



BPS/CFT Correspondence III: Gauge Origami Partition Function and qq -Characters

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Received: 3 January 2017 / Accepted: 28 November 2017

Published online: 13 December 2017 – © Springer-Verlag GmbH Germany, part of Springer Nature 2017

Abstract: We study generalized gauge theories engineered by taking the low energy limit of the Dp branes wrapping $X \times \mathbf{T}^{p-3}$, with X a possibly singular surface in a Calabi–Yau fourfold Z . For toric Z and X the partition function can be computed by localization, making it a statistical mechanical model, called the gauge origami. The random variables are the ensembles of Young diagrams. The building block of the gauge origami is associated with a tetrahedron, whose edges are colored by vector spaces. We show the properly normalized partition function is an entire function of the Coulomb moduli, for generic values of the Ω -background parameters. The orbifold version of the theory defines the qq -character operators, with and without the surface defects. The analytic properties are the consequence of a relative compactness of the moduli spaces $\mathcal{M}(\vec{n}, k)$ of crossed and spiked instantons, demonstrated in “BPS/CFT correspondence II: instantons at crossroads, moduli and compactness theorem”.

Contents

1. Introduction	864
2. Review of Notations	864
2.1 Sets and partitions	864
2.2 Four and six	865
2.3 Finite groups and quiver varieties	866
2.4 The local model data	872
3. Partition Function of Spiked Instantons	874
3.1 The configuration space	875
3.2 The statistical weight	876
3.3 The perturbative prefactor	879
3.4 The main result	880
4. Orbi Folding	881
4.1 Abelian orbifolds	885

4.2 The **abelian** \times **ALE** case 888
 4.3 The **ALE** \times **ALE** case 890
 4.4 The main fact 892
 5. **Conclusions and Outlook** 892

1. Introduction

This paper is a continuation of the series [Ne2,Ne3]. There we proposed a set of observables in quiver $\mathcal{N} = 2$ supersymmetric gauge theories. These observables are useful in organizing the non-perturbative Dyson-Schwinger equations. The latter relate different instanton sectors contributions to the expectation values of gauge invariant chiral ring observables. We also introduced the geometric setting to which these observables belong in a natural way. Namely, we defined the moduli spaces $\mathfrak{M}_{X,G}$ of what might be called supersymmetric gauge fields in the generalized gauge theories, whose space-time X contains several, possibly intersecting, components:

$$X = \bigcup_A X_A. \tag{1}$$

The gauge groups $G|_{X_A} = G_A$ on different components may be different. The intersections $X_A \cup X_B$ lead to the matter fields charged under the product group $G_A \times G_B$ (bi-fundamental multiplets). In this paper we shall be studying the integrals over the moduli space $\mathfrak{M}_{X,G}$, which we shall compute using equivariant localization.

2. Review of Notations

2.1. Sets and partitions.

2.1.1. Sequences. For two sets X and S let $X^S = \text{Maps}(S, X)$ denote the set of maps from S to X . For a map $f : S \rightarrow X$ we sometimes use the notation

$$(x_s)_{s \in S}, \tag{2}$$

with $x_s = f(s) \in X$. For example, a sequence $(a_n), n \in \mathbb{N}$ would be denoted as $(a_n)_{n \in \mathbb{N}}$ or $(a_n)_{n \geq 1}$, if the context is clear.

2.1.2. Non-negative integers. are denoted by $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$.

2.1.3. Finite sets. Let $[n]$ denote the set $\{1, 2, \dots, n\}$ for $n \in \mathbb{N}$. For a finite set X we denote by $\#X$ the number of its elements. Thus, for finite X and S

$$\#X^S = (\#X)^{\#S} \tag{3}$$

2.1.4. Partitions. There are lots of sums over partitions in this paper. Let Λ denote the set of all partitions. An element $\lambda \in \Lambda$ is a non-increasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)} > \lambda_{\ell(\lambda)+1} = \lambda_{\ell(\lambda)+2} = \dots = 0)$ of integers, with a finite number of positive terms, sometimes called *the parts of λ* . The number $\ell(\lambda)$ of positive terms is called the *length of the partition λ* , the sum

$$\sum_{i=1}^{\ell(\lambda)} \lambda_i = |\lambda| \tag{4}$$

is called its *size*. We also identify the partitions λ with the finite subsets of $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, as follows:

$$\lambda = \{ \square \mid \square = (i, j), i, j \geq 1, 1 \leq j \leq \lambda_i \} \tag{5}$$

The size $|\lambda|$ of the partition λ is the number of elements $\#\lambda$ of the corresponding finite set. Not every finite subset of \mathbb{N}^2 corresponds to a partition, only those, for which the complement $\mathbb{N}^2 \setminus \lambda$ is preserved by the action of the semi-group $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ on \mathbb{N}^2 by translations. Equivalently, the partitions are in one-to-one correspondence with finite codimension monomial ideals in the ring of polynomials in two variables: $\lambda \leftrightarrow \mathcal{I}_\lambda$, $\mathcal{I}_\lambda \subset \mathbb{C}[x, y]$, $\mathcal{I}_\lambda = \cup_{i=1}^{\ell_\lambda+1} \mathbb{C}[x, y]x^{i-1}y^{\lambda_i}$.

We denote by $\Lambda[k]$ the set of partitions of k , i.e. the set of all $\lambda \in \Lambda$, such that $|\lambda| = k$. We have:

$$\Lambda = \bigsqcup_{k \geq 0} \Lambda[k] \tag{6}$$

The celebrated Euler formula:

$$\sum_{k=0}^{\infty} \#\Lambda[k] q^k = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} \tag{7}$$

2.2. *Four and six.* Let $\underline{\mathbf{4}}$ denote the set [4] and let $\underline{\mathbf{6}}$ denote the set of 2-element subsets of $\underline{\mathbf{4}}$ (we write ab instead of $\{a, b\}$ to avoid the clutter):

$$\underline{\mathbf{4}} = \{1, 2, 3, 4\}, \quad \underline{\mathbf{6}} = \binom{\underline{\mathbf{4}}}{2} = \{12, 13, 14, 23, 24, 34\} \tag{8}$$

In (8) we exhibit the lexicographic order of the sets $\underline{\mathbf{4}}$ and $\underline{\mathbf{6}}$ which is used below in some formulas. For example, $12 < 14 < 23 < 34$. For $A \in \underline{\mathbf{6}}$ we denote by $\bar{A} = \underline{\mathbf{4}} \setminus A$ its complement. Let $\underline{\mathbf{3}}$ denote the quotient $\underline{\mathbf{6}}/\sim$ where $A \sim \bar{A}$. Identify $\underline{\mathbf{3}} = \{1, 2, 3\} \subset \underline{\mathbf{4}}$ by choosing a representative $A = a4, a \in [3]$. Define the map: $\varphi : \underline{\mathbf{6}} \rightarrow \underline{\mathbf{4}}$ by

$$\varphi(A) = \inf \bar{A} \in \underline{\mathbf{4}}, \tag{9}$$

so that

$$\varphi(12) = 3, \quad \varphi(13) = \varphi(14) = 2, \quad \varphi(24) = \varphi(23) = \varphi(34) = 1. \tag{10}$$

We also define the following map $\varepsilon : \underline{\mathbf{6}} \rightarrow \mathbb{Z}_2$: write $A = \{a, b\}, a < b \in \underline{\mathbf{4}}$, write $\bar{A} = \{c, d\}, c < d \in \underline{\mathbf{4}}$, then $\varepsilon(A) = \varepsilon_{abcd}$. Thus,

$$\varepsilon(12) = \varepsilon(34) = \varepsilon(14) = \varepsilon(23) = +1, \quad \varepsilon(13) = \varepsilon(24) = -1. \tag{11}$$

It may seem surprising that ε takes values $+1$ four times and -1 only two times, but in fact it is natural, since $\varepsilon(A) = \varepsilon(\bar{A})$, therefore ε is defined on $\underline{\mathbf{3}}$. Since a two-valued function on a set of odd cardinality cannot split it equally, more classes are bound to be good rather than bad (assuming the values $+1$ and -1 are identified with “good” and “bad”).

It is useful to view $\underline{\mathbf{4}}$ as the set of faces (or vertices) of the tetrahedron, while $\underline{\mathbf{6}}$ is the set of edges. The edge ab connects the vertices a and b . Alternatively the edge ab is the common boundary of the faces a and b .

2.3. Finite groups and quiver varieties.

2.3.1. Abelian groups. We denote by Γ_{ab} a finite abelian group. It is well-known that any such Γ_{ab} is a product of cyclic groups whose orders are powers of primes:

$$\Gamma_{ab} = \prod_{\kappa=1}^d \left(\mathbb{Z} / p_{\kappa}^{l_{\kappa}} \mathbb{Z} \right), \quad l_{\kappa} \in \mathbb{N}, \quad p_{\kappa} - \text{primes} \tag{12}$$

An element of Γ_{ab} is a string $\mathbf{t} = (t_1, \dots, t_d)$ of integers defined modulo lattice $t_{\kappa} \sim t_{\kappa} + p_{\kappa}^{l_{\kappa}} \mathbb{Z}$. All irreducible representations $\mathcal{L}_{\mathbf{v}}$ of Γ_{ab} are complex one-dimensional, labeled by a string of integers

$$\mathbf{v} = \left(n^1, \dots, n^d \right) \in \Gamma_{ab}^{\vee}, \quad n^{\kappa} \in \mathbb{Z} \tag{13}$$

also defined modulo lattice $n^{\kappa} \sim n^{\kappa} + p_{\kappa}^{l_{\kappa}} \mathbb{Z}$:

$$T_{\mathcal{L}_{\mathbf{v}}}(\mathbf{t}) = \exp 2\pi \sqrt{-1} \sum_{\kappa} \frac{t_{\kappa} n^{\kappa}}{p_{\kappa}^{l_{\kappa}}} \tag{14}$$

We set $\mathbf{v} = \mathbf{0}$ to label the trivial representation with all $n^{\kappa} = 0$,

$$T_{\mathcal{L}_{\mathbf{0}}}(\mathbf{t}) \equiv 1 \tag{15}$$

The set Γ_{ab}^{\vee} is also an abelian group, isomorphic to Γ_{ab} , with multiplication given by the tensor product of irreducible representations. We shall be using the addition symbol for the group law on Γ_{ab}^{\vee} :

$$\mathcal{L}_{\mathbf{v}_1 + \mathbf{v}_2} = \mathcal{L}_{\mathbf{v}_1} \otimes \mathcal{L}_{\mathbf{v}_2}, \quad \mathcal{L}_{\mathbf{v}}^* = \mathcal{L}_{-\mathbf{v}} \tag{16}$$

Let

$$\delta_{\Gamma_{ab}^{\vee}} : \Gamma_{ab}^{\vee} \rightarrow \{0, 1\} \tag{17}$$

be the indicator function of the trivial representation:

$$\delta_{\Gamma_{ab}^{\vee}}(\mathbf{0}) = 1, \quad \delta_{\Gamma_{ab}^{\vee}}(\mathbf{v}) = 0, \quad \mathbf{v} \neq \mathbf{0} \tag{18}$$

2.3.2. Nonabelian subgroups of $SU(2)$. Let γ denote the affine Dynkin diagram of type D , or E , respectively (see the Fig. 1):

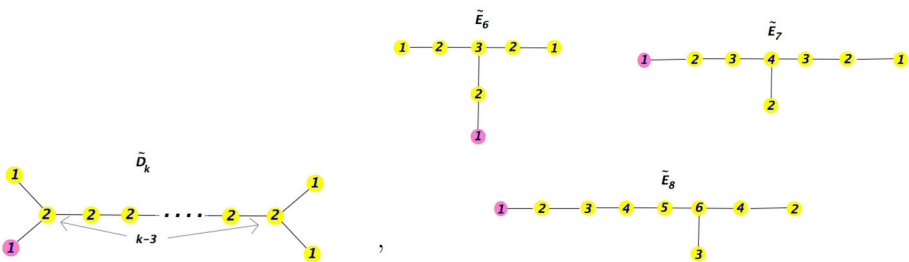


Fig. 1. Affine Dynkin diagrams of the D and E type

Let Vert_γ be the set of vertices of γ , Edge_γ be the set of oriented edges (we pick any orientation). For the edge $e \in \text{Edge}_\gamma$, let $s(e), t(e) \in \text{Vert}_\gamma$ denote its source and target, respectively.

Let $\Gamma_\gamma \subset SU(2)$ denote the corresponding non-abelian finite subgroup. For $\gamma = \tilde{E}_{6,7,8}$ the group Γ_γ is the binary tetrahedral, octahedral, icosahedral group, respectively.

In this correspondence $\mathbf{i} \in \text{Vert}_\gamma$ labels the irreducible representations $R_{\mathbf{i}} \in \Gamma_\gamma^\vee$ of Γ_γ . The edges Edge_γ show up in the tensor products: let $\mathbf{2}$ denote the defining two-dimensional representation of $SU(2)$. Then:

$$\mathbf{2} \otimes R_{\mathbf{i}} = \bigoplus_{e \in s^{-1}(\mathbf{i})} R_{t(e)} \oplus \bigoplus_{e \in t^{-1}(\mathbf{i})} R_{s(e)} \tag{19}$$

where $\mathbf{2}$ is viewed as the representation of $\Gamma_\gamma \subset SU(2)$. The dimensions $\dim R_{\mathbf{i}}$ are indicated on the corresponding nodes in the picture, the vector of dimensions is annihilated by the affine Cartan matrix = 2–incidence matrix of γ , cf. (19):

$$2\dim R_{\mathbf{i}} = \sum_{e \in s^{-1}(\mathbf{i})} \dim R_{t(e)} + \sum_{e \in t^{-1}(\mathbf{i})} \dim R_{s(e)} \tag{20}$$

The trivial representation is colored pink on Fig. 1.

2.3.3. Walks on quivers. Let $\mathbf{i}_s, \mathbf{i}_t \in \text{Vert}_\gamma$. A path p connecting \mathbf{i}_s (the source of p) to \mathbf{i}_t (the target of p) of length ℓ_p on the quiver γ is the ordered sequence of pairs $p_i = (e_i, \sigma_i), i = 1, \dots, \ell_p$ where $e_i \in \text{Edge}_\gamma, \sigma_i = \pm 1$, and

- (1) the source of p : if $\sigma_1 = 1$, then $s(e_1) = \mathbf{i}_s$, otherwise $t(e_1) = \mathbf{i}_s$
- (2) the end-point of p : if $\sigma_{\ell_p} = 1$, then $t(e_{\ell_p}) = \mathbf{i}_t$, otherwise $s(e_{\ell_p}) = \mathbf{i}_t$
- (3) concatenation: if $\sigma_i = 1, \sigma_{i+1} = 1$, then $t(e_i) = s(e_{i+1})$, if $\sigma_i = 1, \sigma_{i+1} = -1$, then $t(e_i) = t(e_{i+1})$, if $\sigma_i = -1, \sigma_{i+1} = 1$, then $s(e_i) = s(e_{i+1})$, if $\sigma_i = -1, \sigma_{i+1} = -1$, then $s(e_i) = t(e_{i+1})$

Let us denote the set of all paths on γ connecting \mathbf{i}_s to \mathbf{i}_t by $\mathcal{P}_{\mathbf{i}_s}^{\mathbf{i}_t}[\gamma]$. There is an obvious associative concatenation map:

$$\star : \mathcal{P}_{\mathbf{i}_1}^{\mathbf{i}_2}[\gamma] \times \mathcal{P}_{\mathbf{i}_2}^{\mathbf{i}_3}[\gamma] \longrightarrow \mathcal{P}_{\mathbf{i}_1}^{\mathbf{i}_3}[\gamma],$$

$$p \times \tilde{p} \mapsto \tilde{p} \star p, \quad (\tilde{p} \star p)_i = \begin{cases} p_i, & 1 \leq i \leq \ell_p \\ \tilde{p}_{i-\ell_p}, & \ell_p < i \leq \ell_p + \ell_{\tilde{p}} = \ell_{\tilde{p} \star p} \end{cases} \tag{21}$$

and the inversion map

$$- : \mathcal{P}_{\mathbf{i}_t}^{\mathbf{i}_s}[\gamma] \longrightarrow \mathcal{P}_{\mathbf{i}_s}^{\mathbf{i}_t}[\gamma],$$

$$p \mapsto \bar{p}, \quad \bar{p}_i = (e_{\ell_p+1-i}, -\sigma_{\ell_p+1-i}), \quad 1 \leq i \leq \ell_p \tag{22}$$

2.3.4. Nakajima varieties. Define the Nakajima varieties $\mathfrak{M}_\gamma(\mathbf{v}, \mathbf{w})$ associated with a quiver γ and two dimension vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}_{\geq 0}^{\text{Vert}_\gamma}$ [Na1, Na2, Na3].

