain. On the one hand, a solves the homogeneous Helmholtz equation in such domain and it also satisfies the interior homogeneous

Dirichlet conditions in S since $a_- = a_+$ on S and $a = 0$ in Ω . Moreover, λ is prevented from being a Dirichlet eigenvalue of the Laplacian in the interior domain, so a also vanishes in G . In particular, the jumps relations (41) yields

$$\lambda g + w \cdot \eta + \operatorname{div}_S \xi \equiv 0. \tag{45}$$

Furthermore, since u is now a solution to the next inhomogeneous Beltrami equation, $\operatorname{curl} u - \lambda u = w$, $x \in \Omega$, taking trace values at S one gets $\eta \cdot (\operatorname{curl} u)_+ - \lambda \eta \cdot u_+ = w \cdot \eta$. Now, one can write the first term in an intrinsic way through $\eta \cdot (\operatorname{curl} u)_+ = -\operatorname{div}_S(\eta \times u_+) = -\operatorname{div}_S \xi$, and, consequently, we have

$$\eta \cdot u_+ + w \cdot \eta + \operatorname{div}_S \xi \equiv 0. \tag{46}$$

Then, comparing (45) and (46) entails the boundary condition $\eta \cdot u_+ = g$.

Finally, let us show the decay and radiation conditions on u . First, since

$$\Gamma_\lambda(x), \nabla \Gamma_\lambda(x) = O(|x|^{-1}), \text{ when } |x| \rightarrow +\infty,$$

and w has compact support, then u enjoys the optimal decay $u = O(|x|^{-1})$ when $|x| \rightarrow +\infty$ according to Theorem 2.6. Second, as u is again a sum of single and volume layer potential associated with the Helmholtz equation along with some partial derivatives, then u satisfies Sommerfeld radiation condition componentwise thanks to Propositions 2.4 and 2.5. Therefore, one can show that u verifies SMH conditions (19) and (20). Since $\operatorname{curl} u - \lambda u = w$ and w is compactly supported, then u actually satisfies the strong SMB radiation condition and this finishes the proof. \square

2.3. Well-posedness of the boundary integral equation. One should also notice that, in addition to the uniqueness result proved in Theorem 2.15, we will also need a study of the regularity of the solution, which is obviously in $C^1(\Omega, \mathbb{C}^3)$ by the decomposition (33). We will prove in this next subsection that the regularity assumptions on the data w and g actually leads to $C^{k+1,\alpha}(\Omega, \mathbb{C}^3)$ regularity on u . Some necessary potential theoretic estimates have been relegated to ‘‘Appendix A’’ to streamline the exposition.

Let us start by studying the well-posedness of (30) using the Riesz–Fredholm theory for compact operators, which follows easily from our previous estimates:

Proposition 2.16. *The linear operator $T_\lambda : \mathfrak{X}^{k+1,\alpha}(S) \longrightarrow \mathfrak{X}^{k+1,\alpha}(S)$ is compact.*

Proof. The gain of regularity proved in Theorem A.10 implies that T_λ defines a continuous linear operator

$$T_\lambda : \mathfrak{X}^{k,\alpha}(S) \longrightarrow \mathfrak{X}^{k+1,\alpha}(S).$$

Since $\mathfrak{X}^{k+1,\alpha}(S) \hookrightarrow \mathfrak{X}^{k,\alpha}(S)$ is compact by the Ascoli–Arzelà theorem, the proposition follows. \square

The proposition ensures that it is possible to apply Riesz–Fredholm theory to the operator $\frac{1}{2}I - T_\lambda$. In particular, $\frac{1}{2}I - T_\lambda$ is one to one if, and only if, it is onto, i.e.,

$$\operatorname{Ker} \left(\frac{1}{2}I - T_\lambda \right) = 0 \iff \operatorname{Im} \left(\frac{1}{2}I - T_\lambda \right) = \mathfrak{X}^{k+1,\alpha}(S).$$

As it is hard to show explicitly that such operator is onto, let us equivalently show that it is one to one. This is a consequence of the uniqueness Lemma 2.14 and the existence Theorem 2.15.

Proposition 2.17. *The bounded linear operator $\frac{1}{2}I - T_\lambda$ on $\mathfrak{X}^{k+1,\alpha}(S)$ is one to one and onto. Consequently, the boundary integral equation (30) has a unique solution $\xi \in \mathfrak{X}^{k+1,\alpha}(S)$ for any $\mu \in \mathfrak{X}^{k+1,\alpha}(S)$.*

Proof. According to the preceding argument, we only have to show that $\text{Ker}(\frac{1}{2}I - T_\lambda) = 0$. To this end, let us consider an arbitrary $\xi \in \text{Ker}(\frac{1}{2}I - T_\lambda)$ and show that $\xi \equiv 0$. By definition, $\xi \in \mathfrak{X}^{k+1,\alpha}(S)$ solves the boundary integral equation $\frac{1}{2}\xi - T_\lambda\xi = 0$ on S . Define $u(x) := \text{curl} A(x) + \lambda A(x)$, where A is the vector potential $A(x) := \int_S \Gamma_\lambda(x - y)\xi(y) d_y S$. Thus, Theorem 2.15 for $w \equiv 0$ and $g \equiv 0$ leads to a solution $u \in C^1(\bar{\Omega}, \mathbb{C}^3)$ to the homogeneous Beltrami equation in Ω

$$\begin{cases} \text{curl } u = \lambda u, & x \in \Omega, \\ \eta \cdot u_+ = g = 0, & x \in S, \end{cases}$$

that satisfies the Dirichlet boundary condition $\eta \times u_+ = \xi$ on S and the SMB radiation condition.

We would like to show that this boundary value problem has a unique solution, but this does not follow directly from Lemma 2.14. However, since $\eta \cdot u_+ = 0$ on S , then $u_+ = -\eta \times (\eta \times u_+)$ on S and we have the following relation between the curl operator on S , curl_S , and the curl operator on \mathbb{R}^3 :

$$\text{curl}_S u_+ = \text{curl}_S (-\eta \times (\eta \times u_+)) = \eta \cdot \text{curl } u_+ = \lambda \eta \cdot u_+ = 0.$$

As S is homeomorphic to a sphere, Poincaré’s lemma shows that u_+ has a potential $\psi \in C^2(S)$ on the surface, $u_+ = \nabla_S \psi$ on S , where ∇_S stands for the Riemannian connection on the surface S . Consequently,

$$\begin{aligned} \Im \left(\int_S \bar{u}_+ \cdot (\eta \times u_+) dS \right) &= \Im \left(\int_S \overline{\nabla_S \psi} \cdot (\eta \times \nabla_S \psi) dS \right) \\ &= -\Im \left(\int_S \overline{\text{curl}_S (\nabla_S \psi)} \psi dS \right) = 0. \end{aligned}$$

The identity follows from an integration by parts on S and the classical property $\text{curl}_S(\nabla_S \psi) = 0$. Therefore, Lemma 2.14 yields the desired result. \square

Remark 2.18. The importance of the above result lies on the following facts.

- (1) First, the existence part of the above result ensures that it is possible to choose some ξ solving (30). Obviously, it is essential to rigorously establish the existence Theorem 2.15.
- (2) Second, the uniqueness result shows that since ξ can be uniquely chosen, then (36) has a unique solution too.
- (3) Finally, it provides a very useful estimate for the subsequent result. Since $\frac{1}{2}I - T_\lambda$ is linear, continuous and bijective, then $(\frac{1}{2}I - T_\lambda)^{-1}$ is continuous by virtue of the Banach isomorphism theorem. Consequently, there exists a positive constant c (which depends on G and λ) such that

$$c \|\xi\|_{C^{k+1,\alpha}(S)} \leq \left\| \left(\frac{1}{2}I - T_\lambda \right) \xi \right\|_{C^{k+1,\alpha}(S)}, \tag{47}$$

for any $\xi \in \mathfrak{X}^{k+1,\alpha}(S)$.

We conclude by proving the following regularity result for the solution u of (36) according to Theorem 2.15. It is an immediate consequence of the decomposition (33), the estimates for the volume and single layer potentials in “Appendix A” (Lemmas A.9 and A.1) and the estimate (47).

Corollary 2.19. *Assume that the hypothesis in Theorem 2.15 are satisfied, fix any $R > 0$ such that $\overline{G} \subseteq B_R(0)$ and assume that the closure of $\Omega_R := B_R(0) \setminus \overline{G}$ contains the support of w . Then, there exists some nonnegative constant $C_0 = C_0(k, \alpha, G, R, \lambda)$ such that the next estimate*

$$\|u\|_{C^{k+1,\alpha}(\Omega)} \leq C_0 \left\{ \|w\|_{C^{k,\alpha}(\Omega)} + \|\operatorname{div} w\|_{C^{k,\alpha}(\Omega)} + \|g\|_{C^{k+1,\alpha}(S)} \right\}. \tag{48}$$

holds. In particular, not only does u belong to $C^1(\overline{\Omega}, \mathbb{C}^3)$, but also to $C^{k+1,\alpha}(\overline{\Omega}, \mathbb{C}^3)$.

2.4. Optimal fall-off in exterior domains. It is worth discussing the differences between the optimal fall-off $|x|^{-1}$ of the solutions to inhomogeneous Beltrami equation and that of the solutions of the div-curl problem. First, it is well known that the exterior Neumann boundary value problem associated with the div-curl system

$$\begin{cases} \operatorname{curl} u = w, & x \in \Omega, \\ \operatorname{div} u = f, & x \in \Omega, \\ u \cdot \eta = g, & x \in S, \\ u = O(|x|^{1-\rho}), & x \in \Omega, \end{cases} \tag{49}$$

where $w, f = O(|x|^{-\rho})$ and $\rho \in (1, 3)$, is uniquely solvable when appropriate regularity spaces are considered (see [27, 38]) and w has zero flux in the exterior domain. Moreover, the solution inherits the optimal fall-off $|x|^{-2}$ when w and f are assumed to have compact support. In particular, any harmonic field ($w = 0, f = 0$) so obtained decays at infinity as $|x|^{-2}$. Such result is an easy consequence of the Helmholtz–Hodge representation formula in [38, Theorem 4.1] and the natural fall-off of the fundamental solution of the Laplace equation, $\Gamma_0(x)$.

In our case, the exterior Neumann boundary value problem associated with the inhomogeneous Beltrami equation (36) has an associated representation formula of Helmholtz–Hodge type (33) that transfers the “optimal fall-off” $|x|^{-1}$ to the solution in Theorem 2.15 when w is assumed to have compact support. Let us show that it is indeed the optimal decay rate. To this end, assume that u solves the equation

$$\operatorname{curl} u - \lambda u = w, \quad x \in \Omega,$$

(not necessarily fulfilling neither (23) nor (22)) for some divergence-free vector field w . Then, the solution u is divergence-free too. Hence, taking curl in the inhomogeneous Beltrami equation, we are led to the vector-valued Helmholtz equation

$$-(\Delta u + \lambda^2 u) = \lambda w + \operatorname{curl} w, \quad x \in \Omega.$$

Consider $K := \operatorname{supp} w \subseteq \overline{\Omega}$ and notice that $\lambda w + \operatorname{curl} w$ is also compactly supported in K . Imagine that u decayed as $|x|^{-(1+\varepsilon)}$ for some small $\varepsilon > 0$. Hence, a straightforward computation leads to

$$\lim_{R \rightarrow +\infty} \int_{\partial B_R(0)} |u(x)|^2 = 0.$$

Consequently, Rellich’s Lemma would show that u vanishes outside some sufficiently large ball centered at the origin and containing K . Then, the unique continuation principle of the Helmholtz equation allow proving that u is also compactly supported in K (see [31] for the study of such property in many other linear PDEs with constants coefficients). In particular, g would vanish outside $K \cap S$. In an equivalent way, the next result holds.

Corollary 2.20. *Let $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ be a solution to*

$$\operatorname{curl} u - \lambda u = w, \quad x \in \Omega,$$

for a divergence-free compactly supported w and some $\lambda \in \mathbb{R} \setminus \{0\}$. If u is transverse to S at some point outside the support of w , then u cannot decay faster than $|x|^{-1}$ at infinity.

The above Corollary can be interpreted in two different ways. First, it establishes the optimal fall-off of a “transverse” strong Beltrami field ($w = 0$). Second, it also deals with some kind of “transverse” generalized Beltrami fields in exterior domains ($w = \varphi u$) that will be of a great interest in our work. We restrict to the second result since it contains the first one as a particular case.

Corollary 2.21. *Let $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ be a generalized Beltrami field, i.e.,*

$$\begin{cases} \operatorname{curl} u - f u = 0, & x \in \Omega, \\ \operatorname{div} u = 0, & x \in \Omega, \end{cases}$$

whose proportionality factor is a compactly supported perturbation of a constant proportionality factor $\lambda \in \mathbb{R} \setminus \{0\}$, i.e., $f = \lambda + \varphi$ for some $\varphi \in C_c^{k,\alpha}(\overline{\Omega})$. If u is transverse to S at some point outside the support of the perturbation φ , then u cannot decay faster than $|x|^{-1}$ at infinity.

Remark 2.22. In particular, the above result leads to the natural counterpart for exterior domain of the Liouville theorem in [7, 36] about the fall-off of entire generalized Beltrami fields. Such theorem states that there is no globally defined generalized Beltrami field decaying faster than $|x|^{-1}$ at infinity. As many others Liouville type results, it strongly depends on the solution being defined in the whole \mathbb{R}^3 . In our case we remove this hypothesis but, in return, we need to argue with generalized Beltrami fields with constant proportionality factor outside a compact set enjoying some transversality condition on the boundary surface of the exterior domain.

3. An Iterative Scheme for Strong Beltrami Fields

Our objective in this section is to set the iterative scheme that we will use to establish the partial stability of strong Beltrami fields that will yield the existence of almost global Beltrami fields with a non-constant factor and complex vortex structures.

3.1. Further notation and preliminaries. On the differentiable surface S , we will consider local charts of the same regularity as S (that is, maps μ covering open subsets $\Sigma \subseteq S$ of the form

$$\mu : D \longrightarrow \mathbb{R}^3,$$

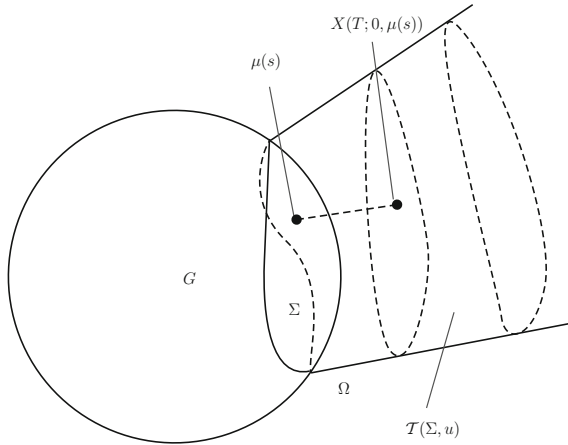


Fig. 3. Stream lines and tubes of the velocity field u

where $\mu(D) = \Sigma$ and D is a disk in the plane). We will assume μ to be a local parametrization up to the boundary so that μ can be homeomorphically extended to the closure \overline{D} , $\overline{\Sigma} = \mu(\overline{D})$ (Fig. 3).

We will also consider the corresponding C^k and $C^{k,\alpha}$ spaces of functions defined on a coordinate neighborhood Σ of S provided with a local chart μ . Up to the degree of smoothness of the surface, by compactness they are known to be independent of the choice of the chart, so one can write

$$C^k(\Sigma) := \{f : \Sigma \rightarrow \mathbb{R} : f \circ \mu \in C^k(D)\}, \text{ and } C^{k,\alpha}(\Sigma) := \{f : \Sigma \rightarrow \mathbb{R} : f \circ \mu \in C^{k,\alpha}(D)\}$$

and similarly for spaces on $\overline{\Sigma}$. These spaces can be respectively endowed with the complete norms

$$\|f\|_{C^k(\Sigma,\mu)} := \|f \circ \mu\|_{C^k(D)}, \quad \|f\|_{C^{k,\alpha}(\Sigma,\mu)} := \|f \circ \mu\|_{C^{k,\alpha}(D)},$$

where the dependence on μ will be removed if it is apparent from the context.

An useful result is *Calderón's extension theorem* for $C^{k,\alpha}$ functions, see e.g. [23, Lemma 6.37]:

Proposition 3.1. *Let $O \subseteq \mathbb{R}^3$ be a $C^{k,\alpha}$ domain with bounded boundary ∂O , and let O' be any open subset such that $\overline{O} \subseteq O'$. Then, there exists a linear operator*

$$\mathcal{P} : C^{k,\alpha}(\overline{O}) \rightarrow C^{k,\alpha}(\overline{O'}),$$

$\mathcal{P}(f) \equiv \overline{f}$, such that

- (1) \mathcal{P} is an extension operator, i.e., $\mathcal{P}(f)|_O = f, \forall f \in C^{k,\alpha}(\overline{O})$.
- (2) The support of $\mathcal{P}(f)$ is contained in the open subset O' for every $f \in C^{k,\alpha}(\overline{O})$.
- (3) \mathcal{P} is continuous in the $C^{k,\alpha}$ topology, i.e.,

$$\|\mathcal{P}(f)\|_{C^{k,\alpha}(O')} \leq C_{\mathcal{P}} \|f\|_{C^{k,\alpha}(O)}, \forall f \in C^{k,\alpha}(\overline{O}).$$

(4) \mathcal{P} is also continuous in the C^m topology for any $0 \leq m \leq k$, i.e.,

$$\|\mathcal{P}(f)\|_{C^m(O')} \leq C_{\mathcal{P}} \|f\|_{C^m(O)}, \quad \forall f \in C^{k,\alpha}(\overline{O}).$$

In the above inequalities, $C_{\mathcal{P}}$ stands for a constant which depends on k , O and O' .

To describe the stream lines and tubes associated with a velocity field $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ in presence of a boundary surface which u is not tangent to, it is convenient to consider an extension of the field to obtain the following characterization from the Picard–Lindelöf theorem on Hölder spaces:

Proposition 3.2. *Let $O \subseteq \mathbb{R}^3$ be a $C^{k+1,\alpha}$ bounded domain, where $k \geq 0$ and $0 < \alpha \leq 1$. Consider any vector field $u \in C^{k+1,\alpha}(\overline{O}, \mathbb{R}^3)$, its associated extension $\bar{u} = \mathcal{P}(u) \in C^{k+1,\alpha}(\mathbb{R}^3, \mathbb{R}^3)$ according to Proposition 3.1, any point $x_0 \in \mathbb{R}^3$ and an initial time $t_0 \in \mathbb{R}$. Consider the associated characteristic system*

$$\begin{cases} \frac{dX}{dt} = \bar{u}(X), & t \in \mathbb{R}, \\ X(t_0) = x_0. \end{cases} \tag{50}$$

Then, such problem is uniquely and globally (in time) solvable, its solution will be denoted $X(t; t_0, x_0)$, $X(t; t_0, \cdot)$ is a C^{k+1} global diffeomorphism of the Euclidean space for every $t, t_0 \in \mathbb{R}$ and its inverse is $X(t_0; t, \cdot)$. The solutions to these problems are the stream lines of the extended velocity field \bar{u} .

Consider any $x_0 \in \overline{O}$ and let $T(x_0) \geq 0$ be the greatest time for which the stream line $X(t; 0, x_0)$, $t > 0$ remains inside the open subset O , i.e., $T(x_0) := \sup\{T > 0 : X(t; 0, x_0) \in O \forall t \in (0, T)\}$. Then, $X(t; 0, x_0)$, $0 < t < T(x_0)$ is a stream line of u , or equivalently, it solves the ODE

$$\begin{cases} \frac{dX}{dt} = u(X), & 0 < t < T(x_0), \\ X(0) = x_0. \end{cases}$$

Notice that when $X(t; 0, x_0) \notin \overline{O}$, $\forall t \in (0, T)$ for some $T > 0$, then $T(x_0) = 0$, i.e., the corresponding stream line of \bar{u} does not originally enter the region O .

We will also consider stream tubes of a velocity field which emanate from an open subset of the surface S . Consider any vector field $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$, $\bar{u} = \mathcal{P}(u) \in C^{k+1,\alpha}(\mathbb{R}^3, \mathbb{R}^3)$ its extension according to Calderón’s extension theorem, $X(t; t_0, x_0)$ its associated flux mapping through Proposition 3.2 and an open subset $\Sigma \subseteq S$ together with a local chart $\mu : D \rightarrow S$. The stream tube of u which emanates from Σ is the collection of all stream lines of u radiating from the points in the open subset Σ , i.e.,

$$\mathcal{T}(\Sigma, u) := \{X(t; 0, \mu(s)) : s \in D, 0 < t < T(\mu(s))\}.$$

It is also useful to consider bounded stream lines with “height” $T > 0$

$$\mathcal{T}(\Sigma, u, T) := \{X(t; 0, \mu(s)) : s \in D, 0 < t < \min\{T, T(\mu(s))\}\}.$$

Notice that in order for a stream line of u to be well defined, it is necessary that the velocity field points towards the exterior domain. The same condition leads to well defined stream tubes emanating from Σ . The regularity in the preceding result follows from Peano’s differentiability theorem. The same regularity result may be used in order to derive the regularity in the stream tubes parametrization. For the proof it can be seen [27, Lemma 5.1] in the case $k = 0$ and [40, Proposición 2.1.7] for arbitrary k .

Proposition 3.3. *Consider G, Σ , and μ verifying the hypothesis (7), $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ be a velocity field in the exterior domain, and assume that the vector field u points towards the exterior domain at any point of Σ , i.e., there exists a positive $\rho_0 > 0$ such that $u \cdot \eta \geq \rho_0$ on Σ . Then, a well defined stream line of u emanates from each point of Σ and they smoothly foliate the whole stream tube $\mathcal{T}(\Sigma, u)$. To make this statement more precise, let us define*

$$\mathcal{D}(\Sigma, u) := \{(t, s) : s \in D, 0 < t < T(\mu(s))\},$$

and the mapping

$$\begin{aligned} \phi : \mathcal{D}(\Sigma, u) &\longrightarrow \mathcal{T}(\Sigma, u) \\ (t, s) &\longmapsto \phi(t, s) := X(t; 0, \mu(s)). \end{aligned}$$

Then,

- (1) $T(\mu(s)) > 0$, for each $s \in D$.
- (2) ϕ is bijective.
- (3) ϕ is a C^{k+1} diffeomorphism.
- (4) $\text{Jac}(\phi)$ and $\text{Jac}(\phi)^{-1}$ belongs to $C^{k,\alpha}$ locally in t , i.e., there exists a function $\kappa : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ which is increasing with respect to each variable, such that if one defines

$$\mathcal{D}(\Sigma, u, T) := \{(t, s) : s \in D, 0 < t < \min\{T, T(\mu(s))\}\}$$

and the mapping

$$\phi|_{\mathcal{D}(\Sigma, u, T)} : \mathcal{D}(\Sigma, u, T) \longrightarrow \mathcal{T}(\Sigma, u, T),$$

then,

$$\|\text{Jac}(\phi)\|_{C^{k,\alpha}(\overline{\mathcal{D}(\Sigma, \mu, T)})}, \|\text{Jac}(\phi)^{-1}\|_{C^{k,\alpha}(\overline{\mathcal{T}(\Sigma, \mu, T)})} \leq \kappa\left(\|u\|_{C^{k+1,\alpha}(\overline{\Omega})}, T\right),$$

for every positive number T .

The analysis in the next sections requires stream tubes of u that are bounded and have both ends on S . These structures were considered (although its existence was not proved) in [27]. In our setting, we will say that the stream tube of u arising from Σ is a (ρ_0, T, δ) -stream tube of u when

- $u \cdot \eta \geq \rho_0$ on Σ .
- For every $s \in D$ there exist two associated positive numbers $0 < T_0(s), T_\delta(s) < \frac{T}{2}$ such that $X(T_0(s); 0, \mu(s)) \in S$ and $X(T_\delta(s); 0, \mu(s)) \in S_\delta$.

Here ρ_0, T, δ are positive constants which measure the initial angle of the streams lines over Σ , the time at which the whole tube has returned to the surface and the depth that the stream lines achieve into the interior domain G , while S_δ stands for the boundary of the subdomain of G made of the points in G at distance at least δ from S , i.e., $G_\delta := \{x \in G : \text{dist}(x, S) > \delta\}$ (see Fig. 4).

