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Classical Affine W-Superalgebras via Generalized Drinfeld–Sokolov Reductions and Related Integrable Systems

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Abstract: The purpose of this article is to investigate relations between W-superalgebras and integrable super-Hamiltonian systems. To this end, we introduce the generalized Drinfel'd–Sokolov (D–S) reduction associated to a Lie superalgebra $\mathfrak{g}$ and its even nilpotent element f, and we find a new definition of the classical affine W-superalgebra $\mathcal{W}(\mathfrak{g}, f, k)$ via the D–S reduction. This new construction allows us to find free generators of $\mathcal{W}(\mathfrak{g}, f, k)$, as a differential superalgebra, and two independent Lie brackets on $\mathcal{W}(\mathfrak{g}, f, k)/\partial\mathcal{W}(\mathfrak{g}, f, k)$. Moreover, we describe super-Hamiltonian systems with the Poisson vertex algebras theory. A W-superalgebra with certain properties can be understood as an underlying differential superalgebra of a series of integrable super-Hamiltonian systems.

1. Introduction

Classical affine W-algebras have been studied in the theory of integrable systems since the 1980s, when Drinfel'd–Sokolov [12] discovered relations between a finite dimensional simple Lie algebra $\mathfrak{g}$ and a sequence of integrable systems.

The main idea of Drinfel'd and Sokolov in [12] is considering Lax operators associated to a Lie algebra $\mathfrak{g}$. Precisely, such Lax operators have the form of

$$L = \frac{\partial}{\partial x} + q(x) + \Lambda \tag{1.1}$$

where (i) q is a differentiable function whose value is in a borel subalgebra $\mathfrak{n}_+ \oplus \mathfrak{h} \subset \mathfrak{g}$ (ii) $\Lambda = f + zs \in \mathfrak{g}[z]$ for the principal nilpotent element $f \in \mathfrak{n}_-$ and $s \in \ker(\mathrm{ad}\mathfrak{n}_+)$. Here, $\mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a triangular decomposition of $\mathfrak{g}$. On the phase space $\mathcal{F}_{\mathfrak{g}}$ consisting of functions q in Lax operators, Drinfel'd–Sokolov defined gauge transformations. As

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a consequence, the W-algebra $\mathcal{W}(\mathfrak{g})$ associated to $\mathfrak{g}$ was introduced as a set of gauge invariant functions.

Furthermore, a Lax operator (1.1) gives rise to a bi-Poisson structure on $\mathcal{W}(\mathfrak{g})$, which has an important role to find related integrable systems [12]. A bi-Poisson structure $(\{,\}_K, \{,\}_H)$ consists of a couple of linearly independent *local* Poisson brackets which involve delta distributions, that is,

$$\{u(x), v(y)\}_X = \sum_{n\in\mathbb{Z}_+\cup\{0\}} W_n(y)\, \partial_y^n \delta(x-y) \quad \text{for } u, v, W_n \in \mathcal{W}(\mathfrak{g}),\ X = K, H$$

satisfy *skew symmetries*, *Jacobi identities* and *Leibniz rules*. After all, a systematic algorithm of getting a sequence of Hamiltonian integrable systems on $\mathcal{W}(\mathfrak{g})$ was discovered. In this algorithm, they used the Lenard–Magri scheme and the bi-Poisson structure, i.e., there are $k_i \in \mathcal{W}(\mathfrak{g})$ for $i \in \mathbb{Z}_{\geq 0}$ such that

$$\frac{d\phi(x,t)}{dt} = \int \{k_i(x), \phi(y)\}_H dy = \int \{k_{i+1}(x), \phi(y)\}_K dy, \quad \text{for } \phi \in \mathcal{W}(\mathfrak{g}),$$

are all distinct integrable systems.

In an algebraic point of view, classical affine W-algebras are Poisson vertex algebras. A Poisson vertex algebra (PVA) is a differential algebra endowed with a Poisson λ-bracket structure, denoted by $\{\,{}_\lambda\,\}$. Here, a λ-bracket can be understood as an algebraic interpretation of a local Poisson bracket. On the other hand, Poisson vertex algebras are closely related to vertex algebras since the quasi-classical limit of a certain family of vertex algebras is a Poisson vertex algebra. As one can expect, there is a vertex algebra called a quantum affine W-algebra that is a quantization of a classical affine W-algebra.

The quantum affine W-(super)algebra associated to $\mathfrak{g}$ and $f \in \mathfrak{g}$ is introduced via the quantum affine BRST complexes (or the quantum Drinfel'd–Sokolov reduction), provided that $\mathfrak{g}$ is a finite dimensional simple *Lie superalgebra* with a non-degenerate even supersymmetric bilinear form and f is an even nilpotent element with an sl_2-triple (e, h, f) [5,13,19,20]. In [24,25], the author proved that the quasi-classical limit of the quantum affine W-(super)algebra associated to $\mathfrak{g}$ and f is

$$\mathcal{W}(\mathfrak{g}, f, k) = \left(\mathcal{P}/\mathcal{I}\right)^{\mathrm{ad}_\lambda \mathfrak{n}}, \tag{1.2}$$

where $\mathcal{P} = S(\mathbb{C}[\partial] \otimes \mathfrak{g})$ is the affine PVA, and the Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ and the differential algebra ideal $\mathcal{I}$ of $\mathcal{P}$ are determined by f. Here the $\mathrm{ad}_\lambda \mathfrak{n}$-action is induced from the λ-bracket on the affine PVA $\mathcal{P}$ and $k \in \mathbb{C}$ is the constant involved in the λ-bracket of $\mathcal{P}$. These results imply that classical affine W-(super)algebras have properties analogous to those of finite W-(super)algebras [14]. Note that the W-algebra $\mathcal{W}(\mathfrak{g})$ introduced by Drinfel'd and Sokolov in [12] is just $\mathcal{W}(\mathfrak{g}, f, k)$, when $f = f_{\mathrm{princ}}$ is a principal nilpotent element and $k = 1$ (see also [7]).

Since $\mathcal{W}(\mathfrak{g})$ is a special case of W-algebras associated to $\mathfrak{g}$, there have been many attempts to understand algebraic structures of $\mathcal{W}(\mathfrak{g}, f, k)$ and to find integrable systems associated to $\mathcal{W}(\mathfrak{g}, f, k)$ for any nilpotent element f (see [1,4] for instance). Regarding these topics, there are plenty of considerable articles, provided that $\mathfrak{g}$ is a *Lie algebra*. In [11,28], using the definition (1.2), De Sole, Kac and Valeri succeeded in explaining generators of $\mathcal{W}(\mathfrak{g}, f, k)$ and the λ-bracket relations between them. Moreover, integrable systems on $\mathcal{W}(\mathfrak{g}, f, k)$ are discovered in [3,6,8,9]. However, in the case when $\mathfrak{g}$ is a Lie superalgebra, there still remain many open problems.

For algebraic structures of W-algebras associated to Lie superalgebras (W-superalgebras), we refer [22,23,27,29], where structures of finite W-superalgebras are considered, and [20,26], where the algebraic structures of affine W-superalgebas associated to minimal nilpotent elements are given. On the other hand, integrable systems have not been yet explored in precise connections with W-superalgebras (1.2) via PVA structures, to the best of the author's knowledge. There are some articles that investigated integrable systems on noncommutative algebras (see [10,16,17,21] and the references therein). In particular, in [16,17,21], the authors described relations between integrable systems and Lie superalgebras, for instance $\mathrm{spo}(2|1)$ and $\mathrm{sl}(n|n)$. However, it is not clear if these integrable systems can be explained by PVA structures of W-superalgebras (1.2).

In this context, a natural question is whether a W-superalgebra can be related to a sequence of integrable systems. In this paper, as the first step toward answering this question, we construct W-superalgebras using Lax operators, mainly inspired by the important papers [1,4,7,12]. The key idea is to consider Lax operators in algebraic languages:

$$L = k\partial + q - \Lambda \otimes 1 \ \in\ \mathbb{C}\partial \ltimes (\mathfrak{g}[z] \otimes \mathcal{P}/\mathcal{I})_{\bar{0}}$$

with even parities, where $\mathcal{P}/\mathcal{I}$ is the differential algebra in (1.2) (see Definition 3.1). In other words, we assume that the entire phase space

$$\mathcal{F}_{\mathfrak{g},f} = \{q \mid L = k\partial + q - \Lambda \otimes 1 \ \text{ is even Lax operator}\} \subset (\mathfrak{g}[z] \otimes \mathcal{P}/\mathcal{I})_{\bar{0}}$$

is even. By considering *even gauge transformations*, we prove that the construction of W-superalgebras via Lax operators is equivalent to (1.2) (see Theorem 3.11). This new construction is particularly useful to find free generators of W-superalgebras as differential algebras (see Proposition 3.14).

Recall that if $\mathfrak{g}$ is a Lie algebra and $f \in \mathfrak{g}$ is a nilpotent element, then the W-algebra $\mathcal{W}(\mathfrak{g}, f, k)$ is endowed with a pair of Poisson λ-brackets, and these brackets play crucial roles in describing integrable Hamiltonian systems associated to $\mathcal{W}(\mathfrak{g}, f, k)$. Therefore, in order to find integrable systems associated to W-superalgebras in an analogous approach, we need to understand the following:

- the definition of *super-Hamiltonian evolution equations* using Poisson λ-brackets of Poisson vertex algebras;
- how to describe two linearly independent Lie (super)brackets on $\mathcal{W}(\mathfrak{g}, f, k)/\partial\mathcal{W}(\mathfrak{g}, f, k)$ with Lax operators and variational derivatives;
- Lenard–Magri scheme associated to super-Hamiltonian integrable systems.

To describe two linearly independent Lie (super)brackets on $\mathcal{W}(\mathfrak{g}, f, k)/\partial\mathcal{W}(\mathfrak{g}, f, k)$, we consider the special operator called the *sign twisted*[1] *universal Lax operator,*

$$L^{\sigma}_{\mathrm{univ}} = k\partial + q^{\sigma}_{\mathrm{univ}} - \Lambda \otimes 1 \in \mathbb{C}\partial \ltimes \mathfrak{g}[z] \otimes \mathcal{P}/\mathcal{I}.$$

(see Proposition 3.18 for the precise definition). Employing this operator, we describe the Lie brackets on $\mathcal{W}(\mathfrak{g}, f, k)/\partial\mathcal{W}(\mathfrak{g}, f, k)$ which are needed to use the Lenard–Magri scheme. Indeed, under the assumption that Λ is semisimple in $\mathfrak{g}((z^{-1}))$, we show that there are super-Hamiltonian integrable systems associated to classical affine W-superalgebras (see Theorem 5.14).[2] As a simplest example, we show one of super-Hamiltonian integrable systems associated to $\mathcal{W}(\mathrm{spo}(2|1), f, k)$ is equivalent to the super KdV equation, which appears in [21], up to constant factors.

[1] The main reason we consider *sign twisted* operators is explained in Remark 4.16.

[2] This is a quite strong assumption (see Remark 5.5). An important open question in this field is to find integrable systems associated to arbitrary W-superalgebras.

2. Classical Affine W-Algebras

In this section, we review some known facts about classical affine W-algebras. For the notions about Poisson vertex algebras, we refer to [2,18]. Properties about W-algebras and Lax operators denoted in this section can be found in [7,12].

2.1. Poisson vertex algebras.
A *vector superspace* is a vector space V with the $\mathbb{Z}/2\mathbb{Z}$-graded decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$. For $i = 0, 1$, we denote the *parity* by $\mathrm{p}(a) = i$ for a homogeneous element $a \in V_{\bar{i}}$. The *superalgebra* $\mathrm{End}(V) = \mathrm{End}(V)_{\bar{0}} \oplus \mathrm{End}(V)_{\bar{1}}$ is a vector superspace such that

$$\text{if } F \in \mathrm{End}(V)_{\bar{i}} \text{ then } F(V_{\bar{j}}) \subset V_{\bar{i}+\bar{j}}.$$

The *supersymmetric algebra* A is a superalgebra with the *supersymmetry*

$$ab = (-1)^{\mathrm{p}(a)\mathrm{p}(b)} ba,$$

for homogeneous elements a and b.

Definition 2.1. A *Lie conformal algebra* (LCA) R is a $\mathbb{C}[\partial]$-module with a $\mathbb{C}$-linear λ-bracket

$$[\,{}_\lambda\,] : R \otimes R \to R[\lambda]$$

satisfying the following properties:

- (sesquilinearity) $[a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b]$, $[\partial a_\lambda b] = -\lambda[a_\lambda b]$,
- (skewsymmetry) $[a_\lambda b] = -(-1)^{\mathrm{p}(a)\mathrm{p}(b)}[b_{-\lambda-\partial} a]$,
- (Jacobi identity) $[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{\mathrm{p}(a)\mathrm{p}(b)}[b_\mu [a_\lambda c]]$.

Here, we assume ∂ is an even operator on R.

Remark 2.2. Let R be a LCA.

(1) The sesquilinearity implies ∂ is a derivation for the λ-bracket on R, i.e., $\partial[a_\lambda b] = [\partial a_\lambda b] + [a_\lambda \partial b]$.
(2) For any $a, b \in R$, we denote by $[a_\lambda b] = \sum_{n\geq 0} \frac{a_{(n)}b}{n!}\lambda^n$ for $a_{(n)}b \in R$. Here, $a_{(n)}b \in R$ is called the nth product of a and b.

