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# Repulsion in Low Temperature $\beta$-Ensembles

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**Abstract:** We prove a result on separation of particles in a two-dimensional Coulomb plasma, which holds provided that the inverse temperature $\beta$ satisfies $\beta > 1$. For large $\beta$, separation is obtained at the same scale as the conjectural Abrikosov lattice optimal separation.

Consider a large but finite system of identical point-charges $\{\zeta_i\}_1^n$ in the plane $\mathbb{C}$, in the presence of an external field $nQ$, such that $Q(\zeta)$ is "large" near $\zeta = \infty$. The system is picked randomly from the Boltzmann-Gibbs distribution at inverse temperature $\beta > 1$,

$$d\mathbf{P}_n^{(\beta)}(\zeta) = \frac{1}{Z_n^{(\beta)}} e^{-\beta H_n(\zeta)} \, dA^{\otimes n}(\zeta), \quad \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n.$$

Here $H_n$ is the total energy

$$H_n(\zeta_1, \ldots, \zeta_n) = \sum_{j \neq k} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_{j=1}^{n} Q(\zeta_j),$$

$dA = dxdy/\pi$ is Lebesgue measure on $\mathbb{C}$ divided by $\pi$. The constant $Z_n^{(\beta)} = \int e^{-\beta H_n} \, dA^{\otimes n}$ is the so-called partition function of the ensemble.

A random sample $\{\zeta_j\}_1^n$ might be termed "Coulomb gas", "one-component plasma", or "$\beta$-ensemble". For brevity, we use "system" as a synonym.

It is well-known that the system tends, on average, to follow Frostman's equilibrium measure in external potential $Q$. The support of the equilibrium measure is a compact set, which we call the droplet.

The rough approximation afforded by the equilibrium measure is too crude to reveal details on a microscopic scale. However, it is believed on physical grounds that the particles should be evenly spread out in the interior of the droplet, with a non-trivial behaviour near the boundary—the Hall effect. Everything of importance goes on in the vicinity of the droplet.

In this note, we prove that the distance between neighbouring particles at a given location in the plane is large with high probability. Further, the distance tends to increase with $\beta$, and as $\beta \to \infty$, we recover formally the separation theorem for Fekete sets from the papers [1,5].

*Remark.* The case of minimum-energy configurations or "Fekete sets" is sometimes referred to as "the case $\beta = \infty$". We will follow this tradition, but we want to emphasize that "$\beta = \infty$" is just a figure of thought, not a rigorous limit.

## 1. Formulation of Results

Let $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ be a suitable function of sufficient increase near $\infty$; precise conditions are given below. We call $Q$ the external potential.

Let $\mu$ be a compactly supported Borel probability measure on $\mathbb{C}$. The weighted logarithmic energy of $\mu$ is defined by

$$I_Q[\mu] = \iint_{\mathbb{C}^2} \log \frac{1}{|\zeta - \eta|} \, d\mu(\zeta) d\mu(\eta) + \int_{\mathbb{C}} Q \, d\mu.$$

Assuming that $Q$ obeys some natural conditions recalled below there is a unique compactly supported probability measure $\sigma$ which minimizes $I_Q$. This is Frostman's equilibrium measure in external potential $Q$. The support $S = \operatorname{supp} \sigma$ is known as the droplet, and the equilibrium measure takes the form (see [26])

$$d\sigma(\zeta) = \chi_S(\zeta) \Delta Q(\zeta) \, dA(\zeta),$$

where we write $\Delta = \partial\bar{\partial}$ for $1/4$ times the standard Laplacian; $\chi_S$ is the characteristic function of the set $S$.

*Remark.* Let $\{\zeta_j\}_1^n$ be a random sample with respect to $\mathbf{P}_n^{(\beta)}$. Write $\mathbf{E}_n^{(\beta)}$ for the expectation with respect to $\mathbf{P}_n^{(\beta)}$. It is well-known that $\mathbf{E}_n^{(\beta)}[\frac{1}{n}\sum_{j=1}^n f(\zeta_j)] \to \sigma(f)$ as $n \to \infty$ for each continuous bounded function $f$ on $\mathbb{C}$. See [18,21].

The preceding remark shows that, in a sense, the equilibrium measure gives a first approximation to the macroscopic behaviour of the system. We here want to study microscopic properties. For this, we could fix a point $p \in \mathbb{C}$, which might depend on $n$, and zoom on it at an appropriate rate. However, for technical reasons it is easier to choose the coordinate system so that $p = 0$. In other words, 0 will in the following denote the origin of an $n$-dependent coordinate system which can be obtained from some static reference system by rigid motion.

