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Length-Two Representations of Quantum Affine Superalgebras and Baxter Operators

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Abstract: Associated to quantum affine general linear Lie superalgebras are two families of short exact sequences of representations whose first and third terms are irreducible: the Baxter TQ relations involving infinite-dimensional representations; the extended T-systems of Kirillov–Reshetikhin modules. We make use of these representations over the *full* quantum affine superalgebra to define Baxter operators as transfer matrices for the quantum integrable model and to deduce Bethe Ansatz Equations, under genericity conditions.

Contents

Introduction

Fix $\mathfrak{g} := \mathfrak{gl}(M|N)$ a general linear Lie superalgebra and q a non-zero complex number that is not a root of unity. Let $U_q(\widehat{\mathfrak{g}})$ be the associated quantum affine superalgebra [48]. This is a Hopf superalgebra neither commutative nor co-commutative, and it can be seen as a q-deformation of the universal enveloping algebra of the affine Lie superalgebra of central charge zero $\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

In this paper we study a tensor category of (finite- and infinite-dimensional) representations of $U_q(\widehat{\mathfrak{g}})$. Its Grothendieck ring turns out to be commutative as is common in Lie Theory. We produce various identities of isomorphism classes of representations, and interpret them as functional relations of transfer matrices in the quantum integrable system attached to $U_q(\widehat{\mathfrak{g}})$, the XXZ spin chain.

1. Baxter operators. In an exactly solvable model a common problem is to find the spectrum of a family $T(z)$ of commuting endomorphisms of a vector space V depending on a complex spectral parameter z, called transfer matrices. The Bethe Ansatz method, initiated by H. Bethe, gives explicit eigenvectors and eigenfunctions of $T(z)$ in terms of solutions to a system of algebraic equations, the Bethe Ansatz equations (BAE). Typical examples are the Heisenberg spin chain and the ice model.

In [2], for the 6-vertex model R. Baxter related $T(z)$ to another family of commuting endomorphisms $Q(z)$ on V by the relation:

$$\text{TQ relation:} \qquad T(z) = a(z)\frac{Q(zq^2)}{Q(z)} + d(z)\frac{Q(zq^{-2})}{Q(z)}.$$

Here $a(z), d(z)$ are scalar functions and q is the parameter of the model. $Q(z)$ is a polynomial in z, called the Baxter operator. The cancellation of poles at the right-hand side becomes Bethe Ansatz equations for the roots of $Q(z)$. A similar operator equation holds for the 8-vertex model [2], where the Bethe Ansatz method fails.

Within the framework of Quantum Inverse Scattering Method, the transfer matrix $T(z)$ is defined in terms of representations of a quantum group $\mathbf{U}$. Let $\mathcal{R}(z) \in \mathbf{U}^{\otimes 2}$ be the universal R-matrix with spectral parameter z and let V, W be two representations of $\mathbf{U}$. Then $t_W(z) := \mathrm{tr}_W(\mathcal{R}(z)_{W\otimes V})$ forms a commuting family of endomorphisms on V, thanks to the quasi-triangularity of $(\mathbf{U}, \mathcal{R}(z))$. As examples, the transfer matrix for the 6-vertex model (resp. XXX spin chain) comes from tensor products of two-dimensional irreducible representations of the affine quantum group $U_q(\widehat{\mathfrak{sl}_2})$ (resp. Yangian $Y_\hbar(\mathfrak{sl}_2)$), while the face-type model of Andrews–Baxter–Forrester, which is equivalent to the 8-vertex model by a vertex-IRF correspondence, requires Felder's elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$ [20,21].

The representation meaning of the $Q(z)$ was understood in the pioneer work of Bazhanov–Lukyanov–Zamolodchikov [3] for $U_q(\widehat{\mathfrak{sl}_2})$, and extended to an arbitrary non-twisted affine quantum group $U_q(\widehat{\mathfrak{a}})$ of a finite-dimensional simple Lie algebra $\mathfrak{a}$ in the recent work of Frenkel–Hernandez [24]. One observes that the first tensor factor of $\mathcal{R}(z)$ lies in a Borel subalgebra $U_q(\mathfrak{b})$ of $U_q(\widehat{\mathfrak{a}})$, so the above transfer-matrix construction makes sense for $U_q(\mathfrak{b})$-modules. Notably the Baxter operators $Q(z)$ are transfer matrices of $L_{i,a}^+$, the *positive prefundamental modules* over $U_q(\mathfrak{b})$, for i a Dynkin node of $\mathfrak{a}$ and $a \in \mathbb{C}^\times$. The $L_{i,a}^+$ are irreducible objects of a category $\mathcal{O}_{\mathrm{HJ}}$ of $U_q(\mathfrak{b})$-modules introduced by Hernandez–Jimbo [34].

Making use of the prefundamental modules, Frenkel–Hernandez [24] solved a conjecture of Frenkel–Reshetikhin [27] on the spectra of the quantum integrable system, which connects eigenvalues of transfer matrices $t_W(z)$, for W finite-dimensional $U_q(\widehat{\mathfrak{a}})$-modules, with polynomials arising as eigenvalues of the Baxter operators.

The two-term TQ relations, as a tool to derive Bethe Ansatz Equations for the roots of Baxter polynomials, are consequences of identities in the Grothendieck ring $K_0(\mathcal{O}_{\mathrm{HJ}})$ of category $\mathcal{O}_{\mathrm{HJ}}$ [18,19,24,25,35]. Such identities are also examples of cluster mutations of Fomin–Zelevinsky [35].

In the elliptic case, the triangular structure of $\mathcal{R}(z)$ is less clear as there is not yet a formulation of Borel subalgebras. Still the eigenvalues of $T(z)$ admit TQ relations by a Bethe Ansatz in [21]. In a joint work with G. Felder [22], we were able to construct elliptic Baxter operator $Q(z)$ for $E_{\tau,\eta}(\mathfrak{sl}_2)$ as a transfer matrix of certain infinite-dimensional representations over the full elliptic quantum group.

Then a natural question is whether the Baxter operators can always be realized from representations of the full quantum group (of type Yangian, affine, or elliptic). Inspired by [22], in the present paper we provide a partial answer for the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$, based on the *asymptotic representations*, which we introduced in a previous work [53].

Let us mention the appearance of quantum affine superalgebras and Yangians in other supersymmetric integrable models like the deformed Hubbard model and anti de Sitter/conformal field theory correspondences; see [7,8] and references therein.

Compared to the intense works on affine quantum groups (see the reviews [13,40]), the representation theory of $U_q(\widehat{\mathfrak{g}})$ is still less understood as the super case poses one essential difficulty, the smallness of Weyl group symmetry.

2. Asymptotic representations. Before stating the main results of this paper, let us recall from [53] the asymptotic modules over $U_q(\widehat{\mathfrak{g}})$.

Let $I_0 := \{1, 2, \ldots, M+N-1\}$ be the set of Dynkin nodes of the Lie superalgebra $\mathfrak{g}$. There are $U_q(\widehat{\mathfrak{g}})$-valued power series $\phi_i^{\pm}(z)$ in $z^{\pm 1}$ for $i \in I_0$ whose coefficients mutually commute; they can be viewed as q-analogs of $A \otimes t^{\pm n} \in \widehat{\mathfrak{g}}$ with A being a diagonal matrix in $\mathfrak{g}$ and n a positive integer. Algebra $U_q(\widehat{\mathfrak{g}})$ admits a triangular decomposition whose Cartan part is generated by the $\phi_i^{\pm}(z)$. The highest weight representation theory built from this decomposition is suitable for the classification of finite-dimensional irreducible representations [49] in terms of rational functions.

Fix a Dynkin node $i \in I_0$ and a spectral parameter $a \in \mathbb{C}^{\times}$. To each positive integer k is attached a *Kirillov–Reshetikhin module*. It is a finite-dimensional irreducible $U_q(\widehat{\mathfrak{g}})$-module generated by a highest weight vector ω such that

$$\phi_j^{\pm}(z)\omega = \omega \quad \text{if } j \neq i, \quad \phi_i^{\pm}(z)\omega = \frac{q_i^k - zaq_i^{-k}}{1 - za}\omega.$$

Here $q_i = q$ for $i \leq M$ and $q_i = q^{-1}$ for $i > M$. In [53], we made an "analytic continuation" by taking q_i^k to be a fixed $c \in \mathbb{C}^{\times}$ as $k \to \infty$ to obtain a $U_q(\widehat{\mathfrak{g}})$-module $\mathscr{W}_{c,a}^{(i)}$. This is what we call an asymptotic module. It is a modification of the limit construction of prefundamental modules over Borel subalgebras in [3,34].

We defined in [53] a category $\mathcal{O}_{\mathfrak{g}}$ of representations of $U_q(\widehat{\mathfrak{g}})$ by imposing the standard weight condition as for Kac–Moody algebras [37] and dropping integrability condition [32,41]. It contains the $\mathscr{W}_{c,a}^{(i)}$ and all the finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules. Category $\mathcal{O}_{\mathfrak{g}}$ is monoidal and abelian.[1]

3. Main results. We prove the following property of Grothendieck ring $K_0(\mathcal{O}_{\mathfrak{g}})$:

(i) If $\mathscr{W}$ is an asymptotic module, then there exist three modules D, S', S'' in category $\mathcal{O}_{\mathfrak{g}}$ such that $[D][\mathscr{W}] = [S'] + [S'']$ and S', S'' are tensor products of asymptotic modules; see Theorem 5.3.

[1] In the main text we also study category $\mathcal{O}$ of representations of a Borel subalgebra of $U_q(\widehat{\mathfrak{g}})$, which admits prefundamental modules as in [34]; see Definition 1.4. Here $\mathcal{O}_{\mathfrak{g}}$ is the full subcategory of $\mathcal{O}$ consisting of $U_q(\widehat{\mathfrak{g}})$-modules.

Consider the XXZ spin chain of $U_q(\widehat{\mathfrak{g}})$. For $i \in I_0$, we define the Baxter operator $Q_i(u)$ to be the transfer matrix of $\mathscr{W}_{u,1}^{(i)}$ evaluated at 1 (Definition 9.6), as in the elliptic case [22]. To justify the definition, we prove the following facts.

(ii) If V is a finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module, then $t_V(z^{-2})$ is a sum of monomials of the $d(z)\frac{Q_i(zac)}{Q_i(za)}$ where $i \in I_0, a, c \in \mathbb{C}^\times$, and the $d(z)$ are scalar functions, the number of terms being dim V; see Corollary 9.7.

(iii) Each $Q_i(z)$ satisfies a two-term TQ relation; see Eq. (9.38).

Note that (ii) reduces the transfer matrix of an *arbitrary* finite-dimensional $U_q(\widehat{\mathfrak{g}})$ to the *finite* set $\{Q_i(u) \mid i \in I_0\}$ up to scalar functions. It forms generalized Baxter TQ relations in the sense of Frenkel–Hernandez [24].

4. Proofs. This requires the q-character map of Frenkel–Reshetikhin [27], which is an injective ring homomorphism from the Grothendieck ring $K_0(\mathcal{O}_{\mathfrak{g}})$ to a commutative ring of I_0-tuples of rational functions with parity (Proposition 1.8).

The q-character of an asymptotic module is fairly easy thanks to its limit construction in [53]. We obtain a separation of variable identity (SOV, Lemma 9.5),

$$[\mathscr{W}_{c,1}^{(i)}][\mathscr{W}_{1,a^2}^{(i)}] = [\mathscr{W}_{ca,a^2}^{(i)}][\mathscr{W}_{a^{-1},1}^{(i)}] \in K_0(\mathcal{O}_{\mathfrak{g}}).$$

This identity puts the parameters $c, a \in \mathbb{C}^\times$ in $\mathscr{W}_{c,a}^{(i)}$ at an equal role. It categorifies

$$\frac{c - zc^{-1}}{1-z} \times \frac{1-za^2}{1-za^2} = \frac{ca - zc^{-1}a}{1-za^2} \times \frac{a^{-1} - za}{1-z}.$$

In [53] we established generalized TQ relations in category $\mathcal{O}_{\mathfrak{g}}$, which together with SOV proves (ii). Similarly (iii) follows from (i) and SOV.

Along the proof of (i) we obtain results of independent interest:

- q-character formulas of four families of finite-dimensional irreducible $U_q(\widehat{\mathfrak{g}})$-modules, including all the Kirillov–Reshetikhin modules (Theorem 2.4);
- a criteria for a tensor product of Kirillov–Reshetikhin modules to admit an irreducible head (i.e. of highest weight, Theorem 6.1);
- short exact sequences of tensor products of Kirillov–Reshetikhin modules (Theorem 3.3).

The third point includes the T-system [31,42,44] as a special case.

5. Perspectives. We expect that our main results (i)–(iii) have analogy in elliptic quantum groups $E_{\tau,\hbar}(\mathfrak{a})$, based on twistor theory relating affine quantum groups to elliptic quantum groups [29,36,39]. For $\mathfrak{a} = \mathfrak{sl}_N$ this has been verified in [22,54]. For $\mathfrak{a}$ of general type, a category of $E_{\tau,\hbar}(\mathfrak{a})$-modules was studied in [30] with well-behaved q-character theory, although its tensor product structure is unclear.

It is possible to adapt the arguments to the case of Yangians (not necessarily of type A) in view of [29]. One could avoid *degenerate Yangian* [4,5,28], whose prefundamental representations lead to Baxter operators but do not carry natural action of the ordinary Yangian. [22, Appendix] discussed the $\mathfrak{gl}_2$ case. The Yangian of centrally extended $\mathfrak{psl}(2|2)$ [7] is of special interest in AdS/CFT. We do not know of any representation category $\mathcal{O}$ with well-behaved highest weight theory, yet there are limit constructions of infinite-dimensional representations [1].

For twisted quantum affine algebras $\mathbf{U}$, there are conjectural TQ relations in category $\mathcal{O}_{\mathrm{HJ}}$ [25]. One may ask for such relations in terms of $\mathbf{U}$-modules. This is interesting from another point of view: the correspondence between twisted quantum affine algebras and non-twisted quantum affine superalgebras [17,55]. (This is different from Langlands duality in that the Cartan matrices for these algebras are identical.) A typical example is the equivalence [17] of categories $\mathcal{O}_{\mathrm{int}}$ of integrable representations over $U_q(A_{2n}^{(2)})$ and $U_q(\widehat{\mathfrak{osp}(1|2n)})$. Let us mention an earlier work of Z. Tsuboi [45] on Bethe Ansatz Equations for orthosymplectic Lie superalgebras, the representation theory meaning of which is to be understood. One should need the Drinfeld second realization of quantum affine superalgebras [47].

The paper is structured as follows. In Sect. 1 we review the quantum affine superalgebra $U_q(\widehat{\mathfrak{g}})$ and its Borel subalgebra $Y_q(\mathfrak{g})$, and study the basic properties of category $\mathcal{O}$ of $Y_q(\mathfrak{g})$-modules. Section 5 presents the main result (i). In Sect. 9, for the $U_q(\widehat{\mathfrak{g}})$ XXZ spin chain, we construct Baxter operators from the $\mathscr{W}_{c,a}^{(i)}$ and derive Bethe Ansatz Equations from (i).

The two basics ingredients are: the q-character formulas in terms of Young tableaux, proved in Sect. 2; cyclicity of tensor products of Kirillov–Reshetikhin modules studied in Sect. 6. The q-characters already lead to TQ relations of positive prefundamental modules over $Y_q(\mathfrak{g})$ in Sects. 3 and 4. The proof of (i) is completed in Sect. 7 upon realizing D as a suitable asymptotic limit.

The extended T-systems of Kirillov–Reshetikhin modules are proved in Sect. 8. Although they are not needed in the proof of the main theorem, we include them here as applications of q-characters and cyclicity.

1. Basics on Quantum Affine Superalgebras

Fix $M, N \in \mathbb{Z}_{>0}$. In this section we collect basic facts on the quantum affine superalgebra associated with the general linear Lie superalgebra $\mathfrak{g} := \mathfrak{gl}(M|N)$ and its representations. The main references are [51–53], some of whose results are modified to be coherent with the non-graded quantum affine algebras.

Set $\kappa := M + N$, $I := \{1, 2, \dots, \kappa\}$ and $I_0 := I \backslash \{\kappa\}$. Let $\mathbb{Z}_2$ denote the ring $\mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$. The *weight lattice* $\mathbf{P}$ is the abelian group freely generated by the ϵ_i for $i \in I$. Let $||$ be the morphism of additive groups $\mathbf{P} \longrightarrow \mathbb{Z}_2$ such that

$$|\epsilon_1| = |\epsilon_2| = \cdots = |\epsilon_M| = \overline{0}, \quad |\epsilon_{M+1}| = |\epsilon_{M+2}| = \cdots = |\epsilon_{M+N}| = \overline{1}.$$

$\mathbf{P}$ is equipped with a symmetric bilinear form $(,) : \mathbf{P} \times \mathbf{P} \longrightarrow \mathbb{Z}$,

$$(\epsilon_i, \epsilon_j) = \delta_{ij}(-1)^{|\epsilon_i|} \quad \text{where } (-1)^{\overline{0}} := 1,\ (-1)^{\overline{1}} := -1.$$

Define $\alpha_i := \epsilon_i - \epsilon_{i+1}$ for $i \in I_0$, and the *root lattice* $\mathbf{Q}$ to be the subgroup of $\mathbf{P}$ generated by the α_i. Set $q_l := q^{(\epsilon_l, \epsilon_l)}$ and $q_{ij} := q^{(\alpha_i, \alpha_j)}$ for $i, j \in I_0$ and $l \in I$.

If W is a vector superspace and $w \in W$ is a $\mathbb{Z}_2$-homogeneous vector, then by abuse of language let $|w| \in \mathbb{Z}_2$ denote the parity of w. (It is not to be confused with the absolute value $|n|$ of an integer n.)

Let $\mathbf{V}$ be the vector superspace with basis $(v_i)_{i \in I}$ and parity $|v_i| := |\epsilon_i|$. Define the elementary matrices $E_{ij} \in \mathrm{End}(\mathbf{V})$ by $E_{ij} v_k = \delta_{jk} v_i$ for $i, j, k \in I$. They form a basis of the vector superspace $\mathrm{End}(\mathbf{V})$ and $|E_{ij}| = |\epsilon_i| + |\epsilon_j|$.

1.1. Quantum superalgebras. Recall the Perk–Schultz matrix [43]

$$R(z,w) = \sum_{i\in I}(zq_i - wq_i^{-1})E_{ii}\otimes E_{ii} + (z-w)\sum_{i\neq j} E_{ii}\otimes E_{jj}$$
$$+ z\sum_{i<j}(q_i - q_i^{-1})E_{ji}\otimes E_{ij} + w\sum_{i<j}(q_j - q_j^{-1})E_{ij}\otimes E_{ji}.$$

It is well-known that $R(z,w)$ satisfies the quantum Yang–Baxter equation:

$$R_{12}(z_1,z_2)R_{13}(z_1,z_3)R_{23}(z_2,z_3) = R_{23}(z_2,z_3)R_{13}(z_1,z_3)R_{12}(z_1,z_2) \in \mathrm{End}(\mathbf{V})^{\otimes 3}.$$

The convention for the tensor subscripts is as usual. Let $n \geq 2$ and $A_1, A_2, \ldots, A_n$ be unital superalgebras. Let $1 \leq i < j \leq n$. If $x \in A_i$ and $y \in A_j$, then

$$(x\otimes y)_{ij} := (\otimes_{k=1}^{i-1} 1_{A_k})\otimes x\otimes(\otimes_{k=i+1}^{j-1}1_{A_k})\otimes y\otimes(\otimes_{k=j+1}^{n}1_{A_k}) \in \otimes_{k=1}^n A_k.$$

Now we can define the quantum affine superalgebra associated to $\mathfrak{g}$.

Definition 1.1 [51, Section 3.1]. $U_q(\widehat{\mathfrak{g}})$ is the superalgebra with presentation:

(R1) RTT-generators $s_{ij}^{(n)}, t_{ij}^{(n)}$ of parity $|\epsilon_i| + |\epsilon_j|$ for $i, j \in I$ and $n \in \mathbb{Z}_{\geq 0}$;
(R2) RTT-relations in $U_q(\widehat{\mathfrak{g}}) \otimes (\mathrm{End}(\mathbf{V})^{\otimes 2})[[z, z^{-1}, w, w^{-1}]]$

$$\begin{aligned} R_{23}(z,w)T_{12}(z)T_{13}(w) &= T_{13}(w)T_{12}(z)R_{23}(z,w),\\ R_{23}(z,w)S_{12}(z)S_{13}(w) &= S_{13}(w)S_{12}(z)R_{23}(z,w),\\ R_{23}(z,w)T_{12}(z)S_{13}(w) &= S_{13}(w)T_{12}(z)R_{23}(z,w); \end{aligned}$$

(R3) $t_{ij}^{(0)} = s_{ji}^{(0)} = 0$ and $s_{kk}^{(0)}t_{kk}^{(0)} = 1$ for $i, j, k \in I$ and $i < j$.

$T(z) \in U_q(\widehat{\mathfrak{g}}) \otimes \mathrm{End}(\mathbf{V})[[z^{-1}]]$ and $S(z) \in U_q(\widehat{\mathfrak{g}}) \otimes \mathrm{End}(\mathbf{V})[[z]]$ are power series

$$T(z) = \sum_{ij} t_{ij}(z)\otimes E_{ij}, \quad t_{ij}(z) = \sum_{n\in\mathbb{Z}_{\geq 0}} t_{ij}^{(n)} z^{-n},$$
$$S(z) = \sum_{ij} s_{ij}(z)\otimes E_{ij}, \quad s_{ij}(z) = \sum_{n\in\mathbb{Z}_{\geq 0}} s_{ij}^{(n)} z^{n}.$$

The *Borel subalgebra* $Y_q(\mathfrak{g})$, also called q-Yangian,[2] is the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by the $s_{ij}^{(n)}$ and $(s_{ii}^{(0)})^{-1}$. The finite-type quantum supergroup $U_q(\mathfrak{g})$ is the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by the $s_{ij}^{(0)}$ and $t_{ij}^{(0)}$.

$U_q(\widehat{\mathfrak{g}})$ has a Hopf superalgebra structure with counit $\varepsilon : U_q(\widehat{\mathfrak{g}}) \longrightarrow \mathbb{C}$ defined by $\varepsilon(s_{ij}^{(n)}) = \varepsilon(t_{ij}^{(n)}) = \delta_{ij}\delta_{n0}$, and coproduct $\Delta : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}})^{\otimes 2}$:

$$\Delta(s_{ij}^{(n)}) = \sum_{m=0}^{n}\sum_{k\in I}\epsilon_{ijk}s_{ik}^{(m)}\otimes s_{kj}^{(n-m)}, \quad \Delta(t_{ij}^{(n)}) = \sum_{m=0}^{n}\sum_{k\in I}\epsilon_{ijk}t_{ik}^{(m)}\otimes t_{kj}^{(n-m)}.$$

[2] This is because the algebra $Y_q(\mathfrak{g})$ admits an $RTT = TTR$ type presentation, as does the ordinary Yangian $Y(\mathfrak{g})$. Here q is a parameter of R.

Here $\epsilon_{ijk} := (-1)^{|E_{ik}||E_{kj}|}$. The antipode $\mathbb{S} : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}})$ is determined by

$$(\mathbb{S} \otimes \mathrm{Id})(S(z)) = S(z)^{-1}, \quad (\mathbb{S} \otimes \mathrm{Id})(T(z)) = T(z)^{-1}.$$

$S(z)^{-1}$ and $T(z)^{-1}$ are well-defined owing to Definition 1.1 (R3). Notice that $Y_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$ are sub-Hopf-superalgebras of $U_q(\widehat{\mathfrak{g}})$.

We shall need $U_{q^{-1}}(\widehat{\mathfrak{g}})$, whose RTT generators are denoted by $\overline{s}_{ij}^{(n)}, \overline{t}_{ij}^{(n)}$.

Recall the following are isomorphisms of Hopf superalgebras ($a \in \mathbb{C}^\times$):

$$\Phi_a : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}}), \qquad s_{ij}^{(n)} \mapsto a^n s_{ij}^{(n)},\ t_{ij}^{(n)} \mapsto a^{-n} t_{ij}^{(n)}, \tag{1.1}$$

$$\Psi : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}})^{\mathrm{cop}}, \qquad s_{ij}^{(n)} \mapsto \varepsilon_{ji} t_{ji}^{(n)}, \quad t_{ij}^{(n)} \mapsto \varepsilon_{ji} s_{ji}^{(n)}, \tag{1.2}$$

$$h : U_{q^{-1}}(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}})^{\mathrm{cop}}, \quad \overline{S}(z) \mapsto S(z)^{-1}, \quad \overline{T}(z) \mapsto T(z)^{-1}. \tag{1.3}$$

Here $\varepsilon_{ij} := (-1)^{|\epsilon_i|+|\epsilon_i||\epsilon_j|}$ and A^{cop} of a Hopf superalgebra A takes the same underlying superalgebra but the twisted coproduct $\Delta^{\mathrm{cop}} := c_{A,A}\Delta$, with $c_{A,A} : x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$ the graded permutation, and antipode $\mathbb{S}^{-1}$. There are superalgebra morphisms for $p(z) \in \mathbb{C}[[z]]^\times$, $p_1(z) \in \mathbb{C}[[z^{-1}]]^\times$ with $p(0)p_1(\infty) = 1$:

$$\mathrm{ev}_a^+ : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\mathfrak{g}),\ s_{ij}(z) \mapsto \frac{s_{ij}^{(0)} - zat_{ij}^{(0)}}{1 - za},\ t_{ij}(z) \mapsto \frac{t_{ij}^{(0)} - z^{-1}a^{-1}s_{ij}^{(0)}}{1 - z^{-1}a^{-1}}, \tag{1.4}$$

$$\phi_{[p,p_1]} : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\widehat{\mathfrak{g}}),\ s_{ij}(z) \mapsto p(z)s_{ij}(z),\ t_{ij}(z) \mapsto p_1(z)t_{ij}(z). \tag{1.5}$$

$\Phi_a, h, \mathrm{ev}_a^+, \phi_{[p,p_1]}$ restrict to $Y_q(\mathfrak{g})$ or $Y_q(\mathfrak{g}')$, denoted by $\Phi_a, h, \mathrm{ev}_a^+, \phi_p$. Let $\overline{\mathrm{ev}}_a^+ : U_{q^{-1}}(\widehat{\mathfrak{g}}) \longrightarrow U_{q^{-1}}(\mathfrak{g})$ be the corresponding morphisms when replacing q by q^{-1}. This gives rise to (notice that $h(U_{q^{-1}}(\mathfrak{g})) = U_q(\mathfrak{g})$):

$$\mathrm{ev}_a^- : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_q(\mathfrak{g}), \quad \mathrm{ev}_a^- = h \circ \overline{\mathrm{ev}}_a^+ \circ h^{-1}. \tag{1.6}$$

$U_q(\widehat{\mathfrak{g}})$ is $\mathbf{Q}$-graded: $x \in U_q(\widehat{\mathfrak{g}})$ is of weight $\lambda \in \mathbf{Q}$ if $s_{ii}^{(0)} x = q^{(\lambda,\epsilon_i)} x s_{ii}^{(0)}$ for all $i \in I$. For example $s_{ij}^{(n)}$ and $t_{ij}^{(n)}$ are of weight $\epsilon_i - \epsilon_j$ [51, (3.14)]. Let $U_q(\widehat{\mathfrak{g}})_\lambda$ be the weight space of weight λ. The $\mathbf{Q}$-grading restricts to $Y_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$.

We recall the *Drinfeld second realization* of $U_q(\widehat{\mathfrak{g}})$ from [51, Section 3.1.4]. Write

$$\begin{cases} S(z) = (\sum\limits_{i<j} e_{ij}^+(z) \otimes E_{ij} + 1)(\sum\limits_l K_l^+(z) \otimes E_{ll})(\sum\limits_{i<j} f_{ji}^+(z) \otimes E_{ji} + 1), \\ T(z) = (\sum\limits_{i<j} e_{ij}^-(z) \otimes E_{ij} + 1)(\sum\limits_l K_l^-(z) \otimes E_{ll})(\sum\limits_{i<j} f_{ji}^-(z) \otimes E_{ji} + 1), \end{cases}$$

as invertible power series in $z^{\pm 1}$ over $U_q(\widehat{\mathfrak{g}}) \otimes \mathrm{End}(\mathbf{V})$; the subscripts $i, j, l \in I$. Notice that $K_\kappa^+(z) = s_{\kappa\kappa}(z)$. For $i \in I_0$, $j \in I$ let us define τ_i, θ_j:

$$\tau_i := q^{M-N+1-i} \quad \text{for } 1 \leq i \leq M, \quad \tau_{M+l} := q^{l+1-N} \quad \text{for } 1 \leq l < N, \tag{1.7}$$

$$\theta_j := q^{2(M-N+1-j)} \quad \text{for } 1 \leq j \leq M, \quad \theta_{M+l} := q^{2(l-N)} \quad \text{for } 1 \leq l \leq N. \tag{1.8}$$

The Drinfeld loop generators are defined by generating series: let $i \in I_0$,

$$x_i^+(z) = \sum_{n\in\mathbb{Z}} x_{i,n}^+ z^n := \frac{e_{i,i+1}^+(z\tau_i) - e_{i,i+1}^-(z\tau_i)}{q_i - q_i^{-1}} \in U_q(\widehat{\mathfrak{g}})[[z, z^{-1}]],$$

$$x_i^-(z) = \sum_{n\in\mathbb{Z}} x_{i,n}^- z^n := \frac{f_{i+1,i}^-(z\tau_i) - f_{i+1,i}^+(z\tau_i)}{q_i^{-1} - q_i} \in U_q(\widehat{\mathfrak{g}})[[z, z^{-1}]],$$

$$\phi_i^\pm(z) = \sum_{n\geq 0} \phi_{i,\pm n}^\pm z^{\pm n} := K_i^\pm(z\tau_i) K_{i+1}^\pm(z\tau_i)^{-1} \in U_q(\widehat{\mathfrak{g}})[[z^{\pm 1}]].$$

From Gauss decomposition we have $K_l^+(z), \phi_i^+(z) \in Y_q(\mathfrak{g})[[z]]$ for $l \in I$ and $i \in I_0$.

