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# Brezis–Gallouet–Wainger Type Inequalities and Blow-Up Criteria for Navier–Stokes Equations in Unbounded Domains

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**Abstract:** We shall find the weakest norm that satisfies the Brezis–Gallouet–Wainger type inequality, under some conditions. As an application of the Brezis–Gallouet–Wainger type inequality, we shall establish Beale–Kato–Majda type blow-up criteria of smooth solutions to the 3-D Navier–Stokes equations in unbounded domains.

## 1. Introduction

Let $\Omega$ be a 3-dimensional domain with $\partial\Omega \in C^\infty$. The motion of a viscous incompressible fluid in $\Omega$ is governed by the Navier–Stokes equations:

$$\text{(N–S)} \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, & \text{div } u = 0 \qquad t \in (0, T), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, & u|_{t=0} = u_0, \end{cases}$$

where $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ and $p = p(x,t)$ denote the velocity vector and the pressure, respectively, of the fluid at $(x,t) \in \Omega \times (0,T)$ and $u_0$ is the given initial velocity vector fields. In this paper, we consider Beale–Kato–Majda type blow-up criteria of classical solutions to (N–S). In the case that $\Omega$ is the whole space, Beale–Kato–Majda [1] and Kato–Ponce [23] showed that the $L^\infty$-norm of the vorticity $\omega = \text{rot } u$ controls the breakdown of smooth solutions to the Euler and Navier–Stokes equations. To be precise, if the smooth solution $u$ in $C([0,T); W^{s,p}(\mathbb{R}^n))(s > n/p + 1)$ breaks down at a finite time $t = T$, then

$$\int_0^t \|\omega(\tau)\|_{L^\infty(\Omega)} d\tau \nearrow \infty \tag{1.1}$$

as $t \nearrow T$. Chemin [9] and Kozono, Ogawa and the second author [24] proved similar blow-up criteria with $\|\omega\|_{L^\infty}$ replaced by $\|u\|_{B^1_{\infty,\infty}}$ and $\|\omega\|_{\dot{B}^0_{\infty,\infty}}$, respectively. Note that Chemin [9] dealt with solutions in $C^\alpha, \alpha > 1$. Chae [8] also proved the same criterion via $\|\omega\|_{\dot{B}^0_{\infty,\infty}}$ for solutions in the Triebel–Lizorkin spaces. It is notable that Planchon [36] improved the criterion given in [24]. He showed that, if the solution $u$ to the Euler equations in $C([0,T);B^s_{p,q}(\mathbb{R}^n))(s>n/p+1, 1\le p,q<\infty)$ cannot continue the solution in $C([0,T');B^s_{p,q}(\mathbb{R}^n))$ for any $T'>T$, then $\lim_{\epsilon\to 0+}\sup_j\int_{T-\epsilon}^T\|\Delta_j\omega\|_\infty dt$ must be greater than an absolute number $M$, where $\Delta_j$ is a frequency localization operator at $\xi\sim 2^j$. For the Navier–Stokes equations in $\mathbb{R}^n$, Fan et al. [11] proved a logarithmically improved blow-up criterion: $\int_0^t\|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)}\Big/\sqrt{1+\log(1+\|\omega(\tau)\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)})}\,d\tau\nearrow\infty$. Recently, for the 3-D Navier–Stokes equations in $\mathbb{R}^3$, Bradshaw et al. [4] proved that the local existence time $T_*$ of a unique smooth solution can be estimated from below as $T_*>c/\|\omega(0)\|_{L^\infty(\mathbb{R}^3)}$, if initial vorticity $\omega(0)\in L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. This estimate implies that if $T$ is the maximal time of existence of the smooth solution, then

$$\|\omega(t)\|_{L^\infty(\mathbb{R}^3)}\ge c/(T-t)\quad \text{for } t<T, \tag{1.2}$$

which directly yields the Beale–Kato–Majda blow-up criterion (1.1) for the 3-D Navier–Stokes equations. See also Kukavica [28].

In the case where $\Omega$ is a bounded domain, for the 3-D Euler equations, Ferrari [14] and Shirota–Yanagisawa [37] succeeded in proving that the same result of breakdown as Beale–Kato–Majda holds. See also Zajaczkowski [46]. In [34], Ogawa and the second author proved a similar blow-up criterion with $\|\omega\|_{L^\infty(\Omega)}$ replaced by $\|\omega\|_{bmo(\Omega)}$. To be precise, in [34] it was shown that, if $\Omega$ is a 3-D bounded domain and if the smooth solution $u$ in $C([0,T);H^m(\Omega))(m>3/2+1)$ breaks down at a finite time $t=T$, then

$$\int_0^t\|\omega(\tau)\|_{bmo(\Omega)}d\tau\nearrow\infty \tag{1.3}$$

as $t\nearrow T$. However, in [34], the blow-up criterion via $\|\omega\|_{bmo(\Omega)}$ was proven only for 3-D Euler equations. In the present paper, we prove the same criterion for 3-D Navier–Stokes equations in domains with $\partial\Omega\in C^\infty$. In [34], the proof was based on the $H^m$-energy method. However, it seems difficult to apply the $H^m$-energy method to solutions of the Navier–Stokes equations in a domain with the no-slip boundary condition $u|_{\partial\Omega}=0$, due to the diffusion term. In the present paper, instead of applying the $H^m$-energy method, we will use the integral equation of (N–S) and the smoothing effect of the Stokes semigroup. Moreover, by using a space of Morrey type we improve the *bmo*-criterion (1.3) to

$$\int_0^t\|\omega(\tau)\|_{M_1^{\log}(\Omega)}d\tau\nearrow\infty, \tag{1.4}$$

as $t\nearrow T$. Here, $M_1^{\log}(\Omega)$ will be defined in Sect. 2.

It is notable that Grujic and Guberovic [18] established a local version of the *bmo*-criterion for the interior regularity of weak solutions to the 3-D Navier–Stokes equations by using the non-homogeneous div-curl lemma. It is also notable that Morrey type norms of $\nabla u$ and vorticity $\omega$ were utilized by Caffarelli et al. [7] and Gustafson et al. [19] for the interior regularity of suitable weak solutions to the Navier–Stokes equations. More precisely, in [7], it was shown that if $u$ is a suitable weak solution and if

$$\limsup_{r\to 0+} r^{-1}\int_{t-r^2}^t\int_{B(x,r)}|\nabla u(y,\tau)|^2\,dyd\tau \text{ is sufficiently small,} \tag{1.5}$$

then $u$ is regular at $(x, t)$, i.e., $u \in L^\infty(B(x,r)\times(t-r^2,t))$ for some $r > 0$. In [19], this interior regularity criterion (1.5) was refined, replacing by the more general condition that

$$\limsup_{r\to 0+} r^{-(3/p+2/q-2)}\left\{\int_{t-r^2}^{t}\left(\int_{B(x,r)}|\omega(y,\tau)|^p\,dy\right)^{q/p}d\tau\right\}^{1/q} \text{ is sufficiently small} \tag{1.6}$$

for some $p, q$ with $2 \le 3/p + 2/q \le 3$, $1 \le q \le \infty$, $(p,q) \ne (1,\infty)$.

In order to prove (1.3) and (1.4) for (N–S), the following Brezis–Gallouet–Wainger type inequalities play important roles:

$$(BGW)_\beta \qquad \|f\|_{L^\infty(\Omega)} \le C(1 + \|f\|_X \log^\beta(e + \|f\|_Y)).$$

When $\Omega = \mathbb{R}^n$, by using the Fourier transform, Brezis–Gallouet–Wainger [5,6] proved $(BGW)_\beta$ in the case

$$\beta = 1 - 1/p, \quad X = W^{n/p,p}(\mathbb{R}^n), \quad Y = W^{n/q+\alpha,q}(\mathbb{R}^n)(\subset \dot{C}^\alpha)(\alpha > 0).$$

Engler [10] proved the same inequality for general domains $\Omega$ without using the Fourier transform. Ozawa [35] proved the Gagliardo–Nirenberg type inequality

$$\|f\|_{L^q(\mathbb{R}^n)} \le C(p,n)q^{1-1/p}\|(-\Delta)^{n/2p}f\|_{L^p(\mathbb{R}^n)}^{1-p/q}\|f\|_{L^p(\mathbb{R}^n)}^{p/q} \text{ for all } q \in [p,\infty) \tag{1.7}$$

with the explicit growth rate with respect to $q$ and that this estimate directly yields $(BGW)_\beta$ with $\beta = 1 - 1/p$. When $\Omega$ is a bounded domain, in [33,34], $(BGW)_\beta$ was proven in the cases

$$\beta = 1, \quad X = bmo(\Omega), \quad Y = \dot{C}^\alpha(\Omega), \text{ or}$$
$$\beta = 1, \quad X = B_\Theta(\Omega) \text{ with } \Theta(q) = q, \quad Y = \dot{C}^\alpha(\Omega),$$

where $B_\Theta(\Omega)$ is introduced by $\|f\|_{B_\Theta(\Omega)} := \sup_{q\ge 1}\frac{\|f\|_{L^q(\Omega)}}{\Theta(q)}$. Furthermore, in [1,9,10, 14,20,23–26,29,32,34,35,37,42,46] several inequalities of Brezis–Gallouet–Wainger type were established. Then, we have one question.

**Question. What is the largest normed space $X$ that satisfies $(BGW)_\beta$ with $Y = \dot{C}^\alpha(\Omega)$?**

In the present paper, we also consider this problem and find the largest normed space $X$ under some additional assumptions.

Throughout this paper we impose the following assumption on the domain.

**Assumption 1.** $\Omega \subset \mathbb{R}^n$ is the half-space $\mathbb{R}^n_+$, the whole space $\mathbb{R}^n$, a bounded domain, an exterior domain, a perturbed half-space, or an aperture domain with $\partial\Omega \in C^\infty$.

For the definitions of perturbed half-spaces and aperture domains, see Kubo–Shibata [27] and Farwig–Sohr [12].

The remainder of the present paper is organized as follows. In Sect. 2, some function spaces are introduced. In Sect. 3, the main results are described. In Sects. 4, 5 and 6, the proofs of main results are presented.

In this paper, we denote by $C$ various constants.