Let $\mathbb{D}_r = \mathbb{D}_r(0)$ denote the disk center 0 radius $r$. By the *microscopic scale* $r_n$ at 0 we mean the radius such that

$$n \int_{\mathbb{D}_{r_n}} \Delta Q \, dA = 1.$$

We allow for any situation such that $r_n$ is well-defined. This is a mild restriction. Indeed, we always have $\Delta Q \geq 0$ on $S$, since $\sigma$ is a probability measure. By our assumptions below, this implies that $r_n$ is always well-defined if 0 is in the interior of $S$. Also, if we have $\Delta Q > 0$ on some portion of $\partial S$, then $r_n$ is well-defined when 0 is in some neighbourhood of that portion. Since the behaviour of the gas is of interest only in a neighbourhood of the droplet, we can thus essentially treat all cases of interest.

Given a sample $\{\zeta_j\}_1^n$, we rescale about 0 and consider the process $\{z_j\}_1^n$ where

$$z_j = r_n^{-1}\zeta_j. \tag{1}$$

We denote by $\mathbb{P}_n^{(\beta)}$ the image of $\mathbf{P}_n^{(\beta)}$ under the map (1) and write $\mathbb{E}_n^{(\beta)}$ for the corresponding expectation. Also let $\mathbb{D} = \mathbb{D}_1$ be the unit disk.

Fix a large $n$ and let $\mathscr{F}_n$ be the event that at least one of the $z_j$ falls in $\mathbb{D}$. Denote $\eta = \eta_n = \mathbb{P}_n^{(\beta)}(\mathscr{F}_n)$.

Given a random sample $\{z_j\}_1^n \in \mathscr{F}_n$ we define a number $s_0$ by

$$s_0 = \min_{z_j \in \mathbb{D}} \min_{k \neq j} |z_j - z_k|, \quad (\{z_j\}_1^n \in \mathscr{F}_n).$$

Thus $s_0$ is the largest rescaled distance from a particle in $\mathbb{D}$ to its nearest neighbour. We refer to $s_0$ as the *spacing* of the sample, in the vicinity of the point 0.

We are now prepared to formulate our main results. The following result shows that the strength of repulsion tends to increase with $\beta$.

**Theorem.** *Suppose that $\beta > 1$ and fix $n_0 \geq 1$. Then there is a constant $c = c(n_0, \beta) > 0$ such that if $n \geq n_0$ and $0 < \epsilon < 1$, then*

$$\mathbb{P}_n^{(\beta)}(\{s_0 \geq c \cdot n^{-\frac{1}{\beta-1}} \cdot (\epsilon\eta)^{\frac{1}{2(\beta-1)}}\} \,|\, \mathscr{F}_n) \geq 1 - m_0\epsilon, \tag{2}$$

*where $m_0 = 16 n^{\frac{2}{\beta-1}} c^{-2} (\epsilon\eta)^{-\frac{1}{\beta-1}}$. Moreover, given any $\beta_0 > 1$, $c$ can be chosen independent of $\beta$ when $\beta \geq \beta_0$.*

The left hand side in (2) should be understood as a conditional probability given that $\mathscr{F}_n$ has occurred.

The next result gives a kind of separation which holds for large $\beta$. To this end, it is natural to assume some kind of lower bound on the probability $\eta_n$. One possibility is to assume that $\inf \eta_n > 0$, which is certainly a reasonable assumption in many cases. However, it will suffice to assume existence of some number $\vartheta \geq 0$ such that

$$\eta_n \geq \text{const.} n^{-2\vartheta}, \quad (\text{const.} > 0). \tag{3}$$

For simplicity, we will also assume that we are zooming on a regular point,

$$\Delta Q(0) \geq \text{const.} > 0. \tag{4}$$

**Corollary.** *Below fix a positive number $c$ with $c < 1/(8\sqrt{e})$.*

*(i) Suppose that (4) holds and let $n$ be a given large integer. Then*

$$\lim_{\beta \to \infty} \mathbb{P}_n^{(\beta)}(\{s_0 > c\} \,|\, \mathscr{F}_n) = 1.$$

*(ii) Suppose that (3) and (4) hold. Also fix a parameter $\mu > 0$. Then*

$$\lim_{n \to \infty} \inf_{\beta \geq \mu \log n} \mathbb{P}_n^{(\beta)}\left(\left\{s_0 > c e^{-(1+\vartheta)/\mu}\right\} \,|\, \mathscr{F}_n\right) = 1.$$

Condition (3) is reasonable when the droplet is "sufficiently present" at 0, see concluding remarks.