Definition 2.3. (1) A *differential (super)algebra* D is an associative (super)algebra with an even operator $\partial : D \to D$ called *derivation* such that

$$\partial(AB) = \partial(A)B + A\partial(B) \quad \text{for } A, B \in D. \tag{2.1}$$

In other words, the derivation ∂ defined on a generating set of D can be extended using the Leibniz rule (2.1).

(2) The *(super)algebra of differential polynomials*

$$\mathbb{C}_{\mathrm{diff}}[\, w_i \mid i \in I_{\bar{0}} \cup I_{\bar{1}} \,]$$

generated by even elements in $\{w_i\}_{i\in I_{\bar{0}}}$ and odd elements $\{w_i\}_{\imath\in I_{\bar{1}}}$ is the algebra isomorphic to the tensor products of a symmetric algebra and a exterior algebra

$$S(V_0) \otimes \bigwedge(V_1)$$

where $V_0 := \mathrm{Span}_{\mathbb{C}}(\partial^n w_i \mid i \in I_{\bar{0}},\ n \in \mathbb{Z}_{\geq 0})$ and $V_1 := \mathrm{Span}_{\mathbb{C}}(\partial^n w_i \mid i \in I_{\bar{1}},\ n \in \mathbb{Z}_{\geq 0})$.

Remark 2.4. For the simplicity of notations, we denote by $S(V)$ the supersymmetric algebra generated by the vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$. In other words,

$$S(V) := S(V_{\bar{0}}) \otimes \bigwedge(V_{\bar{1}}).$$

Using the notion of a LCA and a differential algebra, we can introduce Poisson vertex algebras.

Definition 2.5. A quintuple $(\mathcal{P}, 1, \{\,_\lambda\,\}, \partial, \cdot)$ is a *Poisson vertex algebra* (PVA) if

(1) $(\mathcal{P}, \{\,_\lambda\,\}, \partial)$ is a Lie conformal algebra,
(2) $(\mathcal{P}, 1, \cdot, \partial)$ is a supersymmetric differential algebra with the derivation ∂ and the unity 1,
(3) the Leibniz rule holds:

$$\{A_\lambda BC\} = (-1)^{\mathrm{p}(A)\mathrm{p}(B)} B\{A_\lambda C\} + (-1)^{\mathrm{p}(C)(\mathrm{p}(A)+\mathrm{p}(B))} C\{A_\lambda B\}.$$

Example 2.6. Let $\mathfrak{g}$ be a finite simple Lie superalgebra with the even supersymmetric bilinear form $(\,|\,)$. The *affine LCA* of $\mathfrak{g}$ is $R = \mathbb{C}[\partial] \otimes \mathfrak{g}$ with the λ-bracket defined by

$$[a_\lambda b] = [a, b] + \lambda k(a|b), \quad \text{for } a, b \in \mathfrak{g} \text{ and } k \in \mathbb{C},$$

and sesquilinearity. The *affine PVA* of $\mathfrak{g}$ is the (super)symmetric algebra $S(R)$ generated by R endowed with the λ-bracket induced from the bracket of R and the Leibniz rule.

Proposition 2.7. *Let $\mathcal{P}$ be a PVA and $\partial\mathcal{P}$ be the subspace $\{\,\partial p \mid p \in \mathcal{P}\}$ of P. Then the quotient space $\mathcal{P}/\partial\mathcal{P} = \{p + \partial\mathcal{P} | p \in \mathcal{P}\}$ endowed with the bracket*

$$[a + \partial\mathcal{P}, b + \partial\mathcal{P}] := \{a_\lambda b\}|_{\lambda=0} + \partial\mathcal{P} \quad \textit{for } a, b \in \mathcal{P}$$

is a well-defined Lie superalgebra.

Proof. By the sesquilinearity of λ-brackets, we can see $[\partial a, b]$ and $[a, \partial b]$ are in $\partial\mathcal{P}$. Hence it is a well-defined bilinear map $\mathcal{P}/\partial\mathcal{P} \times \mathcal{P}/\partial\mathcal{P} \to \mathcal{P}/\partial\mathcal{P}$. Skew-symmetry and Jacobi identity of $[\,,\,]$ follow from those properties of $\{\,_\lambda\,\}$. □

Definition 2.8. Let $\mathcal{P}$ be a PVA and $H : \mathcal{P} \to \mathcal{P}$ be a diagonalizable operator. Denote by Δ_a the eigenvalue of homogenous element $a \in \mathcal{P}$ with respect to the operator H. If

$$\Delta_1 = 0, \quad \Delta_{\partial a} = \Delta_a + 1, \quad \Delta_{ab} = \Delta_a + \Delta_b, \quad \Delta_{a_{(n)}b} = \Delta_a + \Delta_b - n - 1,$$

for any homogenous elements $a, b \in \mathcal{P}$ then H is called a *Hamiltonian operator* and Δ_a is called the *conformal weight* of a.

Remark 2.9. Let L be an element of a PVA $\mathcal{P}$. If (i) $L_{(0)} = \partial$, (ii) $L_{(1)}$ is a diagonalizable, and

$$\text{(iii) } \{L_\lambda L\} = (\partial + 2\lambda)L + c\lambda^3, \quad \text{for } c \in \mathbb{C}$$

then L is called an energy momentum field of $\mathcal{P}$. By sesquilinearities and Leibniz rules, $L_{(1)}$ is a Hamiltonian operator. By convention, we denote by

$$L_n = L_{(n+1)} \quad \text{for } n \geq -1.$$

The operator $L_0 = L_{(1)}$ is called the Hamiltonian operator induced from the energy momentum field L.

Hamiltonian operators are useful to describe relations between PVAs (VAs) and Poisson algebras (associative algebras).

2.2. Classical affine W-algebras.
There are some ingredients to construct a classical affine W-algebra. From now on, we fix notations to indicate them.

Setup 2.10. *Let $\mathfrak{g}$ be a finite simple Lie superalgebra and (e, h, f) be an even sl_2-triple in $\mathfrak{g}$. Suppose $(\,|\,)$ is the even supersymmetric bilinear form on $\mathfrak{g}$ such that $(e|f) = \frac{1}{2}(h|h) = 1$ and $\mathfrak{g} = \bigoplus_{i=-d}^{d} \mathfrak{g}(i)$ is the $ad\frac{h}{2}$ eigenspace decomposition. We consider two Lie subalgebras $\mathfrak{n}$ and $\mathfrak{m}$:*

$$\mathfrak{n} = \bigoplus_{i>0} \mathfrak{g}(i) = \bigoplus_{i\geq 1/2} \mathfrak{g}(i) \quad \supset \quad \mathfrak{m} = \bigoplus_{i\geq 1} \mathfrak{g}(i).$$

Recall that the grading by adh is called a Dynkin grading, which is an example of a good grading. Hence we have the following properties:

(1) $ad f \, : \bigoplus_{i\geq 1/2} \mathfrak{g}(i) \to \bigoplus_{i\geq -1/2} \mathfrak{g}(i)$ is injective,
(2) $ad f \, : \bigoplus_{i\leq 1/2} \mathfrak{g}(i) \to \bigoplus_{i\leq -1/2} \mathfrak{g}(i)$ is surjective.

By (1) and (2), we have the bijection $ad f \, : \bigoplus_{i=1/2} \mathfrak{g}(i) \to \bigoplus_{i=-1/2} \mathfrak{g}(i)$.

Using notations in Setup 2.10, we define classical affine W-algebras.

Definition 2.11. Let $\mathcal{I}$ be the differential algebra ideal of the affine PVA $\mathcal{P} = S(\mathbb{C}[\partial]\otimes\mathfrak{g})$ generated by $m + (f|m)$ for $m \in \mathfrak{m}$. The *classical affine W-(super)algebra* $\mathcal{W}(\mathfrak{g}, f, k)$ *associated to* $\mathfrak{g}$, f *and* $k \in \mathbb{C}$ is

$$\mathcal{W}(\mathfrak{g}, f, k) = \big(\mathcal{P}/\mathcal{I}\big)^{\mathrm{ad}_\lambda \mathfrak{n}},$$

where $\mathrm{ad}_\lambda \mathfrak{n}$-action on $\mathcal{P}/\mathcal{I}$ is induced from the λ-bracket on $\mathcal{P}$ in Example 2.6. The W-algebra $\mathcal{W}(\mathfrak{g}, f, k)$ is a PVA with the λ-bracket induced from that of $\mathcal{P}$.

Proposition 2.12 [7]. *Suppose there is an even element $s \in \mathfrak{g}(d)$, where d is the largest integer such that $\mathfrak{g}(d) \neq \{0\}$. (See Remark 2.14.) The W-(super)algebra $\mathcal{W}(\mathfrak{g}, f, k)$ in Definition 2.11 is endowed with another Poisson λ-bracket which is induced from the bracket on the affine PVA $\mathcal{P} := S(\mathbb{C}[\partial] \otimes \mathfrak{g})$ defined by*

$$\{a_\lambda b\}_2 = (s|[a, b]). \tag{2.2}$$

Proof. Consider the one parameter family of Poisson λ-brackets on $S(\mathbb{C}[\partial]\otimes\mathfrak{g})$ defined by

$$\{a_\lambda b\}^t = [a, b] + k\lambda(a|b) + t(s|[a, b]), \quad t \in \mathbb{C}. \tag{2.3}$$

Observe that $[s, n] = 0$ and $\{n_\lambda A\} = \{n_\lambda A\}^0$, where $\{\ _\lambda\ \}$ is the Poisson λ-bracket on the affine PVA in Example 2.6. Hence $\{n_\lambda A\}^t = \{n_\lambda A\}$. Thus, for the ideal $\mathcal{I}$ in Definition 2.11, we have

$$\mathcal{W}^t(\mathfrak{g}, f, k) := \{\bar{A} \in \mathcal{P}/\mathcal{I} \,|\, \{n_\lambda A\}^t \in \mathcal{I}\} \simeq \mathcal{W}(\mathfrak{g}, f, k), \quad A \mapsto A,$$

as differential algebras. One can also check that $\mathcal{W}^t(\mathfrak{g}, f, k)$ is a PVA endowed with the λ-bracket induced from (2.3) and extended via Leibniz rules. For $A, B \in \mathcal{W}(\mathfrak{g}, f, k)$, we have

$$\{A_\lambda B\}_2 := \{A_\lambda B\}^{t+1} - \{A_\lambda B\}^t \in \mathcal{W}(\mathfrak{g}, f, k)$$

which defines another λ-bracket on $\mathcal{W}(\mathfrak{g}, f, k)$. The well-definedness of the bracket $\{\ _\lambda\}_2$ can be shown by the master formula, or Proposition 4.4. This bracket can be understood as the bracket induced from (2.2). □

Remark 2.13. In order to distinguish two λ-brackets on $\mathcal{W}(\mathfrak{g}, f, k)$, we denote by $\{\ _\lambda\ \}_1$ or $\{\ _\lambda\ \}$ the bracket in Definition 2.11 and by $\{\ _\lambda\ \}_2$ the bracket in Proposition 2.12.

Remark 2.14. A nonzero even element $s \in \mathfrak{g}(d)$ exists for a subalgebra of $\mathrm{gl}(m|n)$. In [15], Hoyt showed that Dynkin grading on $\mathfrak{g}_{\bar{0}}$ can be extended to $\mathfrak{g}$. For example, in $\mathrm{sl}(m|n)$ case, it can be shown as follows. A Dynkin grading of $\mathrm{gl}(m|n)$ corresponds to a pair $(\lambda|\mu)$ of partitions of m and n. If $\lambda = (p_1, p_2, \ldots, p_{r_0})$ and $\mu = (q_1, q_2, \ldots, q_{r_1})$ are decreasing sequences then the largest numbers $d_{\bar{0}}$ and $d_{\bar{1}}$ such that $\mathfrak{g}_{\bar{0}}(d_{\bar{0}}) \neq \{0\}$ and $\mathfrak{g}_{\bar{1}}(d_{\bar{1}}) \neq \{0\}$ satisfy

$$d_{\bar{0}} = \max\{2(p_1 - 1),\ 2(q_1 - 1)\}, \quad d_{\bar{1}} = (p_1 - 1) + (q_1 - 1).$$

Hence, for $d = \max\{d_{\bar{0}},\ d_{\bar{1}}\} = d_{\bar{0}}$, there exists a nonzero even element $s \in \mathfrak{g}(d)$. Similar argument works in $\mathrm{spo}(m|n)$ cases.

Proposition 2.15 [7]. *Let $\{u_i\}_{i\in I}$ and $\{u^i\}_{i\in I}$ be dual bases of $\mathfrak{g}$ with respect to the bilinear form $(\cdot|\cdot)$ and let $\{v_i\}_{i\in I_{1/2}}$ and $\{v^i\}_{i\in I_{1/2}}$ be the dual bases of $\mathfrak{g}_{1/2}$ with respect to the bilinear form $\omega(\cdot|\cdot)$ on $\mathfrak{g}_{1/2}$ defined by*

$$\omega(a, b) = (f|[a, b]).$$

Then

$$L = \sum_{i\in I} \frac{1}{2k} u^i\, u_i + \sum_{i\in I_{1/2}} \frac{1}{2} \partial(v^i)\, v_i + \frac{\partial h}{2} \tag{2.4}$$

is an energy momentum field of the affine PVA of $\mathfrak{g}$ with the λ-bracket

$$\{a_\lambda b\} = [a, b] + k\lambda(a|b), \quad \textit{for} \quad a, b \in \mathfrak{g}.$$

The conformal weight Δ_a of $a \in \mathfrak{g}(j_a)$ is $1 - j_a$. Moreover, the element $L \in \mathcal{W}(\mathfrak{g}, f, k)$ which is the quotient of L in (2.4) is an energy momentum field of $\mathcal{W}(\mathfrak{g}, f, k)$.