## 2. Function Spaces and Preliminaries

We first introduce Banach spaces of Morrey type and Besov type which are wider than $L^\infty$. Let $E_0$ be the 0-extension operator from functions defined on $\Omega$ to functions on $\mathbb{R}^n$ and $R_\Omega$ be the restriction operator from functions on $\mathbb{R}^n$ to functions on $\Omega$, i.e.,

$$E_0 f(x) := \begin{cases} f(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \in \mathbb{R}^n \backslash \Omega, \end{cases} \qquad R_\Omega f := f\big|_\Omega ,$$

$B(x,t) := \{y \in \mathbb{R}^n;\ |y-x| < t\}$ and let $L^1_{uloc}(\Omega)$ denote the uniformly local $L^1$ space, i.e.,

$$L^1_{uloc}(\Omega) := \left\{ f \in L^1_{loc}(\bar{\Omega})\ ;\ \|f\|_{L^1_{uloc}(\Omega)} := \sup_{x\in\mathbb{R}^n} \frac{1}{|B(x,1)|} \int_{B(x,1)\cap\Omega} |f(y)|dy < \infty \right\}.$$

**Definition 1.** Let $\beta > 0$ and $\Omega$ be a domain in $\mathbb{R}^n$. Then,

- $M^{\log}_\beta(\Omega) := \{f \in L^1_{uloc}(\Omega); \|f\|_{M^{\log}_\beta} < \infty\}$ is introduced by the norm

$$\|f\|_{M^{\log}_\beta(\Omega)} := \sup_{x\in\Omega,\ 0<t<1} \frac{1}{|B(x,t)| \log^\beta(e+\frac{1}{t})} \int_{B(x,t)} E_0|f(y)|dy.$$

- $\tilde{M}^{\log}_\beta(\Omega)$ is defined by

$$\tilde{M}^{\log}_\beta(\Omega) := \overline{BC(\bar{\Omega})}^{\|\cdot\|_{M^{\log}_\beta(\Omega)}},$$

  where $BC(\bar{\Omega})$ denotes the set of all bounded continuous functions in $\bar{\Omega}$.

  We note that, for any constant $\delta > 0$, $\|f\|_{M^{\log}_\beta(\Omega)}$ is equivalent to the following norms

$$\|f\|_{M^{\log}_{\beta,\delta}(\Omega)} := \sup_{x\in\Omega,\ 0<t<\delta} \frac{1}{|B(x,t)| \log^\beta(e+\frac{1}{t})} \int_{B(x,t)} E_0|f(y)|dy,$$
$$\|f\|'_{M^{\log}_{\beta,\delta}(\Omega)} := \sup_{x\in\mathbf{R^n},\ 0<t<\delta} \frac{1}{|B(x,t)| \log^\beta(e+\frac{1}{t})} \int_{B(x,t)} E_0|f(y)|dy.$$

Indeed, for $0 < \delta < r$, clearly $\|f\|_{M^{\log}_{\beta,\delta}(\Omega)} \le \|f\|_{M^{\log}_{\beta,r}(\Omega)} \le \|f\|'_{M^{\log}_{\beta,r}(\Omega)} \le C(n,\beta,\delta,r)$ $\|f\|'_{M^{\log}_{\beta,\delta}(\Omega)}$ holds, where $C(n,\beta,\delta,r)$ is a constant independent of $f$. By Proposition 4.3 (4.9) in Sect. 4, we observe that $\|f\|'_{M^{\log}_{\beta,\delta}(\Omega)} \le C(n)\|f\|_{M^{\log}_{\beta,\delta}(\Omega)}$. Then we have $\|f\|_{M^{\log}_\beta(\Omega)} \cong \|f\|_{M^{\log}_{\beta,\delta}(\Omega)} \cong \|f\|'_{M^{\log}_{\beta,\delta}(\Omega)}$ for all $\delta > 0$.

**Definition 2** (*Modified Vishik's space*). Let $\beta > 0$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a spherical symmetric function with $\hat{\psi}(\xi) = 1$ in $B(0,1)$ and $\hat{\psi}(\xi) = 0$ in $B(0,2)^c$. Then,

- $V_\beta(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{V_\beta} < \infty\}$ is introduced by the norm

$$\|f\|_{V_\beta} := \sup_{N=1,2,\dots} \frac{\|\psi_N * f\|_\infty}{N^\beta}, \quad \text{where } \psi_N(x) := 2^{nN}\psi(2^N x).$$

- $\tilde{V}_\beta$ is defined by

$$\tilde{V}_\beta := \overline{BUC(\mathbb{R}^n)}^{\|\cdot\|_{V_\beta}},$$

  where $BUC(\mathbb{R}^n)$ denotes the set of all bounded uniformly continuous functions in $\mathbb{R}^n$.

Note that the space $V_\beta$ is a modified version of spaces introduced by Vishik [43]. We also note that the following inclusions hold:

$$M_1^{\log}(\Omega) \supset bmo(\Omega) \supset L^\infty(\Omega), \text{ see Appendix,}$$
$$V_1(\mathbb{R}^n) \supset B^0_{\infty,\infty}(\mathbb{R}^n) \supset bmo(\mathbb{R}^n) \supset L^\infty(\mathbb{R}^n).$$

Here $bmo(\mathbb{R}^n) = BMO(\mathbb{R}^n) \cap L^1_{uloc}(\mathbb{R}^n)$ and $bmo(\Omega)$ is defined by the restriction of $bmo(\mathbb{R}^n)$ on $\Omega$, i.e. $bmo(\Omega) := \{R_\Omega f;\ f \in bmo(\mathbb{R}^n)\}$, where $R_\Omega f$ is the restriction of $f$ on $\Omega$. The norm of $bmo(\Omega)$ is defined by

$$\|f\|_{bmo(\Omega)} := \inf\{\|\tilde{f}\|_{bmo(\mathbb{R}^n)};\ \tilde{f} \in bmo(\mathbb{R}^n) \text{ with } \tilde{f} = f \text{ in } \Omega\}.$$

Let $C_0^\infty(K)$ denote the set of all $C^\infty$ functions with compact support in the set $K$, $BC^\infty(K) := \{g \in C^\infty(K);\ \partial^\alpha g \in L^\infty(K) \text{ for all multi-indices } \alpha\}$ and $C^\infty_{0,\sigma}(\Omega) = C^\infty_{0,\sigma} := \{\phi \in (C_0^\infty(\Omega))^n;\ \operatorname{div}\phi = 0\}$. Then $L^r_\sigma$, $1 < r < \infty$, is the closure of $C^\infty_{0,\sigma}$ with respect to the $L^r$-norm $\|\cdot\|_r$. Concerning Sobolev spaces we use the notations $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$, $k \in \mathbb{N}$, $1 \le p \le \infty$. Note that very often we will simply write $L^r$ and $W^{k,p}$ instead of $L^r(\Omega)$ and $W^{k,p}(\Omega)$, respectively. The symbol $(\cdot,\cdot)$ denotes the $L^2$-inner product and the duality pairing between $L^p$ and $L^{p'}$, where $1/p + 1/p' = 1$.

Let us recall the Helmholtz decomposition: $L^r(\Omega) = L^r_\sigma \oplus G_r$ $(1 < r < \infty)$, where $G_r = \{\nabla p \in L^r;\ p \in L^r_{loc}(\overline{\Omega})\}$, see Fujiwara and Morimoto [15], Miyakawa [30], Simader and Sohr [38], Borchers and Miyakawa [2] and Farwig–Sohr [13]; $P_r$ denotes the projection operator from $L^r$ onto $L^r_\sigma$ along $G_r$. The Stokes operator $A_r$ on $L^r_\sigma$ is defined by $A_r = -P_r\Delta$ with domain $D(A_r) = W^{2,r} \cap W_0^{1,r} \cap L^r_\sigma$. It is known that $(L^r_\sigma)^*$ (the dual space of $L^r_\sigma$) $= L^{r'}_\sigma$ and $A_r^*$ (the adjoint operator of $A_r$) $= A_{r'}$, where $1/r + 1/r' = 1$. It is shown by Giga [16], Borchers and Sohr [3], Borchers and Miyakawa [2], Iwashita [21] and Farwig and Sohr [13] that $-A_r$ generates a holomorphic semigroup $\{e^{-tA_r};\ t \ge 0\}$ of class $C_0$ in $L^r_\sigma$. Since $P_r u = P_q u$ for all $u \in L^r \cap L^q$ $(1 < r, q < \infty)$ and since $A_r u = A_q u$ for all $u \in D(A_r) \cap D(A_q)$, for simplicity, we shall abbreviate $P_r u$, $P_q u$ as $Pu$ for $u \in L^r \cap L^q$ and $A_r u$, $A_q u$ as $Au$ for $u \in D(A_r) \cap D(A_q)$, respectively.

We very often use the notations $⨍$ for integral means, i.e., $⨍_B f(y)dy := \frac{1}{|B|}\int_B f(y)dy$ and $\tau_y$ for the translation operator, i.e., $\tau_y f = f(\cdot - y)$. Let $1_\Omega$ denote the characteristic function on $\Omega$.

## 3. Main Theorems

Now our main results read as follows.

**Theorem 1.** *Let $\Omega (\subset \mathbb{R}^n)$ satisfy Assumption 1.*

(a) *For any* $\alpha \in (0,1)$ *and* $\beta > 0$, *there exists a constant* $C(\Omega, \alpha, \beta, n) > 0$ *such that*

$$\|f\|_{L^\infty(\Omega)} \le C\Big(1+\|f\|_{M_\beta^{\log}(\Omega)} \log^\beta \big(e+\|f\|_{\dot{C}^\alpha(\Omega)}\big)\Big) \quad \text{for all } f \in \dot{C}^\alpha(\Omega)\cap M_\beta^{\log}(\Omega). \tag{3.1}$$

(b) *Let* $\beta > 0$ *and* $X(\Omega)$ *be a normed space. Assume that* $X(\Omega)$ *satisfies the following conditions:*
   (A1) $\overline{C_0(\overline{\Omega})}^{\|\cdot\|_\infty} \hookrightarrow X(\Omega) \subset L^1_{uloc}(\Omega)$ *;*
   (A2) $X(\Omega)$ *is a translation invariant space, i.e.,* $\|R_\Omega(\tau_y f)\|_{X(\Omega)} \le \|f\|_{X(\Omega)}$ *for all* $y \in \mathbb{R}^n$ *and all* $f \in C_0(\Omega)$*;*
   (A3) $X(\Omega)$*-norm has the property:* $\|f\|_{X(\Omega)} \le \|g\|_{X(\Omega)}$ *if* $f, g \in BC(\bar{\Omega})\cap X(\Omega)$ *and* $|f(x)| \le |g(x)|$ *a.e.* $x \in \Omega$*;*
   (A4) *there exist constants* $\alpha \in (0,1)$ *and* $C > 0$ *such that*
   $$\|f\|_{L^\infty(\Omega)} \le C\Big(1+\|f\|_X \log^\beta \big(e+\|f\|_{\dot{C}^\alpha(\Omega)}\big)\Big) \quad \text{for all } f \in \dot{C}^\alpha(\Omega)\cap X(\Omega).$$

   *Then, there exists a constant* $C_0 > 0$ *such that*

$$\|f\|_{M_\beta^{\log}(\Omega)} \le C_0 \|f\|_{X(\Omega)} \quad \text{for all } f \in BC(\bar{\Omega}) \cap X(\Omega).$$

   *In particular, if* $BC(\bar{\Omega})$ *is densely contained in* $X(\Omega)$, *then* $X(\Omega) \hookrightarrow \tilde{M}_\beta^{\log}(\Omega)$.