Since a stream tube consists of integral curves, the diameter of a (ρ_0, T, δ) -stream tube is bounded in terms of the sup norm of the vector field, the flow time T and the diameter at time 0 as

$$\text{diam}(\mathcal{T}(\Sigma, u)) \leq T\|u\|_{C^0(\Omega)} + \text{diam}(\Sigma). \tag{51}$$

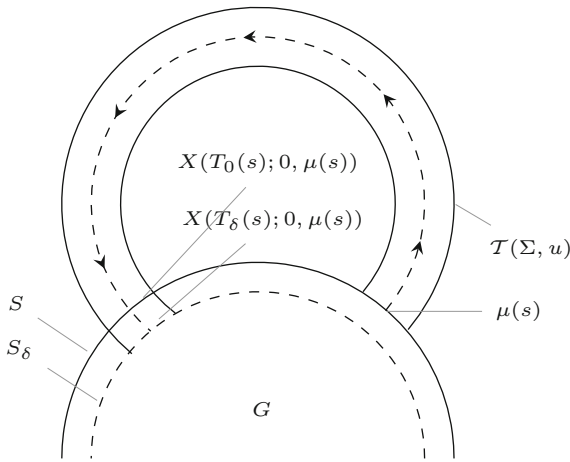


Fig. 4. (ρ_0, T, δ) -stream tube of u

(A detailed proof of this can be found in [27, Lemma 4.6]). In a similar way, [27, Lemma 4.7] provides a criterion to obtain “almost” (ρ_0, T, δ) -stream tubes for velocity fields which are “close enough” to any other given velocity field enjoying this kind of stream tubes. This merely asserts that, as is well known, a C^0 -small perturbation of the initial vector field will not prevent the integral curves of the perturbed field from intersecting a surface to which the initial flow was transverse. This can be written as follows:

Lemma 3.4. *Let G, Σ, μ verify (7) and consider $u_1, u_2 \in C^{k+1, \alpha}(\overline{\Omega}, \mathbb{R}^3)$. Define $\mathcal{T}_i := \mathcal{T}(\Sigma, u_i)$ its stream tubes emanating from Σ and assume that \mathcal{T}_1 is a (ρ_0, T, δ) -stream tube of u_1 and*

- (1) $u_1 \cdot \eta = u_2 \cdot \eta$ on Σ .
- (2) u_1 and u_2 are “close enough” in $C^0(\Omega)$ norm. Specifically, assume

$$\|u_1 - u_2\|_{C^0(\Omega)} < 2 \frac{(1 - \theta)\delta}{C_{\mathcal{P}T}} e^{-\frac{1}{2}C_{\mathcal{P}T}\|u_1\|_{C^1(\Omega)}},$$

for some $0 < \theta < 1$.

Then, \mathcal{T}_2 is also a $(\rho_0, T, \theta\delta)$ -stream tube of u_2 .

3.2. Iterative scheme. In this section we discuss the *Grad–Rubin iterative method* (see, the review [44]) used to obtain nonlinear force-free fields in the magnetohydrodynamical setting. An implementation of the Grad–Rubin method was obtained through the decomposition of the Beltrami equation with small proportionality factor f into a hyperbolic part, which transports the proportionality factor f along the magnetic field lines, and an elliptic one, to correct the magnetic field step by step using Ampere’s law [1]. This method was tried in [5] to obtain small perturbations of harmonic fields in bounded domains, leading to a strategy to generate generalized Beltrami fields with small non-constant proportionality factors. It was also analyzed in [27] to obtain small perturbations of harmonic fields in exterior domains. The $C^{0, \alpha}$ regularity of the small proportionality factors and the $C^{1, \alpha}$ regularity of the magnetic fields were also addressed in such paper. A

natural question is to ascertain whether these results can be adapted to get perturbations of strong Beltrami fields with any constant proportionality factor $\lambda \neq 0$.

Assume that u_0 is a strong Beltrami field with constant proportionality factor in the exterior domain Ω . We will restrict ourselves to strong Beltrami fields u_0 with optimal decay at infinity, say $|x|^{-1}$ (in contrast with the sharp fall-off for harmonic fields $|x|^{-2}$). Now, we would like to solve

$$\begin{cases} \operatorname{curl} u = (\lambda + \varphi)u, & x \in \Omega, \\ \operatorname{div} u = 0, & x \in \Omega, \\ u \cdot \eta = u_0 \cdot \eta, & x \in S, \\ |u(x)| \leq \frac{C}{|x|}, & x \in \Omega, \end{cases} \quad (52)$$

where φ is a “small” perturbation of the constant proportionality factor λ . To solve this problem, we move the term λu in the equation for $\operatorname{curl} u$ from the inhomogeneous side, to the homogeneous one and we propose the following modification of the classical Grad–Rubin iterative method

$$\begin{cases} \operatorname{curl} u_{n+1} - \lambda u_{n+1} = \varphi_n u_n, & x \in \Omega, \\ u_{n+1} \cdot \eta = u_0 \cdot \eta, & x \in S, \\ |u_{n+1}(x)| \leq \frac{C}{|x|}, & x \in \Omega, \end{cases} \quad \begin{cases} \nabla \varphi_n \cdot u_n = 0, & x \in \Omega, \\ \varphi_n = \varphi^0, & x \in \Sigma. \end{cases} \quad (53)$$

We have intentionally removed the divergence-free conditions $\operatorname{div} u_{n+1} = 0$ in the left hand side. The reason is twofold. First, note that if one computes the divergence in the first equation and assumes $\lambda \neq 0$, one recovers $\operatorname{div} u_{n+1} = -\frac{1}{\lambda} \operatorname{div}(\varphi_n u_n)$ from the first equation. Therefore, it is an easy task to check that as soon as u_0 is divergence-free and φ_n is a first integral of u_n , then u_{n+1} is also divergence-free in each step of the iteration. Second, as it has been shown in the preceding section, the exterior inhomogeneous Beltrami equation is generally an overdetermined system if one also prescribes the value of the divergence of the vector field. In particular, for the inhomogeneous Beltrami equation to have nontrivial divergence-free solutions it is necessary that the inhomogeneity is also divergence-free.

The inhomogeneous Beltrami equations in the left hand side was studied in the preceding section through the analysis of the complex-valued solutions satisfying both the L^1 decay condition (23) and the L^1 SMB radiation condition (22). The stationary problem along a (ρ_0, T, δ) -stream tube of u_n in the right hand side of (53) will be studied in the $C^{k+1,\alpha}$ setting in the next subsection and the convergence of the *modified Grad–Rubin iterative method* discussed in the Introduction (Equation (6)) will be analyzed at the end of this section.

3.3. Linear transport problem. We begin with the steady transport equations along (ρ_0, T, δ) -stream tubes in the right hand side of (6). The main idea to find a solution is to transport φ^0 along the foliated stream tube and to check that this definition leads to regular enough factors f_n of u_n due to the regularity of the tube.

Theorem 3.5. *Let G, Σ, μ satisfy the hypotheses (7), consider any $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ such that $T(\Sigma, u)$ is a (ρ_0, T, δ) -stream tube of such a velocity field and assume that $\varphi^0 \in C_c^{k+1,\alpha}(\Sigma)$. Consider the first integral equation associated with u*

$$\begin{cases} u \cdot \nabla \varphi = 0 & \text{in } \Omega \\ \varphi = \varphi^0 & \text{on } \Sigma. \end{cases} \quad (54)$$

Then, there exists an unique solution φ along $\mathcal{T}(\Sigma, u)$, its support lies in the closure of $\mathcal{T}(\Sigma, u)$ and it can be extended to a global solution in Ω with zero value outside $\mathcal{T}(\Sigma, u)$. Moreover, it belongs to $C^{k+1,\alpha}(\bar{\Omega})$ and the estimate

$$\|\varphi\|_{C^{k+1,\alpha}(\Omega)} \leq \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} \kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T)$$

holds, for some continuous and separately increasing function $\kappa : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$.

Proof. The proof of this result can be found for the particular case $k = 0$ in [27, Lemmas 4.8, 4.9 and 5.2]. Let us then sketch the proof of the general case $k \neq 0$. Define the Calderón extension of $u, \bar{u} := \mathcal{P}(u)$, according to Proposition 3.1 and denote its flux mapping by $X(t; t_0, x_0)$. First, let us prove the uniqueness part of our assertion. Notice that as long as φ is a smooth first integral of u , then

$$\frac{d}{dt}\varphi(X(t; 0, \mu(s))) = (\bar{u} \cdot \nabla\varphi)(X(t; 0, \mu(s))) = (u \cdot \nabla\varphi)(X(t; 0, \mu(s))) = 0,$$

for every $(t, s) \in \mathcal{D}(\Sigma, u)$. Therefore, $\varphi(x) = \varphi^0(\mu(s(x)))$ for every $x \in \mathcal{T}(\Sigma, u)$, where $(t(x), s(x)) = \phi^{-1}(x)$. Second, regarding the existence assertion, the previous formula for φ defines a smooth function in $\mathcal{T}(\Sigma, u)$ (by virtue of the bijectivity and regularity of the parametrization ϕ in Proposition 3.2) which obviously solves (54) along the stream tube. Furthermore, with the exception of the endpoints, it is compactly supported in the interior of the tube. The extension of φ by zero outside the tube yields a global smooth solution of (54) in Ω .

To show the bound for $\|\varphi\|_{C^{k+1,\alpha}(\Omega)}$ (equivalently for $\|\varphi\|_{C^{k+1,\alpha}(\mathcal{T}(\Sigma, u))}$), let us fix any multi-index $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ such that $|\gamma| \leq k + 1$ and note that

$$D^\gamma\varphi(x) = \gamma! \sum_{(l, \beta, \delta) \in \mathcal{D}(\gamma)} (D^\delta(\varphi^0 \circ \mu))(s(x)) \prod_{r=1}^l \frac{1}{\delta_r!} \left(\frac{1}{\beta_r!} D^{\beta_r} s(x) \right)^{\delta_r}.$$

for every $x \in \mathcal{T}(\Sigma, u)$. The above formula is nothing but a chain rule for high order partial derivatives in high dimension. Here, $\mathcal{D}(\gamma)$ stands for the set of all possible decompositions of $\gamma = \sum_{r=1}^l |\delta_r| \beta_r$, where δ_r, β_r are multi-indices, $\delta := \sum_{r=1}^l \delta_r$ and for every $r = 1, \dots, l - 1$ there exists some $i_r \in \{1, 2, 3\}$ such that $(\beta_r)_{i_r} = (\beta_{r+1})_{i_r}$ for every $i \neq i_r$ and $(\beta_r)_{i_r} < (\beta_{r+1})_{i_r}$. First of all, it is necessary to know how to handle $D^{\beta_r} s(x)$. To this end, note that $\text{Jac}(\phi^{-1})(x) = \text{Jac}(\phi)^{-1}(\phi^{-1}(x))$, so

$$D^\rho(\text{Jac}(\phi^{-1})_{i,j})(x) = \sum_{n=1}^{n_\rho} \prod_{\substack{\beta \in \Gamma_n \\ 1 \leq p, q \leq 3}} A_{n,p,q}^{i,j}(\rho, \beta) (D^\beta(\text{Jac}(\phi)_{p,q}^{-1}))(\phi^{-1}(x)),$$

for every multi-index ρ such that $|\rho| \leq k$. Here, $A_{n,p,q}^{i,j}(\rho, \beta)$ stand for constant coefficients and Γ_n is a set of 3-multi-indices of order at most $|\rho| \leq k$. Expanding the products of sums by distributivity, each term in $D^\gamma\varphi$ takes the form

$$(D^\delta(\varphi^0 \circ \mu))(s(x)) \prod_{\substack{\beta \in \Gamma \\ 1 \leq p, q \leq 3}} B_{p,q}^{i,j}(\gamma, \beta) (D^\beta(\text{Jac}(\phi)_{p,q}^{-1}))(\phi^{-1}(x)),$$

where Γ is a set of multi-indices with degree at most k . The first factor can be bounded by $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}$ whilst the terms in the second factor are bounded by $\kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T)$ as stated in Proposition 3.2. Hence, it is clear that

$$\|\varphi\|_{C^{k+1}(\Omega)} \leq \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}\kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T).$$

Finally, for any multi-index with maximum order $k + 1$, the α -Hölder seminorm of $D^\gamma \varphi$ can be estimated as follows. Take $x_1, x_2 \in \mathcal{T}(\Sigma, u)$ and appropriately add and subtract the crossed terms. Since $D^\delta(\varphi^0 \circ \mu)$ is bounded by $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}$ and $D^\beta(\text{Jac}(\phi)_{pq}^{-1})$ is bounded by $\kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T)$, then it only remains to obtain estimates for

$$I := (D^\delta(\varphi^0 \circ \mu))(s(x)) \Big|_{x_1}^{x_2},$$

$$II := (D^\beta(\text{Jac}(\phi)_{pq}^{-1}))(\phi^{-1}(x)) \Big|_{x_1}^{x_2}.$$

First, we distinguish the cases $|\delta| < k + 1$ and $|\delta| = k + 1$. In the former case, the mean value theorem, the estimates in Proposition 3.2 for $\text{Jac}(\phi)^{-1}$ and the estimate (51) of the diameter of the stream tube $\mathcal{T}(\Sigma, u)$ yield the upper bound

$$I \leq \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}\kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T)|x_1 - x_2|$$

$$\leq \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}\kappa(\|u\|_{C^{k+1,\alpha}})(T\|u\|_{C^0(\Omega)} + \text{diam}(\Sigma))^{1-\alpha}|x_1 - x_2|^\alpha.$$

In the later case, the α -Hölder continuity of $D^\delta(\varphi^0 \circ \mu)$ gives rise to an analogous estimate

$$I \leq \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)}\kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T)^\alpha |x_1 - x_2|^\alpha.$$

Second, note that $D^\beta(\text{Jac}(\phi)_{pq}^{-1})$ is α -Hölder continuous with Hölder’s constant that can be bounded above by $\kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T)$ by virtue of Proposition 3.2. Thus,

$$II \leq \kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T)|\phi^{-1}(x_1) - \phi^{-1}(x_2)|^\alpha.$$

The mean value theorem then leads to the desired upper estimate

$$|D^\gamma \varphi(x_1) - D^\gamma \varphi(x_2)| \leq \kappa(\|u\|_{C^{k+1,\alpha}(\Omega)}, T)|x_1 - x_2|^\alpha,$$

appropriately modifying the separately increasing function κ . \square

In addition to the existence and uniqueness results of (54), in order to take limits in (6) we will need a compactness result for $\{\varphi_n\}_{n \in \mathbb{N}}$. Once we know that the sequence $\{u_n\}_{n \in \mathbb{N}}$ converges in $C^{k+1,\alpha}(\bar{\Omega}, \mathbb{R}^3)$, a result of stability for the problem (54) leads to the convergence of the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ in $C^{k,\alpha}(\bar{\Omega})$. This stability result was proved in [27, Lemma 5.3] in the $C^{1,\alpha}$ framework and can be easily extended to $C^{k+1,\alpha}$ using the same lines as in Theorem 3.5 (the details, which are straightforward, can be found in [40, Corolario 2.4.4]):

Corollary 3.6. *Let G, Σ, μ satisfy the properties (7). Consider any couple of vector fields $u_1, u_2 \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$, and denote as $\mathcal{T}_1 := \mathcal{T}(\Sigma, u_1)$ and $\mathcal{T}_2 := \mathcal{T}(\Sigma, u_2)$ the associated stream tubes which emanate from Σ . Assume that \mathcal{T}_i is a (ρ_0, T, δ_i) -stream tube of u_i . Consider any boundary data $\varphi^0 \in C_c^{k+1,\alpha}(\Sigma)$ and the solutions φ_1 and φ_2 (according to Theorem 3.5) to each transport problem associated with u_1 and u_2 respectively:*

$$\begin{cases} \nabla\varphi_1 \cdot u_1 = 0, & x \in \Omega, \\ \varphi_1 = \varphi^0, & x \in \Sigma, \end{cases} \quad \begin{cases} \nabla\varphi_2 \cdot u_2 = 0, & x \in \Omega, \\ \varphi_2 = \varphi^0, & x \in \Sigma. \end{cases}$$

Then,

$$\begin{aligned} \|\varphi_1 - \varphi_2\|_{C^{k,\alpha}(\Omega)} &\leq \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} \cdot \kappa \left(\|u_1\|_{C^{k+1,\alpha}(\Omega)}, T \right) \\ &\quad \times \kappa \left(\|u_2\|_{C^{k+1,\alpha}(\Omega)}, T \right) \|u_1 - u_2\|_{C^{k+1,\alpha}(\Omega)}, \end{aligned}$$

where $\kappa : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is continuous, separately increasing and does not depend on u_i, φ^0 or T .

3.4. Limit of the approximate solutions. The existence and uniqueness results in Theorems 3.5 and 2.15 together with the stability result for the transport problem in Corollary 3.6 now allow us to take the limit as $n \rightarrow +\infty$ in the modified Grad–Rubin iterative scheme (6). Therefore, we obtain a generalized Beltrami field which is close to the initial strong Beltrami field and whose proportionality factor is a non-constant small enough perturbation of the initial constant proportionality factor λ :

Theorem 3.7. *Let G, Σ, μ satisfy the hypotheses (7) and assume that $0 \neq \lambda \in \mathbb{R}$ is not a Dirichlet eigenvalue of Laplace operator in the interior domain G . Consider any complex-valued strong Beltrami field $v_0 \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{C}^3)$ which satisfy the L^1 SMB radiation condition (22) and the L^1 decay property (23) in the exterior domain. Consider its real part $u_0 := \Re v_0$ and assume that $\mathcal{T}(\Sigma, u_0)$ is a (ρ_0, T, δ) -stream tube of the velocity field u_0 . Let ε_0 be any positive number. Then, there exists a nonnegative constant δ_0 for which the real parts u_{n+1} of the solutions $v_{n+1} \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{C}^3)$ together with the solutions $\varphi_n \in C^{k+1,\alpha}(\overline{\Omega})$ of the modified Grad–Rubin scheme (6) (Theorems 3.5 and 2.15) have a limit vector field $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and a limit perturbation of the proportionality factor $\varphi \in C^{k,\alpha}(\overline{\Omega})$ such that*

$$u_n \rightarrow u \text{ in } C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3), \quad \varphi_n \rightarrow \varphi \text{ in } C^{k,\alpha}(\overline{\Omega}),$$

as $n \rightarrow +\infty$, for any $\varphi^0 \in C_c^{k+1,\alpha}(\Sigma)$ with $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} < \delta_0$. Also, $(u, \lambda + \varphi)$ solves the boundary value problem (52), u has optimal decay and $\varphi = \varphi^0$ in Σ . Moreover, $\mathcal{T}(\Sigma, u)$ is a $(\rho_0, T, \delta/2)$ -stream tube of u , φ has compact support inside the closure of such stream tube and u is close enough to u_0 , specifically

$$\|u - u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \varepsilon_0 \|u_0\|_{C^{k+1,\alpha}(\Omega)}.$$

Proof. For simplicity of notation, we will denote the stream tubes associated with each vector field u_n which emanates from Σ by $\mathcal{T}_n := \mathcal{T}(\Sigma, u_n)$. First of all, it is necessary to check whether the hypothesis of Theorems 3.5 and 2.15 hold and they can be deduced

in each step from the corresponding hypotheses in the previous step in the iteration. Let us begin with the step $n = 0$:

$$\begin{cases} \nabla\varphi_0 \cdot u_0 = 0, & x \in \Omega, \\ \varphi_0 = \varphi^0, & x \in \Sigma, \end{cases} \quad \begin{cases} \operatorname{curl} v_1 - \lambda v_1 = \varphi_0 u_0, & x \in \Omega, \\ v_1 \cdot \eta = u_0 \cdot \eta, & x \in S, \\ + L^1 \text{ Decay property (3)}, \\ + L^1 \text{ SBM radiation condition (4)}. \end{cases}$$

The hypotheses imply that \mathcal{T}_0 is a (ρ_0, T, δ) -stream tube of u_0 and $\varphi^0 \in C_c^{k+1,\alpha}(\Sigma)$. Hence, there exists a global solution φ_0 to the transport equation (Theorem 3.5). Moreover, $\varphi_0 u_0 \in C_c^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3) \subseteq C_c^{k,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and its compact support is contained in the stream tube $\overline{\mathcal{T}_0}$. In particular, the estimate (51) ensures that $\operatorname{supp}(\varphi_0 u_0) \subseteq \overline{\mathcal{T}_0} \subseteq \overline{\Omega_R}$, where $\Omega_R := B_R(0) \setminus \overline{G}$ and $R := 2T\|u_0\|_{C^{k+1,\alpha}(\Omega)} + \operatorname{diam}(\Sigma)$. On the other hand, as S is regular enough, so η is and, consequently, $u_0 \cdot \eta \in C^{k+1,\alpha}(S)$. An integration by parts leads to the following expression

$$\int_S (\lambda u_0 \cdot \eta + \varphi_0 u_0 \cdot \eta) dS = \lambda \int_S u_0 \cdot \eta dS + \int_{\partial B_{R'}(0)} \varphi_0 u_0 \cdot \eta dS - \int_{\Omega_{R'}} \operatorname{div}(\varphi_0 u_0) dx$$

For $R' > R$, the second term vanishes as a consequence of the previous estimate for the diameter of the initial stream tube. Regarding the third term, notice that the same argument as above leads to

$$\operatorname{div}(\varphi_0 u_0) = \nabla\varphi_0 \cdot u_0 + \varphi_0 \operatorname{div} u_0 = 0.$$

We have $u_0 \cdot \eta = -\frac{1}{\lambda} \operatorname{div}_S(\eta \times u_0)$. Thus, the divergence theorem concludes that the first term vanishes too. Therefore, the hypotheses of Theorem 2.15 are satisfied, so there is a unique solution v_1 to the corresponding complex-valued inhomogeneous Beltrami equation in the right hand side of the step $n = 0$.

Let us prove an estimate for $u_1 - u_0$ that will be useful to prove the Cauchy condition in $C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ for the sequence $\{u_n\}_{n \in \mathbb{N}}$. This vector field is the real part of $v_1 - v_0$, which satisfies the complex-valued exterior Neumann problem

$$\begin{cases} (\operatorname{curl} - \lambda)(v_1 - v_0) = \varphi_0 u_0, & x \in \Omega, \\ (v_1 - v_0) \cdot \eta = 0, & x \in S, \\ + L^1 \text{ decay condition (3)}, \\ + L^1 \text{ SMB radiation condition (4)}. \end{cases}$$

Therefore, the uniqueness of the solution to this problem (Proposition 2.17), the $C^{k+1,\alpha}$ estimates of such solutions (Corollary 2.19), and the $C^{k,\alpha}$ estimates for the solution of the steady transport equation (Theorem 3.5) allow us to obtain the following estimate for $v_1 - v_0$ and, consequently, for $u_1 - u_0$:

$$\|u_1 - u_0\|_{C^{k+1,\alpha}(\Omega)} = \|\Re(v_1 - v_0)\|_{C^{k+1,\alpha}(\Omega)} \leq \|v_1 - v_0\|_{C^{k+1,\alpha}(\Omega)} \leq C_0 \|\varphi_0 u_0\|_{C^{k,\alpha}(\Omega)}.$$

Here $C_0 > 0$ depends on k, α, λ, G and R . The *Leibniz rule* for the derivative of a product reads

$$D^\gamma(\varphi_0 u_0) = \sum_{\beta \leq \gamma} \binom{\gamma}{\beta} D^\beta \varphi_0 D^{\gamma-\beta} u_0,$$

for any multi-index γ . Therefore, the estimates in Theorem 3.5 for the derivatives up to order k of φ_0 and the combination of the mean value theorem and the Calderón’s extension theorem (Proposition 3.1) to estimate the $C^{0,\alpha}$ -norm of the derivatives of u_0 up to order k allow us to arrive at the inequality

$$\|D^\gamma(\varphi_0 u_0)\|_{C^0(\Omega)} \leq C_k \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} \kappa(\|u_0\|_{C^{k+1,\alpha}(\Omega)}, T) \|u_0\|_{C^{k+1,\alpha}(\Omega)},$$

for every multi-index γ with $|\gamma| \leq k$, and

$$\begin{aligned} & \|D^\gamma(\varphi_0 u_0)\|_{C^{0,\alpha}(\Omega)} \\ &= \|D^\gamma(\varphi_0 u_0)\|_{C^{0,\alpha}(\mathcal{T}_0)} \\ &\leq C_k C_{\mathcal{P}} \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} \kappa(\|u_0\|_{C^{k+1,\alpha}}, T) \|u_0\|_{C^{k+1,\alpha}(\Omega)} (T \|u_0\|_{C^{k,\alpha}(\Omega)} + \text{diam}\Sigma)^{1-\alpha}, \end{aligned}$$

for every multi-index γ so that $|\gamma| = k$ and a nonnegative constant C_k depending on k . To derive the last estimate, we have used that

$$\begin{aligned} |D^{\gamma-\beta} u_0(x) - D^{\gamma-\beta} u_0(y)| &\leq \|D^{\gamma-\beta} \overline{u_0}\|_{C^1(\mathbb{R}^3)} |x - y| \\ &\leq C_{\mathcal{P}} \|u_0\|_{C^{k+1,\alpha}(\Omega)} |x - y|^\alpha (\text{diam}\mathcal{T}_0)^{1-\alpha}, \end{aligned}$$

for every $x, y \in \mathcal{T}_0$ and the estimate (51) for the diameter of the (ρ_0, T, δ) -stream tube of \mathcal{T}_0 . Hence the following inequality

$$\begin{aligned} & \|u_1 - u_0\|_{C^{k+1,\alpha}(\Omega)} \\ &\leq K \left\{ 1 + (T \|u_0\|_{C^{k+1,\alpha}(\Omega)} + \text{diam}\Sigma)^{1-\alpha} \right\} \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} \kappa(\|u_0\|_{C^{k+1,\alpha}(\Omega)}, T) \\ &\quad \times \|u_0\|_{C^{k+1,\alpha}(\Omega)} \end{aligned}$$

holds, with a constant $K = K(k, \alpha, \lambda, G, R)$.