About algebraic structures of classical affine W-algebras, the following proposition can be found in [26]. Also we note that, in [5], there is the analogous result for quantum affine W-algebras.

Proposition 2.16 [7,26]. *Let L be the energy momentum field of $\mathcal{W}(\mathfrak{g}, f, k)$ in (2.4). For the Hamiltonian operator L_0, let Δ_a be the conformal weight of the homogenous element $a \in \mathcal{W}(\mathfrak{g}, f, k)$.*

(1) As differential algebras $\mathcal{W}(\mathfrak{g}, f, k) \simeq S(\mathbb{C}[\partial] \otimes \mathfrak{g}_f)$, where $\mathfrak{g}_f = \ker(ad\, f) \subset \mathfrak{g}$.
(2) Let B_f be a basis of $\mathfrak{g}_f$. There is $\Phi_f = \{\phi_g \,|\, g \in B_f\} \subset \mathcal{W}(\mathfrak{g}, f)$ such that

$$\phi_g = g + \psi_g,$$

where $\Delta_{\psi_g} = \Delta_g$ and ψ_g is an element of the algebra of differential polynomials generated by $\bigoplus_{i>1-\Delta_g} \mathfrak{g}(i)$. Note that $g \in \mathfrak{g}(1 - \Delta_g)$. Moreover,

$$\mathcal{W}(\mathfrak{g}, f, k) = \mathbb{C}_{\textit{diff}}[\phi_g | g \in B_f].$$

Example 2.17. Let $\mathfrak{g} = \mathrm{sl}_2$. Then $\mathfrak{n} = \mathfrak{m} = \mathbb{C}e$ and the ideal $\mathcal{I}$ of $S(\mathbb{C}[\partial] \otimes \mathfrak{g})$ is generated by $e + 1$. As differential algebras,

$$\mathcal{W}(\mathfrak{g}, f, k) \quad = \quad \mathbb{C}_{\text{diff}}[\phi_f],$$

where $\phi_f = f - \frac{1}{2}x^2 - k\partial x$ for $x = \frac{h}{2}$. We can check that

$$\{\phi_f \,{}_\lambda\, \phi_f\} = -k(\lambda + 2\partial)\phi_f - \frac{k^3}{2}\lambda^3.$$

Example 2.18. Let $\mathfrak{g} = \mathrm{spo}(2|1) \subset \mathfrak{gl}(2|1)$. Then the even part $\mathfrak{g}_{\bar{0}}$ is generated by an sl_2-triple (e_{ev}, h, f_{ev}) and the odd part $\mathfrak{g}_{\bar{1}}$ is generated by e_{od} and f_{od}. As matrix forms,

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{ev} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_{ev} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_{od} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad f_{od} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Consider the even supersymmetric invariant bilinear form $(\,|\,)$ such that $(h|h) = 2(e_{ev}|f_{ev}) = 2$ and $(e_{od}|f_{od}) = -2$. There are two elements

$$\phi_{od} := f_{od} - \frac{1}{2}e_{od}h - k\partial e_{od}, \quad \phi_{ev} := f_{ev} + \frac{1}{2}f_{od}e_{od} - \frac{1}{4}h^2 + k\frac{1}{4}e_{od}\partial e_{od} - k\frac{1}{2}\partial h,$$

which satisfy

$$\mathrm{ad}_\lambda e_{ev}(\phi_{od}) = \mathrm{ad}_\lambda e_{od}(\phi_{od}) = \mathrm{ad}_\lambda e_{ev}(\phi_{ev}) = \mathrm{ad}_\lambda e_{od}(\phi_{ev}) = 0 + I.$$

Hence

$$\mathcal{W}(\mathfrak{g}, f_{ev}, k) = \mathbb{C}_{\text{diff}}[\phi_{od}, \phi_{ev}]$$

as a differential algebra. By direct computations, we can check that the λ-bracket of $\mathcal{W}(\mathfrak{g}, f_{ev}, k)$ is defined as follows:

$$\begin{aligned} \{\phi_{od} \,{}_\lambda\, \phi_{od}\} &= -2\phi_{ev} - 2k^2\lambda^2, \\ \{\phi_{ev} \,{}_\lambda\, \phi_{od}\} &= -k(\partial + \frac{3}{2}\lambda)\phi_{od}, \\ \{\phi_{ev} \,{}_\lambda\, \phi_{ev}\} &= -k(\partial + 2\lambda)\phi_{ev} - \frac{k^3}{2}\lambda^3. \end{aligned}$$

2.3. Generalized Drinfeld–Sokolov reductions.
In Sect. 2.3, we recall the construction of classical W-algebras associated to Lie algebras via Drinfeld–Sokolov reductions in PVA theories. For the purpose, we assume $\mathfrak{g}$ is a finite dimensional simple Lie algebra (without odd part), in this subsection.

For a symmetric differential algebra $\mathcal{V}$, the vector space $\mathfrak{g} \otimes \mathcal{V}$ is the Lie algebra endowed with the bracket

$$[a \otimes F, b \otimes G] = [a, b] \otimes FG \quad \text{for } a, b \in \mathfrak{g}, \;\; F, G \in \mathcal{V}.$$

The bilinear form $(\,|\,) : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ on $\mathfrak{g}$ can be extended to the map $(\,|\,) : \mathfrak{g}\otimes\mathcal{V}\times\mathfrak{g}\otimes\mathcal{V} \to \mathcal{V}$ by

$$(\, a \otimes F \mid b \otimes G\,) = (a|b)FG.$$

The derivation $\partial : \mathcal{V} \to \mathcal{V}$ on $\mathcal{V}$ can be extended to an endomorphism on $\mathfrak{g} \otimes \mathcal{V}$ such that

$$\partial(a \otimes F) = a \otimes \partial F.$$

Consider $\mathbb{C}\partial$ as the trivial one dimensional Lie algebra. Then $\mathbb{C}\partial \ltimes \mathfrak{g}\otimes\mathcal{V}$ is the semidirect product of Lie algebras $\mathbb{C}\partial$ and $\mathfrak{g} \otimes \mathcal{V}$ endowed with the bracket

$$[\, c_1\partial + a \otimes F,\ c_2\partial + b \otimes G\,] = c_1(b \otimes \partial G) - c_2(a \otimes \partial F) + [a \otimes F, b \otimes G]. \quad (2.5)$$

Recall the notation $\mathfrak{m} = \bigoplus_{i\geq 1} \mathfrak{g}(i)$ in Setup 2.10. If we denote by $V^{\perp} := \{\, w \in \mathfrak{g} \mid (w|v) = 0 \text{ for any } v \in V\,\}$ for a subset $V \subset \mathfrak{g}$ then

$$\mathfrak{m}^{\perp} = \bigoplus_{i>-1} \mathfrak{g}(i) = \bigoplus_{i\geq -1/2} \mathfrak{g}(i). \quad (2.6)$$

Let us consider the subspace

$$\mathfrak{p} = \bigoplus_{i<1} \mathfrak{g}(i) = \bigoplus_{i\leq 1/2} \mathfrak{g}(i) \quad (2.7)$$

of $\mathfrak{g}$. Then we have

$$\text{(i) } \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}, \quad \text{(ii) } \mathfrak{p} \simeq \mathfrak{m}^{\perp} \text{ by the bilinear form } (\,|\,).$$

Definition 2.19. Let $\mathfrak{p}$ be defined as (2.7) and let $\mathcal{V}(\mathfrak{p}) := S(\mathbb{C}[\partial] \otimes \mathfrak{p})$.

(1) Let $\mathcal{F}_{\mathfrak{g},f}$ be the set of elements

$$q = \sum_{i\in I_{\mathfrak{p}}} q_i \otimes P^i \in \mathfrak{m}^{\perp} \otimes \mathcal{V}(\mathfrak{p}) \quad (2.8)$$

for a basis $\{\, q_i \mid i \in I_{\mathfrak{p}}\,\}$ of $\mathfrak{m}^{\perp}$ and a subset $\{\, P^i \mid i \in I_{\mathfrak{p}}\,\} \subset \mathcal{V}(\mathfrak{p})$. The set $\mathcal{F}_{\mathfrak{g},f}$ is called the phase space associated to $\mathfrak{g}$ and f.

(2) For a given q and $k \in \mathbb{C}$, the operator L of the form

$$L = k\partial + q - f \otimes 1 \in \mathbb{C}\partial \ltimes \mathfrak{g} \otimes \mathcal{V}(\mathfrak{p}). \quad (2.9)$$

is called a *Lax operator*

(3) The *gauge transformation* of $q \in \mathcal{F}_{\mathfrak{g},f}$ with $A \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ is $q^A \in \mathcal{F}_{\mathfrak{g},f}$ where

$$e^{\mathrm{ad} A}(k\partial + q - f \otimes 1) = k\partial + q^A - f \otimes 1.$$

On the other hand, for $q' \in \mathcal{F}_{\mathfrak{g},f}$, if there is $A \in \mathfrak{n} \otimes \mathcal{V}(\mathfrak{p})$ such that $q' = q^A$ then we say they are *gauge equivalent* and write $q \sim q'$. Note that it is not hard to check that the equivalence relation is well-defined in $\mathcal{F}_{\mathfrak{g},f}$.

Elements in the differential algebra $\mathcal{V}(\mathfrak{p})$ can be identified with functions $\mathcal{F}_{\mathfrak{g},f} \to \mathcal{V}(\mathfrak{p})$ as follows:

$$p(a\otimes F) = (p|a)F, \quad PQ(a\otimes F) = P(a\otimes F)Q(a\otimes F), \quad \partial P(a\otimes F) = \partial(P(a\otimes F)), \quad (2.10)$$

for $p \in \mathfrak{p}$, $P, Q \in \mathcal{V}(\mathfrak{p})$ and $a \otimes F \in \mathcal{F}_{\mathfrak{g},f}$. An element $P \in \mathcal{V}(\mathfrak{p})$ is said to be a *gauge invariant function* if $P(q) = P(q')$ whenever $q \sim q'$ for $q, q' \in \mathcal{F}_{\mathfrak{g},f}$.

Proposition 2.20 [7]. *The set $\mathcal{W}$ of gauge invariant functions in $\mathcal{V}(\mathfrak{p})$ is a differential subalgebra of $\mathcal{V}(\mathfrak{p})$. Moreover, $\mathcal{W}$ is isomorphic to the classical affine W-algebra $\mathcal{W}(\mathfrak{g}, f, k)$ associated to $\mathfrak{g}$ and f as differential algebras.*

Remark 2.21. In [12], a pair of local Poisson structures in $\mathcal{W}$ is described by a Lax operator. The Poisson structures are equivalent to the PVA structures on the classical affine W-algebra $\mathcal{W}(\mathfrak{g}, f, k)$ which are induced from those in the affine PVA $S(\mathbb{C}[\partial]\otimes\mathfrak{g})$. (See Definition 2.11 and Proposition 2.12.)

The construction of W-algebras in Proposition 2.20 allows to compute generators of the algebras (see Theorem 2.23).

Lemma 2.22 [7]. *Let V be a subspace of $\mathfrak{m}^\perp$ such that $\mathfrak{m}^\perp = [\mathfrak{n}, f] \oplus V$. Take a basis $\{v_i\}_{i\in I_\mathfrak{p}}$ of $\mathfrak{m}^\perp$ such that $\{v_i\}_{i\in J\subset I_\mathfrak{p}}$ is a basis of V and $\{v_i\}_{i\in I_\mathfrak{p}\setminus J}$ is a basis of $[\mathfrak{n}, f]$. If $\{v^i\}_{i\in I_\mathfrak{p}}$ is the dual basis of $\mathfrak{p}$ then $\{v^i\}_{i\in J}$ is a basis of $\mathfrak{g}^f := ker(ad f)$.*

Theorem 2.23 [7].

(1) Let $\{q_i\}_{i\in I_\mathfrak{p}}$ and $\{q^i\}_{i\in I_\mathfrak{p}}$ be bases of $\mathfrak{m}^\perp$ and $\mathfrak{p}$ such that $(q_i|q^j) = \delta_{ij}$. Denote by

$$q_{univ} = \textstyle\sum_{i\in I_\mathfrak{p}} q_i \otimes q^i \quad \text{and} \quad \mathcal{L} = k\partial + q_{univ} - f\otimes 1. \tag{2.11}$$

Then there is unique $X \in \mathfrak{n}\otimes\mathcal{V}(\mathfrak{p})$ such that $q^X_{univ} \in V\otimes\mathcal{V}(\mathfrak{p})$ satisfies

$$e^{adX}\mathcal{L} = k\partial + q^X_{univ} - f\otimes 1. \tag{2.12}$$

(2) As in Lemma 2.22, let $\{q_i\}_{i\in J\subset I_\mathfrak{p}}$ be a basis of V. If

$$q^X_{univ} = \textstyle\sum_{i\in J} q_i \otimes w_i$$

for q^X_{univ} in (1) then w_i are gauge invariant functions in $\mathcal{V}(\mathfrak{p})$. Moreover, by Lemma 2.22, we have $w_i = v^i + (\text{degree} \geq 2 \text{ part})$.