*Remark 1.* (i) The condition (A2) implies that if both of $f$ and $\tau_y f$ belong to $C_0(\Omega)$, then $\|f\|_{X(\Omega)} = \|\tau_y f\|_{X(\Omega)}$, since $f = \tau_{-y}\tau_y f$. Hence we call (A2) the translation invariant property.

(ii) By Definition 1 and Theorem 1(a) we see that $M_\beta^{\log}(\Omega)$ and $\tilde{M}_\beta^{\log}(\Omega)$ satisfy Conditions (A1)–(A4). We emphasize that Theorem 1(b) implies that $\tilde{M}_\beta^{\log}(\Omega)$ is the largest normed space that satisfies Conditions (A1)–(A4) and densely contains $BC(\bar{\Omega})$.

If we do not assume (A3), there is a normed space wider than $M_\beta^{\log}$, when $\Omega = \mathbb{R}^n$ as below.

**Theorem 2.** (a) *For any* $\alpha \in (0,1)$ *and* $\beta > 0$, *there exists a constant* $C(\alpha, \beta, n) > 0$ *such that*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le C\Big(1+\|f\|_{V_\beta(\mathbb{R}^n)} \log^\beta \big(e+\|f\|_{\dot{C}^\alpha(\mathbb{R}^n)}\big)\Big) \quad \text{for all } f \in \dot{C}^\alpha(\mathbb{R}^n)\cap V_\beta. \tag{3.2}$$

(b) *Let* $\beta > 0$ *and* $X$ *be a normed space. Assume that* $X$ *satisfies the following conditions*
   (B1) $BUC(\mathbb{R}^n) \hookrightarrow X$*;*
   (B2) $X \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ *or* $X \subset L^1_{uloc}(\mathbb{R}^n)$*;*
   (B3) $X$ *is a translation invariant space, i.e.,*
   $$\|f(\cdot - y)\|_X = \|f\|_X \quad \text{for all } y \in \mathbb{R}^n;$$
   (B4) *there exist constants* $\alpha \in (0,1)$ *and* $C > 0$ *such that*
   $$\|f\|_{L^\infty(\mathbb{R}^n)} \le C\Big(1+\|f\|_X \log^\beta \big(e+\|f\|_{\dot{C}^\alpha(\mathbb{R}^n)}\big)\Big) \quad \text{for all } f \in BC^\infty(\mathbb{R}^n).$$

   *Then, there exists a constant* $C_0 > 0$ *such that*

$$\|f\|_{V_\beta} \le C_0 \|f\|_X \quad \text{for all } f \in BUC(\mathbb{R}^n).$$

   *In particular, if* $BUC$ *is densely contained in* $X$, *then* $X \hookrightarrow \tilde{V}_\beta$.

(c) *Let* $\beta > 0$ *and* $X$ *be a normed space. Assume that* $X$ *satisfies Conditions (B2)–(B4). Furthermore assume that*

$$(C) \qquad \|\rho * f\|_X \le \|\rho\|_{L^1(\mathbb{R}^n)} \|f\|_X \text{ holds for all } \rho \in \mathcal{S} \text{ and all } f \in X.$$

*Then,* $X \hookrightarrow V_\beta$.

*Remark 2.* (i) From Theorem 2(a) we observe that $V_\beta$ and $\tilde{V}_\beta$ satisfy Conditions (B1)–(B4). Hence, Theorem 2(b) implies that $\tilde{V}_\beta$ is the largest Banach space that satisfies Condition (B1)–(B4) and densely contains $BUC(\mathbb{R}^n)$.

(ii) Since $V_\beta$ satisfies (B2)–(B4) and $(C)$, Theorem 2(c) implies that $V_\beta$ is the largest normed space that satisfies Conditions (B2)–(B4) and $(C)$.

(iii) Since $M_\beta^{\log}(\mathbb{R}^n)$ satisfies (B2)–(B4) and $(C)$, from Theorem 2(c) we observe that $M_\beta^{\log}(\mathbb{R}^n) \hookrightarrow V_\beta$.

(iv) If $X$ satisfies (B1)–(B3) and the following condition

$$(C)' \qquad X \text{ is a Banach space and } BUC(\mathbb{R}^n) \text{ is dense in } X,$$

then $X$ satisfies Condition $(C)$.

Our results on (N–S) read as follows.

**Theorem 3.** *Let the dimension* $n = 3$, $\Omega(\subset \mathbb{R}^3)$ *satisfy Assumption 1,* $p \ge 3$ *and* $u$ *be a solution to (N–S) on* $(0, T)$ *in the class*

$$S_p(0,T) := C([0,T); L^p_\sigma) \cap C^1((0,T); L^p_\sigma) \cap C((0,T); W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)).$$

*Assume that* $T < \infty$ *and* $T$ *is maximal, i.e.,* $u$ *cannot be continued to the solution of (N–S) in the class* $S_p(0, T')$ *for any* $T' > T$. *Then,*

$$\int_s^t \|\omega(\tau)\|_{M_1^{\log}(\Omega)} d\tau \nearrow \infty \quad as \quad t \nearrow T \quad for\ any\ s \in (0,T), \tag{3.3}$$

*where* $\omega = \mathrm{rot}\, u$.

*Remark 3.* (i) Solutions in the class $S_p(0, T)$ are called strong $L^p$ solutions on $(0, T)$. For $p \ge 3$, the existence of strong $L^p$ solutions to (N–S) is proven in [17,21,22,44].

(ii) The blow-up criterion (1.2) given in [4] dose not directly imply (3.3). Indeed, let $f(x,t) := \min\{c/|T-t|, \log^+(1/|x|)\}$, then $\|f(t)\|_\infty = c/|T-t|$, while $\int_0^T \|f\|_{M_1^{\log}} dt \le \int_0^T \|\log^+(1/|x|)\|_{M_1^{\log}} dt = CT < \infty$. Here $\log^+ s := \max\{0, \log s\}$.

(iii) By using Theorem 3 and the standard argument, we can also show the following regularity criterion for weak solutions. If $n = 3$, $u$ is a Leray–Hopf weak solution with the energy inequality in the strong form and if $u$ satisfies

$$\int_0^T \|\omega(\tau)\|_{M_1^{\log}(\Omega)} d\tau < \infty, \tag{3.4}$$

then $u$ is smooth in $(0, T] \times \bar{\Omega}$. Here, for the definitions of Leray–Hopf weak solutions and the energy inequality in the strong form, see e.g. [41, Definition 2.1]. Note that there are no inclusion between Conditions (3.4) and (1.6). See Lemma 7.1 in Appendix.

It holds that $\|f\|_{M_1^{\log}(\Omega)} \le C\|f\|_{bmo(\Omega)}$, see Appendix. Hence, we have:

**Corollary 1.** *Under the same assumptions as in Theorem 3, it holds that*

$$\int_s^t \|\omega(\tau)\|_{bmo(\Omega)} d\tau \nearrow \infty$$

*as $t \nearrow T$ for any $s \in (0, T)$.*

In the same way as in the proof of Theorem 3, we also have:

**Corollary 2.** *Let $\Omega = \mathbb{R}^3$. Then, under the same assumptions as in Theorem 3, it holds that*

$$\int_s^t \|\omega(\tau)\|_{V_1(\mathbb{R}^3)} d\tau \nearrow \infty$$

*as $t \nearrow T$ for any $s \in (0, T)$.*

Before closing this section, we introduce an inequality equivalent to $(BGW)_\beta$.

**Proposition 3.1.** *Let $X$ be a normed space, $Y$ be a semi-normed space, $Z(\subset X \cap Y)$ be a linear space, and $\beta > 0$. Then, $(BGW)_\beta$ holds for all $f \in Z$ if and only if there exists a constant $C > 0$ such that*

$$(BGW)'_\beta \qquad \|f\|_{L^\infty(\Omega)} \le C\big(\epsilon\|f\|_Y + \log^\beta(e + \frac{1}{\epsilon})\|f\|_X\big)$$

*for all $f \in Z$ and all $\epsilon > 0$.*

Indeed, substituting $f = \frac{g}{\epsilon\|g\|_Y}$ into $(BGW)_\beta$, we see that $(BGW)_\beta$ yields $(BGW)'_\beta$. Conversely, letting $\epsilon = \frac{1}{\|f\|_Y}$, we see that $(BGW)'_\beta$ yields $(BGW)_\beta$.

## 4. Proof of Theorem 1

*Proof of Theorem 1(a).* Here, we give the proof of the first part of Theorem 1, using arguments given in Engler [10] and Ozawa [35]. See also [34]. For the sake of simplicity, we assume $n = 3$. By Assumption 1 we see that $\partial\Omega$ satisfies the interior cone condition. Namely there are $\delta \in (0, 1)$ and $\theta \in (\pi/2, \pi)$ depending only on $\Omega$ with the following property: For any point $x \in \Omega$, there exists a spherical sector $C_\delta^\theta(x) = \{x + \xi \in \mathbb{R}^3; 0 < |\xi| < \delta, -|\xi| \le \kappa(x) \cdot \xi < |\xi| \cos\theta\}$ having a vertex at $x$ such that $C_\delta^\theta(x) \subset \Omega$, where $\kappa(x)$ is an appropriate unit vector from $x$. We note that for each $x \in \Omega$, $C_\delta^\theta(x)$ is congruent to $C_\delta^\theta \equiv \{\xi \in \mathbb{R}^3; 0 < |\xi| < \delta, -|\xi| \le \xi_3 < |\xi| \cos\theta\}$. In particular, for any boundary point $x \in \partial\Omega$, $C_\delta^\theta(x)$ can be expressed as $C_\delta^\theta(x) \equiv \{x + \xi \in \mathbb{R}^3; 0 < |\xi| < \delta, -|\xi| \le \xi \cdot \nu(x) < |\xi| \cos\theta\}$, where $\nu(x)$ denotes the unit outward normal at $x$.