Case (i) of the corollary comes close to an unpublished result due to Lieb in the zero temperature case, see [24, Theorem 4] as well as [25]; cf. [5] for an independent proof. Our estimate for the constant $c$ should be compared with the asymptotic lower bound $1/\sqrt{e}$ for the distance between Fekete points obtained in [1, Theorem 1]. In fact, our method of proof is somewhat related to the approach in [1,5], see concluding remarks below.

In the present context, Abrikosov's conjecture states that under the conditions in Corollary, the system $\{z_j\}_1^n$ should more and more resemble a honeycomb lattice as $\beta \to \infty$. The distance between neighbouring particles in this lattice can be computed, leading naturally to the conjecture that the "right" bound for $c$ in Corollary should be $c < 2^{1/2}3^{-1/4}$. Cf. [1].

Here are precise assumptions to be used in the proofs below: (i) $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ is l.s.c.; (ii) the interior of the set $\Sigma := \{Q < \infty\}$ is non-empty; (iii) $Q$ is real-analytic on Int $\Sigma$; (iv) $\liminf_{\zeta\to\infty} Q(\zeta)/\log|\zeta|^2 > 1$; (v) $S \subset \text{Int}\,\Sigma$.

In addition, we freely use the following notation: The $dA$-measure of a subset $\omega \subset \mathbb{C}$ is denoted $|\omega|$. By $\mathscr{W}_n$ we mean the set of weighted polynomials $f = pe^{-nQ/2}$ where $p$ is a holomorphic polynomial of degree at most $n-1$. We denote averages by $\fint_\omega g = \frac{1}{|\omega|}\int_\omega g\,dA$. $\mathbb{D}_r(\zeta)$ denotes the disc center $\zeta$ radius $r$ and we write $\mathbb{D}_r = \mathbb{D}_r(0)$.

## 2. Proofs of the Main Results

Suppose that the Taylor expansion of $\Delta Q$ about 0 takes the form

$$\Delta Q = P + \text{"higher order terms"}$$

where $P \geq 0$, $P \not\equiv 0$, and $P$ is homogeneous of some degree $2k-2$. The existence of such a $P$ is of course a consequence of the real-analyticity of $Q$.

Following [7] we write $\tau_0$ for the positive constant such that

$$\tau_0^{-2k} = \frac{1}{2\pi k}\int_0^{2\pi} P(e^{i\theta})\,d\theta.$$

Note that $\tau_0$ can be cast in the form

$$\tau_0^{-2k} = \frac{\Delta^k Q(0)}{k[(k-1)!]^2}.$$

This follows easily by expressing $P$ as a polynomial in $\zeta$ and $\bar{\zeta}$.

Using $\tau_0$, we conveniently express the microscopic scale to a negligible error, as follows

$$r_n = \tau_0 n^{-1/2k}(1 + O(n^{-1/2k})), \quad (n \to \infty).$$

Note that if $k = 1$ then $\tau_0 = 1/\sqrt{\Delta Q(0)}$.

As in [7] we define a holomorphic polynomial $H$ by

$$H(\zeta) := Q(0) + 2\partial Q(0)\,\zeta + \cdots + \frac{2}{(2k)!}\partial^{2k} Q(0)\,\zeta^{2k}. \tag{5}$$

We will also use the dominant homogeneous part of $Q$ at 0, i.e., the function

$$Q_0(\zeta) = \sum_{i+j=2k,\, i,j\geq 1} \frac{\partial^i \bar{\partial}^j Q(0)}{i!j!} \zeta^i \bar{\zeta}^j.$$

The point is that we have the canonical decomposition (cf. [7])

$$Q(\zeta) = \operatorname{Re} H(\zeta) + Q_0(\zeta) + O(|\zeta|^{2k+1}), \quad (\zeta \to 0).$$

Below we fix a large integer $n_0$. The following Bernstein-type lemma is an elaboration of [1, Lemma 2.1].

**Lemma 1.** *Suppose that* $n \geq n_0$. *If* $f \in \mathscr{W}_n$ *and* $f(0) \neq 0$ *then there is a constant* $K = K(n_0)$ *such that*

$$|\nabla |f|(0)| \leq K r_n^{-1} \fint_{\mathbb{D}_{r_n}} |f|.$$

*If* $\Delta Q(0) \geq \text{const.} > 0$ *then we can take* $K(n_0) = 4\sqrt{e}(1 + o(1))$*,* $(n_0 \to \infty)$.