(3) The set of gauge invariant functions in $\mathcal{V}(\mathfrak{p})$ is the algebra of differential polynomials

$$\mathbb{C}_{diff}[\, w_i \mid i\in J\,].$$

Remark 2.24. We have the differential algebra isomorphism $\mathcal{V}(\mathfrak{p}) \simeq S(\mathbb{C}[\partial]\otimes\mathfrak{g})/\mathcal{I} =: \mathcal{V}_\mathcal{I}(\mathfrak{p})$, $A \mapsto \bar{A}$, where $\mathcal{I}$ is the ideal defined in Definition 2.11. Due to the isomorphism,

(1) we can consider a Lax operator L an element in $\mathbb{C}\partial \ltimes \mathfrak{m}^\perp \otimes \mathcal{V}_\mathcal{I}(\mathfrak{p})$,
(2) since the W-algebra $\mathcal{W}(\mathfrak{g}, f, k)$ is a subalgebra of $\mathcal{V}_\mathcal{I}(\mathfrak{p})$, we prefer to regard $\mathcal{W}(\mathfrak{g}, f, k)$ as a set of functions from $\mathcal{F}_{\mathfrak{g},f}$ to $\mathcal{V}_\mathcal{I}(\mathfrak{p})$.

Theorem 2.25 [7]. *The W-algebra $\mathcal{W}(\mathfrak{g}, f, k)$ is the set of gauge invariant functions in $\mathcal{V}_\mathcal{I}(\mathfrak{p})$. Hence we can find free generators by Theorem 2.23.*

The following is the simplest example of classical affine W-algebras.

Example 2.26. Let $\mathfrak{g} = \mathrm{sl}_2$. Then $q_{\mathrm{univ}} = e\otimes f + h\otimes x$ for $x = \frac{h}{2}$ and

$$\mathcal{L} = k\partial + q_{\mathrm{univ}} - f\otimes 1.$$

If we take $X = e\otimes x$ then $q^X_{\mathrm{univ}} = e\otimes(f - x^2 - k\partial x)$. Hence $\mathbb{C}_{\mathrm{diff}}[f - x^2 - k\partial x]$ is the set of gauge invariant functions. Indeed, we can check that $\phi_f := f - x^2 - k\partial x$ is a gauge invariant function as follows:

Let $Y = e\otimes r$ for $r\in\mathcal{V}_\mathcal{I}(\mathfrak{p})$. Then $q^Y = h\otimes(x-r) + e\otimes(f - k\partial r - 2rx + r^2)$ and

$$\phi_f(q^Y) = (f - k\partial r - 2rx + r^2) - (x-r)^2 - k\partial(x-r) = f - x^2 - k\partial x$$

which is independent on r. Also, we can check that the algebra of differential polynomials $\mathbb{C}_{\mathrm{diff}}[\phi_f]$ is isomorphic to the W-algebra $\mathcal{W}(\mathfrak{g}, f, k)$ in Example 2.17.

2.4. Integrable Hamiltonian systems associated to W-algebras.
Integrable Hamiltonian systems can be investigated by Poisson vertex algebras theories [3]. In this subsection, we briefly review basic notions related to integrable Hamiltonian systems.

Definition 2.27. Let $\mathcal{P}$ be an (even) algebra of differential polynomials with a PVA structure.

(1) An evolution equation is called a *Hamiltonian system* on $\mathcal{P}$ if there is $h \in \mathcal{P}$ such that

$$\frac{du}{dt} = \{h_\lambda u\}|_{\lambda=0}, \quad \text{for } u \in \mathcal{P}.$$

(2) Consider the quotient map $\int : \mathcal{P} \to \mathcal{P}/\partial\mathcal{P}$. The image $\int f$ of $f \in \mathcal{P}$ is called a *local functional.*
(3) A Hamiltonian system is called an *integrable system* if there are infinitely many linearly independent integrals of motion $\int h_i$, $i \in \mathbb{Z}_{\geq 0}$. Here, an integral of motion $\int h_i$ is a local functional such that $\int \frac{dh_i}{dt} = 0$.

In the rest of this subsection, consider the Laurent series $\mathfrak{g}((z^{-1}))$ with the Lie bracket

$$[az^n, bz^m] = [a, b]z^{n+m} \quad \text{for } a, b \in \mathfrak{g}.$$

Recall $\mathcal{W}(\mathfrak{g}, f, k)$ is endowed with a bi-Poisson λ-bracket which is induced from that on $S(\mathbb{C}[\partial] \otimes \mathfrak{g})$:

$$\{a_\lambda b\}_1 = [a, b] + k\lambda(a|b), \quad \{a_\lambda b\}_2 = (s|[a, b]). \tag{2.13}$$

Remark 2.28. (Lenard–Magri Scheme) Let $\mathcal{P}$ be a PVA with the bi-Poisson λ-bracket $(\{\ {}_\lambda\ \}_1, \{\ {}_\lambda\ \}_2)$. Suppose there is a sequence of linearly independent local functionals $\int h_i \in \mathcal{P}/\partial\mathcal{P}$, $i = 0, 1, 2, \ldots$ such that

(i) $\{h_{0\,\lambda}\,\mathcal{P}\}_2|_{\lambda=0} = 0$, (ii) $\{h_{i\,\lambda}\, p\}_1|_{\lambda=0} = \{h_{i+1\,\lambda}\, p\}_2|_{\lambda=0}$ for $i \geq 0$ and $p \in \mathcal{P}$.

Then $\frac{du}{dt} = \{h_{i\,\lambda}\, u\}_K|_{\lambda=0}$ for $i = 0, 1, 2, \ldots$ are Hamiltonian integrable systems.

Theorem 2.29 [7]. *Suppose $\Lambda := f + sz \in \mathfrak{g}((z^{-1}))$ is semisimple for $s \in \ker(ad\,\mathfrak{n})$. There is a sequence of integrable systems on $\mathcal{W}(\mathfrak{g}, f, k)$ which satisfies the assumptions of Lenard–Magri scheme. More specifically, consider*

$$\mathcal{L}(\Lambda) := k\partial + q_{univ} - \Lambda \otimes 1 = \mathcal{L} - zs \otimes 1 \in \mathbb{C}\partial \ltimes \mathfrak{g}((z^{-1})) \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$$

and take $h(z) \in \big(\ker(ad\Lambda) \cap \mathfrak{g}[[z^{-1}]]\big) \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ such that $e^{adS(z)}\mathcal{L}(\Lambda) = k\partial + h(z) + \Lambda \otimes 1$ for some $S(z) \in \mathfrak{g}[[z^{-1}]] \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. Then $h_i = (z^i \Lambda \otimes 1|h(z))$ is an element in $\mathcal{W}(\mathfrak{g}, f, k)$ and the Hamiltonian equation

$$\frac{du}{dt} = \{h_{i\,\lambda}\, u\}_H|_{\lambda=0}$$

is an integrable system on $\mathcal{W}(\mathfrak{g}, f, k)$. Here, the bilinear form on $\mathfrak{g}((z^{-1})) \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ is defined by $(az^n \otimes F|bz^m \otimes G) = (a|b)FG\delta_{n+m,0}$ for $a, b \in \mathfrak{g}$ and $F, G \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$.

Remark 2.30. [7] It is a natural question to ask that if we can find a semisimple element $\Lambda = f + zs$ for a given nilpotent element f. In the case when $\mathfrak{g} = \mathrm{sl}_n$, if f corresponds to one of the following partitions λ of n then we can find such a semisimple element Λ.

(1) $\lambda = (r, r, \ldots, r, 1, 1, \ldots, 1)$, (2) $\lambda = (r, r-1, r, r-1, \ldots, r, r-1, 1, 1, \ldots, 1)$.

3. Classical Affine W-Superalgebras and Generalized Drinfel'd–Sokolov Reductions

In this section, we shall show a set of generators of a classical affine W-superalgebra as a superalgebra of differential polynomials can be obtained by an analogous method to the generalized Drinfeld–Sokolov reduction.

Let $\mathfrak{g}$ be a simple Lie superalgebra with a nondegenerate invariant even supersymmetric bilinear form $(\,|\,)$ and let $\mathcal{V}$ be a supersymmetric differential superalgebra. The vector superspace $\mathfrak{g} \otimes \mathcal{V}$ is endowed with the Lie bracket and the bilinear form defined by

$$[\,a \otimes F,\ b \otimes G] = (-1)^{\mathrm{p}(b)\mathrm{p}(F)}[a, b] \otimes FG,$$
$$(\,a \otimes F \,|\, b \otimes G\,) = (-1)^{\mathrm{p}(b)\mathrm{p}(F)}(a|b)FG$$

for the homogeneous elements $a, b \in \mathfrak{g}$ and $F, G \in \mathcal{V}$.

Due to the invariance of the bilinear form on $\mathfrak{g}$, we get the invariance of the bilinear form $(\,|\,)$ on $\mathfrak{g} \otimes \mathcal{V}$

$$(\,a \otimes F \,|\, [b \otimes G, c \otimes H]\,) = (\,[a \otimes F, b \otimes G] \,|\, c \otimes H\,),$$

for $a, b, c \in \mathfrak{g}$ and $F, G, H \in \mathcal{V}$.

Let us consider an even derivation $\partial : \mathcal{V} \to \mathcal{V}$ on $\mathcal{V}$. Then it can be extended to the map on $\mathfrak{g} \otimes \mathcal{V}$ by $\partial(a \otimes F) = a \otimes \partial F$. The Lie superalgebra

$$\mathbb{C}\partial \ltimes (\mathfrak{g} \otimes \mathcal{V})$$

is the semidirect product of the trivial Lie algebra $\mathbb{C}\partial$ and the Lie superalgebra $\mathfrak{g} \otimes \mathcal{V}$.

Suppose the Lie superalgebra $\mathfrak{g}$ has an sl_2-triple (e, h, f) with the even supersymmetric bilinear form $(\,|\,)$ such that $(e|f) = \frac{1}{2}(h|h) = 1$. As in Sect. 2.3, let $\mathfrak{m} = \bigoplus_{i \geq 1} \mathfrak{g}(i)$, $\mathfrak{m}^\perp = \bigoplus_{i > -1} \mathfrak{g}(i)$ and $\mathfrak{p} = \bigoplus_{i < 1} \mathfrak{g}(i)$, where $\mathfrak{g}(i)$ is the $\mathrm{ad}\frac{h}{2}$ eigenspace with the eigenvalue i. Recall that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}$ and $\mathfrak{p} \simeq \mathfrak{m}^\perp$ as vector superspaces via the bilinear form $(\,|\,)$ on $\mathfrak{g}$. For the superspace $\mathfrak{m}^\perp = \mathfrak{m}^\perp_{\bar{0}} \oplus \mathfrak{m}^\perp_{\bar{1}}$, there is a basis $\{\, q_i \mid i \in I := I_{\bar{0}} \cup I_{\bar{1}} \,\}$ of $\mathfrak{m}^\perp$ such that

$$\text{(i) } \{\, q_i \mid i \in I_{\bar{0}} \,\} \text{ is a basis of } \mathfrak{m}^\perp_{\bar{0}}, \quad \text{(ii) } \{\, q_i \mid i \in I_{\bar{1}} \,\} \text{ is a basis of } \mathfrak{m}^\perp_{\bar{1}}. \tag{3.1}$$

Definition 3.1. Let $\mathcal{V}(\mathfrak{p}) := S(\mathbb{C}[\partial] \otimes \mathfrak{p})$ be the differential superalgebra generated by the vector superspace $\mathfrak{p}$. A *Lax operator* L is an even element in $\mathbb{C}\partial \ltimes \mathfrak{g} \otimes \mathcal{V}(\mathfrak{p})$ such that

$$L = k\partial + \textstyle\sum_{i \in I_{\bar{0}}} q_i \otimes P^i + \sum_{i \in I_{\bar{1}}} q_j \otimes P^j - f \otimes 1 \in \mathbb{C}\partial \ltimes \big(\mathfrak{g} \otimes \mathcal{V}(\mathfrak{p})\big)_{\bar{0}},$$

where $q = \sum_{i \in I_{\bar{0}}} q_i \otimes P^i + \sum_{i \in I_{\bar{1}}} q_j \otimes P^j \in \big(\mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p})\big)_{\bar{0}}$.

Remark 3.2. Let $\mathcal{I}$ be the differential superalgebra ideal of $\mathcal{V}(\mathfrak{g})$ generated by $\{m + (f|m) | m \in \mathfrak{m}\}$. Denote $\mathcal{V}_{\mathcal{I}}(\mathfrak{p}) := \mathcal{V}(\mathfrak{g})/\mathcal{I}$. Then we can check the following facts:

(1) $\mathcal{V}(\mathfrak{p}) \simeq \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ as differential superalgebras by the canonical isomorphism $\iota : \mathcal{V}(\mathfrak{p}) \to \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. If there is no danger of confusion then we denote $\iota(P) \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ by P.

(2) We can regard Lax operators as elements in $\mathbb{C}\partial \ltimes \big(\mathfrak{m}^\perp \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})\big)_{\bar{0}}$ via the isomorphism ι in (1).

Recall that the W-superalgebra $\mathcal{W}(\mathfrak{g}, f, k)$ is a subset of $\mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. In order to see the relation between W-superalgebras and Lax operators, we use (2) in Remark 3.2.