Remark 1.2. In [51, Section 3.1.4] a different Gauss decomposition of $S(z), T(z)$ was considered (f always ahead of e). If $\overline{X_i}^\pm(z), \overline{K_l}^\pm(z)$ with $i \in I_0,\ l \in I$ denote the Drinfeld generating series of $U_{q^{-1}}(\widehat{\mathfrak{g}})$ in *loc. cit.*, then

$$h(\overline{K}_l^\pm(z)) = K_l^\pm(z)^{-1}, \quad h(\overline{X}_i^\pm(z)) = \pm(q_i^{-1} - q_i)x_i^\pm(z\tau_i^{-1}).$$

Let us rewrite [51, Theorem 3.5] in terms of the $x_i^\pm(z), \phi_i^\pm(z), K_l^\pm(z)$. First, the coefficients of these series generate the whole algebra $U_q(\widehat{\mathfrak{g}})$. Second, for $i, j \in I_0,\ l, l' \in I$ and $\eta, \eta' \in \{\pm\}$ we have: (recall $q_{ij} = q^{(\alpha_i,\alpha_j)}$)

$$K_l^\eta(z) K_{l'}^{\eta'}(w) = K_{l'}^{\eta'}(w) K_l^\eta(z), \quad K_l^+(0) K_l^-(\infty) = 1,$$

$$K_{M+N}^\eta(z) x_i^\pm(w) = \left(\frac{zq^{-1} - wq}{z - w}\right)^{\pm\delta_{i+1,M+N}} x_i^\pm(w) K_{M+N}^\eta(z),$$

$$\phi_i^\eta(z) x_j^\pm(w) = \frac{z - wq_{ij}^{\pm 1}}{zq_{ij}^{\pm 1} - w} x_j^\pm(w) \phi_i^\eta(z),$$

$$[x_i^+(z), x_j^-(w)] = \delta_{ij} \frac{\phi_i^+(z) - \phi_i^-(w)}{q_i - q_i^{-1}} \delta(\frac{z}{w}),$$

$$(zq_{ij}^{\pm 1} - w) x_i^\pm(z) x_j^\pm(w) = (z - wq_{ij}^{\pm 1}) x_j^\pm(w) x_i^\pm(z) \quad \text{if } (i, j) \neq (M, M),$$

$$[x_i^\pm(z_1), [x_i^\pm(z_2), x_j^\pm(w)]_q]_{q^{-1}} + \{z_1 \leftrightarrow z_2\} = 0 \quad \text{if } (i \neq M,\ |j - i| = 1),$$

$$x_M^\pm(z) x_M^\pm(w) = -x_M^\pm(w) x_M^\pm(z), \quad x_i^\pm(z) x_j^\pm(w) = x_j^\pm(w) x_i^\pm(z) \quad \text{if } |i - j| > 1,$$

together with the degree 4 oscillator relation when $M, N > 1$:

$$[[[x_{M-1}^\pm(u), x_M^\pm(z_1)]_q, x_{M+1}^\pm(v)]_{q^{-1}}, x_M^\pm(z_2)] + \{z_1 \leftrightarrow z_2\} = 0.$$

Here $[x, y]_a := xy - a(-1)^{|x||y|} yx$ for $x, y \in U_q(\widehat{\mathfrak{g}})$ and $a \in \mathbb{C}$. These relations are coherent with the Drinfeld second realization of quantum affine algebras (e.g. [32, Section 3.2]) and superalgebras [48, Theorem 8.5.1]. For $i \in I_0 \backslash \{M\}$, the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by $(x_{i,n}^\pm, \phi_{i,n}^\pm)_{n\in\mathbb{Z}}$ is a quotient algebra of $U_{q_i}(\widehat{\mathfrak{sl}_2})$.

Let $\mathbf{Q}^+ := \oplus_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i \subset \mathbf{P}$ and $\mathbf{Q}^- := -\mathbf{Q}^+$. By [51, Proposition 3.6]:

$$\Delta(K_i^{\pm}(z)) \in K_i^{\pm}(z) \otimes K_i^{\pm}(z) + \sum_{0 \neq \alpha \in \mathbf{Q}^+} U_q(\widehat{\mathfrak{g}})_{-\alpha} \otimes U_q(\widehat{\mathfrak{g}})_{\alpha}[[z^{\pm 1}]], \tag{1.9}$$

$$\Delta(x_i^+(z)) \in x_i^+(z) \otimes 1 + \sum_{0 \neq \alpha \in \mathbf{Q}^+} U_q(\widehat{\mathfrak{g}})_{\alpha_i - \alpha} \otimes U_q(\widehat{\mathfrak{g}})_{\alpha}[[z, z^{-1}]], \tag{1.10}$$

$$\Delta(x_i^-(z)) \in 1 \otimes x_i^-(z) + \sum_{0 \neq \alpha \in \mathbf{Q}^+} U_q(\widehat{\mathfrak{g}})_{-\alpha} \otimes U_q(\widehat{\mathfrak{g}})_{\alpha - \alpha_i}[[z, z^{-1}]]. \tag{1.11}$$

The coproduct shares the same triangular property as [27, Lemma 1].

1.2. Category $\mathcal{O}$. We first recall the notion of weights from [53, Section 6]. Define

$$\mathfrak{P} := (\mathbb{C}^{\times})^I \times \mathbb{Z}_2, \quad \widehat{\mathfrak{P}} := (\mathbb{C}[[z]]^{\times})^I \times \mathbb{Z}_2.$$

The multiplicative group structure on $\mathbb{C}^{\times}$, $\mathbb{C}[[z]]^{\times}$ and the *additive* group structure on the ring $\mathbb{Z}_2$ make $\mathfrak{P}$, $\widehat{\mathfrak{P}}$ into multiplicative abelian groups. $\mathfrak{P}$ is naturally a subgroup of $\widehat{\mathfrak{P}}$, and $\mathbb{C}[[z]]^{\times} \longrightarrow \mathbb{C}^{\times}$, $f(z) \mapsto f(0)$ induces a projection $\varpi : \widehat{\mathfrak{P}} \longrightarrow \mathfrak{P}$. There is an injective homomorphism of abelian groups (see also [19, Section 3.1])

$$q : \mathbf{P} \longrightarrow \mathfrak{P}, \quad \lambda \mapsto q^{\lambda} := ((q^{(\epsilon_i, \lambda)})_{i \in I}; |\lambda|). \tag{1.12}$$

Elements of $\widehat{\mathfrak{P}}$ will usually be denoted by $\mathbf{f}, \mathbf{g}, \ldots$, or $\mathbf{f}(z), \mathbf{g}(z), \ldots$ when their dependence on z is needed. For instance, if $\mathbf{f} = ((f_i(z))_{i \in I}; s) \in \widehat{\mathfrak{P}}$, then for $a \in \mathbb{C}^{\times}$ we have $\mathbf{f}(za) = ((f_i(za))_{i \in I}; s) \in \widehat{\mathfrak{P}}$. We view $h(z) \in \mathbb{C}[[z]]^{\times}$ as the element $(h(z), \ldots, h(z); \overline{0}) \in \widehat{\mathfrak{P}}$, which makes $\mathbb{C}[[z]]^{\times}$ a subgroup of $\widehat{\mathfrak{P}}$.

Let V be a $Y_q(\mathfrak{g})$-module. For $p = ((p_i)_{i \in I}; s) \in \mathfrak{P}$, define

$$V_p := \{v \in V_s \mid s_{ii}^{(0)} v = p_i v \ \text{ for } i \in I\}.$$

If $V_p \neq 0$, then p is called a *weight* of V, and V_p the weight space of weight p. Let $\mathrm{wt}(V)$ denote the set of weights of V. We have $s_{ij}^{(n)} V_p \subseteq V_{q^{\epsilon_i - \epsilon_j} p}$ for $p \in \mathrm{wt}(V)$. Similarly, for $\mathbf{f} = ((f_i(z))_{i \in I}; s) \in \widehat{\mathfrak{P}}$ define

$$V_{\mathbf{f}} := \{v \in V_s \mid \exists d \in \mathbb{Z}_{>0} \text{ such that } (K_i^+(z) - f_i(z))^d v = 0 \text{ for } i \in I\}.$$

If $V_{\mathbf{f}} \neq 0$, then $\mathbf{f}$ is an ℓ-weight of V, and $V_{\mathbf{f}}$ the ℓ-weight space of ℓ-weight $\mathbf{f}$. Let $\mathrm{wt}_{\ell}(V)$ be the set of ℓ-weights of V.

One should be aware that in [53, Section 6] *the definition of ℓ-weight spaces involves different Drinfeld generators. Nevertheless making use of Remark* 1.2 *and the involution h, we can translate all the results concerning $Y_{q^{-1}}(\mathfrak{g})$- and $U_{q^{-1}}(\mathfrak{g})$-modules in* [53], *so as to obtain parallel results on $Y_q(\mathfrak{g})$-* and *$U_q(\widehat{\mathfrak{g}})$-modules.*

Example 1.3. To $\mathbf{f} = h(z)p \in \widehat{\mathfrak{P}}$ with $h(z) \in 1 + z\mathbb{C}[[z]]$ and $p = ((p_i)_{i \in I}; s) \in \mathfrak{P}$ is associated a representation of $Y_q(\mathfrak{g})$ on the one-dimensional vector superspace $\mathbb{C}_s := \mathbb{C}\mathbf{1}$ of parity $s = |\mathbf{1}|$, defined by $s_{ij}(z)\mathbf{1} = \delta_{ij} h(z) p_i \mathbf{1}$. Let $\mathbb{C}_{\mathbf{f}}$ denote this $Y_q(\mathfrak{g})$-module. We have $\{\mathbf{f}\} = \mathrm{wt}_{\ell}(\mathbb{C}_{\mathbf{f}})$ and $\{p\} = \mathrm{wt}(\mathbb{C}_{\mathbf{f}})$.

Definition 1.4. [53, Definition 6.3] A $Y_q(\mathfrak{g})$-module V is in category $\mathcal{O}$ if:

(i) V has a weight space decomposition $V = \oplus_{p\in\mathfrak{P}} V_p$;
(ii) $\dim V_p < \infty$ for all $p \in \mathfrak{P}$;
(iii) there exist $\mu_1, \mu_2, \dots, \mu_d \in \mathfrak{P}$ such that $\mathrm{wt}(V) \subseteq \cup_{j=1}^d (q^{\mathbf{Q}^-}\mu_j)$.

Let V be a $Y_q(\mathfrak{g})$-module in category $\mathcal{O}$. A non-zero $\omega \in V$ is called a *highest ℓ-weight* vector if it belongs to $V_{\mathbf{f}}$ for certain $\mathbf{f} = ((f_i(z))_{i\in I}; s) \in \widehat{\mathfrak{P}}$ and it is annihilated by the $s_{ij}(z)$ for $i < j$. Necessarily $K_i^+(z)\omega = f_i(z)\omega$. Call V a highest ℓ-weight module if it is generated as a $Y_q(\mathfrak{g})$-module by a highest ℓ-weight vector ω, in which case ω is unique up to scalar multiple and its ℓ-weight is called the highest ℓ-weight of V. Lowest ℓ-weight vector/module is defined similarly by replacing the condition $i < j$ with $i > j$.

In Example 1.3 the vector $\mathbf{1} \in \mathbb{C}_{\mathbf{f}}$ is both of highest and of lowest ℓ-weight.

Attention! If ω is a lowest ℓ-weight vector of ℓ-weight $\mathbf{f} = ((f_i(z))_{i\in I}; s)$, then we have $s_{ii}(z)\omega = f_i(z)\omega$ for $i \in I$; see also [53, Section 6]. This is not necessarily true if "lowest" is replaced by "highest".

Let $\mathbf{R}$ be the subset of $\widehat{\mathfrak{P}}$ consisting of the $\mathbf{f} = ((f_i(z))_{i\in I}; s)$ such that $\frac{f_i(z)}{f_{i+1}(z)}$ is the Taylor expansion at $z = 0$ of a rational function for $i \in I_0$.

Lemma 1.5 [53, Lemma 6.8 & Proposition 6.10]. *Let* $\mathbf{f} = ((f_i(z))_{i\in I}; s) \in \mathbf{R}$.

(1) *In category $\mathcal{O}$ there exists a unique irreducible highest ℓ-weight module $L(\mathbf{f})$ of highest ℓ-weight $\mathbf{f}$ up to isomorphism. The $L(\mathbf{g})$ for $\mathbf{g} \in \mathbf{R}$ form the set of irreducible objects (two-by-two non-isomorphic) of category $\mathcal{O}$.*
(2) $\dim L(\mathbf{f}) = 1$ *if and only if* $\frac{f_i(z)}{f_{i+1}(z)} \in \mathbb{C}^\times$ *for $i \in I_0$, i.e.* $\mathbf{f} \in \mathbb{C}[[z]]^\times \widehat{\mathfrak{P}}$.
(3) $\dim L(\mathbf{f}) < \infty$ *if and only if for $i \in I_0 \setminus \{M\}$ there exist $P_i(z) \in 1 + z\mathbb{C}[z]$ and $c_i \in \mathbb{C}^\times$ such that* $\frac{f_i(z)}{f_{i+1}(z)} = c_i \frac{P_i(zq_i^{-1})}{P_i(zq_i)}$.
(4) $L(\mathbf{f})$ *can be extended to a $U_q(\widehat{\mathfrak{g}})$-module if and only if $\frac{f_i(z)}{f_{i+1}(z)}$ is a product of the $c\frac{1-zac^{-2}}{1-za}$ with $a, c \in \mathbb{C}^\times$ for $i \in I_0$.*

Based on (4), let $\mathbf{R}_U$ be the subset of $\mathbf{R}$ consisting of $\mathbf{f} = ((f_i(z))_{i\in I}; s)$ such that for $i \in I$, the rational function $f_i(z)$ is a product of the $c\frac{1-zac^{-2}}{1-za}$ with $a, c \in \mathbb{C}^\times$. For $\mathbf{f} \in \mathbf{R}_U$, the $Y_q(\mathfrak{g})$-module $L(\mathbf{f})$ is extended *uniquely* to a $U_q(\widehat{\mathfrak{g}})$-module by

$$K_i^+(z)\omega = f_i(z)\omega = K_i^-(z)\omega \quad \text{for } i \in I.$$

Here ω is a highest ℓ-weight vector, and in the second identity one views $f_i(z) \in \mathbb{C}[[z^{-1}]]$ by taking the its Taylor expansion of at $z = \infty$. We continue to let $L(\mathbf{f})$ denote the irreducible $U_q(\widehat{\mathfrak{g}})$-module thus obtained for $\mathbf{f} \in \mathbf{R}_U$.

Example 1.6. For $i \in I_0$ and $a \in \mathbb{C}^\times$ define the *prefundamental ℓ-weight* $\Psi_{i,a} \in \mathbf{R}$, the *fundamental weight* $\varpi_i \in \mathbf{P}$, and $[a]_i \in \mathbf{R}$ by:

	$i \le M$	$i > M$
$\Psi_{i,a}$	$(\underbrace{h(z), \dots, h(z)}_{i}, \underbrace{1, \dots, 1}_{\kappa-i}; \bar{0})$	$(\underbrace{1, \dots, 1}_{i}, \underbrace{h(z)^{-1}, \dots, h(z)^{-1}}_{\kappa-i}; \bar{0})$
$[a]_i$	$(\underbrace{a, \dots, a}_{i}, \underbrace{1, \dots, 1}_{\kappa-i}; \bar{0})$	$(\underbrace{1, \dots, 1}_{i}, \underbrace{a^{-1}, \dots, a^{-1}}_{\kappa-i}; \bar{0})$
ϖ_i	$\epsilon_1 + \epsilon_2 + \cdots + \epsilon_i$	$-\epsilon_{i+1} - \epsilon_{i+2} - \cdots - \epsilon_\kappa$

where $h(z) = 1 - za\tau_i^{-1}$. For $i, j \in I_0$ let us write $i \sim j$ if $|i - j| = 1$. Define

$$a_{ij} := a^{(\alpha_i,\alpha_j)}, \quad \hat{q}_i = q_i \quad \text{if } i \neq M, \quad \hat{q}_M = q^{-1}.$$

Let us introduce the following elements of $\mathbf{R}$ for $c \in \mathbb{C}^\times$ and $m \in \mathbb{Z}_{>0}$:

$$\mathbf{n}_{i,a}^+ := \frac{\Psi_{i,aq_i^{-2}}}{\Psi_{i,a}} \prod_{j \in I_0 : j \sim i} \Psi_{j,aq_{ij}^{-1}}, \quad \mathbf{n}_{i,a}^- := \frac{\Psi_{i,a}}{\Psi_{i,a\hat{q}_i^2}} \prod_{j \in I_0 : j \sim i} \Psi_{j,aq_{ij}}^{-1},$$

$$\boldsymbol{\omega}_{c,a}^{(i)} := [c]_i \frac{\Psi_{i,ac^{-2}}}{\Psi_{i,a}}, \quad \varpi_{m,a}^{(i)} := q^{m\varpi_i} \frac{\Psi_{i,aq_i^{1-2m}}}{\Psi_{i,aq_i}}, \quad Y_{i,a} := q^{\varpi_i} \frac{\Psi_{i,aq_i^{-1}}}{\Psi_{i,aq_i}},$$

$$\mathbf{n}_{c,a}^{(i)} := \boldsymbol{\omega}_{\hat{q}_i,a\hat{q}_i^2}^{(i)} \prod_{j \in I_0 : j \sim i} \boldsymbol{\omega}_{c_{ij}^{-1},aq_{ij}}^{(j)}, \quad \mathbf{m}_{c,a}^{(i)} := \boldsymbol{\omega}_{q_i,a}^{(i)} \prod_{j \in I_0 : j \sim i} \boldsymbol{\omega}_{c_{ij}^{-1},aq_{ij}^{-1}c_{ij}^{-2}}^{(j)},$$

$$A_{i,a} := (\underbrace{1, \ldots, 1}_{i-1}, q_i \frac{1 - za\tau_i q^{-1}\theta_i^{-1} q_i^{-1}}{1 - za\tau_i q^{-1}\theta_i^{-1} q_i}, q_{i+1}^{-1} \frac{1 - za\tau_i q^{-1}\theta_i^{-1} q_i q_{i+1}^2}{1 - za\tau_i q^{-1}\theta_i^{-1} q_i} \underbrace{1, \ldots, 1}_{\kappa-i-1}; |\alpha_i|).$$

The irreducible $Y_q(\mathfrak{g})$-modules $L_{i,a}^\pm := L(\Psi_{i,a}^{\pm 1})$ are called positive/negative *prefundamental modules*. If ω is a highest ℓ-weight vector of $L_{i,a}^+$, then

$$\phi_j^+(z)\omega = \omega \quad \text{for } j \neq i, \quad \phi_i^+(z)\omega = (1 - za)\omega.$$

So $\Psi_{i,a}$ is a super analog of [34, (3.16)]. Define the irreducible $Y_q(\mathfrak{g})$-modules:

$$N_{i,a}^\pm := L(\mathbf{n}_{i,a}^\pm), \quad M_{c,a}^{(i)} := L(\mathbf{m}_{c,a}^{(i)}), \quad W_{m,a}^{(i)} := L(\varpi_{m,a}^{(i)}).$$

Call $W_{m,a}^{(i)}$ a *Kirillov–Reshetikhin module* (KR module). By Lemma 1.5, the M, W are $U_q(\widehat{\mathfrak{g}})$-modules with W finite-dimensional. (In Sects. 7 and 8 $N_{m,a}^{(i)}$ will denote the irreducible module $L(\mathbf{m}_{q^m,a}^{(i)})$ for $m \in \mathbb{Z}_{>0}$, so here we do not use $N_{c,a}^{(i)}$.)

Remark 1.7. Later in Sects. 6 and 7 we work with $U_q(\widehat{\mathfrak{g}})$-modules in category $\mathcal{O}$. Such a module V is called a highest ℓ-weight $U_q(\widehat{\mathfrak{g}})$-module in [52, Section 1.2] if there exists a non-zero $\mathbb{Z}_2$-homogeneous vector ω such that $V = U_q(\widehat{\mathfrak{g}})\omega$ and

$$s_{ij}^{(n)}\omega = t_{ij}^{(n)}\omega = 0, \quad s_{ll}^{(n)}\omega \in \mathbb{C}\omega \ni t_{ll}^{(n)}\omega \quad \text{for } i < j.$$

Indeed V is of highest ℓ-weight as a $U_q(\widehat{\mathfrak{g}})$-module if and only it is of highest ℓ-weight as a $Y_q(\mathfrak{g})$-module. (The "if" part comes from weight grading, while the "only if" part from the Drinfeld relations in Remark 1.2.) It also follows that V is an irreducible $U_q(\widehat{\mathfrak{g}})$-module if and only if it is an irreducible $Y_q(\mathfrak{g})$-module, as in [34, Proposition 3.5]. Therefore when we say V is of highest ℓ-weight or irreducible, we make no reference to $Y_q(\mathfrak{g})$ or $U_q(\widehat{\mathfrak{g}})$.

As in [35, Section 3.2], let $\mathcal{E}_\ell$ be the set of formal sums $\sum_{\mathbf{f} \in \widehat{\mathfrak{P}}} c_{\mathbf{f}} \mathbf{f}$ with integer coefficients $c_{\mathbf{f}} \in \mathbb{Z}$ such that $\oplus_{\mathbf{f} \in \widehat{\mathfrak{P}}} \mathbb{C}_{\mathbf{f}}^{\oplus |c_{\mathbf{f}}|}$ is an object of category $\mathcal{O}$. It is a ring: addition is the usual one of formal sums; multiplication is induced by that of $\widehat{\mathfrak{P}}$. (One views $\mathcal{E}_\ell$ as a completion of the group ring $\mathbb{Z}[\widehat{\mathfrak{P}}]$.)

For V an object of category $\mathcal{O}$, its weight space decomposition can be refined to an ℓ-weight decomposition because of condition (ii) in Definition 1.4. Following [27] we define its q-character and classical character

$$\chi_q(V) = \sum_{\mathbf{f} \in \mathrm{wt}_\ell(V)} \dim(V_\mathbf{f})\mathbf{f}, \quad \chi(V) = \sum_{p \in \mathrm{wt}(V)} \dim(V_p) p \in \mathcal{E}_\ell. \tag{1.13}$$

In Example 1.3 we have $\chi_q(\mathbb{C}_\mathbf{f}) = \mathbf{f}$ and $\chi(\mathbb{C}_\mathbf{f}) = \varpi(\mathbf{f})$.

We shall need the completed Grothendieck group $K_0(\mathcal{O})$. Its definition is the same as that in [35, Section 3.2]: elements are formal sums $\sum_{\mathbf{f} \in \mathbf{R}} c_\mathbf{f}[L(\mathbf{f})]$ with integer coefficients $c_\mathbf{f} \in \mathbb{Z}$ such that $\oplus_{\mathbf{f} \in \mathbf{R}} L(\mathbf{f})^{\oplus |c_\mathbf{f}|}$ is in category $\mathcal{O}$; addition is the usual one of formal sums. For $\mathbf{f} \in \mathbf{R}$ and V in category $\mathcal{O}$, the multiplicity of the irreducible module $L(\mathbf{f})$ in V is well-defined due to Definition 1.4, as in the case of Kac–Moody algebras [37, Section 9.6]; it is denoted by $m_{L(\mathbf{f}),V} \in \mathbb{Z}_{\geq 0}$. Necessarily $[V] := \sum_{\mathbf{f} \in \mathbf{R}} m_{L(\mathbf{f}),V}[L(\mathbf{f})] \in K_0(\mathcal{O})$. In the case $V = L(\mathbf{f})$ the right-hand side is simply $[L(\mathbf{f})]$ because $m_{L(\mathbf{g}),L(\mathbf{f})} = \delta_{\mathbf{gf}}$ for $\mathbf{g} \in \mathbf{R}$.

Make $K_0(\mathcal{O})$ into a ring by $[V][W] := [V \otimes W]$. Equation (1.13) extends uniquely to morphisms of additive groups $\chi_q : K_0(\mathcal{O}) \longrightarrow \mathcal{E}_\ell$ and $\chi : K_0(\mathcal{O}) \longrightarrow \mathcal{E}_\ell$, called q-character map and character map respectively. As in [27, Theorem 3], we have

Proposition 1.8 [53, Corollary 6.9]. *The q-character map χ_q is an injective morphism of rings. Consequently the ring $K_0(\mathcal{O})$ is commutative.*

The tensor product $L(\mathbf{f}) \otimes L(\mathbf{g})$ contains an irreducible sub-quotient $L(\mathbf{fg})$ for $\mathbf{f}, \mathbf{g} \in \mathbf{R}$. Let us define the *normalized q-character* $\widetilde{\chi}_q(L(\mathbf{f})) := \mathbf{f}^{-1}\chi_q(L(\mathbf{f}))$.

For V, W in category $\mathcal{O}$, write $V \simeq W$ if there is a one-dimensional module D in category $\mathcal{O}$ such that $V \cong W \otimes D$ as $Y_q(\mathfrak{g})$-modules. By Lemma 1.5 (2) and Proposition 1.8 we have $L(\mathbf{f}) \simeq L(\mathbf{g})$ if and only if $\mathbf{g}^{-1}\mathbf{f} \in \mathbb{C}[[z]]^\times \widehat{\mathfrak{P}}$, in which case the normalized q-characters of $L(\mathbf{f})$ and $L(\mathbf{g})$ are identical and we write $\mathbf{f} \equiv \mathbf{g}$.

As an example, for the *generalized simple root* $A_{i,a} \in \mathbf{R}_U$ we have

$$A_{i,a} \equiv \frac{\Psi_{i,aq_i^{-2}}}{\Psi_{i,a\hat{q}_i^2}} \prod_{j \in I_0 : j \sim i} \frac{\Psi_{j,aq_{ij}^{-1}}}{\Psi_{j,aq_{ij}}}. \tag{1.14}$$

1.3. Category $\mathcal{O}'$. As in [52, Section 1], let $\mathfrak{gl}(N|M) =: \mathfrak{g}'$ be another Lie superalgebra, which is not to be confused with the derived algebra of $\mathfrak{g}$. Define the Hopf superalgebras $U_q(\widehat{\mathfrak{g}'})$, $Y_q(\mathfrak{g}')$, $U_q(\mathfrak{g}')$ in the same way as for $U_q(\widehat{\mathfrak{g}})$, $Y_q(\mathfrak{g})$, $U_q(\mathfrak{g})$ in Sect. 1.1, except that M, N are interchanged. We start from the same weight/root lattices $\mathbf{P}$, $\mathbf{Q}$ and $\mathfrak{P}$, $\widehat{\mathfrak{P}}$ but with different parity map $|?|' : \mathbf{P} \longrightarrow \mathbb{Z}_2$:

$$|\epsilon_1|' = |\epsilon_2|' = \cdots = |\epsilon_N|' = \overline{0}, \quad |\epsilon_{N+1}|' = |\epsilon_{N+2}|' = \cdots = |\epsilon_{N+M}|' = \overline{1},$$

bilinear form $(\epsilon_i, \epsilon_j)' = \delta_{ij}(-1)^{|\epsilon_i|'}$, and embedding $q'^\lambda := ((q^{(\lambda,\epsilon_i)'})_{i \in I}; |\lambda|')$ of $\mathbf{P}$ in $\mathfrak{P}$. One defines category $\mathcal{O}'$ of $Y_q(\mathfrak{g}')$-modules as in Sect. 1.2. Let us summarize the modifications of notations related to $\mathfrak{g}'$ to be used later on:

$\mathfrak{g},\ U_q(\mathfrak{g}),\ Y_q(\mathfrak{g}),\ U_q(\widehat{\mathfrak{g}})$	$\mathfrak{g}',\ U_q(\mathfrak{g}'),\ Y_q(\mathfrak{g}'),\ U_q(\widehat{\mathfrak{g}'})$	algebras
$s_{ij}^{(n)},\ t_{ij}^{(n)},\ q_i,\ q_{ij},\ \tau_i,\ \theta_j$	$s_{ij}^{\prime(n)},\ t_{ij}^{\prime(n)},\ q_i',\ q_{ij}',\ \tau_i',\ \theta_j'$	RTT
$x_i^\pm(z),\ K_i^\pm(z),\ \phi_i^\pm(z)$	$x_i'^\pm(z),\ K_i'^\pm(z),\ \phi_i'^\pm(z)$	currents
$\mathcal{O},\ L(\mathbf{f}),\ L_{i,a}^\pm,\ N_{i,a}^\pm,\ W_{m,a}^{(i)}$	$\mathcal{O}',\ L'(\mathbf{f}),\ L_{i,a}'^\pm,\ N_{i,a}'^\pm,\ W_{m,a}'^{(i)}$	categories

(1.15)

In case $M = N$ one can simply remove all the primes in the table.

For $i, j \in I$, set $\widehat{i} := \kappa + 1 - i$ and $\varepsilon'_{ij} := (-1)^{|\epsilon_i|'+|\epsilon_i|'|\epsilon_j|'}$. Then

$$\mathcal{F} : U_q(\widehat{\mathfrak{g}'}) \longrightarrow U_q(\widehat{\mathfrak{g}})^{\mathrm{cop}}, \quad s'^{(n)}_{ij} \mapsto \varepsilon'_{ji} s^{(n)}_{\widehat{j}\widehat{i}}, \quad t'^{(n)}_{ij} \mapsto \varepsilon'_{ji} t^{(n)}_{\widehat{j}\widehat{i}}. \tag{1.16}$$

defines a Hopf superalgebra isomorphism. Let $\overline{\mathcal{F}} : U_{q^{-1}}(\widehat{\mathfrak{g}'}) \longrightarrow U_{q^{-1}}(\widehat{\mathfrak{g}})^{\mathrm{cop}}$ and $h' : U_{q^{-1}}(\widehat{\mathfrak{g}'}) \longrightarrow U_q(\widehat{\mathfrak{g}'})^{\mathrm{cop}}$ be analogs of Eqs. (1.16) and (1.3). They induce

$$\mathcal{G} : U_q(\widehat{\mathfrak{g}'}) \longrightarrow U_q(\widehat{\mathfrak{g}})^{\mathrm{cop}}, \quad \mathcal{G} := h \circ \overline{\mathcal{F}} \circ h'^{-1} \tag{1.17}$$

a Hopf superalgebra isomorphism which restricts to $\mathcal{G} : Y_q(\mathfrak{g}') \longrightarrow Y_q(\mathfrak{g})$.

Lemma 1.9. *The pullback by $\mathcal{G}$ is an anti-equivalence of monoidal categories $\mathcal{G}^* : \mathcal{O} \longrightarrow \mathcal{O}'$. If $\mathbf{f} = (f_1(z), f_2(z), \dots, f_\kappa(z); s) \in \mathbf{R}$, then as $Y_q(\mathfrak{g}')$-modules*

$$\mathcal{G}^*(L(\mathbf{f})) \cong L'(f_\kappa(z), f_{\kappa-1}(z), \dots, f_1(z); s).$$

In particular, $\mathcal{G}^*(L^{\pm}_{i,a}) \simeq L'^{\mp}_{M+N-i,aq^{N-M}}$ for $1 \le i < M + N$.