*Proof.* Denote $h = \operatorname{Re} H$ where $H$ is the polynomial in (5). Also write

$$q_0 = \sum_{i+j=2k,\, i,j\geq 1} \frac{|\partial^i \bar{\partial}^j Q(0)|}{i!j!}.$$

Since $r_n = \tau_0 n^{-1/2k}(1 + O(n^{-1/2k}))$ we have

$$|\zeta| \leq r_n \quad \Rightarrow \quad n|Q(\zeta) - h(\zeta)| \leq q_0 n |\zeta|^{2k} + O(n^{-1/2k}) \leq C_n, \tag{6}$$

where $C_n = \tau_0^{2k} q_0 + C n^{-1/2k}$.

Now note that

$$|\nabla |f|(\zeta)| = |p'(\zeta) - n\partial Q(\zeta) p(\zeta)| e^{-nQ(\zeta)/2}$$

and

$$\left|\nabla \left(|p| e^{-nh/2}\right)(\zeta)\right| = \left|\frac{d}{d\zeta}\left(p e^{-nH/2}\right)(\zeta)\right|.$$

Inserting $\zeta = 0$ it is now seen that

$$|\nabla |f|(0)| = \left|\frac{d}{d\zeta}\left(p e^{-nH/2}\right)(0)\right|.$$

We now apply a Cauchy estimate to deduce that if $r_n/2 \leq r \leq r_n$ then

$$\left|\frac{d}{d\zeta}\left(p e^{-H/2}\right)(0)\right| = \frac{1}{2\pi}\left|\int_{|\zeta|=r} \frac{p(\zeta) e^{-nH(\zeta)/2}}{\zeta^2}\, d\zeta\right| \leq \frac{2}{\pi r_n^2} \int_{|\zeta|=r} |p| e^{-nh/2}\, |d\zeta|.$$

By (6) the last integral is dominated by

$$e^{C_n/2} \int_{|\zeta|=r} |p| e^{-nQ/2}\, |d\zeta| = e^{C_n/2} \int_{|\zeta|=r} |f|\, |d\zeta|.$$

It follows that

$$|\nabla|f|(0)| \le \frac{4e^{C_n/2}}{\pi r_n^3}\int_{r_n/2}^{r_n} dr \int_{|\zeta|=r} |f|\,|d\zeta| \le K r_n^{-1} \fint_{\mathbb{D}_{r_n}} |f|,$$

where $K = 4\sup_{n\ge n_0}\{e^{C_n/2}\}$. If $\Delta Q(0) \ge$ const. $> 0$ then $k = 1$ and $\tau_0^2 q_0 = 1$, which gives $K \le 4e^{1/2+C/\sqrt{n_0}}$. □

The weighted Lagrange interpolation polynomials associated with a configuration $\{\zeta_j\}_1^n$ of distinct points are defined by

$$\ell_j(\zeta) = \left(\prod_{i\ne j}(\zeta-\zeta_i)/\prod_{i\ne j}(\zeta_j-\zeta_i)\right)\cdot e^{-n(Q(\zeta)-Q(\zeta_j))/2}, \qquad (j=1,\dots,n).$$

Note that $\ell_j \in \mathscr{W}_n$ and $\ell_j(\zeta_k) = \delta_{jk}$.

Now let $\{\zeta_j\}_1^n$ be a random sample from $\mathbf{P}_n^{(\beta)}$. Then $\ell_j(\zeta)$ is a random variable which depends on the sample and on $\zeta$. In the next few lemmas, we fix an index $j$, $1 \le j \le n$.

**Lemma 2.** *Suppose that $U$ is a measurable subset of $\mathbb{C}$ of finite measure $|U|$. Then*

$$\mathbf{E}_n^{(\beta)}\left[\chi_U(\zeta_j)\cdot\int_{\mathbb{C}}|\ell_j(\zeta)|^{2\beta}\,dA(\zeta)\right] = |U|. \tag{7}$$

*Proof.* We shall use the following identity, whose verification is left to the reader

$$|\ell_j(\zeta)|^{2\beta}e^{-\beta H_n(\zeta_1,\dots,\zeta_j,\dots,\zeta_n)} = e^{-\beta H_n(\zeta_1,\dots,\zeta,\dots,\zeta_n)}.$$

By this and Fubini's theorem, integrating first in $\zeta_j$, we get

$$\int_{\mathbb{C}} dA(\zeta)\,\mathbf{E}_n^{(\beta)}\left[|\ell_j(\zeta)|^{2\beta}\cdot\chi_U(\zeta_j)\right]$$
$$= \int_U dA(\zeta_j)\int_{\mathbb{C}^n} d\mathbf{P}_n^{(\beta)}(\zeta_1,\dots,\zeta,\dots,\zeta_n) = |U|,$$

proving (7). □

In the sequel, we assume that $n \ge n_0$ and recall the constant $K = K(n_0)$ provided by Lemma 1. We will write $K(n_0,\zeta)$ the same constant with 0 replaced by $\zeta$ and let $r_n(\zeta)$ be the microscopic scale at $\zeta$. Finally, we fix a suitable, large enough, constant $M$; we may take $M = 3$ for example.