Let γ be as before. To each vertex $\mathbf{i} \in \text{Vert}_\gamma$ we associate two Hermitian vector spaces $W_{\mathbf{i}}, V_{\mathbf{i}}$ of dimensions $w_{\mathbf{i}}, v_{\mathbf{i}}$, respectively. Let

$$U_\gamma(\mathbf{v}) = \bigotimes_{\mathbf{i} \in \text{Vert}_\gamma} U(v_{\mathbf{i}}) \tag{23}$$

be the group of unitary transformations of $\mathbf{V} = (V_i)_{i \in \text{Vert}_\gamma}$. First, form the Hermitian vector space:

$$\begin{aligned}
 H_\gamma(\mathbf{y}, \mathbf{w}) &= T^* \left(\bigoplus_{e \in \text{Edge}_\gamma} \text{Hom}(V_{s(e)}, V_{t(e)}) \bigoplus \bigoplus_{i \in \text{Vert}_\gamma} \text{Hom}(W_i, V_i) \right) \\
 &= \left\{ (B_e, \tilde{B}_e)_{e \in \text{Edge}_\gamma}, (I_i, J_i)_{i \in \text{Vert}_\gamma} \mid I_i : W_i \rightarrow V_i, J_i : V_i \rightarrow W_i, B_e : V_{s(e)} \right. \\
 &\quad \left. \rightarrow V_{t(e)}, \tilde{B}_e : V_{t(e)} \rightarrow V_{s(e)} \right\} \tag{24}
 \end{aligned}$$

which is acted upon by $U_\gamma(\mathbf{y})$ via:

$$\begin{aligned}
 (u_\omega)_{\omega \in \text{Vert}_\gamma} \cdot \left((B_e, \tilde{B}_e)_{e \in \text{Edge}_\gamma}, (I_i, J_i)_{i \in \text{Vert}_\gamma} \right) \\
 = \left((u_{t(e)} B_e u_{s(e)}^{-1}, u_{s(e)} \tilde{B}_e u_{t(e)}^{-1})_{e \in \text{Edge}_\gamma}, (u_i I_i, J_i u_i^{-1})_{i \in \text{Vert}_\gamma} \right) \tag{25}
 \end{aligned}$$

For a path $p \in \mathcal{P}_i^{\text{is}}[\gamma]$ define its holonomy $\mathcal{B}_p : V_{i_s} \rightarrow V_{i_t}$ in the obvious way:

$$\mathcal{B}_p = \prod_{i=1}^{\leftarrow \ell_p} \begin{cases} B_{e_i}, & \sigma_i = +1 \\ \tilde{B}_{e_i}, & \sigma_i = -1 \end{cases} \tag{26}$$

This definition is compatible with the path multiplication:

$$\mathcal{B}_{p_2} \mathcal{B}_{p_1} = \mathcal{B}_{p_2 * p_1} \tag{27}$$

The action (25) preserves the hyper-Kähler structure of $H_\gamma(\mathbf{y}, \mathbf{w})$, with the three symplectic forms $\varpi_{I,J,K}$ given by:

$$\begin{aligned}
 \varpi_I &= \sum_{e \in \text{Edge}_\gamma} \text{Tr}_{V_{t(e)}} \left(dB_e \wedge dB_e^\dagger - d\tilde{B}_e^\dagger \wedge d\tilde{B}_e \right) \\
 &\quad + \sum_{i \in \text{Vert}_\gamma} \text{Tr}_{W_i} \left(dJ_i \wedge dJ_i^\dagger - dI_i^\dagger \wedge dI_i \right), \\
 \varpi_J + \sqrt{-1} \varpi_K &= \sum_{i \in \text{Vert}_\gamma} \text{Tr}_{W_i} (dJ_i \wedge dI_i) + \sum_{e \in \text{Edge}_\gamma} \text{Tr}_{V_{t(e)}} (dB_e \wedge d\tilde{B}_e) \tag{28}
 \end{aligned}$$

Then perform the hyper-Kähler reduction with respect to the action (25):

$$\mathfrak{M}_\gamma(\mathbf{y}, \mathbf{w}) = \bar{\mu}^{-1}(\vec{\zeta}) / U_\gamma(\mathbf{y}) \tag{29}$$

where $\bar{\mu} = (\mu_{I,i}, \mu_{J,i}, \mu_{K,i})_{i \in \text{Vert}_\gamma}$,

$$\begin{aligned}
 \mu_{I,i} &= I_i I_i^\dagger - J_i^\dagger J_i + \sum_{e \in t^{-1}(i)} (B_e B_e^\dagger - \tilde{B}_e^\dagger \tilde{B}_e) \\
 &\quad + \sum_{e \in s^{-1}(i)} (\tilde{B}_e \tilde{B}_e^\dagger - B_e^\dagger B_e), \\
 \mu_{J,i} + \sqrt{-1} \mu_{K,i} &= I_i J_i + \sum_{e \in t^{-1}(i)} B_e \tilde{B}_e - \sum_{e \in s^{-1}(i)} \tilde{B}_e B_e, \tag{30}
 \end{aligned}$$

and we take (this is not the most general definition)

$$\vec{\zeta} = (\zeta_{\mathbf{i}} \mathbf{1}_{V_{\mathbf{i}}}, 0, 0)_{\mathbf{i} \in \text{Vert}_{\gamma}} \tag{31}$$

with all $\zeta_{\mathbf{i}} > 0$.

Stability. Instead of solving three equations $\vec{\mu} = \vec{\zeta}$ one can actually solve only $\mu_{\mathbb{C}} \equiv \mu_J + \sqrt{-1} \mu_K = 0$, and then take a quotient of the set of stable points in $\mu_{\mathbb{C}}^{-1}(0)$ by the action of

$$G_{\gamma}(\mathbf{v}) = \prod_{\mathbf{i} \in \text{Vert}_{\gamma}} GL(v_{\mathbf{i}}; \mathbb{C}) \tag{32}$$

so that

$$\mathfrak{M}_{\gamma}(\mathbf{v}, \mathbf{w}) = \mu_{\mathbb{C}}^{-1}(0)^{\text{stable}} / G_{\gamma}(\mathbf{v}) \tag{33}$$

The stable points are the $G_{\gamma}(\mathbf{v})$ -orbits of (B_e, \tilde{B}_e, I, J) s.t. the path algebra of γ represented by the products of B_e and \tilde{B}_e acting on the image $\bigoplus_{\mathbf{i} \in \text{Vert}_{\gamma}} I_{\mathbf{i}} W_{\mathbf{i}}$ generates all of $\bigoplus_{\mathbf{i} \in \text{Vert}_{\gamma}} V_{\mathbf{i}}$:

$$V_{\mathbf{i}} = \sum_{\mathbf{i}' \in \text{Vert}_{\gamma}} \sum_{p \in \mathcal{P}_{\mathbf{i}}^1[\gamma]} B_p I_{\mathbf{i}'} W_{\mathbf{i}'}. \tag{34}$$

In other words: any collection $\mathbf{V}' = (V'_{\mathbf{i}})_{\mathbf{i} \in \text{Vert}_{\gamma}} \subset \mathbf{V}$ of vector subspaces $V'_{\mathbf{i}} \subset V_{\mathbf{i}}$, obeying:

- S1) $I_{\mathbf{i}} W_{\mathbf{i}} \subset V'_{\mathbf{i}}$, for all $\mathbf{i} \in \text{Vert}_{\gamma}$,
 - S2) $B_e(V'_{s(e)}) \subset V'_{t(e)}$, $\tilde{B}_e(V'_{t(e)}) \subset V'_{s(e)}$, for all $e \in \text{Edge}_{\gamma}$
- (35)

must coincide with \mathbf{V} : $V'_{\mathbf{i}} = V_{\mathbf{i}}$ for all $\mathbf{i} \in \text{Vert}_{\gamma}$.

A simple proof of the equivalence of (29) and (33) can be found along the lines of the arguments of the section 3.4 and [Ne3]: in one direction, any solution to $\mu_{I, \mathbf{i}} = \zeta_{\mathbf{i}} \cdot \mathbf{1}_{V_{\mathbf{i}}}$ is stable. Indeed, $\mathbf{V}' \subset \mathbf{V}$ as in (35), and let $P_{\mathbf{i}}$ denote the orthogonal projection $V_{\mathbf{i}} \rightarrow V'_{\mathbf{i}}$. By (35) we have:

$$P_{\mathbf{i}} I_{\mathbf{i}} = 0, \quad P_{t(e)} B_e (1 - P_{s(e)}) = 0, \quad P_{s(e)} \tilde{B}_e (1 - P_{t(e)}) = 0 \tag{36}$$

Define $b_e = P_{t(e)} B_e P_{s(e)}$, $\tilde{b}_e = P_{s(e)} \tilde{B}_e P_{t(e)}$, $b'_e = (1 - P_{t(e)}) B_e P_{s(e)}$, $\tilde{b}'_e = (1 - P_{s(e)}) \tilde{B}_e P_{t(e)}$. Then

$$\begin{aligned} \zeta_{\mathbf{i}} \dim(V_{\mathbf{i}} / V'_{\mathbf{i}}) &= \text{Tr}_{V_{\mathbf{i}}} (P_{\mathbf{i}} \mu_{\mathbf{i}} P_{\mathbf{i}}) \\ &= \text{Tr}_{(V'_{\mathbf{i}})^{\perp}} \left(-j_{\mathbf{i}}^{\dagger} j_{\mathbf{i}} + \sum_{e \in t^{-1}(\mathbf{i})} (b_e b_e^{\dagger} - \tilde{b}_e^{\dagger} \tilde{b}_e - \tilde{b}'_e{}^{\dagger} \tilde{b}'_e) \right. \\ &\quad \left. + \sum_{e \in s^{-1}(\mathbf{i})} (\tilde{b}_e \tilde{b}_e^{\dagger} - b_e^{\dagger} b_e - b'_e{}^{\dagger} b'_e) \right), \end{aligned} \tag{37}$$

hence, after obvious cancellations,

$$\begin{aligned}
 0 &\leq \sum_{\mathbf{i} \in \text{Vert}_\gamma} \zeta_{\mathbf{i}} \dim(V_{\mathbf{i}}/V'_{\mathbf{i}}) \\
 &= - \left(\sum_{\mathbf{i} \in \text{Vert}_\gamma} \text{Tr}_{V'_{\mathbf{i}}^\perp} J_{\mathbf{i}}^\dagger J_{\mathbf{i}} + \sum_{e \in \text{Edge}_\gamma} \text{Tr}_{V'_{t(e)}^\perp} \tilde{b}'_{e^\dagger} \tilde{b}'_e + \text{Tr}_{V'_{s(e)}^\perp} b'_{e^\dagger} b'_e \right) \leq 0 \quad (38)
 \end{aligned}$$

which implies $V'_{\mathbf{i}} = V_{\mathbf{i}}$ for all $\mathbf{i} \in \text{Vert}_\gamma$. Conversely, given a stable solution $(B_e, \tilde{B}_e, I_{\mathbf{i}}, J_{\mathbf{i}})$ to $\mu_{\mathbb{C}} = 0$ equations, run the gradient flow of the function:

$$f = \frac{1}{2} \sum_{\mathbf{i} \in \text{Vert}_\gamma} \text{Tr}_{V_{\mathbf{i}}} (\mu_{I,\mathbf{i}} - \zeta_{\mathbf{i}} \mathbf{1}_{V_{\mathbf{i}}})^2 \quad (39)$$

which goes along the $\times_{\mathbf{i}} GL(V_{\mathbf{i}})$ orbits. The end-point of the flow is either at $f = 0$ which would establish the rest of the equations in (29), or at the higher critical point. There, the $\text{End}(V_{\mathbf{i}})$ -matrices $h_{\mathbf{i}} = \mu_{I,\mathbf{i}} - \zeta_{\mathbf{i}} \mathbf{1}_{V_{\mathbf{i}}}$ solve:

$$h_{t(e)} B_e = B_e h_{s(e)}, \quad h_{s(e)} \tilde{B}_e = \tilde{B}_e h_{t(e)}, \quad h_{\mathbf{i}} I_{\mathbf{i}} = 0, \quad J_{\mathbf{i}} h_{\mathbf{i}} = 0 \quad (40)$$

Therefore $V'_{\mathbf{i}} = \ker h_{\mathbf{i}}$ obeys both (S1) and (S2) conditions of (35), therefore $h_{\mathbf{i}} = 0$ for all $\mathbf{i} \in \text{Vert}_\gamma$.

2.3.5. *Framing symmetries of Nakajima varieties.* The Nakajima variety $\mathfrak{M}_\gamma(\underline{\mathbf{v}}, \underline{\mathbf{w}})$ has a symmetry group

$$U_\gamma(\underline{\mathbf{w}}) = \times_{\mathbf{i} \in \text{Vert}_\gamma} U(w_{\mathbf{i}}) \quad (41)$$

acting in an obvious way on the operators $(I_{\mathbf{i}}, J_{\mathbf{i}})$. The maximal torus $T_\gamma(\underline{\mathbf{w}}) \subset U_\gamma(\underline{\mathbf{w}})$ fixed point locus is the union

$$\mathfrak{M}_\gamma(\underline{\mathbf{v}}, \underline{\mathbf{w}})^{T_\gamma(\underline{\mathbf{w}})} = \bigsqcup_{\substack{\underline{\mathbf{v}} = \\ \mathbf{i} \in \text{Vert}_\gamma, \alpha \in [w_{\mathbf{i}}]}} \times_{\mathbf{i} \in \text{Vert}_\gamma} \times_{\alpha \in [w_{\mathbf{i}}]} \mathfrak{M}_\gamma(\underline{\mathbf{v}}^{\mathbf{i},\alpha}, \underline{\delta}_{\mathbf{i}}) \quad (42)$$

where $\underline{\mathbf{v}}^{\mathbf{i},\alpha} \in \mathbb{Z}_{\geq 0}^{\text{Vert}_\gamma}$ for each (\mathbf{i}, α) , i.e.

$$\underline{\mathbf{v}}^{\mathbf{i},\alpha} = \left(v_{\mathbf{i}}^{\mathbf{i},\alpha} \right)_{\mathbf{i} \in \text{Vert}_\gamma}, \quad (43)$$

and

$$\underline{\delta}_{\mathbf{i}} = (\delta_{\mathbf{i},\mathbf{j}})_{\mathbf{j} \in \text{Vert}_\gamma} \quad (44)$$

We define the fundamental Nakajima variety

$$\mathbf{M}_\gamma^{\mathbf{i}}(\underline{\mathbf{v}}) = \mathfrak{M}_\gamma(\underline{\mathbf{v}}, \underline{\delta}_{\mathbf{i}}) \quad (45)$$

The Eq. (42) explains the importance of the fundamental Nakajima varieties.

2.3.6. *Nakajima–Young varieties.* The Nakajima varieties $\mathfrak{M}_\gamma(\mathbf{v}, \mathbf{w})$ with the choice (31) have a holomorphic \mathbb{C}^\times -symmetry (its compact subgroup $U(1)$ acts by an isometry): $u \in \mathbb{C}^\times$ acts via

$$u \cdot (B_e, \tilde{B}_e, I_i, J_i) = (u B_e, u \tilde{B}_e, u I_i, u J_i) \tag{46}$$

Define Nakajima–Young variety $\mathbf{Y}_\gamma^{\mathbf{i}}(\mu)$ to be the connected component of the fixed point set:

$$\mathbf{M}_\gamma^{\mathbf{i}}(\mathbf{v})^{\mathbb{C}^\times} = \bigsqcup_{\mu \in \Lambda_\gamma^{\mathbf{i}}[\mathbf{v}]} \mathbf{Y}_\gamma^{\mathbf{i}}(\mu) \tag{47}$$

with

$$\Lambda_\gamma^{\mathbf{i}}[\mathbf{v}] = \pi_0(\mathbf{M}_\gamma^{\mathbf{i}}(\mathbf{v})^{\mathbb{C}^\times}) \tag{48}$$

denoting the set of connected components. We define the sets $\Lambda_\gamma^{\mathbf{i}}$ for $\mathbf{i} \in \text{Vert}_\gamma$:

$$\Lambda_\gamma^{\mathbf{i}} = \bigsqcup_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^{\text{Vert}_\gamma}} \Lambda_\gamma^{\mathbf{i}}[\mathbf{v}] \tag{49}$$

For $\mu \in \Lambda_\gamma^{\mathbf{i}}[\mathbf{v}]$ we define:

$$|\mu| = \mathbf{v} \in \mathbb{Z}_{\geq 0}^{\text{Vert}_\gamma} \tag{50}$$

Each Nakajima–Young variety $\mathbf{Y}_\gamma^{\mathbf{i}}(\mu)$ carries a set of vector bundles:

$$V_{\mathbf{j},n}^{\mathbf{i}}(\mu) \longrightarrow \mathbf{Y}_\gamma^{\mathbf{i}}(\mu) \tag{51}$$

where $\mathbf{j} \in \text{Vert}_\gamma, n \geq 0$, and

$$V_{\mathbf{j},n}^{\mathbf{i}}(\mu) = \sum_{p \in \mathcal{P}_\mathbf{i}^{\mathbf{j}}[\gamma], \ell_p=n} B_p I(\mathbb{C}). \tag{52}$$

The stability condition (34) implies, for any $\mathbf{j} \in \text{Vert}_\gamma$:

$$V_{\mathbf{j}} = \bigoplus_{n=0}^{\infty} V_{\mathbf{j},n}^{\mathbf{i}} \tag{53}$$