Proof. Let V be a $Y_q(\mathfrak{g})$-module in category $\mathcal{O}$. If $p \in \mathfrak{P}$, then $V_p = (\mathcal{G}^* V)_{p'}$ where $p' = ((p_{\widehat{i}})_{i\in I}; s)$, and so $V_{q^{n\alpha_i} p} = (\mathcal{G}^* V)_{q'^{n\alpha_{\kappa-i}} p'}$ for $i \in I_0$ and $n \in \mathbb{Z}$. This implies that $\mathcal{G}^* V$ is in category $\mathcal{O}'$. The first statement is now clear.

Let $V = L(\mathbf{f})$ and let $\omega \in V$ be a highest ℓ-weight vector. In $h^* V$ we have

$$\overline{K}^+_l(z) h^*\omega = f_l(z)^{-1} h^*\omega, \quad \overline{s}_{ij}(z) h^*\omega = 0 \quad \text{for } i, j, l \in I \text{ with } i < j.$$

From the Gauss decomposition of $h^{-1}(S(z))$ we get $\overline{s}_{ll}(z)h^*\omega = \overline{K}^+_l(z)h^*\omega$. Similar identities hold when replacing $h^*\omega$ by $\overline{\mathcal{F}}^* h^*\omega$. This implies:

$$\begin{aligned}
\overline{K}'^+_i(z)\overline{\mathcal{F}}^* h^*\omega &= \overline{s}'_{ii}(z)\overline{\mathcal{F}}^* h^*\omega = \overline{\mathcal{F}}^* \left(\overline{s}_{\widehat{i},\widehat{i}}(z)h^*\omega\right) \\
&= \overline{\mathcal{F}}^* \left(\overline{K}^+_{\widehat{i}}(z)h^*\omega\right) = f_{\widehat{i}}(z)^{-1}\overline{\mathcal{F}}^* h^*\omega, \\
K'^+_i(z)\mathcal{G}^*\omega &= K'^+_i(z)(h'^{-1})^*\overline{\mathcal{F}}^* h^*\omega = (h'^{-1})^* \left(\overline{K}'^+_i(z)^{-1}\overline{\mathcal{F}}^* h^*\omega\right) \\
&= f_{\widehat{i}}(z)(h'^{-1})^*\overline{\mathcal{F}}^* h^*\omega = f_{\widehat{i}}(z)\mathcal{G}^*\omega,
\end{aligned}$$

leading to the second statement; here the $\overline{s}'_{ii}(z)$, $\overline{K}'^+_i(z)$ denote the RTT generators and Drinfeld generators of $U_{q^{-1}}(\widehat{\mathfrak{g}'})$ arising from [51]; see Remark 1.2. The last statement is a comparison of highest ℓ-weights based on $\tau'_{M+N-i} = \tau_i q^{N-M}$. □

$\mathcal{G}^*$ can be viewed as a categorification of the duality function of Grothendieck rings in [35, Theorem 5.17]. We shall make extensive use of it: to change the signature of the $L^{\pm}_{i,a}$; to pass from Dynkin nodes $i \le M$ to $i \ge M$.

2. Tableau-Sum Formulas of q-Characters

We compute $\chi_q(L(\mathbf{m}))$ for $\mathbf{m} \in \mathbf{R}_U$ coming from Young diagrams.

Definition 2.1 [9, Section 4.2]. $\mathcal{P}$ is the set of $\lambda = \sum_i \lambda_i \epsilon_i \in \mathbf{P}$ such that:

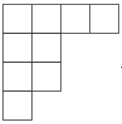
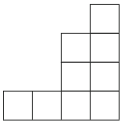
- we have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M \geq 0$ and $\lambda_{M+1} \geq \lambda_{M+2} \geq \cdots \geq \lambda_\kappa \geq 0$;
- if $\lambda_{M+j} > 0$ for some $1 \leq j \leq N$, then $\lambda_M \geq j$.

To $\lambda \in \mathcal{P}$ we attach a subset Y_+^λ of $\mathbb{Z}^2_{>0}$ consisting of (k, l) such that: $l \leq \lambda_k$ for $1 \leq k \leq M$; if $k > M$ then $l \leq N$ and $k \leq M + \lambda_{M+l}$. Let $\mathcal{B}_+(\lambda)$ be the set of functions $T : Y_+^\lambda \longrightarrow I$ such that:

- $T(k, l) \leq T(k', l')$ if $k \leq k'$, $l \leq l'$ and $(k, l), (k', l') \in Y_+^\lambda$;
- $T(k, l) < T(k + 1, l)$ if $(k, l), (k + 1, l) \in Y_+^\lambda$ and $T(k, l) \leq M$;
- $T(k, l) < T(k, l + 1)$ if $(k, l), (k, l + 1) \in Y_+^\lambda$ and $T(k, l) > M$.

Let $Y_-^\lambda = -Y_+^\lambda \subset \mathbb{Z}^2_{<0}$ and define $\mathcal{B}_-(\lambda)$ as the set of functions $Y_-^\lambda \longrightarrow I$ satisfying the above three conditions with Y_+^λ replaced by Y_-^λ.

We view Y_+^λ, Y_-^λ as Young diagrams at the southeast and northwest positions respectively, so that $(k, l) \in Y_\pm^\lambda$ correspond to the box at row $\pm k$ and column $\pm l$. For example, take $\mathfrak{g} = \mathfrak{gl}(2|2)$ and $\lambda = 4\epsilon_1 + 2\epsilon_2 + 2\epsilon_3 + \epsilon_4 \in \mathcal{P}$:

$Y_+^\lambda =$  , $Y_-^\lambda =$ 

Definition 2.2. Let $i \in I_0,\ j \in I$ and $a \in \mathbb{C}^\times$. Define the ℓ-weights in $\mathbf{R}_U$:

$$\boxed{j}_a := (\underbrace{1, \ldots, 1}_{j-1}, q_j \frac{1 - za\theta_j^{-1}q_j^{-1}}{1 - za\theta_j^{-1}q_j}, \underbrace{1, \ldots, 1}_{\kappa-j}; |\epsilon_j|),$$

Define the $\boxed{j}_a^*, \boxed{j}_a'$ inductively by $\boxed{1}_a^* := \boxed{1}_{a\theta_1}^{-1}$, $\boxed{\kappa}_a' := \boxed{\kappa}_a^{-1}$ and

$$\boxed{i+1}_a^* := \boxed{i}_a^* A_{i,a\tau_i q^{-1}}, \quad \boxed{i+1}_a' =: \boxed{i}_a' A_{i,a\tau_i q^{-1}}.$$

Call a the *spectral parameter* of the boxes $\boxed{j}_a, \boxed{j}_a^*, \boxed{j}_a'$.

One checks that $A_{i,a} = \boxed{i}_{a\tau_i q^{-1}} \boxed{i+1}_{a\tau_i q^{-1}}^{-1}$ using $\theta_{i+1} = \theta_i q_i^{-1} q_{i+1}^{-1}$.

Example 2.3. If $\mathfrak{g} := \mathfrak{gl}(2|3)$, then $\tau_1 = q^{-1}$ and (compare with [27, Section 5.4.1])

$$\boxed{1}_a \xrightarrow{A^{-1}_{1,aq^2}} \boxed{2}_a \xrightarrow{A^{-1}_{2,aq^3}} \boxed{3}_a \xrightarrow{A^{-1}_{3,aq^2}} \boxed{4}_a \xrightarrow{A^{-1}_{4,aq}} \boxed{5}_a,$$

$$\boxed{1}_a^* \xrightarrow{A_{1,aq^{-2}}} \boxed{2}_a^* \xrightarrow{A_{2,aq^{-3}}} \boxed{3}_a^* \xrightarrow{A_{3,aq^{-2}}} \boxed{4}_a^* \xrightarrow{A_{4,aq^{-1}}} \boxed{5}_a^*.$$

To $p = ((p_i)_{i\in I}; s) \in \mathfrak{P}$ is associated a unique irreducible $U_q(\mathfrak{g})$-module $V_q(p)$, which is generated by a vector v of parity s subject to the following relations:

$$s_{ii}^{(0)} v = p_i v, \quad s_{jk}^{(0)} v = 0 \quad \text{for } i, j, k \in I \text{ with } j < k.$$

For $\lambda \in \mathbf{P}$, set $V_q(\lambda) := V_q(q^\lambda)$. (It was denoted by $V(\lambda)$ in [9, Section 3.3].)

For $\lambda \in \mathcal{P}$, the $U_q(\mathfrak{g})$-module $V_q(\lambda)$ is finite-dimensional [9, Section 3.3]; its dual space $V_q^*(\lambda) := \mathrm{Hom}_{\mathbb{C}}(V_q(\lambda), \mathbb{C})$ is equipped with a $U_q(\mathfrak{g})$-module structure:

$$\langle x\varphi, v\rangle := (-1)^{|\varphi||x|}\langle \varphi, \mathbb{S}(x)v\rangle \quad \text{for } x \in U_q(\mathfrak{g}),\ \varphi \in V_q^*(\lambda),\ v \in V_q(\lambda).$$

Theorem 2.4. *Let $a \in \mathbb{C}^\times$ and $\lambda \in \mathcal{P}$. Let $V_q^{\pm}(\lambda; a)$, $V_q^{\pm *}(\lambda; a)$ be the pullbacks of the $U_q(\mathfrak{g})$-modules $V_q(\lambda)$, $V_q^*(\lambda)$ by $\mathrm{ev}_a^{\pm}$ respectively. Then we have*

$$\chi_q\left(V_q^+(\lambda; a)\right) = \sum_{T\in\mathcal{B}_-(\lambda)} \prod_{(i,j)\in Y_-^\lambda} \boxed{T(i,j)}_{aq^{2(j-i)+1}}, \tag{2.18}$$

$$\chi_q\left(V_q^{+*}(\lambda; a)\right) = \sum_{T\in\mathcal{B}_-(\lambda)} \prod_{(i,j)\in Y_-^\lambda} \boxed{T(i,j)}^*_{aq^{2(i-j)+1}}, \tag{2.19}$$

$$\chi_q\left(V_q^-(\lambda; a)\right) = \sum_{T\in\mathcal{B}_+(\lambda)} \prod_{(i,j)\in Y_+^\lambda} \boxed{T(i,j)}_{aq^{2(j-i+M-N)+1}}, \tag{2.20}$$

$$\chi_q\left(V_q^{-*}(\lambda; a)\right) = \sum_{T\in\mathcal{B}_+(\lambda)} \prod_{(i,j)\in Y_+^\lambda} \boxed{T(i,j)}'_{aq^{2(i-j)+1}}. \tag{2.21}$$

In particular, $V_q^{\pm}(\lambda; a)$ and $V_q^{\pm *}(\lambda; a)$ have multiplicity free q-characters.

Remark 2.5. Applying $\varpi : \widehat{\mathfrak{P}} \longrightarrow \mathfrak{P}$ to Eq. (2.20) recovers the character formula of $V_q(\lambda)$ in [9, Theorem 5.1].

We shall prove Eqs. (2.18)–(2.19); the idea is similar to [26, Lemma 4.7]. The proof of Eqs. (2.20)–(2.21) is parallel and will be omitted.

For $i \in I$, let $U_q^{\geq i}(\widehat{\mathfrak{g}})$ (resp. $U_q^{\geq i}(\mathfrak{g})$) be the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by the $s_{jk}^{(n)}, t_{jk}^{(n)}$ (resp. for $n = 0$) with $j, k \geq i$. Define

$$C_i(z) := \prod_{j\geq i} K_j^+(z\theta_j)^{(\epsilon_j,\epsilon_j)} \in Y_q(\mathfrak{g})[[z]]. \tag{2.22}$$

The coefficients of $C_i(z)$ are central elements of $U_q^{\geq i}(\widehat{\mathfrak{g}})$; see [53, Proposition 6.1].

Lemma 2.6. *Let $i, l \in I$. The spectra of $C_i(z)$ on ℓ-weight spaces of ℓ-weights $\boxed{l}_a$, $\boxed{l}^*_a$ are $(q\frac{1-zaq^{-1}}{1-zaq})^{\delta_{i\leq l}}$ and $(q^{-1}\frac{1-zat_iq}{1-zat_iq^{-1}})^{\delta_{i\leq l}}$ respectively, where $t_1 = \theta_1$ and $t_i = \tau_{i-1}^2 q^{-2}$ for $i > 1$. Moreover $\boxed{l}^*_a = \frac{(1-zaq^{-3})(1-zaq)}{(1-zaq^{-1})^2}\boxed{l}'_a$.*

Proof. The $\boxed{l}$-case is from Definition 2.2. In particular the $A_{j,b}$ for $j \neq i-1$ do not contribute to the spectra of $C_i(z)$. The $\boxed{l}^*$-case is now clear from $\boxed{l}^*_a = \boxed{1}^{-1}_{a\theta_1} A_{1,a\tau_1 q^{-1}} A_{2,a\tau_2 q^{-1}} \cdots A_{l-1,a\tau_{l-1}q^{-1}}$. To compare $\boxed{l}^*$ with $\boxed{l}'$ one may assume $l = \kappa$ by Definition 2.2; the spectrum of $C_i(z)$ associated to the ℓ-weight $\boxed{\kappa}'_a$ is $q^{-1}\frac{1-zaq}{1-zaq^{-1}}$, leading to the last identity. □

Let S be $V_q^+(\lambda; a)$ or $V_q^{+*}(\lambda; a)$. If $\mu \in \mathbf{P}$ and $v \in S$ are such that $s_{ii}^{(0)} v = q^{(\mu,\epsilon_i)} v$ for all $i \in I$, then $|v| = |\mu|$. To compute the q-character of S, it is enough to determine the action of the $C_i(z)$ since it in turn implies the parity.

Let S_1 be an irreducible sub-$U_q^{\geq i}(\mathfrak{g})$-module of S and $0 \neq v_1 \in S_1$, $\mu \in \mathbf{P}$ with

$$t_{jk}^{(0)} v_1 = 0, \quad s_{ll}^{(0)} v_1 = q^{(\mu,\epsilon_l)} v_1 \quad \text{for } j,k,l \in I,\ j > k.$$

Call μ the lowest weight of S_1. By Schur Lemma and Gauss decomposition,

$$C_i(z) v = \prod_{j \geq i} \left(\frac{q^{(\mu,\epsilon_j)} - za\theta_j q^{-(\mu,\epsilon_j)}}{1 - za\theta_j} \right)^{(\epsilon_j,\epsilon_j)} v \quad \text{for } v \in S_1. \tag{2.23}$$

The strategy is to find all such triples (i, S_1, μ).

Following Table (1.15) and Definition 2.1, define for $\mathfrak{g}'$ the similar objects

$$\mathcal{P}' \subset \mathbf{P}, \quad Y_\pm'^\lambda \subset \mathbb{Z}^2, \quad \mathcal{B}'_\pm(\lambda), \ V'_q(\lambda), \ V_q'^*(\lambda)$$

with (M, N) replaced by (N, M). The transpose of Young diagrams induces a bijection $\mathcal{P} \longrightarrow \mathcal{P}'$, $\lambda \mapsto \lambda^\sharp$ such that $(k,l) \in Y_+^\lambda$ if and only if $(l,k) \in Y_+'^{\lambda^\sharp}$.

Lemma 2.7. *Let $\lambda \in \mathcal{P}$.*

(1) *As $U_q(\mathfrak{g}')$-modules $\mathcal{F}^*\left(V_q(\lambda)\right) \cong V_q'^*(\lambda^\sharp)$ and $\mathcal{F}^*\left(V_q^*(\lambda)\right) \cong V'_q(\lambda^\sharp)$.*

(2) *If $T \in \mathcal{B}_-(\lambda)$, then $T'(k,l) := M + N + 1 - T(-l,-k)$ defines an element $T' \in \mathcal{B}'_+(\lambda^\sharp)$. Moreover $T \mapsto T'$ is a bijection $\mathcal{B}_-(\lambda) \longrightarrow \mathcal{B}'_+(\lambda^\sharp)$.*

Proof. (2) is a lengthy but straightforward check by Definition 2.2. For (1), it suffices to establish the second isomorphism since $\mathcal{F}$ respects Hopf superalgebra structures. Let μ be the lowest weight of $V_q(\lambda)$ and define

$$\begin{aligned} r_i &:= \sharp\{j \in \mathbb{Z}_{>0} \mid (i,j) \in Y_+^\lambda\}, \quad c_j := \sharp\{i \in \mathbb{Z}_{>0} \mid (i,j) \in Y_+^\lambda\}; \\ r'_i &:= \max(r_i - N, 0), \quad c'_j := \max(c_j - M, 0). \end{aligned}$$

Then from [9, (4.1)–(4.2)] we have

$$\lambda = \sum_{i=1}^M r_i \epsilon_i + \sum_{j=1}^N c'_j \epsilon_{M+j}, \quad \mu = \sum_{i=1}^M r'_{M+1-i} \epsilon_i + \sum_{j=1}^N c_{N+1-j} \epsilon_{M+j}.$$

If v is a lowest weight vector of $V_q(\lambda)$, then $V_q^*(\lambda)$ contains a highest weight vector v^* of weight $-\mu$, and $\mathcal{F}^*(v^*) \in \mathcal{F}^*\left(V_q^*(\lambda)\right)$ is a highest weight vector of weight

$$c_1\epsilon_1 + c_2\epsilon_2 + \cdots + c_N\epsilon_N + r'_1\epsilon_{N+1} + r'_2\epsilon_{N+2} + \cdots + r'_M\epsilon_{M+N},$$

which is exactly $\lambda^\sharp$, leading to the desired isomorphism. □

For $i \in I$ let $U_q^{\le i}(\mathfrak{g}') := \mathcal{F}^{-1}(U_q^{\ge \kappa+1-i}(\mathfrak{g}))$; it is the subalgebra of $U_q(\mathfrak{g}')$ generated by the $s'^{(0)}_{jk}, t'^{(0)}_{jk}$ with $j, k \le i$. To decompose $V_q(\lambda)$ (resp. $V_q^*(\lambda)$) with respect to lowest weights along the ascending chain of subalgebras of $U_q(\mathfrak{g})$

$$U_q^{\ge \kappa}(\mathfrak{g}) \subset U_q^{\ge \kappa-1}(\mathfrak{g}) \subset \cdots \subset U_q^{\ge 2}(\mathfrak{g}) \subset U_q^{\ge 1}(\mathfrak{g}) = U_q(\mathfrak{g})$$

is to decompose $V'_q(\lambda^\sharp)$ with respect to highest (resp. lowest) weights along

$$U_q^{\le 1}(\mathfrak{g}') \subset U_q^{\le 2}(\mathfrak{g}') \subset \cdots \subset U_q^{\le \kappa-1}(\mathfrak{g}') \subset U_q^{\le \kappa}(\mathfrak{g}') = U_q(\mathfrak{g}'),$$

Remark 2.8. By [9], $V'_q(\lambda^\sharp)$ is an irreducible submodule of a tensor power of $V'_q(\epsilon_1)$, and all such tensor powers are semi-simple $U_q(\mathfrak{g}')$-modules. So the decomposition for $V'_q(\lambda^\sharp)$ is equivalent to that for the character formula in Remark 2.5, and then to the branching rule of $\mathfrak{g}'$-modules in [10, Section 5]. We reformulate the latter in terms of $\mathcal{B}'_+(\lambda^\sharp)$, equivalently $\mathcal{B}_-(\lambda)$ by Lemma 2.7, as follows.

(1) $V_q(\lambda)$ admits a basis $(v_T : T \in \mathcal{B}_-(\lambda))$ such that v_T is contained in an irreducible sub-$U_q^{\ge i}(\mathfrak{g})$-module of lowest weight $\mu_T^{\ge i}$ for $i \in I$.
(2) $V_q^*(\lambda)$ admits a basis $(w_T : T \in \mathcal{B}_-(\lambda))$ such that w_T is contained in an irreducible sub-$U_q^{\ge i}(\mathfrak{g})$-module of lowest weight $-\nu_T^{\ge i}$ for $i \in I$.

$\mu_T^{\ge i}$ and $\nu_T^{\ge i}$ are defined as follows. Set $Y_T^{\ge i} := \{(k,l) \in Y_-^\lambda \mid T(k,l) \ge i\}$ and

$$r_k := \sharp\{l \in \mathbb{Z} \mid (-k,-l) \in Y_T^{\ge i}\}, \quad c_l := \sharp\{k \in \mathbb{Z} \mid (-k,-l) \in Y_T^{\ge i}\}.$$

If $i > M$, then $\begin{cases} \mu_T^{\ge i} &= c_1\epsilon_{M+N} + c_2\epsilon_{M+N-1} + \cdots + c_{M+N+1-i}\epsilon_i, \\ \nu_T^{\ge i} &= c_1\epsilon_i + c_2\epsilon_{i+1} + \cdots + c_{M+N+1-i}\epsilon_{M+N}. \end{cases}$ If $i \le M$, then

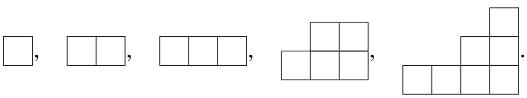
$$\begin{aligned} \mu_T^{\ge i} &= c_1\epsilon_{M+N} + c_2\epsilon_{M+N-1} + \cdots + c_N\epsilon_{M+1} + r'_1\epsilon_M + r'_2\epsilon_{M-1} + \cdots + r'_{M+1-i}\epsilon_i, \\ \nu_T^{\ge i} &= r_1\epsilon_i + r_2\epsilon_{i+1} + \cdots + r_{M+1-i}\epsilon_M + c'_1\epsilon_{M+1} + c'_2\epsilon_{M+2} + \cdots + c'_N\epsilon_{M+N}, \end{aligned}$$

where $r'_k := \max(r_k - N, 0)$ and $c'_l := \max(c_l - M + i - 1, 0)$.

Example 2.9. To illustrate Lemma 2.7 (2) and Remark 2.8, let $\mathfrak{g} = \mathfrak{gl}(2|3)$ and $\lambda = 4\epsilon_1 + 2\epsilon_2 + \epsilon_3 \in \mathcal{P}$. We represent elements in $\mathcal{B}_-(\lambda)$ and $\mathcal{B}'_+(\lambda^\sharp)$ by Young tableaux of shapes $\lambda, \lambda^\sharp$ respectively. Let $T \in \mathcal{B}_-(\lambda)$ be such that

$$\mathcal{B}_-(4\epsilon_1 + 2\epsilon_2 + \epsilon_3) \ni T = \begin{array}{cccc} & & & 1 \\ & & 2 & 2 \\ 1 & 3 & 4 & 5 \end{array} \mapsto \begin{array}{ccc} 1 & 4 & 5 \\ 2 & 4 & \\ 3 & & \\ 5 & & \end{array} = T' \in \mathcal{B}'_+(3\epsilon_1 + 2\epsilon_2 + \epsilon_3 + \epsilon_4).$$

The Young diagrams $Y_T^{\ge i}$ with descending order on $5 \ge i \ge 1$ become:



Correspondingly, the pairs $(\mu_T^{\ge i}, \nu_T^{\ge i})$ from $i = 5$ to $i = 1$ are:

$$\begin{aligned} &(\epsilon_5,\ \epsilon_5), \quad (\epsilon_4 + \epsilon_5,\ \epsilon_4 + \epsilon_5), \quad (\epsilon_3 + \epsilon_4 + \epsilon_5,\ \epsilon_3 + \epsilon_4 + \epsilon_5), \\ &(\epsilon_3 + 2\epsilon_4 + 2\epsilon_5,\ 3\epsilon_2 + \epsilon_3 + \epsilon_4), \quad (\epsilon_2 + \epsilon_3 + 2\epsilon_4 + 3\epsilon_5,\ 4\epsilon_1 + 2\epsilon_2 + \epsilon_3). \end{aligned}$$

Proof of Equations. (2.18)–(2.19)**.** Let us define $g_i(z), g_i^*(z) \in \mathbb{C}[[z]]^\times$ for $i \in I$:

$$g_i(z) := \prod_{(k,l)\in Y_T^{\geq i}} q\frac{1-zaq^{2(l-k)}}{1-zaq^{2(l-k+1)}}, \quad g_i^*(z) := \prod_{(k,l)\in Y_T^{\geq i}} \left(q^{-1}\frac{1-zat_iq^{2(k-l+1)}}{1-zat_iq^{2(k-l)}}\right).$$

By Lemma 2.6, it suffices to prove that: for $i \in I$,

$$C_i(z)v_T = g_i(z)v_T \quad \text{in } V_q^+(\lambda; a), \quad C_i(z)w_T = g^*(z)w_T \quad \text{in } V_q^{+*}(\lambda; a).$$

This is divided into two cases: $i > M$ or $i \leq M$.

Assume $i > M$. Then $T(-k,-l) \geq i$ if and only if $1 \leq l \leq M+N-i+1$ and $1 \leq k \leq c_l$. It follows from Eq. (1.8) that

$$g_i(z) = \prod_{l=1}^{M+N-i+1}\prod_{k=1}^{c_l} q\frac{1-zaq^{2(k-l)}}{1-zaq^{2(k-l+1)}} = \prod_{l=1}^{M+N-i+1}\frac{1-zaq^{2(1-l)}}{q^{-c_l}-zaq^{2(1-l)+c_l}}$$

$$= \prod_{j=i}^{M+N}\left(\frac{q^{(\mu_T^{\geq i},\epsilon_j)}-za\theta_jq^{-(\mu_T^{\geq i},\epsilon_j)}}{1-za\theta_j}\right)^{(\epsilon_j,\epsilon_j)},$$

$$g_i^*(z) = \prod_{l=1}^{M+N-i+1}\prod_{k=1}^{c_l} q^{-1}\frac{1-zat_iq^{2(l-k+1)}}{1-zat_iq^{2(l-k)}} = \prod_{l=1}^{M+N-i+1}\frac{1-zat_iq^{2l}}{q^{c_l}-zat_iq^{2l-c_i}}$$

$$= \prod_{j=i}^{M+N}\left(\frac{q^{-(\nu_T^{\geq i},\epsilon_i)}-za\theta_jq^{(\mu_T^{\geq i},\epsilon_i)}}{1-za\theta_j}\right)^{(\epsilon_j,\epsilon_j)}.$$

Here in the last equation we used $t_iq^{2l} = \tau_{i-1}^2q^{2l-2} = \theta_iq^{2l-2} = \theta_{i+l-1}$.

Assume $i \leq M$. Then $T(-k,-l) \geq i$ if and only if $(1 \leq l \leq N,\ 1 \leq k \leq c_l)$ or $(1 \leq k \leq M+1-i,\ N+1 \leq l \leq N+r_k')$. This gives

$$g_i(z) = \left(\prod_{l=1}^{N}\prod_{k=1}^{c_l} q\frac{1-zaq^{2(k-l)}}{1-zaq^{2(k-l+1)}}\right) \times \left(\prod_{k=1}^{M+1-i}\prod_{l=1}^{r_k'} q\frac{1-zaq^{2(k-l-N)}}{1-zaq^{2(k-l-N+1)}}\right)$$

$$= \prod_{j=i}^{M+N}\left(\frac{q^{(\mu_T^{\geq i},\epsilon_j)}-za\theta_jq^{-(\mu_T^{\geq i},\epsilon_j)}}{1-za\theta_j}\right)^{(\epsilon_j,\epsilon_j)}.$$

Notice that $T(-k,-l) \geq i$ if and only if $(1 \leq k \leq M+1-i,\ 1 \leq l \leq r_k)$ or $(1 \leq l \leq N,\ M-i+2 \leq k \leq M-i+1+c_l')$. This gives

$$g_i^*(z) = \left(\prod_{k=1}^{M+1-i}\prod_{l=1}^{r_k} q^{-1}\frac{1-zat_iq^{2(l-k+1)}}{1-zat_iq^{2(l-k)}}\right)\left(\prod_{l=1}^{N}\prod_{k=1}^{c_l'} q^{-1}\frac{1-zat_iq^{2(l-k-M+i)}}{1-zat_iq^{2(l-k-M+i-1)}}\right)$$

$$= \left(\prod_{k=1}^{M+1-i}\frac{q^{-r_k}-zat_iq^{2(1-k+r_k)}}{1-zat_iq^{2(1-k)}}\right)\left(\prod_{l=1}^{N}\frac{1-zat_iq^{2(l-1-M+i)}}{q^{c_l'}-zat_iq^{2(l-1-M+i-c_l')}}\right)$$

$$= \prod_{j=i}^{M+N}\left(\frac{q^{-(\nu_T^{\geq i},\epsilon_i)}-za\theta_jq^{(\mu_T^{\geq i},\epsilon_i)}}{1-za\theta_j}\right)^{(\epsilon_j,\epsilon_j)}.$$

The last identity comes from $t_i q^{2(l-1-M+i)} = \theta_{M+l}$ and $t_i q^{2(1-k)} = \theta_{i+k-1}$.

In both cases, $g_i(z)$ and $g_i^*(z)$ become Eq. (2.23) with $\mu = \mu_T^{\geq i}$ and $-\nu_T^{\geq i}$ respectively, and this completes the proof of Eqs. (2.18)–(2.19). □

Let $\widehat{\mathcal{Q}}^-$ be the submonoid of $\mathbf{R}$ generated by the $A_{i,a}^{-1}$ with $i \in I_0$ and $a \in \mathbb{C}^\times$.

Corollary 2.10. *Let $i \in I_0$, $a \in \mathbb{C}^\times$ and $m \in \mathbb{C}^\times$. We have*

$$W_{m,a}^{(i)} \cong V_q^+(m\varpi_i; aq^{M-N-i}) \cong V_q^-(m\varpi_i; aq^{N-M+i-2m}) \text{ if } i \leq M, \tag{2.24}$$

$$W_{m,a}^{(i)} \cong V_q^{-*}(\lambda_m^{(i)}; aq^{M+N-2-i}) \simeq V_q^{+*}(\lambda_m^{(i)}; aq^{i-M-N+2m-2}) \text{ if } i > M. \tag{2.25}$$

Here for $i > M$, the Young diagram of $\lambda_m^{(i)} \in \mathcal{P}$ is a rectangle with m rows and $\kappa - i$ columns. An ℓ-weight of $W_{m,a}^{(i)}$ different from $\varpi_{m,a}^{(i)}$ must belong to $\varpi_{m,a}^{(i)} A_{i,aq_i}^{-1} \widehat{\mathcal{Q}}^-$.