It is easy to see that there is a constant $T = T(M, n_0) \ge 1$ such that if $\zeta \in \mathbb{D}_{Mr_n}$ and $n \ge n_0$ then $T^{-1}r_n(\zeta) \le r_n(0) \le Tr_n(\zeta)$. If $\Delta Q(0) \ge$ const. $> 0$ we might take $T = 1 + o(1)$ as $n_0 \to \infty$.

**Lemma 3.** *We have that*

$$\frac{1}{r_n^2}\mathbf{E}_n^{(\beta)}\left[\chi_{\mathbb{D}_{r_n}}(\zeta_j)\cdot\int_{\mathbb{C}}|\ell_j(\zeta)|^{2\beta}\,dA(\zeta)\right] = 1. \tag{8}$$

*Now suppose that $\beta \ge 1/2$, $n \ge n_0$, $K = \sup_{\zeta\in\mathbb{D}_{Mr_n}} K(\zeta)$, and $r_n = r_n(0)$. Then*

$$\frac{1}{r_n^2}\mathbf{E}_n^{(\beta)}\left[\chi_{\mathbb{D}_{r_n}}(\zeta_j)\cdot\int_{\mathbb{D}_{Mr_n}}|\nabla|\ell_j|(\zeta)|^{2\beta}\,dA(\zeta)\right] \le T^{2\beta+4}K^{2\beta}r_n^{-2\beta}. \tag{9}$$

*Proof.* The identity (8) follows from Lemma 2 with $U = \mathbb{D}_{r_n}$.

To prove (9) we fix a non-zero $f \in \mathscr{W}_n$ and assume that $f(\zeta) \neq 0$ where $\zeta \in \mathbb{D}_{Mr_n}$. By Lemma 1 and Jensen's inequality, we have for all $\beta \geq 1/2$ that

$$|\nabla |f|(\zeta)|^{2\beta} \leq K^{2\beta} r_n(\zeta)^{-2\beta} \fint_{\mathbb{D}_{r_n(\zeta)}(\zeta)} |f|^{2\beta}.$$

Applying this with $f = \ell_j$ and taking expectations, we get

$$\begin{aligned}
&\mathbf{E}_n^{(\beta)} \left[ \int_{\mathbb{D}_{Mr_n}} |\nabla|\ell_j|(\zeta)|^{2\beta} \, dA(\zeta) \cdot \chi_{\mathbb{D}_{r_n}}(\zeta_j) \right] \\
&\leq K^{2\beta} \int_{\mathbb{D}_{Mr_n}} dA(\zeta) \, r_n(\zeta)^{-2\beta-2} \int_{\mathbb{D}_{r_n(\zeta)}(\zeta)} dA(\eta) \, \mathbf{E}_n^{(\beta)} \left[ \chi_{\mathbb{D}_{r_n}}(\zeta_j) \, |\ell_j(\eta)|^{2\beta} \right] \\
&\leq T^{2\beta+2} K^{2\beta} r_n^{-2\beta-2} \int_{\mathbb{C}} dA(\eta) \, \mathbf{E}_n^{(\beta)} \left[ \chi_{\mathbb{D}_{r_n}}(\zeta_j) \, |\ell_j(\eta)|^{2\beta} \right] \int_{\mathbb{D}_{Tr_n}(\eta)} dA(\zeta) \\
&= T^{2\beta+4} K^{2\beta} r_n^{-2\beta+2},
\end{aligned}$$

where we used (8) in the last step. $\square$

In the following, we let $z$ and $\zeta$ denote complex variables related via

$$z = r_n^{-1} \zeta.$$

We shall use the random functions $\varrho_j$ defined by

$$\varrho_j(z) = |\ell_j(\zeta)| = |\ell_j(r_n z)|.$$

Thus $\varrho_j(z_k) = \delta_{jk}$ where $\{z_k\}_1^n$ is the rescaled process.