A Lax operator L acts on $\mathfrak{g} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ by

$$L(a \otimes F) := [L, a \otimes F].$$

Consider the phase space

$$\mathcal{F}_{\mathfrak{g},f} := \left(\mathfrak{m}^{\perp} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})\right)_{\bar{0}}.$$

Then for any $q \in \mathcal{F}_{\mathfrak{g},f}$, there is the corresponding Lax operator $L = k\partial + q - f \otimes 1$. Note that, for a Lax operator L and an element $X \in \left(\mathfrak{n} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})\right)_{\bar{0}}$, there is $q^X \in \mathcal{F}_{\mathfrak{g},f}$ such that

$$e^{\mathrm{ad}X} L = e^{\mathrm{ad}X}(k\partial + q - f \otimes 1) = k\partial + q^X - f \otimes 1. \tag{3.2}$$

Hence $e^{\mathrm{ad}X} L$ is again a Lax operator.

Definition 3.3. (1) Let $q \in \mathcal{F}_{\mathfrak{g},f}$ and $X \in \left(\mathfrak{n} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})\right)_{\bar{0}}$. Then $q^X \in \mathcal{F}_{\mathfrak{g},f}$ defined as in (3.2) is said to be the *gauge transformation* of $q \in \mathcal{F}_{\mathfrak{g},f}$ by X.

(2) For two elements $q, q' \in \mathcal{F}_{\mathfrak{g},f}$, if there is an element $Y \in \left(\mathfrak{n} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})\right)_{\bar{0}}$ such that $q^Y = q'$ then we say q and q' are *gauge equivalent* and write $q \sim q'$.

(3) The *universal Lax operator* associated to $\mathfrak{g}$ and f is

$$\mathcal{L} = k\partial + q_{\mathrm{univ}} - 1 \otimes f = k\partial + \textstyle\sum_{i \in I = I_{\bar{0}} \cup I_{\bar{1}}} q_i \otimes q^i - f \otimes 1, \tag{3.3}$$

where $\{q_i\}_{i \in I}$ and $\{q^i\}_{i \in I}$ are bases of $\mathfrak{m}^{\perp}$ and $\mathfrak{p}$, such that $(q_i | q^j) = \delta_{ij}$.

Remark 3.4. Since the bilinear form $(\,|\,)$ defined on the Lie superalgebra $\mathfrak{g}$ is *even*, the universal Lax operator in (3.3) is even. Hence it is a Lax operator.

Now we identify an element in $\mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ with a linear map $\partial \ltimes (\mathfrak{g} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}} \to \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ defined by (3.4) and (3.5):

$$p(\partial) = 0, \quad c(q) = c, \quad p(a \otimes F) = F(a|p) = (a|p)F, \tag{3.4}$$

for $c \in \mathbb{C}$, $p \in \mathfrak{p} \subset \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ and $a \otimes F \in (\mathfrak{g} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}}$. For $P, Q \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$, we have

$$PQ(a \otimes F) = P(a \otimes F)Q(a \otimes F), \quad \partial P(a \otimes F) = \partial(P(a \otimes F)). \tag{3.5}$$

Remark 3.5. If $\mathfrak{g}$ is even then $p(a \otimes F) = (p|a)F = (a|p)F$. Hence (3.4) and (3.5) define the same functions as those in (2.10). If $\mathfrak{g}$ is not even and $a \otimes F$ is an even element in $\mathfrak{g} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ then

$$p(a \otimes F) = (a|p)F = (-1)^{\mathrm{p}(p)}(p|a)F.$$

Here, the last equality holds since $(p|a) \neq 0$ implies $\mathrm{p}(p) = \mathrm{p}(a)$. The reason we consider the definition $p(a \otimes F) := (a|p)F$ instead of $p(a \otimes F) := (p|a)F$ can be explained by the proof of Lemma 3.8 and Proposition 3.9.

Definition 3.6. A function $P \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ is said to be *gauge invariant* if $P(q) = P(q')$ for any gauge equivalent elements q and q' in $\mathcal{F}_{\mathfrak{g},f}$.

Proposition 3.7. *The subset of $\mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ consisting of gauge invariant functions is a differential superalgebra .*

Proof. It is clear that if $P, Q \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ are gauge invariant then PQ and ∂P are also gauge invariant. □

We note that

$$P(\mathcal{L}) = P \quad \text{for } P \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \tag{3.6}$$

since an element $p \in \mathfrak{p} \subset \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ has properties $(f|p) = 0$ and $p(\partial) = 0$ so that

$$p(\mathcal{L}) = p(q_{\text{univ}}) = \textstyle\sum_{i \in I_{\bar{0}} \cup I_{\bar{1}}} q^i (q_i|p) = p.$$

Also, a Lax operator $L = k\partial + Q - f \otimes 1$ with $Q = \sum_{i \in I} q_i \otimes Q^i \in \mathcal{F}_{\mathfrak{g},f}$ satisfies

$$P(Q) = P(L) = P(\mathcal{L})|_{q^i = Q^i} \quad \text{for any } P \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p}). \tag{3.7}$$

Here, the subscript $q^i = Q^i$ means that we substitute q^i by Q^i.

Now, the following lemma is useful to see detailed computations in the proof of Proposition 3.9.

Lemma 3.8. *Let $X = n \otimes r \in (\mathfrak{n} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}}$ and $p \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. Then we have*

$$p([X, \mathcal{L}]) = -k\partial r(n|p) - r[n, p] \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p}).$$

Proof. We have

$$[X, \mathcal{L}] = -n \otimes k\partial r + [n \otimes r, q_{\text{univ}} - f \otimes 1] = -n \otimes k\partial r - \textstyle\sum_{i \in I} [q_i, n] \otimes r q^i - [n, f] \otimes r \tag{3.8}$$

since $\mathrm{p}(q_i) = \mathrm{p}(p^i)$ and $\mathrm{p}(n) = \mathrm{p}(r)$. Hence

$$\begin{aligned} p([X, \mathcal{L}]) &= -k\partial r(n|p) - r\, q^i\, (\,[q_i, n]\,|\,p\,) + r\,(\,[n, f]\,|\,p\,) \\ &= -k\partial r(n|p) - r[n, p] \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p}). \end{aligned} \tag{3.9}$$

□

Proposition 3.9. *If $W \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ is a gauge invariant function then $W \in \mathcal{W}(\mathfrak{g}, f, k)$.*

Proof. Let $X = n \otimes r \in (\mathfrak{n} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}}$ and $\epsilon \in \mathbb{C}$. For $W \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$, we denote

$$W(\mathcal{L} + \epsilon[X, \mathcal{L}] + \tfrac{1}{2}\epsilon^2[X, [X, \mathcal{L}]] + \cdots) = \textstyle\sum_{t \geq 0} \epsilon^t P_t^W. \tag{3.10}$$

for some $P_t^W \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$.

If W is a gauge invariant function, we have

$$W(e^{\mathrm{ad}\epsilon X}(\mathcal{L})) = W \tag{3.11}$$

for any $\epsilon \in \mathbb{C}$. Since (3.11) implies $P_t^W = 0$ for any $t \geq 1$, it is enough to show that

$$P_1^W = 0 \text{ implies } W \in \mathcal{W}(\mathfrak{g}, f, k).$$

In order to show that $P_1^W = 0$, let us denote by $[A_\lambda B]_{\mathcal{I}} \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})[\lambda]$, for $A, B \in \mathcal{V}(\mathfrak{g})$, the image of $[A_\lambda B] \in \mathcal{V}(\mathfrak{g})[\lambda]$. In other words,

$$[A_\lambda B]_{\mathcal{I}} = [A_\lambda B] + \mathcal{I}[\lambda]. \tag{3.12}$$

Note that $[\ {}_\lambda\ ]_{\mathcal{I}}$ induces the well-defined λ-bracket on $\mathcal{W}(\mathfrak{g}, f, k)$ since

$$[n_\lambda P(m + (f|m))]_{\mathcal{I}} = 0 + \mathcal{I}[\lambda] \ \text{ for } \ P \in \mathcal{V}(\mathfrak{g}) = S(\mathbb{C}[\partial] \otimes \mathfrak{g}) \text{ and } m \in \mathfrak{m} \tag{3.13}$$

so that $[n_\lambda \mathcal{I}] = \mathcal{I}[\lambda]$.

By the definition of $\mathcal{W}(g, f, k)$, an element $W \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ is in $\mathcal{W}(\mathfrak{g}, f, k)$ if and only if $[n_\lambda \widetilde{W}]_{\mathcal{I}} = 0$, where $\widetilde{W}$ is an element in $\mathcal{V}(\mathfrak{p})$ such that $\iota(\widetilde{W}) = W$ for the map ι in Remark 3.2. Thus, it is enough to show that

$$P_1^W = -\sum_{t\geq 0} (\partial^t r) P_{1t}^W \quad \text{if and only if} \quad [n_\lambda \widetilde{W}]_{\mathcal{I}} = \sum_{t\geq 0} \lambda^t \cdot P_{1t}^W. \tag{3.14}$$

Now, observe the following facts.

(i) If we substitute W with $p \in \mathfrak{p} \subset \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ in (3.10) then, by Lemma 3.8, we have $P_1^W = -k\partial r(n|p) - r[n, p] \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. On the other hand, take $\widetilde{W} \in \mathcal{V}(\mathfrak{p})$ such that $\iota(\widetilde{W}) = p$. Since $[n_\lambda \widetilde{W}]_{\mathcal{I}} = [n, p] + k\lambda(n|p) \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})[\lambda]$, we have (3.14).

(ii) If $W = \partial^m p$ then $P_1^W = -\partial^m(k\partial r(n|p) - r[n, p])$. In this case, the sesquilinearity $[n_\lambda \partial^m p] = (\lambda + \partial)^m [n_\lambda p]$ implies (3.14).

(iii) Suppose $W = BC$ for homogeneous elements $B, C \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. Since

$$W(e^{\mathrm{ad}\epsilon X}\mathcal{L}) = B(e^{\mathrm{ad}\epsilon X}\mathcal{L}) \cdot C(e^{\mathrm{ad}\epsilon X}\mathcal{L}),$$

we have $P_1^W = BP_1^C + P_1^B C$. Denote $P_1^B = \sum_{t\geq 0}(\partial^t r) P_{1t}^B$ and $P_1^C = \sum_{t\geq 0}(\partial^t r) P_{1t}^C$. Then

$$P_1^W = \sum_{t\geq 0} \big[(-1)^{\mathrm{p}(r)\mathrm{p}(B)} (\partial^t r) B\, P_{1t}^C + (\partial^t r) P_{1t}^B\, C\big]. \tag{3.15}$$

On the other hand, by the Leibniz rule, we have

$$\begin{aligned}[n_\lambda \widetilde{W}]_{\mathcal{I}} = [n_\lambda \widetilde{B}\widetilde{C}]_{\mathcal{I}} &= (-1)^{\mathrm{p}(r)\mathrm{p}(B)} B[n_\lambda \widetilde{C}]_{\mathcal{I}} + [n_\lambda \widetilde{B}]_{\mathcal{I}} C \\ &= -\sum_{t\geq 0} \big[(-1)^{\mathrm{p}(r)\mathrm{p}(B)} B\, \lambda^t P_{1t}^C + \lambda^t P_{1t}^B\, C\big],\end{aligned} \tag{3.16}$$

for $\widetilde{B}, \widetilde{C} \in \mathcal{V}(\mathfrak{p})$ such that $\iota(\widetilde{B}) = B$ and $\iota(\widetilde{C}) = C$. It is not hard to see (3.15) and (3.16) imply (3.14).

By (i), (ii), (iii) and the induction, we prove the proposition. □

Proposition 3.10. *If $W \in \mathcal{W}(\mathfrak{g}, f, k)$ then W is a gauge invariant function in $\mathcal{V}_{\mathcal{I}}(\mathfrak{p})$.*

Proof. Let W be an element in $\mathcal{W}(\mathfrak{g}, f, k)$. Note that

(i) in the proof of Proposition 3.9, we showed that $W \in \mathcal{W}(\mathfrak{g}, f, k)$ if and only if $W([X, \mathcal{L}]) = 0$ for any $X \in (\mathfrak{n} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}}$,

(ii) for $P \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ and a Lax operator $L = k\partial + \sum_{i\in I} q_i \otimes Q^i - f \otimes 1$, we have

$$P([X, L]) = P([X, \mathcal{L}])|_{q^i = Q^i}.$$

Hence $W([X, L]) = 0$ for any $X \in (\mathfrak{n} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}}$ and any Lax operator L. Moreover, since $\mathrm{ad}^{n-1} X(L)$ is a Lax operator for $n \geq 1$, we have $W(\mathrm{ad}^n X(L)) = 0$. Thus,

$$W(e^{\mathrm{ad} X}(L)) = W(L),$$

which means that W is gauge invariant. □

Theorem 3.11. *The set of gauge invariant functions in $\mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ is the classical affine W-superalgebra associated to $\mathfrak{g}$ and f.*

Proof. It directly follows from Propositions 3.9 and 3.10. □

Now the following propositions are useful to find generators of W-superalgebras.