It is easy to show that $J \equiv 0$ on all $\mathbf{Y}_\gamma^{\mathbf{i}}(\mathbf{v}; \mu)$, and $V_{\mathbf{j},0}^{\mathbf{i}} = \mathbb{C}\delta_{\mathbf{i},\mathbf{j}}$. Let us clarify the origin of the direct sum decomposition (53). The \mathbb{C}^\times -invariance of the $G_\gamma(\mathbf{v})$ -orbit of (B_e, \tilde{B}_e, I, J) means that the transformation (46) can be compensated by an element $(g_{\mathbf{j}}(u))_{\mathbf{j} \in \text{Vert}_\gamma}$:

$$\begin{aligned} g_{t(e)}(u) B_e g_{s(e)}(u)^{-1} &= u B_e, & g_{s(e)}(s) \tilde{B}_e g_{t(e)}^{-1}(u) &= u \tilde{B}_e, \\ g_{\mathbf{i}}(u) I &= u I, & J g_{\mathbf{i}}(u)^{-1} &= u J \end{aligned} \tag{54}$$

Then

$$V_{\mathbf{j},n}^{\mathbf{i}} = \text{Ker} \left(g_{\mathbf{j}}(u) - u^{n+1} \right) \subset V_{\mathbf{j}} \tag{55}$$

are obviously mutually orthogonal for different n 's. The ranks $v_{j,n}^i(\mu) = \text{rk} V_{j,n}^i(\mu)$ are important local invariants of $\mathbf{Y}_\gamma^i(\mu)$. By definition:

$$\sum_{n=0}^{\infty} v_{j,n}^i(\mu) = v_j \tag{56}$$

The K -theory class of the tangent bundle $\mathcal{T}_{\mathbf{Y}_\gamma^i(\mu)}$ to $\mathbf{Y}_\gamma^i(\mu)$ can be expressed in terms of those of $V_{j,n}^i$:

$$\begin{aligned} [\mathcal{T}_{\mathbf{Y}_\gamma^i(\mu)}] &= [V_{i,0}^i] \\ &+ \sum_{n \geq 0, e \in \text{Edge}_\gamma} [\text{Hom}(V_{t(e),n}^i, V_{s(e),n+1}^i) \oplus \text{Hom}(V_{s(e),n}^i, V_{t(e),n+1}^i)] \\ &- \sum_{n \geq 0, j \in \text{Vert}_\gamma} [\text{Hom}(V_{j,n}^i, V_{j,n}^i) \oplus \text{Hom}(V_{j,n}^i, V_{j,n+2}^i)]. \end{aligned} \tag{57}$$

Remark. In the case of $\gamma = \widehat{A}_0$, where $\text{Edge}_\gamma = \{e\}$, $\text{Vert}_\gamma = \{0\}$, $s(e) = t(e) = 0$, the fundamental Nakajima variety is the Hilbert scheme of v points on \mathbb{C}^2 , a.k.a. the moduli space of noncommutative $U(1)$ instantons on \mathbb{R}^4 , while the Nakajima–Young varieties are the connected components of the so-called graded Hilbert scheme of v points.

2.4. *The local model data.* To specify the basic local model data we fix:

- (1) The string

$$\bar{\varepsilon} = (\varepsilon_a)_{a \in \underline{4}} \tag{58}$$

of 4 complex numbers which sum to zero:

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 0 \tag{59}$$

- (2) The string \bar{n} of 6 non-negative integers $n_A \geq 0$, $A \in \underline{6}$. Let

$$\mathbf{N} = \bigsqcup_{A \in \underline{6}} [n_A] \approx \{(A, \alpha) \mid A \in \underline{6}, \alpha \in [n_A]\}. \tag{60}$$

- (3) The string $\bar{\mathbf{a}} \in \mathbb{C}^{\mathbf{N}}$ of

$$\sum_{A \in \underline{6}} n_A$$

complex numbers $\mathbf{a}_{A,\alpha} \in \mathbb{C}$, $\alpha = 1, \dots, n_A$, also denoted as

$$\mathbf{a}_A = (\mathbf{a}_{A,\alpha})_{\alpha \in [n_A]} \equiv (\mathbf{a}_{A,1}, \dots, \mathbf{a}_{A,n_A}) \in \mathbb{C}^{n_A}. \tag{61}$$

- (4) The fugacity

$$\mathfrak{q} \in \mathbb{C}, \tag{62}$$

$|\mathfrak{q}| < 1$.

We also use the notations: for any $a \in \underline{4}$,

$$q_a(\beta) = e^{\beta \varepsilon_a}, \quad P_a(\beta) = 1 - q_a(\beta), \quad q_a^*(\beta) = e^{-\beta \varepsilon_a}, \quad P_a^*(\beta) = 1 - q_a^*(\beta), \quad (63)$$

and for any $S \subset \underline{4}$

$$q_S(\beta) = \prod_{a \in S} q_a(\beta), \quad q_S^*(\beta) = \prod_{a \in S} q_a^*(\beta), \quad P_S(\beta) = \prod_{a \in S} P_a(\beta), \quad P_S^*(\beta) = \prod_{a \in S} P_a^*(\beta) \quad (64)$$

We shall often skip the argument β in the notations for q_a, P_S^* , etc. The notation (64), in particular, implies [cf. (59)]

$$q_{\underline{4}} = q_{\emptyset} = 1, \quad P_{\underline{4}} = P_1 P_2 P_3 P_4 = P_{\underline{4}}^*, \quad q_{\bar{A}} = q_A^* \quad (65)$$

and

$$P_S^* = (-1)^{|S|} q_S^* P_S \quad (66)$$

We shall also encounter the relation

$$P_{\underline{4}} = P_{\underline{3}} + P_{\underline{3}}^* \quad (67)$$

in what follows.

2.4.1. Geometry of the local model data. The meaning of the parameters $\underline{a}, \bar{\varepsilon}$ is the following. Define the gauge group G_A corresponding to the stratum $X_A \approx \mathbb{C}_A^2$ of the singular toric surface X to be

$$G_A = U(n_A) \quad (68)$$

Let $T_A \subset G_A$ denote its maximal torus. Let $U(1)_{\bar{\varepsilon}}^3 \subset SU(4)$ be the maximal torus of the $(4, 0)$ -volume preserving unitary symmetries of $Z = \mathbb{C}^4$. The $U(1)_{\bar{\varepsilon}}^3$ -action preserves X . The Lie algebra $\text{Lie} T_A \otimes \mathbb{C}$ is parametrized by diagonal matrices $\underline{a}_A = \text{diag}(a_{A,1}, \dots, a_{A,n_A})$ with complex entries $a_{A,\alpha} \in \mathbb{C}$. The Lie algebra $\text{Lie} U(1)_{\bar{\varepsilon}}^3 \otimes \mathbb{C}$ is parametrized by $\text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 0$. Let

$$T_H = U(1)_{\bar{\varepsilon}}^3 \times \bigtimes_{A \in \underline{6}} T_A \quad (69)$$

2.4.2. Additive to multiplicative. Let I_{\pm} be two finite sets, $I = I_+ \sqcup I_-$. Let \mathbf{M} be a space with an action of a Lie group \mathbf{G} , and let $\mathcal{E}_i, i \in I$ be a collection of \mathbf{G} -equivariant vector bundles over \mathbf{M} . Let $w_i \in \mathbb{C}$. We combine them into the $\mathbf{G} \times (\mathbb{C}^{\times})^I$ -equivariant virtual bundle $\mathcal{E} = \left[\bigoplus_{i \in I_+} \mathcal{E}_i \ominus \bigoplus_{i \in I_-} \mathcal{E}_i \right]$. Let

$$\text{Ch}_{\beta}(\mathcal{E}_i) = \sum_{\alpha} e^{\beta \xi_{i,\alpha}} \quad (70)$$

be the refined Chern character (with $\xi_{i,\alpha}$ equivariant Chern roots of \mathcal{E}_i), so that in the non-equivariant setting

$$\text{Ch}_{\beta}(E) = \sum_{k \geq 0} \beta^k \text{ch}_k(E),$$

and define

$$f(\beta) = \sum_{i \in I_+} e^{\beta w_i} \text{Ch}_\beta(\mathcal{E}_i) - \sum_{i \in I_-} e^{\beta w_i} \text{Ch}_\beta(\mathcal{E}_i) \tag{71}$$

To $f(\beta)$ we associate the equivariant characteristic class, a rational function of w_i 's:

$$\epsilon[f] = \prod_{i \in I_+} \mathbf{c}_{w_i}(\mathcal{E}_{+,i}) \prod_{i \in I_-} \mathbf{c}_{w_i}(\mathcal{E}_i)^{-1}. \tag{72}$$

where

$$\mathbf{c}_w(\mathcal{E}) = \sum_{k=0}^{\text{rk}\mathcal{E}} w^k c_{\text{rk}\mathcal{E}-k}(\mathcal{E}) \tag{73}$$

is the usual \mathbf{G} -equivariant Chern polynomial of \mathcal{E} , evaluated at $w \in H_{\mathbb{C}^\times}^\bullet(pt) = \mathbb{C}$, equivalently, it is the top $\mathbb{C}^\times \times \mathbf{G}$ -equivariant Chern class of \mathcal{E} . We define the $*$ -operation on the expressions $f(\beta)$:

$$f^*(\beta) \equiv \sum_{i \in I_+} e^{-\beta w_i} \text{Ch}_\beta(\mathcal{E}_i^*) - \sum_{i \in I_-} e^{-\beta w_i} \text{Ch}_\beta(\mathcal{E}_i^*) = f(-\beta) \tag{74}$$

This definition is consistent with the notations (63).

We have:

$$\epsilon[f] = (-1)^{f(0)} \epsilon[f^*] \tag{75}$$

where

$$f(0) = \sum_{i \in I_+} \text{rk}(\mathcal{E}_i) - \sum_{i \in I_-} \text{rk}(\mathcal{E}_i) \tag{76}$$

Therefore,

$$\epsilon[PSf] = \epsilon[PSq_{\bar{S}}f^*]^{(-1)^{|S|}}, \tag{77}$$

The definition (72) is the generalization of the notation used in [Ne3], where we defined ϵ as a map from the space of \mathbb{Z} -linear combinations of exponents to rational functions:

$$\epsilon \left[\sum_{i \in I_+} e^{\beta w_i} - \sum_{i \in I_-} e^{\beta w_i} \right] = \left(\prod_{i \in I_+} w_i \right) \cdot \left(\prod_{i \in I_-} w_i \right)^{-1}. \tag{78}$$

3. Partition Function of Spiked Instantons

In this section we define the statistical mechanical model. The random variables are the strings of Young diagrams and the complex Boltzmann weights are rational functions of the complex numbers (58), (61). The definition might look first a bit artificial. Its origin is geometric. Namely, in [Ne3] the moduli space of spiked instantons $\mathcal{M}(k, \vec{n})$ is introduced, with $\vec{n} = (n_A)_{A \in \mathfrak{g}}$. It has an action of the group $\mathbf{H} = \times_{A \in \mathfrak{g}} U(n_A) \times U(1)_{\epsilon}^3$. The fixed points of the maximal torus $T_{\mathbf{H}}$ are in one-to-one correspondence with the strings $\underline{\lambda}$ of partitions described below. The Boltzmann weight is simply the localization contribution to the integral of 1 over $\mathcal{M}(k, \vec{n})$, multiplied by q^k . This contribution is the product of the weights of the $T_{\mathbf{H}}$ -action on the virtual tangent space to $\mathcal{M}(k, \vec{n})$, which is the difference of the kernel and the cokernel of the linearization of the equations defining $\mathcal{M}(k, \vec{n})$ at the fixed point. The kernel is always a complex vector space, henceforth it is naturally oriented and the product of weights is well-defined. The cokernel (the obstruction space) is only a real vector space, hence the product of the weights depends on the choice of its orientation. In what follows we specify the choice of the orientation with the help of the choice of the order on **4** and **6**. The resulting measure will not depend on this choice.

3.1. *The configuration space.* The basic local model is a statistical ensemble. The random variables are the strings

$$\underline{\lambda} = \left(\lambda^{(A,\alpha)} \right)_{A \in \mathfrak{6}, \alpha \in [n_A]} \in \Lambda^{\mathbf{N}} \tag{79}$$

of [cf. (60)]

$$\sum_{A \in \mathfrak{6}} n_A = \#\mathbf{N}$$

partitions $\lambda^{(A,\alpha)} \in \Lambda$. In other words, the configuration space is

$$\Lambda^{\mathbf{N}}. \tag{80}$$

Define, cf. (5):

$$N_A(\beta) = \sum_{\alpha=1}^{n_A} e^{\beta \mathfrak{a}_{A,\alpha}}, \quad K_A(\beta) = \sum_{\alpha=1}^{n_A} \sum_{\square \in \lambda^{(A,\alpha)}} e^{\beta c_{A,\alpha}(\square)}, \tag{81}$$

with

$$c_{A,\alpha}(\square) = \mathfrak{a}_{A,\alpha} + \varepsilon_a(i-1) + \varepsilon_b(j-1), \quad \text{for } \square = (i, j) \tag{82}$$

and [cf. (74)]

$$T_A = N_A K_A^* + q_A N_A^* K_A - P_A K_A K_A^* \tag{83}$$

Let

$$k_A = K_A(0) = \sum_{\alpha=1}^{[n_A]} |\lambda^{(A,\alpha)}| \tag{84}$$

and

$$|\underline{\lambda}| = \sum_{A \in \mathfrak{6}} k_A \tag{85}$$

It is well-known [Nel,NY,AGT] that

$$T_A = q_A T_A^* \tag{86}$$

is a pure character, i.e.

$$T_A = \sum_{I=1}^{2n_A k_A} e^{t_{A,I}} \tag{87}$$

where $t_{A,I}$ are integral linear combinations of $\mathfrak{a}_{A,\alpha}$, $\alpha \in [n_A]$, $\varepsilon_a, \varepsilon_b$, $a, b \in A$. Let us assume $\underline{\mathfrak{a}}_A, \bar{\varepsilon}$ are sufficiently generic, so that $t_{A,I} \neq 0$, $t_{A,I} + \varepsilon_{\bar{a}} \neq 0$ for any $\bar{a} \in \bar{A}$, $I \in [2n_A k_A]$.

Define, finally,

$$K(\beta) = \sum_{A \in \mathfrak{6}} K_A(\beta) \tag{88}$$

3.2. *The statistical weight.* The complex Boltzmann weight of $\underline{\lambda}$ is given by the following expression:

$$Z_{\underline{\lambda}} = q^{|\underline{\lambda}|} \epsilon [-T_{\underline{\lambda}}], \tag{89}$$

where [cf. (9)]:

$$T_{\underline{\lambda}}(\beta) = \sum_{A \in \underline{\mathbf{6}}} \left(P_{\varphi(\bar{A})} T_A + P_{\bar{A}} N_A \sum_{B \neq A} K_B^* \right) - P_{\underline{\mathbf{4}}} \sum_{A < B} K_A K_B^* \tag{90}$$

The definition (89) depends explicitly on the choice of the ordering of the sets $\underline{\mathbf{4}}$ and $\underline{\mathbf{6}}$, since it enters the definition of the maps $\varphi : \underline{\mathbf{6}} \rightarrow \underline{\mathbf{4}}$ and the meaning of $A < B$ in (90). Morally,

$$\epsilon [-T_{\underline{\lambda}}] \sim \epsilon \left[- \sum_{A \in \underline{\mathbf{6}}} P_{\bar{A}} N_A K^* \right] \sqrt{\epsilon [P_{\underline{\mathbf{4}}} K K^*]} \tag{91}$$

so the Boltzmann weight is defined canonically up to a sign.