Let $0 < t \le \delta$ and $C_t^\theta(x) := C_\delta^\theta(x) \cap B(x, t)$. For any fixed $x \in \Omega$ and $y \in C_t^\theta(x) \subset \Omega$,

$$|f(x)| \le |f(x) - f(y)| + |f(y)| \le \|f\|_{\dot{C}^\alpha}|x - y|^\alpha + |f(y)| \le \|f\|_{\dot{C}^\alpha} t^\alpha + |f(y)|.$$

Integrating both sides of the above inequality with respect to $y$ over $C_t^\theta(x)$,

$$\begin{aligned}|f(x)||C_t^\theta(x)| &\le t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)}|C_t^\theta(x)| + \int_{y\in C_t^\theta(x)} |f(y)|dy \\ &\le t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)}|C_t^\theta(x)| + \int_{y\in B(x,t)\cap\Omega} |f(y)|dy \\ &\le t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)}|C_t^\theta(x)| + |B(x,t)| \log^\beta(\frac{1}{t}+e)\|f\|_{M_\beta^{\log}(\Omega)}. \end{aligned} \quad (4.1)$$

Since $|B(x,t)|/|C_t^\theta(x)|(=: K_\theta)$ is a constant independent of $x, t$, we have

$$|f(x)| \le t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)} + K_\theta \log^\beta(\frac{1}{t}+e)\|f\|_{M_\beta^{\log}(\Omega)} \quad (4.2)$$

for all $0 < t \le \delta$. Then we optimize $t$ by letting $t = (1/\|f\|_{\dot{C}^\alpha(\Omega)})^{1/\alpha}$ if $\|f\|_{\dot{C}^\alpha(\Omega)} \ge \delta^{-\alpha}$ and letting $t = \delta$ if $\|f\|_{\dot{C}^\alpha(\Omega)} \le \delta^{-\alpha}$ to obtain (3.1). □

In order to prove the second part of Theorem 1, we introduce the following propositions and lemmata. Although the propositions are elementary, for readers' convenience, we write proofs of those.

**Proposition 4.1.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then, it holds that*

$$⨍_{B(x,t)} |f(y)|dy \le 2^n ⨍_{B(x,2t)} \left(⨍_{B(y,t)} |f(z)|dz\right) dy \quad (4.3)$$

*for all $x \in \mathbb{R}^n$.*

*Proof of Proposition 4.1.* Clearly, by Fubini's theorem we have

$$\begin{aligned}\text{R.H.S.} &= \frac{2^n}{|B(0,2t)|\,|B(0,t)|}\int_{|y'|<2t}\left(\int_{|z'|<t}|f(z'+y'+x)|dz'\right)dy' \\ &= \frac{2^n}{|B(0,2t)|\,|B(0,t)|}\int_{|z'|<t}\left(\int_{|y'|<2t}|f(z'+y'+x)|dy'\right)dz' \\ &= \frac{2^n}{|B(0,2t)|\,|B(0,t)|}\int_{|z'|<t}\left(\int_{y''\in B(z',2t)}|f(y''+x)|dy''\right)dz' =: I. \end{aligned}\quad (4.4)$$

Since $B(0,t) \subset B(z',2t)$ for $|z'| < t$,

$$I \ge \frac{2^n}{|B(0,2t)|\,|B(0,t)|}\int_{|z'|<t}\left(\int_{y''\in B(0,t)}|f(y''+x)|dy''\right)dz' = \text{L.H.S.}, \quad (4.5)$$

which proves Proposition 4.1. □

**Proposition 4.2.** *Let $g \in L^\infty(\mathbb{R}^n)$. Then, the following inequalities hold:*

$$\left|⨍_{B(x,t)} |g|dz - ⨍_{B(y,t)} |g|dz\right| \le C(n)\frac{|x-y|}{t}\|g\|_\infty, \quad (4.6)$$

$$\left\|\nabla_x ⨍_{B(x,t)} |g|dz\right\|_\infty \le C(n)\frac{1}{t}\|g\|_\infty, \quad (4.7)$$

$$\left\|⨍_{B(x,t)} |g|dz\right\|_{\dot{C}^\alpha(\mathbb{R}^n)} \le C(n,\alpha)\frac{1}{t^\alpha}\|g\|_\infty \textit{ for } 0<\alpha<1. \quad (4.8)$$

*Proof of Proposition 4.2.* For the simplicity, we assume $n = 3$. Let $B(x,t) \ominus B(y,t) := (B(x,t) \cup B(y,t)) \backslash (B(x,t) \cap B(y,t))$. Since

$$\left| ⨍_{B(x,t)} |g| dz - ⨍_{B(y,t)} |g| dz \right| \leq \frac{1}{|B(0,t)|} |B(x,t) \ominus B(y,t)| \, \|g\|_\infty$$

and

$$\begin{aligned} |B(x,t) \ominus B(y,t)| &\leq \begin{cases} 2\pi t^2 |x-y| & \text{if} \quad |x-y| \leq 2t \\ 2|B(0,t)| & \text{if} \quad |x-y| > 2t \end{cases} \\ &\leq C|B(0,t)| \frac{|x-y|}{t}, \end{aligned}$$

we obtain (4.6). (4.7) is a direct consequence of (4.6). The interpolation inequality $\|f\|_{\dot{C}^\alpha} \leq C \|f\|_\infty^{1-\alpha} \|\nabla f\|_\infty^\alpha$ and (4.7) yield (4.8). □

**Proposition 4.3.** *There exists a constant* $C = C(n) > 0$ *such that*

$$\|m_t * |E_0 f|\|_{L^\infty(\mathbb{R}^n)} \leq C \|R_\Omega (m_t * |E_0 f|)\|_{L^\infty(\Omega)} \tag{4.9}$$

*for all* $t > 0$ *and* $f \in L^1_{uloc}(\Omega)$*, where* $m_t(x) := \frac{1}{|B(0,t)|} 1_{B(0,t)}(x)$.

*Proof of Proposition 4.3.* Clearly, $B(x,t)$ can be covered by $N$ balls $\{B_1, B_2, \dots B_N\}$ of radius $t/2$, where the natural number $N \in \mathbb{N}$ depends only on $n$. Thus,

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} \int_{B(x,t)} |E_0 f| dy &\leq N \sup_{x \in \mathbb{R}^n} \int_{B(x,t/2)} |E_0 f| dy \\ &= N \sup_{x \in \mathbb{R}^n, B(x,t/2) \cap \Omega \neq \emptyset} \int_{B(x,t/2) \cap \Omega} |E_0 f| dy. \end{aligned}$$

If $B(x,t/2) \cap \Omega \neq \emptyset$, then there is $z_x \in B(x,t/2) \cap \Omega$. Since $B(x,t/2) \subset B(z_x,t)$ and $z_x \in \Omega$, it holds that

$$\sup_{x \in \mathbb{R}^n, B(x,t/2) \cap \Omega \neq \emptyset} \int_{B(x,t/2) \cap \Omega} |E_0 f| dy \leq \sup_{z \in \Omega} \int_{B(z,t) \cap \Omega} |E_0 f| dy.$$

Therefore we obtain

$$\sup_{x \in \mathbb{R}^n} \int_{B(x,t)} |E_0 f| dy \leq N \sup_{z \in \Omega} \int_{B(z,t) \cap \Omega} |E_0 f| dy, \tag{4.10}$$

which yields the desired estimate (4.9). □

**Lemma 4.1.** *Let* $\Omega$ *satisfy Assumption 1. Assume that* $X(\Omega)$ *is a normed space and satisfies Conditions (A1)–(A3) given in Theorem 1. Then, it holds that*

$$\|R_\Omega (1_B * E_0 h))\|_{X(\Omega)} \leq |B| \|h\|_{X(\Omega)} \tag{4.11}$$

*for all* $h \in C_0(\Omega)$ *and all* $B = B(0,r)$.

*Proof of Lemma 4.1.* By Condition (A2), we have $\|R_\Omega \tau_y(E_0 h)\|_{X(\Omega)} \le \|E_0 h\|_{X(\Omega)} = \|h\|_{X(\Omega)}$ for all $y \in \mathbb{R}^n$. Since for all $x \in \Omega$ it holds that $R_\Omega(1_B * E_0 h)(x) = \int_{\mathbb{R}^n} 1_B(y)\tau_y(E_0 h)(x)dy = \int_{\mathbb{R}^n} 1_B(y) R_\Omega \tau_y(E_0 h)(x)dy$, we formally obtain

$$\begin{aligned}\|R_\Omega(1_B * E_0 h)\|_{X(\Omega)} &\le \int_{\mathbb{R}^n} |1_B(y)| \|R_\Omega \tau_y(E_0 h)(\cdot)\|_{X(\Omega)} dy \\ &\le \int_{\mathbb{R}^n} |1_B(y)| dy \ \|h\|_{X(\Omega)}.\end{aligned}$$

More precisely, since $E_0 h \in C_0(\mathbb{R}^n)$, we easily observe that the Riemann sum

$$r_k(x) := \frac{1}{2^{nk}} \sum_{z_m \in (\mathbb{Z}/2^k)^n} 1_B(z_m)(E_0 h)(x - z_m) (\in C_0(\mathbb{R}^n))$$

converges to $\int_{\mathbb{R}^n} 1_B(y)(E_0 h)(x - y)dy$ in $L^\infty(\Omega)$ as $k \to \infty$. Since $\overline{C_0(\overline{\Omega})}^{\|\cdot\|_\infty} \hookrightarrow X(\Omega)$, this convergence holds in $X(\Omega)$. Hence, we have

$$\begin{aligned}\|R_\Omega(1_B * E_0 h)\|_{X(\Omega)} &= \lim_{k\to\infty} \|R_\Omega r_k(x)\|_{X(\Omega)} \\ &\le \limsup_{k\to\infty} \frac{1}{2^{nk}} \sum |1_B(z_m)| \|R_\Omega((E_0 h)(\cdot - z_m))\|_{X(\Omega)} \\ &\le \lim_{k\to\infty} \frac{1}{2^{nk}} \sum |1_B(z_m)| \|h\|_{X(\Omega)} = |B| \|h\|_{X(\Omega)}\end{aligned}$$

for all $h \in C_0(\Omega)$. This proves Lemma 4.1. □

**Lemma 4.2.** *Let $\Omega$ satisfy Assumption 1 and $X(\Omega)$ satisfy the assumption in Lemma 4.1. Then there exist positive constants $\epsilon_0 = \epsilon_0(\Omega)$ and $C = C(\Omega)$ such that*

$$\|R_\Omega(m_{2t} * m_t * |E_0 f|)\|_{X(\Omega)} \le C\|f\|_{X(\Omega)} \tag{4.12}$$

*for all $f \in C_0(\Omega)$ and all $0 < t < \epsilon_0$, where $m_t(x) := \frac{1}{|B(0,t)|} 1_{B(0,t)}(x)$.*