Now, we can fix the small parameter δ_0 such that it satisfies

$$\left\{ \begin{aligned} & K \left\{ 1 + (4T \|u_0\|_{C^{k+1,\alpha}(\Omega)} + \text{diam}\Sigma)^{1-\alpha} \right\} \\ & \quad \times \left\{ \kappa(2\|u_0\|_{C^{k+1,\alpha}(\Omega)}, T) + \|u_0\|_{C^{k+1,\alpha}(\Omega)} \kappa(2\|u_0\|_{C^{k+1,\alpha}(\Omega)}, T)^2 \right\} \delta_0 < \frac{1}{2} \min\{\varepsilon_0, 1\}. \\ & K \left\{ 1 + (4T \|u_0\|_{C^{k+1,\alpha}(\Omega)} + \text{diam}\Sigma)^{1-\alpha} \right\} \\ & \quad \times \left\{ \kappa(2\|u_0\|_{C^{k+1,\alpha}(\Omega)}, T) + \|u_0\|_{C^{k+1,\alpha}(\Omega)} \kappa(2\|u_0\|_{C^{k+1,\alpha}(\Omega)}, T)^2 \right\} \|u_0\|_{C^{k+1,\alpha}(\Omega)} \delta_0 \\ & < \frac{1}{4} \frac{2\delta}{C_{\mathcal{P}} T} e^{-\frac{1}{2} C_{\mathcal{P}} \|u_0\|_{C^{k+1,\alpha}(\Omega)} T}. \end{aligned} \right. \tag{55}$$

Then we infer

$$\left\{ \begin{aligned} & \|u_1 - u_0\|_{C^{k+1,\alpha}(\Omega)} < \min\{\varepsilon_0, 1\} \frac{1}{2} \|u_0\|_{C^{k+1,\alpha}(\Omega)}, \\ & \|u_1 - u_0\|_{C^{k+1,\alpha}(\Omega)} < \frac{1}{4} \frac{2\delta}{C_{\mathcal{P}} T} e^{-\frac{1}{2} C_{\mathcal{P}} T \|u_0\|_{C^{k+1,\alpha}(\Omega)}}, \\ & \|u_1\|_{C^{k+1,\alpha}(\Omega)} \leq \frac{3}{2} \|u_0\|_{C^{k+1,\alpha}(\Omega)}. \end{aligned} \right. \tag{56}$$

To obtain similar estimates for the remaining terms we will use induction to show that

$$\left\{ \begin{aligned} \|u_{n+1} - u_n\|_{C^{k+1,\alpha}(\Omega)} &\leq \frac{1}{2^n} \|u_1 - u_0\|_{C^{k+1,\alpha}(\Omega)} < \min\{\varepsilon_0, 1\} \frac{1}{2^{n+1}} \|u_0\|_{C^{k+1,\alpha}(\Omega)}, \\ \|u_{n+1} - u_n\|_{C^{k+1,\alpha}(\Omega)} &< \frac{1}{2} \frac{1}{2^{n+1}} \frac{2\delta}{C_{\mathcal{P}T}} e^{-\frac{1}{2}C_{\mathcal{P}T}\|u_0\|_{C^{k+1,\alpha}(\Omega)}}, \\ \|u_{n+1} - u_0\|_{C^{k+1,\alpha}(\Omega)} &< \min\{\varepsilon_0, 1\} \sum_{i=1}^{n+1} \frac{1}{2^i} \|u_0\|_{C^{k+1,\alpha}(\Omega)}, \\ \|u_{n+1} - u_0\|_{C^{k+1,\alpha}(\Omega)} &< \frac{1}{2} \sum_{i=1}^{n+1} \frac{1}{2^i} \frac{2\delta}{C_{\mathcal{P}T}} e^{-\frac{1}{2}C_{\mathcal{P}}\|u_0\|_{C^{k+1,\alpha}(\Omega)}}, \\ \|u_{n+1}\|_{C^{k+1,\alpha}(\Omega)} &< \sum_{i=0}^{n+1} \frac{1}{2^i} \|u_0\|_{C^{k+1,\alpha}(\Omega)}. \end{aligned} \right. \tag{57}$$

This is true for $n = 0$ due to (56), so we can assume that the inductive hypotheses holds for all indices less than n . Specifically, we assume that φ_m, v_{m+1} are well defined, i.e., the corresponding problems have a unique solution, that u_{m+1} are divergence-free and (57) hold for indices $m < n$.

Let us now prove that the result is verified for $m = n$. The inductive hypotheses imply the existence of a vector field $v_n \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{C}^3)$ and $\varphi_{n-1} \in C^{k,\alpha}(\overline{\Omega})$. Moreover, \mathcal{T}_n is a $(\rho_0, T, (1 - \frac{1}{2} \sum_{i=1}^n \frac{1}{2^i})\delta)$ -stream tube of the real part $u_n = \Re v_n$ because of the third inequality in (57). Consequently, there exists a unique solution $\varphi_n \in C^{k,\alpha}(\overline{\Omega})$ to the transport problem in the left hand side of (6) according to Theorem 3.5. The last estimate in (57) along with (51) lead to $\mathcal{T}_n \subseteq \overline{\Omega}_R$. Therefore, φ_n is compactly supported in $\overline{\Omega}_R \subseteq \Omega$ and the same argument as in the step $n = 0$ ensures the existence and uniqueness of a solution $v_{n+1} \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{C}^3)$ to the complex-valued exterior Neumann problem for the inhomogeneous Beltrami equation in the right hand side of (6).

Notice that the vanishing flux hypothesis in Theorem 2.15 is satisfied. To check it we get

$$\int_S (\lambda u_0 \cdot \eta + \varphi_n u_n \cdot \eta) dS = \lambda \int_S u_0 \cdot \eta dS + \int_{\partial B_{R'}(0)} \varphi_n u_n \cdot \eta dS - \int_{\Omega_{R'}} \operatorname{div}(\varphi_n u_n) dx.$$

The first term is zero as before, the second one also vanishes for a choice $R' > R$ and the last one is zero too because φ_n is a first integral of u_n and u_n is divergence-free according to the induction hypothesis. Consequently, it is easy to verify that u_{n+1} is also divergence-free.

To conclude, let us prove the inductive hypothesis (57) for $u_{n+1} - u_n$. Taking the difference of the corresponding complex-valued exterior boundary value problems we have that $v_{n+1} - v_n$ solves

$$\begin{cases} (\operatorname{curl} - \lambda)(v_{n+1} - v_n) = \varphi_n u_n - \varphi_{n-1} u_{n-1}, & x \in \Omega, \\ (v_{n+1} - v_n) \cdot \eta = 0, & x \in S, \\ + L^1 \text{ decay conditions (3),} \\ + L^1 \text{ SMB radiation condition (4).} \end{cases}$$

Again, thanks to the uniqueness property (Proposition 2.17), the $C^{k+1,\alpha}$ estimates for these solutions (Corollary 2.19) and the $C^{k,\alpha}$ estimates for the solution of the steady transport equation (Theorem 3.5), we obtain the following estimate for $v_{n+1} - v_n$ and, consequently, for $u_{n+1} - u_n$

$$\begin{aligned} \|u_{n+1} - u_n\|_{C^{k+1,\alpha}(\Omega)} &= \|\mathfrak{R}(v_{n+1} - v_n)\|_{C^{k+1,\alpha}(\Omega)} \\ &\leq \|v_{n+1} - v_n\|_{C^{k+1,\alpha}(\Omega)} \\ &\leq C_0 \|\varphi_n u_n - \varphi_{n-1} u_{n-1}\|_{C^{k,\alpha}(\Omega)}. \end{aligned}$$

Now, $\varphi_n u_n - \varphi_{n-1} u_{n-1}$ has compact support inside $\overline{\mathcal{T}_n} \cup \overline{\mathcal{T}_{n-1}} \subseteq \overline{\Omega_R}$ (see estimate (51) and the last inequalities for the $C^{k+1,\alpha}$ norms of u_n and u_{n-1} in the inductive hypothesis). Thus, Theorem A.9 asserts that the constant $C_0 = C_0(k, \alpha, \lambda, G, R)$ is the same as in the basic step because all the supports of the inhomogeneous terms in the complex-valued exterior Neumann problems are contained in the same bounded subset $\overline{\Omega_R}$ of the exterior domain. This is a crucial fact because it prevents those constants from depending on the iteration number n and avoids the blowup when $n \rightarrow +\infty$. Notice that

$$\|\varphi_n u_n - \varphi_{n-1} u_{n-1}\|_{C^{k,\alpha}(\Omega)} \leq \|(\varphi_n - \varphi_{n-1})u_n\|_{C^{k,\alpha}(\Omega)} + \|\varphi_{n-1}(u_n - u_{n-1})\|_{C^{k,\alpha}(\Omega)}.$$

Since \mathcal{T}_n is a $(\rho_0, T, (1 - \frac{1}{2} \sum_{i=1}^n \frac{1}{2^i})\delta)$ -stream tube of u_n , \mathcal{T}_{n-1} is a $(\rho_0, T, (1 - \frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{2^i})\delta)$ -stream tube of u_{n-1} and $u_{n-1} \cdot \eta = u_0 \cdot \eta = u_n \cdot \eta$ on S , we can apply both estimates in Theorem 3.5 and Corollary 3.6 to obtain the inequality

$$\begin{aligned} \|\varphi_n u_n - \varphi_{n-1} u_{n-1}\|_{C^{k+1,\alpha}(\Omega)} &\leq K \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} \left\{ 1 + (4T \|u_0\|_{C^{k+1,\alpha}(\Omega)} + \text{diam} \Sigma)^{1-\alpha} \right\} \\ &\quad \times \left\{ \kappa (2\|u_0\|_{C^{k+1,\alpha}(\Omega)}, T) + \|u_0\|_{C^{k+1,\alpha}(\Omega)} \kappa (2\|u_0\|_{C^{k+1,\alpha}(\Omega)}, T)^2 \right\} \\ &\quad \|u_n - u_{n-1}\|_{C^{k+1,\alpha}(\Omega)}. \end{aligned}$$

Consequently, the estimate

$$\begin{aligned} \|u_{n+1} - u_n\|_{C^{k+1,\alpha}(\Omega)} &\leq K \|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} \left\{ 1 + (4T \|u_0\|_{C^{k+1,\alpha}(\Omega)} + \text{diam} \Sigma)^{1-\alpha} \right\} \\ &\quad \times \left\{ \kappa (2\|u_0\|_{C^{k+1,\alpha}(\Omega)}, T) + \|u_0\|_{C^{k+1,\alpha}(\Omega)} \kappa (2\|u_0\|_{C^{k+1,\alpha}(\Omega)}, T)^2 \right\} \\ &\quad \|u_n - u_{n-1}\|_{C^{k+1,\alpha}(\Omega)} \end{aligned}$$

holds, with K independent of n . Since $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} < \delta_0$ and δ_0 is small enough to ensure (55), one has

$$\|u_{n+1} - u_n\|_{C^{k+1,\alpha}(\Omega)} < \frac{1}{2} \|u_n - u_{n-1}\|_{C^{k+1,\alpha}(\Omega)},$$

and the inductive hypothesis for indices less than n leads to the first two inequalities in (57).

The last three estimates can be obtained as follows. Firstly, the preceding two estimates together with the induction hypotheses lead to

$$\|u_{n+1} - u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \sum_{i=0}^n \|u_{i+1} - u_i\|_{C^{k+1,\alpha}(\Omega)} \leq \min\{\varepsilon_0, 1\} \sum_{i=1}^{n+1} \frac{1}{2^i} \|u_0\|_{C^{k+1,\alpha}(\Omega)}.$$

Similarly, we have

$$\|u_{n+1} - u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \sum_{i=0}^n \|u_{i+1} - u_i\|_{C^{k+1,\alpha}(\Omega)} \leq \frac{1}{2} \sum_{i=1}^{n+1} \frac{1}{2^i} \frac{2\delta}{C_{\mathcal{P}T}} e^{-\frac{1}{2}C_{\mathcal{P}T}\|u_0\|_{C^{k+1,\alpha}(\Omega)}}.$$

The last inequality in (57) is obvious by the triangle inequality:

$$\|u_{n+1}\|_{C^{k+1,\alpha}(\Omega)} \leq \|u_0\|_{C^{k+1,\alpha}(\Omega)} + \|u_{n+1} - u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \sum_{i=0}^{n+1} \frac{1}{2^i} \|u_0\|_{C^{k+1,\alpha}(\Omega)}.$$

Using the above inequalities in (57) one can show that $\{u_n\}_{n \in \mathbb{N}}$ and $\{\varphi_n\}_{n \in \mathbb{N}}$ are Cauchy sequences in $C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and $C^{k,\alpha}(\overline{\Omega})$, respectively. On the one hand, we find

$$\begin{aligned} \|u_{n+m} - u_n\|_{C^{k+1,\alpha}(\Omega)} &\leq \sum_{i=n}^{n+m-1} \|u_{i+1} - u_i\|_{C^{k+1,\alpha}(\Omega)} \\ &< \sum_{i=n}^{n+m-1} \frac{1}{2^{i+1}} \|u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \frac{1}{2^n} \|u_0\|_{C^{k+1,\alpha}(\Omega)}. \end{aligned}$$

Likewise, the third inequality in (57) along with the property $u_n \cdot \eta = u_0 \cdot \eta$ on S , shows that \mathcal{T}_n are $(\rho_0, T, (1 - \frac{1}{2} \sum_{i=0}^n \frac{1}{2^i}) \delta)$ -stream tubes of u_n . Therefore, $\{\varphi_n\}_{n \in \mathbb{N}}$ also satisfies the Cauchy condition in $C^{k,\alpha}(\overline{\Omega})$ due to Corollary 3.6. Thus, it converges in $C^{k,\alpha}$ to some $\varphi \in C^{k,\alpha}(\overline{\Omega})$.

Let us now take the limit as $n \rightarrow +\infty$ in the iterative scheme to deduce

$$\begin{array}{ccc} \operatorname{div} u_{n+1} = 0 & \operatorname{curl} u_{n+1} - \lambda u_{n+1} = \varphi_n u_n & u_{n+1} \cdot \eta = u_0 \cdot \eta \\ \downarrow \quad \downarrow & \downarrow \quad \downarrow & \downarrow \quad \downarrow \\ \operatorname{div} u = 0 & \operatorname{curl} u - \lambda u = \varphi u & u \cdot \eta = u_0 \cdot \eta. \end{array}$$

Moreover, the L^1 SMB radiation condition (22) and the decay property (23) lead to complex-valued solutions v_n to the exterior Neumann problem for the inhomogeneous Beltrami equations in the iterative scheme with the asymptotic behavior $|v_n(x)| \leq C|x|^{-1}$, $x \in \Omega$, for every n and C independent of n . To check it, notice that Theorem 2.15 provides a decomposition of v_{n+1} into generalized volume and single layer potentials whose densities are $u_0 \cdot \eta$, $\varphi_n u_n$ and the sequence ξ_n of solutions to the boundary integral equations (30). The single layer potentials and its first order partial derivatives are dominated by the corresponding integral kernels Γ_λ and $\nabla \Gamma_\lambda$ for x far enough from the surface S . This leads to an upper bound $C|x|^{-1}$ where C depends on the C^0 norm of $u_0 \cdot \eta$ and ξ_n . Both quantities can be bounded above by $\|u_0 \cdot \eta\|_{C^{k,\alpha}(S)}$ and $\|\varphi_n u_n\|_{C^{k,\alpha}(\Omega)}$, which are uniformly bounded with respect to n . Furthermore, the volume layer potentials and its first order partial derivatives can be bounded by $C|x|^{-1}$ for an n -independent constant thanks to Theorem 2.6 and the above argument. Consequently, we get the same asymptotic behavior at infinity for the limit vector field u .

Let us show now that $\mathcal{T}(\Sigma, u)$ is a $(\rho_0, T, \delta/2)$ -stream tube of u and that the support of φ lies in it. Since, by taking limits in the fourth inequality in (57),

$$\|u - u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \frac{1}{2} \frac{2\delta}{C_{\mathcal{P}T}} e^{-\frac{1}{2}C_{\mathcal{P}T}\|u_0\|_{C^{k+1,\alpha}(\Omega)}},$$

Corollary 3.6 yields the first assertion. The second one is clear by taking into account that $\text{supp}\varphi_n \subseteq \overline{T}_n$, for every $n \in \mathbb{N}$. Finally, to check that the limit solution is close to the initial strong Beltrami field u_0 , it suffices to take limits in the third inequality in (57) to get

$$\|u - u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \min\{\varepsilon_0, 1\} \sum_{i=1}^{+\infty} \frac{1}{2^i} \|u_0\|_{C^{k+1,\alpha}(\Omega)} \leq \varepsilon_0 \|u_0\|_{C^{k+1,\alpha}(\Omega)}.$$

□

Remark 3.8. The generalized Beltrami field $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ obtained in the preceding Theorem has proportionality factor $f = \lambda + \varphi$, for some compactly supported perturbation $\varphi \in C^{k,\alpha}(\overline{\Omega})$. Moreover, it decays as $|x|^{-1}$ at infinity. To check that it is optimal note first that $\text{div}(\varphi u) = 0$ and consider any open subset $\Sigma' \subseteq S$ such that $\text{supp}\varphi^0 \subseteq \underline{\Sigma}' \subseteq \overline{\Sigma}' \subseteq \Sigma$. Then, the preceding proof shows that φ is compactly supported in $\overline{T}(\Sigma', u)$, that is indeed a $(\rho_0, T, \delta/2)$ -stream tube. Take $x \in \Sigma \setminus \overline{\Sigma}'$ and note that $u(x) \cdot \eta(x) \geq \rho_0 > 0$. Hence, $u = O(|x|^{-1})$ is optimal by Corollary 2.21.

A related remark in the harmonic case ($\lambda = 0$) is in order now.

Remark 3.9. Recall that a similar result to that in Theorem 3.7 was previously proved in [27] to obtain generalized Beltrami fields $u \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ (nonlinear force-free fields), i.e., solutions to

$$\text{curl } u = fu, \quad x \in \Omega,$$

with compactly supported small proportionality factors $f \in C^{0,\alpha}(\overline{\Omega})$.

On the one hand, the low regularity $C^{1,\alpha}$ and $C^{0,\alpha}$ is not a weakness in such result since despite not being directly considered in [27], our results in Sect. A provide the necessary background to promote the existence theorem in [27] to a high regularity setting. On the other hand, such generalized Beltrami fields decay as $|x|^{-2}$ at infinity. There is no contradiction neither with Corollary 2.21 (since it holds under the assumption $\lambda \neq 0$) nor with the Liouville theorem in [36] (since it just holds for globally defined generalized Beltrami fields).

On the contrary, the latter can be used to show an interesting property of such generalized Beltrami fields obtained as perturbations of harmonic fields. Specifically: they cannot be globally extended to the whole space by virtue of the fall-off obstructions in [36]. Nevertheless, the same cannot be directly said for generalized Beltrami fields obtained as perturbations of strong Beltrami fields.

4. Knotted and Linked Stream Lines and Tubes in Generalized Beltrami Fields

Our objective in this section is to apply the convergence result for the modified Grad–Rubin method (6) that we established in the previous section (Theorem 3.7) to show the existence of almost global Beltrami fields of class $C^{k+1,\alpha}$ with a nonconstant factor that realize any given configuration of vortex tubes and vortex lines, modulo a small diffeomorphism. Here k is an arbitrary integer.

4.1. Knots and links in almost global generalized Beltrami fields. Our goal here is to show that the partial stability result for almost global Beltrami fields allows us to conclude the existence of Beltrami fields with a non-constant proportionality factor that are defined in all of \mathbb{R}^3 but, say, in the complement of an arbitrarily small ball, and which

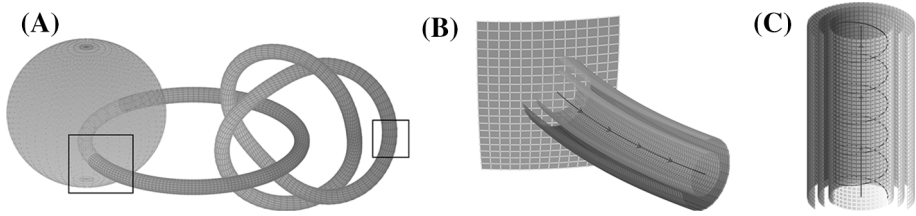


Fig. 5. **a** Collection of knotted and linked vortex tubes of the strong Beltrami field u_0 , $\{\Phi'(\mathcal{T}_\varepsilon(\Gamma_0)), \Phi'(\mathcal{T}_\varepsilon(\Gamma_1))\}$, respectively homeomorphic to the unknot and to the trefoil. **b** Transverse intersection of the vortex tube $\Phi'(\mathcal{T}_\varepsilon(\Gamma_0))$ and the interior domain G . Here we have zoomed in the squared region on the left side of the above figure, showing the smaller outward pointing (ρ_0, T, δ) -stream tube of u_0 that emerges from Σ . The perturbation φ of λ will be supported there. **c** Zoom of the vortex tube $\Phi'(\mathcal{T}_\varepsilon(\Gamma_1))$ with trefoil knot. It shows the internal structure of such vortex tube of u_0 , which contains uncountably many nested tori and knotted vortex lines

have a collection of vortex tubes and vortex lines of arbitrary topology. Let us recall that, as mentioned in the Introduction, a stream tube (invariant torus) of a divergence-free velocity field u is structurally stable if any divergence-free field that is close enough to u in $C^{3,\alpha}$ has an invariant torus given by a $C^{0,\alpha}$ -small diffeomorphism of the initial tube. Although we shall not state these properties explicitly, just as in [18] the vortex tubes that we construct are accumulated on by a positive-measure set of invariant tori on which the vortex lines are ergodic.

Theorem 4.1. *Let G be an exterior domain satisfying (7) and consider any collection of disjoint knotted and linked thin tubes $\mathcal{T}_\varepsilon(\Gamma_1), \dots, \mathcal{T}_\varepsilon(\Gamma_n)$ whose closure is contained in the exterior domain Ω . Then, for ε small enough and any k, α there exists a nonzero constant λ , an open subset $\Sigma \subseteq S$ and some $\delta_0 > 0$ with the following property: for any function $\varphi^0 \in C_c^{k+1,\alpha}(\Sigma)$ with $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} < \delta_0$ there is a Beltrami field $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ with factor $\lambda + \varphi$, where φ is a function in $C^{k,\alpha}(\overline{\Omega})$ satisfying $\varphi|_\Sigma = \varphi^0$:*

$$\begin{cases} \operatorname{curl} u = (\lambda + \varphi)u, & x \in \Omega, \\ \operatorname{div} u = 0, & x \in \Omega. \end{cases}$$

Furthermore, $u = O(|x|^{-1})$ as $|x| \rightarrow +\infty$, the support of φ is compact and lies in the (ρ_0, T, δ) -stream tube $\mathcal{T}(\Sigma, u)$ of u radiating from Σ (with the exception of the endpoints) and $\mathcal{T}_\varepsilon(\Gamma_1), \dots, \mathcal{T}_\varepsilon(\Gamma_n)$ can be modified by a diffeomorphism Φ close enough to the identity in any C^m norm into a collection of structurally stable vortex tubes of u , $\Phi(\mathcal{T}_\varepsilon(\Gamma_1)), \dots, \Phi(\mathcal{T}_\varepsilon(\Gamma_n))$, (possibly) knotted and linked with $\mathcal{T}(\Sigma, u)$.