Note that for generic \underline{a}_A, \bar{e} the measure (89) does not depend on the choice of the order on $\underline{\mathbf{4}}$ or $\underline{\mathbf{6}}$:

$$\begin{aligned} \epsilon [q_{\bar{a}} T_A] &= \epsilon [q_{\bar{a}}^* T_A^*] = \epsilon [q_{\bar{a}}^* q_A^* T_A] = \epsilon [q_{\bar{b}} T_A], \\ \epsilon \left[P_{\underline{\mathbf{4}}} \sum_{A < B} K_A K_B^* \right] &= \epsilon \left[P_{\underline{\mathbf{3}}} \sum_{A \neq B} K_A K_B^* \right] \end{aligned} \tag{92}$$

where we used (86), (87), (67), and $q_{\bar{a}} q_{\bar{b}} q_A = 1$ for $\bar{A} = \{\bar{a}, \bar{b}\}$. Define,

$$Z^{\text{inst}} = \sum_{\underline{\lambda} \in \Lambda^N} Z_{\underline{\lambda}} = \sum_{k=0}^{\infty} q^k Z_k^{\text{inst}} \tag{93}$$

3.2.1. *The origins: spiked instantons, tori and characters.* The partition function Z^{inst} is the $T_{\mathbb{H}}$ -equivariant integral of 1 over the virtual fundamental cycle of the moduli space of spiked instantons [Ne3]. The latter is the space of solutions to certain quadratic matrix equations, generalizing the ADHM equations [ADHM], on four complex $k \times k$ matrices B_a , their Hermitian conjugates B_a^\dagger , $a \in \underline{\mathbf{4}}$, and twelve rectangular matrices I_A, J_A , of sizes $n_A \times k$ and $k \times n_A$, $A \in \underline{\mathbf{6}}$, and their Hermitian conjugates. The definition (90) stems from the equivariant localization. The strings of partitions $\underline{\lambda}$ are the $T_{\mathbb{H}}$ -fixed points. The matrices (B_a, I_A, J_A) of the construction [Ne3] obey, for such a fixed point:

$$[B_a, B_b] = 0, \quad a, b \in \underline{\mathbf{4}}, \quad J_A = 0, \quad A \in \underline{\mathbf{6}} \tag{94}$$

the vectors

$$|i, j; \alpha; ab\rangle = B_a^{i-1} B_b^{j-1} I_{ab}(N_{ab, \alpha}) \tag{95}$$

with $\alpha \in [n_{ab}]$, $1 \leq j \leq \lambda^{(ab, \alpha)}$ forming the basis of the vector space K , $N_{ab, \alpha}$ being the eigenspace of T_{ab} action on the framing space N_{ab} (see [Ne3] for the notations and more explanations).

The equivariant weights of the matrices contribute

$$T_+ = \sum_{a \in \underline{4}} q_a K K^* + \sum_{A \in \underline{6}} (K^* N_A + q_A K N_A^*) \tag{96}$$

with

$$K = \sum_{A \in \underline{6}} K_A \tag{97}$$

while the equivariant weights of the equations they obey, and the symmetries one divides by, contribute (with the minus sign)

$$T_- = \left(1 + \sum_{c \in \underline{3}} q_c q_4 \right) K K^* + \sum_{A \in \underline{6}} \sum_{\bar{a} \in \bar{A}} q_{\bar{a}} K^* N_A \tag{98}$$

Moreover, the T_+ part is defined canonically by using the complex structure of the space of matrices (B_a, I_A, J_A) . The T_- part is defined non-canonically, as the expression (98) does not respect the symmetry between q_a 's. The real (i.e. such that $\chi^* = \chi$) character $T_- + T_-^*$ is defined canonically. This subtlety has to do with the real, as opposed to complex, nature of the equations defining the spiked instantons [Ne3]. So, $\epsilon[T_-]$ may have a sign ambiguity, as $\sqrt{\epsilon[T_- + T_-^]}$. Also, $\epsilon[T_-]$ and $\epsilon[T_+]$ separately may vanish, as some of the weights in (96) and (98) may vanish. It is easy to show that formally $\epsilon[T_- - T_+] = \epsilon[-T_{\underline{\lambda}}]$. One simply uses (75) several times. The details of the choice of the sign will be clarified elsewhere (it uses the residue definition of the localization contribution, which was worked out in [MNS], it is similar to what sometimes is referred to as the Jeffrey–Kirwan residue [JK] in the mathematical literature, see also [W]).

The resulting measure factor

$$\epsilon[T_- - T_+] = \frac{\epsilon[Obs_{\underline{\lambda}}]}{\epsilon[Def_{\underline{\lambda}}]} = \epsilon[-T_{\underline{\lambda}}] \tag{99}$$

where $Def_{\underline{\lambda}}, Obs_{\underline{\lambda}}$ are the T_H -characters of $\ker D_{\underline{\lambda}}, \text{coker } D_{\underline{\lambda}}$, respectively. Here $D_{\underline{\lambda}}$ is the linearization of the spiked instanton equations at the solution, corresponding to $\underline{\lambda}$.

The expressions $N_A, K_A, T_{\underline{\lambda}}(\beta)$ etc. are the elements of the K-group $K[T_H]$, i.e. the abelian group whose elements are the formal linear combinations

$$\sum_{w \in T_H^\vee} n_w L_w \tag{100}$$

where $n_w \in \mathbb{Z}$,

$$L_w \tag{101}$$

are the irreducible representations of the torus T_H , i.e. the elements of the lattice $T_H^\vee = \text{Hom}(T_H, U(1))$. We assign to the weight $w = (w_{A,\alpha}) \oplus (w_a)$ a function of $(\underline{\alpha}, \underline{\varepsilon})$, the character of T_H^C in the representation L_w :

$$L_w \mapsto \exp \beta \left(\sum_{A,\alpha} w_{A,\alpha} \underline{\alpha}_{A,\alpha} + \sum_a w_a \underline{\varepsilon}_a \right) \tag{102}$$

Here $w_{A,\alpha} \in \mathbb{Z}, w_a \in \mathbb{Z}$ are defined up to a shift $w_a \mapsto w_a + \mathbf{w}, \mathbf{w} \in \mathbb{Z}$.

3.2.2. *More general definition.* The definition (93) is fine as long as \underline{a} and $\bar{\varepsilon}$ are generic.

However, e.g. if for some $ab \in \underline{\mathfrak{6}}$ the ratio $\varepsilon_a/\varepsilon_b \in \mathbf{Q}_+$ is a positive rational number, or if for some $\alpha \neq \beta \in [n_A]$, $\mathbf{a}_{A,\alpha} = \mathbf{a}_{A,\beta}$, the individual contributions $Z_{\underline{a}}$ to the formula (93) have apparent poles. Actually, the poles cancel. Let us give the presentation of the formula (93) which is applicable in these cases.

$$Z_k^{\text{inst}} = \sum_{(k_A)_{A \in \underline{\mathfrak{6}}}, \sum_A k_A = k} \int_{\times_{A \in \underline{\mathfrak{6}}} \mathcal{M}_{k_A}(n_A)} S_{\bar{n}, \bar{k}}(\underline{a}, \bar{\varepsilon}) \tag{103}$$

where Gieseker-Nakajima moduli spaces $\mathcal{M}_k(n)$ parametrize the charge k noncommutative $U(n)$ instantons on \mathbb{R}^4 and framed rank n torsion free sheaves \mathcal{E} on $\mathbb{C}P^2$ with $\text{ch}_2(\mathcal{E}) = k$, while $S_{\bar{n}, \bar{k}}(\underline{a}, \bar{\varepsilon})$ is the equivariant characteristic class, given by [cf. (90)]:

$$\begin{aligned} S_{\bar{n}, \bar{k}}(\underline{a}, \bar{\varepsilon}) &= \prod_A \mathbf{c}_{m_A}(T\mathcal{M}_{k_A}(n_A)) \\ &\times \prod_{A \neq B \in \underline{\mathfrak{6}}} \prod_{\alpha \in [n_A]} \frac{\prod_{\bar{a} \in \bar{A}} \mathbf{c}_{\mathbf{a}_{A,\alpha} + \varepsilon_{\bar{a}}}(K_B)}{\mathbf{c}_{\mathbf{a}_{A,\alpha}}(K_B) \mathbf{c}_{\mathbf{a}_{A,\alpha} - \varepsilon_A}(K_B)} \\ &\times \prod_{A < B \in \underline{\mathfrak{6}}} \frac{(\mathbf{c}_0(\text{Hom}(K_B, K_A)))^2 \prod_{C \in \underline{\mathfrak{6}}} \mathbf{c}_{\varepsilon_C}(\text{Hom}(K_B, K_A))}{\prod_{C \in \underline{\mathfrak{4}}} \mathbf{c}_{\varepsilon_C}(\text{Hom}(K_B, K_A)) \mathbf{c}_{-\varepsilon_C}(\text{Hom}(K_B, K_A))} \end{aligned} \tag{104}$$

with K_A being the tautological rank k_A bundle over $\mathcal{M}_{k_A}(n_A)$. Finally, $m_A = \varepsilon_{\bar{a}}$ for either $\bar{a} \in \bar{A}$. The choice of \bar{a} is immaterial. Indeed, the moduli space $\mathcal{M}_{k_A}(n_A)$ is a complex symplectic manifold, as reflected by the symmetry $T_A = q_A T_A^*$. It implies

$$\mathbf{c}_{\varepsilon_{\bar{a}}}(T\mathcal{M}_{k_A}(n_A)) = \mathbf{c}_{\varepsilon_{\bar{b}}}(T\mathcal{M}_{k_A}(n_A)) \tag{105}$$

for both $\bar{a}, \bar{b} \in \bar{A}$.

3.2.3. *One-instanton case.* As an illustration, let us consider the case $k = 1$. There are

$$\sum_A n_A$$

possibilities, with

$$K_B = \delta_{A,B} e^{\beta \mathbf{a}_{A,\alpha}}, \quad A \in \underline{\mathfrak{6}}, \alpha \in [n_A] \tag{106}$$

Thus, the 1-instanton partition function is given by:

$$Z_1^{\text{inst}} = \sum_{A \in \underline{\mathfrak{6}}} \sum_{\alpha \in [n_A]} Z_{A,\alpha}, \tag{107}$$

with

$$\begin{aligned} Z_{A,\alpha} &= \frac{E_{\bar{A}} - E_A}{E_A} \prod_{\alpha' \in [n_A], \alpha' \neq \alpha} \left(1 + \frac{E_{\bar{A}}}{(\mathbf{a}_{A,\alpha'} - \mathbf{a}_{A,\alpha})(\mathbf{a}_{A,\alpha'} - \mathbf{a}_{A,\alpha} - \varepsilon_A)} \right) \\ &\times \prod_{B \neq A} \prod_{\gamma \in [n_B]} \left(1 + \frac{E_{\bar{B}}}{(\mathbf{a}_{B,\gamma} - \mathbf{a}_{A,\alpha})(\mathbf{a}_{B,\gamma} - \mathbf{a}_{A,\alpha} - \varepsilon_B)} \right) \end{aligned} \tag{108}$$

where

$$\varepsilon_S = \sum_{s \in S} \varepsilon_s = -\varepsilon_{\bar{S}}, \quad E_S = \prod_{s \in S} \varepsilon_s \tag{109}$$

3.3. *The perturbative prefactor.* We introduce a common, i.e. $\underline{\lambda}$ -independent prefactor in the statistical weight. The so completed statistical weight is equal to:

$$W_{\underline{\lambda}} = Z^{\text{pert}}(\bar{\mathbf{a}}, \bar{\varepsilon}) Z_{\underline{\lambda}} \tag{110}$$

with

$$\begin{aligned} Z^{\text{pert}}(\bar{\mathbf{a}}, \bar{\varepsilon}) &= \prod_{A \in \underline{\mathbf{6}}} Z_{N=2*}^{\text{pert},A}(\mathbf{a}_A, \bar{\varepsilon}) \\ &\times \prod_{\{a,b,c\} \subset \underline{\mathbf{4}}} Z_{\text{fold}}^{\text{pert},a|bc}(\mathbf{a}_{ab}, \mathbf{a}_{ac}, \bar{\varepsilon}) \\ &\times \prod_{A \in \underline{\mathbf{6}}, A < \bar{A}} Z_{\text{cross}}^{\text{pert},A}(\mathbf{a}_A, \mathbf{a}_{\bar{A}}, \bar{\varepsilon}) \end{aligned} \tag{111}$$

where

- for $A = ab$: define $m_A = \varepsilon_{\varphi(A)}$, and

$$\begin{aligned} Z_{N=2*}^{\text{pert},A}(\mathbf{a}_A, \bar{\varepsilon}) &= \prod_{\alpha, \beta=1}^{n_A} \Gamma_2(\mathbf{a}_{A,\alpha} - \mathbf{a}_{A,\beta}; \varepsilon_a, \varepsilon_b) \\ &\times \prod_{\alpha, \beta=1}^{n_A} \Gamma_2^{-1}(\mathbf{a}_{A,\alpha} - \mathbf{a}_{A,\beta} + m_A; \varepsilon_a, \varepsilon_b) \end{aligned} \tag{112}$$

with the Barnes double gamma functions $\Gamma_2(x; \varepsilon_a, \varepsilon_b)$ normalized in such a way so as to have simple zeroes on a quadrant of the integral lattice spanned by $\varepsilon_a, \varepsilon_b$:

$$\Gamma_2(x; \varepsilon_a, \varepsilon_b) \sim \prod_{i,j \geq 1} (x + \varepsilon_a(i-1) + \varepsilon_b(j-1)) \quad , \tag{113}$$

it is defined by the analytic continuation of the integral formula

$$\Gamma_2(x; \varepsilon_a, \varepsilon_b) = \exp - \frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s \frac{e^{-tx}}{(1 - e^{-t\varepsilon_a})(1 - e^{-t\varepsilon_b})} \tag{114}$$

from the domain $\text{Re}(x), \text{Re}(\varepsilon_a), \text{Re}(\varepsilon_b) > 0$.

-

$$Z_{\text{fold}}^{\text{pert},a|bc}(\mathbf{a}_{ab}, \mathbf{a}_{ac}, \bar{\varepsilon}) = \prod_{\alpha=1}^{n_{ab}} \prod_{\beta=1}^{n_{ac}} \Gamma_1(\mathbf{a}_{ab,\alpha} - \mathbf{a}_{ac,\beta} + \varepsilon_a + \varepsilon_c; \varepsilon_a) \tag{115}$$

where $\Gamma_1(x; \varepsilon_a)$ is essentially the ordinary Γ -function:

$$\Gamma_1(x; y) \sim \prod_{i=1}^\infty (x + y(i-1)) \quad , \tag{116}$$

Again, it can be defined by the analytic continuation of the integral

$$\Gamma_1(x; y) = \exp - \frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s \frac{e^{-tx}}{(1 - e^{-ty})} \tag{117}$$

from the domain $\text{Re}(x), \text{Re}(y) > 0$, giving

$$\Gamma_1(x; y) = \frac{\sqrt{2\pi/y}}{y^{\frac{x}{y}} \Gamma\left(\frac{x}{y}\right)}, \tag{118}$$

•

$$Z_{\text{cross}}^{\text{pert},A}(\mathfrak{a}_A, \mathfrak{a}_{\bar{A}}, \bar{\varepsilon}) = \prod_{\alpha=1}^{n_A} \prod_{\beta=1}^{n_{\bar{A}}} \left(\mathfrak{a}_{A,\alpha} - \mathfrak{a}_{\bar{A},\beta} + \varepsilon_{\bar{A}} \right) \tag{119}$$

3.3.1. Anomalies and other definitions of perturbative factors. In [AGT] another normalization for the perturbative prefactor is used: the second line in (112) would read, in our notation, as

$$\prod_{1 \leq \alpha < \beta \leq n_A} \prod_{\bar{a} \in \bar{A}} \Gamma_2^{-1}(\mathfrak{a}_{A,\alpha} - \mathfrak{a}_{A,\beta} + \varepsilon_{\bar{a}}; \varepsilon_a, \varepsilon_b)$$

This normalization makes explicit the symmetry between $\bar{a} \in \bar{A}$, however the gauge invariance, i.e. the Weyl symmetry of $U(n_A)$ acting on $\mathfrak{a}_{A,\alpha}$ is partly broken.

Unlike the instanton partition function Z^{inst} the perturbative factor does depend on the choice of the order on $\underline{4}$ which is used in the definition of $m_A = \varepsilon_{\varphi(A)}$. This dependence will be analyzed elsewhere.

3.3.2. Subtleties for tuned parameters. When the equivariant parameters $\underline{a}, \bar{\varepsilon}$ are rationally dependent, the torus $\tilde{T}_H \subset T_H$ they generate is strictly smaller than T_H . Accordingly, the fixed points on the moduli space of spiked instantons need not be isolated, and the formula (103) is used. It can be further localized to the set of torus-fixed points on $\mathcal{M}_{k_A}(n_A)$, which are relatively well-understood in the case $n_A = 1$ [I, IY, L]. We shall encounter these complications when dealing with gauge theories on \mathbb{C}^2/Γ spaces, or on the complex surfaces in the $\mathbb{C}^4/\Gamma' \times \Gamma''$ spaces, with finite $SU(2)$ subgroups $\Gamma, \Gamma', \Gamma'' \subset SU(2)$ of D or E type.

3.4. The main result. Here is the main fact about the partition function of spiked instantons: the compactness theorem proven in [Ne3] implies $Z_{\text{spiked}}(\underline{a}, \bar{\varepsilon})$ defined by

$$Z_{\text{spiked}}(\underline{a}, \bar{\varepsilon}) = \sum_{\underline{\lambda}} W_{\underline{\lambda}} \tag{120}$$

has no singularities in the variables:

$$x_A = \frac{1}{n_A} \sum_{\alpha=1}^{n_A} \mathfrak{a}_{A,\alpha} \tag{121}$$

with fixed

$$\tilde{\mathfrak{a}}_{A,\alpha} = \mathfrak{a}_{A,\alpha} - x_A \tag{122}$$

Remark. The reason we have to keep the majority of our variables fixed is the denominator Γ_2^{-1} in the perturbative prefactors $Z_{N=2*}^{\text{pert},A}$. Without it the partition function would have been an entire function of all $\mathfrak{a}_{A,\alpha}$'s.