Proof. Assume $i \leq M$. The Young diagram $Y_-^{m\varpi_i}$ is a rectangle with i rows and m columns. Let $H \in \mathcal{B}_-(m\varpi_i)$ be such that $H(-k,-l) = i+1-k$ for $1 \leq k \leq i$. Then $v_H \in V_q^+(m\varpi_i; a\tau_i q^{-1})$ in Remark 2.8 is a highest ℓ-weight vector of ℓ-weight

$$m_H = \prod_{l=1}^{m} \prod_{k=1}^{i} \boxed{k}_{a\tau_i q^{2(i+1-k-l)}} = \prod_{l=1}^{m} Y_{i,aq^{2-2l}} = \varpi_{m,a}^{(i)}.$$

Here we used $\prod_{k=1}^{i} \boxed{k}_{a\tau_i q^{2(i+1-k-l)}} = Y_{i,aq^{2-2l}}$ and $\theta_i = \tau_i^2 = q^{2(M-N+1-i)}$ for $1 \leq i \leq M$, based on Example 1.6. This proves the first isomorphism of (2.24); the second one is a consequence of Eqs. (2.18) and (2.20). If $T \in \mathcal{B}_-(m\varpi_i)$ and $T \neq H$, then $T(-k,-l) \geq i+1-k$ and $T(-1,-1) > i$. The ℓ-weight property of $W_{m,a}^{(i)}$ follows from Definition 2.2 and Eq. (2.18):

$$m_T m_H^{-1} \in \boxed{i+1}_{a\tau_i} \boxed{i}_{a\tau_i}^{-1} \widehat{\mathcal{Q}}^- = A_{i,aq}^{-1} \widehat{\mathcal{Q}}^-.$$

Assume $i > M$. Let v be the highest ℓ-weight of $V_q^{-*}(\lambda_m^{(i)}; b)$. By Eq. (1.6),

$$K_p^+(z)v = v \quad \text{for } p \leq i, \quad K_p^+(z)v = \frac{1-zb}{q^{-m} - zbq^m} v \qquad \text{for } p > i.$$

v is of ℓ-weight $\varpi_{m,b\tau_j q}^{(j)}$, proving the first isomorphism of (2.25).Since $\boxed{l}_a^* \equiv \boxed{l}_a'$ for $l \in I$, the second isomorphism of (2.25) is deduced from Eqs. (2.19) and (2.21). let $H \in \mathcal{B}_+(\lambda_m^{(i)})$ be such that $H(k,l) = i+l$ for $1 \leq l \leq M+N-i$. The monomial m_H' associated to H in Eq. (2.21) is the highest ℓ-weight. If $T \in \mathcal{B}_+(\lambda_m^i)$ and $T \neq H$, then $T(k,l) \leq i+l$ and $T(1,1) \leq i$. By Definition 2.2 and Eq. (2.21):

$$m_T' m_H'^{-1} \in \boxed{i}_{a\tau_i^{-1}}' \boxed{i+1}_{a\tau_i^{-1}}'^{-1} \widehat{\mathcal{Q}}^- = A_{i,aq^{-1}}^{-1} \widehat{\mathcal{Q}}^-,$$

proving the ℓ-weight property of $W_{m,a}^{(i)}$. □

The ℓ-weight property is similar to [31, Lemma 4.4]; $W_{m,a}^{(i)}$ in [31] is $W_{m,aq_i^{2m-2}}^{(i)}$ here.

Let $\varpi_{m,a}^{(M-)} := \prod_{l=1}^{m} Y_{M,aq^{2l-2}}^{-1}$ and $W_{m,a}^{(M-)} := L(\varpi_{m,a}^{(M-)})$. Similarly we have

$$W_{m,a}^{(M-)} \cong V_q^{-*}(\lambda_m; aq^{N-2}) \simeq V_q^{+*}(\lambda_m; aq^{2m-2-N}). \tag{2.26}$$

where $\lambda_m \in \mathcal{P}$ is such that its Young diagram is a rectangle with m rows and N columns. If $\varpi_{m,a}^{(M-)}\mathbf{n} \in \mathrm{wt}_\ell(W_{m,a}^{(M-)})$ and $\mathbf{n} \neq 1$, then $\mathbf{n} \in A_{M,aq^{-1}}^{-1} \widehat{\mathcal{Q}}^-$.

3. Length-Two Representations

A $Y_q(\mathfrak{g})$-module V in category $\mathcal{O}$ is *of length-two* if it admits a Jordan–Hölder series of length two, namely, it fits in a short exact sequence $0 \to S \to V \to S' \to 0$ in category $\mathcal{O}$ such that both S' and S' are irreducible. We shall simply write such a sequence as $S \hookrightarrow V \twoheadrightarrow S'$.

In this section we describe length-two modules by tensor products.

For $i \in I_0, a \in \mathbb{C}^\times, m \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}_{\geq 0}$, let us define $\mathbf{d}_{m,a}^{(i,s)} \in \mathbf{R}_U$ to be

$$\varpi^{(i)}_{m,aq_i^{2m+1}} \varpi^{(i)}_{m+s,aq_i^{2m-1}} \prod_{l=1}^{m} A^{-1}_{i,aq_i^{2l}} \text{ if } i \neq M, \quad \varpi^{(M)}_{s,aq^{-1}} \prod_{j \in I_0 : j \sim M} \varpi^{(j)}_{m,aq_j^{2m}} \text{ if } i = M.$$

Let $D_{m,a}^{(i,s)} := L(\mathbf{d}_{m,a}^{(i,s)})$ be the irreducible $U_q(\widehat{\mathfrak{g}})$-module.

Remark 3.1. Let us rewrite $\mathbf{d}_{m,a}^{(i,s)}$ in terms of the Ψ using Eq. (1.14):

$$\mathbf{d}_{m,a}^{(i,s)} \equiv \frac{\Psi_{i,aq_i^{-2s}}}{\Psi_{i,a}} \prod_{j \in I_0 : j \sim i} \frac{\Psi_{j,aq_{ij}^{-1}}}{\Psi_{j,aq_{ij}^{-2m-1}}}.$$

In the non-graded case $N = 0$, we can identify $\mathbf{n}_{i,a}^+$ with Ψ in [35, (6.13)] and $\mathfrak{m}_{i,a}^{(2)}$ in [19, (6.2)], $\mathbf{d}_{m,a}^{(i,s)}$ with $\widetilde{\Psi}_i^{(-s,2m-1)}$ in [25, Section 4.3]. Notice that $\mathbf{d}_{m,a}^{(i,s)}$ satisfies the condition of "minimal affinization by parts" in [14, Theorem 2].

Theorem 3.2. *Let $i \in I_0$ and $a \in \mathbb{C}^\times$. The $Y_q(\mathfrak{g})$-module $N_{i,a}^+ \otimes L_{i,a}^+$ has a Jordan–Hölder series of length two and in the Grothendieck ring $K_0(\mathcal{O})$:*

$$[N_{i,a}^+ \otimes L_{i,a}^+] = [L^+_{i,aq_i^{-2}}] \prod_{j \in I_0 : j \sim i} [L^+_{j,aq_{ij}^{-1}}] + [D][L^+_{i,a\hat{q}_i^2}] \prod_{j \in I_0 : j \sim i} [L^+_{j,aq_{ij}}]. \tag{3.27}$$

Here $D = L(\mathbf{n}_{i,a}^+ \Psi_{i,a} \Psi^{-1}_{i,a\hat{q}_i^2} A^{-1}_{i,a} \prod_{j \sim i} \Psi^{-1}_{j,aq_{ij}})$ is one-dimensional.

When $i = M$, the two monomials at the right-hand side of Eq. (3.27) has a common factor $[L^+_{M,aq^{-2}}]$. This is a special feature of quantum affine superalgebras.

Theorem 3.3. *Let $i \in I_0 \setminus \{M\}$, $a \in \mathbb{C}^\times$ and $m, s \in \mathbb{Z}_{>0}$. There are short exact sequences of $U_q(\widehat{\mathfrak{g}})$-modules whose first and third terms are irreducible:*

$$D_{m,a}^{(i,s)} \hookrightarrow W^{(i)}_{m,aq_i^{2m+1}} \otimes W^{(i)}_{m+s,aq_i^{2m-1}} \twoheadrightarrow W^{(i)}_{m+s+1,aq_i^{2m+1}} \otimes W^{(i)}_{m-1,aq_i^{2m-1}},$$

$$D^{(i,0)}_{m+s,aq_i^{-2s}} \otimes W^{(i)}_{m,aq_i^{2m+1}} \hookrightarrow W^{(i)}_{m+s,aq_i^{2m+1}} \otimes D^{(i,s)}_{m,a} \twoheadrightarrow W^{(i)}_{m+s+1,aq_i^{2m+1}} \otimes D^{(i,s-1)}_{m,a}.$$

The assumption $i \neq M$ is necessary because $\dim W_{m,a}^{(M)} = 2^{MN}$ for $m \geq N$. Equation (3.27) corresponds to [35, (6.14)] and [19, Proposition 6.8], and can be thought of as a two-term Baxter TQ relation for $Y_q(\mathfrak{g})$. The exact sequences of Theorem 3.3 are extended T-systems [31,42], the initial case $s = 0$ being the T-system in [44]; see Theorem 8.3.

The proof of Theorem 3.2, given in Sect. 4, is similar to [35, (6.14)], based on q-characters. Theorem 3.3 is more involved and requires cyclicity of tensor products of KR modules; its proof is postponed to Sect. 8.

We make crucial use of the idea that $D_{m,a}^{(i,s)}$ admits an injective resolution by tensor products of KR modules of the same Dynkin node for $i \neq M$.

Lemma 3.4. *Let $m \in \mathbb{Z}_{>0}$, $a \in \mathbb{C}^\times$ and $i \in I_0 \setminus \{M\}$. If $\varpi_{m,a}^{(i)} \mathbf{n} \in \mathrm{wt}_\ell(W_{m,a}^{(i)})$ and $\mathbf{n} \neq 1$, then either $\mathbf{n} = A_{i,aq_i}^{-1} A_{i,aq_i^{-1}}^{-1} \cdots A_{i,aq_i^{3-2l}}^{-1}$ for some $1 \leq l \leq m$, or $\mathbf{n}$ belongs to $A_{i,aq_i}^{-1} A_{j,aq_i^2}^{-1} \widehat{\mathcal{Q}}^-$ where $j \in I_0$ and $j \sim i$.*

Proof. We only consider the case $i < M$; the other case is similar. Let us be in the situation of the proof of Corollary 2.10. By Eq. (2.18), $\mathbf{n} = m_T m_H^{-1}$ for a unique $T \in \mathcal{B}_-(m\varpi_i)$ with $T(-1,-1) > i$ and $T(-k,-l) \geq i+1-k$. If $T(-1,-1) > i+1$, then using $\tau_{i+1} = q^{-1}\tau_i$ we obtain

$$m_T m_H^{-1} \in \boxed{i+2}_{a\tau_i} \boxed{i}_{a\tau_i}^{-1} \widehat{\mathcal{Q}}^- = A_{i,aq}^{-1} A_{i+1,aq^2}^{-1} \widehat{\mathcal{Q}}^-.$$

If $T(-2,-1) > i-1$, then together with $T(-1,-1) > i$ we have

$$m_T m_H^{-1} \in \boxed{i+1}_{a\tau_i} \boxed{i}_{a\tau_i}^{-1} \boxed{i}_{a\tau_i q^2} \boxed{i-1}_{a\tau_i q^2}^{-1} \widehat{\mathcal{Q}}^- = A_{i,aq}^{-1} A_{i-1,aq^2}^{-1} \widehat{\mathcal{Q}}^-.$$

Suppose $T(-1,-1) = i+1$ and $T(-2,-1) = i-1$. There exists $1 \leq l \leq m$ such that the only difference between T, H is at $(-1,-j)$ with $1 \leq j \leq l$, and

$$m_T m_H^{-1} = \prod_{j=1}^{l} \boxed{i+1}_{a\tau_i q^{2-2j}} \boxed{i}_{a\tau_i q^{2-2j}}^{-1} = \prod_{j=1}^{l} A_{i,aq^{3-2j}}^{-1}.$$

This completes the proof of the lemma. □

Corollary 3.5. *Let $m, s \in \mathbb{Z}_{>0}$, $a \in \mathbb{C}^\times$ and $i \in I_0 \setminus \{M\}$.*

(1) *For $1 \leq l \leq s$, we have $\mathbf{d}_{m,a}^{(i,s)} A_{i,a}^{-1} A_{i,aq_i^{-2}}^{-1} \cdots A_{i,aq_i^{2-2l}}^{-1} \in \mathrm{wt}_\ell(D_{m,a}^{(i,s)})$ and its associated ℓ-weight space is one-dimensional.*

(2) *If $\mathbf{d}_{m,a}^{(i,s)} \mathbf{n} \in \mathrm{wt}_\ell(D_{m,a}^{(i,s)})$ is not of the form of (1) and $\mathbf{n} \neq 1$, then $\mathbf{n} \in \{A_{j,aq_i^{2m+1}}^{-1}, A_{i,aq_i^{2m+2}}^{-1} \mid j \in I_0,\ j \sim i\} \widehat{\mathcal{Q}}^-$.*

Proof. For non-graded quantum affine algebras this corollary is [25, Lemma 4.8], the proof utilized a delicate elimination theorem of ℓ-weights [33, Theorem 5.1]. Here our proof is a weaker version of elimination based on the restriction to the diagram subalgebra U_i of $U_q(\widehat{\mathfrak{g}})$ generated by $(x_{i,n}^\pm, \phi_{i,n}^\pm)_{n \in \mathbb{Z}}$. By Remark 1.2, the algebra U_i is a quotient of $U_{q_i}(\widehat{\mathfrak{sl}_2})$.

Set $T := W_{m,aq_i^{2m+1}}^{(i)} \otimes W_{m+s,aq_i^{2m-1}}^{(i)}$ and $S := L(\varpi_{m,aq_i^{2m+1}}^{(i)} \varpi_{m+s,aq_i^{2m-1}}^{(i)})$. Then S is a sub-quotient of T. Let $\lambda := (2m+s)\varpi_i$. By Corollary 2.10,

(A) $\dim T_{q^{\lambda - k\alpha_i}} = \min(m+1, k+1)$ for $0 \leq k \leq m+s$.

1. Let $v_0 \in S$ be a highest ℓ-weight vector and $S^i := U_i v_0 \subseteq S$. Viewed as a $U_{q_i}(\widehat{\mathfrak{sl}_2})$-module, S^i is of highest ℓ-weight [27, Section 2]

$$\mathbf{m}_i := (Y_{aq_i^{2m+1}} Y_{aq_i^{2m-1}} \cdots Y_{aq_i^3})(Y_{aq_i^{2m-1}} Y_{i,aq_i^{2m-3}} \cdots Y_{aq_i^{1-2s}}).$$

S^i is spanned by the $x_{i,n_1}^- x_{i,n_2}^- \cdots x_{i,n_k}^- v_0$. If $w \in S^i$ is annihilated by the $x_{i,n}^+$, then $x_{j,n}^+ w = 0 \in S$ for all $j \in I_0 \backslash \{i\}$ (because $[x_{j,n}^+, x_{i,k}^-] = 0$) and $w \in \mathbb{C}v_0$. The $U_{q_i}(\widehat{\mathfrak{sl}_2})$-module S^i is irreducible and has a factorization [15, Theorem 4.8]:

$$S^i \cong L^i(Y_{aq_i^{2m+1}} Y_{aq_i^{2m-1}} \cdots Y_{aq_i^{1-2s}}) \otimes L^i(Y_{aq_i^{2m-1}} Y_{aq_i^{2m-3}} \cdots Y_{aq_i^3}),$$

where $L^i(\mathbf{n})$ denotes the irreducible $U_{q_i}(\widehat{\mathfrak{sl}_2})$-module of highest ℓ-weight $\mathbf{n}$ (for $\mathbf{n}$ a product of the Y_b). For $k \in \mathbb{Z}_{>0}$, let $V_k \subseteq S^i$ be the subspace spanned by the $x_{i,n_1}^- x_{i,n_2}^- \cdots x_{i,n_k}^- v_0$ with $n_l \in \mathbb{Z}$ for $1 \leq l \leq k$. Then $V_k = S_{q^{\lambda - k\alpha_i}}$. Based on the q-character of S^i with respect to the spectra of $\phi_i^+(z)$ in [27, Section 4.1], for $-1 \leq l < s$ we have:

(B) $\dim S_{q^{\lambda-k\alpha_i}} = \min(m, k+1)$ for $1 \leq k \leq m+s$;

(C) $\mathbf{m}_i \prod_{t=-l}^{m} (Y_{aq_i^{2t+1}}^{-1} Y_{aq_i^{2t-1}}^{-1})$ is not an ℓ-weight of the $U_{q_i}(\widehat{\mathfrak{sl}_2})$-module S^i.

2. By (A)–(B), $\{\mathbf{n} \in \mathrm{wt}_\ell(T) \backslash \mathrm{wt}_\ell(S) \mid \varpi(\mathbf{n}) = \lambda - (m+l)\alpha_i\} = \{\mathbf{n}_l\}$ for $0 \leq l \leq s$, the multiplicity of $\mathbf{n}_l$ in $\chi_q(T) - \chi_q(S)$ is one, and $L(\mathbf{n}_0)$ is a sub-quotient of T. Comparing the spectra of $\phi_i^+(z)$ by (C) and Lemma 3.4, we obtain: $\mathbf{n}_0 = \mathbf{d}_{m,a}^{(i,s)}$ and $\mathbf{n}_l = \mathbf{d}_{m,a}^{(i,s)} A_{i,a}^{-1} A_{i,aq_i^{-2}}^{-1} \cdots A_{i,aq_i^{2-2l}}^{-1}$. Part (2) follows by viewing $D_{m,a}^{(i,s)}$ as a sub-quotient of T. If $(D_{m,a}^{(i,s)})_{q^{\lambda-(m+l)\alpha_i}} \neq 0$ for $1 \leq l \leq s$, then necessarily $\mathbf{n}_l \in \mathrm{wt}_\ell(D_{m,a}^{(i,s)})$ and its ℓ-weight space is one-dimensional, proving (1).

3. Let $w_0 \in D_{m,a}^{(i,s)}$ be a highest ℓ-weight vector. Then $x_{i,0}^+ w_0 = 0$ and $\phi_{i,0}^+ w_0 = q_i^s w_0$. Since the triple $(x_{i,0}^+,\ x_{i,0}^-,\ \phi_{i,0}^+)$ generates a quotient algebra of $U_{q_i}(\mathfrak{sl}_2)$, we have $(D_{m,a}^{(i,s)})_{q^{\lambda-(m+l)\alpha_i}} \ni (x_{i,0}^-)^l w_0 \neq 0$ for $1 \leq l \leq s$. $\square$

The case $i = M$ is distinguished since U_M is not related to $U_q(\widehat{\mathfrak{sl}_2})$.

Corollary 3.6. *Let $m, s \in \mathbb{Z}_{>0}$ and $a \in \mathbb{C}^\times$.*

(1) $\mathbf{d}_{m,a}^{(M,s)} A_{M,a}^{-1} \in \mathrm{wt}_\ell(D_{m,a}^{(M,s)})$ *and the ℓ-weight space is one-dimensional.*

(2) $(\mathbf{d}_{m,a}^{(M,s)})^{-1} \mathrm{wt}_\ell(D_{m,a}^{(M,s)}) \subset \left(\{A_{j,aq_j^{2m+1}}^{-1} \mid j \in I_0,\ j \sim M\} \widehat{\mathcal{Q}}^- \right) \cup \{1, A_{M,a}^{-1}\}.$

Proof. Assume $M, N > 1$ without loss of generality. Let $\mathbf{n} \in (\mathbf{d}_{m,a}^{(M,s)})^{-1} \mathrm{wt}_\ell(D_{m,a}^{(M,s)})$ with $\mathbf{n} \notin \{A_{M+1,aq^{-2m-1}}^{-1},\ A_{M-1,aq^{2m+1}}^{-1}\} \widehat{\mathcal{Q}}^-$ and $\mathbf{n} \neq 1$.

Firstly, set $\lambda := s\varpi_M + m\varpi_{M-1}$. Then $\lambda \in \mathcal{P}$ and its Young diagram Y_+^λ is formed of (k,l) where either $(1 \leq k < M,\ 1 \leq l \leq s+m)$ or $(k = M,\ 1 \leq l \leq s)$. Consider the evaluation module $S := V_q^-(\lambda; aq^{N-2s-1})$. Let $H \in \mathcal{B}_+(\lambda)$ be such that $H(k,l) = k$. The monomial m_H attached to H in Eq. (2.20) is the highest ℓ-weight of S. From the proof of Corollary 2.10 we see that

$$m_H = (Y_{M,aq^{1-2s}} \cdots Y_{M,aq^{-3}} Y_{M,aq^{-1}})(Y_{M-1,aq^2} Y_{M-1,aq^4} \cdots Y_{M-1,aq^{2m}}).$$

In particular, the spectral parameters at the boxes (M,s) and $(M-1, s+m)$ of H are $a\tau_M q^{-1}$ and $a\tau_{M-1} q^{2m}$ respectively. Let $T \in \mathcal{B}_+(\lambda)$ and $T \neq H$. If $T(M-1, s+m) \geq M$, then by Definition 2.2 and Eq. (2.20),

$$m_T m_H^{-1} \in \boxed{M}_{a\tau_{M-1}q^{2m}} \boxed{M-1}_{a\tau_{M-1}q^{2m}}^{-1} \widehat{\mathcal{Q}}^- = A_{M-1,aq^{2m+1}}^{-1} \widehat{\mathcal{Q}}^-.$$

If $T(M-1, s+m) < M$, then $T(k,l) = k$ for $k < M$ and by Eq. (2.20):

(i) the ℓ-weight space S_{m_T} is also the one-dimensional weight space $S_{\varpi(m_T)}$;
(ii) $m_T m_H^{-1} A_{M,a}$ is a product of the $A_{j,b}^{-1}$ with $j \geq M$;
(iii) if $m_T m_H^{-1} A_{M,a}$ is a product of the $A_{M,b}^{-1}$, then $m_T m_H^{-1} A_{M,a} = 1$.

Here we used Definition 2.1 and $T(M,l) \geq M,\ T(M,s) > M$.

Secondly, viewing $D_{m,a}^{(M,s)}$ as a sub-quotient of $S \otimes W_{m,aq^{-2m}}^{(M+1)}$ gives $\mathbf{n} = \mathbf{n}_1\mathbf{n}_2$ with $m_H\mathbf{n}_1 \in \mathrm{wt}_\ell(S)$ and $\mathbf{n}_2\varpi_{m,aq^{-2m}}^{(M+1)} \in \mathrm{wt}_\ell(W_{m,aq^{-2m}}^{(M+1)})$. Since $\mathbf{n} \notin A_{M+1,aq^{-2m-1}}^{-1}\widehat{\mathcal{Q}}^-$, by Corollary 2.10, $\mathbf{n}_2 = 1$ and $m_H\mathbf{n} \in \mathrm{wt}_\ell(S)$. Since $\mathbf{n} \notin A_{M-1,aq^{2m+1}}^{-1}\widehat{\mathcal{Q}}^-$, (ii)–(iii) hold by replacing $m_T m_H^{-1}$ with $\mathbf{n}$, and $\dim(D_{m,a}^{M,s})_{\mathbf{d}_{m,a}^{(M,s)}\mathbf{n}} = 1$.

Thirdly, for $t \in \mathbb{Z}_{>0}$, let $\mu_t \in \mathcal{P}$ be such that its Young diagram $Y_-^{\mu_t}$ is formed of $(-k,-l)$ where either $(1 \leq l < N,\ 1 \leq k \leq m+t)$ or $(l = N,\ 1 \leq k \leq t)$. Consider the evaluation module $S_t := V_q^{+*}(\mu_t; aq^{2t-1-N})$. Let $H_t \in \mathcal{B}_-(\mu_t)$ be such that $H_t(-k,-l) = M+N+1-l$. The monomial $m_{H_t}^*$ in Eq. (2.19) is the highest ℓ-weight of S_t and by Corollary 2.10 and Eq. (2.26):

$$m_{H_t}^* \equiv \varpi_{m,aq^{-2m}}^{(M+1)}\varpi_{t,aq}^{(M-)}.$$

The spectral parameters at the boxes $(-t,-N)$ and $(-t-m, 1-N)$ of H_t are $a\tau_M^{-1}q$ and $a\tau_{M+1}^{-1}q^{-2m}$ respectively. Let $T \in \mathcal{B}_-(\mu_t)$ and $T \neq H_t$. If $T(-t-m, 1-N) < M+2$, then by Definition 2.2 and Eq. (2.19),

$$m_T^* m_{H_t}^{*-1} \in \boxed{M+1}^*_{a\tau_{M+1}^{-1}q^{-2m}} \boxed{M+2}^{*-1}_{a\tau_{M+1}^{-1}q^{-2m}} \widehat{\mathcal{Q}}^- = A_{M+1,aq^{-2m-1}}^{-1}\widehat{\mathcal{Q}}^-.$$

If $T(-t-m, 1-N) = M+2$, then $T(-k,-l) = M+N+1-l$ for $1 \leq l < N$. Equation (2.19) implies that $m_T^* m_{H_t}^{*-1} A_{M,a}$ is a product of the $A_{j,b}^{-1}$ with $j \leq M$.

Lastly, viewing $D_{m,a}^{(M,s)}$ (after tensoring with a one-dimensional module) as a sub-quotient of $S_t \otimes W_{t+s,aq^{2t-1}}^{(M)} \otimes W_{m,aq^{2m}}^{(M-1)}$ and choosing $t \in \mathbb{Z}_{>0}$ so large that $\mathbf{n} \notin A_{M,aq^{2t}}^{-1}\widehat{\mathcal{Q}}^-$, we obtain $m_{H_t}^*\mathbf{n} \in \mathrm{wt}_\ell(S_t)$, and so $\mathbf{n}A_{M,a}$ is a product of the $A_{j,b}^{-1}$ with $j \leq M$. From (ii)–(iii) it follows that $\mathbf{n}A_{M,a} = 1$.

It remains to show that $\mathbf{d}_{m,a}^{(M,s)}A_{M,a}^{-1} \in \mathrm{wt}_\ell(D_{m,a}^{(M,s)})$. Indeed, as a $U_q(\mathfrak{g})$-module, $D_{m,a}^{(M,s)}$ has a highest weight vector of highest weight $q^{m\varpi_{M-1}+s\varpi_M+m\varpi_{M+1}}$, and so $q^{m\varpi_{M-1}+s\varpi_M+m\varpi_{M+1}-\alpha_M} \in \mathrm{wt}(D_{m,a}^{(M,s)})$. This means that there exists $\mathbf{n} \in (\mathbf{d}_{m,a}^{(M,s)})^{-1}\mathrm{wt}_\ell(D_{m,a}^{(M,s)})$ with $\varpi(\mathbf{n}) = q^{-\alpha_M}$, which forces $\mathbf{n} = A_{M,a}^{-1}$. $\square$

As an illustration, for $\mathfrak{g} = \mathfrak{gl}(3|4)$ and $(m,s,t) = (2,3,1)$ we have

$$H = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\\hline 2&2&2&2&2\\\hline 3&3&3 \\\cline{1-3}\end{array} \in \mathcal{B}_+(3\varpi_3 + 2\varpi_2), \quad H_t = \begin{array}{cccc}\cline{2-4} &\multicolumn{1}{|c|}{5}&6&\multicolumn{1}{c|}{7}\\\cline{2-4} &\multicolumn{1}{|c|}{5}&6&\multicolumn{1}{c|}{7}\\\hline \multicolumn{1}{|c|}{4}&5&6&\multicolumn{1}{c|}{7}\\\hline\end{array} \in \mathcal{B}_-(3\varpi_3 + \varpi_1).$$

4. Proof of TQ Relations: Theorem 3.2

The crucial part in the proof is the irreducibility of arbitrary tensor products of positive prefundamental modules. In the case of quantum affine algebras this was proved in [24, Theorem 4.11] and [19, Lemma 5.1]. Our approach is similar to [19], based on the duality functor $\mathcal{G}^*$ in Lemma 1.9.

Lemma 4.1. *Let $a \in \mathbb{C}^\times$ and $i \in I_0$. We have*

$$\chi_q(L^+_{i,a}) = \Psi_{i,a} \times \chi(L^+_{i,a}).$$

Proof. We can adapt the proof of [24, Theorem 4.1]. Essentially we just need a weaker version of [24, Lemma 4.5]: any ℓ-weight of $W^{(i)}_{m,a}$ different from $\varpi^{(i)}_{m,a}$ belongs to $\varpi^{(i)}_{m,a} A^{-1}_{i,aq_i} \widehat{\mathcal{Q}}^-$, which is Corollary 2.10. □

For negative prefundamental modules we recall the main results of [53].

Lemma 4.2 [53, Lemma 6.7 & Corollary 7.4]. *Let $a, c \in \mathbb{C}^\times$ and $i \in I_0$.*

(i) *The $\widetilde{\chi}_q(W^{(i)}_{m,aq_i^{-1}})$ for $m \in \mathbb{Z}_{>0}$ are polynomials in $\mathbb{Z}[A^{-1}_{j,b}]_{(j,b)\in I_0\times aq^{\mathbb{Z}}}$, and as $m \to \infty$ they converge to a formal power series in $\mathbb{Z}[[A^{-1}_{j,b}]]_{(j,b)\in I_0\times aq^{\mathbb{Z}}}$, which is exactly the normalized q-character $\widetilde{\chi}_q(L^-_{i,a})$.*

(ii) *There exists a $U_q(\widehat{\mathfrak{g}})$-module $\mathscr{W}^{(i)}_{c,a}$ in category $\mathcal{O}$ such that*

$$\chi_q(\mathscr{W}^{(i)}_{c,a}) = \boldsymbol{\omega}^{(i)}_{c,a} \times \widetilde{\chi}_q(L^-_{i,a}).$$

It is irreducible if $c \notin \pm q^{\mathbb{Z}}$.

In particular, any ℓ-weight of $L^-_{i,a}$ different from $\Psi_{i,a}$ belongs to $\Psi_{i,a} A^{-1}_{i,a} \widehat{\mathcal{Q}}^-$.

By [53, Section 4], the $U_q(\widehat{\mathfrak{g}})$-module $\mathscr{W}^{(i)}_{c,a}$ is a "generic asymptotic limit" of the KR modules $W^{(i)}_{m,aq_i^{-1}}$; see also the proof of Lemma 9.4.

Corollary 4.3. *Any tensor product of positive (resp. negative) prefundamental modules in category $\mathcal{O}$ is irreducible.*

Proof. In view of Lemmas 4.1–4.2, the proof of [19, Lemma 5.1] works here by replacing the duality of [19, Lemma 3.5] with the functor $\mathcal{G}^*$ in Lemma 1.9. □

Proof of Theorem 3.2. In the non-graded case this was sketched in [35, Section 6.1.3]. Here our proof is in the spirit of [25, Lemma 4.8], by replacing the elimination theorem of ℓ-weights therein with Corollaries 3.5–3.6.