Proof. Take a curve Γ_0 intersecting S transversally and such that $\mathcal{T}_\varepsilon(\Gamma_0) \cap \Omega$ has only a connected component. We also assume that Γ_0 does not intersect any of the other curves Γ_j , so that the setup is then as depicted in Fig. 5. For $\varepsilon > 0$ small enough, [18] asserts the existence of some diffeomorphism Φ' arbitrarily close to the identity map in any C^m norm such that $\Phi'(\mathcal{T}_\varepsilon(\Gamma_0)), \dots, \Phi'(\mathcal{T}_\varepsilon(\Gamma_n))$ are vortex tubes of a strong Beltrami field u_0 which satisfies the equation $\operatorname{curl} u_0 = \lambda u_0$ in \mathbb{R}^3 for some non-zero constant λ (of order ε^3). By construction, these tubes are structurally stable and Φ' can be assumed to be arbitrarily close to the identity in any C^m norm, so the new thin tubes enjoy the same geometric features as we had assumed on the initial ones. Let $x_0 \in S \cap \Phi'(\Gamma_0)$ be where u_0 points outwards and consider any open and connected neighborhood Σ of x_0 in S such that $\Sigma \subseteq S \cap \Phi'(\mathcal{T}_\varepsilon(\Gamma_0))$.

Recall that u_0 is of the form

$$u_0 = \frac{\operatorname{curl}(\operatorname{curl} + \lambda)}{2\lambda^2} \sum_{l=0}^L \sum_{m=-l}^l c_l^m j_l(\lambda|x|) Y_l^m \left(\frac{x}{|x|} \right).$$

as stated in [18]. Since u_0 is obviously real-valued, it is the real part of the vector field

$$v_0 = \frac{\operatorname{curl}(\operatorname{curl} + \lambda)}{2\lambda^2} \sum_{l=0}^L \sum_{m=-l}^l c_l^m h_l^{(1)}(\lambda|x|) Y_l^m \left(\frac{x}{|x|} \right),$$

where $h_l^{(1)} := j_l + iy_l$ is the spherical Hankel function of l -th order and y_l denotes the spherical Bessel function of the second kind and l -th order. By construction, v_0 satisfies the Beltrami equation (and in particular is smooth) in $\mathbb{R}^3 \setminus \{0\}$, while it diverges at the origin due to the presence of a Bessel function of the second kind. In particular, it is a Beltrami field in Ω .

As the Hankel function $h_l^{(1)}$ has been chosen to satisfy the scalar radiation condition

$$(\partial_r - i\lambda)h_l^{(1)}(\lambda r) = o(r^{-1}),$$

it is straightforward to check that $v_0 \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{C}^3)$ is a complex-valued solution to the Beltrami equation in the exterior domain Ω , which satisfies the L^1 SMB radiation condition (22) and the weak L^1 decay property (23) (see [9, Equation 2.41] along with Remark 2.9 and Fig. 1). It is also apparent that $\mathcal{T}(\Sigma, u_0) \subseteq \Phi'(\mathcal{T}_\varepsilon(\Gamma_0))$ is a (ρ_0, T, δ) -stream tube of u_0 by construction (see Fig. 5), and that $\lambda \sim \varepsilon^3$ can be prevented from being a Dirichlet eigenvalue of the Laplace operator in the interior domain G as long as ε is taken small enough. Then, we are ready to apply the convergence Theorem 3.7 for the modified Grad–Rubin method starting up with the strong Beltrami field u_0 . This result ensures the existence of $\delta_0 > 0$ so that whenever $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma)} \leq \delta_0$, then there exists a generalized Beltrami field $u \in C^{k+1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ and a perturbation $\varphi \in C^{k,\alpha}(\overline{\Omega})$ solving the exterior boundary value problem (52) with $\varphi = \varphi^0$, $x \in \Sigma$. $\mathcal{T}(\Sigma, u)$ is a $(\rho_0, T, \delta/2)$ stream tube of u , φ is compactly supported in the closure of such stream tube and $\|u - u_0\|_{C^{k+1,\alpha}(\Omega)}$ can be made arbitrarily small. In view of the structural stability of the vortex tubes of u_0 , the theorem follows. \square

5. Local Stability of Generalized Beltrami Fields

Our objective in this section is to show that, in fact, any generalized Beltrami field possesses a local partial stability property which can be essentially regarded as a local version of Theorem 3.7. We recall that, in view of the results in [20], one cannot prove a full stability result even in arbitrarily small open sets, so we regard this partial stability (where partial is understood in a very precise sense) as a satisfactory counterpart to the results in this paper.

5.1. A local stability theorem. We shall next present the local stability result that constitutes the core of this section. The philosophy of this result is that, as one is able to perturb strong Beltrami fields, one should also be able to perturb generalized Beltrami

fields in small domains, since in a small region a $C^{k,\alpha}$ function behaves as a constant plus a small perturbation. Somehow, this reduces our effort to estimates similar to the ones that we have already obtained, so our presentation of the proof of this result will be a little sketchier than before. The gist will be to show that, although the strong convergence of the modified Grad–Rubin scheme cannot be granted in $C^{k+1,\alpha}$ for u_n and $C^{k,\alpha}$ for f_n , we can pass to the limit in $C^{1,\alpha}$ and $C^{0,\alpha}$ provided that both the domain and the perturbation of the proportionality factor are small enough. Elliptic regularity will then yield the high order regularity by a bootstrap argument.

In order to support our argument, let us first sketch the effect of the size of the domain on the solutions of the next Neumann boundary value problem associated with the inhomogeneous Beltrami equation in some open ball $B_R(x_0)$

$$\begin{cases} \operatorname{curl} u - \lambda u = w, & x \in B_R(x_0), \\ u \cdot \eta = 0, & x \in \partial B_R(x_0), \end{cases} \tag{58}$$

where $w \in C^{0,\alpha}(\overline{B_R(x_0)}, \mathbb{R}^3)$ has zero flux. We will be interested in the case where R becomes very small.

This problem has being carefully analyzed in [43] for bounded domains and in [27] for exterior unbounded domains in the harmonic case ($\lambda = 0$). The non-harmonic counterpart was studied in [29] and Sect. 2 for the inhomogeneous Beltrami equation in bounded and exterior domains respectively. In the bounded setting, λ has to be assumed “regular” (see [29]). To this end, notice that taking $|\lambda| < c/R$ (for an appropriate universal constant $c > 0$) prevents λ from being an eigenvalue of the Laplacian in $B_R(x_0)$. Hence, $|\lambda| < c/R$ is a sufficient condition ensuring the well-posedness of (58). All the above results provide an estimate for the unique solution u to (58) in terms of w of the form

$$\|u\|_{C^{1,\alpha}(B_R(x_0))} \leq C_{\lambda,R} \|w\|_{C^{0,\alpha}(B_R(x_0))},$$

where the dependence of the constant $C_{\lambda,R}$ on λ and R is not explicit. The next technical result aims to provide some explicit R -dependent estimate for u in some space.

Lemma 5.1. *Let $u \in C^{1,\alpha}(\overline{B_R(x_0)}, \mathbb{R}^3)$ be the unique solution to the Neumann boundary value problem associated with the Beltrami equation (58) for $|\lambda| < c/R$ and $R \in (0, 1)$. Then,*

$$\|u\|_{C^{1,\alpha}(B_R(x_0))} \leq C R^{-\alpha} \|w\|_{C^{0,\alpha}(B_R(x_0))}, \tag{59}$$

for some positive constant C depending on α but not on u, w, x_0 or R .

Proof. To obtain an explicit R -dependent estimate of u in some space, let us perform the change of variables $y = \frac{x-x_0}{R}$. Then, one obtains the following vector fields in the unit ball centered at the origin:

$$U(y) = u(x), \quad W(y) = w(x),$$

solving the Neumann boundary value problem for the Beltrami equation in $B_1(0)$:

$$\begin{cases} \operatorname{curl} U - \lambda R U = R W, & y \in B_1(0), \\ U \cdot \eta = 0, & y \in \partial B_1(0). \end{cases}$$

Thus, the above-mentioned results yield the following bound for some R -independent $C > 0$

$$\|U\|_{C^{1,\alpha}(B_1(0))} \leq C R \|W\|_{C^{0,\alpha}(B_1(0))},$$

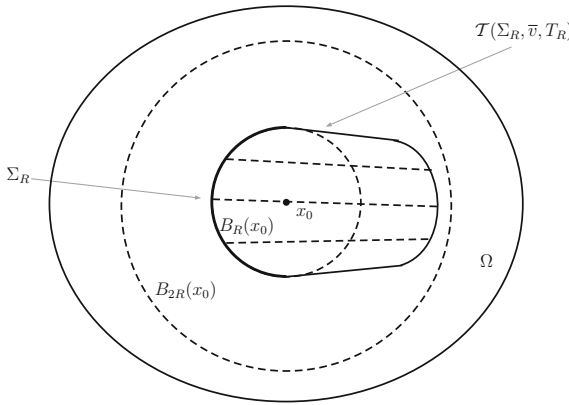


Fig. 6. Flow box $\mathcal{T}(\Sigma_R, \bar{v}, T_R)$ covering the small ball $B_R(x_0)$

where $|\lambda| < c/R$ has been used to avoid the λ -dependence of the constant C . Note that by definition

$$\begin{aligned} \|W\|_{C^{0,\alpha}(B_1(0))} &= \|w\|_{C^0(B_R(x_0))} + R^\alpha [w]_{\alpha, B_R(x_0)}, \\ \|U\|_{C^{1,\alpha}(B_1(0))} &= \|u\|_{C^0(B_R(x_0))} + R \sum_{i=1}^3 \|\partial_{x_i} u\|_{C^0(B_R(x_0))} + R^{1+\alpha} \sum_{i=1}^3 [\partial_{x_i} u]_{\alpha, B_R(x_0)}. \end{aligned}$$

Since $R \in (0, 1)$, then we are led to (59). \square

Another key ingredient is to show that $C^{1,\alpha}$ vector fields near a non-equilibrium point verify a “structurally stable” flow box theorem, to be understood in the next precise sense.

Lemma 5.2. *Let $u \in C^{1,\alpha}(\Omega, \mathbb{R}^3)$ be a (nontrivial) vector field and consider some $x_0 \in \Omega$ such that $u(x_0) \neq 0$. There exist $R_0 > 0$ and $\delta_0 > 0$ such that $\bar{B}_{2R_0}(x_0) \subseteq \Omega$, u vanishes nowhere in the ball and for every $0 < R < R_0$ there exists some surface $\Sigma_R \subseteq \partial B_R(x)$ and a positive function $T_R \in C(\Sigma_R)$ such that for every $v \in C^{1,\alpha}(\bar{B}_R(x_0), \mathbb{R}^3)$ with $\|u - v\|_{C^{1,\alpha}(B_R(x_0))} < \delta_0$, then*

$$B_R(x_0) \subseteq \mathcal{T}(\Sigma_R, \bar{v}, T_R) \subseteq B_{2R}(x_0).$$

Here, the above stream tube reads

$$\mathcal{T}(\Sigma_R, \bar{v}, T_R) := \{X^{\bar{v}}(t; 0, x) : x \in \Sigma_R, t \in (0, T_R(x))\},$$

\bar{v} is the Calderón extension of v from $B_R(x_0)$ to $\bar{B}_{2R_0}(x_0)$ (Proposition 3.1) and the height T_R of the stream tube is not constant but it continuously depends, stream line by stream line, on the base point $x \in \Sigma_R$ (see Fig. 6). Furthermore, the parametrizations μ_R of Σ_R can be normalized by choosing

$$\mu_R(s) = R\mu(s), \quad s \in D_R,$$

for some open subset $D_R \subseteq D_1(0)$ of the unit disc centered at 0, and some local parametrization of the unit sphere $\mu : D_1(0) \rightarrow \partial B_1(x_0)$. Since the proof follows the same lines as Lemma 3.4 in Sect. 3, we skip it and pass to the central result of this section.

Theorem 5.3. *Let u_0 be a nontrivial generalized Beltrami field of class $C^{k+1,\alpha}(\Omega, \mathbb{R}^3)$, where $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, and consider its (nonconstant) proportionality factor $f_0 \in C^{k,\alpha}(\Omega)$. Take some nonequilibrium point $x_0 \in \Omega$ of u_0 and fix some $\varepsilon_0 > 0$. Then, for each small enough radius $R > 0$ there is some surface $\Sigma_R \subseteq \partial B_R(x_0)$ and some constant $\delta_R > 0$ so that for every $\varphi^0 \in C^{k+1,\alpha}(\Sigma_R)$ with $\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma_R, \mu_R)} < \delta_R$ there exist $\varphi \in C^{k,\alpha}(\overline{B}_R(x_0))$ and $u \in C^{k+1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ such that $\varphi = \varphi^0$ on Σ_R and u is a strong Beltrami field with proportionality factor $f_0 + \varphi$ enjoying the same normal component as u_0 in $\partial B_R(x_0)$:*

$$\begin{cases} \operatorname{curl} u = (f_0 + \varphi)u, & x \in B_R(x_0), \\ \operatorname{div} u = 0, & x \in B_R(x_0), \\ u \cdot \eta = u_0 \cdot \eta, & x \in \partial B_R(x_0). \end{cases}$$

Furthermore,

$$\|u - u_0\|_{C^{k+1,\alpha}(B_R(x_0))} \leq \varepsilon_0 \|u_0\|_{C^{k+1,\alpha}(B_R(x_0))}.$$

Proof. The proof has two steps. First, we will prove the theorem for low Hölder exponents and regularity (namely, $\alpha \in (0, 1/2)$ and $k = 0$). Second, we will show a bootstrap argument based on elliptic gain of regularity that will raise the estimates in the first step to its full strength and will conclude the proof of the theorem for general regularity and Hölder exponents.

Then, let us first assume that $\alpha \in (0, 1/2)$, define $\lambda_0 := f_0(x_0)$ and fix some radius $R_0 > 0$ so that $\overline{B}_{2R_0}(x_0) \subseteq \Omega$, u_0 vanishes nowhere in $\overline{B}_{2R_0}(x_0)$ and the assertions in Lemma 5.2 fulfil. Without loss of generality, we can assume that $R_0 < \min\{1, c/|\lambda_0|\}$. Moreover, note that the homogeneous generalized Beltrami equation can be restated as an inhomogeneous Beltrami equation with constant proportionality factor and an inhomogeneous term taking the form of a small remainder, i.e.,

$$\operatorname{curl} u_0 - \lambda_0 u_0 = \mathcal{R}(x - x_0)u_0, \quad x \in \Omega, \tag{60}$$

where $f_0(x) = \lambda_0 + \mathcal{R}(x - x_0)$ for every $x \in \overline{B}_{2R_0}(x_0)$, i.e.,

$$\mathcal{R}(z) := \left(\int_0^1 \nabla f_0(x_0 + \theta z) d\theta \right) \cdot z, \quad z \in \overline{B}_{2R_0}(0).$$

Next, consider the following modified iterative scheme of Grad–Rubin type. It consists of a sequence of transport equations

$$\begin{cases} \nabla \varphi_n \cdot u_n = -\nabla f_0 \cdot u_n, & x \in B_R(x_0), \\ \varphi_n = \varphi^0, & x \in \Sigma_R, \end{cases} \tag{61}$$

along with a sequence of boundary value problems associated with the inhomogeneous Beltrami equation

$$\begin{cases} \operatorname{curl} u_{n+1} - \lambda_0 u_{n+1} = \mathcal{R}(x - x_0)u_n + \varphi_n u_n, & x \in B_R(x_0), \\ u_{n+1} \cdot \eta = u_0 \cdot \eta, & x \in \partial B_R(x_0). \end{cases} \tag{62}$$

Note that they have been chosen in a consistent way so that as long as $\{u_n\}_{n \in \mathbb{N}}$ and $\{\varphi_n\}_{n \in \mathbb{N}}$ have limits (in some sense), then the limits u and φ give rise to a generalized Beltrami field whose proportionality factor is a perturbation $f_0 + \varphi$ of the initial factor

f_0 . Without loss of generality, we can assume that $\lambda_0 \neq 0$ (in the case $\lambda_0 = 0$ would need the additional condition $\operatorname{div} u_{n+1} = 0$).

Let us show that both $u_{n+1} \in C^{1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ and $f_n \in C^{0,\alpha}(\overline{B}_R(x_0))$ are well defined and that

$$\left\{ \begin{array}{l} \|u_{n+1} - u_n\|_{C^{1,\alpha}(B_R(x_0))} \leq \frac{1}{2^n} \|u_1 - u_0\|_{C^{1,\alpha}(B_R(x_0))} < \min\{\varepsilon_0, 1\} \frac{1}{2^{n+1}} \|u_0\|_{C^{1,\alpha}(B_R(x_0))}, \\ \|u_{n+1} - u_0\|_{C^{1,\alpha}(B_R(x_0))} \leq \min\{\varepsilon_0, 1\} \sum_{i=1}^{n+1} \frac{1}{2^i} \|u_0\|_{C^{1,\alpha}(B_R(x_0))}, \\ \|u_{n+1}\|_{C^{1,\alpha}(B_R(x_0))} \leq \min\{\varepsilon_0, 1\} \sum_{i=0}^{n+1} \|u_0\|_{C^{1,\alpha}(B_R(x_0))}, \end{array} \right. \tag{63}$$

for every $n \in \mathbb{N}$. Let us start with $n = 0$. On the one hand, the transport problem (61) with $n = 0$ can be solved in $B_R(x_0)$ as $B_R(x_0) \subseteq \mathcal{T}(\Sigma_R, u_0, T_R) \subseteq B_{2R}(x_0)$ by virtue of Lemma 5.2. Indeed,

$$\varphi_0(X^{u_0}(t; 0, x)) = \varphi^0(x) - \int_0^t (\nabla f_0 \cdot u_0)(X^{u_0}(\tau, 0, x)) d\tau, \quad x \in \Sigma_R, t \in (0, T_R(x))$$

defines a solution in $\mathcal{T}(\Sigma_R, u_0, T_R)$ and, in particular, in $B_R(x_0)$. Now, notice that

$$\begin{aligned} & \int_{\partial B_R(x_0)} (\mathcal{R}(\cdot - x_0)u_0 + \varphi_0 u_0 + \lambda_0 u_0) \cdot \eta dS \\ &= \int_{B_R(x_0)} (\nabla(f_0 + \varphi_0) \cdot u_0 + (f_0 + \varphi_0) \operatorname{div} u_0) dx = 0, \end{aligned}$$

and λ is regular (see [29]) with respect to the inhomogeneous problem (62) with $n = 0$ because $R < R_0 < c/|\lambda_0|$. Hence, (62) has an unique solution $u_1 \in C^{1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ by virtue of the existence theorem in [29]. Notice that since $\operatorname{div} u_0 = 0$ and the first integral equations in (61) hold, then

$$-\lambda_0 \operatorname{div} u_1 = (f_0 + \varphi_0) \operatorname{div} u_0 + \nabla(f_0 + \varphi_0) \cdot u_0 = 0.$$

Furthermore, $u_1 - u_0$ solves the Neumann boundary value problem

$$\left\{ \begin{array}{l} (\operatorname{curl} - \lambda_0)(u_1 - u_0) = \mathcal{R}(x - x_0)u_0 + \varphi_0 u_0, \quad x \in B_R(x_0), \\ (u_1 - u_0) \cdot \eta = 0, \quad x \in B_R(x_0). \end{array} \right.$$

Consequently,

$$\begin{aligned} \|u_1 - u_0\|_{C^{1,\alpha}(B_R(x_0))} &\leq \frac{C}{R^\alpha} (\|\mathcal{R}(\cdot - x_0)\|_{C^{0,\alpha}(B_R(x_0))} \\ &\quad + \|\varphi_0\|_{C^{0,\alpha}(B_R(x_0))}) \|u_0\|_{C^{0,\alpha}(B_R(x_0))}. \end{aligned}$$

A similar result to that in Theorem 3.5 yields the estimate

$$\begin{aligned} \|\varphi_0\|_{C^{0,\alpha}(B_R(x_0))} &\leq \left(\|\varphi^0\|_{C^{1,\alpha}(\Sigma_R, \mu_R)} + R^{1-\alpha} + \|T_R\|_{C^0(\Sigma_R)} \right) \\ &\quad \times \kappa (\|u_0\|_{C^{1,\alpha}(B_R(x_0))}, \|T_R\|_{C^0(\Sigma_R)}, \|\mu_R\|_{C^{1,\alpha}(B_R(x_0))}), \end{aligned}$$

for some separately increasing function κ . Regarding the remainder, it is clear that

$$\|\mathcal{R}(\cdot - x_0)\|_{C^{0,\alpha}(B_R(x_0))} \leq CR^{1-\alpha}, \tag{64}$$

which is indeed the reason behind the estimate for φ_0 that we stated above. Notice that although \mathcal{R} is clearly bounded above by R in $B_R(0)$, the α -Hölder constant is $O(R^{1-\alpha})$. Specifically, take $z_1, z_2 \in B_R(0)$ and split \mathcal{R} as follows

$$\mathcal{R}(z_1) - \mathcal{R}(z_2) = I + II,$$

where

$$I := \left(\int_0^1 \nabla f_0(x_0 + \theta z_1) d\theta \right) \cdot (z_1 - z_2),$$

$$II := \left(\int_0^1 (\nabla f_0(x_0 + \theta z_1) - \nabla f_0(x_0 + \theta z_2)) d\theta \right) \cdot z_2.$$

By virtue of the α -Hölder continuity of ∇f_0 , II can be bounded as follows:

$$|II| \leq \|f_0\|_{C^{1,\alpha}(B_R(x_0))} |z_2| \int_0^1 |z_1 - z_2|^\alpha \theta^\alpha d\theta \leq \frac{\|f_0\|_{C^{1,\alpha}(B_R(x_0))}}{\alpha + 1} R |z_1 - z_2|^\alpha.$$

The first term enjoys the bound

$$|I| \leq \|\nabla f_0\|_{C^0(B_R(x_0))} |z_1 - z_2| \leq 2^{1-\alpha} \|\nabla f_0\|_{C^0(B_R(x_0))} R^{1-\alpha} |z_1 - z_2|^\alpha,$$

which then leads to the desired estimate (64). Notice that one could have raised the $R^{1-\alpha}$ power to R if one assumed that $\nabla f_0(x_0) = 0$.

Also, note that $\|\mu_R\|_{C^{1,\alpha}(D_R)}, \|T_R\|_{C^0(\Sigma_R)} \leq C_0 R$ for some universal constant $C_0 > 0$. Then, the above estimate for $u_1 - u_0$ can be written as

$$\begin{aligned} & \|u_1 - u_0\|_{C^{0,\alpha}(B_R(x_0))} \\ & \leq \frac{C}{R^\alpha} \left(\|\varphi^0\|_{C^{1,\alpha}(\Sigma_R, \mu_R)} + 2R^{1-\alpha} \right) \left\{ 1 + \kappa \left(\|u_0\|_{C^{1,\alpha}(B_R(x_0))}, C_0, \|\mu\|_{C^{1,\alpha}(D_1(0))} \right) \right\} \\ & \quad \|u_0\|_{C^{0,\alpha}(B_R(x_0))}. \end{aligned}$$

Hereafter we will assume that

$$\begin{aligned} C \left(\frac{\delta_R}{R^\alpha} + 2R^{1-2\alpha} \right) & \left\{ 1 + \kappa \left(2\|u_0\|_{C^{1,\alpha}(B_R(x_0))}, C_0, \|\mu\|_{C^{1,\alpha}(D_1(0))} \right) \right. \\ & \left. + 2\kappa \left(2\|u_0\|_{C^{1,\alpha}(B_R(x_0))}, C_0, \|\mu\|_{C^{1,\alpha}(D_1(0))} \right)^2 \|u_0\|_{C^{1,\alpha}(B_R(x_0))} \right\} \\ & < \frac{\varepsilon_0}{2}, \end{aligned} \tag{65}$$

with $\varepsilon_0 \in (0, 1)$ small enough so that $\varepsilon_0 \|u_0\|_{C^{0,\alpha}(B_R(x_0))} < \delta_0$. Since we are considering low Hölder exponents $\alpha \in (0, 1/2)$, then we can ensure the existence of small enough $R \in (0, R_0)$ and $\delta_R > 0$ enjoying the above property.