4. Orbi Folding

In this section we discuss the partition function of the generalized gauge theories defined on the orbifolds with respect to a discrete (finite) group Γ . Both the worldvolume and the transverse space of the theory may be subject to the orbifold projection. Geometrically, the action of Γ factors through the linear action in \mathbb{C}^4 , which we assume to preserve the Calabi–Yau fourfold structure:

$$\rho_{\text{geom}} : \Gamma \longrightarrow SU(4) \tag{123}$$

This construction is motivated by the consideration of D-branes on \mathbb{C}^4/Γ . As is explained in [DM], the orbifold projection involves an action of Γ on the Chan-Paton spaces:

$$\rho_{\text{CP}} : \Gamma \longrightarrow \bigtimes_{A \in \mathfrak{6}} U(n_A) \tag{124}$$

which amounts to the decomposition:

$$N_A = \bigoplus_{\varpi \in \Gamma^\vee} N_{A, \varpi} \otimes \mathcal{R}_\varpi \tag{125}$$

The global symmetry group H is reduced to the Γ -centralizer: the subgroup $H_\Gamma \subset H$ which commutes with Γ . The particular cases of this construction are the quiver gauge theories, the theories in the presence of special surface operators, possibly intersecting, and the theories on the ALE spaces.

The parameters of the partition function of the orbifolded theory are $(\tilde{\mathfrak{a}}, \tilde{\mathfrak{e}}) \in \text{Lie } T_{H_\Gamma} \otimes \mathbb{C}$, where $\tilde{\mathfrak{a}}$ is in the Cartan subalgebra of the centralizer of the image $\rho_{\text{CP}}(\Gamma)$ in (124), while $\tilde{\mathfrak{e}}$ is in the Cartan subalgebra of the centralizer of the image of $\rho_{\text{geom}}(\Gamma)$ in $SU(4)$. In addition, the fugacity q of the original model fractionalizes:

$$q \longrightarrow \underline{q} = (q_\varpi)_{\varpi \in \Gamma^\vee} \tag{126}$$

4.0.1. Choices of discrete groups. Since we want the action of Γ to admit the invariant complex two-planes supporting the strata (X_A, G_A) of the generalized gauge theories, at least for one $A \in \mathfrak{6}$, the choice of Γ is reduced to the following three possibilities:

(1) The **abelian** case:

$$\Gamma = \Gamma_{\text{ab}}, \tag{127}$$

represented in $U(1)_\varepsilon^3 \subset SU(4)$ with the help of the homomorphism $\rho_{\text{geom}} = \text{diag}(\rho_a)_{a \in \mathfrak{4}}$:

$$\rho_a \left[\left(e^{\frac{2\pi \sqrt{-1} m_\kappa}{p_\kappa^{l_\kappa}}} \right)_\kappa \right] = \exp \left(2\pi \sqrt{-1} \sum_\kappa \frac{m_\kappa n_a^\kappa}{p_\kappa^{l_\kappa}} \right) \tag{128}$$

where $n_a^\kappa = 0, \dots, p_\kappa^{l_\kappa} - 1$, and

$$\sum_{a \in \mathfrak{4}} n_a^\kappa = p_\kappa^{l_\kappa} n^\kappa \quad n^\kappa \in \mathbb{Z}, \tag{129}$$

The Chan-Paton representation (124) amounts to the choice of multiplicities $n_{A,\mathbf{v}}$:

$$\rho_{\text{CP}} : \left(e^{2\pi\sqrt{-1}m_\kappa p_\kappa^{-l_\kappa}} \right)_\kappa \mapsto \text{diag} \left(\prod_\kappa e^{2\pi\sqrt{-1}m_\kappa n^\kappa p_\kappa^{-l_\kappa}} \cdot \mathbf{1}_{n_{A,\mathbf{v}}} \right)_{\mathbf{v} \in \Gamma_{\text{ab}}^\vee} \in U(n_A) \tag{130}$$

where $\mathbf{v} = (n^\kappa)_\kappa \in \Gamma_{\text{ab}}^\vee$ labels the irreducible (one-dimensional) representations of Γ_{ab} . The centralizer \mathbf{H}_Γ is equal to

$$\mathbf{H}_\Gamma = U(1)_\varepsilon^3 \times \prod_{\mathbf{v} \in \Gamma_{\text{ab}}^\vee} \prod_{A \in \underline{\mathfrak{6}}} U(n_{A,\mathbf{v}}), \tag{131}$$

its maximal torus

$$T_{\mathbf{H}_\Gamma} = U(1)_\varepsilon^3 \times \prod_{\mathbf{v} \in \Gamma_{\text{ab}}^\vee} \prod_{A \in \underline{\mathfrak{6}}} T_{n_{A,\mathbf{v}}}, \tag{132}$$

with $T_{n_{A,\mathbf{v}}} \subset U(n_{A,\mathbf{v}})$ the maximal torus of diagonal matrices. Define [cf. (60)]:

$$\mathbf{N}_{\Gamma_{\text{ab}}} = \bigsqcup_{A \in \underline{\mathfrak{6}}, \mathbf{v} \in \Gamma_{\text{ab}}^\vee} [n_{A,\mathbf{v}}] = \{ (A, \mathbf{v}, \alpha) \mid A \in \underline{\mathfrak{6}}, \mathbf{v} \in \Gamma_{\text{ab}}^\vee, \alpha \in [n_{A,\mathbf{v}}] \} \tag{133}$$

(2) The **abelian** \times **ALE** case:

$$\Gamma = \Gamma_{\text{ab}} \times \Gamma_\gamma \tag{134}$$

represented in $SU(4)$ with the help of the homomorphism

$$\rho_{\text{geom}} = \begin{pmatrix} \rho_1 = \rho_l \rho_r & & \\ & \rho_2 = \rho_l \rho_r^{-1} & \\ & & \rho_{34} = \rho_l^{-1} T_2 \in U(2)_{34} \end{pmatrix}, \tag{135}$$

with

$$\rho_\alpha \left[\left(e^{2\pi\sqrt{-1}m_\kappa p_\kappa^{-l_\kappa}} \right)_\kappa \times h \right] = \prod_\kappa e^{2\pi\sqrt{-1}m_\kappa \rho_\alpha^\kappa p_\kappa^{-l_\kappa}} \in U(1), \quad \alpha = l, r \tag{136}$$

where $\rho_\alpha^\kappa = 0, \dots, p_\kappa^{l_\kappa} - 1$, and

$$\rho_{34} \left[\left(e^{2\pi\sqrt{-1}m_\kappa p_\kappa^{-l_\kappa}} \right)_\kappa \times h \right] = \prod_\kappa e^{-2\pi\sqrt{-1}m_\kappa \rho_1^\kappa p_\kappa^{-l_\kappa}} T_2(h) \tag{137}$$

with T_2 the defining two-dimensional representation of $SU(2) \ni \Gamma_\gamma$.

The irreducible representations of the group $\Gamma = \Gamma_{\text{ab}} \times \Gamma_\gamma$ are the tensor products:

$$\mathcal{R}_\mathfrak{w} = \mathcal{L}_\mathbf{v} \otimes R_\mathbf{i}, \tag{138}$$

labelled by the pairs $\mathfrak{w} = (\mathbf{v}, \mathbf{i})$, $\mathbf{v} \in \Gamma_{\text{ab}}^\vee$, $\mathbf{i} \in \text{Vert}_\gamma$. With the choice (134) of Γ the only non-trivial Chan-Paton spaces are N_{12} and N_{34} . The choice of the Chan-Paton representation ρ_{CP} in this case amounts to the choice of multiplicity spaces $N_{12,\mathfrak{w}}, N_{34,\mathfrak{w}}$, i.e. the dimension vectors

$$n_\mathfrak{w} = \dim N_{12,\mathfrak{w}}, \quad w_\mathfrak{w} = \dim N_{34,\mathfrak{w}}, \tag{139}$$

The centralizer

$$H_\Gamma = U(1)_\gamma^2 \times \bigtimes_{\varpi \in \Gamma^\vee} U(n_\varpi) \times U(w_\varpi) \tag{140}$$

where $U(1)_\gamma^2 \subset U(1)_\varepsilon^3$ consists of the diagonal matrices of the form:

$$\begin{pmatrix} e^{\sqrt{-1}\vartheta_1} & & & \\ & e^{\sqrt{-1}\vartheta_2} & & \\ & & e^{-\frac{\sqrt{-1}}{2}(\vartheta_1+\vartheta_2)} \cdot \mathbf{1}_2 & \\ & & & \end{pmatrix} \in SU(4), \tag{141}$$

Define [cf. (60), (133)], for $\Gamma = \Gamma_{ab} \times \Gamma_\gamma$:

$$N_\Gamma = N_\Gamma^+ \sqcup N_\Gamma^- \tag{142}$$

with

$$\begin{aligned} N_\Gamma^+ &= \bigsqcup_{\varpi \in \Gamma^\vee} [n_\varpi] = \{ (\varpi, \alpha) \mid \varpi \in \Gamma^\vee, \alpha \in [n_\varpi] \}, & N_\Gamma^- &= \bigsqcup_{\mathbf{i} \in \text{Vert}_\gamma} N_\Gamma^{\mathbf{i}, -}, \\ N_\Gamma^{\mathbf{i}, -} &= \bigsqcup_{\mathbf{v} \in \Gamma_{ab}^\vee} [w_{\mathbf{i}, \mathbf{v}}] = \{ (\mathbf{i}, \mathbf{v}, \beta) \mid \mathbf{v} \in \Gamma_{ab}^\vee, \beta \in [w_{\mathbf{i}, \mathbf{v}}] \} \end{aligned} \tag{143}$$

(3) The **ALE** \times **ALE** case:

$$\Gamma = \Gamma_{ab} \times \Gamma_{\gamma'} \times \Gamma_{\gamma''} \tag{144}$$

represented in $SU(4)$ with the help of the homomorphism

$$\rho_{\text{geom}} [t \times h' \times h''] = \begin{pmatrix} \rho(t) \cdot T_2(h') & & 0 \\ & 0 & \rho(t)^{-1} \cdot T_2(h'') \\ & & \end{pmatrix} \in SU(4),$$

with $h' \in \Gamma_{\gamma'}$, $h'' \in \Gamma_{\gamma''}$, $\rho \in \Gamma_{ab}^\vee$. The irreducible representations of Γ are the tensor products

$$\mathcal{R}_\varpi = \mathcal{L}_\mathbf{v} \otimes R'_{\mathbf{i}'} \otimes R''_{\mathbf{i}''} \tag{145}$$

labelled by $\varpi = (\mathbf{v}, \mathbf{i}', \mathbf{i}'')$, where $\mathbf{v} \in \Gamma_{ab}^\vee$, $\mathbf{i}' \in \text{Vert}_{\gamma'}$, $\mathbf{i}'' \in \text{Vert}_{\gamma''}$, and R', R'' are the irreps of $\Gamma_{\gamma'}, \Gamma_{\gamma''}$, respectively. Again, with the choice (144) of Γ the only non-trivial Chan-Paton spaces are N_{12} and N_{34} . The choice of the Chan-Paton representation ρ_{CP} in this case amounts to the choice of multiplicity spaces $N_{12, \varpi}, N_{34, \varpi}$, i.e. the dimension vectors

$$n_\varpi = \dim N_{12, \varpi}, \quad w_\varpi = \dim N_{34, \varpi}, \tag{146}$$

for $\varpi = (\mathbf{v}, \mathbf{i}', \mathbf{i}'') \in \Gamma^\vee$. The centralizer

$$H_\Gamma = U(1)_{\gamma', \gamma''} \times \bigtimes_{\varpi \in \Gamma^\vee} U(n_\varpi) \times U(w_\varpi) \tag{147}$$

where $U(1)_{\gamma', \gamma''} \subset U(1)_\varepsilon^3$ consists of diagonal matrices of the form:

$$\begin{pmatrix} e^{\sqrt{-1}\vartheta} \cdot \mathbf{1}_2 & & & \\ & e^{-\sqrt{-1}\vartheta} \cdot \mathbf{1}_2 & & \\ & & & \end{pmatrix} \in SU(4), \tag{148}$$

Define [cf. (60), (133), (142)], for $\Gamma = \Gamma_{\text{ab}} \times \Gamma_{\gamma'} \times \Gamma_{\gamma''}$:

$$\mathbf{N}_{\Gamma} = \mathbf{N}_{\Gamma}^+ \sqcup \mathbf{N}_{\Gamma}^- \tag{149}$$

with $\mathbf{N}_{\Gamma}^+ = \bigsqcup_{\mathbf{i}' \in \text{Vert}_{\gamma'}} \mathbf{N}_{\Gamma}^{\mathbf{i}',+}$, $\mathbf{N}_{\Gamma}^- = \bigsqcup_{\mathbf{i}'' \in \text{Vert}_{\gamma''}} \mathbf{N}_{\Gamma}^{\mathbf{i}'',-}$, and

$$\begin{aligned} \mathbf{N}_{\Gamma}^{\mathbf{i}',+} &= \bigsqcup_{\mathbf{v} \in \Gamma_{\text{ab}}^{\vee}, \mathbf{i}'' \in \text{Vert}_{\gamma''}} [n_{\mathbf{v}, \mathbf{i}', \mathbf{i}''}] = \{ (\mathbf{v}, \mathbf{i}', \mathbf{i}'', \alpha) \mid \mathbf{v} \in \Gamma_{\text{ab}}^{\vee}, \mathbf{i}'' \in \text{Vert}_{\gamma''}, \alpha \in [n_{\mathbf{v}, \mathbf{i}', \mathbf{i}''}] \} , \\ \mathbf{N}_{\Gamma}^{\mathbf{i}'',-} &= \bigsqcup_{\mathbf{v} \in \Gamma_{\text{ab}}^{\vee}, \mathbf{i}' \in \text{Vert}_{\gamma'}} [w_{\mathbf{v}, \mathbf{i}', \mathbf{i}''}] = \{ (\mathbf{v}, \mathbf{i}', \mathbf{i}'', \beta) \mid \mathbf{v} \in \Gamma_{\text{ab}}^{\vee}, \mathbf{i}' \in \text{Vert}_{\gamma'}, \beta \in [w_{\mathbf{v}, \mathbf{i}', \mathbf{i}''}] \} \end{aligned} \tag{150}$$

In what follows the expressions N_A, K_A etc. are promoted to $\mathcal{N}_A, \mathcal{K}_A$ etc. which are valued in $K[T_{\text{H}\Gamma}] \otimes K[\Gamma]$, i.e. they are the formal linear combinations:

$$\sum_{w \in T_{\text{H}\Gamma}^{\vee}, \varpi \in \Gamma^{\vee}} n_{w, \varpi} L_w \otimes \mathcal{R}_{\varpi} \tag{151}$$

where L_w are the characters of $T_{\text{H}\Gamma}$, and \mathcal{R}_{ϖ} are the irreducible representations of Γ . Likewise, the ‘‘tangent space’’ character T_A is promoted to $\mathcal{T}_A \in K[T_{\text{H}\Gamma}] \otimes K[\Gamma]$.

4.0.2. Orbifold partition functions. The definition of the partition function in the orbifold situation is the following. The random variable is a string $\underline{\lambda}$ of objects, which now involve both Young diagrams and connected components of the Nakajima–Young varieties, specifically:

- (1) In the **abelian** case the random variables are again the Young diagrams (partitions) $\lambda^{(A, \mathbf{v}, \alpha)} \in \Lambda$, now labeled by triples :

$$\underline{\lambda}_{\text{ab}} = \left(\lambda^{(A, \mathbf{v}, \alpha)} \right)_{A \in \underline{6}, \mathbf{v} \in \Gamma_{\text{ab}}^{\vee}, \alpha \in [n_{A, \mathbf{v}}]} \in \Lambda^{\mathbf{N}_{\Gamma, \text{ab}}} \tag{152}$$

- (2) In the **abelian** \times **ALE** case the random variables are the collections of two types of objects: Young diagrams as before, and the connected components of Nakajima–Young varieties:

$$\begin{aligned} \underline{\lambda}_{\text{ab} \times \text{ale}} &= \left(\left(\lambda^{(\varpi, \alpha)} \right)_{\varpi \in \Gamma^{\vee}, \alpha \in [n_{\varpi}]} ; \left(\mu^{(\varpi, \beta)} \right)_{\varpi \in \Gamma^{\vee}, \beta \in [w_{\varpi}]} \right) \\ &\in \Lambda^{\mathbf{N}_{\Gamma}^+} \times \prod_{\mathbf{i} \in \text{Vert}_{\gamma}} \left(\Lambda_{\gamma}^{\mathbf{i}} \right)^{\mathbf{N}_{\Gamma}^{\mathbf{i},-}} \end{aligned} \tag{153}$$

with $\lambda^{(\varpi, \alpha)} \in \Lambda$, $\varpi \in \Gamma^{\vee}$, $\mu^{(\mathbf{i}, \mathbf{v}, \beta)} \in \Lambda_{\gamma}^{\mathbf{i}}$, for $\mathbf{i} \in \text{Vert}_{\gamma}$, $\mathbf{v} \in \Gamma_{\text{ab}}^{\vee}$.