*Proof of Lemma 4.2.* We first prove Lemma 4.2 in the case where $\Omega$ is unbounded. Let

$$\Omega_\epsilon := \{x \in \Omega\,;\ \mathrm{dist}(x, \partial\Omega) < \epsilon\} \quad \text{and} \quad \Omega^\epsilon := \Omega \backslash \Omega_\epsilon.$$

Since $\Omega$ is unbounded, by Assumption 1, there is $y_0 \in \mathbb{R}^n$ such that

$$\Omega_1 + y_0 \subset \Omega \quad \text{and} \quad \mathrm{dist}(\Omega_1 + y_0, \partial\Omega) > 1.$$

Let $\varphi_1, \varphi_2 \in C(\bar{\Omega})$ satisfy

$$\begin{aligned}&\varphi_1 + \varphi_2 = 1 \text{ in } \Omega, \quad 0 \le \varphi_1, \varphi_2 \le 1 \\ &\varphi_1 = 1 \text{ in } \Omega_{1/2}, \quad \varphi_1 = 0 \text{ in } \Omega^1.\end{aligned} \tag{4.13}$$

For $f \in C_0(\Omega)$ we decompose $|E_0 f|$ into two functions as follows.

$$|E_0 f| = E_0(|f|\varphi_1) + E_0(|f|\varphi_2).$$

Let $0 < t < 1/3$. Since $\operatorname{dist}(\Omega_1 + y_0, \partial\Omega) > 1$ implies $\tau_{y_0}(m_{2t} * m_t * E_0(|f|\varphi_1)) \in C_0(\Omega)$, by Condition (A2) and Lemma 4.1 we have

$$\begin{aligned}
&\|R_\Omega(m_{2t} * m_t * E_0(|f|\varphi_1))\|_{X(\Omega)} \\
&\quad = \|R_\Omega \tau_{-y_0}[\tau_{y_0}(m_{2t} * m_t * E_0(|f|\varphi_1))]\|_{X(\Omega)} \\
&\quad \le \|\tau_{y_0}(m_{2t} * m_t * E_0(|f|\varphi_1)\|_{X(\Omega)} \\
&\quad = \|m_{2t} * m_t * \tau_{y_0} E_0(|f|\varphi_1)\|_{X(\Omega)} \\
&\quad = \|R_\Omega\Big\{m_{2t} * E_0 R_\Omega\Big(m_t * E_0 R_\Omega \tau_{y_0} E_0(|f|\varphi_1)\Big)\Big\}\|_{X(\Omega)} \\
&\quad \le \|R_\Omega \tau_{y_0}(E_0(|f|\varphi_1))\|_{X(\Omega)}.
\end{aligned} \tag{4.14}$$

Since $E_0(|f|\varphi_1) \in C_0(\Omega)$, Conditions (A2)–(A3) yield $\|R_\Omega \tau_{y_0}(E_0(|f|\varphi_1))\|_{X(\Omega)} \le \||f|\varphi_1\|_{X(\Omega)} \le \|f\|_{X(\Omega)}$. Hence, by (4.14) we have

$$\|R_\Omega(m_{2t} * m_t * E_0(|f|\varphi_1))\|_{X(\Omega)} \le \|f\|_{X(\Omega)}. \tag{4.15}$$

Similarly, since $\operatorname{supp} \varphi_2 \subset \Omega^{1/2}$ and $\operatorname{dist}(\Omega^{1/2}, \partial\Omega) = 1/2$, by Lemma 4.1 and Condition (A3) we have, for $0 < t < \frac{1}{6}$,

$$\begin{aligned}
&\|R_\Omega(m_{2t} * m_t * E_0(|f|\varphi_2))\|_{X(\Omega)} \\
&\quad = \|R_\Omega\left\{m_{2t} * E_0 R_\Omega\big(m_t * E_0(|f|\varphi_2)\big)\right\}\|_{X(\Omega)} \\
&\quad \le \||f|\varphi_2\|_{X(\Omega)} \le \|f\|_{X(\Omega)}.
\end{aligned} \tag{4.16}$$

From (4.15) and (4.16) we obtain the desired estimate (4.12) for $0 < t < \frac{1}{6}$.

Next, we prove (4.12) in the case where $\Omega$ is bounded. We choose $r_0 > 0$ and $z_0 \in \Omega$ such that $B(z_0, 2r_0) \subset \Omega$. Now we consider partitions of unity. Since $\bar{\Omega}$ is bounded, there are a finite collection of open sets $\{U_j\}_{j=1}^N$ and smooth functions $\{\varphi_j\}_{j=1}^N$ such that

$$\begin{gathered}
\Omega \subset \bigcup_{j=1}^N U_j, \quad \operatorname{diam} U_j < r_0, \quad \sum_{j=1}^N \varphi_j = 1 \quad \text{in} \quad \Omega \\
\operatorname{supp} \varphi_j \subset U_j, \quad 0 \le \varphi_j \le 1 \quad \text{for} \quad j = 1, 2, \ldots, N.
\end{gathered}$$

Here $\operatorname{diam} A := \sup_{x,y\in A} |x - y|$. Let $f \in C_0(\Omega)$. Since $|E_0 f(x)| = \sum_{j=1}^N E_0(|f|\varphi_j)(x)$, we have

$$\|R_\Omega(m_{2t} * m_t * |E_0 f|)\|_{X(\Omega)} \le \sum_{j=1}^N \|R_\Omega\big(m_{2t} * m_t * E_0(|f|\varphi_j)\big)\|_{X(\Omega)}. \tag{4.17}$$

Since $\operatorname{diam} U_j < r_0$, for each $j = 1, 2, \ldots N$, there exists $y_j \in \mathbb{R}^n$ such that

$$U_j + y_j \subset B(z_0, r_0)(\subset B(z_0, 2r_0) \subset \Omega).$$

Let $0 < t < \frac{r_0}{3}$. Then, since $\operatorname{dist}(U_j + y_j, \partial\Omega) \ge r_0$, we see that $\tau_{y_j}\big(m_{2t} * m_t * E_0(|f|\varphi_j)\big) \in C_0(\Omega)$. Hence, in the same way as in the proof of (4.14) from Condition (A2) and Lemma 4.1, we obtain

$$\|R_\Omega\big(m_{2t} * m_t * E_0(|f|\varphi_j)\big)\|_{X(\Omega)} \le \|R_\Omega \tau_{y_j} E_0(|f|\varphi_j)\|_{X(\Omega)}. \tag{4.18}$$

Since $f \in C_0(\Omega)$ implies $E_0(|f|\varphi_j) \in C_0(\Omega)$, Conditions (A2)-(A3) yield

$$\|R_\Omega \tau_{y_j} E_0(|f|\varphi_j)\|_{X(\Omega)} \le \|E_0(|f|\varphi_j)\|_{X(\Omega)} \le \|f\|_{X(\Omega)}.$$

Hence, by (4.18),

$$\|R_\Omega\big(m_{2t} * m_t * E_0(|f|\varphi_j)\big)\|_{X(\Omega)} \le \|f\|_{X(\Omega)}. \tag{4.19}$$

This estimate and (4.17) yield the desired estimate (4.12) for all $0 < t < r_0/3$, which proves Lemma 4.2. □

We are now in the position to prove the second part of Theorem 1.

*Proof of Theorem 1 (b).* By Condition (A4) and Proposition 3.1 we have

$$\|g\|_{L^\infty(\Omega)} \le C\Big(\epsilon\|g\|_{\dot{C}^\alpha(\Omega)} + \log^\beta\big(e + \frac{1}{\epsilon}\big)\|g\|_{X(\Omega)}\Big) \tag{4.20}$$

for all $g \in X(\Omega) \cap \dot{C}^\alpha(\Omega)$ and all $\epsilon > 0$. Let $f \in C_0(\Omega)$, $0 < t < \epsilon_0$ where $\epsilon_0$ is the constant given in Lemma 4.2, and let $h$ be a function on $\mathbb{R}^n$ defined by

$$h(x) := (m_{2t} * m_t * |E_0 f|)(x) = \fint_{B(x,2t)} \left(\fint_{B(y,t)} |E_0 f|(z)dz\right) dy$$

for $x \in \mathbb{R}^n$. From Proposition 4.2 we see that $R_\Omega h \in \dot{C}^\alpha \cap C_0(\overline{\Omega})(\subset \dot{C}^\alpha \cap X(\Omega))$. Substituting $g = R_\Omega h$ into (4.20), from Propositions 4.2 and 4.3, and Lemma 4.2 we obtain

$$\begin{aligned}
\|R_\Omega h\|_{L^\infty(\Omega)} &\le C\Big(\epsilon\|R_\Omega h\|_{\dot{C}^\alpha(\Omega)} + \log^\beta\big(e + \frac{1}{\epsilon}\big)\|R_\Omega h\|_{X(\Omega)}\Big)\\
&\le C\Big(\epsilon\|h\|_{\dot{C}^\alpha(\mathbb{R}^n)} + \log^\beta\big(e + \frac{1}{\epsilon}\big)\|R_\Omega h\|_{X(\Omega)}\Big)\\
&\le C\Big(\epsilon(2t)^{-\alpha}\|m_t * |E_0 f|\|_{L^\infty(\mathbb{R}^n)} + \log^\beta\big(e + \frac{1}{\epsilon}\big)\|f\|_{X(\Omega)}\Big)\\
&\le C\Big(\epsilon(2t)^{-\alpha}\|R_\Omega(m_t * |E_0 f|)\|_{L^\infty(\Omega)} + \log^\beta\big(e + \frac{1}{\epsilon}\big)\|f\|_{X(\Omega)}\Big) \qquad (4.21)
\end{aligned}$$

for all $\epsilon > 0$. By taking $L^\infty(\Omega)$-norm of the both sides of inequality (4.3) with $f$ replaced by $|E_0 f|$, we have

$$\|R_\Omega(m_t * |E_0 f|)\|_{L^\infty(\Omega)} \le 2^n\|R_\Omega(m_{2t} * m_t * |E_0 f|)\|_{L^\infty(\Omega)} = 2^n\|R_\Omega h\|_{L^\infty(\Omega)}. \tag{4.22}$$