**Lemma 4.** *Let $\beta \geq 1/2$. Then with notation as above*

$$\mathbb{E}_n^{(\beta)} \left[ \chi_{\mathbb{D}}(z_j) \, \|\nabla \varrho_j\|_{L^{2\beta}(\mathbb{D}_M)}^{2\beta} \right] \leq T^4 (TK)^{2\beta}. \tag{10}$$

*Proof.* The inequality (9) says that

$$\frac{1}{r_n^2} \mathbf{E}_n^{(\beta)} \left[ \chi_{\mathbb{D}_{r_n}}(\zeta_j) \int_{\mathbb{D}_{Mr_n}} (r_n |\nabla|\ell_j|(\zeta)|)^{2\beta} \, dA(\zeta) \right] \leq T^4 (TK)^{2\beta}.$$

Rescaling we immediately obtain (10). $\square$

Suppose that $\beta > 1$. We will use Morrey's inequality, which asserts that for all real-valued $f$ in the Sobolev space $W^{1,2\beta}(\mathbb{D}_M)$, all $z, w \in \mathbb{D}_{M/\sqrt{2}}$, we have

$$|f(z) - f(w)| \leq C \|\nabla f\|_{L^{2\beta}(\mathbb{D}_M)} |z - w|^{1-1/\beta}. \tag{11}$$

See [10, Corollary 9.12] and its proof. In fact, the proof in [10, p. 283] shows that (11) holds with $C = C_0(1 - 1/\beta)^{-1}$ where $C_0 \leq 2\pi^{1/2\beta}$.

We are now ready to finish the proof of Theorem.

Recall that $\mathscr{F}_n$ denotes the event that at least one particle hits $\mathbb{D}$ and fix an arbitrary $j$, $1 \le j \le n$. Assuming that $\mathbb{P}_n^{(\beta)}(\mathscr{F}_n) \ge \eta > 0$, we deduce from Lemma 4 the following inequality for the conditional expectation

$$\mathbb{E}_n^{(\beta)}\left(\chi_{\mathbb{D}}(z_j)\,\|\nabla \varrho_j\|^{2\beta}_{L^{2\beta}(\mathbb{D}_M)} \;\Big|\; \mathscr{F}_n\right) \le \frac{T^4(TK)^{2\beta}}{\eta}. \tag{12}$$

Fix $\epsilon > 0$ and recall that $s_0$ denotes the distance from a point in $\{z_j\}_1^n \cap \mathbb{D}$ to its closest neighbour, where we assume that $\{z_j\}_1^n \in \mathscr{F}_n$. We must prove that $s_0 \ge c(\epsilon\eta)^{1/2(\beta-1)}$ with (conditional) probability at least $1-\epsilon$.

For each $\lambda > 0$ we have by Chebyshev's inequality and (12)

$$\begin{aligned}\mathbb{P}_n^{(\beta)}\left(\left\{\chi_{\mathbb{D}}(z_j)\|\nabla \varrho_j\|^{2\beta}_{L^{2\beta}(\mathbb{D}_M)} > \lambda\right\} \;\Big|\; \mathscr{F}_n\right) &\le \frac{1}{\lambda}\mathbb{E}_n^{(\beta)}\left(\chi_{\mathbb{D}}(z_j)\|\nabla \varrho_j\|^{2\beta}_{L^{2\beta}(\mathbb{D}_M)} \;\Big|\; \mathscr{F}_n\right)\\ &\le \frac{T^4(TK)^{2\beta}}{\eta\lambda},\end{aligned}$$

which implies

$$\mathbb{P}_n^{(\beta)}\left(\left\{\sum_{j=1}^n \chi_{\mathbb{D}}(z_j)\|\nabla \varrho_j\|^{2\beta}_{L^{2\beta}(\mathbb{D}_M)} > n\lambda\right\} \;\Big|\; \mathscr{F}_n\right) \le n\frac{T^4(TK)^{2\beta}}{\eta\lambda}.$$

Given a random sample $\{z_j\}_1^n \in \mathscr{F}_n$ we let $I_n = I_n(\{z_j\}_1^n)$ be the random, nonempty set of indices $j$ for which $z_j \in \mathbb{D}$; we then have

$$\mathbb{P}_n^{(\beta)}\left(\left\{\sum_{j\in I_n} \|\nabla \varrho_j\|^{2\beta}_{L^{2\beta}(\mathbb{D}_M)} > n\lambda\right\} \;\Big|\; \mathscr{F}_n\right) \le n\frac{T^4(TK)^{2\beta}}{\eta\lambda}. \tag{13}$$

We now set

$$\lambda = n \cdot T^4(TK)^{2\beta}\eta^{-1}\epsilon^{-1}.$$

Consider the event $A_n$ consisting of all samples $\{z_j\}_1^n \in \mathscr{F}_n$ such that there is a $j \in I_n(\{z_j\}_1^n)$ for which $\|\nabla \varrho_j\|^{2\beta}_{L^{2\beta}(\mathbb{D}_M)} > n\lambda$. By (13) and our choice of $\lambda$ we have $\mathbb{P}_n^{(\beta)}(A_n|\mathscr{F}_n) \le \epsilon$. Hence, with conditional probability at least $1-\epsilon$,

$$j \in I_n(\{z_j\}_1^n) \quad \Rightarrow \quad \|\nabla \varrho_j\|_{L^{2\beta}(\mathbb{D}_M)} \le (n\lambda)^{1/2\beta} = n^{1/\beta}T^{2/\beta}(TK)(\epsilon\eta)^{-1/2\beta}. \tag{14}$$