- (3) Finally, in the **ALE** \times **ALE** case the random variables are the collections of connected components of Nakajima–Young varieties:

$$\underline{\lambda}_{\text{ale} \times \text{ale}} = \left(\left(\mu^{(\varpi, \alpha)} \right)_{\varpi \in \Gamma^{\vee}, \alpha \in [n_{\varpi}]} ; \left(\tilde{\mu}^{(\varpi, \beta)} \right)_{\varpi \in \Gamma^{\vee}, \beta \in [w_{\varpi}]} \right) \tag{154}$$

with $\mu^{(\varpi, \alpha)} \in \Lambda_{\gamma'}^{\mathbf{i}'}$, $\tilde{\mu}^{(\varpi, \beta)} \in \Lambda_{\gamma''}^{\mathbf{i}''}$, with $\varpi = (\mathbf{v}, \mathbf{i}', \mathbf{i}'')$, $\mathbf{v} \in \Gamma_{\text{ab}}^{\vee}$, $\mathbf{i}' \in \text{Vert}_{\gamma'}$, $\mathbf{i}'' \in \text{Vert}_{\gamma''}$.

We first describe the case of abelian orbifolds, and then proceed with the somewhat more restricted case of the non-abelian orbifolds. In the latter case our formulas are less explicit.

4.1. *Abelian orbifolds.* We define the statistical model, which is parametrized by the following generalization of the data of the Sect. 2:

- (1) The string $\bar{\varepsilon} = (\varepsilon_a)_{a \in \underline{4}}$ of 4 complex numbers $\varepsilon_a, a \in \underline{4}$ which sum up to zero, as in (59)
- (2) The string $\rho_{\text{geom}} = (\rho_a)_{a \in \underline{4}}$ of 4 irreducible Γ_{ab} -representations $\rho_a \in \Gamma_{\text{ab}}^\vee$ obeying

$$\sum_{a \in \underline{4}} \rho_a = 0 \in \Gamma_{\text{ab}}^\vee \tag{155}$$

In other words, ρ is a homomorphism $\Gamma_{\text{ab}} \rightarrow U(1)_\varepsilon^3 \subset SU(4)$, so that

$$\square_{SU(4)} = \bigoplus_{a \in \underline{4}} \mathcal{L}_{\rho_a} \tag{156}$$

We shall also use the notation ρ_S with $S \subset \underline{4}$ for the sum:

$$\rho_S = \sum_{s \in S} \rho_s. \tag{157}$$

so that $\rho_\emptyset = \rho_{\underline{4}} = \rho_0$ and

$$\wedge^\bullet \square_{SU(4)} = \bigoplus_{S \subset \underline{4}} \mathcal{L}_{\rho_S} \tag{158}$$

- (3) The string \bar{n} of 6 Γ_{ab} -representations

$$N_A = \bigoplus_{\mathfrak{v} \in \Gamma_{\text{ab}}^\vee} N_{A, \mathfrak{v}} \otimes \mathcal{L}_{\mathfrak{v}}, \quad A \in \underline{6} \tag{159}$$

with the multiplicity spaces $N_{A, \mathfrak{v}} \approx \mathbb{C}^{n_{A, \mathfrak{v}}}$ of dimensions $n_{A, \mathfrak{v}} = \dim N_{A, \mathfrak{v}}$.

- (4) The string $\bar{\alpha} = (\alpha_{A, \mathfrak{v}, \alpha})_{\alpha \in [n_{A, \mathfrak{v}}], \mathfrak{v} \in \Gamma_{\text{ab}}^\vee} \in \mathbb{C}^{\#\mathbf{N}_{\Gamma_{\text{ab}}}}$ of

$$\sum_{A \in \underline{6}, \mathfrak{v} \in \Gamma_{\text{ab}}^\vee} n_{A, \mathfrak{v}} = \#\mathbf{N}_{\Gamma_{\text{ab}}} \tag{160}$$

complex numbers $\alpha_{A, \mathfrak{v}, \alpha}, \alpha \in [n_{A, \mathfrak{v}}]$.

The data $(\bar{\alpha}; \bar{\varepsilon})$ parametrizes the Cartan subalgebra of the centralizer $\mathbf{H}_{\Gamma_{\text{ab}}}$. Define, for $A = (ab) \in \underline{6}, a, b \in \underline{4}, a < b$:

$$\mathcal{N}_A = \sum_{\mathfrak{v} \in \Gamma_{\text{ab}}^\vee} \sum_{\alpha \in [n_{A, \mathfrak{v}}]} e^{\beta \alpha_{A, \mathfrak{v}, \alpha}} \mathcal{L}_{\mathfrak{v}} = \sum_{\mathfrak{v} \in \Gamma_{\text{ab}}^\vee} N_{A, \mathfrak{v}}(\beta) \mathcal{L}_{\mathfrak{v}} \tag{161}$$

- (5) The string $\underline{q} = (q_{\mathfrak{v}})_{\mathfrak{v} \in \Gamma}$ of $|\Gamma_{\text{ab}}| = \prod_k p_k^k$ fugacities

$$q_{\mathfrak{v}} \in \mathbb{C}, \quad \mathfrak{v} \in \Gamma_{\text{ab}}^\vee \tag{162}$$

obeying $|q_{\mathfrak{v}}| < 1$.

Define, for $S \subset \underline{4}$:

$$P_{S, \mathbf{v}}(\beta) = \sum_{J \subset S} \prod_{a \in J} (-e^{\beta \varepsilon_a}) \delta_{\Gamma^{\mathbf{v}}} \left(-\mathbf{v} + \sum_{a \in J} \boldsymbol{\rho}_a \right), \quad \mathcal{P}_S = \sum_{\mathbf{v} \in \Gamma^{\mathbf{v}}} P_{S, \mathbf{v}} \mathcal{L}_{\boldsymbol{\rho}_S} \quad (163)$$

Define, for $\underline{\lambda} \in \Lambda^{N_{\Gamma_{ab}}}$, $A \in \underline{6}$, $A = (ab)$ as before:

$$\begin{aligned} \mathcal{K}_A &= \sum_{\mathbf{v} \in \Gamma_{ab}^{\mathbf{v}}} \sum_{\alpha \in [n_{A, \boldsymbol{\alpha}}]} \sum_{(i, j) \in \lambda^{(A, \mathbf{v}, \alpha)}} e^{\beta(a_{A, \mathbf{v}, \alpha} + \varepsilon_a(i-1) + \varepsilon_b(j-1))} \mathcal{L}_{\mathbf{v} + \boldsymbol{\rho}_a(i-1) + \boldsymbol{\rho}_b(j-1)} \\ &= \sum_{\mathbf{v} \in \Gamma_{ab}^{\mathbf{v}}} K_{A, \mathbf{v}}(\beta) \mathcal{L}_{\mathbf{v}} \end{aligned} \quad (164)$$

4.1.1. *The abelian orbifold model statistical weights.* Define, for $A = \{a, b\}$, $a < b \in \underline{4}$, cf. (163)

$$\mathcal{T}_A = \sum_{\mathbf{v} \in \Gamma_{ab}^{\mathbf{v}}} T_{A, \mathbf{v}} \mathcal{L}_{\mathbf{v}} = \mathcal{N}_A \mathcal{K}_A^* + q_A \mathcal{K}_A \mathcal{N}_A^* \otimes \mathcal{L}_{\boldsymbol{\rho}_A} - \mathcal{K}_A \mathcal{K}_A^* \mathcal{P}_A. \quad (165)$$

The statistical weight of $\underline{\lambda}$ is given by the following expression:

$$Z_{\underline{\lambda}}^{\Gamma_{ab}} = \left(\prod_{\mathbf{v} \in \Gamma_{ab}^{\mathbf{v}}} q_{\mathbf{v}}^{k_{\mathbf{v}}} \right) \epsilon \left[-T_{\underline{\lambda}}^{\Gamma_{ab}} \right] \quad (166)$$

where

$$\begin{aligned} k_{\mathbf{v}} &= \sum_{A \in \underline{6}, \alpha \in [n_{A, \mathbf{v}}]} |\lambda^{(A, \mathbf{v}, \alpha)}|, \quad (167) \\ T_{\underline{\lambda}}^{\Gamma_{ab}} &= \sum_{A \in \underline{6}, \mathbf{v} \in \Gamma_{ab}^{\mathbf{v}}} \left(P_{\varphi(A), -\mathbf{v}} T_{A, \mathbf{v}} + \sum_{\mathbf{v}' \in \Gamma_{ab}^{\mathbf{v}'}} \left(P_{\bar{A}, \mathbf{v}} N_{A, \mathbf{v}'} \left(\sum_{B \neq A} K_{B, \mathbf{v} + \mathbf{v}'} \right)^* \right. \right. \\ &\quad \left. \left. - P_{\underline{4}, \mathbf{v}} K_{A, \mathbf{v}'} \left(\sum_{B > A} K_{B, \mathbf{v} + \mathbf{v}'} \right)^* \right) \right), \end{aligned} \quad (168)$$

and the full abelian orbifold partition function is defined as

$$Z_{\text{spiked}}^{\Gamma_{ab}}(\tilde{\mathbf{a}}, \boldsymbol{\rho}, \bar{\varepsilon}; \tilde{\mathbf{q}}) = Z^{\text{pert}, \Gamma_{ab}}(\tilde{\mathbf{a}}, \boldsymbol{\rho}, \bar{\varepsilon}) \sum_{\underline{\lambda}} Z_{\underline{\lambda}}^{\Gamma_{ab}} \quad (169)$$

where the prefactor is given by the following formulas:

4.1.2. *The abelian orbifold gauge origami perturbative factors.* Define:

$$\begin{aligned}
 Z^{\text{pert}, \Gamma_{\text{ab}}}(\bar{\mathbf{a}}, \boldsymbol{\rho}, \bar{\varepsilon}) &= \prod_{A \in \underline{\mathbf{6}}} Z_{\mathcal{N}=2^*}^{\text{pert}, A, \Gamma_{\text{ab}}}(\mathbf{a}_A, \boldsymbol{\rho}_A, \boldsymbol{\rho}_B, \bar{\varepsilon}) \\
 &\times \prod_{\{a,b,c\} \subset \underline{\mathbf{4}}} Z_{\text{fold}}^{\text{pert}, a|bc, \Gamma_{\text{ab}}}(\mathbf{a}_{ab}, \mathbf{a}_{ac}, \boldsymbol{\rho}_a, \boldsymbol{\rho}_b, \boldsymbol{\rho}_c, \bar{\varepsilon}) \\
 &\times \prod_{A \in \underline{\mathbf{6}}, A < \bar{A}} Z_{\text{cross}}^{\text{pert}, A, \Gamma_{\text{ab}}}(\mathbf{a}_A, \mathbf{a}_{\bar{A}}, \boldsymbol{\rho}, \bar{\varepsilon})
 \end{aligned} \tag{170}$$

where $(A = ab)$:

$$\begin{aligned}
 &Z_{\mathcal{N}=2^*}^{\text{pert}, A, \Gamma_{\text{ab}}}(\mathbf{a}_A, \boldsymbol{\rho}, \bar{\varepsilon}) \\
 &= \prod_{\mathbf{v}, \mathbf{v}' \in \Gamma_{\text{ab}}^{\vee}} \prod_{\alpha \in [n_{A, \mathbf{v}}], \alpha' \in [n_{A, \mathbf{v}'}]} \frac{\Gamma_{2, \Gamma_{\text{ab}}} \left(\begin{matrix} \mathbf{a}_{A, \mathbf{v}, \alpha} - \mathbf{a}_{A, \mathbf{v}', \alpha'} ; \varepsilon_a, \varepsilon_b \\ \mathbf{v} - \mathbf{v}' ; \boldsymbol{\rho}_a, \boldsymbol{\rho}_b \end{matrix} \right)}{\Gamma_{2, \Gamma_{\text{ab}}} \left(\begin{matrix} \mathbf{a}_{A, \mathbf{v}, \alpha} - \mathbf{a}_{A, \mathbf{v}', \alpha'} + \varepsilon_{\varphi(A)} ; \varepsilon_a, \varepsilon_b \\ \mathbf{v} - \mathbf{v}' + \boldsymbol{\rho}_{\varphi(A)} ; \boldsymbol{\rho}_a, \boldsymbol{\rho}_b \end{matrix} \right)}
 \end{aligned} \tag{171}$$

where the projected double gamma

$$\Gamma_{2, \Gamma_{\text{ab}}} \left(\begin{matrix} x ; y', y'' \\ \mathbf{v} ; \boldsymbol{\rho}', \boldsymbol{\rho}'' \end{matrix} \right) \sim \prod_{i, j \geq 1} (x + y'(i-1) + y''(j-1))^{\delta_{\Gamma_{\text{ab}}}^{\vee}(\mathbf{v} + \boldsymbol{\rho}'(i-1) + \boldsymbol{\rho}''(j-1))} \tag{172}$$

can be easily expressed in terms of the ordinary Γ_2 's,

•

$$\begin{aligned}
 &Z_{\text{fold}}^{\text{pert}, a|bc, \Gamma_{\text{ab}}}(\mathbf{a}_{ab}, \mathbf{a}_{ac}, \boldsymbol{\rho}, \bar{\varepsilon}) \\
 &= \prod_{\mathbf{v}, \mathbf{v}' \in \Gamma_{\text{ab}}^{\vee}} \prod_{\alpha \in [n_{ab, \mathbf{v}}]} \prod_{\beta \in [n_{ac, \mathbf{v}'}]} \Gamma_{1, \Gamma_{\text{ab}}} \left(\begin{matrix} \mathbf{a}_{ab, \mathbf{v}, \alpha} - \mathbf{a}_{ac, \mathbf{v}', \beta} + \varepsilon_a + \varepsilon_c ; \varepsilon_a \\ \mathbf{v} - \mathbf{v}' + \boldsymbol{\rho}_a + \boldsymbol{\rho}_c ; \boldsymbol{\rho}_a \end{matrix} \right)
 \end{aligned} \tag{173}$$

where

$$\Gamma_{1, \Gamma_{\text{ab}}} \left(\begin{matrix} x ; y \\ \mathbf{v} ; \boldsymbol{\rho} \end{matrix} \right) \sim \prod_{i=1}^{\infty} (x + y(i-1))^{\delta_{\Gamma_{\text{ab}}}^{\vee}(\mathbf{v} + \boldsymbol{\rho}(i-1))} \tag{174}$$

can be easily expressed in terms of the ordinary gamma-functions,

•

$$\begin{aligned}
 &Z_{\text{cross}}^{\text{pert}, A, \Gamma_{\text{ab}}}(\mathbf{a}_A, \mathbf{a}_{\bar{A}}, \boldsymbol{\rho}, \bar{\varepsilon}) \\
 &= \prod_{\mathbf{v}, \mathbf{v}' \in \Gamma_{\text{ab}}^{\vee}} \prod_{\alpha \in [n_{A, \mathbf{v}}]} \prod_{\beta \in [n_{B, \mathbf{v}'}]} \left(\mathbf{a}_{A, \mathbf{v}, \alpha} - \mathbf{a}_{\bar{A}, \mathbf{v}', \beta} + \varepsilon_{\bar{A}} \right)^{\delta_{\Gamma_{\text{ab}}}^{\vee}(\mathbf{v} - \mathbf{v}' - \boldsymbol{\rho}_A)}
 \end{aligned} \tag{175}$$

4.2. **The abelian \times ALE case.** Fix $\Gamma_{\text{ab}}, \gamma$ of D or E type. Let $\Gamma = \Gamma_{\text{ab}} \times \Gamma_{\gamma}$. The irreps of Γ are labelled by the pairs $\overline{\omega} = (\mathbf{v}, \mathbf{i}), \mathbf{v} \in \Gamma_{\text{ab}}^{\vee}, \mathbf{i} \in \text{Vert}_{\gamma}$. Fix the discrete data: the dimension vectors $\underline{n} = (n_{\overline{\omega}})_{\overline{\omega} \in \Gamma^{\vee}}, \underline{\mathbf{w}} = (w_{\overline{\omega}})_{\overline{\omega} \in \Gamma^{\vee}}$, and the two characters $\rho_l, \rho_r : \Gamma_{\text{ab}} \rightarrow U(1)$. The orbifold gauge origami in this case depends on the following continuous data:

- (1) Two complex numbers $\varepsilon_1, \varepsilon_2$, and $\varepsilon = \varepsilon_1 + \varepsilon_2$.
- (2) Two sets of Coulomb parameters: $\underline{\mathbf{a}} = (\mathbf{a}_{\mathbf{v}, \mathbf{i}, \alpha}) \in \mathbb{C}^{\mathbf{N}_{\Gamma}^+}, \underline{\mathbf{b}} = (\mathbf{b}_{\mathbf{v}, \mathbf{i}, \beta}) \in \mathbb{C}^{\mathbf{N}_{\Gamma}^-}$

$$\mathbf{a}_{\mathbf{v}, \mathbf{i}, \alpha} \in \mathbb{C}, \quad \alpha \in [n_{\mathbf{v}, \mathbf{i}}], \quad \mathbf{b}_{\mathbf{v}, \mathbf{i}, \beta} \in \mathbb{C}, \quad \beta \in [w_{\mathbf{v}, \mathbf{i}}] \tag{176}$$