Let $T := N^+_{i,a} \otimes L^+_{i,a}$. We need to prove that T has exactly two irreducible sub-quotient $S' := L(\mathbf{n}^+_{i,a}\Psi_{i,a})$ and $S'' := L(\mathbf{n}^+_{i,a}\Psi_{i,a}A^{-1}_{i,a})$ of multiplicity one, which implies Theorem 3.2 since S' and S'' are irreducible tensor products of positive prefundamental modules with D. Clearly S' is an irreducible sub-quotient of T, and $\chi_q(S') + \chi_q(S'') = \mathbf{n}^+_{i,a}\Psi_{i,a}(1 + A^{-1}_{i,a})\chi(L^+_{i,1})\prod_{j\sim i}\chi(L^+_{j,1})$ by Corollary 4.3.

That S'' is a sub-quotient of T, i.e. $\chi_q(T)$ is bounded below by $\chi_q(S') + \chi_q(S'')$, is proved in the same way as in the first half of the comment after [35, (6.13)]. For the reverse inequality, it suffices to show that $\chi_q(N^+_{i,a})$ is bounded above by $\mathbf{n}^+_{i,a}(1 + A^{-1}_{i,a})\prod_{j\sim i}\chi(L^+_{j,1})$.

Assume $\mathbf{n}^+_{i,a}\mathbf{n} \in \mathrm{wt}_\ell(N^+_{i,a})$ and $\mathbf{n} \neq 1$. For $m \in \mathbb{Z}_{>0}$ let $S_m := L(\mathbf{n}^+_{i,a}(\mathbf{d}^{(i,1)}_{m,a})^{-1})$ and view $N^+_{i,a}$ as a sub-quotient of $D^{(i,1)}_{m,a} \otimes S_m$. Write

$$\mathbf{n} = \mathbf{n}'_m \mathbf{n}''_m, \quad \mathbf{n}'_m \mathbf{d}^{(i,1)}_{m,a} \in \mathrm{wt}_\ell(D^{(i,1)}_{m,a}), \quad \mathbf{n}''_m \mathbf{n}^+_{i,a} (\mathbf{d}^{(i,1)}_{m,a})^{-1} \in \mathrm{wt}_\ell(S_m).$$

By Remark 3.1, we have $\mathbf{n}^+_{i,a}(\mathbf{d}^{(i,1)}_{m,a})^{-1} \equiv \prod_{j\sim i} \Psi_{j,aq_{ij}^{-2m-1}}$. It follows from Corollary 4.3 that $\mathbf{n}''_m \in q^{\mathbf{Q}^-}$, $\chi(S_m) = \prod_{j\sim i} \chi(L^+_{j,1})$, and so $\mathbf{n} \in \widehat{\mathcal{Q}}^- q^{\mathbf{Q}^-}$.

Choose $t \in \mathbb{Z}_{>0}$ large enough so that $\mathbf{n} \in \widehat{\mathcal{Q}}_t^- q^{\mathbf{Q}^-}$ where $\widehat{\mathcal{Q}}_t^-$ is the submonoid of $\widehat{\mathcal{Q}}$ generated by the A^{-1}_{j,aq^l} with $-t < l < t$. Then for $m > t$, we must have $\mathbf{n}'_m \in \{1, A^{-1}_{i,a}\}$ by Corollaries 3.5–3.6. This implies that $\mathbf{n}''_m$ is uniquely determined by $\mathbf{n}$ and $\dim(N^+_{i,a})_{\mathbf{n}} \leq \dim(S_m)_{\mathbf{n}''_m}$. As a consequence, the coefficient of any $\mathbf{f} \in \widehat{\mathfrak{P}}$ in $\mathbf{n}^+_{i,a}(1 + A^{-1}_{i,a}) \prod_{j\sim i} \chi(L^+_{j,1}) - \chi_q(N^+_{i,a})$ is non-negative. □

5. Main Result: Asymptotic TQ Relations

We replace the L, N in Eq. (3.27) by $U_q(\widehat{\mathfrak{g}})$-modules using the functor $\mathcal{G}^*$.

Corollary 5.1. *Let $i \in I_0$ and $a \in \mathbb{C}^\times$. In the Grothendieck ring $K_0(\mathcal{O})$:*

$$[N^-_{i,a}][L^-_{i,a}] = [L^-_{i,a\hat{q}_i^2}] \prod_{j\in I_0: j\sim i} [L^-_{j,aq_{ij}}] + [D][L^-_{i,aq_i^{-2}}] \prod_{j\in I_0: j\sim i} [L^-_{j,aq_{ij}^{-1}}] \tag{5.28}$$

where $D = L(\mathbf{n}^-_{i,a} \Psi^{-1}_{i,a} A^{-1}_{i,a} \Psi_{i,aq_i^{-2}} \prod_{j\sim i} \Psi_{j,aq_{ij}^{-1}})$ is one-dimensional.

Proof. Applying $\mathcal{G}^{*-1}$ to Eq. (3.27) in $K_0(\mathcal{O}')$ gives (5.28) by Lemma 1.9. Take q-characters in Eq. (5.28). By Lemma 4.2, $\mathbf{n}^-_{i,a}\Psi^{-1}_{i,a}A^{-1}_{i,a}$ appears at the left-hand side, but in none of the $\chi_q(L^-_{j,b})$ at the right-hand side. This forces $\chi_q(D)\Psi^{-1}_{i,aq_i^{-2}} \prod_{j\sim i} \Psi^{-1}_{j,aq_{ij}^{-1}} = \mathbf{n}^-_{i,a}\Psi^{-1}_{i,a}A^{-1}_{i,a}$ and proves the second statement. □

Equation (5.28) becomes [35, Example 7.8] when $N = 0$.

Proposition 5.2. *Let $i \in I_0$ and $a, c \in \mathbb{C}^\times$. There exists a $U_q(\widehat{\mathfrak{g}})$-module $\mathcal{N}^{(i)}_{c,a}$ in category $\mathcal{O}$ whose q-character is*

$$\chi_q(\mathcal{N}^{(i)}_{c,a}) = \mathbf{n}^{(i)}_{c,a} \times \widetilde{\chi}_q(N^-_{i,a}).$$

If $c^2 \notin q^{\mathbb{Z}}$, then $\mathcal{N}^{(i)}_{c,a}$ is irreducible.

The proof of this proposition will be given in Sect. 7. Assuming this proposition, we are able to prove the main result of the paper.

Theorem 5.3. *Let $i \in I_0$ and $a, c, d \in \mathbb{C}^\times$. In the Grothendieck ring $K_0(\mathcal{O})$:*

$$\begin{aligned}[][\mathcal{N}^{(i)}_{c,a}][\mathscr{W}^{(i)}_{d,a}] = [\mathscr{W}^{(i)}_{d\hat{q}_i, a\hat{q}_i^2}] \prod_{j\in I_0: j\sim i} [\mathscr{W}^{(j)}_{c_{ij}^{-1}, aq_{ij}}] \\ + [D_i^-][\mathscr{W}^{(i)}_{dq_i^{-1}, aq_i^{-2}}] \prod_{j\in I_0: j\sim i} [\mathscr{W}^{(j)}_{c_{ij}^{-1}q_{ij}^{-1}, aq_{ij}^{-1}}]\end{aligned} \tag{5.29}$$

where $D_i^- = L(\mathbf{n}_{c,a}^{(i)}\boldsymbol{\omega}_{d,a}^{(i)}A_{i,a}^{-1}(\boldsymbol{\omega}_{dq_i^{-1},aq_i^{-2}}^{(i)}\prod_{j\sim i}\boldsymbol{\omega}_{c_{ij}^{-1}q_{ij}^{-1},aq_{ij}^{-1}}^{(j)})^{-1})$ *is a one-dimensional* $U_q(\widehat{\mathfrak{g}})$*-module. If* $c^2\notin q^{\mathbb{Z}}$*, then in* $K_0(\mathcal{O})$

$$[M_{c,a}^{(i)}][\mathscr{W}_{d,ad^2}^{(i)}] = [\mathscr{W}_{dq_i,ad^2}^{(i)}]\prod_{j\in I_0:j\sim i}[\mathscr{W}_{c_{ij}^{-1},aq_{ij}^{-1}c_{ij}^{-2}}^{(j)}] + [D_i][\mathscr{W}_{d\hat{q}_i^{-1},ad^2}^{(i)}]\prod_{j\in I_0:j\sim i}[\mathscr{W}_{c_{ij}^{-1}q_{ij}^{-1},aq_{ij}^{-1}c_{ij}^{-2}}^{(j)}] \tag{5.30}$$

with $D_i = L(\mathbf{m}_{c,a}^{(i)}\boldsymbol{\omega}_{d,ad^2}^{(i)}A_{i,a}^{-1}(\boldsymbol{\omega}_{d\hat{q}_i^{-1},ad^2}^{(i)}\prod_{j\sim i}\boldsymbol{\omega}_{c_{ij}^{-1}q_{ij}^{-1},aq_{ij}^{-1}c_{ij}^{-2}}^{(j)})^{-1})$ *one-dimensional.*

The advantage of Eq. (5.30) over (5.29) is that for fixed $j\in I_0$ the spectral parameter a in $\mathscr{W}_{c,a}^{(j)}$ is also fixed. This is crucial in deriving BAE in Sect. 9.

Proof. D_i^- is one-dimensional by the formulas in Example 1.6:

$$\mathbf{n}_{c,a}^{(i)}\boldsymbol{\omega}_{d,a}^{(i)}A_{i,a}^{-1} \equiv \left(\frac{\Psi_{i,a}}{\Psi_{i,a\hat{q}_i^2}}\prod_{j\sim i}\frac{\Psi_{j,aq_{ij}c_{ij}^2}}{\Psi_{j,aq_{ij}}}\right)\times\frac{\Psi_{i,ad^{-2}}}{\Psi_{i,a}}\times\left(\frac{\Psi_{i,aq_i^{-2}}}{\Psi_{i,a\hat{q}_i^2}}\prod_{j\sim i}\frac{\Psi_{j,aq_{ij}^{-1}}}{\Psi_{j,aq_{ij}}}\right)^{-1}$$
$$\equiv \frac{\Psi_{i,ad^{-2}}}{\Psi_{i,aq_i^{-2}}}\prod_{j\sim i}\frac{\Psi_{j,aq_{ij}c_{ij}^2}}{\Psi_{j,aq_{ij}^{-1}}} \equiv \boldsymbol{\omega}_{dq_i^{-1},aq_i^{-2}}^{(i)}\prod_{j\in I_0:j\sim i}\boldsymbol{\omega}_{c_{ij}^{-1}q_{ij}^{-1},aq_{ij}^{-1}}^{(j)}.$$

Dividing the q-characters of both sides of (5.29) by $\mathbf{n}_{c,a}^{(i)}\boldsymbol{\omega}_{d,a}^{(i)}$, we obtain the normalized q-characters of (5.28) by Lemma 4.2 and Proposition 5.2. This proves (5.29). For (5.30), let us assume first $d\notin\pm q^{\mathbb{Z}}$.

As in Table (1.15), let $\mathcal{N}_{c,a}^{\prime(i)}$, $\mathscr{W}_{c,a}^{\prime(i)}$ be the corresponding $U_q(\widehat{\mathfrak{g}'})$-modules in category $\mathcal{O}'$. Since $c^2,\pm d\notin q^{\mathbb{Z}}$, by Lemma 4.2, Proposition 5.2 and Lemma 1.9, $\mathcal{G}^*(M_{c,a}^{(i)})\simeq\mathcal{N}_{c,aq^{N-M}}^{\prime(M+N-i)}$ and $\mathcal{G}^*(\mathscr{W}_{c,a}^{(i)})\simeq\mathscr{W}_{c^{-1},ac^{-2}q^{N-M}}^{\prime(M+N-i)}$ as irreducible $U_q(\widehat{\mathfrak{g}'})$-modules in category $\mathcal{O}'$. Applying $\mathcal{G}^{*-1}$ to (5.29) in $K_0(\mathcal{O}')$ gives (5.30). The ℓ-weight of D_i is fixed similarly as in the proof of Corollary 5.1. This implies

$$\chi_q(M_{c,a}^{(i)}) = \mathbf{m}_{c,a}^{(i)}(1+A_{i,a}^{-1})\prod_{j\in I_0:j\sim i}\widetilde{\chi}_q(L_{j,aq_{ij}^{-1}c_{ij}^{-2}}^{-}),$$

from which follows (5.30) for arbitrary $c\in\mathbb{C}^\times$. □

One can give an alternative proof to Eq. (5.30), by slightly modifying that of Theorem 3.2; see a closer situation in [54, Theorem 6.1]. This approach is independent of Theorem 3.3 and results in Sects. 6, 7 and 8.

6. Cyclicity of Tensor Products

We provide a criteria for a tensor product of Kirillov–Reshetikhin modules to be of highest ℓ-weight, which is needed to prove Theorem 3.3 and Proposition 5.2.

For $i,j\in\mathbb{Z}_{>0}$ let us define the q-segment

$$\mathcal{S}(i,j) := \{q^{-i-j+2r}\mid 0\leq r<\min(i,j)\}\subset\mathbb{C}^\times.$$

It is $q^{j-i}\Sigma(i,j)^{-1}$ in [52, Section 5] and is symmetric in i, j.

Theorem 6.1. *Let $s \in \mathbb{Z}_{>0}$. For $1 \le l \le s$ let $1 \le i_l \le M$ and $(m_l, a_l) \in \mathbb{Z}_{>0} \times \mathbb{C}^\times$. The $U_q(\widehat{\mathfrak{g}})$-module $W^{(i_1)}_{m_1,a_1} \otimes W^{(i_2)}_{m_2,a_2} \otimes \cdots \otimes W^{(i_s)}_{m_s,a_s}$ is of highest ℓ-weight if*

$$\frac{a_j}{a_k} \notin \bigcup_{p=1}^{m_j} q^{2p-2m_k} \mathcal{S}(i_j, i_k) \quad \text{for } 1 \le j < k \le s. \tag{6.31}$$

The idea is similar to [51,52], which in turn was inspired by [12], by restricting to diagram subalgebras. Let A, B be Hopf superalgebras and let $\iota : A \longrightarrow B$ be a morphism of superalgebras. If W is a B-module and W' is a sub-A-module of the A-module $\iota^*(W)$, then let $\iota^\bullet(W')$ denote the A-module structure on W'.

For $1 \le p \le 3$, define the quantum affine superalgebra U_p with RTT generators $s^{(n)}_{ij;p}, t^{(n)}_{ij;p}$ and the superalgebra morphism $\iota_p : U_p \longrightarrow U_q(\widehat{\mathfrak{g}})$ as follows: $U_1 := U_q(\widehat{\mathfrak{gl}(1|1)})$, $U_2 := U_{q^{-1}}(\widehat{\mathfrak{gl}(1|1)})$ and $U_3 := U_q(\widehat{\mathfrak{gl}(M-1|N)})$, so that in $s^{(n)}_{ij;p}, t^{(n)}_{ij;p}$ we understand either $(1 \le i, j, p \le 2)$ or $(1 \le i, j < M+N,\ p = 3)$;

$$\begin{aligned}
&\iota_1 : U_1 \longrightarrow U_q(\widehat{\mathfrak{g}}), \quad s^{(n)}_{ij;1} \mapsto s^{(n)}_{i'j'}, \quad t^{(n)}_{ij;1} \mapsto t^{(n)}_{i'j'};\\
&\iota_2 : U_2 \longrightarrow U_q(\widehat{\mathfrak{g}}), \quad s^{(n)}_{ij;2} \mapsto h(\overline{s}^{(n)}_{i'j'}), \quad t^{(n)}_{ij;2} \mapsto h(\overline{t}^{(n)}_{i'j'});\\
&\iota_3 : U_3 \longrightarrow U_q(\widehat{\mathfrak{g}}), \quad s^{(n)}_{ij;3} \mapsto s^{(n)}_{i+1,j+1}, \quad t^{(n)}_{ij;3} \mapsto t^{(n)}_{i+1,j+1}.
\end{aligned}$$

Here h is the involution in Eq. (1.3) and $1' = 1,\ 2' = M+N$.

Lemma 6.2 [51, Lemma 3.7]. *The tensor product of a lowest ℓ-weight $U_q(\widehat{\mathfrak{g}})$-module with a highest ℓ-weight module is generated, as a $U_q(\widehat{\mathfrak{g}})$-module, by a tensor product of a lowest ℓ-weight vector with a highest ℓ-weight vector.*

Let $1 \le p \le 2$. We recall the notion of *Weyl module* over U_p from [52]. Let $f(z) \in \mathbb{C}(z)$ be a product of the $c\frac{1-za}{1-zac^2}$ with $a, c \in \mathbb{C}^\times$ and let $P(z) \in 1 + z\mathbb{C}[z]$ be such that $\frac{P(z)}{f(z)} \in \mathbb{C}[z]$. The Weyl module $\mathcal{W}_p(f; P)$ is the U_p-module generated by a highest ℓ-weight vector w of even parity such that

$$s_{11;p}(z)w = f(z)w = t_{11;p}(z)w, \quad s_{22;p}(z)w = w = t_{22;p}(z)w,$$

and $\frac{P(z)}{f(z)} s_{21;p}(z)w$, as a formal power series in z with coefficients in $\mathcal{W}_p(f; P)$, is a polynomial in z of degree $\le \deg P$. Given another pair (g, Q), if the polynomials $\frac{P(z)}{f(z)}$ and $Q(z)$ are co-prime, then $\mathcal{W}_p(f; P) \otimes \mathcal{W}_p(g; Q)$ is a quotient of $\mathcal{W}_p(fg; PQ)$ and is of highest ℓ-weight; see [52, Proposition 14].

Example 6.3. In the situation of Theorem 6.1, fix $v_l \in W^{(i_l)}_{m_l,a_l}$ a highest ℓ-weight vector. Let W_p be the sub-U_p-module of $\iota_p^*(\otimes_{l=1}^s W^{(i_l)}_{m_l,a_l})$ generated by $\otimes_{l=1}^s v_l$. Then $\iota_p^\bullet(W_p)$ is a quotient of the Weyl module $\mathcal{W}_p$ for $1 \le p \le 2$ where

$$\mathcal{W}_1 := \mathcal{W}_1\left(\prod_{l=1}^s \frac{q^{m_l} - za_l q^{M-N-i_l-m_l}}{1 - za_l q^{M-N-i_l}};\ \prod_{l=1}^s (1 - za_l q^{M-N-i_l-2m_l})\right),$$

$$\mathcal{W}_2 := \mathcal{W}_2\left(\prod_{l=1}^s \frac{q^{-m_l} - za_l q^{N-M+i_l-m_l}}{1 - za_l q^{N-M+i_l-2m_l}};\ \prod_{l=1}^s (1 - za_l q^{N-M+i_l})\right).$$

$\iota_3^\bullet(W_3)$ is the tensor product $\otimes_{l=1}^s W_{m_l,a_l}^{3(i_l-1)}$ of KR modules over U_3. The proof is the same as [52, Lemmas 18 & 20], based on Corollary 2.10.

For $p \in \mathbb{Z}_{>0}$, let $\mathfrak{g}_p := \mathfrak{gl}(1|p)$ and let $U_q(\widehat{\mathfrak{g}}_p)$ be the quantum affine superalgebra with RTT generators $s_{ij|p}^{(n)}, t_{ij|p}^{(n)}$ for $1 \leq i, j \leq p+1$. Similarly $U_{q^{-1}}(\widehat{\mathfrak{g}}_p)$ with RTT generators $\bar{s}_{ij|p}^{(n)}, \bar{t}_{ij|p}^{(n)}$ and the involution $h_p : U_{q^{-1}}(\widehat{\mathfrak{g}}_p) \longrightarrow U_q(\widehat{\mathfrak{g}}_p)$ are defined. For $1 \leq p \leq N$, the following extends uniquely to a superalgebra morphism

$$\vartheta_p : U_q(\widehat{\mathfrak{g}}_p) \longrightarrow U_q(\widehat{\mathfrak{g}}), \quad s_{ij|p}^{(n)} \mapsto s_{i'j'}^{(n)}, \quad t_{ij|p}^{(n)} \mapsto t_{i'j'}^{(n)} \tag{6.32}$$

where $1' = 1$ and $i' = M + N - p - 1 + i$ for $2 \leq i \leq p+1$.

Definition 6.4. Let $s \in \mathbb{Z}_{>0}$ and $(m_l, a_l) \in \mathbb{Z}_{>0} \times \mathbb{C}^\times$ for $1 \leq l \leq s$. The *Weyl module* $\mathcal{W}^p(\prod_{l=1}^s \varpi_{m_l,a_l})$ is the $U_q(\widehat{\mathfrak{g}}_p)$-module generated by a highest ℓ-weight vector w of even parity such that for $2 \leq j \leq p+1$,

$$s_{11|p}(z)w = w \prod_{l=1}^s \frac{q^{m_l} - za_lq^{-p-m_l}}{1 - za_lq^{-p}} = t_{11|p}(z)w,$$
$$h_p(\bar{s}_{11|p}(z))w = w \prod_{l=1}^s \frac{q^{-m_l} - za_lq^{p-m_l}}{1 - za_lq^{p-2m_l}} = h_p(\bar{t}_{11|p}(z))w,$$
$$s_{jj|p}(z)w = t_{jj|p}(z)w = h_p(\bar{s}_{jj|p}(z))w = h_p(\bar{t}_{jj|p}(z))w = w,$$

and the following vector-valued polynomials in z are of degree $\leq s$:

$$\prod_{l=1}^s (1 - za_lq^{-p}) \times s_{j1|p}(z)w, \quad \prod_{l=1}^s (1 - za_lq^{p-2m_l}) \times h_p(\bar{s}_{j1|p}(z))w.$$

Let $L^p(\prod_{l=1}^s \varpi_{m_l,a_l})$ denote its irreducible quotient of $\mathcal{W}^p(\prod_{l=1}^s \varpi_{m_l,a_l})$.

Example 6.5. Let $1 \leq p \leq N$. In Example 6.3, let W^p be the sub-$U_q(\widehat{\mathfrak{g}}_p)$-module of $\vartheta_p^*(\otimes_{l=2}^s W_{m_l,a_l}^{(i_l)})$ generated by $\otimes_{l=2}^s v_l$. Then $\vartheta_p^\bullet(W^p)$ is a quotient of the Weyl module $\mathcal{W}^p\left(\prod_{l=2}^s \varpi_{m_l,a_lq^{M-N-i_l+p}}\right)$ over $U_q(\widehat{\mathfrak{g}}_p)$.

Example 6.6. Suppose $m_1 \leq N$ and take $p = m_1$. In $W_{m_1,a_1}^{(i_1)}$ there is a non-zero vector v_1^1 whose ℓ-weight corresponds to the tableau $T_1^1 \in \mathcal{B}_-(m_1\varpi_{i_1})$ such that: $T_1^1(-i_1, -j) = 1$ for $1 \leq j \leq m_1$ and $T_1^1(-i, -j) = N + M - j + 1$ for $1 \leq i < i_1$ and $1 \leq j \leq m_1$. Let X be the sub-$U_q(\widehat{\mathfrak{g}}_{m_1})$-module of $\vartheta_{m_1}^*(W_{m_1,a_1}^{(i_1)})$ generated by v_1^1. By comparing the character formulas in Remark 2.5 we see that the $U_q(\widehat{\mathfrak{g}}_{m_1})$-module $\vartheta_{m_1}^\bullet(X)$ is irreducible and in terms of evaluation modules:

$$\begin{aligned}
\vartheta_{m_1}^\bullet(X) &\cong V_q^+(m_1\epsilon_1 + \sum_{j=1}^{m_1}(i_1 - 1)\epsilon_{j+1}; a_1q^{M-N-i_1}) \\
&\simeq V_q^+((m_1 + i_1 - 1)\epsilon_1; a_1q^{M-N+i_1-2}) \cong V_q^-((m_1 + i_1 - 1)\epsilon_1; a_1q^{M-N-i_1}) \\
&\cong L^{m_1}(\varpi_{m_1+i_1-1,a_1q^{M-N+i_1+m_1-2}}).
\end{aligned}$$

Let v_1^2 be a lowest ℓ-weight vector of the $U_q(\widehat{\mathfrak{g}}_{m_1})$-module $\vartheta^\bullet_{m_1}(X)$. Then v_1^2 corresponds to the tableau $T_1^2 \in \mathcal{B}_-(m_1\varpi_{i_1})$ such that $T_1^2(-i,-j) = N+M-j+1$ for $1 \leq i \leq i_1$ and $\leq j \leq m_1$; it is a lowest ℓ-weight vector of the $U_q(\widehat{\mathfrak{g}})$-module $W^{(i_1)}_{m_1,a_1}$. Notice that $s_{ij}^{(n)} X = 0$ if $2 \leq j \leq M+N-m_1$. Combining with Example 6.5, we observe that $X \otimes W^{m_1}$ is stable by $\vartheta_{m_1}(U_q(\widehat{\mathfrak{g}}_{m_1}))$ and the identity map is an isomorphism of $U_q(\widehat{\mathfrak{g}}_{m_1})$-modules $\vartheta^\bullet_{m_1}(X \otimes W^{m_1}) \cong \vartheta^\bullet_{m_1}(X) \otimes \vartheta^\bullet_{m_1}(W^{m_1})$.

Lemma 6.7. *Let $p, s \in \mathbb{Z}_{>0}$ and let $(m_l, a_l) \in \mathbb{Z}_{>0} \times \mathbb{C}^\times$ for $1 \leq l \leq s$. Assume $m_1 \geq p$. The $U_q(\widehat{\mathfrak{g}}_p)$-module $L^p(\varpi_{m_1,a_1}) \otimes \mathcal{W}^p(\prod_{l=2}^s \varpi_{m_l,a_l})$ is of highest ℓ-weight if $a_1 \neq a_l q^{2t-2m_l-2}$ for $2 \leq l \leq s$ and $1 \leq t \leq p$.*

Proof. By induction on p: for $p = 1$ we are led to consider the tensor product

$$\mathcal{W}_1\left(\frac{q^{m_1} - za_1q^{-1-m_1}}{1 - za_1q^{-1}}; 1 - za_1q^{-1-2m_1}\right) \otimes$$
$$\mathcal{W}_1\left(\prod_{l=2}^s \frac{q^{m_l} - za_lq^{-1-m_l}}{1 - za_lq^{-p}}; \prod_{l=2}^s (1 - za_lq^{-1-2m_l})\right)$$

of Weyl modules over $U_1 = U_q(\widehat{\mathfrak{g}}_1)$, which is of highest ℓ-weight if $a_1 \neq a_l q^{-2m_l}$ for $2 \leq l \leq s$. Assume therefore $p > 1$. In Eq. (6.32) let us take (p, M, N) to be $(p-1, 1, p)$. This defines a superalgebra morphism

$$\vartheta_{p-1} : U_q(\widehat{\mathfrak{g}}_{p-1}) \longrightarrow U_q(\widehat{\mathfrak{g}}_p), \quad s^{(n)}_{ij|p-1} \mapsto s^{(n)}_{i'j'|p}, \quad t^{(n)}_{ij|p-1} \mapsto t^{(n)}_{i'j'|p}$$

where $1' = 1$ and $i' = i+1$ for $1 < i \leq p$. Let v_1, w be highest ℓ-weight vectors of the $U_q(\widehat{\mathfrak{g}}_p)$-modules $L^p(\varpi_{m_1,a_1})$ and $\mathcal{W}^p(\prod_{l=2}^s \varpi_{m_l,a_l})$ respectively. Set

$$X_1 := \vartheta_{p-1}(U_q(\widehat{\mathfrak{g}}_{p-1}))v_1, \quad Y_1 := \vartheta_{p-1}(U_q(\widehat{\mathfrak{g}}_{p-1}))w.$$

Using evaluation modules over $U_q(\widehat{\mathfrak{g}}_p)$ we have by Corollary 2.10 and Definition 6.4:

$$L^p(\varpi_{m_1,a_1}) \cong V_q^+(m_1\epsilon_1; a_1q^{-p}) \cong V_q^-(m_1\epsilon_1; a_1q^{p-2m_1}).$$

It follows that $s^{(n)}_{i2|p} X_1 = 0$ if $i \neq 2$. This implies that $X_1 \otimes Y_1$ is stable by $\vartheta_{p-1}(U_q(\widehat{\mathfrak{g}}_{p-1}))$ and the identity map is an isomorphism of $U_q(\widehat{\mathfrak{g}}_{p-1})$-modules:

$$\vartheta^\bullet_{p-1}(X_1 \otimes Y_1) \cong \vartheta^\bullet_{p-1}(X_1) \otimes \vartheta^\bullet_{p-1}(Y_1).$$

As in Example 6.6, the $U_q(\widehat{\mathfrak{g}}_{p-1})$-module $\vartheta^\bullet_{p-1}(X_1)$ is irreducible and isomorphic to $L^{p-1}(\varpi_{m_1,a_1q^{-1}})$. By Definition 6.4, $\vartheta^\bullet_{p-1}(Y_1)$ is a quotient of the Weyl module $\mathcal{W}^{p-1}(\prod_{l=2}^s \varpi_{m_l,a_lq^{-1}})$. The induction hypothesis applied to $p-1$ shows that L^{p-1} $(\varpi_{m_1,a_1q^{-1}}) \otimes \mathcal{W}^{p-1}(\prod_{l=2}^s \varpi_{m_l,a_lq^{-1}})$ and so $\vartheta^\bullet_{p-1}(X_1) \otimes \vartheta^\bullet_{p-1}(Y_1)$ are of highest ℓ-weight. Let v_1' be the lowest ℓ-weight vector of the $U_q(\widehat{\mathfrak{g}}_{p-1})$-module $\vartheta^\bullet_{p-1}(X_1)$; it corresponds to the tableau $T \in \mathcal{B}_-(m_1\epsilon_1)$ such that $T(-1,-j) = p+2-j$ for $1 \leq j \leq p-1$ and $T(-1,-j) = 1$ for $p \leq j \leq m_1$. We have

$$v_1' \otimes w \in \vartheta_{p-1}(U_q(\widehat{\mathfrak{g}}_{p-1}))(v_1 \otimes w) = X_1 \otimes Y_1. \tag{*}$$

Notice that $s_{ij;2}^{(n)} \mapsto h_p(\overline{s}_{ij|p}^{(n)})$ and $t_{ij;2}^{(n)} \mapsto h_p(\overline{t}_{ij|p}^{(n)})$ extend uniquely to a superalgebra morphism $\iota : U_2 \longrightarrow U_q(\widehat{\mathfrak{g}}_p)$. Let $X_2 := \iota(U_2)v_1'$ and $Y_2 := \iota(U_2)w$. The identification $L^p(\varpi_{m_1,a_1}) \cong V_q^-(m_1\epsilon_1; a_1q^{p-2m_1})$ gives $X_2 := \mathbb{C}v_1' + \mathbb{C}v_1''$ where v_1'' is a lowest ℓ-weight vector of $L^p(\varpi_{m_1,a_1})$. This implies $h_p(\overline{s}_{ij|p}^{(n)})X_2 = 0$ if $i \notin \{1,2\}$, meaning that $X_2 \otimes Y_2$ is stable by $\iota(U_2)$ and the graded permutation map is an isomorphism of U_2-modules $\iota^\bullet(X_2 \otimes Y_2) \cong \iota^\bullet(Y_2) \otimes \iota^\bullet(X_2)$. By Definition 6.4 the tensor product $\iota^\bullet(Y_2) \otimes \iota^\bullet(X_2)$ of U_2-modules is a quotient of

$$\mathcal{W}_2\left(\prod_{l=2}^{s} \frac{q^{-m_l} - za_lq^{p-m_l}}{1 - za_lq^{p-2m_l}}; \prod_{l=2}^{s}(1 - za_lq^p)\right) \otimes$$
$$\mathcal{W}_2\left(\frac{q^{-m_1+p-1} - za_1q^{-m_1+1}}{1 - za_1q^{p-2m_1}}; 1 - za_1q^{2-p}\right),$$

which is of highest ℓ-weight since $a_lq^{p-2m_l} \neq a_1q^{2-p}$ for $2 \le l \le s$. The U_2-module $\iota^\bullet(X_2 \otimes Y_2)$ is of highest ℓ-weight and $v_1'' \otimes w \in \iota(U_2)(v_1' \otimes w)$, which together with $(*)$ implies $v_1'' \otimes w \in U_q(\widehat{\mathfrak{g}}_p)(v_1 \otimes w)$. The $U_q(\widehat{\mathfrak{g}}_p)$-module $L^p(\varpi_{m_1,a_1})$ being generated by the lowest ℓ-weight vector v_1'', we conclude by Lemma 6.2. $\square$

For $\mathfrak{gl}(1|3)$ we related the highest/lowest ℓ-weight vectors of $L^3(\varpi_{5,a})$ by:

$$v_1 = \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1} \xrightarrow{\vartheta_2:(134)q} v_1' = \boxed{1}\boxed{1}\boxed{1}\boxed{3}\boxed{4} \xrightarrow{\iota:(12)q^{-1}} v_1'' = \boxed{1}\boxed{1}\boxed{2}\boxed{3}\boxed{4}.$$

Proof of Theorem 6.1. Let us assume first that $m_l \le N$ for all $1 \le l \le s$. We use a double induction on (M, s) with Lemma 6.7 being the initial case $M = 1$. Under Condition (6.31), the induction hypothesis on M applied to the tensor product of KR modules over U_3 in Example 6.3 shows that $\iota_3^\bullet(W_3)$ is of highest ℓ-weight and $v_1^1 \otimes (\otimes_{l=2}^s v_l) \in \iota_3(U_3)(\otimes_{l=1}^s v_l)$. It suffices to prove that the $U_q(\widehat{\mathfrak{g}}_{m_1})$-module $\vartheta_{m_1}^\bullet(X) \otimes \vartheta_{m_1}^\bullet(W^{m_1})$ in Example 6.6 is of highest ℓ-weight, from which follows $v_1^2 \otimes (\otimes_{l=2}^s v_l) \in \vartheta_{m_1}(U_q(\widehat{\mathfrak{g}}_{m_1}))\iota_3(U_3)(\otimes_{l=1}^s v_l)$. The $U_q(\widehat{\mathfrak{g}})$-module $W_{m_1,a_1}^{(i_1)}$ being generated by the lowest ℓ-weight vector v_1^2, we can use the second induction on s and Lemma 6.2 to conclude.