Assume that we have already defined $f_m \in C^{0,\alpha}(\overline{B}_R(x_0))$ and $u_{m+1} \in C^{1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ for every $m < n$ such that they verify (61)–(63) and $u_m = 0$ is divergence-free for every index $m < n$. To close the inductive argument let us prove the result for

$m = n$. First, the transport problem (61) can be uniquely solved in $B_R(x_0)$ by virtue of Lemma 5.2, the inductive hypothesis (63) and the assumption on ε_0 since

$$\|u_n - u_0\|_{C^{1,\alpha}(B_R(x_0))} \leq \varepsilon_0 \|u_0\|_{C^{1,\alpha}(B_R(x_0))} < \delta_0.$$

Second, the boundary value problem (62) can also be uniquely solved since

$$\begin{aligned} & \int_{\partial B_R(x_0)} (\mathcal{R}(\cdot - x_0)u_n + \varphi_n u_n) \cdot \eta \, dS + \lambda_0 \int_{\partial B_R(x_0)} u_0 \cdot \eta \, dS \\ &= \int_{B_R(x_0)} (\nabla(f_0 + \varphi_n) \cdot u_n + (f_0 + \varphi_n) \operatorname{div} u_n) \, dx = 0, \end{aligned}$$

by the inductive hypothesis and λ is assumed to be a regular value. Furthermore, a similar argument to that in the step $n = 0$ shows that u_{n+1} is divergence-free again. Let us finally obtain the desired estimates for $u_{n+1} - u_n$. To this end, note that $u_{n+1} - u_n$ solves the boundary value problem

$$\begin{cases} (\operatorname{curl} - \lambda_0)(u_{n+1} - u_n) = \mathcal{R}(\cdot - x_0)(u_n - u_{n-1}) + (\varphi_n - \varphi_{n-1})u_n + \varphi_{n-1}(u_n - u_{n-1}), & x \in B_R(x_0), \\ (u_{n+1} - u_n) \cdot \eta = 0, & x \in \partial B_R(x_0). \end{cases}$$

Hence, we arrive at the following bound

$$\begin{aligned} \|u_{n+1} - u_n\|_{C^{1,\alpha}(B_R(x_0))} &\leq \frac{C}{R^\alpha} (\|\mathcal{R}(\cdot - x_0)\|_{C^{0,\alpha}(B_R(x_0))} \|u_n - u_{n-1}\|_{C^{0,\alpha}(B_R(x_0))} \\ &\quad + \|\varphi_n - \varphi_{n-1}\|_{C^{0,\alpha}(B_R(x_0))} \|u_n\|_{C^{0,\alpha}(B_R(x_0))} \\ &\quad + \|\varphi_{n-1}\|_{C^{0,\alpha}(B_R(x_0))} \|u_n - u_{n-1}\|_{C^{0,\alpha}(B_R(x_0))}). \end{aligned}$$

On the one hand, the remainder can be bounded above as in (64). On the other hand, $\|\varphi_n\|_{C^{0,\alpha}(B_R(x_0))}$ and $\|\varphi_n - \varphi_{n-1}\|_{C^{0,\alpha}(B_R(x_0))}$ can be estimated as

$$\begin{aligned} \|\varphi_{n-1}\|_{C^{0,\alpha}(B_R(x_0))} &\leq \left(\|\varphi^0\|_{C^{1,\alpha}(\Sigma_R, \mu_R)} + R^{1-\alpha} + \|T_R\|_{C^0(\Sigma_R)} \right) \\ &\quad \times \kappa \left(\|u_{n-1}\|_{C^{1,\alpha}(B_R(x_0))}, \|T_R\|_{C^0(\Sigma_R)}, \|\mu_R\|_{C^{1,\alpha}(B_R(x_0))} \right), \\ \|\varphi_n - \varphi_{n-1}\|_{C^{0,\alpha}(B_R(x_0))} &\leq \left(\|\varphi^0\|_{C^{1,\alpha}(\Sigma_R, \mu_R)} + R^{1-\alpha} + \|T_R\|_{C^0(\Sigma_R)} \right) \\ &\quad \times \kappa \left(\|u_n\|_{C^{1,\alpha}(B_R(x_0))}, \|T_R\|_{C^0(\Sigma_R)}, \|\mu_R\|_{C^{1,\alpha}(B_R(x_0))} \right) \\ &\quad \times \kappa \left(\|u_{n-1}\|_{C^{1,\alpha}(B_R(x_0))}, \|T_R\|_{C^0(\Sigma_R)}, \|\mu_R\|_{C^{1,\alpha}(B_R(x_0))} \right) \\ &\quad \times \|u_n - u_{n-1}\|_{C^{1,\alpha}(B_R(x_0))}. \end{aligned}$$

Consequently, the inductive hypothesis along with our choice (65) leads to the first inequality in (63) and the remaining two inequalities obviously follows from the first one by virtue of the triangle inequality.

As in Sect. 3, the first inequality in (63) shows that $\{u_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $C^{1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$. By completeness, consider $u \in C^{1,\alpha}(\overline{B}_R(x_0))$ such that $u_n \rightarrow u$ in $C^{1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$. Moreover, the same reasoning as above yields the estimate

$$\begin{aligned} \|\varphi_n - \varphi_m\|_{C^{0,\alpha}(B_R(x_0))} &\leq \left(\|\varphi^0\|_{C^{1,\alpha}(\Sigma_R, \mu_R)} + R^{1-\alpha} + \|T_R\|_{C^0(\Sigma_R)} \right) \\ &\quad \times \kappa \left(\|u_n\|_{C^{1,\alpha}(B_R(x_0))}, \|T_R\|_{C^0(\Sigma_R)}, \|\mu_R\|_{C^{1,\alpha}(B_R(x_0))} \right) \\ &\quad \times \kappa \left(\|u_m\|_{C^{1,\alpha}(B_R(x_0))}, \|T_R\|_{C^0(\Sigma_R)}, \|\mu_R\|_{C^{1,\alpha}(B_R(x_0))} \right) \\ &\quad \times \|u_n - u_m\|_{C^{1,\alpha}(B_R(x_0))}, \end{aligned}$$

for every indices $n, m \in \mathbb{N}$. Then, there exists some constant $K = K(\delta_R, R, \|u_0\|_{C^{0,\alpha}}) > 0$ so that

$$\|\varphi_n - \varphi_m\|_{C^{0,\alpha}(\overline{B}_R(x_0))} \leq K \|u_n - u_m\|_{C^{1,\alpha}(B_R(x_0))}.$$

Hence, $\{\varphi_n\}_{n \in \mathbb{N}}$ is also a Cauchy sequence in $C^{0,\alpha}(\overline{B}_R(x_0))$ and one can consider $\varphi \in C^{0,\alpha}(\overline{B}_R(x_0))$ such that $\varphi_n \rightarrow \varphi$ in $C^{0,\alpha}(\overline{B}_R(x_0))$. Taking limits in (61)–(62) we are led to a generalized Beltrami field $u \in C^{1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ solving

$$\begin{cases} \operatorname{curl} u = (f_0 + \varphi)u, & x \in B_R(x_0), \\ \operatorname{div} u = 0, & x \in B_R(x_0), \\ u \cdot \eta = u_0 \cdot \eta, & x \in \partial B_R(x_0), \end{cases}$$

for a perturbation $\varphi \in C^{0,\alpha}(\overline{B}_R(x_0))$ of the factor such that $\varphi = \varphi^0$ on Σ_R .

Let us finally show that $u \in C^{k+1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ and $\varphi \in C^{k,\alpha}(\overline{B}_R(x_0))$ by a bootstrap argument based on the elliptic gain of regularity. The key observation now is that, by acting with the curl operator on the equation for u , it follows that

$$\begin{cases} \Delta u = -\operatorname{curl}((f_0 + \varphi)u), & x \in B_R(x_0), \\ u \cdot \eta = u_0 \cdot \eta, & x \in \partial B_R(x_0), \\ \operatorname{curl} u \times \eta = (f_0 + \varphi)u \times \eta, & x \in \partial B_R(x_0). \end{cases}$$

Then, the next hierarchy of inequalities

$$\begin{aligned} & \|u\|_{C^{l+1,\alpha}(B_R(x_0))} \\ & \leq C(\|(f_0 + \varphi)u\|_{C^{l,\alpha}(B_R(x_0))} + \|u_0 \cdot \eta\|_{C^{l+1,\alpha}(\partial B_R(x_0))} \\ & \quad + \|(f_0 + \varphi)u \times \eta\|_{C^{l,\alpha}(B_R(x_0))}). \end{aligned}$$

hold for every $l \geq 0$. We then get that the fact that u is of class $C^{1,\alpha}$ implies that φ is of class $C^{0,\alpha}$. In turns, it ensures that u is in $C^{2,\alpha}$ and, repeating the argument as many times as necessary (up to the regularity on φ^0 and u_0 , i.e., $C^{k+1,\alpha}$) we derive the desired gain of regularity. Indeed, the estimate

$$\|u - u_0\|_{C^{1,\alpha}(B_R(x_0))} \leq \varepsilon_0 \|u_0\|_{C^{1,\alpha}(B_R(x_0))},$$

can be promoted to its $C^{k+1,\alpha}$ version, i.e.,

$$\|u - u_0\|_{C^{k+1,\alpha}(B_R(x_0))} \leq \varepsilon_0 \|u_0\|_{C^{k+1,\alpha}(B_R(x_0))}.$$

So far, we have only taken low Hölder exponents $\alpha \in (0, 1/2)$. Assume now that $u_0 \in C^{k+1,\alpha'}(\Omega, \mathbb{R}^3)$ and $f_0 \in C^{k,\alpha'}(\Omega)$ for some $\alpha' \in (\alpha, 1)$. In particular, $u_0 \in C^{k+1,\alpha}(\overline{B}_{2R_0}(x_0), \mathbb{R}^3)$ and $\varphi_0 \in C^{k,\alpha}(\overline{B}_{2R}(x_0))$. The above argument, yields a strong Beltrami field $u \in C^{k+1,\alpha}(\overline{B}_R(x_0), \mathbb{R}^3)$ with proportionality factor $f_0 + \varphi$ for some perturbation $\varphi \in C^{k,\alpha}(\overline{B}_R(x_0))$ such that $\varphi = \varphi^0$ on Σ_R as long as R is small enough and $\|\varphi^0\|_{C^{k+1,\alpha'}(\Sigma_R)} < \delta_R$. Since

$$\|\varphi^0\|_{C^{k+1,\alpha}(\Sigma_R)} = \|\varphi^0 \circ \mu_R\|_{C^{k+1,\alpha}(D_R)} \leq \|\varphi^0 \circ \mu_R\|_{C^{k+1,\alpha'}(D_R)} = \|\varphi^0\|_{C^{k+1,\alpha'}(\Sigma_R)},$$

then, the above smallness assumption on the $C^{k+1,\alpha}(\Sigma_R)$ norm φ^0 follows from the corresponding assumption on the $C^{k+1,\alpha'}(\Sigma_R)$ norm, i.e., $\|\varphi^0\|_{C^{k+1,\alpha'}(\Sigma_R)} < \delta_R$. Since φ solves

$$\begin{cases} \nabla\varphi \cdot u = -\nabla f_0 \cdot u, & x \in B_R(x_0), \\ \varphi = \varphi^0, & x \in \Sigma_R, \end{cases}$$

then, a similar result to that in Theorem 3.5 leads to $\varphi \in C^{1,\alpha}(\overline{B}_R(x_0))$ because so is u , f_0 and φ^0 . In particular $\varphi \in C^{0,\alpha'}(\overline{B}_R(x_0))$ and $u \in C^{0,\alpha'}(\overline{B}_R(x_0), \mathbb{R}^3)$. Then, the above bootstrap in the Beltrami equation yields $\varphi \in C^{k,\alpha'}(\overline{B}_R(x_0))$ and $u \in C^{k+1,\alpha'}(\overline{B}_R(x_0))$, thereby concluding the proof. \square

Acknowledgements. This work has been partially supported by the MINECO-Feder (Spain) research grant, the project MTM2014-53406-R (D.P. and J.S.), the Junta de Andalucía (Spain) Project FQM 954 (D.P. and J.S.), the MEC (Spain) research grant FPU2014/06304 (D.P.) and by the ERC Starting Grant 633152 and the ICMAT-Severo Ochoa grant SEV-2011-0087 (A.E.).

Appendix A: Potential Theory Techniques for Inhomogeneous Integral Kernels

Our goal in this section is to extend some results of classical potential theory to inhomogeneous kernels like the fundamental solution of the Helmholtz equation $\Gamma_\lambda(x)$ (see e.g. [8, 13, 23, 24, 30, 32, 33, 41, 42] in the case of homogeneous kernels). While there are some previous results concerning the inhomogeneous case (see [9, 10, 37] for a study of $\Gamma_\lambda(x)$ with non-zero λ), only low order Hölder estimates have been obtained. Our approach roughly follows the treatment of [25, 38] for the harmonic case (i.e., $\lambda = 0$), and we will introduce nontrivial modifications to derive higher order Hölder estimates of generalized volume and single layer potentials in the inhomogeneous setting. These results were used in Sect. 2 and, of course, the main point throughout is to be able to consider exterior (unbounded) domains.

A.1. Inhomogeneous volume and single layer potentials. In our context, all the integral kernels to be considered come from the fundamental solution of the 3-dimensional Helmholtz equation (12)

$$\Gamma_\lambda(z) = \frac{e^{i\lambda|z|}}{4\pi|z|} = \frac{1}{4\pi} \left(\frac{\cos(\lambda|z|)}{|z|} + i \frac{\sin(\lambda|z|)}{|z|} \right), \quad z \in \mathbb{R}^3 \setminus \{0\}.$$

For $\lambda = 0$ we recover the Newtonian potential associated with the Laplace equation in \mathbb{R}^3 , [23, 24, 32, 33]. As it is not longer homogeneous, the classical theory cannot be directly applied.

Fortunately, this kernel can be thought to be “almost homogeneous” in the following sense. Let us consider the functions

$$\begin{cases} \phi_\lambda(r) := \frac{e^{i\lambda r}}{4\pi r}, & r > 0, \\ \psi_\lambda(r) := \phi_\lambda(r) - \frac{1}{4\pi r} \equiv \frac{e^{i\lambda r} - 1}{4\pi r}, & r > 0. \end{cases} \tag{66}$$

From the definition one has the following splitting

$$\Gamma_\lambda(z) = \phi_\lambda(|z|) = \frac{1}{4\pi|z|} + \psi_\lambda(|z|) =: \Gamma_0(z) + R_\lambda(z), \tag{67}$$

that amounts to a decomposition of the inhomogeneous kernel $\Gamma_\lambda(z)$ into the homogeneous part $\Gamma_0(z)$ and an inhomogeneous remainder $R_\lambda(z)$ exhibiting a lower order singularity at the origin. The main argument supporting our subsequent results is that we do not need our whole kernel to be purely homogeneous, but only the principal (or more singular) part. While high order derivatives of harmonic potentials can be directly controlled through the harmonic kernel $\Gamma_0(z)$ and the classical results [23,24,32,33], it is also important to control the behavior of the higher order derivatives of the remainder $R_\lambda(z)$.

To this end, let us compute the derivative of $\psi_\lambda(r)$

$$\psi'_\lambda(r) = i\lambda \frac{1}{4\pi r} + \left(i\lambda - \frac{1}{r}\right) \psi_\lambda(r).$$

and note that since $\psi_\lambda(r)$ is locally bounded near $r = 0$ and decay as r^{-1} at infinity, it is globally bounded. Hence, a recursive reasoning leads to estimates for high order derivatives:

$$|\psi_\lambda^{(m)}(r)| \leq C \left(1 + \frac{1}{r^m}\right), \quad r > 0, \tag{68}$$

for a nonnegative constant $C = C(\lambda, m)$. It obviously turns into

$$|D^\gamma R_\lambda(z)| \leq C \left(1 + \frac{1}{|z|^{|\gamma|}}\right), \tag{69}$$

for every $z \in \mathbb{R}^3 \setminus \{0\}$ and each multi-index γ , in contrast with the analogous bounds for $\Gamma_0(z)$:

$$|D^\gamma \Gamma_0(z)| \leq C \frac{1}{|z|^{|\gamma|+1}}. \tag{70}$$

A basic fact is that, being inhomogeneous, $R_\lambda(z)$ is one degree less singular than $\Gamma_0(z)$. Thus, we will combine results of Calderón–Zygmund type for singular integrals (e.g. $D^2\Gamma_0(z)$) with a treatment in the spirit of Hardy–Littlewood–Sobolev theorem for weakly singular integral kernels (e.g. $D^2R_\lambda(z)$). See also [37] for a treatment of pseudo-homogeneous kernels.

For the sake of completeness, we shall next introduce the kind of kernels to be faced in this section. Let us consider a bounded domain $D \subseteq \mathbb{R}^N$. A continuous function $K = K(x, z)$, $x \in \overline{D}$, $z \in \mathbb{R}^N \setminus \{0\}$ is a *weakly singular kernel of exponent β* if there is a nonnegative constant C such that

$$|K(x, z)| \leq \frac{C}{|z|^\beta}, \quad x \in \overline{D}, \quad z \in \mathbb{R}^N \setminus \{0\},$$

for a given $0 \leq \beta \leq N - 1$. In this paper, the singular kernels that will appear are first order partial derivatives of positively homogeneous kernel of degree $-(N - 1)$, i.e.,

$$\frac{\partial}{\partial z_i} K(x, z), \quad x \in \overline{D}, \quad z \in \mathbb{R}^3 \setminus \{0\},$$

where $K(x, z)$ satisfies $K(x, \lambda z) = \lambda^{-(N-1)} K(x, z)$ for all $x \in \overline{D}$, $z \in \mathbb{R}^N \setminus \{0\}$, $\lambda > 0$ and $K(x, \sigma)$ is continuous for $x \in \overline{D}$ and $\sigma \in \partial B_1(0)$.

The same lines as in the classical result [33, Teorema 2.I] can be used to achieve bounds for the single layer potential associated with $\Gamma_\lambda(z)$ both in bounded and unbounded domains:

Theorem A.1 (Generalized single layer potential). *Let $G \subseteq \mathbb{R}^3$ be a bounded domain with regularity $C^{k+1,\alpha}$, $\Omega := \mathbb{R}^3 \setminus \overline{G}$ its outer domain and $S = \partial G$ the boundary surface. Consider the generalized single layer potential associated with the Helmholtz equation and generated by a density ζ along S ,*

$$(\mathcal{S}_\lambda \zeta)(x) := \int_S \Gamma_\lambda(x - y) \zeta(y) d_y S, \quad x \in \mathbb{R}^3 \setminus S.$$

Then, the restrictions of $\mathcal{S}_\lambda \zeta$ to the interior and exterior domain defines bounded linear operators

$$\mathcal{S}_\lambda^- : C^{k,\alpha}(S) \longrightarrow C^{k+1,\alpha}(\overline{G}), \quad \mathcal{S}_\lambda^+ : C^{k,\alpha}(S) \longrightarrow C^{k+1,\alpha}(\overline{\Omega}).$$

We omit the proof of this theorem since we are interested in a more singular regularity result that follows similar ideas. Specifically, we will study the regularity along the boundary surface S of these generalized single layer potentials and other related potentials with inhomogeneous kernels that arose in previous sections via similar arguments to those in [33, Teorema 2.I]. The main goal of the next results is to derive the classical Hölder–Korn–Lichtenstein–Giraud inequality for high order estimates of Hölder type in the inhomogeneous case, i.e., the regularity of generalized volume (or Newtonian) potentials with compactly supported densities both for interior and exterior domains.

Lemma A.2. *Let $G \subseteq \mathbb{R}^3$ be a bounded domain with regularity $C^{k+1,\alpha}$ and $S = \partial G$ the boundary surface. The generalized volume potential on G associated with the Helmholtz equation and a density ζ in G ,*

$$(\mathcal{N}_\lambda^- \zeta)(x) = \int_G \Gamma_\lambda(x - y) \zeta(y) dy, \quad x \in G,$$

defines a bounded linear map $\mathcal{N}_\lambda^- : C^{k,\alpha}(\overline{G}) \longrightarrow C^{k+2,\alpha}(\overline{G})$.

Proof. The proof follows the lines of [33, Teorema 3.II] for the harmonic case $\lambda = 0$, that we extend to the inhomogeneous case.

A C^1 estimate of $\mathcal{N}_\lambda^- \zeta$ can be achieved by taking derivatives under the integral sign

$$\frac{\partial}{\partial x_i} (\mathcal{N}_\lambda^- \zeta)(x) = \int_G \frac{\partial}{\partial x_i} \Gamma_\lambda(x - y) \zeta(y) dy, \quad x \in G,$$

and using the local integrability of $\Gamma_\lambda(z)$, $\nabla \Gamma_\lambda(z)$, along with the boundedness of G and the fact that $\zeta \in C^0(G)$:

$$\|\mathcal{N}_\lambda^- \zeta\|_{C^1(G)} \leq C \|\zeta\|_{C^0(G)} \leq C \|\zeta\|_{C^{k,\alpha}(G)}.$$

Now, fix any multi-index γ with $|\gamma| \leq k$ and takes derivatives again under the integral sign to get

$$D^\gamma \frac{\partial}{\partial x_i} (\mathcal{N}_\lambda^- \zeta)(x) = \int_G D_x^\gamma \frac{\partial}{\partial x_i} \Gamma_\lambda(x - y) \zeta(y) dy.$$

A recursive reasoning supported by some chained integrations by parts leads to

$$\begin{aligned}
 D^\gamma \frac{\partial}{\partial x_i} (\mathcal{N}_\lambda^- \zeta)(x) &= - \sum_{m_1=1}^{\gamma_1} \int_S D_x^{\gamma-m_1 e_1} \frac{\partial}{\partial x_i} \Gamma_\lambda(x-y) D^{(m_1-1)e_1} \zeta(y) \eta_1(y) dy S \\
 &\quad - \sum_{m_2=1}^{\gamma_2} \int_S D_x^{\gamma-\gamma_1 e_1 - \gamma_2 e_2} \frac{\partial}{\partial x_i} \Gamma_\lambda(x-y) D^{\gamma_1 e_1 + (m_2-1)e_2} \zeta(y) \eta_2(y) dy S \\
 &\quad - \sum_{m_3=1}^{\alpha_3} \int_S D_x^{\gamma-\gamma_1 e_1 - \gamma_2 e_2 - m_3 e_3} \frac{\partial}{\partial x_i} \Gamma_\lambda(x-y) D^{\gamma_1 e_1 + \gamma_2 e_2 + (m_3-1)e_3} \zeta(y) \eta_3(y) dy S \\
 &\quad + \int_G \frac{\partial}{\partial x_i} \Gamma_\lambda(x-y) D^\gamma \zeta(y) dy.
 \end{aligned}$$

Combining the preceding arguments with Theorem A.1 we arrive at

$$\left\| D^\gamma \frac{\partial}{\partial x_i} (\mathcal{N}_\lambda^- \zeta) \right\|_{C^0(G)} \leq K \|\zeta\|_{C^{k,\alpha}(G)}.$$

To complete the proof, we consider the derivatives of order $k + 2$. For $1 \leq j \leq 3$ we then have

$$\begin{aligned}
 D^\gamma \frac{\partial^2}{\partial x_i \partial x_j} (\mathcal{N}_\lambda^- \zeta)(x) &= - \sum_{m_1=1}^{\gamma_1} \int_S D_x^{\gamma-m_1 e_1 + e_j} \frac{\partial}{\partial x_i} \Gamma_\lambda(x-y) D^{(m_1-1)e_1} \zeta(y) \eta_1(y) dy S \\
 &\quad - \sum_{m_2=1}^{\gamma_2} \int_S D_x^{\gamma-\gamma_1 e_1 - \gamma_2 e_2 + e_j} \frac{\partial}{\partial x_i} \Gamma_\lambda(x-y) D^{\gamma_1 e_1 + (m_2-1)e_2} \zeta(y) \eta_2(y) dy S \\
 &\quad - \sum_{m_3=1}^{\alpha_3} \int_S D_x^{\gamma-\gamma_1 e_1 - \gamma_2 e_2 - m_3 e_3 + e_j} \frac{\partial}{\partial x_i} \Gamma_\lambda(x-y) D^{\gamma_1 e_1 + \gamma_2 e_2 + (m_3-1)e_3} \zeta(y) \eta_3(y) dy S \\
 &\quad + \int_G \frac{\partial^2}{\partial x_i \partial x_j} \Gamma_\lambda(x-y) D^\gamma \zeta(y) dy.
 \end{aligned}$$

Similar estimates for the boundary terms can be obtained in $C^{0,\alpha}(G)$ by virtue of Theorem A.1, while the last term requires an adaptation of the ideas in the harmonic case [33, Teorema 3.II]. We first split it into two parts and use again integration by parts in the second term

$$\begin{aligned}
 &\int_G \frac{\partial^2}{\partial x_i \partial x_j} \Gamma_\lambda(x-y) D^\gamma \zeta(y) dy \\
 &= \int_G \frac{\partial^2}{\partial x_i \partial x_j} \Gamma_\lambda(x-y) (D^\gamma \zeta(y) - D^\gamma \zeta(x)) dy \\
 &\quad + D^\gamma \zeta(x) \int_G \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \Gamma_\lambda(x-y) dy
 \end{aligned}$$

$$\begin{aligned}
 &= \int_G \frac{\partial^2}{\partial x_i \partial x_j} \Gamma_\lambda(x-y)(D^\nu \zeta(y) - D^\nu \zeta(x)) dy \\
 &\quad - D^\nu \zeta(x) \int_S \frac{\partial}{\partial x_i} \Gamma_\lambda(x-y) \eta_j(y) dy S \\
 &=: F(x) - H(x).
 \end{aligned}$$

The idea behind such decomposition is that Theorem A.1 yields

$$\|H\|_{C^{0,\alpha}(G)} \leq K \|\eta\|_{C^{0,\alpha}(G)} \|\zeta\|_{C^{k,\alpha}(S)}$$

and we can cancel an α power of the singularity in $F(x)$:

$$|F(x)| \leq [D^\nu \zeta]_{\alpha,G} \int_G \left| \frac{\partial^2}{\partial x_i \partial x_j} \Gamma_\lambda(x-y) \right| |x-y|^\alpha dy.$$

Bearing the estimates (69) and (70) in mind, along with the local integrability of $|z|^{\alpha-3}$ and the boundedness of G , we find the following C^0 estimate

$$\|F\|_{C^0(G)} \leq K \|\zeta\|_{C^{k,\alpha}(G)}.$$

Let us finally show the local α -Hölder property for F , i.e.,

$$|F(x^1) - F(x^2)| \leq C \|\zeta\|_{C^{k,\alpha}(G)} |x^1 - x^2|^\alpha,$$

for every $x^1, x^2 \in G$ such that $|x^1 - x^2| < \delta$ and some small $\delta > 0$ (the global one follows from the boundedness of F). To this end, consider a neighborhood U of x^1 with $B_{2d}(x^1) \subseteq U \subseteq B_{7d}(x^1)$ an taking Euclidean norms, we finally arrive at

$$\begin{aligned}
 |F(x^1) - F(x^2)| &\leq \int_{G \cap B_{7d}(x^1)} \left| \frac{\partial^2 \Gamma_\lambda(x^1-y)}{\partial x_i \partial x_j} \right| |(D^\nu \zeta(y) - D^\nu \zeta(x^1))| dy \\
 &\quad + \int_{G \cap B_{8d}(x^2)} \left| \frac{\partial^2 \Gamma_\lambda(x^2-y)}{\partial x_i \partial x_j} \right| |(D^\nu \zeta(y) - D^\nu \zeta(x^2))| dy \\
 &\quad + \int_{G \setminus B_{2d}(x^1)} \left| \frac{\partial^2 \Gamma_\lambda(x^1-y)}{\partial x_i \partial x_j} - \frac{\partial^2 \Gamma_\lambda(x^2-y)}{\partial x_i \partial x_j} \right| \\
 &\quad \times |D^\nu \zeta(y) - D^\nu \zeta(x^1)| dy \\
 &\quad + |D^\nu \zeta(x^1) - D^\nu \zeta(x^2)| \int_{G \setminus U} \left| \frac{\partial^2 \Gamma_\lambda(x^2-y)}{\partial x_i \partial x_j} \right| dy, \tag{71}
 \end{aligned}$$

where in the last three terms we have respectively used that $G \cap B_{7d}(x^1) \subseteq G \cap B_{8d}(x^2)$, $G \setminus B_{7d}(x^1) \subseteq G \setminus B_{2d}(x^1)$ and $G \setminus B_{7d}(x^1) \subseteq G \setminus U$ (Fig. 7).