- (3) The string $\mathbf{q} = (\mathbf{q}_{\overline{\omega}})_{\overline{\omega} \in \Gamma^{\vee}}$ of $|\Gamma|$ fugacities:

$$\mathbf{q}_{\overline{\omega}} \in \mathbb{C}, \quad |\mathbf{q}_{\overline{\omega}}| < 1 \tag{177}$$

The geometric action $\Gamma \rightarrow SU(4)$ defines the following three representations:

$$\mathbb{C}_1^1 \equiv \mathcal{L}_{\rho_l + \rho_r}, \quad \mathbb{C}_2^1 \equiv \mathcal{L}_{\rho_l - \rho_r}, \quad \mathbb{C}_{34}^2 = \mathcal{L}_{-\rho_l} \otimes \mathbf{2}, \tag{178}$$

which obey

$$\mathbb{C}_1^1 \otimes \mathbb{C}_2^1 \otimes \Lambda^2 \mathbb{C}_{34}^2 = \mathcal{R}_0, \tag{179}$$

the trivial representation. Write $\mathcal{P}_4 = \mathcal{P}_{12} \mathcal{P}_{34}$, with:

$$\mathcal{P}_{12} = 1 - q_1 \mathbb{C}_1^1 - q_2 \mathbb{C}_2^1 + e^{\beta \varepsilon} \mathcal{L}_{2\rho_l}, \quad \mathcal{P}_{34} = 1 - q^{-\frac{1}{2}} \mathbb{C}_{34}^2 + q^{-1} \mathcal{L}_{-2\rho_l}. \tag{180}$$

Finally, define

$$\mathcal{N} = \sum_{\overline{\omega} \in \Gamma^{\vee}} \sum_{\alpha \in [n_{\overline{\omega}}]} e^{\beta \mathbf{a}_{\overline{\omega}, \alpha}} \mathcal{R}_{\overline{\omega}}, \quad \mathcal{W} = \sum_{\overline{\omega} \in \Gamma^{\vee}} \sum_{\tilde{\alpha} \in [w_{\overline{\omega}}]} e^{\beta \mathbf{b}_{\overline{\omega}, \tilde{\alpha}}} \mathcal{R}_{\overline{\omega}} \tag{181}$$

• The random variables $\underline{\lambda}_{\text{ab} \times \text{ale}}$ in the Γ -orbifold gauge origami model were defined in (153). The statistical weight of $\underline{\lambda}_{\text{ab} \times \text{ale}}$ is an integral over the product of Nakajima–Young varieties:

$$\mathcal{X}_{\underline{\lambda}_{\text{ab} \times \text{ale}}} = \prod_{\mathbf{v} \in \Gamma_{\text{ab}}^{\vee}} \prod_{\mathbf{i} \in \text{Vert}_{\gamma}} \prod_{\tilde{\alpha} \in [w_{\mathbf{v}, \mathbf{i}}]} \mathbf{Y}_{\gamma}^{\mathbf{i}}(\mu_{\mathbf{v}, \mathbf{i}, \tilde{\alpha}}), \quad \mu_{\mathbf{v}, \mathbf{i}, \tilde{\alpha}} \in \Lambda_{\gamma}^{\mathbf{i}} \tag{182}$$

Define

$$\mathcal{K} = \sum_{\mathbf{v} \in \Gamma_{\text{ab}}^{\vee}, \mathbf{i} \in \text{Vert}_{\gamma}, \alpha \in [n_{\mathbf{v}, \mathbf{i}}]} \sum_{(i, j) \in \lambda^{(\mathbf{i}, \overline{\omega}, \alpha)}} e^{\beta \mathbf{a}_{\mathbf{v}, \mathbf{i}, \alpha}} q_1^{i-1} q_2^{j-1} \mathcal{L}_{\mathbf{v} + \rho_l(i+j-2) + \rho_r(i-j)} \otimes R_{\mathbf{i}}, \tag{183}$$

$$\mathcal{T}_{12} = \mathcal{N} \mathcal{K}^* + \mathcal{N}^* \mathcal{K} q \mathcal{L}_{2\rho_l} - \mathcal{P}_{12} \mathcal{K} \mathcal{K}^* = \sum_{\overline{\omega} \in \Gamma^{\vee}} \mathcal{T}_{12, \overline{\omega}} \mathcal{R}_{\overline{\omega}} \tag{184}$$

so that, in particular

$$\begin{aligned} \mathcal{T}_{12, \mathbf{v}, \mathbf{0}} = & \sum_{\mathbf{i} \in \text{Vert}_{\gamma}} \sum_{\mathbf{v}' \in \Gamma_{\text{ab}}^{\vee}} \left(\mathcal{N}_{\mathbf{v} + \mathbf{v}', \mathbf{i}} \mathcal{K}_{\mathbf{v}', \mathbf{i}}^* + q \mathcal{N}_{2\rho_l + \mathbf{v}', \mathbf{i}}^* \mathcal{K}_{\mathbf{v} + \mathbf{v}', \mathbf{i}} - \mathcal{K}_{\mathbf{v} + \mathbf{v}', \mathbf{i}} \mathcal{K}_{\mathbf{v}', \mathbf{i}}^* \right. \\ & \left. + q_1 \mathcal{K}_{\mathbf{v} + \mathbf{v}', \mathbf{i}} \mathcal{K}_{\rho_l + \rho_r + \mathbf{v}', \mathbf{i}}^* + q_2 \mathcal{K}_{\mathbf{v} + \mathbf{v}', \mathbf{i}} \mathcal{K}_{\rho_l - \rho_r + \mathbf{v}', \mathbf{i}}^* + q \mathcal{K}_{\mathbf{v} + \mathbf{v}', \mathbf{i}} \mathcal{K}_{2\rho_l + \mathbf{v}', \mathbf{i}}^* \right). \end{aligned} \tag{185}$$

Define:

$$\mathcal{V} = \sum_{\mathbf{i}, \mathbf{i} \in \text{Vert}_\gamma} \sum_{\mathbf{v} \in \Gamma_{\text{ab}}^\vee} \sum_{\tilde{\alpha} \in [w_{\mathbf{v}, \mathbf{i}}]} \sum_{n \geq 0} e^{\beta \mathbf{b}_{\mathbf{v}, \mathbf{i}, \tilde{\alpha}}} q^{-\frac{n}{2}} \text{Ch} \left(V_{\mathbf{i}, n}^{\mathbf{i}}(\mu_{\mathbf{v}', \mathbf{i}', \tilde{\alpha}}) \right) \mathcal{L}_{\mathbf{v} - n \boldsymbol{\rho}_l} \otimes R_{\mathbf{i}} \quad (186)$$

$$\mathcal{T}_{34} = \mathcal{W} \mathcal{V}^* + q^{-1} \mathcal{W}^* \mathcal{V} \mathcal{L}_{-2 \boldsymbol{\rho}_l} - \mathcal{P}_{34} \mathcal{V} \mathcal{V}^* = \sum_{\boldsymbol{\omega} \in \Gamma^\vee} \mathcal{T}_{34, \boldsymbol{\omega}} \mathcal{R}_{\boldsymbol{\omega}} \quad (187)$$

We view $\mathcal{N}, \mathcal{W}, \mathcal{K}, \mathcal{V}, \mathcal{T}_{12}, \mathcal{T}_{34}$ as the $K[T_{\text{H}_\Gamma}] \otimes K(\Gamma)$ -valued linear combinations of Chern characters of vector bundles over $\mathcal{X}_{\underline{\lambda}_{\text{ab} \times \text{ale}}}$, as well as $\mathcal{P}_1 = 1 - q_1 \mathbb{C}_1^1$, $\mathcal{P}_2 = 1 - q_2 \mathbb{C}_2^1$.

The measure (89) dressed with a partial perturbative contribution, the orbifold version of (175), is generalized to

$$\mathcal{Z}_{\Gamma, \text{cross}}^{\text{pert}} \mathcal{Z}_{\underline{\lambda}_{\text{ab} \times \text{ale}}} = \left(\prod_{\boldsymbol{\omega} \in \Gamma^\vee} q^{k_{\boldsymbol{\omega}}} \right) \int_{\mathcal{X}_{\underline{\lambda}_{\text{ab} \times \text{ale}}}} \epsilon \left[-[\mathcal{R}_0] \mathcal{T}_{\underline{\lambda}_{\text{ab} \times \text{ale}}} + [L_0 \mathcal{R}_0] \mathcal{T}_{\underline{\lambda}_{\text{ab} \times \text{ale}}} \right], \quad (188)$$

where [cf. (57)]:

$$\begin{aligned} [\mathcal{R}_0] \mathcal{T}_{\underline{\lambda}_{\text{ab} \times \text{ale}}} &= [\mathcal{R}_0] \left(-q \mathcal{L}_{2 \boldsymbol{\rho}_l} \mathcal{N}^* \mathcal{W} + \mathcal{T}_{12} + \mathcal{P}_{34} \mathcal{N} \mathcal{V}^* + \mathcal{P}_1 \mathcal{T}_{34} + \mathcal{P}_{12} \mathcal{W} \mathcal{K}^* - \mathcal{P}_4 \mathcal{K} \mathcal{V}^* \right) \\ &\quad - q^{-\frac{1}{2}} [\mathcal{R}_0] \sum_{\mathbf{v} \in \Gamma_{\text{ab}}^\vee} \sum_{e \in \text{Edge}_\gamma} \left(\mathcal{N}_{\mathbf{v} + \boldsymbol{\rho}_l, t(e)} \mathcal{K}_{\mathbf{v}, s(e)}^* \right. \\ &\quad \left. - \mathcal{K}_{\mathbf{v} + \boldsymbol{\rho}_l, t(e)} \mathcal{K}_{\mathbf{v}, s(e)}^* - q \mathcal{K}_{\mathbf{v} - \boldsymbol{\rho}_l, t(e)} \mathcal{K}_{\mathbf{v}, s(e)}^* \right. \\ &\quad \left. + q_1 \mathcal{K}_{\mathbf{v} - \boldsymbol{\rho}_r, t(e)} \mathcal{K}_{\mathbf{v}, s(e)}^* + q_2 \mathcal{K}_{\mathbf{v} + \boldsymbol{\rho}_r, t(e)} \mathcal{K}_{\mathbf{v}, s(e)}^* \right), \\ [L_0 \mathcal{R}_0] \mathcal{T}_{\underline{\lambda}_{\text{ab} \times \text{ale}}} &= [L_0 \mathcal{R}_0] \mathcal{T}_{34} = \mathcal{T}_{\mathcal{X}_{\underline{\lambda}_{\text{ab} \times \text{ale}}}} \end{aligned} \quad (189)$$

and $[\mathcal{R}_0](\dots)$ denotes taking the Γ -invariant part in (\dots) , i.e. the contribution of the trivial representation of Γ , while $[L_0 \mathcal{R}_0] \mathcal{T}$ [cf. (101)] denotes the $T_{\text{H}_\Gamma} \times \Gamma$ -invariant part. Geometrically, the $T_{\text{H}_\Gamma} \times \Gamma$ -invariant $[L_0 \mathcal{R}_0] \mathcal{T}_{34}$ is the tangent space to the variety $\mathcal{X}_{\underline{\lambda}_{\text{ab} \times \text{ale}}}$ so its contribution is subtracted from the measure as the rest is being integrated over $\mathcal{X}_{\underline{\lambda}_{\text{ab} \times \text{ale}}}$ (note that $L_0 \mathcal{R}_0 = L_0 \mathbf{R}_0$ as Γ_{ab} action is contained in T_{H_Γ}). Finally,

$$\begin{aligned} k_{\mathbf{v}, \mathbf{i}} &= \sum_{\mathbf{v}' \in \Gamma_{\text{ab}}^\vee} \left(\sum_{\alpha \in [n_{\mathbf{v}', \mathbf{i}}]} \sum_{(i, j) \in \lambda(\mathbf{v}', \mathbf{i}, \alpha)} \delta_{\Gamma_{\text{ab}}^\vee}(\mathbf{v}' + \boldsymbol{\rho}_l(i + j - 2) + \boldsymbol{\rho}_r(i - j) - \mathbf{v}) \right. \\ &\quad \left. + \sum_{\mathbf{i}' \in \text{Vert}_\gamma} \sum_{\tilde{\alpha} \in [w_{\mathbf{v}', \mathbf{i}'}]} \sum_{n \geq 0} \delta_{\Gamma_{\text{ab}}^\vee}(\mathbf{v}' - n \boldsymbol{\rho}_l - \mathbf{v}) \nu_{\mathbf{i}, n}^{\mathbf{i}'}(\mu_{\mathbf{v}', \mathbf{i}', \tilde{\alpha}}) \right) \end{aligned} \quad (190)$$

Of course, this formalism also applies to $\gamma = \widehat{A}_k$. In this case the formulas (190), (188) reduce to the $\varepsilon_3 = \varepsilon_4$ limit of the **abelian** orbifold case of crossed instantons [Ne3].

4.3. *The ALE × ALE case.* Fix Γ_{ab} , and two quivers γ', γ'' of D or E type. In this section $\Gamma = \Gamma_{\text{ab}} \times \Gamma_{\gamma'} \times \Gamma_{\gamma''}$, with its irreps $\varpi = (\nu, \mathbf{i}', \mathbf{i}'')$, $\nu \in \Gamma_{\text{ab}}^\vee$, $\mathbf{i}' \in \text{Vert}_{\gamma'}$, $\mathbf{i}'' \in \text{Vert}_{\gamma''}$.

Fix the discrete data: the dimension vectors $\mathbf{n} = (n_\varpi)_{\varpi \in \Gamma^\vee}$, $\mathbf{w} = (w_\varpi)_{\varpi \in \Gamma^\vee}$, one character $\rho : \Gamma_{\text{ab}} \rightarrow U(1)$, equivalently a representation $\mathcal{L}_\rho \in \Gamma_{\text{ab}}^\vee$. The orbifold gauge origami in this case depends on the following continuous data:

- (1) A complex number $\varepsilon \in \mathbb{C}$.
- (2) Two sets of Coulomb parameters [cf. (149)]:

$$\underline{\mathbf{a}} = (\mathbf{a}_{\varpi, \alpha}) \in \mathbb{C}^{\mathbf{N}_\Gamma^+}, \quad \underline{\mathbf{b}} = (\mathbf{b}_{\varpi, \beta}) \in \mathbb{C}^{\mathbf{N}_\Gamma^-}, \tag{191}$$

where $\mathbf{a}_{\varpi, \alpha} \in \mathbb{C}$, $\alpha \in [n_\varpi]$, $\mathbf{b}_{\varpi, \beta} \in \mathbb{C}$, $\beta \in [w_\varpi]$.