By Examples 6.5 and 6.6, $\vartheta_{m_1}^\bullet(X) \otimes \vartheta_{m_1}^\bullet(W^{m_1})$ is, up to tensor product by one-dimensional modules, a quotient of the $U_q(\widehat{\mathfrak{g}}_{m_1})$-module

$$L^{m_1}(\varpi_{m_1+i_1-1,a_1q^{M-N+i_1+m_1-2}}) \otimes \mathcal{W}^{m_1}\left(\prod_{l=1}^{s} \varpi_{m_l,a_lq^{M-N-i_l+m_1}}\right),$$

which by Lemma 6.7 is of highest ℓ-weight if for $2 \le l \le s$ and $1 \le t \le m_1$:

$$a_1q^{M-N+i_1+m_1-2} \neq a_lq^{M-N-i_l+m_1} \times q^{2t-2-2m_l},$$

namely, $a_1 \neq a_lq^{2t-2m_l-i_1-i_l}$. This is included in Condition (6.31).

Suppose $m_l > N$ for some $1 \le l \le s$. Let $m := \max(m_l : 1 \le l \le s)$ and let $U_4 := U_q(\widehat{\mathfrak{gl}(M|N+m)})$ be the quantum affine superalgebra with RTT generators $s_{ij;4}^{(n)}, t_{ij;4}^{(n)}$ for $1 \le i, j \le M + N + m$. There is a unique superalgebra morphism

$$\iota_4 : U_q(\widehat{\mathfrak{g}}) \longrightarrow U_4, \quad s_{ij}^{(n)} \mapsto s_{ij;4}^{(n)}, \quad t_{ij}^{(n)} \mapsto t_{ij;4}^{(n)}.$$

Under Condition (6.31), the tensor product $\otimes_{l=1}^{s} W_{m_l,a_l}^{4(i_l)}$ of KR modules over U_4 is of highest ℓ-weight. For $1 \le l \le s$, let $X_l := \iota_4(U_q(\widehat{\mathfrak{g}}))v_l$ where $v_l \in W_{m_l,a_l}^{4(i_l)}$ is a highest ℓ-weight vector. Then a weight argument and Corollary 2.10 show that

$$\iota_4(U_q(\widehat{\mathfrak{g}}))(\otimes_{l=1}^{s} v_l) = \otimes_{l=1}^{s} X_l,$$

and as $U_q(\widehat{\mathfrak{g}})$-modules $\iota_4^{\bullet}(\otimes_{l=1}^{s} X_l) \cong \otimes_{l=1}^{s} W_{m_l,a_l q^{-m}}^{(i_l)}$. This implies that the $U_q(\widehat{\mathfrak{g}})$-module $\otimes_{l=1}^{s} W_{m_l,a_l q^{-m}}^{(i_l)}$ is of highest ℓ-weight, proving the theorem. □

For $\mathfrak{gl}(3|6)$ we related the highest/lowest ℓ-weight vectors of $W_{4,a}^{(3)}$ by:

$$v_1 = \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline 2&2&2&2\\ \hline 3&3&3&3\\ \hline\end{array} \xrightarrow{\iota_3:(23456789)q} v_1^1 = \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline 6&7&8&9\\ \hline 6&7&8&9\\ \hline\end{array} \xrightarrow{\vartheta_4:(16789)q} v_1^2 = \begin{array}{|c|c|c|c|}\hline 6&7&8&9\\ \hline 6&7&8&9\\ \hline 6&7&8&9\\ \hline\end{array}.$$

For $\lambda \in \mathcal{P}$ and $a \in \mathbb{C}^{\times}$ define the $U_{q^{-1}}(\widehat{\mathfrak{g}})$-module $V_{q^{-1}}^{+}(\lambda; a)$ to be the pullback of the $U_{q^{-1}}(\mathfrak{g})$-module $V_{q^{-1}}(\lambda)$ by $\overline{\mathrm{ev}}_a^{+}$, as in Theorem 2.4. By Eq. (1.6),

$$h^*\left(V_q^{-}(\lambda; a)\right) \cong V_{q^{-1}}^{+}(\lambda; a).$$

Corollary 6.8. *The tensor product in Theorem 6.1 is of highest ℓ-weight if*

$$\frac{a_j}{a_k} \notin \bigcup_{p=1}^{m_k} q^{2p-2m_k} \mathcal{S}(i_j, i_k) \quad \text{for } 1 \le j < k \le s. \tag{6.33}$$

Proof. The tensor product T in Theorem 6.1 is of highest ℓ-weight if and only if so is the $U_{q^{-1}}(\widehat{\mathfrak{g}})$-module $h^*(T)$. By Corollary 2.10 we have

$$h^*(T) \cong \otimes_{l=s}^{1} V_{q^{-1}}^{+}(m_l \varpi_{i_l}; a_l q^{N-M-2m_l+i_l}).$$

Applying Theorem 6.1 to $U_{q^{-1}}(\widehat{\mathfrak{g}})$, by viewing $W_{m,a}^{(i)}$ first as $V_q^{+}(m\varpi_i; aq^{M-N-i})$ and then as $V_{q^{-1}}^{+}(m\varpi_i; aq^{N-M+i})$, we have that $h^*(T)$ is of highest ℓ-weight if

$$\frac{a_k q^{-2m_k}}{a_j q^{-2m_j}} \notin \bigcup_{p=1}^{m_k} q^{-2p+2m_j} \mathcal{S}(i_k, i_j)^{-1} \quad \text{for } 1 \le j < k \le s.$$

This is Condition (6.33) since $\mathcal{S}(i_k, i_j) = \mathcal{S}(i_j, i_k)$. □

Let V be a finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module. Its *twisted dual* is the dual space $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) =: V^{\vee}$ endowed with the $U_q(\widehat{\mathfrak{g}})$-module structure [52, Section 6]:

$$\langle x\varphi, v\rangle := (-1)^{|\varphi||x|}\langle \varphi, \mathbb{S}\Psi(x)\rangle \quad \text{for } x \in U_q(\widehat{\mathfrak{g}}),\ \varphi \in V^{\vee},\ v \in V.$$

By Eq. (1.2), $(V \otimes W)^{\vee} \cong V^{\vee} \otimes W^{\vee}$ if W is another finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module. V is irreducible if and only if both V and $V^{\vee}$ are of highest ℓ-weight.

We recall the notion of *fundamental representations* from [52]. Let $1 \le r \le M$ and $1 \le s < N$. Define (compare [52, Lemmas 5 & 6] with Corollary 2.10)

$$V_{r,a}^{+} := W_{1,aq^{N-M-r}}^{(r)}, \quad V_{s,a}^{-} := W_{1,aq^{s+2}}^{(M+N-s)}, \quad V_{N,a}^{-} := W_{1,aq^{N+2}}^{(M-)}. \tag{6.34}$$

Lemma 6.9. *Let* $1 \le i \le M < j < M+N$ *and* $(m,a) \in \mathbb{Z}_{>0} \times \mathbb{C}^\times$. *We have:*

$$(W_{m,a}^{(i)})^\vee \simeq W_{m,a^{-1}q^{2m}}^{(i)}, \quad (W_{m,a}^{(j)})^\vee \simeq W_{m,a^{-1}q^{4-2m}}^{(j)}, \quad (W_{m,a}^{(M-)})^\vee \simeq W_{m,a^{-1}q^{4-2m}}^{(M-)}.$$

Proof. The twisted dual of a fundamental module is known [52, Lemma 27]:

$$(V_{i,a}^+)^\vee \simeq V_{i,a^{-1}q^{2(M-N+i+1)}}^+, \quad (V_{M+N-j,a}^-)^\vee \simeq V_{M+N-j;a^{-1}q^{-2(M+N+1-j)}}^-.$$

By Eq. (6.34), $(W_{1,a}^{(i)})^\vee \cong W_{1,a^{-1}q^2}^{(i)}$ and $(W_{1,a}^{(j)})^\vee \simeq W_{m,a^{-1}q^2}^{(j)}$. Viewing $W_{m,a}^{(i)}$ as the unique irreducible sub-quotient of $\otimes_{l=1}^m W_{1,aq_i^{2-2l}}^{(i)}$ of highest ℓ-weight $\varpi_{m,a}^{(i)}$, and taking twisted duals, we obtain the desired formulas. $\square$

Corollary 6.10. *Let* $1 < i < M$, $a \in \mathbb{C}^\times$ *and* $m \in \mathbb{Z}_{>0}$. *The* $U_q(\widehat{\mathfrak{g}})$*-module* $W_{m,a}^{(i-1)} \otimes W_{m,a}^{(i+1)}$ *is irreducible.*

Proof. The tensor product and its twisted dual, which is $\simeq W_{m,a^{-1}q^{2m}}^{(i-1)} \otimes W_{m,a^{-1}q^{2m}}^{(i+1)}$ by Lemma 6.9, satisfy Condition (6.33) and are of highest ℓ-weight. $\square$

The following special result on Dynkin node M is needed in Sect. 7.

Lemma 6.11. [52] *For* $m \in \mathbb{Z}_{>0}$, *the* $U_q(\widehat{\mathfrak{g}})$*-module* $V_{N,aq^{-3}}^- \otimes (\otimes_{l=1}^m V_{N-1,aq^{2l-1}}^-) \otimes (\otimes_{k=1}^m V_{M-1,aq^{2M-2k-1}}^+)$ *is of highest* ℓ*-weight. Moreover for* $1 \le k,l \le m$ *we have* $V_{N-1,aq^{2l-1}}^- \otimes V_{M-1,aq^{2M-2k-1}}^+ \cong V_{M-1,aq^{2M-2k-1}}^+ \otimes V_{N-1,aq^{2l-1}}^-$.

Proof. The first statement is induced from [52, Theorem 15] by the involution h as in [52, Remarks 3 & 4], and the second is a particular case of [52, Example 5]. $\square$

7. Asymptotic Representations

In this section we construct the $U_q(\widehat{\mathfrak{g}})$-module $\mathcal{N}_{c,a}^{(i)}$ of Proposition 5.2 for $i \in I_0$ and $a, c \in \mathbb{C}^\times$ from finite-dimensional representations.

For $m \in \mathbb{Z}_{>0}$, let $N_{m,a}^{(i)} := L(\mathbf{n}_{q^m,a}^{(i)})$ be the irreducible $U_q(\widehat{\mathfrak{g}})$-module; it is finite-dimensional by Lemma 1.5 (3). Fix $v^m \in N_{m,a}^{(i)}$ to be a highest ℓ-weight vector.

The main step is to construct an inductive system $(N_{m,a}^{(i)})_{m \in \mathbb{Z}_{>0}}$ compatible with (normalized) q-characters, as in [34, Section 4.2] and [35, Theorem 7.6]. We shall need the cyclicity results in Sect. 6 to adapt the arguments of [34,35].

Lemma 7.1. *If* $\mathbf{n}_{q^m,a}^{(i)}\mathbf{m} \in \mathrm{wt}_\ell(N_{m,a}^{(i)})$, *then* $\mathbf{n}_{i,a}^-\mathbf{m} \in \mathrm{wt}_\ell(N_{i,a}^-)$ *and*

$$\dim(N_{m,a}^{(i)})_{\mathbf{n}_{q^m,a}^{(i)}\mathbf{m}} \le \dim(N_{i,a}^-)_{\mathbf{n}_{i,a}^-\mathbf{m}}.$$

Proof. The first paragraph of the proof of [35, Theorem 7.6] can be copied here, based on Lemma 4.1 and the fact that $\mathbf{m}$ is a product of the $A_{j,b}^{-1}$ with $j \in I_0$ and $b \in aq^{\mathbb{Z}}$. For the latter fact, we realize $N_{m,a}^{(i)}$ as a tensor product of KR modules with one-dimensional modules and apply Corollary 2.10. $\square$

Lemma 7.2. *Let $c \in \mathbb{C}^\times$ be such that $c^2 \notin q^{\mathbb{Z}}$. If $\mathbf{n}_{i,a}^-\mathbf{m} \in \mathrm{wt}_\ell(N_{i,a}^-)$, then $\mathbf{n}_{c,a}^{(i)}\mathbf{m} \in \mathrm{wt}_\ell(L(\mathbf{n}_{c,a}^{(i)}))$ and* $\dim(N_{i,a}^-)_{\mathbf{n}_{i,a}^-\mathbf{m}} \leq \dim L(\mathbf{n}_{c,a}^{(i)})_{\mathbf{n}_{c,a}^{(i)}\mathbf{m}}$.

Proof. From Example 1.6 we obtain

$$\mathbf{n}_{i,a}^- \equiv \mathbf{n}_{c,a}^{(i)} \prod_{j\sim i} \Psi_{j,aq_{ij}c_{ij}^2}^{-1}.$$

Viewing $N_{i,a}^-$ as a sub-quotient of $L(\mathbf{n}_{c,a}^{(i)}) \otimes (\otimes_{j\sim i} L_{j,aq_{ij}c_{ij}^2}^-) \otimes D$ with D being a one-dimensional $Y_q(\mathfrak{g})$-module, we have $\mathbf{m} = \mathbf{m}' \prod_{j\sim i} \mathbf{m}^j$ with

$$\mathbf{n}_{c,a}^{(i)}\mathbf{m}' \in \mathrm{wt}_\ell(L(\mathbf{n}_{c,a}^{(i)})), \quad \Psi_{j,aq_{ij}c_{ij}^2}^{-1}\mathbf{m}^j \in \mathrm{wt}_\ell(L_{j,aq_{ij}c_{ij}^2}^-) \quad \text{for } j \sim i.$$

By Corollary 5.1 and Lemmas 4.1–4.2 we have:

(1) $\mathbf{m}, \mathbf{m}' \in \widehat{\mathcal{Q}}^- q^{\mathbf{Q}^-}$ and $\mathbf{m}$ is a monomial in the $A_{i',b}^{-1}$ with $i' \in I_0$ and $b \in aq^{\mathbb{Z}}$;
(2) $\mathbf{m}^j$ is a monomial in the $A_{i',b'}^{-1}$ with $i' \in I_0$ and $b' \in \{ac^2, ac^{-2}\}q^{\mathbb{Z}}$ for $j \sim i$.

Since $\{ac^2, ac^{-2}\}q^{\mathbb{Z}}$ and $aq^{\mathbb{Z}}$ do not intersect, $\mathbf{m}^j = 1$ and $\mathbf{m}' = \mathbf{m}$. □

For $m_1, m_2 \in \mathbb{Z}_{>0}$ with $m_1 < m_2$, let $Z_{i,a}^{m_1,m_2}$ be the irreducible $U_q(\widehat{\mathfrak{g}})$-module of highest ℓ-weight $\mathbf{n}_{q^{m_2},a}^{(i)}(\mathbf{n}_{q^{m_1},a}^{(i)})^{-1} = \prod_{j\sim i} \boldsymbol{\omega}_{q_{ij}^{m_1-m_2},aq_{ij}^{1+2m_1}}^{(j)}$; by Lemma 1.5 (3) it is finite-dimensional. Fix $v^{m_1,m_2} \in Z_{i,a}^{m_1,m_2}$ to be a highest ℓ-weight vector.

Lemma 7.3. *The $U_q(\widehat{\mathfrak{g}})$-module $N_{m_1,a}^{(i)} \otimes Z_{i,a}^{m_1,m_2} \otimes Z_{i,a}^{m_2,m_3}$ is of highest ℓ-weight for $0 < m_1 < m_2 < m_3$.*

Proof. We shall assume $1 \leq i \leq M$. The case $M + 1 < i < M + N$ can be deduced from $1 \leq i < M$ using $\mathcal{G}^*$. (See typical arguments in the proof of Lemma 8.2.)

Suppose $1 \leq i < M$. By Corollary 6.10, $Z_{i,a}^{m_1,m_2} \simeq \otimes_{j\sim i} W_{m_2-m_1,aq^{-2m_1-2}}^{(j)}$. The tensor product $W_{1,aq}^{(i)} \otimes (\otimes_{j\sim i} W_{m_1,aq^{-2}}^{(j)})$ satisfies Condition (6.33) and is of highest ℓ-weight. Its irreducible quotient is $\simeq N_{m_1,a}^{(i)}$. Next,

$$W_{1,aq}^{(i)} \otimes (\otimes_{j\sim i} W_{m_1,aq^{-2}}^{(j)}) \otimes (\otimes_{j\sim i} W_{m_2-m_1,aq^{-2m_1-2}}^{(j)}) \otimes (\otimes_{j\sim i} W_{m_3-m_2,aq^{-2m_2-2}}^{(j)})$$

also satisfies Condition (6.33), and is of highest ℓ-weight, implying that $N_{m_1,a}^{(i)} \otimes Z_{i,a}^{m_1,m_2} \otimes Z_{i,a}^{m_2,m_3}$ is of highest ℓ-weight.

Suppose $i = M$. Consider the tensor product of fundamental modules:

$$T := V_{N,aq^{-N-3}}^- \otimes (\otimes_{l=1}^{m_3} V_{N-1,aq^{-N+2l-1}}^-) \otimes (\otimes_{k=1}^{m_3} V_{M-1,aq^{-N+2M-2k-1}}^+).$$

By Lemma 6.11, T is of highest ℓ-weight and

$$\begin{aligned} T \cong\, & V_{N,aq^{-N-3}}^- \otimes (\otimes_{l=1}^{m_1} V_{N-1,aq^{-N+2l-1}}^-) \otimes (\otimes_{k=1}^{m_1} V_{M-1,aq^{-N+2M-2k-1}}^+) \otimes \\ & (\otimes_{l=m_1+1}^{m_2} V_{N-1,aq^{-N+2l-1}}^-) \otimes (\otimes_{k=m_1+1}^{m_2} V_{M-1,aq^{-N+2M-2k-1}}^+) \\ & (\otimes_{l=m_2+1}^{m_3} V_{N-1,aq^{-N+2l-1}}^-) \otimes (\otimes_{k=m_2+1}^{m_3} V_{M-1,aq^{-N+2M-2k-1}}^+). \end{aligned}$$

Let T_1, T_2, T_3 denote the above tensor products of the first, second, and third row at the right-hand side. They are of highest ℓ-weight. By Eq. (6.34),

$$V^-_{N,aq^{-N-3}} \simeq W^{(M-)}_{1,aq^{-1}}, \quad V^-_{N-1,aq^{-N+1}} \simeq W^{(M+1)}_{1,aq^2}, \quad V^-_{M-1,aq^{-N+2M-3}} \simeq W^{(M-1)}_{1,aq^{-2}}.$$

By Example 1.6, the irreducible quotients of T_1, T_2, T_3 are $\simeq N^{(M)}_{m_1,a}$, $Z^{m_1,m_2}_{M,a}$ and $Z^{m_2,m_3}_{M,a}$, proving the cyclicity statement. □

Let $0 < m_1 < m_2$. The tensor product $N^{(i)}_{m_1,a} \otimes Z^{m_1,m_2}_{i,a}$ being of highest ℓ-weight, its irreducible quotient is isomorphic to $N^{(i)}_{m_2,a}$. There exists a unique morphism of $U_q(\widehat{\mathfrak{g}})$-modules $\mathscr{F}_{m_2,m_1} : N^{(i)}_{m_1,a} \otimes Z^{m_1,m_2}_{i,a} \longrightarrow N^{(i)}_{m_2,a}$ which sends $v^{m_1} \otimes v^{m_1,m_2}$ to v^{m_2}. As in [34, Section 4.2], define

$$F_{m_2,m_1} : N^{(i)}_{m_1,a} \longrightarrow N^{(i)}_{m_2,a}, \quad w \mapsto \mathscr{F}_{m_2,m_1}(w \otimes v^{m_1,m_2}).$$

Then $(\{N^{(i)}_{m,a}\}, \{F_{m_2,m_1}\})$ constitutes an inductive system of vector superspaces: $F_{m_3,m_2} F_{m_2,m_1} = F_{m_3,m_1}$ for $0 < m_1 < m_2 < m_3$. The proof is the same as that of [53, Proposition 4.1 (2)], based on Lemma 7.3.

Lemma 7.4. *Let $0 < m_1 < m_2$. We have $F_{m_2,m_1} x^+_{j,n} = x^+_{j,n} F_{m_2,m_1}$ for $j \in I_0$ and $n \in \mathbb{Z}$. The linear map F_{m_2,m_1} is injective.*

Proof. This is [34, Theorem 3.15]. For a proof independent of ℓ-weights, we refer to the first two paragraphs of the proof of [53, Proposition 4.3]; the coproduct $\Delta(e^+_i)$ therein should be replaced by the $\Delta(x^+_{j,n})$ in Eq. (1.10). □

Lemma 7.5. *Let us write $(h_1(z), h_2(z), \dots, h_\kappa(z); \overline{0}) := \mathbf{n}^{(i)}_{q^{m_2},a}(\mathbf{n}^{(i)}_{q^{m_1},a})^{-1} \in \mathbf{R}_U$ for $m_2 > m_1 > 0$. Then for $j \in I_0$ we have*

$$K^\pm_j(z) F_{m_2,m_1} = h_j(z) \times F_{m_2,m_1} K^\pm_j(z) \in \mathrm{Hom}_{\mathbb{C}}(N^{(i)}_{m_1,a}, N^{(i)}_{m_2,a})[[z^{\pm 1}]].$$

Here for $\pm$ we take Taylor expansions of $h_j(z)$ at $z = 0, z = \infty$ respectively.

Proof. The same as [34, Proposition 4.2] in view of Eq. (1.9). □

All the $h_j(z) \in \mathbb{C}[[z]]$ are of the form $A(z)q^{-m_2} + B(z) + C(z)q^{m_2}$ where $A(z), B(z), C(z) \in \mathbb{C}[[z]]$ are independent of m_2. Let $j \in I_0$. If $j \sim i$, then

$$\phi^\pm_j(z) F_{m_2,m_1} = q_{ij}^{m_1-m_2} \frac{1 - zaq_{ij}^{1+2m_2}}{1 - zaq_{ij}^{1+2m_1}} \times F_{m_2,m_1} \phi^\pm_j(z).$$

Otherwise, F_{m_2,m_1} commutes with $\phi^\pm_j(z)$ for $|j - i| \neq 1$.

From Lemmas 7.1 and 7.4–7.5, we conclude that: the normalized q-characters $\widetilde{\chi}_q(N^{(i)}_{m,a})$ for $m \in \mathbb{Z}_{>0}$ are polynomials in $\mathbb{Z}[A^{-1}_{j,b}]_{(j,b)\in I_0 \times aq^{\mathbb{Z}}}$, and as $m \to \infty$ they converge to a formal power series $\lim_{m\to\infty} \widetilde{\chi}_q(N^{(i)}_{m,a}) \in \mathbb{Z}[[A^{-1}_{j,b}]]_{(j,b)\in I_0 \times aq^{\mathbb{Z}}}$, which is bounded above by the normalized q-character $\widetilde{\chi}_q(N^-_{i,a})$.

Lemma 7.6. *For $j \in I_0$ and $m_2 - 1 > m > 0$ we have*

$$x_{j,0}^- F_{m_2,m} = F_{m_2,m} x_{j,0}^- \quad \text{if } |j-i| \neq 1,$$
$$x_{j,0}^- F_{m_2,m} = F_{m_2,m+1}(q^{m_2} A_{j,m} + q^{-m_2} C_{j,m}) \quad \text{if } |j-i| = 1.$$

Here $A_{j,m}, C_{j,m} : N_{m,a}^{(i)} \longrightarrow N_{m+1,a}^{(i)}$ are linear maps of parity $|\alpha_j|$.

Proof. This corresponds to [34, Lemma 4.4 & Proposition 4.5]. Here we give a straightforward proof without induction arguments.

By Lemma 7.3, the $U_q(\widehat{\mathfrak{g}})$-module $Z_{i,a}^{m,m+1} \otimes Z_{i,a}^{m+1,m_2}$ is of highest ℓ-weight with irreducible quotient $Z_{i,a}^{m,m_2}$; let $\mathscr{G}_{m_2,m}$ be the quotient map sending $v^{m,m+1} \otimes v^{m+1,m_2}$ to v^{m,m_2}. We claim that for $v \in N_{m,a}^{(i)},\ v' \in Z_{i,a}^{m,m+1}$ and $j \in I_0$:

(i) $\mathscr{F}_{m_2,m}(v \otimes \mathscr{G}_{m_2,m}(v' \otimes v^{m+1,m_2})) = F_{m_2,m+1} \mathscr{F}_{m+1,m}(v \otimes v')$;
(ii) $x_{j,0}^- v^{m,m_2} = [(m_2 - m)\delta_{|j-i|,1}]_q \times \mathscr{G}_{m_2,m}(x_{j,0}^- v^{m,m+1} \otimes v^{m+1,m_2})$.

Here $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$ for $n \in \mathbb{Z}$. Assume the claim for the moment. For $v \in N_{m,a}^{(i)}$, based on $\Delta(x_{j,0}^-) = 1 \otimes x_{j,0}^- + x_{j,0}^- \otimes \phi_{j,0}^-$ we compute $x_{j,0}^- F_{m_2,m}(v)$

$$\begin{aligned}
&= x_{j,0}^- \mathscr{F}_{m_2,m}(v \otimes v^{m,m_2}) = \mathscr{F}_{m_2,m} \Delta(x_{j,0}^-)(v \otimes v^{m,m_2}) \\
&= \mathscr{F}_{m_2,m}(x_{j,0}^- v \otimes \phi_{j,0}^- v^{m,m_2}) + (-1)^{|v||\alpha_j|} \mathscr{F}_{m_2,m}(v \otimes x_{j,0}^- v^{m,m_2}) \\
&= q_{ij}^{(m_2-m)\delta_{|j-i|,1}} F_{m_2,m}(x_{j,0}^- v) + \\
&\quad (-1)^{|v||\alpha_j|}[(m_2 - m)\delta_{|j-i|,1}]_q \mathscr{F}_{m_2,m}(v \otimes \mathscr{G}_{m_2,m}(x_{j,0}^- v^{m,m+1} \otimes v^{m+1,m_2})) \\
&= q_{ij}^{(m_2-m)\delta_{|j-i|,1}} F_{m_2,m}(x_{j,0}^- v) + \\
&\quad (-1)^{|v||\alpha_j|}[(m_2 - m)\delta_{|j-i|,1}]_q F_{m_2,m+1} \mathscr{F}_{m+1,m}(v \otimes x_{j,0}^- v^{m,m+1}),
\end{aligned}$$

which proves the lemma. The third and fourth identities used (ii) and (i).