Now fix a sample $\{z_j\}_1^n \in \mathscr{F}_n$ and an index $j \in I_n(\{z_j\}_1^n)$. Let $z_k$ be a closest neighbour to $z_j$. By Morrey's inequality (11) and (14) there is another constant $C = C_0(1-1/\beta)^{-1}$ such that, with conditional probability at least $1-\epsilon$, we have either $|z_j - z_k| \ge M/\sqrt{2}$ or

$$1 = |\varrho_j(z_j) - \varrho_j(z_k)| \le n^{1/\beta} \cdot CT^{1+2/\beta}K(\epsilon\eta)^{-1/2\beta}|z_j - z_k|^{1-1/\beta}, \tag{15}$$

i.e., $|z_j - z_k| \ge cn^{-\frac{1}{\beta-1}}(\epsilon\eta)^{\frac{1}{2(\beta-1)}}$ where we may chose

$$c = (CT^{1+2/\beta}K)^{-\beta/(\beta-1)} = (1-1/\beta)^{\beta/(\beta-1)}(C_0T^{1+2/\beta}K)^{-\beta/(\beta-1)}. \tag{16}$$

We have shown that

$$\min_{k\neq j}|z_j - z_k| \geq cn^{-\frac{1}{\beta-1}}(\epsilon\eta)^{\frac{1}{2(\beta-1)}} \tag{17}$$

with probability at least $1-\epsilon$. The formula (16) shows that $c$ can be chosen independent of $\beta$ when $\beta \geq \beta_0 > 1$.

**Lemma 5.** *Let $N_{\mathbb{D}}$ be the number of particles which fall in $\mathbb{D}$. Also define $r_0 = cn^{-\frac{1}{\beta-1}}(\epsilon\eta)^{\frac{1}{2(\beta-1)}}/2$ and $m_0 = 4/r_0^2$. Then $N_{\mathbb{D}} \leq m_0$ and $s_0 \geq cn^{-\frac{1}{\beta-1}}(\epsilon\eta)^{\frac{1}{2(\beta-1)}}$ with conditional probability at least $1 - m_0\epsilon$.*

*Proof.* Suppose that at least $m$ particles, denoted $z_1, \ldots, z_m$, fall in $\mathbb{D}$. For $1 \leq j \leq m$ let $E_j$ be the event that the disk $\mathbb{D}_{r_0}(z_j)$ contains no point $z_k$ with $1 \leq k \leq n$, $k \neq j$. Then $\mathbb{P}_n^{(\beta)}(E_j^c|\mathscr{F}_n) \leq \epsilon$, where $E_j^c$ is the complementary event, so

$$\mathbb{P}_n^{(\beta)}(\cap_{j=1}^m E_j|\mathscr{F}_n) = 1 - \mathbb{P}_n^{(\beta)}(\cup_{j=1}^m E_j^c|\mathscr{F}_n) \geq 1 - m\epsilon.$$

It follows that if at least $m$ particles fall in $\mathbb{D}$ then with probability at least $1 - m\epsilon$ there are $m$ disjoint disks of radius $r_0$ inside $\mathbb{D}_2$. Comparing areas we see that $mr_0^2 \leq 4$, i.e., $m \leq m_0$. □

The lemma says that if $m_0 = m_0(\beta, \epsilon, \eta, n) = 16n^{\frac{2}{\beta-1}}c^{-2}(\epsilon\eta)^{-\frac{1}{\beta-1}}$ then

$$\mathbb{P}_n^{(\beta)}(\{s_0 \geq cn^{-\frac{1}{\beta-1}}(\epsilon\eta)^{\frac{1}{2(\beta-1)}}\}|\mathscr{F}_n) \geq 1 - \epsilon m_0. \tag{18}$$

This proves Theorem.