The first and second terms in (71) can be bounded as desired by virtue of the α -Hölder property for $D^\nu \zeta$ and the fact that $D^2 \Gamma_\lambda(z) = O(|z|^{-3})$. For both cases, note that the underlying kernel $|z|^{-(3-\alpha)}$ is integrable “near the origin”. Regarding the third term in (71), the mean value theorem shows

$$\left| \frac{\partial^2 \Gamma_\lambda}{\partial z_i \partial z_j}(x^1-y) - \frac{\partial^2 \Gamma_\lambda}{\partial z_i \partial z_j}(x^2-y) \right| \leq C \frac{|x^1-x^2|}{|x^1-y|^4}, \quad \forall y \in G \setminus B_{2d}(x^1).$$

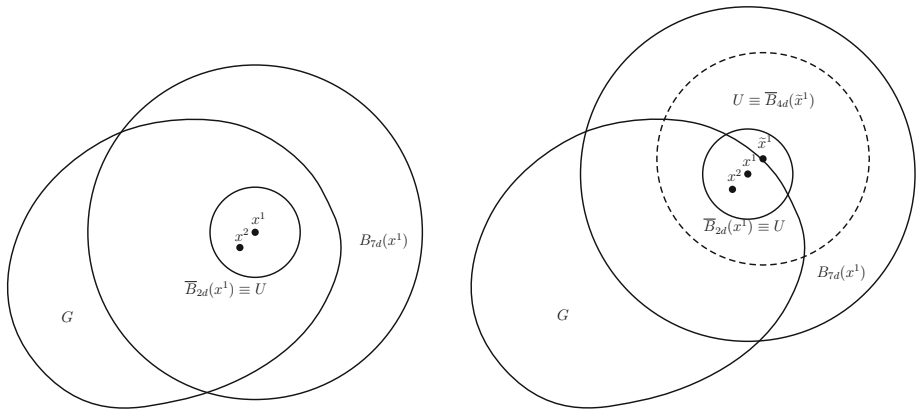


Fig. 7. Right: $U = \overline{B_{2d}(x^1)}$ in the first case. Left: $U = \overline{B_{4d}(\tilde{x}^1)}$ in the second case

In this case, the same ideas bring to light the underlying kernel $|z|^{-(4-\alpha)}$ that is “integrable at infinity” and gives rise to the desired estimate for the third term. Concerning the last term in (71), we are done as long as one notices that $D^\gamma \zeta \in C^{0,\alpha}(G)$ and shows

$$\int_{G \setminus U} \left| \frac{\partial^2 \Gamma_\lambda(x^2 - y)}{\partial x_i \partial x_j} \right| dy \leq C,$$

for some positive constant C depending on δ but not on $d = |x^1 - x^2|$. For that, assume first that $2d \leq \text{dist}(x^1, S)$ and define $U := \overline{B_{2d}(x^1)}$. In such case, $\partial(G \setminus U) = S \cup \partial B_{2d}(x^1)$. Then,

$$\begin{aligned} \int_{G \setminus U} \frac{\partial^2 \Gamma_\lambda(x^2 - y)}{\partial x_i \partial x_j} dy &= \int_S \frac{\partial \Gamma_\lambda(x^2 - y)}{\partial x_i} \eta_j(y) dy - \int_{\partial B_{2d}(x^1)} \frac{\partial \Gamma_\lambda(x^2 - y)}{\partial x_i} \frac{(y - x^1)_j}{|y - x^1|} dy. \end{aligned} \tag{72}$$

On the one hand, Theorem A.1 provides a bound for the first term of (72). On the other hand, note that a combination of the control of $\nabla \Gamma_\lambda(z)$ at infinity (see Eqs. (69)–(70)) along with the estimate

$$\sup_{y \in \partial B_{2d}(x^1)} \frac{1}{|x^2 - y|^2} \leq \frac{1}{|x^1 - x^2|^2},$$

entail the aforementioned bound for the second term in (72).

Secondly, let us consider the opposite case $2d > \text{dist}(x^1, S)$. Now the configuration is slightly different. Let us fix some $\tilde{x}^1 \in S$ so that $|x^1 - \tilde{x}^1| = \text{dist}(x^1, S)$ and define $U := \overline{B_{4d}(\tilde{x}^1)}$.

Since $x^2 \in B_{3d}(\tilde{x}^1)$ then, $B_{2d}(x^1) \subseteq U \subseteq B_{7d}(x^1)$. This time,

$$\int_{G \setminus U} \frac{\partial^2 \Gamma_\lambda(x^2 - y)}{\partial x_i \partial x_j} dy = \int_S \frac{\partial \Gamma_\lambda(x^2 - y)}{\partial x_i} \eta_j(y) dy - \int_{\partial(U \cap G)} \frac{\partial \Gamma_\lambda(x^2 - y)}{\partial x_i} \nu_j(y) dy. \tag{73}$$

The first term in (73) can be bounded through the same reasonings as above, thus we focus on the second term that will follow the idea in [33, Lemma 2.IV]. To this end, define some cut-off function $\xi \left(\frac{|y - \tilde{x}^1|}{d} \right)$ for $\xi \in C_c^\infty(\mathbb{R}_0^+)$ such that

$$\begin{cases} \xi(r) = 1, & r \in [0, \frac{7}{2}], \\ \xi(r) \in (0, 1), & r \in (\frac{7}{2}, 4), \\ \xi(r) = 0, & r \geq 4, \end{cases}$$

and consider the splitting

$$\begin{aligned} & \int_{\partial(G \cap B_{4d}(\tilde{x}^1))} \frac{\partial \Gamma_\lambda(x^2 - y)}{\partial x_i} v_j(y) d_y S \\ &= \int_{G \cap \partial B_{4d}(\tilde{x}^1)} \frac{\partial \Gamma_\lambda(x^2 - y)}{\partial x_i} v_j(y) d_y S \\ &+ \int_{S \cap B_{4d}(\tilde{x}^1)} \frac{\partial \Gamma_\lambda(x^2 - y)}{\partial x_i} \left[1 - \xi \left(\frac{|y - \tilde{x}^1|}{d} \right) \right] v_j(y) d_y S \\ &+ \int_{S \cap B_{4d}(\tilde{x}^1)} \frac{\partial \Gamma_\lambda(x^2 - y)}{\partial x_i} \xi \left(\frac{|y - \tilde{x}^1|}{d} \right) v_j(y) d_y S. \end{aligned} \tag{74}$$

Bear in mind again that $x^2 \in B_{3d}(\tilde{x}^1)$ and $\nabla \Gamma_\lambda(z) = O(|z|^{-2})$ when $|z| \rightarrow +\infty$. In the first term, note that $y \in G \cap \partial B_{4d}(\tilde{x}^1)$ and consequently, $|y - x^2| \geq d$, what shows the boundedness of such term. For the second term, we have that $y \in S \cap B_{4d}(\tilde{x}^1)$ but, in order that y belongs to the support of the cut-off function, one has to assume $|y - \tilde{x}^1| \geq \frac{7}{2}d$. Thus, $|y - x^2| \geq \frac{d}{2}$ and a similar reasoning now yields

$$\left| \int_{S \cap B_{4d}(\tilde{x}^1)} \frac{\partial \Gamma_\lambda(x^2 - y)}{\partial x_i} v_j(y) d_y S \right| \leq \frac{\tilde{C}}{|x^1 - x^2|^2} |S \cap B_{4d}(\tilde{x}^1)|.$$

The upper bound for the second term is done once we note that for regular surfaces $|S \cap B_{4d}(\tilde{x}^1)| \leq Cd^2$. To prove the corresponding bound for the third term in (74), we consider the potential

$$S(x) = \int_S \frac{\partial \Gamma_\lambda(x - y)}{\partial x_i} \xi \left(\frac{|y - \tilde{x}^1|}{d} \right) v_j(y) d_y S, \quad x \in G,$$

whose $C^{0,\alpha}$ estimate follows again from Lemma A.1:

$$\|S\|_{C^{0,\alpha}(\bar{G})} \leq C \left\| \xi \left(\frac{|\cdot - \tilde{x}^1|}{d} \right) v_j \right\|_{C^{0,\alpha}(S)} \leq C \left(1 + \frac{1}{d^\alpha} \right).$$

Let us now fix $0 < \delta < 1$ small enough so that $x - \theta \eta(x) \in G$ for every couple $x \in S$ and $0 < \theta < 4\delta$. Thus, $S(\tilde{x}^1 - 4d\eta(\tilde{x}^1)) = 0$ and consequently,

$$\begin{aligned} |S(x^2)| &= |S(x^2) - S(\tilde{x}^1 - 4d\eta(\tilde{x}^1))| \leq \frac{C}{d^\alpha} |x^2 - \tilde{x}^1 + 4d\eta(\tilde{x}^1)|^\alpha \\ &\leq \frac{C}{d^\alpha} (3d + 4d)^\alpha \leq \tilde{C}. \end{aligned}$$

□

Lemma A.3. *Let $G \subseteq \mathbb{R}^3$ be a bounded domain with regularity $C^{k+1,\alpha}$, $\Omega := \mathbb{R}^3 \setminus \overline{G}$ its exterior domain and $S = \partial G$ the boundary surface. The generalized volume potential on Ω associated with the Helmholtz equation and generated by a density ζ in G*

$$(\mathcal{N}_\lambda^+ \zeta)(x) = \int_G \Gamma_\lambda(x - y) \zeta(y) dy, \quad x \in \Omega,$$

defines a bounded linear map $\mathcal{N}_\lambda^+ : C^{k,\alpha}(\overline{G}) \longrightarrow C^{k+2,\alpha}(\overline{\Omega})$.

Proof. Our argument is based on some ideas of [33, Teorema 3.II]. Consider $R > 0$ such that $\overline{G} \subseteq B_R(0)$ and let us estimate $\|\mathcal{N}_\lambda^+ \zeta\|_{C^{k+2,\alpha}(\Omega)}$ in $\mathbb{R}^3 \setminus \overline{B}_R(0)$ and $\overline{\Omega}_{2R}$ where $\Omega_{2R} := B_{2R}(0) \setminus \overline{G}$. Set

$$d_R := \min\{|x - y| : x \in \mathbb{R}^3 \setminus B_R(0), y \in \overline{G}\} > 0,$$

and assume that $d_R < 1$.

Equations (67), (69) and (70) yield

$$|D_x^\gamma \Gamma_\lambda(x - y)| \leq \tilde{C} \frac{1}{d_R^{|\gamma|+1}},$$

for every multi-index γ and every $x \in \mathbb{R}^3 \setminus B_R(0)$ and $y \in G$. One can then take derivatives under the integral sign and obtain the desired estimate for the $C^{k+2,\alpha}$ norm in $\mathbb{R}^3 \setminus \overline{B}_R(0)$. On the other hand, consider $\bar{\zeta} \in C^{k,\alpha}(\mathbb{R}^3)$ through Proposition 3.1. Then,

$$(\mathcal{N}_\lambda^+ \zeta)(x) = \int_{B_{2R}(0)} \Gamma_\lambda(x - y) \bar{\zeta}(y) dy - \int_{\Omega_{2R}} \Gamma_\lambda(x - y) \bar{\zeta}(y) dy,$$

for every $x \in \Omega_{2R}$. Since $\Omega_{2R} \subseteq B_{2R}(0)$, the triangle inequality yields:

$$\begin{aligned} \|\mathcal{N}_\lambda^+ \zeta\|_{C^{k+2,\alpha}(\overline{\Omega}_{2R})} &\leq \left\| \int_{B_{2R}(0)} \Gamma_\lambda(\cdot - y) \bar{\zeta}(y) dy \right\|_{C^{k+2,\alpha}(\overline{B}_{2R}(0))} \\ &\quad + \left\| \int_{\Omega_{2R}} \Gamma_\lambda(\cdot - y) \bar{\zeta}(y) dy \right\|_{C^{k+2,\alpha}(\overline{\Omega}_{2R})}. \end{aligned}$$

Finally, note that both domains are bounded and, consequently, Lemma A.2 and Proposition 3.1 apply and yield the desired estimate

$$\|\mathcal{N}_\lambda^+ \zeta\|_{C^{k+2,\alpha}(\overline{\Omega}_{2R})} \leq M \|\zeta\|_{C^{k,\alpha}(B_{2R}(0))} \leq M C_{\mathcal{P}} \|\zeta\|_{C^{k,\alpha}(\overline{G})}.$$

□

Now, we focus on similar bounds for singular and weakly singular kernels in the whole space \mathbb{R}^N . This results are classical in the homogeneous harmonic case, $\Gamma_0(z)$, and can be found in [24, 32, 33]. However, not only will we need harmonic potentials, but we will also deal with general singular and weakly singular kernels. To this end, we remind [25, Satz 3.4, Satz 5.4].

Theorem A.4 (Weakly singular kernels). *Let us consider $0 \leq \beta \leq N - 1$, $0 < \alpha < 1$ and $K(x, z)$, $x \in \overline{D}$, $z \in \mathbb{R}^N \setminus \{0\}$ a weakly singular integral kernel of exponent β satisfying the following three hypothesis:*

(1) For each $x \in \overline{D}$

$$K(x, \cdot) \in C^1(\mathbb{R}^N \setminus \{0\}).$$

(2) For each $x \in \overline{D}$ and $z \in \mathbb{R}^N \setminus \{0\}$

$$|\nabla_z K(x, z)| \leq \frac{C}{|z|^{\beta+1}}.$$

(3) For all $x_1, x_2 \in \overline{D}$ and $z \in \mathbb{R}^N \setminus \{0\}$ one has

$$|K(x_1, z) - K(x_2, z)| \leq C \frac{|x_1 - x_2|^\alpha}{|z|^\beta}.$$

Then, the generalized volume potential generated by a density ζ in \mathbb{R}^N ,

$$(\mathcal{N}_K \zeta)(x) := \int_{\mathbb{R}^N} K(x, x - y) \zeta(y) dy, \quad x \in \overline{D},$$

defines a bounded linear map for each positive radius R

$$\mathcal{N}_K : C_c^{0,\alpha}(B_R(0)) \longrightarrow C^{0,\alpha}(\overline{D}).$$

Theorem A.5 (Singular kernels). Consider $0 < \alpha < 1$ and $K(x, z)$, $x \in \overline{D}$, $z \in \mathbb{R}^N \setminus \{0\}$ a kernel satisfying the following hypotheses:

- (1) $K(x, z)$ is positively homogeneous of degree $-(N - 1)$ with respect to the second variable, i.e., $K(x, \lambda z) = \lambda^{-(N-1)} K(x, z)$ for all $x \in \overline{D}$, $z \in \mathbb{R}^N \setminus \{0\}$ and $\lambda > 0$.
- (2) $K(x, z)$ has the following regularity properties for every $x \in \overline{D}$ and each indices $1 \leq i, j \leq N$:

$$\begin{aligned} K &\in C^1(\overline{D} \times (\mathbb{R}^N \setminus \{0\})), & K(x, \cdot) &\in C^2(\mathbb{R}^N \setminus \{0\}), \\ \frac{\partial K}{\partial x_i} &\in C(\overline{D} \times (\mathbb{R}^N \setminus \{0\})), & \frac{\partial K}{\partial x_i}(x, \cdot) &\in C^1(\mathbb{R}^N \setminus \{0\}), \\ \frac{\partial K}{\partial z_i} &\in C(\overline{D} \times (\mathbb{R}^N \setminus \{0\})), & & \\ \frac{\partial^2 K}{\partial z_i \partial x_j} &\in C(\overline{D} \times (\mathbb{R}^N \setminus \{0\})), & \frac{\partial^2 K}{\partial z_i \partial z_j} &\in C(\overline{D} \times (\mathbb{R}^N \setminus \{0\})). \end{aligned}$$

- (3) The first derivatives of $K(x, z)$ are Hölder-continuous with exponent α with respect to x in the sense that, for each $x_1, x_2 \in \overline{D}$, $z \in \mathbb{R}^N \setminus \{0\}$ and for all index $1 \leq i \leq N$,

$$\begin{aligned} \left| \frac{\partial K}{\partial x_i}(x_1, z) - \frac{\partial K}{\partial x_i}(x_2, z) \right| &\leq C \frac{|x_1 - x_2|^\alpha}{|z|^{N-1}}, \\ \left| \frac{\partial K}{\partial z_i}(x_1, z) - \frac{\partial K}{\partial z_i}(x_2, z) \right| &\leq C \frac{|x_1 - x_2|^\alpha}{|z|^N}. \end{aligned}$$

Then, the generalized volume potential defines a bounded linear map for every positive radius $R > 0$

$$\mathcal{N}_K : C_c^{0,\alpha}(B_R(0)) \longrightarrow C^{1,\alpha}(\overline{D}).$$

Moreover, for every $1 \leq i \leq N$,

$$\frac{\partial}{\partial x_i} (\mathcal{N}_K \zeta) = \mathcal{N}_{\frac{\partial K}{\partial x_i}} \zeta + \mathcal{N}_{\frac{\partial K}{\partial z_i}} \zeta.$$

Notice that the singular integral kernel $\frac{\partial K}{\partial z_i}$ has an associated singular integral operator $\mathcal{N}_{\frac{\partial K}{\partial z_i}}$, where the integrals require to be understood in the sense of Cauchy principal values by virtue of the cancellation properties arising from the homogeneity in z of the original kernel $K(x, z)$. Another interesting remark, that explains some differences between volume potentials in the whole \mathbb{R}^N and volume potentials in a bounded domain, is the change of variables formula

$$(\mathcal{N}_K \xi)(x) = \int_{\mathbb{R}^N} K(x, x - y)\xi(y) dy = \int_{\mathbb{R}^N} K(x, z)\xi(x - z) dz, \tag{75}$$

which lets us take derivatives in any of the two factors. When the kernel is not sufficiently well behaved, we can put the derivatives on the density, or the other way round. Obviously, it is no longer valid for densities on G , where the integration by part argument in the proof of Lemma A.2 is required, producing new boundary term that must be studied via Theorem A.1.

As a consequence, one can prove the next two corollaries, where higher order derivatives of these generalized volume potentials can be considered.

Corollary A.6. *Let us consider $0 \leq \beta \leq N - 1$, $0 < \alpha < 1$, $k, m \in \mathbb{N}$ so that $\beta + m \leq N - 1$ and $K(x, z)$, $x \in \overline{D}$, $z \in \mathbb{R}^N \setminus \{0\}$, a weakly singular integral kernel of exponent β verifying the next hypothesis for each couple of multi-indices γ_1, γ_2 with $|\gamma_1| \leq k$ and $|\gamma_2| \leq m$:*

(1) $D_x^{\gamma_1 + \gamma_2} K(x, z)$ is weakly singular with exponent β and $D_x^{\gamma_1} D_z^{\gamma_2} K(x, z)$ is the sum of weakly singular integral kernels with exponents ranging from β to $\beta + |\gamma_2|$, i.e.,

$$\left| D_x^{\gamma_1 + \gamma_2} K(x, z) \right| \leq \frac{C}{|z|^\beta}, \quad \left| D_x^{\gamma_1} D_z^{\gamma_2} K(x, z) \right| \leq C \left(\frac{1}{|z|^\beta} + \frac{1}{|z|^{\beta + |\gamma_2|}} \right).$$

(2) For every $x \in \overline{D}$,

$$(D_x^{\gamma_1 + \gamma_2} K)(x, \cdot), (D_x^{\gamma_1} D_z^{\gamma_2} K)(x, \cdot) \in C^1(\mathbb{R}^N \setminus \{0\}).$$

(3) For all $x \in \overline{D}$, $z \in \mathbb{R}^N \setminus \{0\}$,

$$\begin{aligned} \left| \nabla_z D_x^{\gamma_1 + \gamma_2} K(x, z) \right| &\leq C \left(\frac{1}{|z|^\beta} + \frac{1}{|z|^{\beta + 1}} \right), \quad \left| \nabla_z D_x^{\gamma_1} D_z^{\gamma_2} K(x, z) \right| \\ &\leq C \left(\frac{1}{|z|^\beta} + \frac{1}{|z|^{\beta + |\gamma_2| + 1}} \right). \end{aligned}$$

(4) For any $x_1, x_2 \in \overline{D}$, $z \in \mathbb{R}^N \setminus \{0\}$,

$$\begin{aligned} |D_x^{\gamma_1 + \gamma_2} K(x_1, z) - D_x^{\gamma_1 + \gamma_2} K(x_2, z)| &\leq \frac{C}{|z|^\beta} |x_1 - x_2|^\alpha, \\ |D_x^{\gamma_1} D_z^{\gamma_2} K(x_1, z) - D_x^{\gamma_1} D_z^{\gamma_2} K(x_2, z)| &\leq C \left(\frac{1}{|z|^\beta} + \frac{1}{|z|^{\beta + |\gamma_2|}} \right) |x_1 - x_2|^\alpha. \end{aligned}$$

Then, the generalized volume potential defines a bounded linear operator for every positive radius R

$$\mathcal{N}_K : C_c^{k, \alpha}(B_R(0)) \longrightarrow C^{k+m, \alpha}(\overline{D}).$$

Moreover, for every multi-index $\gamma = \gamma_1 + \gamma_2$ so that $|\gamma_1| \leq k$ and $|\gamma_2| \leq m$

$$D^\gamma (\mathcal{N}_K \zeta) = \sum_{\delta \leq \gamma_1} \binom{\gamma_1}{\delta} \left(\mathcal{N}_{D_x^{\delta+\gamma_2} K} D^{\gamma_1-\delta} \zeta + \mathcal{N}_{D_x^\delta D_z^{\gamma_2} K} D^{\gamma_1-\delta} \zeta \right).$$

Corollary A.7. Let $0 < \alpha < 1$, $k \in \mathbb{N}$, $x \in \overline{D}$, $z \in \mathbb{R}^N \setminus \{0\}$ and $K(x, z)$ be a weakly singular kernel, which has the following properties:

- (1) $K(x, z)$ is positively homogeneous of degree $-(N - 1)$ in the second variable.
- (2) $K(x, z)$ has the regularity properties

$$\begin{aligned} D_x^\gamma K &\in C^1(\overline{D} \times (\mathbb{R}^N \setminus \{0\})), & (D_x^\gamma K)(x, \cdot) &\in C^2(\mathbb{R}^N \setminus \{0\}), \\ \frac{\partial}{\partial x_i} D_x^\gamma K &\in C(\overline{D} \times (\mathbb{R}^N \setminus \{0\})), & \left(\frac{\partial}{\partial x_i} D_x^\gamma K \right)(x, \cdot) &\in C^1(\mathbb{R}^N \setminus \{0\}), \\ \frac{\partial}{\partial z_i} D_x^\gamma K &\in C(\overline{D} \times (\mathbb{R}^N \setminus \{0\})), \\ \frac{\partial^2}{\partial z_i \partial x_j} D_x^\gamma K &\in C(\overline{D} \times (\mathbb{R}^N \setminus \{0\})), & \frac{\partial^2}{\partial z_i \partial z_j} D_x^\gamma K &\in C(\overline{D} \times (\mathbb{R}^N \setminus \{0\})), \end{aligned}$$

for each couple of indices $1 \leq i, j \leq N$ and each multi-index γ with $|\gamma| \leq k$.