Note that $\mathscr{F}_{m_2,m}(1_{N_{m,a}^{(i)}} \otimes \mathscr{G}_{m_2,m})$ and $\mathscr{F}_{m_2,m+1}(\mathscr{F}_{m+1,m} \otimes 1_{Z_{i,a}^{m+1,m_2}})$, as $U_q(\widehat{\mathfrak{g}})$-linear maps from the highest ℓ-weight module $N_{m,a}^{(i)} \otimes Z_{i,a}^{m,m+1} \otimes Z_{i,a}^{m+1,m_2}$ to $N_{m_2,a}^{(i)}$, are identical because they both send the highest ℓ-weight vector $v^m \otimes v^{m,m+1} \otimes v^{m+1,m_2}$ to v^{m_2}. Applying them to $v \otimes v' \otimes v^{m+1,m_2}$ gives (i).

From the proof of Lemma 7.3 it follows that $Z_{i,a}^{m,m_2}$ is $\simeq$ irreducible quotient of a tensor product of KR modules associated to $j' \in I_0$ with $j' \sim i$. Let μ be the weight of v^{m,m_2}. If $|j - i| \neq 1$, then by Lemma 3.4, $\mu q^{-\alpha_j} \notin \mathrm{wt}(Z_{i,a}^{m,m_2})$ and $x_{j,0}^- v^{m,m_2} = 0$. Suppose $j \sim i$. Then $(Z_{i,a}^{m,m_2})_{\mu q^{-\alpha_j}} = \mathbb{C} x_{j,0}^- v^{m,m_2}$ and $q_{ij} = q^{\pm 1}$. The equation $x_{j,0}^- \mathscr{G}_{m_2,m} = \mathscr{G}_{m_2,m} x_{j,0}^-$ applied to $v^{m,m+1} \otimes v^{m+1,m_2}$ gives

$$x_{j,0}^- v^{m,m_2} = \mathscr{G}_{m_2,m}(q_{ij}^{m_2-m-1} x_{j,0}^- v^{m,m+1} \otimes v^{m+1,m_2} + v^{m,m+1} \otimes x_{j,0}^- v^{m+1,m_2}).$$

Consider the following vector in $Z_{i,a}^{m,m+1} \otimes Z_{i,a}^{m+1,m_2}$ of weight $\mu q^{-\alpha_j}$:

$$w := q_{ij}^{-1} \frac{q_{ij}^{m+1-m_2} - q_{ij}^{m_2-m-1}}{q_{ij}^{-1} - q_{ij}} x_{j,0}^- v^{m,m+1} \otimes v^{m+1,m_2} - v^{m,m+1} \otimes x_{j,0}^- v^{m+1,m_2}.$$

We have $\mathscr{G}_{m_2,m}(w) \in \mathbb{C}x_{j,0}^- v^{m,m_2}$ and $x_{j,0}^+ w = 0$. So $x_{j,0}^+ \mathscr{G}_{m_2,m}(w) = 0$. Now $x_{j,0}^+ x_{j,0}^- v^{m,m_2} \neq 0$ forces $\mathscr{G}_{m_2,m}(w) = 0$. We express $x_{j,0}^- v^{m,m_2}$

$$\begin{aligned}
&= \mathscr{G}_{m_2,m}(q_{ij}^{m_2-m-1} x_{j,0}^- v^{m,m+1} \otimes v^{m+1,m_2} + v^{m,m+1} \otimes x_{j,0}^- v^{m+1,m_2}) + \mathscr{G}_{m_2,m}(w) \\
&= \left(q_{ij}^{m_2-m-1} + \frac{q_{ij}^{m-m_2} - q_{ij}^{m_2-m-2}}{q_{ij}^{-1} - q_{ij}} \right) \times \mathscr{G}_{m_2,m}(x_{j,0}^- v^{m,m+1} \otimes v^{m+1,m_2}) \\
&= [m_2 - m]_{q_{ij}} \times \mathscr{G}_{m_2,m}(x_{j,0}^- v^{m,m+1} \otimes v^{m+1,m_2}),
\end{aligned}$$

which proves (ii) because $[n]_{q_{ij}} = [n]_q$ for $n \in \mathbb{Z}$. □

Proof of Proposition 5.2. For $r \in \mathbb{Z}_{\geq 0}$ and $l \in I$ let $K_{l,\pm r}^{\pm}$ be the coefficient of $z^{\pm r}$ in $K_l^{\pm}(z) \in U_q(\widehat{\mathfrak{g}})[[z^{\pm 1}]]$. The superalgebra $U_q(\widehat{\mathfrak{g}})$ is generated by:

$$\mathcal{S} := \{K_{l,\pm r}^{\pm},\ x_{j,0}^-,\ x_{j,n}^+ \mid r \in \mathbb{Z}_{\geq 0},\ n \in \mathbb{Z},\ j \in I_0,\ l \in I\}.$$

By Lemmas 7.4–7.6, there are $\mathrm{Hom}_{\mathbb{C}}(N_{m,a}^{(i)}, N_{m+1,a}^{(i)})$-valued Laurent polynomials $P_{s;m}(u)$ for $m \in \mathbb{Z}_{>0}$ and $s \in \mathcal{S}$ such that

$$sF_{m_2,m} = F_{m_2,m+1} P_{s;m}(q^{m_2}) \in \mathrm{Hom}_{\mathbb{C}}(N_{m,a}^{(i)}, N_{m_2,a}^{(i)}) \quad \text{for } m_2 > m+1.$$

These polynomials have non-zero coefficients only at $u, 1, u^{-1}$. Since q is not a root of unity, the generic asymptotic construction of [53, Section 2] can be applied to the inductive system $(\{N_{m,a}^{(i)}\}, \{F_{m_2,m_1}\})$. Let N_∞ be its inductive limit. Fix $c \in \mathbb{C}^\times$. There exists a unique representation of $U_q(\widehat{\mathfrak{g}})$ on N_∞ on which $s \in \mathcal{S}$ acts as

$$\lim_{m \to \infty} P_{s;m}(c) \in \mathrm{End}(N_\infty)$$

Here the $P_{s;m}(c) : N_{m,a}^{(i)} \longrightarrow N_{m+1,a}^{(i)}$ for $m \in \mathbb{Z}_{>0}$ form a morphism of the inductive system, so their inductive limit $\lim_{m\to\infty} P_{s;m}(c)$ makes sense. As in the proof of [53, Lemma 6.7], the resulting $U_q(\widehat{\mathfrak{g}})$-module $\mathcal{N}_{c,a}^{(i)}$ is in category $\mathcal{O}$ with q-character

$$\chi_q(\mathcal{N}_{c,a}^{(i)}) = \mathbf{n}_{c,a}^{(i)} \times \lim_{m \to \infty} \widetilde{\chi}_q(N_{m,a}^{(i)}).$$

Let us prove $\lim_{m \to \infty} \widetilde{\chi}_q(N_{m,a}^{(i)}) = \widetilde{\chi}_q(N_{i,a}^-)$. We have seen above Lemma 7.6 that the left-hand side is bounded above by the right-hand side. View $L(\mathbf{n}_{c,a}^{(i)})$ as a sub-quotient of $\mathcal{N}_{c,a}^{(i)}$. If $c^2 \notin q^{\mathbb{Z}}$, then by Lemma 7.2, $\widetilde{\chi}_q(N_{i,a}^-)$ is bounded above by $\widetilde{\chi}_q(L(\mathbf{n}_{c,a}^{(i)}))$ and so by $(\mathbf{n}_{c,a}^{(i)})^{-1} \chi_q(\mathcal{N}_{c,a}^{(i)})$, which is the left-hand side. This implies the reverse inequality and the irreducibility of $\mathcal{N}_{c,a}^{(i)}$ for $c^2 \notin q^{\mathbb{Z}}$. □

One can have asymptotic modules $\mathcal{M}_{c,a}^{(i)}$ over $U_q(\widehat{\mathfrak{g}})$ as limits of $M_{q^m,a}^{(i)}$ (as in [34, Section 7.2], which is slightly different from the limit construction of $\mathcal{N}_{c,a}^{(i)}$). Then Eq. (5.30) holds with M replaced by $\mathcal{M}$ for all $c, d \in \mathbb{C}^\times$.

Proposition 7.7. *The $U_q(\widehat{\mathfrak{g}})$-module $\mathscr{W}_{c_1,a_1}^{(i_1)} \otimes \mathscr{W}_{c_2,a_2}^{(i_2)} \otimes \cdots \otimes \mathscr{W}_{c_s,a_s}^{(i_s)}$, with $i_l \in I_0$ and $c_l, a_l \in \mathbb{C}^\times$, is irreducible if $a_l c_l^{-2} \notin a_k q^{\mathbb{Z}}$ for all $1 \leq l, k \leq s$.*

Proof. Let $L := \otimes_{l=1}^{s} L^{-}_{i_l,a_l}$ and $S = L(\prod_{l=1}^{s} \boldsymbol{\omega}^{(i_l)}_{c_l,a_l})$, viewed as irreducible $Y_q(\mathfrak{g})$-modules by Corollary 4.3. S is a sub-quotient of the tensor product T in the proposition. Let $\boldsymbol{\omega}, \boldsymbol{\omega}'$ be the highest ℓ-weights of L, S respectively. Then $\chi_q(T) = \boldsymbol{\omega}'\widetilde{\chi}_q(L)$ by Lemma 4.2. It suffices to prove that $\dim L_{\mathbf{n}\boldsymbol{\omega}} \leq \dim S_{\mathbf{n}\boldsymbol{\omega}'}$ for all $\mathbf{n}\boldsymbol{\omega} \in \mathrm{wt}_\ell(L)$. Viewing L as a sub-quotient of $S \otimes D$ where $D \simeq \otimes_{l=1}^{s} L^{-}_{i_l,a_l c_l^{-2}}$, we can adapt the proof of Lemma 7.2 to the present situation. $\square$

It follows that the tensor products of the $\mathscr{W}$ at the right-hand side of Eqs. (5.29)–(5.30) are irreducible $U_q(\widehat{\mathfrak{g}})$-modules for $c^2, d^2 \notin q^{\mathbb{Z}}$.

8. Proof of Extended T-Systems: Theorem 3.3

The idea is to provide lower and upper bounds for $\dim(D^{(i,s)}_{m,a})$. We recall from the proof of Corollary 3.5 that the $U_q(\widehat{\mathfrak{g}})$-module $W^{(i)}_{m,aq_i^{2m+1}} \otimes W^{(i)}_{m+s,aq_i^{2m-1}}$ has at least two irreducible sub-quotients: $L(\varpi^{(i)}_{m+s+1,aq_i^{2m+1}}\varpi^{(i)}_{m-1,aq_i^{2m-1}})$ and $D^{(i,s)}_{m,a}$.

Lemma 8.1. *For $i \in I_0\backslash\{M\}$, the $U_q(\widehat{\mathfrak{g}})$-module $W^{(i)}_{m+s,aq_i^{2m+1}} \otimes D^{(i,s)}_{m,a}$ has at least two sub-quotients: $L(\mathbf{d}^{(i,s-1)}_{m,a}\varpi^{(i)}_{m+s+1,aq_i^{2m+1}})$ and $L(\mathbf{d}^{(i,0)}_{m+s,aq_i^{-2s}}\varpi^{(i)}_{m,aq_i^{2m+1}})$.*

Proof. Set $T := W^{(i)}_{m+s,aq_i^{2m+1}} \otimes D^{(i,s)}_{m,a}$ and $S := L(\mathbf{d}^{(i,s-1)}_{m,a}\varpi^{(i)}_{m+s+1,aq_i^{2m+1}})$. By Example 1.6, S is an irreducible sub-quotient of T. By Corollary 3.5,

$$\mathbf{m}' := \mathbf{m}\prod_{l=1}^{s} A^{-1}_{i,aq_i^{2-2l}} = \mathbf{d}^{(i,0)}_{m+s,aq_i^{-2s}}\varpi^{(i)}_{m,aq_i^{2m+1}} \in \mathrm{wt}_\ell(T).$$

Viewing S as an irreducible sub-quotient of $W^{(i)}_{m+s+1,aq_i^{2m+1}} \otimes D^{(i,s-1)}_{m,a}$ and using Lemma 3.4 and Corollary 3.5, we have $\mathbf{m}' \notin \mathrm{wt}_\ell(S)$. Let $\mu := (3m+2s)\varpi_i - m\alpha_i$ so that $\varpi(\mathbf{m}) = q^\mu$ and $\varpi(\mathbf{m}') = q^{\mu - s\alpha_i}$. Then $\dim T_{q^{\mu-t\alpha_i}} = t+1$ for $0 \leq t \leq s$.

Let $v_0 \in S$ be a highest ℓ-weight vector and let U_i be the subalgebra in the proof of Corollary 3.5. Then $U_i v_0$ is an irreducible $U_{q_i}(\widehat{\mathfrak{sl}_2})$-module of highest ℓ-weight

$$\mathbf{m}_i := (Y_{aq_i^{-1}}Y_{i,aq_i^{-3}}\cdots Y_{i,aq_i^{1-2s}})(Y_{i,aq_i^{2m+1}}Y_{i,aq_i^{2m-1}}\cdots Y_{i,aq_i^{3-2s}})$$

and factorizes as $L^i(Y_{aq_i^{-1}}Y_{i,aq_i^{-3}}\cdots Y_{i,aq_i^{3-2s}}) \otimes L^i(Y_{i,aq_i^{2m+1}}Y_{i,aq_i^{2m-1}}\cdots Y_{i,aq_i^{1-2s}})$; if $s=1$ then the first tensor factor is trivial. For $1 \leq t \leq s$, the weight space $S_{q^{\mu-t\alpha_i}}$ is spanned by the $x^-_{i,n_1}x^-_{i,n_2}\cdots x^-_{i,n_t}v_0 \in U_i v_0$ with $n_l \in \mathbb{Z}$ for $1 \leq l \leq t$ and is therefore of dimension $\min(s, t+1)$. Since $\mathbf{m}_i\prod_{l=1}^{s}(Y_{aq_i^{1-2l}}Y_{aq_i^{3-2l}})^{-1}$ is not an ℓ-weight of $L^i(\mathbf{m}_i)$, we must have $\mathbf{m}' \notin \mathrm{wt}_\ell(S)$, as in the proof of Corollary 3.5.

It follows that $\chi_q(T) - \chi_q(S)$ is $\mathbf{m}'$ plus terms of the form $\mathbf{m}'' \in \mathbf{R}$ with $\varpi(\mathbf{m}'') \notin \varpi(\mathbf{m}')q^{\mathbf{Q}^+}$, forcing $L(\mathbf{m}')$ to be an irreducible sub-quotient of T. $\square$

Lemma 8.2. *Let $i \in I_0\backslash\{M\}$. The $U_q(\widehat{\mathfrak{g}})$-modules $W^{(i)}_{m,aq_i^{2m+1}} \otimes W^{(i)}_{m+s,aq_i^{2m-1}}$ and $W^{(i)}_{m+s,aq_i^{2m+1}} \otimes D^{(i,s)}_{m,a}$ are of highest ℓ-weight, while $W^{(i)}_{m+s+1,aq_i^{2m+1}} \otimes W^{(i)}_{m-1,aq_i^{2m-1}}$, $D^{(i,0)}_{m+s,aq_i^{-2s}} \otimes W^{(i)}_{m,aq_i^{2m+1}}$ and $W^{(i)}_{m+s+1,aq_i^{2m+1}} \otimes D^{(i,s-1)}_{m,a}$ are irreducible.*

Proof. Assume $i < M$. Notice that $T_{m,a}^{(i,s)} := W_{m,aq^{2m}}^{(i+1)} \otimes W_{m,aq^{2m}}^{(i-1)} \otimes W_{s,aq^{-1}}^{(i)}$ satisfies Condition (6.33) and is of highest ℓ-weight. By Remark 3.1, the irreducible quotient of $T_{m,a}^{(i,s)}$ is $\simeq D_{m,a}^{(i,s)}$. To prove that the five tensor products in the lemma are of highest ℓ-weight, we can replace D by T and show that the resulting tensor products of KR modules satisfy Condition (6.33). For example the last tensor product corresponds to $W_{m+s+1,aq^{2m+1}}^{(i)} \otimes W_{m,aq^{2m}}^{(i+1)} \otimes W_{m,aq^{2m}}^{(i-1)} \otimes W_{s-1,aq^{-1}}^{(i)}$.

Next, $S_{m,a}^{(i,s)} := W_{s,a^{-1}q^{2s+1}}^{(i)} \otimes W_{m,a^{-1}}^{(i-1)} \otimes W_{m,a^{-1}}^{(i+1)}$ also satisfies Condition (6.33) and is of highest ℓ-weight, the irreducible quotient of which is $\simeq (T_{m,a}^{(i,s)})^{\vee}$. To establish the irreducibility of the last three tensor products in the lemma, we take twisted duals as in Lemma 6.9, replace $D^{\vee}$ by S, and check Condition (6.33) for the resulting tensor products of KR modules. Take the fourth as an example: $W_{m+s,a^{-1}q^{2s}}^{(i-1)} \otimes W_{m+s,a^{-1}q^{2s}}^{(i+1)} \otimes W_{m,a^{-1}q^{-1}}^{(i)}$ is of highest ℓ-weight.

This proves the lemma in the case $i < M$.

Assume $i > M$. By Lemma 1.9, $\mathcal{G}^*(W_{m,a}^{(i)}) \simeq W'^{(M+N-i)}_{m,aq^{N-M-2+2m}}$ as $U_q(\widehat{\mathfrak{g}'})$-modules. Applying $\mathcal{G}^{*-1}$ to the $U_q(\widehat{\mathfrak{g}'})$-modules $T'^{(M+N-i,s)}_{m,a}$, $S'^{(M+N-i,s)}_{m,a}$ we obtain that $D_{m,a}^{(i,s)}$ and $(D_{m,a}^{(i,s)})^{\vee}$ are $\simeq$ the irreducible quotients of the highest ℓ-weight modules

$$W_{m,aq^{-2m}}^{(i+1)} \otimes W_{m,aq^{-2m}}^{(i-1)*} \otimes W_{s,aq}^{(i)}, \quad W_{s,a^{-1}q^{3-2s}}^{(i)} \otimes W_{m,a^{-1}q^4}^{(i-1)*} \otimes W_{m,a^{-1}q^4}^{(i+1)}$$

respectively. Here $W_{m,a}^{(M)*} := W_{m,a}^{(M-)}$ and $W_{m,a}^{(j)*} = W_{m,a}^{(j)}$ for $j > M$. By replacing $D, D^{\vee}$ with these tensor products, we obtain eight tensor products of KR modules $W_{m,b}^{(j)}, W_{m,b}^{(M-)}$ with $j > M$ and need to show that they are of highest ℓ-weight. Applying $\mathcal{G}^*$ gives tensor products of KR modules $W'^{(j)}_{m,b}$ with $j \leq M$ over $U_q(\widehat{\mathfrak{g}'})$, which are shown to satisfy Condition (6.31). Consider the last tensor product in the lemma as an example. Let us prove that the $U_q(\widehat{\mathfrak{g}})$-modules

$$T_1 := W_{m+s+1,aq^{-2m-1}}^{(i)} \otimes W_{m,aq^{-2m}}^{(i+1)} \otimes W_{m,aq^{-2m}}^{(i-1)*} \otimes W_{s-1,aq}^{(i)},$$
$$T_2 := W_{m+s+1,a^{-1}q^{3-2s}}^{(i)} \otimes W_{s-1,a^{-1}q^{5-2s}}^{(i)} \otimes W_{m,a^{-1}q^4}^{(i-1)*} \otimes W_{m,a^{-1}q^4}^{(i+1)}$$

are of highest ℓ-weight. Applying $\mathcal{G}^*$ to T_1, T_2 give ($c = q^{N-M-2}$, $j = M+N-i$):

$$T_1' = W'^{(j)}_{s-1,acq^{2s-1}} \otimes W'^{(j+1)}_{m,ac} \otimes W'^{(j-1)}_{m,ac} \otimes W'^{(j)}_{m+s+1,ac^{2s+1}},$$
$$T_2' = W'^{(j-1)}_{m,a^{-1}cq^{2m+4}} \otimes W'^{(j+1)}_{m,a^{-1}cq^{2m+4}} \otimes W'^{(j)}_{s-1,a^{-1}cq^3} \otimes W'^{(j)}_{m+s+1,a^{-1}cq^{2m+5}}.$$

The $U_q(\widehat{\mathfrak{g}'})$-modules T_1', T_2' satisfy Condition (6.31). □

For $i \in I_0$ and $m \in \mathbb{Z}_{>0}$ let $d_m^{(i)} := \dim(W_{m,a}^{(i)})$; it is independent of $a \in \mathbb{C}^{\times}$ because $\Phi_a^*(W_{m,1}^{(i)}) \cong W_{m,a}^{(i)}$ by Eq. (1.1).

Theorem 8.3 [44]. $(d_m^{(i)})^2 = d_{m+1}^{(i)} d_{m-1}^{(i)} + d_m^{(i-1)} d_m^{(i+1)}$ *for* $1 \leq i < M$.

Proof. For $\mu \in \mathcal{P}$, up to normalization $\mathcal{T}_{\emptyset \subset \mu}(u)$ in [44, (2.15)] can be identified with $\chi_q(V_q^-(\mu; a))$ in Eq. (2.20). The dimension identity is a consequence of [44, (3.2)], which in turn comes from Jacobi identity of determinants. □

Proof of Theorem 3.3. By Lemma 8.2, the surjective morphisms of $U_q(\widehat{\mathfrak{g}})$-modules in Theorem 3.3 exist (because the third terms are irreducible quotients of the second terms) and their kernels admit irreducible sub-quotients $D_{m,a}^{(i,s)}$ and $D_{m+s,aq_i^{-2s}}^{(i,0)} \otimes W_{m,aq_i^{2m+1}}^{(i)}$ respectively. This gives:

(1) $\dim(D_{m,a}^{(i,s)}) \leq d_m^{(i)} d_{m+s}^{(i)} - d_{m+s+1}^{(i)} d_{m-1}^{(i)}$;
(2) $\dim(D_{m+s,aq_i^{-2s}}^{(i,0)}) d_m^{(i)} \leq d_{m+s}^{(i)} \dim(D_{m,a}^{(i,s)}) - d_{m+s+1}^{(i)} \dim(D_{m,a}^{(i,s-1)})$.

We prove the equality in (1)–(2) by induction on s. Suppose $s = 0$; (2) is trivial. If $i < M$, then by Example 1.6 and Corollary 6.10,

$$D_{m,a}^{(i,0)} \simeq W_{m,aq^{2m}}^{(i+1)} \otimes W_{m,aq^{2m}}^{(i-1)}.$$

This together with Theorem 8.3 shows that equality holds in (1). Making use of $\mathcal{G}^*$, we can remove the assumption $i < M$, as in the proof of Lemma 8.2.

Suppose $s > 0$. In (2) the induction hypothesis applied to $0, s-1$ indicates that

$$\begin{aligned}((d_{m+s}^{(i)})^2 - d_{m+s+1}^{(i)} d_{m+s-1}^{(i)}) d_m^{(i)} \leq d_{m+s}^{(i)} \dim(D_{m,a}^{(i,s)}) \\ - d_{m+s+1}^{(i)} (d_m^{(i)} d_{m+s-1}^{(i)} - d_{m+s}^{(i)} d_{m-1}^{(i)});\end{aligned}$$

namely, $\dim(D_{m,a}^{(i,s)}) \geq d_m^{(i)} d_{m+s}^{(i)} - d_{m+s+1}^{(i)} d_{m-1}^{(i)}$. This implies that in (1), and henceforth in the above inequality and in (2), $\leq$ can be replaced by $=$. $\square$

Remark 8.4. Let $1 \leq i < M$. Apply $\mathcal{G}^{*-1}$ to the second exact sequence in category $\mathcal{O}'$ of Theorem 3.3 involving $D_{m,a}^{\prime(M+N-i,1)}$ and take normalized q-characters:

$$\begin{aligned}\widetilde{\chi}_q(N_{m,a}^{(i)}) \widetilde{\chi}_q(W_{m+1,aq^{-1}}^{(i)}) = \widetilde{\chi}_q(W_{m+2,aq}^{(i)}) \prod_{j \in I_0 : j \sim i} \widetilde{\chi}_q(W_{m,aq^{-2}}^{(j)}) \\ + A_{i,a}^{-1} \times \widetilde{\chi}_q(W_{m,aq^{-3}}^{(i)}) \prod_{j \in I_0 : j \sim i} \widetilde{\chi}_q(W_{m+1,a}^{(j)}).\end{aligned}$$

Setting $m \to \infty$ recovers the normalized q-characters of Eq. (5.28). The second exact sequence of Theorem 3.3 is likely to be true for $i = M$.

Theorem 3.3 together with its proof could be adapted to quantum affine algebras, in view of the cyclicity results of [12] and T-system [31,42]. The second and third terms of the first exact sequence appeared in the proof of [23, Theorem 4.1] as V', V by setting $(a, m, s) = (q_i^{-3}, m_2+1, m_1-m_2-2)$. In the context of graded representations of current algebras [16, Theorem 2] by taking $(\ell, \lambda) = (m+s, m\omega_i)$ so that $\nu = (2m+s)\omega_i - m\alpha_i$, the exact sequence therein is an injective resolution of the Demazure module $D(\ell, \nu)$ by fusion products of KR modules. It is natural to expect that $D_{m,1}^{(i,s)}$ admits a classical limit $(q = 1)$ as $D(\ell, \nu)$; this is true when $m = s = 1$, as a particular case of [11, Theorem 1].

9. Transfer Matrices and Baxter Operators

Let us fix an integer $\ell \in \mathbb{Z}_{>0}$ (length of spin chain) and complex numbers $b_j \in \mathbb{C}^\times \setminus q^{\mathbb{Z}}$ for $1 \leq j \leq \ell$ (inhomogeneity parameters). We shall construct an action of $K_0(\mathcal{O})$ on the vector superspace $\mathbf{V}^{\otimes \ell}$ as in [22, Section 5]. This is the XXZ spin chain with twisted periodic boundary condition, with $\mathbf{V}^{\otimes \ell}$ referred to as the quantum space and objects of category $\mathcal{O}$ auxiliary spaces.

Following Definition 1.4, let $\mathcal{E}$ be the subset of $\mathcal{E}_\ell$ consisting of the $\sum_{\mathbf{f} \in \mathfrak{P}} c_{\mathbf{f}} \mathbf{f} \in \mathcal{E}_\ell$. Note that $\mathcal{E}$ is a sub-ring and $\chi(W) \in \mathcal{E}$ for W in category $\mathcal{O}$.

We identify $\underline{i} = i_1 i_2 \cdots i_\ell \in I^\ell$, an I-string of length ℓ, with the basis vector $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_\ell}$ of $\mathbf{V}^{\otimes \ell}$. Let $E_{\underline{i}\underline{j}} \in \mathrm{End}(\mathbf{V}^{\otimes \ell})$ be the elementary matrix $\underline{k} \mapsto \delta_{\underline{j}\underline{k}} \underline{i}$ for $\underline{i}, \underline{j} \in I^\ell$, and let $\epsilon_{\underline{i}} := \epsilon_{i_1} + \epsilon_{i_2} + \cdots + \epsilon_{i_\ell} \in \mathbf{P}$.

To a $Y_q(\mathfrak{g})$-module W in category $\mathcal{O}$ is by definition attached an matrix $S^W(z)$, a power series in z with values in $\mathrm{End}(W) \otimes \mathrm{End}(\mathbf{V})$. We decompose

$$S_{1,\ell+1}^W(zb_\ell) \cdots S_{13}^W(zb_2) S_{12}^W(zb_1) = \sum_{\underline{i}, \underline{j} \in I^\ell} S_{\underline{i}\underline{j}}^W(z) \otimes E_{\underline{i}\underline{j}} \in \mathrm{End}(W) \otimes \mathrm{End}(\mathbf{V})^{\otimes \ell}[[z]].$$

Then $S_{\underline{i}\underline{j}}^W(z) = \pm s_{i_\ell j_\ell}^W(zb_\ell) \cdots s_{i_2 j_2}(zb_2) s_{i_1 j_1}^W(zb_1)$ and it sends one weight space W_p for $p \in \mathfrak{P}$ to another of weight $pq^{\epsilon_{\underline{i}} - \epsilon_{\underline{j}}}$. Its trace over W_p is well-defined: either 0 if $\epsilon_{\underline{i}} \neq \epsilon_{\underline{j}}$; or the usual non-graded trace of $S_{\underline{i}\underline{j}}^W(z)|_{W_p} \in \mathrm{End}(W_p)$ if $\epsilon_{\underline{i}} = \epsilon_{\underline{j}}$.

Definition 9.1. Let W be in category $\mathcal{O}$. Its associated *transfer matrix* is

$$t_W(z) := \sum_{\underline{i}, \underline{j} \in I^\ell} \left(\sum_{p \in \mathrm{wt}(W)} p \times \mathrm{Tr}_{W_p}(S_{\underline{i}\underline{j}}^W(z)) \right) E_{\underline{i}\underline{j}},$$

viewed as a power series in z with values in $\mathrm{End}(\mathbf{V}^{\otimes \ell}) \otimes_{\mathbb{Z}} \mathcal{E}$.

In [6,46] (for $U_q(\widehat{\mathfrak{g}})$) and [24] (for an arbitrary non-twisted quantum affine algebra), transfer matrices are partial traces of universal R-matrices $\mathcal{R}(z)$. Since the existence of $\mathcal{R}(z)$ for $U_q(\widehat{\mathfrak{g}})$ is not clear to the author (except the simplest case $\mathfrak{gl}(1|1)$ in [50]), we use a different transfer matrix based on RTT. One should imagine $S^W(z)$ as the specialization of $\mathcal{R}(z)$ at $W \otimes \mathbf{V}$.

As in [24], the transfer matrix $t_W(z)$ is a twisted trace of $S^W(z)$ due to the presence of $p \in \mathrm{wt}(W)$. In [6,46] p is related to an auxiliary field.