Proposition 3.12. *Let us fix an ad h-invariant complementary subspace $V_f \subset \mathfrak{g}$ of $[f, \mathfrak{n}]$ in $\mathfrak{m}^{\perp}$:*

$$\mathfrak{m}^{\perp} = V_f \oplus [f, \mathfrak{n}].$$

Consider a Lax operator $L = k\partial + Q - f \otimes 1$ for $Q \in \mathcal{F}_{\mathfrak{g},f}$. Then there exists unique $Q^{can} \in V_f \otimes (\mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}}$ and unique $X \in (\mathfrak{n} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}}$ such that

$$e^{adX} L = k\partial + Q^{can} - f \otimes 1. \tag{3.17}$$

Proof. We can write $Q = \sum_{i \geq -\frac{1}{2}} Q_i$ where $Q_i \in \mathfrak{g}(i) \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. Similarly, let $X = \sum_{i \geq \frac{1}{2}} X_i$ and $Q^{\text{can}} = \sum_{i \geq 0} Q_i^{can}$ for $X_i, Q_i^{\text{can}} \in \mathfrak{g}(i) \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. Then the ad $\frac{h}{2}$-decomposition of (3.17) implies the following equalities:

$$\begin{aligned} &Q_{-1/2} + [X_{1/2}, -f \otimes 1] = 0, \\ &Q_0 + [X_1, -f \otimes 1] + [X_{1/2}, Q_{-1/2}] = Q_0^{\text{can}}, \\ &Q_{1/2} + [X_{3/2}, -f \otimes 1] + [X_1, Q_{-1/2}] + [X_{1/2}, \partial + Q_0] = Q_{1/2}^{\text{can}}, \\ &\vdots \end{aligned} \tag{3.18}$$

Then we can determine $X_{1/2}$ uniquely by the first equation in (3.18) and it is even. Also, X_1 and Q_0^{can} can be uniquely determined by $X_{1/2}$ and the second equation in (3.18). Since $[X_{1/2}, Q_{-1/2}] + Q_0$ is even, both X_1 and Q_0^{can} are even. The Inductively, even elements X_{i+1} and Q_i^{can} are determined uniquely by $X_{j+1}, Q_j^{\text{can}}$ for $j < i$ and (3.18). □

Lemma 3.13. *Let us take the universal Lax operator $\mathcal{L} = k\partial + q_{univ} - 1 \otimes f$ in Proposition 3.12 and let $\{ q_i \mid i \in \mathcal{J} \}$ be a basis of V_f. If we denote $q_{univ}^{can} = \sum_{i \in \mathcal{J}} q_i \otimes w^i$ then we have the following properties.*

(1) For any $q \in \mathcal{F}_{\mathfrak{g},f}$, we have

$$q^{can} = \textstyle\sum_{i \in \mathcal{J}} q_i \otimes (w^i(q)). \tag{3.19}$$

(2) Let $P \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ and $q \in \mathcal{F}_{\mathfrak{g},f}$. Then we have $P(q) = P(q^{can})$ if and only if $P(q_{univ}) = P(q_{univ}^{can})$.

Proof. (1) For any $q \in \mathcal{F}_{\mathfrak{g},f}$ and the basis $\{q^i\}_{i \in I}$ of $\mathfrak{p}$ such that $(q_i|q^j) = \delta_{ij}$, the Lax operator

$$L = k\partial + q - f \otimes 1 = k\partial + \textstyle\sum_{i \in I} q_i \otimes (q^i(q)) - f \otimes 1.$$

Also, for $X = n \otimes r \in \mathfrak{n} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ such that $e^{\text{ad}X}(\mathcal{L}) = k\partial + q_{\text{univ}}^{\text{can}} - f \otimes 1$, if we let $X_q := n \otimes (r(q))$ then $e^{\text{ad}X_q}(L) = k\partial + q^{X_q} - 1 \otimes f$ is obtained from $e^{\text{ad}X}(\mathcal{L})$ by substituting q^i in $q_{\text{univ}}^{\text{can}}$ by $q^i(q)$. In other words, $q^{X_q} \in V_f \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ and

$$q^{X_q} = q^{\text{can}} = \textstyle\sum_{i \in \mathcal{J}} q_i \otimes (w^i(q)).$$

(2) It is enough to show that $P(q_{\text{univ}}) = P(q_{\text{univ}}^{\text{can}})$ implies $P(q) = P(q^{\text{can}})$ for any $q \in \mathcal{F}_{\mathfrak{g},f}$. Suppose $P(q_{\text{univ}}) = P(q_{\text{univ}}^{\text{can}})$. Then, by (1), we have

$$P(q^{can}) = P\big(\textstyle\sum_{i\in\mathcal{J}} q_i \otimes w^i\big)|_{q^i=Q^i} = P(q_{\text{univ}}^{\text{can}})|_{q^i=Q^i} = P(q_{\text{univ}})|_{q^i=Q^i} = P(q)$$

where $|_{q^i=Q^i}$ denotes we substitute q^i by Q^i. □

Proposition 3.14. *For $q_{univ}^{can} = \sum_{i\in\mathcal{J}} q_i \otimes w^i \in V_f \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ in Lemma 3.13, we have*

$$\mathcal{W}(\mathfrak{g}, f, k) = \mathbb{C}_{diff}[\, w^i \mid i \in \mathcal{J}\,] \subset \mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \tag{3.20}$$

as differential superalgebras.

Proof. Note that we have

$$\text{(i) } w^i = w^i(q_{\text{univ}}) \text{ since } q_{\text{univ}} = \textstyle\sum_{i\in I} q_i \otimes q^i, \quad \text{(ii) } w^i = q^i(q_{\text{univ}}^{\text{can}}).$$

Consider the subset $\{q^i \mid i \in \mathcal{J}\}$ of the basis $\{q^i\}_{i\in I}$ of $\mathfrak{p}$ and take an element $\Phi \in \mathcal{W}(\mathfrak{g}, f, k)$. Then

$$\Phi = \Phi(q_{\text{univ}}) = \Phi(q_{\text{univ}}^{\text{can}}) \in \mathbb{C}_{\text{diff}}[\, q^i(q_{\text{univ}}^{\text{can}}) \mid i \in \mathcal{J}\,] = \mathbb{C}_{\text{diff}}[\, w_i \mid i \in \mathcal{J}\,]$$

and $\Phi \in \mathbb{C}_{\text{diff}}[\, w^i \mid i \in \mathcal{J}\,]$. Hence, by Lemma 3.13, we have $\mathcal{W}(\mathfrak{g}, f, k) \subset \mathbb{C}_{\text{diff}}[\, w_i \mid i \in \mathcal{J}\,]$.

Conversely, for $q, q' \in \mathcal{F}_{\mathfrak{g},f}$ such that $q \sim q'$, we have $q^{\text{can}} = q'^{\text{can}}$. Since $w^i(q) = q^i(q^{\text{can}}) = q^i(q'^{\text{can}}) = w^i(q'^{\text{can}})$, we get $\mathbb{C}_{\text{diff}}[\, w^i \mid i \in \mathcal{J}\,] \subset \mathcal{W}(\mathfrak{g}, f, k)$. □

By Proposition 3.14, we can find generators of $\mathcal{W}(\text{spo}(2|1), f, k)$ as follows (cf. Example 2.18).

Example 3.15. Let $\mathfrak{g} = \text{spo}(2|1)$ and let

$$h = e_{11} - e_{22}, \quad e = e_{12}, \quad f = e_{21}, \quad e_{od} = e_{1\bar{1}} + e_{\bar{1}2}, \quad f_{od} = e_{2\bar{1}} - e_{\bar{1}1}$$

where e_{ij} is the matrix in $\mathfrak{gl}(2|1)$ which has 1 in the ij-entry and 0 in other entries. Then h, e, f are even elements and e_{od}, f_{od} are odd elements. Lie brackets between generators are

$$[h, e_{od}] = e_{od},\ [h, f_{od}] = -f_{od},\ [e_{od}, f_{od}] = [f_{od}, e_{od}] = -h,\ [e_{od}, f] = -f_{od},$$
$$[f_{od}, e] = -e_{od},\ [e_{od}, e_{od}] = 2e,\ [f_{od}, f_{od}] = -2f.$$

The even supersymmetric bilinear form we consider satisfies

$$(h|h) = 2(e|f) = 2,\ (e_{od}|f_{od}) = -2.$$

Take the Lax operator

$$\mathcal{L} = k\partial + \frac{1}{2} f_{od} \otimes e_{od} + \frac{1}{2} h \otimes h - \frac{1}{2} e_{od} \otimes f_{od} + e \otimes f - f \otimes 1$$

and $V_f = \mathbb{C}e \oplus \mathbb{C}e_{od}$. We can check that $\mathfrak{m}^\perp = V_f \oplus [\mathfrak{n}, f]$. Suppose $X = e_{od} \otimes r_{od} + e \otimes r$ for $r_{od} \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})_{\bar{1}}$ and $r \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})_{\bar{0}}$ satisfies

$$e^{\text{ad}X}\mathcal{L} = \mathcal{L}^{can} \in \mathbb{C}\partial \ltimes V_f \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}). \tag{3.21}$$

The terms with ad$h$-eigenvalue -1 in (3.21) are

$$\frac{1}{2} f_{od} \otimes e_{od} - [e_{od}, f] \otimes r_{od} = 0.$$

Hence $r_{od} = -\frac{1}{2} e_{od}$. The terms with ad$h$-eigenvalue 0 in (3.21) are

$$\frac{1}{2} h \otimes h + h \otimes (-\frac{1}{2} e_{od} r_{od} - r) + \frac{1}{2} [e_{od}, f_{od}] \otimes r_{od}^2 = 0.$$

Since $e_{od} r_{od} = r_{od}^2 = 0$, we have $r = \frac{1}{2} h$. Hence $X = -\frac{1}{2} e_{od} \otimes e_{od} + \frac{1}{2} e \otimes h$ and, by direct computations,

$$\begin{aligned} e^{\mathrm{ad}X} \mathcal{L} &= k\partial + e_{od} \otimes (-\frac{1}{2} f_{od} + \frac{k}{2} \partial e_{od} + \frac{1}{4} h e_{od}) \\ &\quad + e \otimes (f + \frac{1}{2} f_{od} e_{od} - \frac{1}{4} h^2 + \frac{k}{4} e_{od} \partial e_{od} - \frac{k}{2} \partial h) - f \otimes 1. \end{aligned} \tag{3.22}$$

If we denote $\phi_{od} = -\frac{1}{2} f_{od} + \frac{k}{2} \partial e_{od} + \frac{1}{4} h e_{od}$ and $\phi_{ev} = f + \frac{1}{2} f_{od} e_{od} - \frac{1}{4} h^2 + \frac{k}{4} e_{od} \partial e_{od} - \frac{k}{2} \partial h$ then $\mathcal{W}(\mathfrak{g}, f, k) = \mathbb{C}_{\mathrm{diff}}[\phi_{od}, \phi_{ev}]$.

Example 3.16. Let $\mathfrak{g} = \mathrm{sl}(2|1)$ and take $e = e_{12}$ and $f = e_{21}$. Consider the Lax operator

$$\mathcal{L} = k\partial + Q - f \otimes 1 = k\partial + e \otimes f + e_{1\bar{1}} \otimes e_{\bar{1}1} - e_{\bar{1}2} \otimes e_{2\bar{1}} + \frac{1}{2} h \otimes h - \frac{1}{2} \tau \otimes \tau - f \otimes 1,$$

where $h = [e, f] = e_{11} - e_{22}$ and $\tau = e_{11} + e_{22} + 2e_{\bar{1}\bar{1}} \in \ker \mathrm{ad} f$. Fix

$$V_f = \mathbb{C} e_{12} \oplus \mathbb{C} e_{1\bar{1}} \oplus \mathbb{C} e_{\bar{1}2} \oplus \mathbb{C} \tau$$

so that $[\mathfrak{n}, f] \oplus V_f = \mathfrak{m}^{\perp}$. Let us take

$$X = e_{1\bar{1}} \otimes X_{1\bar{1}} + e_{\bar{1}2} \otimes X_{\bar{1}2} + e \otimes X_{12}$$

such that $e^{\mathrm{ad}X} \mathcal{L} \in \partial \ltimes V_f \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. In order to vanish degree $\frac{1}{2}$-part of $e^{\mathrm{ad}X} \mathcal{L}$, which is

$$-e_{\bar{1}1} \otimes e_{1\bar{1}} + e_{2\bar{1}} \otimes e_{\bar{1}2} + e_{2\bar{1}} \otimes X_{1\bar{1}} - e_{\bar{1}1} \otimes X_{\bar{1}2},$$

set $X_{1\bar{1}} = -e_{\bar{1}2}$ and $X_{\bar{1}2} = -e_{1\bar{1}}$. Moreover, we want degree 0-part of $e^{\mathrm{ad}X} \mathcal{L}$ lies in $\partial \ltimes \mathbb{C}\tau \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. In other words,

$$\begin{aligned} &\mathcal{L}_0 + [X_{1/2}, \mathcal{L}_{-1/2}] + [X_1, L_{-1}] + \frac{1}{2} [X_{1/2}, [X_{1/2}, f]] \\ &= k\partial + \frac{1}{2} h \otimes h - \frac{1}{2} \tau \otimes \tau - \frac{1}{2} (h + \tau) \otimes e_{1\bar{1}} X_{1\bar{1}} - \frac{1}{2} (h - \tau) \otimes e_{\bar{1}2} X_{\bar{1}2} \\ &\quad - h \otimes X_{12} - \frac{1}{2} [e_{\bar{1}2}, [e_{1\bar{1}}, f]] \otimes X_{1\bar{1}} X_{\bar{1}2} - \frac{1}{2} [e_{1\bar{1}}, [e_{\bar{1}2}, f]] \otimes X_{\bar{1}2} X_{1\bar{1}} \\ &= k\partial + h \otimes (\frac{1}{2} h - X_{12}) + \tau \otimes (-\frac{1}{2} \tau + \frac{1}{2} e_{1\bar{1}} e_{\bar{1}2}) \in \partial \ltimes \mathbb{C}\tau \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}). \end{aligned}$$