- (3) The derivatives of $K(x, z)$ with respect to x up to order k are α -Hölder continuous,

$$\begin{aligned} \left| \left(\frac{\partial}{\partial x_i} D_x^\gamma K \right)(x_1, z) - \left(\frac{\partial}{\partial x_i} D_x^\gamma K \right)(x_2, z) \right| &\leq C \frac{|x_1 - x_2|^\alpha}{|z|^{N-1}}, \\ \left| \left(\frac{\partial}{\partial z_i} D_x^\gamma K \right)(x_1, z) - \left(\frac{\partial}{\partial z_i} D_x^\gamma K \right)(x_2, z) \right| &\leq C \frac{|x_1 - x_2|^\alpha}{|z|^N}, \end{aligned}$$

for each $x_1, x_2 \in \overline{D}$, $z \in \mathbb{R}^N \setminus \{0\}$, each index $1 \leq i \leq N$ and $|\gamma| \leq k$.

Then, the generalized volume potential defines a bounded linear operator for every positive radius R

$$\mathcal{N}_K : C_c^{k,\alpha}(B_R(0)) \longrightarrow C^{k+1,\alpha}(\overline{D}).$$

Moreover, for every multi-index γ with $|\gamma| \leq k$ and any index $1 \leq i \leq N$:

$$\frac{\partial}{\partial x_i} D_x^\gamma (\mathcal{N}_K \zeta) = \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} \left(\mathcal{N}_{\frac{\partial}{\partial x_i} D_x^\gamma K} D^{\delta-\gamma} \zeta + \mathcal{N}_{\frac{\partial}{\partial z_i} D_x^\gamma K} D^{\delta-\gamma} \zeta \right).$$

When the constants C appearing in the statements of the above results do not depend on the chosen bounded domain D , the above estimates can be extended from Hölder estimates over \overline{D} , to global estimates in \mathbb{R}^N . This is the case for the integral kernels which do not depend on the variable x (e.g., $\Gamma_0(z)$, $R_\lambda(z)$ and $\Gamma_\lambda(z)$). In this way, we get the next result in the spirit of Lemmas A.2 and A.3.

Lemma A.8. The generalized volume potential in \mathbb{R}^3 associated with the Helmholtz equation

$$(\mathcal{N}_\lambda \zeta)(x) := \int_{\mathbb{R}^3} \Gamma_\lambda(x - y) \zeta(y) dy, \quad x \in \mathbb{R}^3,$$

defines a bounded linear operator for every positive radius R

$$\mathcal{N}_\lambda : C_c^{k,\alpha}(B_R(0)) \longrightarrow C^{k+2,\alpha}(\mathbb{R}^3).$$

Combining the above results, we can estimate generalized volume potentials in Ω whose densities have compact support in $\overline{\Omega}$ by means of an appropriate splitting. Using Calderón’s extension theorem (Proposition 3.1), for every $\zeta \in C_c^{k,\alpha}(\overline{\Omega})$ there exists an extension $\overline{\zeta} \in C_c^{k,\alpha}(\mathbb{R}^3)$, so

$$\mathcal{N}_\lambda^+ \zeta = (\mathcal{N}_\lambda \overline{\zeta})|_\Omega - \mathcal{N}_\lambda^+ (\overline{\zeta}|_G) \text{ in } \Omega.$$

Then, Lemmas A.3 and A.8 lead to the following result:

Theorem A.9 (Generalized volume potential). *Let $G \subseteq \mathbb{R}^3$ be a bounded domain with regularity $C^{k+1,\alpha}$, $\Omega := \mathbb{R}^3 \setminus \overline{G}$ its exterior domain and $S = \partial G$ the boundary surface. The generalized volume potential associated with the Helmholtz equation and generated by a density ζ in Ω ,*

$$(\mathcal{N}_\lambda^+ \zeta)(x) = \int_\Omega \Gamma_\lambda(x - y) \zeta(y) dy, \quad x \in \Omega,$$

defines a bounded linear operator for every positive radius R

$$\mathcal{N}_\lambda^+ : C_c^{k,\alpha}(B_R(0) \setminus G) \longrightarrow C^{k+2,\alpha}(\overline{\Omega}).$$

A.2. Regularity of the boundary integral operator T_λ . The next step is to analyze the regularity properties of the boundary integral operator T_λ (31) arising in the boundary integral equation (30) associated with the boundary data $\eta \times u$ in Theorem 2.15. Firstly, we split the operator T_λ into

$$T_\lambda = \mathcal{M}_\lambda^T + \lambda \mathcal{S}_\lambda^T.$$

$\mathcal{M}_\lambda^T \zeta$ is known as the *magnetic dipole operator*, which is the tangent component of the electric field generated by a dipole distribution with density $\zeta \in \mathfrak{X}(S)$, i.e.,

$$(\mathcal{M}_\lambda^T \zeta)(x) := \int_S \eta(x) \times \text{curl}_x (\Gamma_\lambda(x - y) \zeta(y)) dy, \quad x \in S.$$

\mathcal{S}_λ^T is the tangential component of the generalized single layer potential generated by ζ ,

$$(\mathcal{S}_\lambda^T \zeta)(x) = \int_S \Gamma_\lambda(x - y) \eta(x) \times \zeta(y) dy, \quad x \in S.$$

The integral kernel of \mathcal{S}_λ^T is weakly singular over S , so this integral is absolutely convergent under suitable hypotheses for ζ . The integral in \mathcal{M}_λ^T is absolutely convergent under minimal assumption on ζ . Indeed, although the integral kernel looks singular over S let us see this it is again weakly singular when ζ is a tangent vector field on S . Notice that, given any tangent field along S , $\zeta \in \mathfrak{X}^{k,\alpha}(S)$, one can split

$$\begin{aligned} \eta(x) \times (\nabla_x \Gamma_\lambda(x - y) \times \zeta(y)) &= (\eta(x) - \eta(y)) \cdot \zeta(y) \nabla_x \Gamma_\lambda(x - y) \\ &\quad - \eta(x) \cdot \nabla_x \Gamma_\lambda(x - y) \zeta(y). \end{aligned}$$

Consequently, the j -th coordinates of the integrands read

$$\begin{aligned}
 (\eta(x) \times (\nabla_x \Gamma_\lambda(x - y) \times \zeta(y)))_j &= \sum_{i=1}^3 (\eta_i(x) - \eta_i(y)) \zeta_i(y) \partial_{x_j} \Gamma_\lambda(x - y) \\
 &\quad - \eta(x) \cdot \nabla_x \Gamma_\lambda(x - y) \zeta_j(y), \\
 (\Gamma_\lambda(x - y) \eta(x) \times \zeta(y))_j &= \sum_{i=1}^3 \Gamma_\lambda(x - y) (e_i \times e_j) \cdot \eta(x) \zeta_i(y).
 \end{aligned}$$

Consider any extension $\tilde{\eta} \in C_c^{k+4, \alpha}(\mathbb{R}^3)$ of the outward unit normal vector field η to the compact surface S and define the kernels

$$\begin{aligned}
 K_\lambda^{\mathcal{D}}(x, z) &= \tilde{\eta}(x) \cdot \nabla \Gamma_\lambda(z), \\
 K_\lambda^{i,j}(x, z) &= (\tilde{\eta}_i(x) - \tilde{\eta}_i(x - z)) \partial_{z_j} \Gamma_\lambda(z), \quad \tilde{K}_\lambda^{i,j}(x, z) = (e_i \times e_j) \cdot \tilde{\eta}(x) \Gamma_\lambda(z).
 \end{aligned} \tag{76}$$

Then, we have the associated splitting of the operators \mathcal{M}_λ^T and \mathcal{S}_λ^T

$$(\mathcal{M}_\lambda^T \zeta)_j(x) = \sum_{i=1}^3 T_{K_\lambda^{i,j}} \zeta_i - T_{K_\lambda^{\mathcal{D}}} \zeta_j, \quad (\mathcal{S}_\lambda^T \zeta)_j(x) = \sum_{i=1}^3 T_{\tilde{K}_\lambda^{i,j}} \zeta_i, \tag{77}$$

where the integral operators in the above decomposition are

$$\begin{aligned}
 (T_{K_\lambda^{\mathcal{D}}} \zeta_j)(x) &= \int_S K_\lambda^{\mathcal{D}}(x, x - y) \zeta_j(y) d_y S, \\
 (T_{K_\lambda^{i,j}} \zeta_i)(x) &= \int_S K_\lambda^{i,j}(x, x - y) \zeta_i(y) d_y S, \\
 (T_{\tilde{K}_\lambda^{i,j}} \zeta_i)(x) &= \int_S \tilde{K}_\lambda^{i,j}(x, x - y) \zeta_i(y) d_y S.
 \end{aligned} \tag{78}$$

Since every C^2 compact surface satisfies

$$|\eta(x) \cdot (x - y)| \leq L|x - y|^2, \quad |\eta(x) - \eta(y)| \leq L|x - y|,$$

for each $x, y \in S$, then all the preceding integral kernels are weakly singular. In particular, it prevents these integrals from being considered in the Cauchy principal value sense.

The study of Hölder estimates for all these potentials can be performed along the same lines as in [25, Satz 4.3, Satz 4.4]. In that work, the author dealt with the homogeneous harmonic case $\lambda = 0$, where the kernels have a simpler form. In our case $\lambda \neq 0$, we will decompose the 3-dimensional kernels into a homogeneous part and an inhomogeneous but less singular part as in (67). Then, we will consider a coordinate system over S which allows transforming the integrals over S into integrals over planar domains by means of a change of variables. The homogeneous and more singular parts will satisfy the hypothesis in Corollary A.7 and the terms in the remainder will verify those in Corollary A.6. We will need C^{k+5} boundaries for the operators in (78) of first and second type to be bounded from $C^{k, \alpha}(S)$ to $C^{k+1, \alpha}(S)$ whilst assuming C^{k+4} boundaries suffices to ensure the corresponding result for the third kind of operators in (78) (see [25, Satz 4.3, Satz 4.4] for the homogeneous harmonic case $\lambda = 0$). Our regularity result then reads as follows:

Theorem A.10. *Let G be a bounded domain of class C^{k+5} , $S = \partial G$ the boundary surface, $\eta \in C^{k+4}(S, \mathbb{R}^3)$ the outward unit normal vector field along S and any extension $\tilde{\eta} \in C_c^{k+4}(\mathbb{R}^3, \mathbb{R}^3)$ of η . Let $K_\lambda^{\mathcal{D}}(x, z)$, $K_\lambda^{i,j}(x, z)$ and $\tilde{K}_\lambda^{i,j}(x, z)$ be the kernels given by (76) Then, the associated boundary operators $T_{K_\lambda^{\mathcal{D}}}$, $T_{K_\lambda^{i,j}}$ and $T_{\tilde{K}_\lambda^{i,j}}$ given by (78) are bounded*

$$T_{K_\lambda^{\mathcal{D}}} : C^{k,\alpha}(S) \longrightarrow C^{k+1,\alpha}(S),$$

$$T_{K_\lambda^{i,j}} : C^{k,\alpha}(S) \longrightarrow C^{k+1,\alpha}(S), \quad T_{\tilde{K}_\lambda^{i,j}} : C^{k,\alpha}(S) \longrightarrow C^{k+1,\alpha}(S).$$

As a consequence, the next linear operators are also bounded

$$\mathcal{M}_\lambda^T : \mathfrak{X}^{k,\alpha}(S) \longrightarrow \mathfrak{X}^{k+1,\alpha}(S), \quad \mathcal{S}_\lambda^T : \mathfrak{X}^{k,\alpha}(S) \longrightarrow \mathfrak{X}^{k+1,\alpha}(S).$$

Proof. Since the kernel $\tilde{K}_\lambda^{i,j}(x, z)$ can be analyzed through a similar reasoning (as shown in [25] for the case $\lambda = 0$), we will restrict our analysis to the kernels $K_\lambda^{i,j}(x, z)$ and $K_\lambda^{\mathcal{D}}(x, z)$, which were not explicitly studied in [25]. Let us then split these inhomogeneous kernels into a homogeneous part and some less singular part [see the decomposition (67) and the functions ϕ_λ and ψ_λ in (66)]. To this end, notice that

$$K_\lambda^{\mathcal{D}}(x, z) = \frac{\phi'_\lambda(|z|)}{|z|} \tilde{\eta}(x) \cdot z, \quad K_\lambda^{i,j}(x, z) = (\tilde{\eta}_i(x) - \tilde{\eta}_i(x - z)) \frac{\phi'_\lambda(|z|)}{|z|} z_j, \quad (79)$$

Consequently, one can decompose,

$$K_\lambda^{i,j}(x, z) = K_{\lambda,0}^{i,j} + K_{\lambda,1}^{i,j}, \quad K_\lambda^{\mathcal{D}}(x, z) = K_{\lambda,0}^{\mathcal{D}} + K_{\lambda,1}^{\mathcal{D}}, \quad (80)$$

where,

$$K_{\lambda,0}^{i,j}(x, z) := -\frac{1}{4\pi} (\tilde{\eta}_i(x) - \tilde{\eta}_i(x - z)) \frac{z_j}{|z|^3}, \quad K_{\lambda,1}^{i,j}(x, z) := (\tilde{\eta}_i(x) - \tilde{\eta}_i(x - z)) \frac{\psi'_\lambda(|z|)}{|z|} z_j,$$

$$K_{\lambda,0}^{\mathcal{D}}(x, z) := -\frac{1}{4\pi} \tilde{\eta}(x) \cdot \frac{z}{|z|^3}, \quad K_{\lambda,1}^{\mathcal{D}}(x, z) := \tilde{\eta}(x) \cdot \frac{\psi'_\lambda(|z|)z}{|z|}. \quad (81)$$

Notice that the associated integral operators only involve values $x, y \in S$. Define $d_S := 2 \max_{x,y \in S} |x - y|$ and take $x \in S, z \in B_{d_S}(0)$. Thus, an easy computation that will be steadily used along the proof is

$$|z|^{-\beta_1} + |z|^{-\beta_2} \leq \left(1 + d_S^{M-m}\right) |z|^{-M}, \quad |z|^{\beta_1} + |z|^{\beta_2} \leq \left(1 + d_S^{M-m}\right) |z|^m, \quad (82)$$

for any couple of exponents $\beta_1, \beta_2 \geq 0$ and any $z \in B_{d_S}(0)$. Here m and M stand for the minimum and maximum values i.e., $m := \min\{\beta_1, \beta_2\}$, $M := \max\{\beta_1, \beta_2\}$.

Another useful remark is that $f_\lambda(r) := \psi'_\lambda(r)/r$, arising in (81), can be controlled by (68) as follows

$$|f_\lambda^{(m)}(r)| \leq C \left(\frac{1}{r} + \frac{1}{r^{m+2}} \right), \quad r > 0,$$

$$|f_\lambda^{(m)}(r)| \leq \tilde{C} \frac{1}{r^{m+2}}, \quad r \in (0, d_S). \quad (83)$$

for some $C > 0$ that does not depend on m and some \tilde{C} depending on m and d_S .

Let us study the boundedness of the integral operators associated with the integral kernels $K_{\lambda,n}^{i,j}$ and $K_{\lambda,n}^{\mathcal{D}}$ for $n = 0, 1$. To this end, let us consider a finite covering of S by M coordinate neighborhoods $\Sigma_1, \dots, \Sigma_M \subseteq S$ endowed with the associated local charts $\mu_m \in C^{k+5}(\overline{D}_m, \Sigma_m)$ that enjoy homeomorphic extensions up to the boundary of the planar disks $D_m \subseteq \mathbb{R}^2$. Also consider the associated partition of unity of class C^{k+5} , $\{\varphi_m\}_{m=1}^M \subseteq C^{k+5}(S)$, subordinated to the above open covering. The Jacobian of each chart will be denoted by

$$J_m(s) := \left| \left(\frac{\partial \mu_m}{\partial s_1} \times \frac{\partial \mu_m}{\partial s_2} \right) (s) \right|, \quad s \in D_m.$$

All the above notation then yields the decompositions

$$\begin{aligned} (T_{K_{\lambda,n}^{i,j}} \zeta)(\mu_m(s)) &= \sum_{m'=1}^M \int_{D_{m'}} K_{\lambda,n}^{i,j}(\mu_m(s), \mu_m(s) - \mu_{m'}(t)) \\ &\quad \times \varphi_{m'}(\mu_{m'}(t)) \zeta(\mu_{m'}(t)) J_{m'}(t) dt, \end{aligned} \tag{84}$$

$$\begin{aligned} (T_{K_{\lambda,n}^{\mathcal{D}}} \zeta)(\mu_m(s)) &= \sum_{m'=1}^M \int_{D_{m'}} K_{\lambda,n}^{\mathcal{D}}(\mu_m(s), \mu_m(s) - \mu_{m'}(t)) \\ &\quad \times \varphi_{m'}(\mu_{m'}(t)) \zeta(\mu_{m'}(t)) J_{m'}(t) dt. \end{aligned} \tag{85}$$

We will study the most singular case $m' = m$ and then show how the case $m' \neq m$ follows from it. An important fact is that we will extract the most singular homogeneous parts of $K_{\lambda,0}^{i,j}(x, z)$ and $K_{\lambda,0}^{\mathcal{D}}(x, z)$ by virtue of the splitting (80). However, the change of variables in the coordinate neighborhoods Σ_m gives rise to new inhomogeneous planar kernels, $K_{\lambda,0}^{i,j}(\mu_m(s), \mu_m(s) - \mu_m(t))$ and $K_{\lambda,0}^{\mathcal{D}}(\mu_m(s), \mu_m(s) - \mu_m(t))$. To solve this difficulty, we will decompose them again into the more singular homogeneous part, which stands for a planar homogeneous kernel of degree -1 , and some inhomogeneous but less singular term. Then, we will prove the corresponding regularity results for each term through Corollaries A.6 and A.7.

Since both $K_{\lambda,0}^{i,j}(x, z)$ and $K_{\lambda,0}^{\mathcal{D}}(x, z)$ can be studied by means of a similar reasoning, we will just analyze one of them, e.g. $K_{\lambda,0}^{i,j}(x, z)$. In fact, $K_{\lambda,0}^{\mathcal{D}}(x, z)$ stands for the integral kernel of the adjoint operator of the harmonic Neumann–Poincaré operator, that was studied in [25, Satz 4.4]. Inspired by [25, Lemma 4.2], let us expand $\mu_m(s) - \mu_m(t)$ though Taylor’s theorem up to second order; that is,

$$|\mu_m(s) - \mu_m(t)| = (P_m(s, s - t) + Q_m(s, s - t))^{1/2}, \tag{86}$$

where,

$$P_m(s, u) := \sum_{p,q=1}^2 \frac{\partial \mu_m}{\partial s_p}(s) \cdot \frac{\partial \mu_m}{\partial s_q}(s) u_p u_q = \sum_{p,q=1}^2 g_m^{pq}(s) u_p u_q = ((g_m^{pq}(s))u) \cdot u, \tag{87}$$

$$\begin{aligned}
 Q_m(s, u) := & -2 \sum_{p,q,r=1}^2 \frac{\partial \mu_m}{\partial s_p}(s) \cdot \left(\int_0^1 (1-\theta) \frac{\partial^2 \mu_m}{\partial s_q \partial s_r}(s-\theta u) d\theta \right) u_p u_q u_r \\
 & + \sum_{p,q,r,l=1}^2 \left(\int_0^1 (1-\theta) \frac{\partial^2 \mu_m}{\partial s_p \partial s_q}(s-\theta u) d\theta \right) \\
 & \times \left(\int_0^1 (1-\theta) \frac{\partial^2 \mu_m}{\partial s_r \partial s_l}(s-\theta u) d\theta \right) u_p u_q u_r u_l.
 \end{aligned} \tag{88}$$

First, $P_m(s, u)$ is positively homogeneous on u of degree 2 with respect to u and (see [25, Satz 4.2])

$$\left. \begin{aligned}
 \frac{1}{C} |u|^2 \leq & \left. \begin{aligned} |P_m(s, u)| & \leq C|u|^2, \\ |Q_m(s, u)| & \leq C|u|^3, \end{aligned} \right\} \\
 \frac{1}{C} |u|^2 \leq & |P_m(s, u) + Q_m(s, u)| \leq C|u|^2.
 \end{aligned} \right\} \left. \begin{aligned}
 \left| D_s^\gamma P_m(s, u) \right| & \leq C|u|^2, \\
 \left| D_s^\gamma Q_m(s, u) \right| & \leq C|u|^3, \\
 \left| D_s^\gamma (P(s, u) + Q(s, u)) \right| & \leq C|u|^2, \\
 \left| \frac{\partial}{\partial u_i} D_s^\gamma P_m(s, u) \right| & \leq C|u|, \\
 \left| \frac{\partial}{\partial u_i} D_s^\gamma Q_m(s, u) \right| & \leq C|u|^2, \\
 \left| \frac{\partial}{\partial u_i} D_s^\gamma (P(s, u) + Q(s, u)) \right| & \leq C|u|, \\
 \left| \frac{\partial^2}{\partial u_i \partial u_j} D_s^\gamma P_m(s, u) \right| & \leq C|u|^0, \\
 \left| \frac{\partial^2}{\partial u_i \partial u_j} D_s^\gamma Q_m(s, u) \right| & \leq C|u|, \\
 \left| \frac{\partial^2}{\partial u_i \partial u_j} D_s^\gamma (P(s, u) + Q(s, u)) \right| & \leq C|u|^0.
 \end{aligned} \right\} \tag{89}$$

hold for each $s \in D_m, u \in \mathbb{R}^2$ such that $s - u \in D_m$ and every multi-index with $|\gamma| \leq k$. Our homogenization procedure follows from the next splitting

$$K_{\lambda,0}^{i,j}(\mu_m(s), \mu_m(s) - \mu_m(t)) = H_{\lambda,0}^{i,j}(s, s - t) + R_{\lambda,0}^{i,j}(s, s - t),$$

where the homogeneous part $H_{\lambda,0}^{i,j}(s, u)$ and the remainder $R_{\lambda,0}^{i,j}(s, u)$ take the form

$$\begin{aligned}
 H_{\lambda,0}^{i,j}(s, u) & := -\frac{1}{4\pi} P_m(s, u)^{-3/2} \sum_{p,q=1}^2 \frac{\partial(\tilde{\eta} \circ \mu_m)_i}{\partial s_p}(s) \frac{\partial(\mu_m)_j}{\partial s_q}(s) u_p u_q, \\
 R_{\lambda,0}^{i,j}(s, u) & := \tilde{R}_{\lambda,0}^{i,j}(s, u) + \hat{R}_{\lambda,0}^{i,j}(s, u).
 \end{aligned}$$

Above, the remainder is split into

$$\begin{aligned}
 \tilde{R}_{\lambda,0}^{i,j}(s, u) & := -\frac{1}{4\pi} \left((P_m(s, u) + Q_m(s, u))^{-3/2} - P_m(s, u)^{-3/2} \right) \\
 & \times \left(\sum_{p,q=1}^2 \frac{\partial(\tilde{\eta} \circ \mu_m)_i}{\partial s_p}(s) \frac{\partial(\mu_m)_j}{\partial s_q}(s) u_p u_q \right),
 \end{aligned}$$

$$\begin{aligned} \widehat{R}_{\lambda,0}^{i,j}(s, u) &:= -\frac{1}{4\pi} (P_m(s, u) + Q_m(s, u))^{-3/2} \\ &\times \left\{ -\sum_{p,q,r=1}^2 \left(\int_0^1 (1-\theta) \frac{\partial^2(\widetilde{\eta} \circ \mu_m)_i}{\partial s_p \partial s_q}(s-\theta u) d\theta \right) \frac{\partial(\mu_m)_j}{\partial s_r}(s) u_p u_q u_r \right. \\ &\quad - \sum_{p,q,r=1}^2 \frac{\partial(\widetilde{\eta} \circ \mu_m)_i}{\partial s_p}(s) \left(\int_0^1 (1-\theta) \frac{\partial(\mu_m)_j}{\partial s_q \partial s_r}(s-\theta u) d\theta \right) u_p u_q u_r \\ &\quad + \sum_{p,q,r,l=1}^2 \left(\int_0^1 (1-\theta) \frac{\partial^2(\widetilde{\eta} \circ \mu_m)_i}{\partial s_p \partial s_q}(s-\theta u) d\theta \right) \\ &\quad \left. \times \left(\int_0^1 (1-\theta) \frac{\partial^2(\mu_m)_j}{\partial s_r \partial s_l}(s-\theta u) d\theta \right) u_p u_q u_r u_l \right\}. \end{aligned}$$

Note again that small values of $u = s - t$ are involved here, thus leading to estimates like (82) for u .