Substituting $\epsilon := (2t)^\alpha/(2^{n+1}C)$ into (4.21), from (4.22) we observe that

$$\|R_\Omega(m_t * |E_0 f|)\|_{L^\infty(\Omega)} \le C_1 \log^\beta\big(e + \frac{1}{t}\big)\|f\|_{X(\Omega)}$$

holds for all $0 < t < \epsilon_0$ and all $f \in C_0(\Omega)$, where the constant $C_1$ is independent of $t$ and $f$. This implies $\|f\|_{M^{\log}_{\beta,\epsilon_0}(\Omega)} \le C_1\|f\|_{X(\Omega)}$ for all $f \in C_0(\Omega)$. Therefore, since $M^{\log}_{\beta,\epsilon_0}(\Omega)$-norm is equivalent to $M^{\log}_\beta(\Omega)$-norm, we conclude that

$$\|f\|_{M^{\log}_\beta(\Omega)} \le C\|f\|_{X(\Omega)} \quad \text{for all } f \in C_0(\Omega).$$

Now, we consider the case where $f \in BC(\bar{\Omega}) \cap X(\Omega)$. Clearly, there exists a sequence $\{f_n\}$ such that

$$f_n \in C_0(\Omega), \quad 0 \le f_n(x) \le |f(x)| \ \ a.e.\ x \in \Omega \ \text{ and } \ f_n(x) \to |f(x)| \text{ a.e. } x \in \Omega.$$

By the definition of $M_\beta^{\log}$-norm, we see that for arbitrary $\epsilon > 0$, there are $x \in \Omega$ and $t \in (0, 1)$ such that

$$\|f\|_{M_\beta^{\log}(\Omega)} - \epsilon < \frac{1}{|B(x,t)| \log^\beta(e + 1/t)} \int_{B(x,t)} |E_0 f(y)| dy.$$

Since Lebesgue's theorem yields $\lim_{n\to\infty} \int_{B(x,t)} |E_0 f_n(y)| dy = \int_{B(x,t)} |E_0 f(y)| dy$, there is $n_0$ such that

$$\|f\|_{M_\beta^{\log}(\Omega)} - \epsilon < \frac{1}{|B(x,t)| \log^\beta(e + 1/t)} \int_{B(x,t)} |E_0 f_{n_0}(y)| dy \le \|f_{n_0}\|_{M_\beta^{\log}(\Omega)}.$$

As shown above, it holds that $\|f_{n_0}\|_{M_\beta^{\log}(\Omega)} \le C\|f_{n_0}\|_{X(\Omega)} \le C\|f\|_{X(\Omega)}$, since $f_{n_0} \in C_0(\Omega)$ and since $|f_n(x)| \le |f(x)|$ a.e. $x \in \Omega$. Therefore, we get

$$\|f\|_{M_\beta^{\log}(\Omega)} \le C\|f\|_{X(\Omega)} \quad \text{for all } f \in BC(\bar{\Omega}) \cap X(\Omega).$$

This proves Theorem 1(b). □

## 5. Proof of Theorem 2

*Proof of Theorem 2.* (a) We now recall the Littlewood–Paley decomposition. Let $\psi$ be the function given in Definition 2 and let $\phi_j \in \mathcal{S}$ be the functions defined by

$$\hat{\phi}(\xi) := \hat{\psi}(\xi) - \hat{\psi}(2\xi) \quad \text{and} \quad \hat{\phi}_j(\xi) := \hat{\phi}(\xi/2^j) \quad \text{for } \xi \in \mathbb{R}^n.$$

Then, $\operatorname{supp} \hat{\phi}_j \subset \{2^{j-1} \le |\xi| \le 2^{j+1}\}$ and

$$1 = \hat{\psi}(\xi/2^N) + \sum_{j=N+1}^{\infty} \hat{\phi}(\xi/2^j) = \hat{\psi}_N(\xi) + \sum_{j=N+1}^{\infty} \hat{\phi}_j(\xi) \quad \text{for } \xi \in \mathbb{R}^n, N = 1, 2, \ldots. \tag{5.1}$$

Using (5.1), we decompose $f$ into two parts such that

$$f(x) = \psi_N * f(x) + \sum_{j=N+1}^{\infty} \phi_j * f(x). \tag{5.2}$$

By Definition 2,

$$\|\psi_N * f\|_\infty \le N^\beta \|f\|_{V_\beta} \tag{5.3}$$

holds. Since $\dot{B}^\alpha_{\infty,\infty}(\mathbb{R}^n) = \dot{C}^\alpha(\mathbb{R}^n)$ for $0 < \alpha < 1$, we have

$$\begin{aligned}\sum_{j=N+1}^{\infty} \|\phi_j * f\|_\infty &= \sum_{j=N+1}^{\infty} 2^{\alpha j} \|\phi_j * f\|_\infty 2^{-\alpha j} \\ &\le \|f\|_{\dot{B}^\alpha_{\infty,\infty}} \sum_{j=N+1}^{\infty} 2^{-\alpha j} \le C\|f\|_{\dot{C}^\alpha} 2^{-\alpha N}.\end{aligned} \tag{5.4}$$

Gathering (5.3) and (5.4) with (5.2), we obtain

$$\|f\|_\infty \leq C(2^{-\alpha N}\|f\|_{\dot{C}^\alpha} + N^\beta \|f\|_{V_\beta}). \tag{5.5}$$

Now we take $N = \left[\frac{\log(\|f\|_{\dot{C}^\alpha}+e)}{\alpha \log 2}\right] + 1$, where $[\cdot]$ denotes the Gauss symbol. Then we have the desired estimate (3.2)

(b) Now we prove the second part of Theorem 2. By Proposition 3.1, we see that Condition (B4) is equivalent to the inequality:

$$\|f\|_\infty \leq C(\epsilon\|f\|_{\dot{C}^\alpha} + \left(\log(e + \frac{1}{\epsilon})\right)^\beta \|f\|_X) \tag{5.6}$$

for all $f \in BC^\infty$ and all $\epsilon > 0$.

Let $g \in BUC(\mathbb{R}^n)$. Then, $\psi_N * g \in BC^\infty$. Substituting $f = \psi_N * g$ into (5.6), we obtain

$$\|\psi_N * g\|_\infty \leq C\epsilon\|\psi_N * g\|_{\dot{C}^\alpha} + C\Big(\log(e + \frac{1}{\epsilon})\Big)^\beta \|\psi_N * g\|_X. \tag{5.7}$$

Since $\psi_N * f = \psi_{N+1} * \psi_N * f$, we have

$$\begin{aligned}\|\psi_N * g\|_{\dot{C}^\alpha} &= \|\psi_{N+1} * \psi_N * g\|_{\dot{C}^\alpha} \leq C\|\psi_{N+1} * \psi_N * g\|_{\dot{B}^\alpha_{\infty,\infty}} \\ &= C \sup_j 2^{\alpha j}\|\phi_j * \psi_{N+1} * \psi_N * g\|_\infty \\ &\leq C \sup_{j \leq N+2} 2^{\alpha j}\|\psi_N * g\|_\infty \leq C2^{\alpha N}\|\psi_N * g\|_\infty. \end{aligned} \tag{5.8}$$

Concerning the second term on the right-hand side of (5.7), Condition (B3) yields

$$\begin{aligned}\|\psi_N * g\|_X &\leq \int_{\mathbb{R}^n} \|\psi_N(y)g(\cdot - y)\|_X dy = \int_{\mathbb{R}^n} |\psi_N(y)|\|g\|_X dy \\ &= \|\psi_N\|_{L^1}\|g\|_X = C\|g\|_X, \end{aligned} \tag{5.9}$$

where the constant $C(= \|\psi\|_{L^1})$ is independent of $N$. (More precisely, the above estimate is justified by the fact that the Riemann sum $\frac{1}{2^{nk}}\sum_{z_m \in (\mathbb{Z}/2^k)^n} \psi_N(z_m)g(x - z_m)$ converges to $\psi_N * g(x)$ in $BUC(\mathbb{R}^n)(\hookrightarrow X)$ as $k \to \infty$.) Thus, gathering (5.8) and (5.9) with (5.7) we obtain

$$\|\psi_N * g\|_\infty \leq C\epsilon 2^{\alpha N}\|\psi_N * g\|_\infty + C\Big(\log(e + \frac{1}{\epsilon})\Big)^\beta \|g\|_X$$

for all $N = 1, 2, \ldots$ and all $\epsilon > 0$. Letting $\epsilon = \frac{1}{2C2^{\alpha N}}$, from the above inequality we get

$$\|\psi_N * g\|_\infty \leq C_0\ (N+1)^\beta \|g\|_X \quad \text{for all } N = 1, 2, \ldots,$$

where the constant $C_0 > 0$ is independent of $N$ and $g$. This implies

$$\|g\|_{V_\beta} \leq C_0\|g\|_X$$

for all $g \in BUC(\mathbb{R}^n)$.

**(c)** Next, we prove the third part of Theorem 2. From (B2)–(B3), we can observe that

$$\psi_N * g \in L^\infty \quad \text{for all } g \in X. \tag{5.10}$$

Indeed, if $X \subset L^1_{uloc}(\mathbb{R}^n)$, (5.10) clearly holds. If $X \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, by the standard contradiction argument, we easily see that for all $\phi \in \mathcal{S}$ there exists a constant $K_\phi > 0$ such that

$$| < g, \phi > | \leq K_\phi \|g\|_X \quad \text{for all } g \in X.$$

See e.g. the proof of Proposition 1.4 in [40, Chap.3]. Since $\psi_N$ is spherical symmetric, we have $\psi_N * g(x) =< g, \tau_x \psi_N >=< \tau_{-x} g, \psi_N >$. Thus, from Condition (B3), we obtain

$$|\psi_N * g(x)| \leq K_{\psi_N} \|\tau_{-x} g\|_X = K_{\psi_N} \|g\|_X \quad \text{for all } x \in \mathbb{R}^n,$$

which implies (5.10). Then, since $\psi_N * g = \psi_{N+1} * \psi_N * g$, we have $\psi_N * g \in BC^\infty$. By Condition $(C)$ we have $\|\psi_N * g\|_X \leq \|\psi_N\|_{L^1} \|g\|_X = \|\psi\|_{L^1} \|g\|_X$. Therefore, by using the same argument as in the proof of the second part of Theorem 2, we get

$$\|g\|_{V_\beta} \leq C_0 \|g\|_X$$

for all $g \in X$, which means $X \hookrightarrow V_\beta$. □

## 6. Proof of Theorem 3

*Proof of Theorem 3.* For the sake of simplicity, we prove Theorem 3 only in the case

$$p > 3.$$

Since $u \in C((0, T); D(A_p))$, without loss of generality, we may assume that $u_0 \in D(A_p)$. It is well-known that the local existence time $T_*$ of strong $L^p$ solutions can be estimated from below as

$$T_* > C(p, \Omega)/\|u_0\|_p{}^{2p/(p-3)},$$

see e.g. [17]. Hence, if $\sup_{0 \leq \tau < T} \|u(\tau)\|_p < \infty$, then $u$ can be continued to the solution in the class $S_p(0, T')$ for some $T' > T$. Therefore, in order to prove Theorem 3, it suffices to show that

$$\sup_{0 \leq \tau \leq t} \|u(\tau)\|_p \leq C \|u_0\|_p \exp\left( C \exp C \int_0^t \|\omega(\tau)\|_{M_1^{\log}} d\tau \right) \tag{6.1}$$

for all $0 < t < T$ with some constant $C = C(u_0, p, \Omega, T) > 0$ which is independent of $t$. Recall that $u \cdot \nabla u = \omega \times u + \frac{1}{2} \nabla |u|^2$ and hence $P(u \cdot \nabla u) = P(\omega \times u)$. Then, $u$ satisfies the following integral equation:

$$(I.E.) \quad u(t) = e^{-tA} u_0 - \int_0^t e^{-(t-s)A} P(\omega \times u)(s) ds \quad \text{for all } 0 < t < T.$$