Here, $\mathcal{L}_l$ and X_l denote the degree l-parts of $\mathcal{L}$ and X, respectively. Hence, we fix $X_{12} = \frac{1}{2} h$. In other words, we have

$$X = e_{12} \otimes \frac{1}{2} h - e_{\bar{1}2} \otimes e_{1\bar{1}} - e_{1\bar{1}} \otimes e_{\bar{1}2}.$$

Now, we compute degree 1/2-part and 1-part of $e^{\mathrm{ad}X}\mathcal{L}$ as follows. For degree 1/2-part, we have

$$
\begin{aligned}
&\mathcal{L}_{1/2} = e_{1\bar{1}} \otimes e_{\bar{1}1} - e_{\bar{1}2} \otimes e_{2\bar{1}},\\
&[X_{1/2}, \mathcal{L}_0] = e_{\bar{1}2} \otimes (k\partial e_{1\bar{1}} + \frac{1}{2}e_{1\bar{1}}(h-\tau)) + e_{1\bar{1}} \otimes (k\partial e_{\bar{1}2} + \frac{1}{2}e_{\bar{1}2}(h+\tau)),\\
&[X_1, \mathcal{L}_{-1/2}] = e_{\bar{1}2} \otimes \frac{1}{2}e_{1\bar{1}}h + e_{1\bar{1}} \otimes \frac{1}{2}e_{\bar{1}2}h,\\
&\frac{1}{2}[X_{1/2}, [X_{1/2}, \mathcal{L}_{-1/2}]] + \frac{1}{2}[X_{1/2}, [X_1, \mathcal{L}_{-1}]] = -\frac{1}{4}e_{\bar{1}2} \otimes he_{1\bar{1}} - \frac{1}{4}e_{1\bar{1}} \otimes he_{\bar{1}2},\\
&\frac{1}{2}[X_1, [X_{1/2}, \mathcal{L}_{-1}]] = -\frac{1}{4}e_{\bar{1}2} \otimes he_{1\bar{1}} - \frac{1}{4}, e_{1\bar{1}} \otimes he_{\bar{1}2}.
\end{aligned}
$$

By adding all the formulas, we get 1/2-degree part of $e^{\mathrm{ad}X}\mathcal{L}$:

$$
e_{1\bar{1}} \otimes (e_{\bar{1}1} + k\partial e_{\bar{1}2} + \frac{1}{2}e_{\bar{1}2}(h+\tau)) + e_{\bar{1}2} \otimes (-e_{2\bar{1}} + k\partial e_{1\bar{1}} + \frac{1}{2}e_{1\bar{1}}(h-\tau)).
$$

Similarly, degree 1-part of $e^{\mathrm{ad}X}\mathcal{L}$ is obtained by adding the following fomulas:

$$
\begin{aligned}
&\mathcal{L}_1 = e \otimes f,\\
&[X_{1/2}, \mathcal{L}_{1/2}] = e \otimes (-e_{1\bar{1}}e_{1\bar{1}} + e_{2\bar{1}}e_{\bar{1}2}),\\
&[X_1, \mathcal{L}_0] = e \otimes (-\frac{k}{2}\partial h - \frac{1}{2}h^2),\\
&\frac{1}{2}([X_{1/2}, [X_{1/2}, \mathcal{L}_0]] + [X_{1/2}, [X_1, \mathcal{L}_{-1/2}]])\\
&\quad = e \otimes \frac{1}{2}(-k\partial e_{1\bar{1}}e_{\bar{1}2} - k\partial e_{\bar{1}2}e_{1\bar{1}} + \tau e_{1\bar{1}}e_{\bar{1}2}),\\
&\frac{1}{2}([X_1, [X_{1/2}, \mathcal{L}_{-1/2}]] + [X_1, [X_1, \mathcal{L}_{-1}]]) = e \otimes \frac{1}{4}h^2,\\
&\frac{1}{6}([X_{1/2}, [X_{1/2}, [X_{1/2}, \mathcal{L}_{-1/2}]]] + [X_{1/2}, [X_{1/2}, [X_1, \mathcal{L}_{-1}]]]\\
&\quad + [X_{1/2}, [X_1, [X_{1/2}, \mathcal{L}_{-1}]]] + [X_1, [X_{1/2}, [X_{1/2}, \mathcal{L}_{-1}]]]) = 0.
\end{aligned}
$$

Hence the degree 1 part of $e^{\mathrm{ad}X}\mathcal{L}$ is

$$
e \otimes (f - e_{\bar{1}1}e_{1\bar{1}} + e_{2\bar{1}}e_{\bar{1}2} - \frac{1}{4}h^2 - \frac{k}{2}\partial h - \frac{k}{2}\partial e_{1\bar{1}}e_{\bar{1}2} - \frac{k}{2}\partial e_{\bar{1}2}e_{1\bar{1}} + \frac{1}{2}\tau e_{1\bar{1}}e_{\bar{1}2}).
$$

If we denote

$$
\begin{aligned}
\phi_t &= -\frac{1}{2}\tau + \frac{1}{2}e_{1\bar{1}}e_{\bar{1}2},\\
\phi_{1\bar{1}} &= e_{\bar{1}1} + k\partial e_{\bar{1}2} + \frac{1}{2}e_{\bar{1}2}(h+\tau),\\
\phi_{\bar{1}2} &= -e_{2\bar{1}} + k\partial e_{1\bar{1}} + \frac{1}{2}e_{1\bar{1}}(h-\tau),\\
\phi_e &= f - e_{\bar{1}1}e_{1\bar{1}} + e_{2\bar{1}}e_{\bar{1}2} - \frac{1}{4}h^2 - \frac{k}{2}\partial h - \frac{k}{2}\partial e_{1\bar{1}}e_{\bar{1}2} - \frac{k}{2}\partial e_{\bar{1}2}e_{1\bar{1}} + \frac{1}{2}\tau e_{1\bar{1}}e_{\bar{1}2}
\end{aligned}
\tag{3.23}
$$

then

$$\mathcal{W}(\mathfrak{g}, f, k) = \mathbb{C}_{\text{diff}}[\phi_t, \phi_{1\bar{1}}, \phi_{\bar{1}2}, \phi_e].$$

Here, we note that

$$\phi_e = -kL - 2\phi_\tau^2,$$

for the energy momentum field L, which is defined in Proposition 2.15.

Now, we introduce another equivalent way to get a generating set of a W-superalgebra, which will be used in the later sections to describe bi-Poisson structures via Lax operators. We consider the *sign twisting* linear map

$$\sigma : \mathfrak{g} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \to \mathfrak{g} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}), \quad a \otimes F \mapsto (a \otimes F)^\sigma, \tag{3.24}$$

where $(a \otimes F)^\sigma := (-1)^{\mathrm{p}(a)\mathrm{p}(F)} a \otimes F$.

Remark 3.17. One can check that the map (3.24) induces a Lie algebra automorphism on $(\mathfrak{g} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}}$. Moreover, if we consider the Lie algebra $(\mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \otimes \mathfrak{g})_{\bar{0}}$ with the bracket $[F \otimes a, G \otimes b] = (-1)^{\mathrm{p}(a)\mathrm{p}(b)} FG \otimes [a, b]$ then there is a natural isomorphism

$$\phi : (\mathfrak{g} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}} \to (\mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \otimes \mathfrak{g})_{\bar{0}}$$

defined by $\phi((a \otimes F)^\sigma) = F \otimes a$. In other words, applying the map σ to $(\mathfrak{g} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}}$ is equivalent to consider the space $(\mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \otimes \mathfrak{g})_{\bar{0}}$.

Proposition 3.18. *For a Lax operator $L = k\partial + \sum_{i \in I} q_i \otimes Q^i - f \otimes 1$, we denote*

$$L^\sigma = k\partial + \textstyle\sum_{i \in I} (-1)^{p(i)} q_i \otimes Q^i - f \otimes 1 \in (\mathfrak{g}[z] \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}},$$

where $p(i)$ denotes the parity $p(q_i)$.

(1) Two elements $q, q' \in \mathcal{F}_{\mathfrak{g},f}$ are gauge equivalent if and only if q^σ, q'^σ are gauge equivalent.
(2) Recall that we denote the universal Lax operator by $\mathcal{L} = k\partial + q_{univ} - f \otimes 1$ and the operator gives rise to generators of $\mathcal{W}(\mathfrak{g}, f, k)$ (see Proposition 3.14). By (1), we can find generators of $\mathcal{W}(\mathfrak{g}, f, k)$ using sign twisted universal Lax operator

$$\mathcal{L}^\sigma = k\partial + q^\sigma_{univ} - f \otimes 1.$$

Proof. The proof follows from the fact that

$$e^{\mathrm{ad}(n \otimes r)^\sigma}(L^\sigma) = L'^\sigma \quad \text{if and only if} \quad e^{\mathrm{ad}(n \otimes r)}(L) = L'.$$

For (2), we note that

$$q^{\text{can}}_{\text{univ}} = \textstyle\sum_{i \in \mathcal{J}} q_i \otimes w^i \text{ if and only if } (q^\sigma_{\text{univ}})^{\text{can}} = \sum_{i \in \mathcal{J}} \mathrm{p}(i) q_i \otimes w^i.$$

Since $\mathbb{C}_{\text{diff}}[w^i | i \in \mathcal{J}] = \mathbb{C}_{\text{diff}}[\mathrm{p}(i) w^i | i \in \mathcal{J}]$, we proved (2). □

4. Lax Operators and Lie Brackets on $\mathcal{W}(\mathfrak{g}, f, k)/\partial\mathcal{W}(\mathfrak{g}, f, k)$

4.1. Derivatives on a differential superalgebra.
Fix the superalgebra of differential polynomials

$$\mathcal{P} = \mathbb{C}_{\text{diff}}[\, u_i \mid i \in I\,].$$

The index set $I = I_{\bar{0}} \cup I_{\bar{1}}$ consists of two subindex sets $I_{\bar{0}}$ and $I_{\bar{1}}$ such that u_i is even if $i \in I_{\bar{0}}$ and is odd if $i \in I_{\bar{1}}$.

Definition 4.1. Take $j_1, j_2, \ldots, j_k \in I_{\bar{1}}$ and $n_1, n_2, \ldots, n_k \in \mathbb{Z}_{\geq 0}$ such that $(j_{k_1}, n_{k_1}) \neq (j_{k_2}, n_{k_2})$ for distinct numbers $k_1, k_2 \in \{1, \ldots, k\}$. Consider an element

$$\phi_1 = u_{j_1}^{(n_1)} u_{j_2}^{(n_2)} \cdots u_{j_k}^{(n_k)} \in \mathbb{C}_{\text{diff}}[u_i | i \in I_{\bar{1}}],$$

where $u_j^{(n)} = \partial^n u_j$ and a monomial

$$\phi = \phi_0 \phi_1 \in \mathcal{P}, \quad \text{for } \phi_0 \in \mathbb{C}_{\text{diff}}[u_i | i \in I_{\bar{0}}].$$

(1) The *(left) derivative* of ϕ with respect to $u_{j_t}^{(n_t)}$ for $t = 1, \ldots, k$ is

$$\frac{\partial \phi}{\partial u_{j_t}^{(n_t)}} = (-1)^{t-1}\, \phi_0 \cdot u_{j_1}^{(i_1)} u_{j_2}^{(i_2)} \cdots u_{j_{t-1}}^{(n_{t-1})} u_{j_{t+1}}^{(n_{t+1})} \cdots u_{j_{k-1}}^{(n_{k-1})} u_{j_k}^{(n_k)}.$$

If $u_j^{(n)} \neq u_{j_t}^{(n_t)}$ is an odd element then $\frac{\partial \phi}{\partial u_j^{(n)}} = 0$. If $j \in I_{\bar{0}}$, we let $\frac{\partial \phi}{\partial u_j^{(n)}} = \frac{\partial \phi_0}{\partial u_j^{(n)}} \cdot \phi_1$.

(2) The *right derivative* of ϕ with respect to $u_{j_t}^{(n_t)}$ for $t = 1, \ldots, k$ is

$$\frac{\partial_R \phi}{\partial_R u_{j_t}^{(n_t)}} = (-1)^{k-t}\, \phi_0 \cdot u_{j_1}^{(i_1)} u_{j_2}^{(i_2)} \cdots u_{j_{t-1}}^{(n_{t-1})} u_{j_{t+1}}^{(n_{t+1})} \cdots u_{j_{k-1}}^{(n_{k-1})} u_{j_k}^{(n_k)}.$$

If $u_j^{(n)} \neq u_{j_t}^{(n_t)}$ is an odd element then $\frac{\partial_R \phi}{\partial_R u_j^{(n)}} = 0$. If $j \in I_{\bar{0}}$, we let $\frac{\partial_R \phi}{\partial_R u_j^{(n)}} = \frac{\partial \phi}{\partial u_j^{(n)}}$.