Example 9.2. Consider the one-dimensional module $\mathbb{C}_{\mathbf{f}}$ in Example 1.3:

$$t_{\mathbb{C}_{\mathbf{f}}}(z)\underline{i} = \underline{i} \times p \times \prod_{l=1}^{\ell} h(zb_l) p_{i_l} \quad \text{for } \underline{i} \in I^\ell.$$

Proposition 9.3. *For X, Y in category $\mathcal{O}$ and $a \in \mathbb{C}^\times$, we have:*

$$t_{\Phi_a^* X}(z) = t_X(za), \quad t_X(z) t_Y(z) = t_{X \otimes Y}(z), \quad t_X(z) t_Y(w) = t_Y(w) t_X(z).$$

Proof. We mainly prove the second equation; the first one is almost clear from Definition 9.1 and Eq. (1.1), and the third one in the same way as [24, Theorem 5.3] based on the commutativity of $K_0(\mathcal{O})$. For $\underline{i}, \underline{j} \in I^\ell$:

$$S_{\underline{i}\underline{j}}^{X\otimes Y}(z) \otimes E_{\underline{i}\underline{j}} = \prod_{r=\ell}^{1} s_{i_r j_r}^{X\otimes Y}(zb_r) \otimes E_{i_1 j_1} \otimes E_{i_2 j_2} \otimes \cdots \otimes E_{i_\ell j_\ell}$$

$$= \sum_{\underline{k}\in I^\ell} \prod_{r=\ell}^{1} \left((-1)^{|E_{i_r k_r}||E_{k_r j_r}|} s_{i_r k_r}^X(zb_r) \otimes s_{k_r i_r}^Y(zb_r)\right) \otimes E_{i_1 j_1} \otimes E_{i_2 j_2} \otimes \cdots \otimes E_{i_\ell j_\ell}$$

$$= \sum_{\underline{k}\in I^\ell} (S_{\underline{i}\underline{k}}^X(z) \otimes 1 \otimes E_{\underline{i}\underline{k}})(1 \otimes S_{\underline{k}\underline{j}}^Y(z) \otimes E_{\underline{k}\underline{j}}).$$

After taking trace over $X_p \otimes Y_{p'}$, only the terms with $\epsilon_{\underline{i}} = \epsilon_{\underline{k}} = \epsilon_{\underline{j}}$ survive and so all the tensor components are of even parity, implying the second equation. $\square$

Let $\varphi : \mathfrak{P} \longrightarrow \mathbb{C}^\times$ be a morphism of multiplicative groups (typical examples are $((p_i)_{i\in I}; s) \mapsto (-1)^s$ and $((p_i)_{i\in I}; s) \mapsto (-1)^s \times \prod_{i\in I} p_i$). If W is a finite-dimensional $Y_q(\mathfrak{g})$-module in category $\mathcal{O}$, then the *twisted transfer matrix* is:

$$t_W(z;\varphi) := \sum_{\underline{i},\underline{j}\in I^\ell} \left(\sum_{p\in \mathrm{wt}(W)} \varphi(p) \times \mathrm{Tr}_{W_p}(S_{\underline{i}\underline{j}}^W(z)) \right) E_{\underline{i}\underline{j}} \in \mathrm{End}(\mathbf{V}^{\otimes \ell})[[z]]. \quad (9.35)$$

If W is infinite-dimensional and the second summation above converges (for a generic choice of φ), then $t_W(z;\varphi)$ is still well-defined.

Lemma 9.4. *Let $i \in I_0$, $a, c \in \mathbb{C}^\times$. The power series $f_{c,a}^{(i)}(z) s_{jk}(z) \in Y_q(\mathfrak{g})[[z]]$ for $j, k \in I$ act on the module $\mathscr{W}_{c,a}^{(i)}$ as polynomials in z of degree ≤ 1, where*

	$i \leq M$	$i > M$
$f_{c,a}^{(i)}$	$1 - zaq^{M-N-i-1}$	$\frac{(1-zac^{-2}q^{M+N-i-1})(1-zaq^{i-M-N-1})}{1-zaq^{M+N-i-1}}$

Proof. Let us recall the generic limit construction of $\mathscr{W}_{c,a}^{(i)}$ in [53]. For $m > 0$ set $V_m := W_{m,aq_i^{-1}}^{(i)} \otimes \mathbb{C}_{(1,\ldots,1;m|\varpi_i|)}$, so that its highest ℓ-weights is of even parity. Let $\mathcal{T} := \{s_{ij}^{(n)}, t_{ij}^{(n)}\}$ be the set of RTT generators for $U_q(\widehat{\mathfrak{g}})$. By [53, Lemma 5.1], their exists an inductive system of vector superspaces $(\{V_m\}, \{F_{m_2,m_1}\})$ with Laurent polynomials $Q_{t;m}(u) \in \mathrm{Hom}_{\mathbb{C}}(V_m, V_{m+1})[u, u^{-1}]$ for $t \in \mathcal{T}$ and $m > 0$ such that

$$tF_{m_2,m} = F_{m_2,m+1} Q_{t;m}(q_i^{m_2}) \in \mathrm{Hom}_{\mathbb{C}}(V_m, V_{m_2}) \quad \text{for } m_2 > m+1. \quad (\blacktriangle)$$

Its inductive limit admits a $U_q(\widehat{\mathfrak{g}})$-module structure where $t \in \mathcal{T}$ acts as the inductive limit $\lim_{m\to\infty} Q_{t;m}(c)$. This is exactly the module $\mathscr{W}_{c,a}^{(i)}$.

Suppose $i > M$. By comparing the highest ℓ-weights of the modules in Eq. (2.25) based on (2.19), (2.21) and Lemma 2.6, we have:

$$W^{(i)}_{m,aq} \cong V_q^{-*}(\lambda^{(i)}_m; aq^{M+N-1-i}) \cong \phi^*_{h_m(z)}\left(V_q^{+*}(\lambda^{(i)}_m; aq^{i-M-N+2m-1})\right),$$
$$h_m(z) = \prod_{l=1}^{m}\prod_{j=1}^{M+N-i} \frac{(1-zaq^{2l-2j+M+N-i-1})^2}{(1-zaq^{2l-2j+M+N-i-3})(1-zaq^{2l-2j+M+N-i+1})}$$
$$= \frac{(1-zaq^{2m+i-M-N-1})(1-zaq^{-i+M+N-1})}{(1-zaq^{2m-i+M+N-1})(1-zaq^{i-M-N-1})}.$$

It follows that $h_{m_2}(z)^{-1}(1-zaq^{2m_2+i-M-N-1})s_{jk}(z)F_{m_2,m}$ is a polynomial in z of degree ≤ 1 for all $m_2 > m$. By Equation (▲) above, this is equal to

$$F_{m_2,m+1}\frac{(1-zaq^{2m_2+i-M-N-1})}{h_{m_2}(z)}\sum_{n\geq 0} z^n Q_{s^{(n)}_{jk};m}(q^{-m_2}).$$

Since $h_{m_2}(z)^{-1}(1-zaq^{2m_2+i-M-N-1}) = f^{(i)}_{q^{-m_2},a}(z)$, from the injectivity of $F_{m_2,m+1}$ and the polynomial dependence on q^{m_2}, we obtain that $f^{(i)}_{c,a}(z)\sum_{n\geq 0} z^n Q_{s^{(n)}_{jk};m}(c)$ is a polynomial in z of degree ≤ 1. By taking the inductive limit $m \to \infty$, the same holds for the action of $f^{(i)}_{c,a}(z)s_{jk}(z)$ on $\mathscr{W}^{(i)}_{c,a}$.

The case $i \leq M$ is much simpler, since $V_m \cong V_q^+(m\varpi_i; aq^{M-N-i-1})$. We omit the details. □

Based on the lemma, let us define the $Y_q(\mathfrak{g})$-module $\mathbb{W}^{(i)}_{c,a} := \phi^*_{f^{(i)}_{c,a}(z)}(\mathscr{W}^{(i)}_{c,a})$. (Indeed it can be equipped with a $U_q(\widehat{\mathfrak{g}})$-module structure.)

Lemma 9.5. *For $i \in I_0$ and $a, c \in \mathbb{C}^\times$ we have:*

$$[\mathbb{W}^{(i)}_{c,1} \otimes \mathbb{W}^{(i)}_{1,a^2}] = [\mathbb{W}^{(i)}_{ca,a^2} \otimes \mathbb{W}^{(i)}_{a^{-1},1}] \in K_0(\mathcal{O}). \tag{9.36}$$

Let X be a finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module in category $\mathcal{O}$. In a fractional ring of $K_0(\mathcal{O})$ we have $[X] = \sum_{l=1}^{\dim X}[D_l]\mathbf{m}_l$ where for each l, D_l is a one-dimensional $U_q(\widehat{\mathfrak{g}})$-module in category $\mathcal{O}$, and $\mathbf{m}_l$ is a product of the $\frac{[\mathbb{W}^{(i)}_{b,a}]}{[\mathbb{W}^{(i)}_{c,a}]}$ with $i \in I_0,\ a, b, c \in \mathbb{C}^\times$.

Proof. For the first statement, by Example 1.6 and Lemma 9.4 we have:

$$\boldsymbol{\omega}^{(i)}_{c,1}\boldsymbol{\omega}^{(i)}_{1,a^2} = \boldsymbol{\omega}^{(i)}_{ca,a^2}\boldsymbol{\omega}^{(i)}_{a^{-1},1}, \quad f^{(i)}_{c,1}(z)f^{(i)}_{1,a^2}(z) = f^{(i)}_{ca,a^2}(z)f^{(i)}_{a^{-1},1}(z).$$

Together with Lemma 4.2, this implies that the q-characters of the two tensor products in Eq. (9.36) coincide. For the second statement, we argue as [24, Theorem 4.8] based on $\frac{\boldsymbol{\omega}^{(i)}_{b,a}}{\boldsymbol{\omega}^{(i)}_{c,a}} = \frac{\chi_q(\mathscr{W}^{(i)}_{b,a})}{\chi_q(\mathscr{W}^{(i)}_{c,a})} \equiv \frac{\chi_q(\mathbb{W}^{(i)}_{b,a})}{\chi_q(\mathbb{W}^{(i)}_{c,a})}$; see also [53, Theorem 6.11]. □

Equation (9.36) is a *separation of variables* identity; see also [22, Theorem 3.11]. The same identity holds when replacing $\mathbb{W}$ by $\mathscr{W}$. Since $t_{\mathbb{W}^{(i)}_{c,a}}(z)$ is a polynomial in z of degree $\leq \ell$, the following definition makes sense.

Definition 9.6. For $i \in I_0$ the *Baxter operator* is $Q_i(z) := t_{\mathbb{W}^{(i)}_{z,1}}(1)$.

Let $p_c^{(i)} = \varpi(\boldsymbol{\omega}_{c,a}^{(i)})$. Then $\mathrm{wt}(\mathbb{W}_{c,a}^{(i)}) \subset p_c^{(i)} q^{\mathbf{Q}^-}$ and $\overline{Q}_i(z) := (p_c^{(i)})^{-1} Q_i(z)$ is a power series in the $q^{-\alpha_j}$ with $j \in I_0$ whose coefficients are in $\mathrm{End}(\mathbf{V}^{\otimes \ell})[z, z^{-1}]$. Let $\overline{Q}_i^0(z)$ be its leading term. Since $(\mathbb{W}_{1,1}^{(i)})_{p_1^{(i)}}$ is the one-dimensional simple socle of $\mathbb{W}_{1,1}^{(i)}$, by Definition 9.1, $\underline{i}$ is an eigenvector of $\overline{Q}_i^0(1)$ with non-zero eigenvalue. (Here we used the overall assumption $b_l \notin q^{\mathbb{Z}}$.) The formal power series $\overline{Q}_i^0(z)$ and $Q_i(z)$ in the $q^{-\alpha_j}$ can therefore be inverted for $z \in \mathbb{C}$ generic.

Corollary 9.7 (Generalized Baxter TQ relations). *For $b, c \in \mathbb{C}^\times$, we have:*

$$\frac{t_{\mathbb{W}_{b,1}^{(i)}}(z^{-2})}{t_{\mathbb{W}_{c,1}^{(i)}}(z^{-2})} = \frac{Q_i(zb)}{Q_i(zc)}, \quad \frac{t_{\mathscr{W}_{b,1}^{(i)}}(z^{-2})}{t_{\mathscr{W}_{c,1}^{(i)}}(z^{-2})} = \prod_{l=1}^{\ell} \frac{f_{c,1}^{(i)}(z^{-2}b_l^{-2})}{f_{b,1}^{(i)}(z^{-2}b_l^{-2})} \times \frac{Q_i(zb)}{Q_i(zc)}. \tag{9.37}$$

If X is a finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module in category $\mathcal{O}$, then $t_X(z^{-2})$ is a sum of monomials in the $\frac{Q_i(zb)}{Q_i(zc)} t_D(z^{-2})$ with $i \in I_0$, $b, c \in \mathbb{C}^\times$ and with D one-dimensional $U_q(\widehat{\mathfrak{g}})$-modules in category $\mathcal{O}$, the number of terms being $\dim X$.

Proof. In Eq. (9.36) let us set $(a, c) = (z^{-1}, bz)$:

$$[\mathbb{W}_{b,z^{-2}}^{(i)}][\mathbb{W}_{z,1}^{(i)}] = [\mathbb{W}_{zb,1}^{(i)}][\mathbb{W}_{1,z^{-2}}^{(i)}].$$

Taking transfer matrices and evaluating them at 1 gives the special case $c = 1$ of Eq. (9.37), which in turn implies the general case $c \in \mathbb{C}^\times$. The second statement is a translation of that of Lemma 9.5. $\square$

Example 9.8. Let $\mathfrak{g} = \mathfrak{gl}(2|2)$ and $X = W_{1,1}^{(1)} = V_q^+(\epsilon_1; q^{-1})$. By Eq. (2.18):

$$\chi_q(X) = \boxed{1}_1 + \boxed{2}_1 + \boxed{3}_1 + \boxed{4}_1.$$

If $s \in \mathbb{Z}_2$, $g(z) \in \mathbb{C}[[z]]^\times$ and $c \in \mathbb{C}^\times$, for simplicity let $sg(z) := (g(z)^4; s) \in \widehat{\mathfrak{P}}$, $[s, g(z)] := [L(g(z)^4; s)] \in K_0(\mathcal{O})$ and $\langle s, c\rangle := (c^4; s) \in \mathfrak{P}$. Set $w_{c,a}^{(i)} := f_{c,a}^{(i)}(z)\boldsymbol{\omega}_{c,a}^{(i)}$. By Definition 2.2, Example 1.6 and Lemma 9.4:

$$\boxed{1}_1 = \left(\frac{q-z}{1-zq}, 1, 1, 1; \bar{0}\right), \quad \boxed{2}_1 = \left(1, \frac{q-zq^2}{1-zq^3}, 1, 1; \bar{0}\right),$$

$$\boxed{3}_1 = \left(1, 1, \frac{1-zq^3}{q-zq^2}, 1; \bar{1}\right), \quad \boxed{4}_1 = \left(1, 1, 1, \frac{1-zq}{q-z}; \bar{1}\right),$$

$$\frac{w_{c,a}^{(1)}}{w_{1,a}^{(1)}} = \left(\frac{c-zac^{-1}}{1-za}, 1, 1, 1; \bar{0}\right), \quad \frac{w_{c,a}^{(2)}}{w_{1,a}^{(2)}} = \left(\frac{c-zaqc^{-1}}{1-zaq}, \frac{c-zaqc^{-1}}{1-zaq}, 1, 1; \bar{0}\right),$$

$$\frac{w_{c,a}^{(3)}}{w_{1,a}^{(3)}} = \left(\frac{1-zac^{-2}}{1-za}, \frac{1-zac^{-2}}{1-za}, \frac{1-zac^{-2}}{1-za}, c^{-1}; \bar{0}\right), \quad \boxed{1}_1 = \frac{w_{q,q}^{(1)}}{w_{1,q}^{(1)}},$$

$$\boxed{2}_1 = \frac{w_{q^{-1},q}^{(1)}}{w_{1,q}^{(1)}} \frac{w_{q,q^2}^{(2)}}{w_{1,q^2}^{(2)}}, \quad \boxed{3}_1 = \bar{1}q^{-1} \frac{w_{q,q^2}^{(2)}}{w_{1,q^2}^{(2)}} \frac{w_{q^{-1},q}^{(3)}}{w_{1,q}^{(3)}}, \quad \boxed{4}_1 = \bar{1}\frac{1-zq}{1-zq^{-1}} \frac{w_{q,q}^{(3)}}{w_{1,q}^{(3)}}.$$

It follows that in the fractional ring of $K_0(\mathcal{O})$:

$$[X] = \frac{[\mathbb{W}^{(1)}_{q,q}]}{[\mathbb{W}^{(1)}_{1,q}]} + \frac{[\mathbb{W}^{(1)}_{q^{-1},q}]}{[\mathbb{W}^{(1)}_{1,q}]}\frac{[\mathbb{W}^{(2)}_{q,q^2}]}{[\mathbb{W}^{(2)}_{1,q^2}]} + [\overline{1}, q^{-1}]\frac{[\mathbb{W}^{(2)}_{q,q^2}]}{[\mathbb{W}^{(2)}_{1,q^2}]}\frac{[\mathbb{W}^{(3)}_{q^{-1},q}]}{[\mathbb{W}^{(3)}_{1,q}]}$$
$$+ [\overline{1}, \frac{1-zq}{1-zq^{-1}}]\frac{[\mathbb{W}^{(3)}_{q,q}]}{[\mathbb{W}^{(3)}_{1,q}]}.$$

Let $q^{\frac{1}{2}}$ be a square root of q. By Example 9.2 and Eq. (9.37):

$$t_X(z^{-2}) = \frac{Q_1(zq^{\frac{1}{2}})}{Q_1(zq^{-\frac{1}{2}})} + \frac{Q_1(zq^{-\frac{3}{2}})}{Q_1(zq^{-\frac{1}{2}})}\frac{Q_2(z)}{Q_2(zq^{-1})} + \langle\overline{1}, q^{-1}\rangle \times \frac{Q_2(z)}{Q_2(zq^{-1})}\frac{Q_3(zq^{-\frac{3}{2}})}{Q_3(zq^{-\frac{1}{2}})}q^{-\ell}$$
$$+ \langle\overline{1}, 1\rangle \times \frac{Q_3(zq^{\frac{1}{2}})}{Q_3(zq^{-\frac{1}{2}})}\prod_{l=1}^{\ell}\frac{z^2 - b_l q}{z^2 - b_l q^{-1}}.$$

Example 9.9. Let $\mathfrak{g} = \mathfrak{gl}(2|0)$ and $X = W^{(1)}_{1,1} = V_q^+(\epsilon_1; q)$. Then

$$\boxed{1}_{q^2} + \boxed{2}_{q^2} = \left(\frac{q - zq^{-2}}{1 - zq^{-1}}, 1; \overline{0}\right) + \left(1, \frac{q-z}{1-zq}; \overline{0}\right) = \frac{w^{(1)}_{q,q}}{w^{(1)}_{1,q}} + \frac{q-z}{1-zq}\frac{w^{(1)}_{q^{-1},q}}{w^{(1)}_{1,q}},$$
$$t_X(z^{-2}) = \frac{Q_1(zq^{\frac{1}{2}})}{Q_1(zq^{-\frac{1}{2}})} + \langle\overline{0}, q\rangle \times \frac{Q_1(zq^{-\frac{3}{2}})}{Q_1(zq^{-\frac{1}{2}})}\prod_{l=1}^{\ell}\frac{qz^2 - b_l}{z^2 - b_l q}.$$

Example 9.10. Let $\mathfrak{g} = \mathfrak{gl}(1|1)$ and $X = W^{(1)}_{1,1} = V_q^+(\epsilon_1; q^{-1})$. We have

$$\chi_q(X) = \boxed{1}_1 + \boxed{2}_1 = \left(\frac{q-z}{1-zq}, 1; \overline{0}\right) + \left(1, \frac{1-zq}{q-z}; \overline{1}\right) = \frac{w^{(1)}_{q,q}}{w^{(1)}_{1,q}}\left(1 + \overline{1}\frac{1-zq}{q-z}\right),$$
$$t_X(z^{-2}) = \frac{Q_1(zq^{\frac{1}{2}})}{Q_1(zq^{-\frac{1}{2}})} + \langle\overline{1}, q^{-1}\rangle \times \frac{Q_1(zq^{\frac{1}{2}})}{Q_1(zq^{-\frac{1}{2}})}\prod_{l=1}^{\ell}\frac{z^2 - b_l q}{z^2 q - b_l}.$$

One can view Examples 9.9–9.10 as degenerate cases of Example 9.8.

We are ready to deduce three-term functional relations of the Baxter operators $Q_i(z)$. Fix $a = 1$. Let $c, d \in \mathbb{C}^\times$ be such that $c^2 \notin q^{\mathbb{Z}}$. In Eq. (5.30) let us evaluate transfer matrices at z^{-2} making use of Proposition 9.3:

$$t_{M^{(i)}_{c,1}}(z^{-2})t_{\mathscr{W}^{(i)}_{d,d^2}}(z^{-2}) = t_{\mathscr{W}^{(i)}_{dq_i,d^2}}(z^{-2})\prod_{j\in I_0: j\sim i} t_{\mathscr{W}^{(j)}_{c_{ij}^{-1}, q_{ij}^{-1}c_{ij}^{-2}}}(z^{-2})$$
$$+ t_{D_i}(z^{-2})t_{\mathscr{W}^{(i)}_{d\hat{q}_i^{-1},d^2}}(z^{-2})\prod_{j\in I_0: j\sim i} t_{\mathscr{W}^{(j)}_{c_{ij}^{-1}q_{ij}^{-1}, q_{ij}^{-1}c_{ij}^{-2}}}(z^{-2}).$$

Dividing both sides by the term at the second row without $t_{D_i}(z^{-2})$ and making use of Eq. (9.37), we obtain the *Baxter TQ relation*:

$$X^{(i)}_c(z)\frac{Q_i(z)}{Q_i(z\hat{q}_i^{-1})} = y_i(z)\frac{Q_i(zq_i)}{Q_i(z\hat{q}_i^{-1})}\prod_{j\in I_0: j\sim i}\frac{Q_j(zq_{ij}^{\frac{1}{2}})}{Q_j(zq_{ij}^{-\frac{1}{2}})} + t_{D_i}(z^{-2}), \qquad (9.38)$$

where $X_c^{(i)}(z)$ (depending on $c \in \mathbb{C}^\times \backslash q^{\mathbb{Z}}$) and $y_i(z)$ are given by

$$X_c^{(i)}(z) = \frac{t_{M_{c,1}^{(i)}}(z^{-2})}{\prod_{j \in I_0 : j \sim i} t_{\mathscr{W}^{(j)}_{c_{ij}^{-1} q_{ij}^{-1}, aq_{ij}^{-1} c_{ij}^{-2}}}(z^{-2})} \times \prod_{l=1}^{\ell} \frac{f^{(i)}_{d\hat{q}_i^{-1}, d^2}(z^{-2} b_l)}{f^{(i)}_{d,d^2}(z^{-2} b_l)},$$

$$y_i(z) = \prod_{l=1}^{\ell} \left(\frac{f^{(i)}_{d\hat{q}_i^{-1}, d^2}(z^{-2} b_l)}{f^{(i)}_{dq_i, d^2}(z^{-2} b_l)} \times \prod_{j \in I_0 : j \sim i} \frac{f^{(j)}_{c_{ij}^{-1} q_{ij}^{-1}, q_{ij}^{-1} c_{ij}^{-2}}(z^{-2} b_l)}{f^{(j)}_{c_{ij}^{-1}, q_{ij}^{-1} c_{ij}^{-2}}(z^{-2} b_l)} \right).$$

Note that $y_i(z)$, D_i are independent of c, d by Lemma 9.4 and Theorem 5.3.

Let us assume that the twisted transfer matrices in Eq. (9.35) are well-defined for all the $M_{c,1}^{(i)}$ and $\mathscr{W}_{c,a}^{(i)}$, upon a generic choice of $\varphi : \widehat{\mathfrak{P}} \longrightarrow \mathbb{C}^\times$; this corresponds to the convergence assumption in [24, Remark 5.12 (ii)]. Then Eq. (9.38) is an operator equation in $\mathrm{End}(\mathbf{V}^{\otimes \ell})[[z^{-2}]]$.

Based on the asymptotic construction of $\mathscr{W}_{c,a}^{(i)}$, one can show that there exists $n \in \mathbb{Z}$ such that $z^n Q_i(z)$ is a polynomial in z with values in $\mathrm{End}(\mathbf{V}^{\otimes \ell})$.

As in [25, Section 5], we expect that the $t_{M_{c,1}^{(i)}}(z^{-2})$ are polynomials in z^{-2} (up to multiplication by an integer power of z). Suppose that w is a zero of $Q_i(z)$ that is neither a zero of $Q_i(z\hat{q}_i^{-1})$, $Q_j(zq_{ij}^{-\frac{1}{2}})$ nor a pole of $X_c^{(i)}(z)$. Then we have the *Bethe Ansatz Equation*: (see [44, (2.6a)] and [5,38])

$$y_i(w) \frac{Q_i(wq_i)}{Q_i(w\hat{q}_i^{-1})} \prod_{j \in I_0 : j \sim i} \frac{Q_j(wq_{ij}^{\frac{1}{2}})}{Q_j(wq_{ij}^{-\frac{1}{2}})} = -t_{D_i}(w^{-2}). \tag{9.39}$$

Example 9.11. Following Example 9.8, we determine the highest ℓ-weight (still denoted by D_i) of the one-dimensional $U_q(\widehat{\mathfrak{g}})$-module D_i and the $y_i(z)$ in Eq. (9.39) for $\mathfrak{g} = \mathfrak{gl}(2|2)$. First by Definition 2.2 and Example 1.6:

$$\boldsymbol{\omega}_{c,a}^{(1)} = \left(\frac{c - zac^{-1}}{1 - za}, 1, 1, 1; \bar{0} \right), \quad \boldsymbol{\omega}_{c,a}^{(2)} = \left(\frac{c - zaqc^{-1}}{1 - zaq}, \frac{c - zaqc^{-1}}{1 - zaq}, 1, 1; \bar{0} \right),$$
$$\boldsymbol{\omega}_{c,a}^{(3)} = \left(1, 1, 1, \frac{1 - za}{c - zac^{-1}}; \bar{0} \right), \quad A_{1,a} = \left(\frac{q - zaq^{-1}}{1 - za}, \frac{1 - zaq^2}{q - zaq}, 1, 1; \bar{0} \right),$$
$$A_{2,a} = \left(1, \frac{q - za}{1 - zaq}, \frac{q - za}{1 - zaq}, 1; \bar{1} \right), \quad A_{3,a} = \left(1, 1, \frac{1 - zaq^2}{q - zaq}, \frac{q - zaq^{-1}}{1 - za}; \bar{0} \right).$$

The relations between A and $\boldsymbol{\omega}$ are as follows: $A_{1,a} = \boldsymbol{\omega}^{(1)}_{q^2, aq^2} \boldsymbol{\omega}^{(2)}_{q^{-1}, aq^{-1}}$ and

$$A_{2,a} = \bar{1} \frac{q - za}{1 - zaq} \boldsymbol{\omega}^{(1)}_{q^{-1}, aq^{-1}} \boldsymbol{\omega}^{(3)}_{q, aq}, \quad A_{3,a} = \frac{1 - zaq^2}{q - zaq} \boldsymbol{\omega}^{(2)}_{q, aq} \boldsymbol{\omega}^{(3)}_{q^{-2}, aq^{-2}}.$$

It it follows that $D_1 = 1,\ D_2 = \overline{1}\frac{1-zq}{q-z},\ D_3 = \frac{q-zq}{1-zq^2}$ and so ($D_i(z) := t_{D_i}(z^{-2})$)

$$D_1(z) = 1, \quad D_2(z) = \langle \overline{1}, q^{-1}\rangle \times \prod_{l=1}^{\ell} \frac{z^2 - b_l q}{z^2 q - b_l}, \quad D_3(z) = \langle \overline{0}, q\rangle \times \prod_{l=1}^{\ell} \frac{z^2 q - b_l q}{z^2 - b_l q^2},$$

$$y_1(z) = 1, \quad y_2(z) = \prod_{l=1}^{\ell} \frac{z^2 - b_l q}{z^2 - b_l q^{-1}}, \quad y_3(z) = \prod_{l=1}^{\ell} \frac{z^2 - b_l q^{-2}}{z^2 - b_l q^2}.$$

The Bethe Ansatz Equations become in this case:

$$\frac{Q_1(w_1 q)}{Q_1(w_1 q^{-1})} \frac{Q_2(w_1 q^{-\frac{1}{2}})}{Q_2(w_1 q^{\frac{1}{2}})} = -1, \quad \frac{Q_1(w_2 q^{-\frac{1}{2}})}{Q_1(w_2 q^{\frac{1}{2}})} \frac{Q_3(w_2 q^{\frac{1}{2}})}{Q_3(w_2 q^{-\frac{1}{2}})} = -\langle \overline{1}, q^{-1}\rangle \times q^{-\ell},$$

$$\frac{Q_3(w_3 q^{-1})}{Q_3(w_3 q)} \frac{Q_2(w_3 q^{\frac{1}{2}})}{Q_2(w_3 q^{-\frac{1}{2}})} = -\langle \overline{0}, q\rangle \times \prod_{l=1}^{\ell} \frac{w_3^2 q - b_l q}{w_3^2 - b_l q^{-2}},$$

where w_i is a zero of $Q_i(z)$ for $1 \leq i \leq 3$.

The generalized Baxter relations in Lemma 9.5 and Bethe Ansatz Equations (9.39) for the Baxter operators $Q_i(z)$ are based on asymptotic $U_q(\widehat{\mathfrak{g}})$-modules: $\mathscr{W}_{c,a}^{(i)}, \mathcal{N}_{c,a}^{(i)}, M_{c,a}^{(i)}$, whereas in recent parallel works [18,19,25,35] representations of Borel subalgebras ($Y_q(\mathfrak{g})$ in our situation) play a key role.

In [5,38], for the Yangian of $\mathfrak{gl}(M|N)$ the Baxter operators $\mathbf{Q}_J(z)$ are labeled by the subsets J of I. In addition to TQ relations, there are algebraic relations among the $\mathbf{Q}_J(z)$ called QQ relations. Our $Q_i(z)$ with $i \in I_0$ seem to be algebraically independent by Proposition 7.7; see also [24, Theorem 4.11].

Remark 9.12. Following [6,24] define $\mathbf{Q}_i(z) := t_{L_{i,1}^+}(z)$ for $i \in I_0$. We have

$$t_{L([c]_i)}(z^{-2}) \frac{\mathbf{Q}_i(z^{-2}c^{-2})}{\mathbf{Q}_i(z^{-2})} = \prod_{l=1}^{\ell} \frac{f_{1,1}^{(i)}(z^{-2}b_l^{-2})}{f_{c,1}^{(i)}(z^{-2}b_l^{-2})} \times \frac{Q_i(zc)}{Q_i(z)} \tag{9.40}$$

based on the q-character formula $\frac{\chi_q(\mathscr{W}_{c,1}^{(i)})}{\chi_q(\mathscr{W}_{1,1}^{(i)})} = [c]_i \frac{\chi_q(L_{i,c^{-2}}^+)}{\chi_q(L_{i,1}^+)}$ and Eq. (9.37). See [22, Remark A.7] for a similar comparison in the Yangian case.

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