By (I.E.) we have

$$\|u(t)\|_p \leq C \|u_0\|_p + C \int_0^t \|\omega(s)\|_\infty \|u(s)\|_p ds,$$

which yields

$$\sup_{0 \leq s \leq t} \|u(s)\|_p \leq C \|u_0\|_p \exp\left( C \int_0^t \|\omega(\tau)\|_\infty d\tau \right) \tag{6.2}$$

for all $0 < t < T$. In order to estimate $\int_0^t \|\omega(\tau)\|_\infty d\tau$, we use the Brezis–Gallouet–Wainger type inequality (3.1). Let $0 < \alpha < 1 - 3/p$. Substituting $f = \frac{\omega(s)}{\epsilon\|\omega(s)\|_{\dot{C}^\alpha}}$ into (3.1) with $\beta = 1$, we obtain

$$\begin{aligned} \|\omega(s)\|_\infty &\le C\Big(\epsilon\|\omega(s)\|_{\dot{C}^\alpha(\Omega)} + \log(e + \frac{1}{\epsilon})\|\omega(s)\|_{M_1^{\log}}\Big) \\ &\le C\Big(\epsilon\|u(s)\|_{C^{1+\alpha}(\Omega)} + \log(e + \frac{1}{\epsilon})\|\omega(s)\|_{M_1^{\log}}\Big) \end{aligned} \tag{6.3}$$

for all $\epsilon > 0$, where $C$ is a constant independent of $s$ and $\epsilon$. Let

$$h(t) := \sup_{0\le\tau\le t} \|u(\tau)\|_p,$$
$$g(t) := \int_0^t \|\omega(\tau)\|_\infty d\tau$$

for $0 < t < T$. Then, from (6.3), for any positive function $\epsilon(s)$ on $(0, T)$ we see that

$$\begin{aligned} g(t) \;&\le\; C\int_0^t \epsilon(s)\|u(s)\|_{C^{1+\alpha}(\Omega)}ds + C\int_0^t \log(e + \frac{1}{\epsilon(s)})\|\omega(s)\|_{M_1^{\log}}ds \\ &=: I_1(t) + I_2(t). \end{aligned} \tag{6.4}$$

Let $\theta = 1 - \frac{3}{p} - \alpha$. Since it holds that

$$\begin{aligned} \|e^{-tA}f\|_{C^{1+\alpha}(\Omega)} &\le C\|e^{-tA}f\|^{\theta}_{C^{1-3/p}(\Omega)}\|e^{-tA}f\|^{1-\theta}_{C^{2-3/p}(\Omega)} \\ &\le C\|e^{-tA}f\|^{\theta}_{W^{1,p}(\Omega)}\|e^{-tA}f\|^{1-\theta}_{W^{2,p}(\Omega)} \\ &\le C\|e^{-tA}f\|^{\theta/2}_{L^p(\Omega)}\|e^{-tA}f\|^{\theta/2}_{W^{2,p}(\Omega)}\|e^{-tA}f\|^{1-\theta}_{W^{2,p}(\Omega)} \\ &\le C\|e^{-tA}f\|^{\theta/2}_{L^p(\Omega)}\|(1+A)e^{-tA}f\|^{1-\theta/2}_{L^p(\Omega)} \le C(1 + t^{-\frac{1+\alpha}{2}-\frac{3}{2p}})\|f\|_p, \end{aligned}$$

from (I.E.) we obtain

$$\begin{aligned} \|u(s)\|_{C^{1+\alpha}(\Omega)} &\le \|e^{-sA}u_0\|_{C^{1+\alpha}(\Omega)} + C\int_0^s \|e^{-(s-\tau)A}P(\omega\times u)(\tau)\|_{C^{1+\alpha}(\Omega)}d\tau \\ &\le C\|e^{-sA}u_0\|_{D(A_p)} + C\int_0^s (1 + (s-\tau)^{-\frac{1+\alpha}{2}-\frac{3}{2p}})\|\omega\times u(\tau)\|_p d\tau \\ &\le C\|u_0\|_{D(A_p)} + Ch(s)\int_0^s (1 + (s-\tau)^{-\frac{1+\alpha}{2}-\frac{3}{2p}})\|\omega(\tau)\|_\infty d\tau. \end{aligned} \tag{6.5}$$

Hence, for $0 < t < T$ we have

$$\begin{aligned} I_1(t) &\le CT\big(\sup_{0<s<T} \epsilon(s)\big)\|u_0\|_{D(A_p)} \\ &\quad + C\int_0^t \epsilon(s)h(s)\int_0^s (1 + (s-\tau)^{-\frac{1+\alpha}{2}-\frac{3}{2p}})\|\omega(\tau)\|_\infty d\tau\, ds. \end{aligned} \tag{6.6}$$

We now choose $\epsilon(s)$ such that

$$\epsilon(s) := \frac{\delta}{Ch(s) + 1},$$

where $\delta > 0$ is a sufficiently small constant. Then, since $-\frac{1+\alpha}{2} - \frac{3}{2p} > -1$, by Fubini's Theorem we have

$$\begin{aligned} I_1(t) &\leq CT\delta\|u_0\|_{D(A_p)} + \delta\int_0^t\int_0^s (1+(s-\tau)^{-\frac{1+\alpha}{2}-\frac{3}{2p}})\|\omega(\tau)\|_\infty d\tau\, ds \\ &\leq CT\delta\|u_0\|_{D(A_p)} + \delta\int_0^t\Big(\int_\tau^t (1+(s-\tau)^{-\frac{1+\alpha}{2}-\frac{3}{2p}})ds\Big)\|\omega(\tau)\|_\infty d\tau \\ &\leq CT\delta\|u_0\|_{D(A_p)} + C_1(T)\delta\int_0^t \|\omega(\tau)\|_\infty d\tau. \end{aligned} \tag{6.7}$$

Since (6.2) yields $h(s) + e \leq C(\|u_0\|_p + e)\exp(Cg(s))$, we see

$$\begin{aligned} \log(e + \frac{1}{\epsilon(s)}) &= \log(e + \frac{Ch(s)+1}{\delta}) \\ &\leq \log(1 + \frac{C+1}{\delta}) + \log(h(s)+e) \leq C(\|u_0\|_p, \delta)\big(1+g(s)\big). \end{aligned}$$

Hence, we have

$$I_2(t) \leq C(\|u_0\|_p, \delta)\int_0^t \|\omega(s)\|_{M_1^{\log}}\big(1+g(s)\big)ds. \tag{6.8}$$

Gathering (6.7) and (6.8) with (6.4), we obtain

$$g(t) \leq CT\delta\|u_0\|_{D(A_p)} + C_1(T)\delta g(t) + C(\|u_0\|_p, \delta)\int_0^t \|\omega(s)\|_{M_1^{\log}}\big(1+g(s)\big)ds.$$

Therefore, letting $\delta = 1/(2C_1(T))$, by the Gronwall lemma, we get

$$g(t) \leq C(\|u_0\|_{D(A_p)}, T, \|u_0\|_p)\exp\Big(C\int_0^t \|\omega(s)\|_{M_1^{\log}} ds\Big)$$

for all $0 < t < T$. This estimate and (6.2) yield the desired estimate (6.1). □

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## 7. Appendix

*7.1. Relation between bmo and $M_1^{\log}$.* In this section, we consider the inclusion between $bmo(\Omega)$ and $M_1^{\log}(\Omega)$. We first recall the space $Y_{uloc}$

$$Y_{uloc} := \{f \in L^1_{uloc}(\mathbb{R}^n);\ \|f\|_{Y_{uloc}} := \sup_{p\geq 1}\frac{\|f\|_{L^p_{uloc}(\mathbb{R}^n)}}{p} < \infty\},$$

which is a modified version of Yudovich's space, see [45]. We will show

$$bmo(\mathbb{R}^n) \hookrightarrow Y_{uloc}(\mathbb{R}^n) \hookrightarrow M_1^{\log}(\mathbb{R}^n). \tag{7.1}$$

*Proof of (7.1).* It is known that

$$\|f\|_{L^p(B(x,1))} \leq C \cdot p \cdot \|f\|_{bmo(\mathbb{R}^n)}$$

for all $f \in bmo(\mathbb{R}^n)$, $p \in [1, \infty)$ and all $x \in \mathbb{R}^n$, where the constant $C$ depends only on $n$, see Stein [39, Ch.IV, Sect.1.3]. This estimate yields $\|f\|_{Y_{uloc}} \leq C\|f\|_{bmo(\mathbb{R}^n)}$ and $bmo(\mathbb{R}^n) \hookrightarrow Y_{uloc}(\mathbb{R}^n)$. Using a similar argument to that in [35], we can observe that $Y_{uloc} \hookrightarrow M_1^{\log}(\mathbb{R}^n)$. Indeed, for all $0 < t < 1$ and all $p \geq 1$, we see

$$\begin{aligned}\fint_{B(x,t)} |f(y)|dy &\leq |B(x,t)|^{-1/p}\|f\|_{L^p(B(x,1))} \leq Ct^{-n/p} \cdot p \cdot \|f\|_{Y_{uloc}} \\ &\leq Ce^{\frac{n}{p}\log(1/t)} \cdot p \cdot \|f\|_{Y_{uloc}}.\end{aligned}$$

Thus, letting $p = \log(e + 1/t)$, we have

$$\fint_{B(x,t)} |f(y)|dy \leq C\log(e + 1/t)\|f\|_{Y_{uloc}}$$

for all $0 < t < 1$, which implies $\|f\|_{M_1^{\log}(\mathbb{R}^n)} \leq C\|f\|_{Y_{uloc}}$ and $Y_{uloc} \hookrightarrow M_1^{\log}(\mathbb{R}^n)$. □