- (3) The string $\mathbf{q} = (\mathbf{q}_\varpi)_{\varpi \in \Gamma^\vee}$ of $|\Gamma|$ fugacities:

$$\mathbf{q}_\varpi \in \mathbb{C}, \quad |\mathbf{q}_\varpi| < 1 \tag{192}$$

The geometric action $\Gamma \rightarrow SU(4)$ defines the following two representations:

$$\mathbb{C}_{12}^2 \equiv \mathcal{L}_\rho \otimes \mathbf{2}', \quad \mathbb{C}_{34}^2 = \mathcal{L}_{-\rho} \otimes \mathbf{2}'' \tag{193}$$

Define:

$$\mathcal{P}_{12} = 1 - q^{\frac{1}{2}} \mathbb{C}_{12}^2 + q \mathcal{L}_{2\rho}, \quad \mathcal{P}_{34} = 1 - q^{-\frac{1}{2}} \mathbb{C}_{34}^2 + q^{-1} \mathcal{L}_{-2\rho} \tag{194}$$

- The random variables $\underline{\lambda}_{\text{ale} \times \text{ale}}$ in the Γ -orbifold gauge origami model were defined in (154). The statistical weight of $\underline{\lambda}_{\text{ale} \times \text{ale}}$ is given by the integral over the product of Nakajima–Young varieties

$$\mathcal{X}_{\underline{\lambda}_{\text{ale} \times \text{ale}}} = \prod_{\nu \in \Gamma_{\text{ab}}^\vee} \prod_{\mathbf{i}' \in \text{Vert}_{\gamma'}, \mathbf{i}'' \in \text{Vert}_{\gamma''}} \left(\prod_{\alpha \in [n_{\nu, \mathbf{i}', \mathbf{i}''}]} \mathbf{Y}_{\gamma'}^{\mathbf{i}'}(\mu^{(\nu, \mathbf{i}', \mathbf{i}''); \alpha}) \times \prod_{\beta \in [w_{\nu, \mathbf{i}', \mathbf{i}''}]} \mathbf{Y}_{\gamma''}^{\mathbf{i}''}(\tilde{\mu}^{(\nu, \mathbf{i}', \mathbf{i}''); \beta}) \right) \tag{195}$$

with $\mu^{(\varpi, \alpha)} \in \Lambda_{\gamma'}^{\mathbf{i}'}$, $\tilde{\mu}^{(\varpi, \beta)} \in \Lambda_{\gamma''}^{\mathbf{i}''}$. Define

$$\mathcal{N} = \sum_{\nu \in \Gamma_{\text{ab}}^\vee, \mathbf{i}' \in \text{Vert}_{\gamma'}, \mathbf{i}'' \in \text{Vert}_{\gamma''}} \sum_{\alpha \in [n_{\nu, \mathbf{i}', \mathbf{i}''}]} e^{\beta \mathbf{a}_{\nu, \mathbf{i}', \mathbf{i}''; \alpha}} \mathcal{L}_\nu \otimes R'_{\mathbf{i}'} \otimes R''_{\mathbf{i}''}, \tag{196}$$

$$\begin{aligned} \mathcal{K} &= \sum_{\substack{\mathbf{i}', \mathbf{i}'' \in \text{Vert}_{\gamma'} \\ \mathbf{i}'' \in \text{Vert}_{\gamma''} \\ \nu \in \Gamma_{\text{ab}}}} \sum_{\substack{n \geq 0 \\ \alpha \in [n_{\nu, \mathbf{i}', \mathbf{i}''}]} } e^{\beta \mathbf{a}_{\nu, \mathbf{i}', \mathbf{i}''; \alpha}} q^{\frac{n}{2}} \text{Ch} \left(V_{\mathbf{i}', n}^{\mathbf{i}'} \left(\mu^{(\nu, \mathbf{i}', \mathbf{i}''); \alpha} \right) \right) \mathcal{L}_{\nu+n\rho} \otimes R'_{\mathbf{i}'} \otimes R''_{\mathbf{i}''} \\ &= \sum_{\varpi \in \Gamma^\vee} \mathcal{K}_\varpi \mathcal{R}_\varpi, \end{aligned} \tag{197}$$

$$\mathcal{I}_{12} = \mathcal{N} \mathcal{K}^* + \mathcal{N}^* \mathcal{K} q \mathcal{L}_{2\rho} - \mathcal{P}_{12} \mathcal{K} \mathcal{K}^*, \tag{198}$$

and

$$\mathcal{W} = \sum_{\mathbf{v} \in \Gamma_{\text{ab}}^{\vee}, \mathbf{i}' \in \text{Vert}_{\gamma'}, \tilde{\mathbf{i}}'' \in \text{Vert}_{\gamma''}, \tilde{\alpha} \in [w_{\mathbf{v}, \mathbf{i}', \tilde{\mathbf{i}}''}] } e^{\beta \mathbf{b}_{\mathbf{v}, \mathbf{i}', \tilde{\mathbf{i}}''; \tilde{\alpha}}} \mathcal{L}_{\mathbf{v}} \otimes R'_{\mathbf{i}'} \otimes R''_{\tilde{\mathbf{i}}''}, \tag{199}$$

$$\begin{aligned} \mathcal{V} &= \sum_{\substack{\mathbf{i}' \in \text{Vert}_{\gamma'} \\ \tilde{\mathbf{i}}, \tilde{\mathbf{i}}'' \in \text{Vert}_{\gamma''} \\ \mathbf{v} \in \Gamma_{\text{ab}}^{\vee}}} \sum_{\substack{n \geq 0 \\ \tilde{\alpha} \in [w_{\mathbf{v}, \mathbf{i}', \tilde{\mathbf{i}}''}]} } e^{\beta \mathbf{b}_{\mathbf{v}, \mathbf{i}', \tilde{\mathbf{i}}; \tilde{\alpha}}} q^{-\frac{n}{2}} \text{Ch} \left(\tilde{V}_{\tilde{\mathbf{i}}, n}^{\mathbf{i}''} \left(\tilde{\mu}^{(\mathbf{v}, \mathbf{i}', \tilde{\mathbf{i}}''; \beta)} \right) \right) \mathcal{L}_{\mathbf{v} - n\rho} \otimes R'_{\mathbf{i}'} \otimes R''_{\tilde{\mathbf{i}}} \\ &= \sum_{\boldsymbol{\omega} \in \Gamma^{\vee}} \mathcal{V}_{\boldsymbol{\omega}} \mathcal{R}_{\boldsymbol{\omega}}, \end{aligned} \tag{200}$$

$$\mathcal{T}_{34} = \mathcal{W}\mathcal{V}^* + \mathcal{W}^*\mathcal{V}q^{-1}\mathcal{L}_{-2\rho} - \mathcal{P}_{34}\mathcal{V}\mathcal{V}^*, \tag{201}$$

a $K[T_{\text{HF}}] \otimes K(\Gamma)$ -valued linear combination of vector bundles over $\mathcal{X}_{\underline{\lambda}_{\text{ale}} \times \text{ale}}$, where the vector bundles over $\mathcal{X}_{\underline{\lambda}_{\text{ale}} \times \text{ale}}$ denoted with some abuse of notation by $V_{\mathbf{i}', n}^{\mathbf{i}} \left(\mu^{(\mathbf{v}, \mathbf{i}', \tilde{\mathbf{i}}''; \alpha)} \right)$, $\tilde{V}_{\tilde{\mathbf{i}}, n}^{\tilde{\mathbf{i}}''} \left(\tilde{\mu}^{(\mathbf{v}, \mathbf{i}', \tilde{\mathbf{i}}''; \beta)} \right)$ are the pullbacks of the bundles $V_{\mathbf{i}', n}^{\mathbf{i}} \left(\mu^{(\mathbf{v}, \mathbf{i}', \tilde{\mathbf{i}}''; \alpha)} \right) \rightarrow \mathbf{Y}_{\gamma'}^{\mathbf{i}} \left(\mu^{(\mathbf{v}, \mathbf{i}', \tilde{\mathbf{i}}''; \alpha)} \right)$, $V_{\tilde{\mathbf{i}}, n}^{\tilde{\mathbf{i}}''} \left(\tilde{\mu}^{(\mathbf{v}, \mathbf{i}', \tilde{\mathbf{i}}''; \beta)} \right) \rightarrow \mathbf{Y}_{\gamma''}^{\tilde{\mathbf{i}}''} \left(\tilde{\mu}^{(\mathbf{v}, \mathbf{i}', \tilde{\mathbf{i}}''; \beta)} \right)$ under the projections to the respective factors in (195).

The measure (89) dressed with a partial perturbative contribution, the Γ -orbifold version of (175), is now generalized to

$$\mathcal{Z}_{\Gamma, \text{cross}}^{\text{pert}} \mathcal{Z}_{\underline{\lambda}_{\text{ale}} \times \text{ale}} = \left(\prod_{\boldsymbol{\omega} \in \Gamma^{\vee}} q^{k_{\boldsymbol{\omega}}} \right) \int_{\mathcal{X}_{\underline{\lambda}_{\text{ale}} \times \text{ale}}} \epsilon \left[-[\mathcal{R}_0] \mathcal{T}_{\underline{\lambda}_{\text{ale}} \times \text{ale}} + [L_0 \mathcal{R}_0] \mathcal{T}_{\underline{\lambda}_{\text{ale}} \times \text{ale}} \right], \tag{202}$$

where [cf. (57)]:

$$\begin{aligned} [\mathcal{R}_0] \mathcal{T}_{\underline{\lambda}_{\text{ale}} \times \text{ale}} &= [\mathcal{R}_0] \left(-q \mathcal{L}_{2\rho} \mathcal{N}^* \mathcal{W} + \mathcal{T}_{12} + \mathcal{P}_{34} \mathcal{N} \mathcal{V}^* + \mathcal{T}_{34} + \mathcal{P}_{12} \mathcal{W} \mathcal{K}^* - \mathcal{P}_{\mathbf{4}} \mathcal{K} \mathcal{V}^* \right) \\ &\quad - [\mathcal{R}_0] \left(q^{-\frac{1}{2}} \mathbb{C}_{34}^2 \mathcal{N} \mathcal{K}^* + q^{\frac{1}{2}} \mathbb{C}_{12}^2 \mathcal{W} \mathcal{V}^* - q^{-\frac{1}{2}} \mathbb{C}_{34}^2 \mathcal{K} \mathcal{K}^* - q^{\frac{1}{2}} \mathbb{C}_{12}^2 \mathcal{V} \mathcal{V}^* \right) \\ &\quad - \sum_{\mathbf{v} \in \Gamma_{\text{ab}}^{\vee}} \sum_{e' \in \text{Edge}_{\gamma'}} \sum_{e'' \in \text{Edge}_{\gamma''}} \left(\mathcal{K}_{\mathbf{v}, t(e'), s(e'')} \mathcal{K}_{\mathbf{v}, s(e'), t(e'')}^* + \mathcal{K}_{\mathbf{v}, t(e'), t(e'')} \mathcal{K}_{\mathbf{v}, s(e'), s(e'')}^* \right) \\ &\quad - \sum_{\mathbf{v} \in \Gamma_{\text{ab}}^{\vee}} \sum_{e' \in \text{Edge}_{\gamma'}} \sum_{e'' \in \text{Edge}_{\gamma''}} \left(\mathcal{V}_{\mathbf{v}, s(e'), t(e'')} \mathcal{V}_{\mathbf{v}, t(e'), s(e'')}^* + \mathcal{V}_{\mathbf{v}, t(e'), t(e'')} \mathcal{V}_{\mathbf{v}, s(e'), s(e'')}^* \right) \\ [L_0 \mathcal{R}_0] \mathcal{T}_{\underline{\lambda}_{\text{ale}} \times \text{ale}} &= [L_0 \mathcal{R}_0] (\mathcal{T}_{12} + \mathcal{T}_{34}) = \mathcal{T}_{\mathcal{X}_{\underline{\lambda}_{\text{ale}} \times \text{ale}}} \end{aligned} \tag{203}$$

and

$$\begin{aligned} k_{\mathbf{v}, \mathbf{i}, \tilde{\mathbf{i}}} &= \sum_{\mathbf{v}' \in \Gamma_{\text{ab}}^{\vee}} \sum_{n \geq 0} \left(\sum_{\mathbf{i}' \in \text{Vert}_{\gamma'}} \sum_{\alpha \in [n_{\mathbf{v}', \mathbf{i}', \tilde{\mathbf{i}}}] } \delta_{\Gamma_{\text{ab}}^{\vee}} (\mathbf{v}' + n\rho - \mathbf{v}) v_{\mathbf{i}', n}^{\mathbf{i}} (\mu_{\mathbf{v}', \mathbf{i}', \tilde{\mathbf{i}}; \alpha}) + \right. \\ &\quad \left. + \sum_{\tilde{\mathbf{i}}'' \in \text{Vert}_{\gamma''}} \sum_{\tilde{\alpha} \in [w_{\mathbf{v}', \mathbf{i}, \tilde{\mathbf{i}}''}]} \delta_{\Gamma_{\text{ab}}^{\vee}} (\mathbf{v}' - n\rho - \mathbf{v}) v_{\tilde{\mathbf{i}}, n}^{\tilde{\mathbf{i}}''} (\tilde{\mu}_{\mathbf{v}', \mathbf{i}, \tilde{\mathbf{i}}''; \tilde{\alpha}}) \right) \end{aligned} \tag{204}$$

To compute the measure (204) we use (19) to write:

$$\begin{aligned}
 & [\mathcal{R}_0] (\mathbf{2}' \otimes \mathbf{2}'' \otimes \mathcal{K}\mathcal{K}^*) \\
 &= \sum_{\mathbf{v} \in \Gamma_{\text{ab}}^\vee} \sum_{e' \in \text{Edge}_{\gamma'}} \sum_{e'' \in \text{Edge}_{\gamma''}} \left(\mathcal{K}_{\mathbf{v},s(e'),s(e'')} \mathcal{K}_{\mathbf{v},t(e'),t(e'')}^* + \mathcal{K}_{\mathbf{v},t(e'),s(e'')} \mathcal{K}_{\mathbf{v},s(e'),t(e'')}^* \right. \\
 & \left. + \mathcal{K}_{\mathbf{v},s(e'),t(e'')} \mathcal{K}_{\mathbf{v},t(e'),s(e'')}^* + \mathcal{K}_{\mathbf{v},t(e'),t(e'')} \mathcal{K}_{\mathbf{v},s(e'),s(e'')}^* \right) \tag{205}
 \end{aligned}$$

and similarly for $\mathbf{2}' \otimes \mathbf{2}'' \otimes \mathcal{V}\mathcal{V}^*$. To compute, e.g. the contribution $[\mathcal{R}_0] (\mathcal{P}_{12}\mathcal{W}\mathcal{K}^*)$ to (204) we also use (19):

$$\begin{aligned}
 & [\mathcal{R}_0] (\mathcal{P}_{12}\mathcal{W}\mathcal{K}^*) \\
 &= \sum_{\mathbf{v} \in \Gamma_{\text{ab}}^\vee} \sum_{i' \in \text{Vert}_{\gamma'}} \left\{ \sum_{i'' \in \text{Vert}_{\gamma'}} \left(\mathcal{W}_{\mathbf{v},i',i''} \mathcal{K}_{\mathbf{v}+\rho,i',i''}^* + q \mathcal{W}_{\mathbf{v},i',i''} \mathcal{K}_{\mathbf{v}+2\rho,i',i''}^* \right) \right. \\
 & \left. - q^{\frac{1}{2}} \sum_{e \in \text{Edge}_{\gamma'}} \left(\mathcal{W}_{\mathbf{v},t(e),i''} \mathcal{K}_{\mathbf{v}+\rho,s(e),i''}^* + \mathcal{W}_{\mathbf{v},s(e),i''} \mathcal{K}_{\mathbf{v}+\rho,t(e),i''}^* \right) \right\} \tag{206}
 \end{aligned}$$

4.4. *The main fact.* For all Γ , let us denote by

$$x_A = \frac{1}{\sum_{\varpi \in \Gamma^\vee} n_{A,\varpi}} \sum_{\varpi \in \Gamma^\vee} \sum_{\alpha \in [n_{A,\varpi}]} \alpha_{A,\varpi} \tag{207}$$

The partition function of the orbifold gauge origami, defined by (169) in the **abelian** case, by

$$\mathcal{Z}_{\text{cross}}^\Gamma(\underline{\mathbf{a}}, \underline{\mathbf{b}}; \varepsilon_1, \varepsilon_2; \underline{\mathbf{q}}) = \sum_{\underline{\lambda}_{\text{ab} \times \text{ale}}} \widehat{\mathcal{Z}}_{\Gamma, \text{cross}}^{\text{pert}} z_{\underline{\lambda}_{\text{ab} \times \text{ale}}} \tag{208}$$

in the **abelian** \times **ALE** case,

$$\mathcal{Z}_{\text{cross}}^\Gamma(\underline{\mathbf{a}}, \underline{\mathbf{b}}; \varepsilon; \underline{\mathbf{q}}) = \sum_{\underline{\lambda}_{\text{ale} \times \text{ale}}} \widehat{\mathcal{Z}}_{\Gamma, \text{cross}}^{\text{pert}} z_{\underline{\lambda}_{\text{ale} \times \text{ale}}} \tag{209}$$

in the **ALE** \times **ALE** case, has no singularities in the x_A variables, with $\tilde{\mathbf{a}}_{A,\varpi} = \alpha_{A,\varpi} - x_A$ fixed. Again, this follows from the compactness theorem proven in [Ne3].

5. Conclusions and Outlook

The partition function of the gauge origami model, can be viewed as the expectation value in the $\mathcal{N} = 2^* U(n_A)$ theory on \mathbb{C}_A^2 of an operator. In the crossed case, $N_A N_B = 0$, $A \cap B \neq \emptyset$, this operator is the qq-character of the \widehat{A}_0 -type [Ne2]. In the orbifolded crossed case this operator is the qq-character of the $\widehat{\mathfrak{g}}_\gamma$ -type. The orbifold partition functions in the **abelian** case describe the \widehat{A} -type quiver gauge theories on the A -type ALE spaces in the presence of various surface defects invariant under the rotational symmetries of the maximal Ω -deformation. In the **abelian** \times **ALE** case these partition functions describe either the qq-characters of the \widehat{D} or \widehat{E} -type quiver gauge theories,

possibly with the surface defects, or the \widehat{A} -type quiver gauge theory on the D or E -type ALE space, possibly with a novel type of surface defect (which collapses to a point-like defect in the orbifold limit of the ALE space), and a qq-character. Finally, in the $\mathbf{ALE} \times \mathbf{ALE}$ case we are dealing with the \widehat{D} or \widehat{E} -type quiver gauge theories, on the D or E -type ALE space, with the qq-characters and novel surface defects.

The physics of these defects will be discussed in the companion paper [Ne7].

The regularity of these expectation values will be used in the forthcoming publications [Ne8, SX] to derive the KZ and BPZ equations [BPZ, KZ] on the partition functions of supersymmetric gauge theories with and without surface operators.

Acknowledgements. I am grateful to H. Nakajima for patient explanations about the quiver varieties and to the anonymous referee for very useful comments on the manuscript. The results of this paper were reported at various seminars [Ne4, Ne5]. The research was carried out at the IITP RAS with the support of the Russian Foundation for Sciences (Project No. 14-50-00150).

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Communicated by X. Yin