Now assume that $\Delta Q(0) \geq$ const. $> 0$. Then by Lemma 1, the constant $K$ there might be taken as $K = 4\sqrt{e}(1+o(1))$ while we may take $T = 1+o(1)$ as $n \to \infty$. Hence $C_0 T^{1+2/\beta} K \leq 8\sqrt{e}(\pi T^4)^{1/2\beta}(1 + o(1))$, and so, if $c(\beta)$ denotes the largest constant such that the estimate (17) holds asymptotically, as $n_0 \to \infty$, then

$$\liminf_{\beta\to\infty} c(\beta) \geq 1/(8\sqrt{e}).$$

Applying the assumptions that $\eta \geq \text{const.}n^{-2\vartheta}$ and $\beta \geq \mu \log n$ we deduce that (as $n \to \infty$)

$$n^{-\frac{1}{\beta-1}}(\epsilon\eta)^{\frac{1}{2(\beta-1)}} \geq e^{-\frac{1}{\mu}-\frac{\vartheta}{\mu}}(1 + o(1))$$

and

$$m_0 \leq 2^{10}e^{1+\frac{2}{\mu}+\frac{2\vartheta}{\mu}}(1 + o(1)).$$

It is now clear from (18) that Corollary is a consequence of Theorem. □

## 3. Concluding Remarks

It is natural to ask for conditions implying that the probabilities $\eta_n = \mathbb{P}_n^{(\beta)}(\mathscr{F}_n)$ satisfy something like

$$\limsup_{n\to\infty} \frac{\log(1/\eta_n)}{\log n} < \infty.$$

In the case $\beta = 1$, the $\eta_n$ are bounded below if $\liminf_{n\to\infty} r_n^2 |\mathbb{D}_{r_n} \cap S| > 0$. ("The proportion of the area of $\mathbb{D}_{r_n}$ which falls inside the droplet is bounded below.") Proofs depending on estimates for the Bergman function can be found in [3,4,6]. On the other hand, if 0 is well in the exterior, or if 0 is a singular boundary point, $\eta_n$ drops off to zero quickly as $n \to \infty$. It is natural to expect a similar behaviour for any given $\beta$.

The analysis of Fekete configurations in [1,5] depends on the inequality $|\ell_j| \le 1$ for the associated weighted Lagrange polynomials $\ell_j$. This bound plays a similar role when $\beta = \infty$ as the $L^{2\beta}$-estimate in Lemma 2 does in the present case. The idea of using an $L^{2\beta}$-bound on Lagrange sections occurs in [14]. The context there is different, but in a way, we have elaborated on this idea here.

When $\beta = 1$, limiting point fields $\{z_j\}_1^\infty$ have been identified in many cases, [3]. When $\beta > 1$, the determinantal structure is lost, and the problem of calculating limiting point fields remains a challenge. The question is perhaps especially intriguing when we rescale about a regular boundary point of $S$, or about some other kind of special point, cf. [4,6,23].

At a regular boundary point, it seems plausible that the distribution should be translation invariant in the direction tangent to the boundary, i.e., in a suitable coordinate system, the distribution depends only on $x = \operatorname{Re} z$. The Hall effect is believed to give rise to certain irregularities in the distribution, which are to be located slightly to the inside of the boundary, see [12]. While our results provide more and more information when $\beta$ gets very large, the results in [12], by contrast, seem to be more accurate when $\beta$ is close to 1. A corresponding analysis was performed earlier in the bulk in [19]; see [13,15,20] for more recent developments.

In the case of "moderately sized" $\beta$, $1 \ll \beta \ll \infty$, neither of the methods seem to give very clear pictures of the situation. However, the recent paper [16] gives some results for the case $\beta = 2$. Moreover, the paper [11] suggests that a phase-transition ("freezing") should take place after a certain finite value $\beta = \beta_0$. The study of existence and possible size of melting temperature $1/\beta_0$ is currently an active area of research.

By the "hard edge $\beta$-ensemble" in external potential $Q$, we mean the ensemble obtained by redefining $Q$ to be $+\infty$ outside of the droplet. Cf. [3,4,27] for the case $\beta = 1$. The question of spacings in this setting will be taken up elsewhere.

Ward's identity (or "loop equation", "fundamental relation") is a relation connecting the one- and two-point functions of a $\beta$-ensemble. In the present context, it was used systematically by Wiegmann and Zabrodin and their school, and it is an important tool in conformal field theory (CFT). In fact a whole family of Ward identities is known, see [22].

In the paper [2], Ward's identity was used to give a relatively simple proof of Gaussian field convergence of linear statistics of a $\beta = 1$ ensemble. A similar statement is believed to hold for general $\beta$-ensembles. There has been progress on $\beta$-ensembles recently: the paper [8] seems to prove Gaussian field convergence in the *bulk* of the droplet. To the best of our knowledge, the full plane field convergence for general $\beta$ still seems to be an open problem. (Cf. however [9,17,21] for results in dimension 1.)

The microscopic version of Ward's identity was introduced fairly recently in [3]. It is called *Ward's equation*. See [3, Section 7.7] for the general case of $\beta$-ensembles. It is natural to ask how Ward's equation fits into the present context. We hope to come back to this later on.

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