Since $\|f\|_{M_1^{\log}(\Omega)} \leq \|\tilde{f}\|_{M_1^{\log}(\mathbb{R}^n)} \leq C\|\tilde{f}\|_{bmo(\mathbb{R}^n)}$ for any extension $\tilde{f}$ of $f$ from $\Omega$ to $\mathbb{R}^n$, we also have

$$\|f\|_{M_1^{\log}(\Omega)} \leq C\|f\|_{bmo(\Omega)}, \quad i.e., \quad bmo(\Omega) \hookrightarrow M_1^{\log}(\Omega).$$

*7.2. Relation between the regularity conditions (1.6) and (3.4).* As mentioned in Introduction and Remark 3 (iii), both of (1.6) and (3.4) guarantee the regularity of weak solutions. Here we will show that there are no inclusion between (1.6) and (3.4) in the following sense. There exists a function in the energy class which satisfies (3.4), but not (1.6). On the other hand, there also exists a function in the energy class which satisfies (1.6), but not (3.4). More precisely, we have:

**Lemma 7.1.** (a) *There exists a vector function $u_1$ on $\mathbb{R}^3 \times [-1/2, 0)$ such that*

$$u_1 \in L^\infty(-1/2, 0; L^2_\sigma(\mathbb{R}^3)) \cap L^2(-1/2, 0; W^{1,2}(\mathbb{R}^3)),$$

$$\int_{-1/2}^0 \|\omega_1(t)\|_{M_1^{\log}(\mathbb{R}^3)}\, dt < \infty, \tag{7.2}$$

$$\limsup_{r\to 0+} r^{-(3/p+2/q-2)}\left\{\int_{-r^2}^0 \left(\int_{B(0,r)} |\omega_1(y,t)|^p\, dy\right)^{q/p} dt\right\}^{1/q} = \infty \tag{7.3}$$

*for all $1 \leq p, q \leq \infty$. Here $\omega_1 = \mathrm{rot}\, u_1$.*

(b) *There exists a vector function $u_2$ on $\mathbb{R}^3 \times [-1/2, 0)$ such that*

$$u_2 \in L^\infty(-1/2, 0; L^2_\sigma(\mathbb{R}^3)) \cap L^2(-1/2, 0; W^{1,2}(\mathbb{R}^3)),$$

$$\int_{-1/2}^0 \|\omega_2(t)\|_{M_1^{\log}(B(0,\epsilon))}\, dt = \infty \quad \text{for all } \epsilon > 0, \tag{7.4}$$

$$\limsup_{r\to 0+} r^{-(3/p+2/q-2)}\left\{\int_{-r^2}^0 \left(\int_{B(0,r)} |\omega_2(y,t)|^p\, dy\right)^{q/p} dt\right\}^{1/q} = 0 \tag{7.5}$$

*for $p = q = 2$. Here $\omega_2 = \mathrm{rot}\, u_2$.*

*Proof.* (a) Clearly, there exists a function $\phi = (\phi^1, \phi^2, \phi^3) \in C^\infty_{0,\sigma}(\mathbb{R}^3)$ with $\phi^1 \geq 1$ in $B(0,1)$. Let $0 < \delta < 1$,

$$\omega_1(x,t) := \frac{\phi(x/|t|^{2/5})}{|t|(\log \frac{1}{|t|})^\delta} \quad \text{for } (x,t) \in \mathbb{R}^3 \times [-1/2, 0).$$

Fix $-1/2 < t < 0$. For $0 < r \leq |t|^{1/5}$, we have

$$\frac{1}{|B(x,r)|\log(e+1/r)} \int_{B(x,r)} |\omega_1(y,t)|dy \leq \frac{1}{\log(1/|t|^{1/5})}\|\omega_1\|_\infty \leq \frac{5\|\phi\|_\infty}{|t|(\log \frac{1}{|t|})^{1+\delta}}.$$

Let $|t|^{1/5} < r < 1$. Since $|t|^{3/5} \leq \frac{C}{\log \frac{1}{|t|}}$, we have

$$\begin{aligned}
\frac{1}{|B(x,r)|\log(e+1/r)} \int_{B(x,r)} |\omega_1(y,t)|dy &\leq \frac{1}{|B(x,r)|}\int_{\mathbb{R}^3} |\omega_1(y,t)|dy \\
&\leq \frac{|t|^{6/5}}{|B(x,r)|}\frac{\|\phi\|_{L^1}}{|t|(\log \frac{1}{|t|})^\delta} = C\frac{(|t|^{6/5}/r^3)}{|t|(\log \frac{1}{|t|})^\delta} \leq \frac{C|t|^{3/5}}{|t|(\log \frac{1}{|t|})^\delta} \leq C\frac{1}{|t|(\log \frac{1}{|t|})^{1+\delta}},
\end{aligned}$$

where $C$ is a constant independent of $t$. Hence, we obtain

$$\|\omega_1(t)\|_{M_1^{\log}(\mathbb{R}^3)} \leq C\frac{1}{|t|(\log \frac{1}{|t|})^{1+\delta}} \quad \text{for all } -1/2 < t < 0,$$

which implies (7.2).

Let $r > 0$ be sufficiently small and $1 \leq p \leq \infty$. Since $\|\phi(\cdot/|t|^{2/5})\|_{L^p(B(0,r))} \geq \|\phi^1(\cdot/|t|^{2/5})\|_{L^p(B(0,r))} \geq \|1_{B(0,|t|^{2/5})}\|_{L^p(B(0,r))}$, we have

$$\begin{aligned}
\|\omega_1(\cdot,t)\|_{L^p(B(0,r))} &\geq \frac{\|1_{B(0,|t|^{2/5})}\|_{L^p(B(0,r))}}{|t|(\log \frac{1}{|t|})^\delta} \\
&= C\begin{cases} \frac{1}{|t|^{1-6/(5p)}(\log \frac{1}{|t|})^\delta} & \text{for } -r^{5/2} \leq t < 0 \\ \frac{r^{3/p}}{|t|(\log \frac{1}{|t|})^\delta} & \text{for } -r^2 < t < -r^{5/2}. \end{cases}
\end{aligned}$$

Hence, for $1 < q < \infty$ we have

$$\begin{aligned}
&r^{-(3/p+2/q-2)}\left\{\int_{-r^2}^0 \|\omega_1(\cdot,t)\|^q_{L^p(B(0,r))}dt\right\}^{1/q} \\
&\geq r^{-(3/p+2/q-2)}\left\{\int_{-r^{5/2}}^{-\frac{r^{5/2}}{2}} \frac{1}{|t|^{q(1-6/(5p))}(\log \frac{1}{|t|})^{q\delta}}dt\right\}^{1/q} \\
&\geq \frac{C}{r^{\frac{1}{2}(1-\frac{1}{q})}(\log \frac{1}{r})^\delta} \to \infty \quad \text{as } r \to 0+.
\end{aligned}$$

In the case $q = 1$, we have

$$r^{-(3/p+2/1-2)}\left\{\int_{-r^2}^{0}\|\omega_1(\cdot,t)\|_{L^p(B(0,r))}dt\right\} \geq r^{-3/p}\int_{-r^2}^{-r^{5/2}}\frac{r^{3/p}}{|t|(\log\frac{1}{|t|})^{\delta}}dt$$

$$\geq \frac{C}{(\log\frac{1}{r^{5/2}})^{\delta}}\int_{-r^2}^{-r^{5/2}}\frac{1}{|t|}dt = C(\log\frac{1}{r})^{1-\delta} \to \infty \quad \text{as } r \to 0+,$$

since $1-\delta > 0$. In the case $q = \infty$, we have

$$r^{-(3/p-2)}\sup_{-r^2<t<0}\|\omega_1(\cdot,t)\|_{L^p(B(0,r))} \geq r^{-(3/p-2)}\sup_{-r^{5/2}<t<0}\frac{C}{|t|^{1-6/(5p)}(\log\frac{1}{|t|})^{\delta}} =: L(r).$$

For $6/5 < p \leq \infty$, clearly $L(r) = \infty$. For $1 \leq p \leq 6/5$, $L(r) = C\frac{r^{-(3/p-2)}}{r^{5/2-3/p}(\log\frac{1}{r^{5/2}})^{\delta}} = C/(r^{1/2}(\log(1/r))^{\delta}) \to \infty$ as $r \to 0+$. Therefore, for all $1 \leq p, q \leq \infty$, (7.3) holds. Since $\|\omega_1(\cdot,t)\|_{L^p(\mathbb{R}^3)} = \frac{\|\phi\|_{L^p(\mathbb{R}^3)}}{|t|^{1-6/(5p)}(\log\frac{1}{|t|})^{\delta}}$, it is straightforward to see that

$$\omega_1 \in L^{\infty}(-1/2,0;L^{6/5}_{\sigma}(\mathbb{R}^3)) \cap L^2(-1/2,0;L^2_{\sigma}(\mathbb{R}^3)).$$

Hence, letting $u_1 := \mathrm{rot}\,(-\Delta)^{-1}\omega_1$, we have $u_1 \in L^{\infty}(-1/2,0;L^2_{\sigma}(\mathbb{R}^3)) \cap L^2(-1/2,0;W^{1,2}(\mathbb{R}^3))$, which proves the assertion (a).

(b) Let

$$\omega_2(x,t) := \frac{\phi(x/|t|)}{|t|},$$

where $\phi$ is the same vector function as in the proof of (a). Then, for $r = |t|$,

$$\frac{\int_{B(0,r)}|\omega_2(y,t)|dy}{|B(0,r)|\log(e+1/r)} \geq \frac{\int_{B(0,|t|)}\frac{1}{|t|}dy}{|B(0,|t|)|\log(e+1/|t|)} = \frac{1}{|t|\log(e+1/|t|)}$$

which yields $\|\omega_2(\cdot,t)\|_{M_1^{\log}(B(0,\epsilon))} \geq \frac{1}{|t|\log(e+1/|t|)}$ for all $-\epsilon < t < 0$. Hence we obtain (7.4). Since $\|\omega_2(\cdot,t)\|_{L^p(\mathbb{R}^3)} = |t|^{3/p-1}\|\phi\|_{L^p(\mathbb{R}^3)}$, it is straightforward to see that $\omega_2$ satisfies (7.5) for $p = q = 2$ and belongs to $L^{\infty}(-1/2,0;L^{6/5}_{\sigma}(\mathbb{R}^3)) \cap L^2(-1/2,0;L^2_{\sigma}(\mathbb{R}^3))$. Hence, $u_2 := \mathrm{rot}\,(-\Delta)^{-1}\omega_2$ is the desired function. □

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