Let us next analyze each term in the above decomposition for $K_{\lambda,0}^{i,j}(\mu_m(s), \mu_m(s) - \mu_m(t))$. Firstly, since $P_m(s, u)$ is positively homogeneous on u with degree 2, then $H_{\lambda,0}^{i,j}(s, u)$ is positively homogeneous on u with degree -1 . The regularity properties in the second part in Corollary A.7 can be straightforwardly checked. Let us then concentrate on the regularity properties in the third part of such corollary and, to this end, let us compute the next partial derivative

$$\begin{aligned} D_s^\gamma H_{\lambda,0}^{i,j}(s, u) &= -\frac{1}{4\pi} \sum_{\sigma \leq \gamma} \binom{\gamma}{\sigma} D_s^\sigma \left(P_m(s, u)^{-3/2} \right) \\ &\times \left[\sum_{p,q=1}^2 D_s^{\gamma-\sigma} \left(\frac{\partial(\widetilde{\eta} \circ \mu_m)_i}{\partial s_p}(s) \frac{\partial(\mu_m)_j}{\partial s_q}(s) \right) u_p u_q \right]. \end{aligned}$$

Define the homogeneous function $h(t) := t^{-3/2}$ and use the chain rule to arrive at

$$D_s^\sigma \left(P_m(s, u)^{-3/2} \right) = \sum_{(l,\beta,\delta) \in \mathcal{D}(\sigma)} (D^\delta h)(P_m(s, u)) \prod_{r=1}^l \frac{1}{\delta_r!} \left(\frac{1}{\beta_r!} D_s^{\beta_r} P_m(s, u) \right)^{\delta_r}.$$

(See the proof of Theorem 3.5 for the definition of $\mathcal{D}(\sigma)$). By virtue of (89),

$$\left| D_s^\gamma H_{\lambda,0}^{i,j}(s, u) \right| \leq C |u|^{-1}.$$

Let us take derivatives with respect to u and arrive at

$$\begin{aligned} \nabla_u D_s^\gamma H_{\lambda,0}^{i,j}(s, u) &= -\frac{1}{4\pi} \sum_{\sigma \leq \gamma} \binom{\gamma}{\sigma} \nabla_u D_s^\sigma \left(P_m(s, u)^{-3/2} \right) \\ &\times \left[\sum_{p,q=1}^2 D_s^{\sigma-\gamma} \left(\frac{\partial(\widetilde{\eta} \circ \mu_m)_i}{\partial s_p}(s) \frac{\partial(\mu_m)_j}{\partial s_q}(s) \right) u_p u_q \right] \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{4\pi} \sum_{\sigma \leq \gamma} \binom{\gamma}{\sigma} D_s^\sigma \left(P_m(s, u)^{-3/2} \right) \\
 & \times \left[\sum_{p,q=1}^2 D_s^{\sigma-\gamma} \left(\frac{\partial(\tilde{\eta} \circ \mu_m)_i}{\partial s_p}(s) \frac{\partial(\mu_m)_j}{\partial s_q}(s) \right) \nabla_u(u_p u_q) \right],
 \end{aligned}$$

that can be similarly estimated by means of (89):

$$\left| D_s^\gamma \nabla_u H_{\lambda,0}^{i,j}(s, u) \right| \leq C |u|^{-2}.$$

Thus, $H_{\lambda,0}^{i,j}$ has the regularity properties required in Corollary A.7, so

$$\left\| \int_{D_m} H_{\lambda,0}^{i,j}(\cdot, \cdot - t) \varphi_m(\mu_m(t)) \zeta(\mu_m(t)) J_m(t) dt \right\|_{C^{k+1,\alpha}(D_m)} \leq M \|\zeta\|_{C^{k,\alpha}(\Sigma_m)}.$$

Let us now move to the remainder $R_{\lambda,0}^{i,j}(s, u)$ and show that the hypothesis in Corollary A.6 are satisfied too. On the one hand, in the first term $\tilde{R}_{\lambda,0}^{i,j}(s, u)$ in $R_{\lambda,0}^{i,j}(s, u)$ one can arrange terms by Barrow’s rule as

$$\begin{aligned}
 & (P_m(s, u) + Q_m(s, u))^{-3/2} - P_m(s, u)^{-3/2} \\
 & = -\frac{3}{2} Q_m(s, u) \int_0^1 (P_m(s, u) + \theta Q_m(s, u))^{-5/2} d\theta.
 \end{aligned}$$

Therefore, a D_s^γ derivative of $\tilde{R}_{\lambda,0}^{i,j}(s, u)$ takes the form

$$\begin{aligned}
 D_s^\gamma \tilde{R}_{\lambda,0}^{i,j}(s, u) & = \frac{1}{4\pi} \frac{3}{2} \sum_{\sigma \leq \gamma} \binom{\gamma}{\sigma} D_s^\sigma \left(Q_m(s, u) \int_0^1 (P_m(s, u) + \theta Q_m(s, u))^{-5/2} d\theta \right) \\
 & \times \sum_{p,q=1}^2 D_s^{\gamma-\sigma} \left(\frac{\partial(\tilde{\eta} \circ \mu_m)_i}{\partial s_p}(s) \frac{\partial(\mu_m)_j}{\partial s_q}(s) \right) u_p u_q.
 \end{aligned}$$

Define the homogeneous function $\tilde{h}(t) = t^{-5/2}$, a similar argument shows that

$$\begin{aligned}
 & D_s^\sigma \left(Q_m(s, u) \int_0^1 (P_m(s, u) + \theta Q_m(s, u))^{-5/2} d\theta \right) \\
 & = \sum_{\rho \leq \sigma} \binom{\sigma}{\rho} D_s^\rho(Q_m(s, u)) \int_0^1 D_s^{\sigma-\rho} \left((P_m(s, u) + \theta Q_m(s, u))^{-5/2} \right) d\theta \\
 & = \sum_{\rho \leq \sigma} \binom{\sigma}{\rho} D_s^\rho(Q_m(s, u)) \int_0^1 \sum_{(l,\beta,\delta) \in \mathcal{D}(\sigma-\rho)} (D^\delta \tilde{h})(P_m(s, u) + \theta Q_m(s, u)) \\
 & \times \prod_{r=1}^l \frac{1}{\delta_r!} \left(\frac{1}{\beta_r!} D_s^{\beta_r}(P_m(s, u) + \theta Q_m(s, u)) \right)^{\delta_r} d\theta.
 \end{aligned}$$

Now, the estimates in (89) yields

$$\begin{aligned} \left| D_s^\gamma \tilde{R}_{\lambda,0}^{i,j}(s,u) \right| &\leq C|u|^0, \\ \left| \partial_{u_l} D_s^\gamma \tilde{R}_{\lambda,0}^{i,j}(s,u) \right| &\leq C|u|^{-1}, \quad \left| \partial_{u_1 u_2} D_s^\gamma \tilde{R}_{\lambda,0}^{i,j}(s,u) \right| \leq C|u|^{-2}. \end{aligned}$$

These estimates ensure that all the hypotheses in Corollary A.6 are satisfied, so

$$\left\| \int_{D_m} \tilde{R}_{\lambda,0}^{i,j}(\cdot, \cdot - t) \varphi_m(\mu_m(t)) \zeta(\mu_m(t)) J_m(t) dt \right\|_{C^{k+1,\alpha}(D_m)} \leq M \|\zeta\|_{C^{k,\alpha}(\Sigma_m)}.$$

Regarding the second term $\hat{R}_{\lambda,0}^{i,j}(s,u)$ of $R_{\lambda,0}^{i,j}(s,u)$ we can use a similar argument. First,

$$\begin{aligned} D_s^\gamma \hat{R}_{\lambda,0}^{i,j}(s,u) &= \frac{1}{4\pi} \sum_{\sigma \leq \gamma} \binom{\gamma}{\sigma} D_s^\sigma \left((P_m(s,u) + Q_m(s,u))^{-3/2} \right) \\ &\times \left\{ \sum_{p,q,r=1}^2 D_s^{\gamma-\sigma} \left(\left(\int_0^1 (1-\theta) \frac{\partial^2(\tilde{\eta} \circ \mu_m)_i}{\partial s_p \partial s_q}(s-\theta u) d\theta \right) \frac{\partial(\mu_m)_j}{\partial s_r}(s) \right) u_p u_q u_r \right. \\ &+ \sum_{p,q,r=1}^2 D_s^{\gamma-\sigma} \left(\frac{\partial(\tilde{\eta} \circ \mu_m)_i}{\partial s_p}(s) \left(\int_0^1 (1-\theta) \frac{\partial(\mu_m)_j}{\partial s_q \partial s_r}(s-\theta u) d\theta \right) \right) u_p u_q u_r \\ &- \sum_{p,q,r,l=1}^2 D_s^{\gamma-\sigma} \left(\left(\int_0^1 (1-\theta) \frac{\partial^2(\tilde{\eta} \circ \mu_m)_i}{\partial s_p \partial s_q}(s-\theta u) d\theta \right) \right. \\ &\left. \times \left(\int_0^1 (1-\theta) \frac{\partial^2(\mu_m)_j}{\partial s_r \partial s_l}(s-\theta u) d\theta \right) \right) u_p u_q u_r u_l \left. \right\}, \end{aligned}$$

and the higher-order chain formula leads again to

$$\begin{aligned} &D_s^\sigma \left((P_m(s,u) + Q_m(s,u))^{-3/2} \right) \\ &= \sum_{(l,\beta,\delta) \in \mathcal{D}(\sigma)} (D^\delta h)(P_m(s,u) + Q_m(s,u)) \\ &\times \prod_{r=1}^l \frac{1}{\delta_r!} \left(\frac{1}{\beta_r!} D_s^{\beta_r} (P_m(s,u) + Q_m(s,u)) \right)^{\delta_r}. \end{aligned}$$

Consequently, the estimates in (89) show that

$$\begin{aligned} \left| D_s^\gamma \hat{R}_{\lambda,0}^{i,j}(s,u) \right| &\leq C|u|^0, \\ \left| \partial_{u_l} D_s^\gamma \hat{R}_{\lambda,0}^{i,j}(s,u) \right| &\leq C|u|^{-1}, \quad \left| \partial_{u_1 u_2} D_s^\gamma \hat{R}_{\lambda,0}^{i,j}(s,u) \right| \leq C|u|^{-2}, \end{aligned}$$

and Corollary A.6 yields

$$\left\| \int_{D_m} \hat{R}_{\lambda,0}^{i,j}(\cdot, \cdot - t) \varphi_m(\mu_m(t)) \zeta(\mu_m(t)) J_m(t) dt \right\|_{C^{k+1,\alpha}(D_m)} \leq M \|\zeta\|_{C^{k,\alpha}(\Sigma_m)}$$

Now we move to $K_{\lambda,1}^{i,j}(x, z)$ that can be expanded as

$$\begin{aligned} &K_{\lambda,1}^{i,j}(\mu_m(s), \mu_m(s) - \mu_m(s - u)) \\ &= f_\lambda(|P_m(s, u) + Q_m(s, u)|^{1/2}) \sum_{p,q=1}^2 \left(\int_0^1 \frac{\partial(\tilde{\eta}_i \circ \mu_m)}{\partial s_q}(s - \theta u) d\theta \right) \\ &\quad \times \left(\int_0^1 \frac{\partial(\mu_m)_j}{\partial s_q}(s - \theta u) d\theta \right) u_p u_q. \end{aligned}$$

Then, the D_s^γ derivative of $K_{\lambda,1}^{i,j}(\mu_m(s), \mu_m(s) - \mu_m(t))$ takes the form

$$\begin{aligned} &D_s^\gamma K_{\lambda,1}^{i,j}(\mu_m(s), \mu_m(s) - \mu_m(s - u)) \\ &= \sum_{\sigma \leq \gamma} \binom{\gamma}{\sigma} D_s^\sigma \left(f_\lambda(|P_m(s, u) + Q_m(s, u)|^{1/2}) \right) \\ &\quad \times \left[\sum_{p,q=1}^2 D_s^{\gamma-\sigma} \left(\left(\int_0^1 \frac{\partial(\tilde{\eta}_i \circ \mu_m)}{\partial s_q}(s - \theta u) d\theta \right) \right. \right. \\ &\quad \left. \left. \times \left(\int_0^1 \frac{\partial(\mu_m)_j}{\partial s_q}(s - \theta u) d\theta \right) \right) u_p u_q \right]. \end{aligned}$$

Again, by the chain derivative formula we arrive at

$$\begin{aligned} &D_s^\sigma \left(f_\lambda(|P_m(s, u) + Q_m(s, u)|^{1/2}) \right) = \sum_{(l,\beta) \in \mathcal{D}(\sigma)} D^\delta (f_\lambda(\cdot^{1/2})) \Big|_{P_m(s,u)+Q_m(s,u)} \\ &\quad \times \prod_{r=1}^l \frac{1}{\delta_r!} \left(\frac{1}{\beta_r!} D_s^{\beta_r} (P_m(s, u) + Q_m(s, u)) \right)^{\delta_r}. \end{aligned}$$

Notice that (83) leads to

$$\left| \frac{d^k}{dr^k} \left(f_\lambda(r^{1/2}) \right) \right| \leq \tilde{C} \frac{1}{r^{k+1}}, \quad \forall r \in (0, d_s^m).$$

Consequently, (89) proves the upper bounds

$$\begin{aligned} &\left| D_s^\gamma K_{\lambda,1}^{i,j}(\mu_m(s), \mu_m(s) - \mu_m(s - u)) \right| \leq C|u|^0, \\ &\left| \partial_{u_1} D_s^\gamma K_{\lambda,1}^{i,j}(\mu_m(s), \mu_m(s) - \mu_m(s - u)) \right| \leq C|u|^{-1}, \\ &\left| \partial_{u_1 u_2} D_s^\gamma K_{\lambda,1}^{i,j}(\mu_m(s), \mu_m(s) - \mu_m(s - u)) \right| \leq C|u|^{-2}, \end{aligned}$$

so the hypotheses in Corollary A.6 are satisfied and

$$\begin{aligned} &\left\| \int_{D_m} K_{\lambda,1}^{i,j}(\mu_m(\cdot), \mu_m(\cdot) - \mu_m(t)) \varphi_m(\mu_m(t)) \zeta(\mu_m(t)) J_m(t) dt \right\|_{C^{k+1,\alpha}(D_m)} \\ &\leq M \|\zeta\|_{C^{k,\alpha}(\Sigma_m)}. \end{aligned}$$

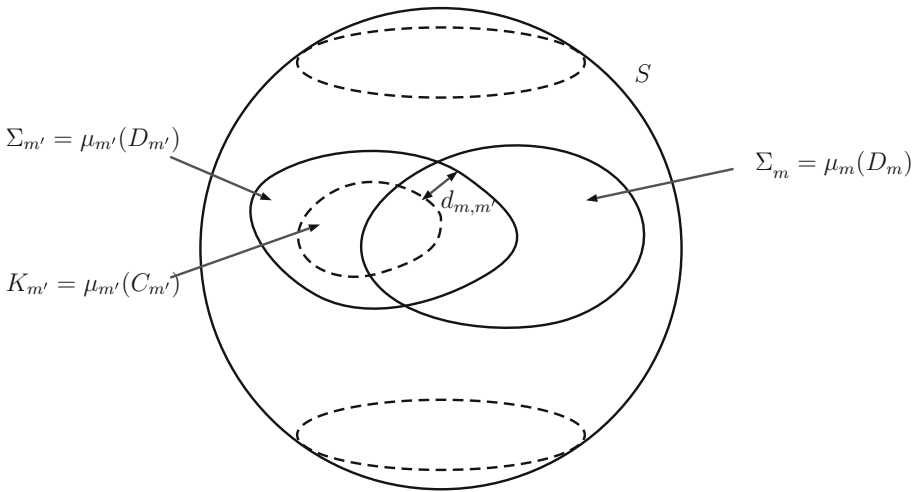


Fig. 8. Overlapping coordinate neighborhoods Σ_m and $\Sigma_{m'}$

In order to complete the proof of the theorem, let us show how to deal with the terms $m' \neq m$ in (84) and (85). The idea is to obtain estimates over $\Sigma_m \cap \Sigma_{m'}$ and $\Sigma_m \setminus \overline{\Sigma_{m'}}$ separately. First,

$$\begin{aligned} & \left\| \int_{D_{m'}} K_\lambda^{i,j}(\mu_m(\cdot), \mu_m(\cdot) - \mu_{m'}(t)) \varphi_{m'}(\mu_{m'}(t)) \zeta(\mu_{m'}(t)) J_{m'}(t) dt \right\|_{C^{k+1,\alpha}(\mu_m^{-1}(\Sigma_m \cap \Sigma_{m'}))} \\ & \leq C \left\| \int_{D_{m'}} K_\lambda^{i,j}(\mu_{m'}(\cdot), \mu_{m'}(\cdot) - \mu_{m'}(t)) \varphi_{m'}(\mu_{m'}(t)) \zeta(\mu_{m'}(t)) J_{m'}(t) dt \right\|_{C^{k+1,\alpha}(D_{m'})} \\ & \leq \tilde{C} \|\zeta\|_{C^{k,\alpha}(\Sigma_{m'})}. \end{aligned}$$

Second, define $C_{m'} := \mu_{m'}^{-1}(\text{supp } \varphi_{m'})$, $K_{m'} := \mu_{m'}(C_{m'})$ and $d_{m,m'} := \text{dist}(\Sigma_m \setminus \overline{\Sigma_{m'}}, K_{m'}) > 0$ as in Fig. 8. This avoids the singularity near $z = 0$ in the preceding kernels. Hence,

$$\begin{aligned} & D_s^\gamma \int_{D_{m'}} K_\lambda^{i,j}(\mu_m(s), \mu_m(s) - \mu_{m'}(t)) \varphi_{m'}(\mu_{m'}(t)) \zeta(\mu_{m'}(t)) J_{m'}(t) dt \\ & = \sum_{(l, (\delta^1, \delta^2), \beta) \in \mathcal{D}(\gamma)} \int_{C_{m'}} \left(D_x^{\delta^1} D_z^{\delta^2} K_\lambda^{i,j} \right) (\mu_m(s), \mu_m(s) - \mu_{m'}(t)) \\ & \quad \times \left[\prod_{r=1}^l \frac{1}{\delta_r^1! \delta_r^2!} \left(\frac{1}{\beta_r!} D^{\beta_r} \mu_m(s) \right)^{\delta_r^1} \left(\frac{1}{\beta_r!} D^{\beta_r} \mu_m(s) \right)^{\delta_r^2} \right] \\ & \quad \times \varphi_{m'}(\mu_{m'}(t)) \zeta(\mu_{m'}(t)) J_{m'}(t) dt. \end{aligned}$$

for each $s \in \mu_m^{-1}(\Sigma_m \setminus \overline{\Sigma_{m'}})$. Since $|D_x^{\delta^1} D_z^{\delta^2} K_\lambda^{i,j}(x, z)| \leq \tilde{C} |z|^{-|\delta^2|}$ for every $z \in B_{d_{m,m'}}(0)$, then

$$\left| D_s^\gamma \int_{D_{m'}} K_\lambda^{i,j}(\mu_m(s), \mu_m(s) - \mu_{m'}(t)) \varphi_{m'}(\mu_{m'}(t)) \zeta(\mu_{m'}(t)) J_{m'}(t) dt \right| \leq \frac{C}{d^{|\gamma|}} |C_{m'}| \|\zeta\|_{C^0(\Sigma_{m'})}.$$

Here, $0 < d < 1$ is such that $d < d_{m,m'}$ for every $m' \neq m$. Since one can take any $|\gamma| \leq k + 2$ by the regularity of S , then we obtain the desired estimate for $m' \neq m$ and the result follows. \square

Appendix B: Obstructions to the Existence of Generalized Beltrami Fields

In this Appendix we will review the main results on the non-existence of Beltrami fields with a non-constant factor proved in [20], as they are of direct interest for the theorems that we have presented in this paper.

Hence, let us consider in this ‘‘Appendix’’ a solution to the Beltrami field equation with a factor f :

$$\operatorname{curl} u = fu, \quad \operatorname{div} u = 0. \tag{90}$$

We will not specify the domain of the solution as the results that we will review are mostly local. The key observation is that, as the divergence of u is zero, f is a first integral of u , i.e., $u \cdot \nabla f = 0$. Since this first integral condition is very restrictive, it stands to reason that Eq. (90) should not admit any nontrivial solutions for most functions f . Before we make this idea precise in the next paragraphs, let us point out that the (well established) idea of constructing the iterations starting by dragging a function along the integral curves of a field, as we have done in the main body of this work, is fully consistent with the intuition that the first integral condition is the heart of the matter.

The first obstruction to the existence of solutions to the Beltrami equation (90) is the following:

Theorem B.1. *Let $D \subseteq \mathbb{R}^3$ be a domain and assume that the function f is nonconstant and of class $C^{6,\alpha}$. Suppose that the vector field u satisfies Eq. (90) in D . Then, there is a sixth order nonlinear partial differential operator $P \neq 0$, which can be computed explicitly, such that $u \equiv 0$ unless $P[f]$ is identically zero in D . In particular, $u \equiv 0$ for all f in a set of infinite codimension in $C^{k,\alpha}(U)$ with any $k \geq 6$.*

It should be noticed that Theorem B.1 is of a purely local nature, as it provides obstructions for the existence of nontrivial Beltrami fields in any open set and most proportionality factors.

A less powerful but more conceivable obstruction is that if f has a regular level set homeomorphic to the sphere, then Eq. (90) does not have nontrivial solutions. Then, there are no Beltrami fields whenever f has strict local extrema or is radially symmetric. This is related to the classical theorem of Cowling on the nonexistence of poloidal Beltrami fields with nonconstant factor and axial symmetry [4]:

Theorem B.2. *Suppose that the function f is of class $C^{2,\alpha}$ in a domain $D \subseteq \mathbb{R}^3$. If a regular level set $f^{-1}(c)$ has a connected component in D homeomorphic to the sphere, then any solution to Eq. (90) in D is identically zero.*

Although we will not repeat here the proof of these results, which can be found in [20], let us give a few words on the main idea. The proof of these theorems is based on formulating the Beltrami equation (90) as a constrained evolution problem. Indeed,

one can show that (90) is locally equivalent, in a precise sense, to the assertion that there is a time-dependent 1-form $\beta(t)$ on a surface Σ satisfying

$$\partial_t \beta = T(t) \beta \quad (91)$$

together with the differential constraint

$$d\beta = 0. \quad (92)$$

Here $T(t)$ is a time-dependent tensor field that depends on f and the exterior differential d is computed with respect to the coordinates on the surface Σ , which, in turn, is a regular level set of f . It should be stressed that this formulation depends strongly on the choice of coordinates.

This formulation lays bare the reason for which the Beltrami equation does not generally admit nonzero solutions: the evolution (91) is not generally compatible with the constraint (92), and the resulting compatibility conditions translate into equations that f and its derivatives must satisfy. In Theorems B.1 and B.2 we have presented the first two of these compatibility conditions, but in fact the method of proof yields a whole hierarchy of explicitly computable obstructions (with increasingly cumbersome expressions) to the existence of solutions. To ascertain how many of these obstructions are actually independent remains an interesting open problem.

Furthermore, the above formulation provides an appealing explanation of the reason for which the attempts at constructing solutions to (90) using variational techniques have failed: while the regularity of the equation is indeed determined by an elliptic system, its existence is in fact controlled by a constrained evolution problem for which the existence theory is ill posed.

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