(3) The *(left) variational derivative* and *right variational derivative* of ϕ with respect to u_i are

$$\frac{\delta \phi}{\delta u_i} = \sum_{n \in \mathbb{Z}_{\geq 0}} (-\partial)^n \frac{\partial \phi}{\partial u_i^{(n)}}, \quad \frac{\delta_R \phi}{\delta_R u_i} = \sum_{n \in \mathbb{Z}_{\geq 0}} (-\partial)^n \frac{\partial_R \phi}{\partial_R u_i^{(n)}}. \tag{4.1}$$

(4) The *variational derivative* of ϕ with respect to $\{\, u_i \mid i \in I\,\}$ is

$$\frac{\delta \phi}{\delta u} = \sum_{i \in I} u_i \otimes \frac{\delta \phi}{\delta u_i}.$$

The derivatives defined in (1), (2), (3), (4) have linearities so that they are well-defined on the set of differential algebra $\mathcal{P} = \mathbb{C}_{\text{diff}}[u_i | i \in I]$.

Remark 4.2. We have

$$\frac{\delta \phi}{\delta u_i} = (-1)^{\mathrm{p}(u_i)\mathrm{p}(\phi \cdot u_i)} \frac{\delta_R \phi}{\delta_R u_i}.$$

Hence, for the map σ in (3.24),

$$\left(\frac{\delta \phi}{\delta u}\right)^{\sigma} = \sum_{i \in I} \left(u_i \otimes \frac{\delta \phi}{\delta u_i}\right)^{\sigma} = \sum_{i \in I} u_i \otimes \frac{\delta_R \phi}{\delta_R u_i}.$$

Remark 4.3. If $u_i \in \mathcal{P}$ is an odd element then $\frac{\partial}{\partial u_i^{(n)}}$ for any $n \in \mathbb{Z}_{\geq 0}$ is an odd derivation, i.e.

$$\frac{\partial}{\partial u_i^{(n)}}(FG) = \left(\frac{\partial}{\partial u_i^{(n)}} F\right) \cdot G + (-1)^{\mathrm{p}(F)} F \cdot \left(\frac{\partial}{\partial u_i^{(n)}} G\right).$$

Proposition 4.4. *For homogeneous elements $f, g \in \mathcal{P}$, we have*

$$\{f_\lambda g\} = \sum_{\substack{i,j\in I\\ m,n\in\mathbb{Z}_{\geq 0}}} C_{i,j}^{f,g} \frac{\partial_R g}{\partial_R u_j^{(n)}} (\lambda+\partial)^n \{u_{i\,\lambda+\partial}\, u_j\}_{\to} (-\lambda-\partial)^m \frac{\partial f}{\partial u_i^{(m)}} \tag{4.2}$$

where $C_{i,j}^{f,g} = (-1)^{p(f)p(g)+p(i)p(j)}$. Note that

$$\{a_{\lambda+\partial} b\}_{\to} c := \sum_{i\in\mathbb{Z}_{\geq 0}} \frac{1}{n!} a_{(n)} b (\lambda+\partial)^n c \quad \textit{for} \quad a, b, c \in \mathcal{P}.$$

Proof. The formula (4.2) follows from sesquilinearities and Leibniz rules of λ-brackets. The only part we have to be careful is the constant factor $C_{i,j}^{f,g}$ in (4.2). One can see that

$$C_{i,j}^{f,g} = (-1)^{(\mathrm{p}(i)+\mathrm{p}(f))(\mathrm{p}(j)+\mathrm{p}(g))} \cdot C_{i,j}^{f} \cdot C_{j,i}^{g}$$

for $C_{i,j}^{f} = (-1)^{(\mathrm{p}(f)+\mathrm{p}(i))\mathrm{p}(j)}$ and $C_{j,i}^{g} = (-1)^{(\mathrm{p}(g)+\mathrm{p}(j))\mathrm{p}(i)}$. Note that switching the position of $\frac{\partial f}{\partial u_i^{(m)}}$ and $\frac{\partial_R g}{\partial_R u_j^{(n)}}$ gives rise to the $(-1)^{(\mathrm{p}(i)+\mathrm{p}(f))(\mathrm{p}(j)+\mathrm{p}(g))}$ in $C_{i,j}^{f,g}$ and switching the position of u_j and $\frac{\partial f}{\partial u_i^{(m)}}$ (resp. u_i and $\frac{\partial_R g}{\partial_R u_j^{(n)}}$) gives rise to the constant factor $C_{i,j}^{f}$ (resp. $C_{j,i}^{g}$). □

Proposition 4.5. *(1) For any variable $u_i^{(n)}$ in $\mathcal{P}$, we have*

$$\left[\frac{\partial}{\partial u_i^{(n)}}, \partial\right] = \frac{\partial}{\partial u_i^{(n-1)}}. \tag{4.3}$$

(2) Let $\phi \in \mathcal{P}$. Then $\frac{\delta}{\delta u_i}\partial\phi = 0$ for any $i \in I$.

Proof. (1) For any $j \in I$ and $m \in \mathbb{Z}_{\geq 0}$, we can check that $\left[\frac{\partial}{\partial u_i^{(n)}}, \partial\right](u_j^{(m)}) = \frac{\partial}{\partial u_i^{(n-1)}}(u_j^{(m)})$. Suppose we get the same element in $\mathcal{P}$ when we apply F (resp. G) to the LHS and RHS of Eq. (4.3). If $i \in I_{\bar{1}}$ then

$$\begin{aligned}
\left[\frac{\partial}{\partial u_i^{(n)}}, \partial\right](FG) &= \left(\frac{\partial}{\partial u_i^{(n)}} \partial F\right) G + (-1)^{\mathrm{p}(F)} \partial F \frac{\partial}{\partial u_i^{(n)}} G \\
&\quad + \frac{\partial}{\partial u_i^{(n)}} F \partial G + (-1)^{\mathrm{p}(F)} F \frac{\partial}{\partial u_i^{(n)}} \partial G \\
&\quad - \left(\partial \frac{\partial}{\partial u_i^{(n)}} F\right) G - (-1)^{\mathrm{p}(F)} \partial F \frac{\partial}{\partial u_i^{(n)}} G \\
&\quad - \frac{\partial}{\partial u_i^{(n)}} F \partial G - (-1)^{\mathrm{p}(F)} F \partial \frac{\partial}{\partial u_i^{(n)}} G \\
&= \left[\frac{\partial}{\partial u_i^{(n)}}, \partial\right](F) \cdot G + (-1)^{\mathrm{p}(F)} F \cdot \left[\frac{\partial}{\partial u_i^{(n)}}, \partial\right](G) \\
&= \frac{\partial}{\partial u_i^{(n-1)}} F \cdot G + (-1)^{\mathrm{p}(F)} F \frac{\partial}{\partial u_i^{(n-1)}}(G) = \frac{\partial}{\partial u_i^{(n-1)}}(FG).
\end{aligned}$$

For $i \in I_{\bar{0}}$, the same argument works. By induction, we proved (1).

(2) By (1), we have

$$\sum_{i\in\mathbb{Z}_{\geq 0}}(-\partial)^n \frac{\partial}{\partial u_i^{(n)}}\partial = -\sum_{i\in\mathbb{Z}_{\geq 0}}(-\partial)^{n+1}\frac{\partial}{\partial u_i^{(n)}} + \sum_{i\in\mathbb{Z}_{\geq 1}}(-\partial)^n \frac{\partial}{\partial u_i^{(n-1)}} = 0.$$

□

4.2. Lie brackets on $\mathcal{W}(\mathfrak{g}, f, k)/\partial\mathcal{W}(\mathfrak{g}, f, k)$.
Now we are ready to find Lie superalgebra structures on a quotient space of W-algebra via Lax operators. In this section, we assume $\mathfrak{g}$ is a finite dimensional simple Lie superalgebra with even supersymmetric bilinear invariant form $(\,|\,)$.

Let us denote by $\mathfrak{g}((z^{-1})) := \mathfrak{g}[z] \oplus z^{-1}\mathfrak{g}[[z^{-1}]]$ the Lie superalgebra endowed with the bracket $[az^m, bz^n] = [a, b]z^{m+n}$ where $[a, b]$ is the Lie bracket on $\mathfrak{g}$. The vector superspace $\mathfrak{g}((z^{-1})) \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ is a Lie superalgebra endowed with the bracket such that

$$[az^m \otimes F, bz^n \otimes G] = (-1)^{\mathrm{p}(b)\mathrm{p}(F)}[az^m, bz^n] \otimes FG$$

for $a, b \in \mathfrak{g}$, $n, m \in \mathbb{Z}$ and $F, G \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. We extend the Lie bracket to that on $\mathbb{C}\partial \ltimes \mathfrak{g}((z^{-1})) \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ by considering $\mathbb{C}\partial$ as the trivial Lie algebra.

Consider the universal Lax operator associated to $\mathfrak{g}$ and Λ:

$$\mathcal{L}(\Lambda) = k\partial + q_{\mathrm{univ}} - \Lambda \otimes 1 \in \mathbb{C}\partial \ltimes \mathfrak{g}((z^{-1})) \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}), \tag{4.4}$$

where q_{univ} is that in (3.3) and $\Lambda = f + zs$ for a nonzero even element s in $\mathfrak{g}(d)$. Here, d is the largest eigenvalue such that $\mathfrak{g}(d) \neq 0$. The operator acts on $\mathfrak{g}((z^{-1})) \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ by the adjoint action. Note that we assume the even part of $\mathfrak{g}(d) \neq 0$ (see Remark 2.14).

Remark 4.6. For the operator $\mathcal{L}$ associated to $\mathfrak{g}$ and f in (3.3), we have $\mathcal{L}(\Lambda) = \mathcal{L} - zs \otimes 1$. Denote by

$$\mathcal{L}_{[1]}(\Lambda) := \mathcal{L} \quad \text{and} \quad \mathcal{L}_{[2]}(\Lambda) := -s \otimes 1. \tag{4.5}$$

The term $\mathcal{L}_{[2]}(\Lambda)$ do the crucial role to define bi-Poisson structures on $\mathcal{W}(\mathfrak{g}, f, k)$. However, this part is not important when we find generators of $\mathcal{W}(\mathfrak{g}, f, k)$.

Also, using the map (3.24), we denote the *sign twisted universal Lax operator* by

$$\mathcal{L}^{\sigma}(\Lambda) = k\partial + q^{\sigma}_{\mathrm{univ}} - \Lambda \otimes 1 \in \mathbb{C}\partial \ltimes \mathfrak{g}((z^{-1})) \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}). \tag{4.6}$$

Now, via $\mathcal{L}^{\sigma}(\Lambda)$, we aim to define Lie brackets on $\mathcal{W}(\mathfrak{g}, f, k)/\partial\mathcal{W}(\mathfrak{g}, f, k)$ for the subspace $\partial\mathcal{W}(\mathfrak{g}, f, k) := \{\,\partial W \mid W \in \mathcal{W}(\mathfrak{g}, f, k)\,\} \subset \mathcal{W}(\mathfrak{g}, f, k)$. (In Remark 4.16, we explain why we consider $\mathcal{L}^{\sigma}(\Lambda)$ instead of $\mathcal{L}(\Lambda)$.)

The following notion is analogous to the notion in (2) of Definition 2.27.

Definition 4.7. Let $\mathcal{V}$ be a differential superalgebra with the derivation ∂ and denote $\partial\mathcal{V} := \{\,\partial V \mid V \in \mathcal{V}\,\}$. For the map

$$\int : \mathcal{V} \to \mathcal{V}/\partial\mathcal{V}, \quad V \mapsto V + \partial\mathcal{V} =: \int V,$$

we call $\int V$ the *local functional* of $V \in \mathcal{V}$.

Let the bilinear form $(\cdot|\cdot) : \mathfrak{g}((z^{-1})) \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \times \mathfrak{g}((z^{-1})) \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \to \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ be defined by

$$(az^m \otimes F|bz^n \otimes G) = (-1)^{\mathrm{p}(b)\mathrm{p}(F)}(a|b)\delta_{m+n,0}FG.$$

Define two bilinear brackets $\{,\}_i : \mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \times \mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \to \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$, for $i = 1, 2$, by

$$\begin{aligned}\{\phi,\psi\}_1 &= \left(\frac{\delta\phi}{\delta q}\,\middle|\left[\frac{\delta\psi}{\delta q}, \mathcal{L}^\sigma(\Lambda)\right]\right) = \left(\frac{\delta\phi}{\delta q}\,\middle|\left[\frac{\delta\psi}{\delta q}, \mathcal{L}^\sigma_{[1]}(\Lambda)\right]\right),\\ \{\phi,\psi\}_2 &= -\left(\frac{\delta\phi}{\delta q}\,\middle|z^{-1}\left[\frac{\delta\psi}{\delta q}, \mathcal{L}^\sigma(\Lambda)\right]\right) = -\left(\frac{\delta\phi}{\delta q}\,\middle|\left[\frac{\delta\psi}{\delta q}, \mathcal{L}^\sigma_{[2]}(\Lambda)\right]\right),\end{aligned} \tag{4.7}$$

where $q = (q^i)_{i\in I}$ is the basis of $\mathfrak{p}$ in (3.3) and $z^{-1}(az^n \otimes F) := az^{n-1} \otimes F \in \mathfrak{g}((z^{-1})) \otimes \mathcal{V}(\mathfrak{p})$. By Proposition 4.5 (2), we can consider the induced bilinear brackets on $\mathcal{V}_{\mathcal{I}}(\mathfrak{p})/\partial\mathcal{V}_{\mathcal{I}}(\mathfrak{p})$

$$[\ ,\ ]_i : \mathcal{V}_{\mathcal{I}}(\mathfrak{p})/\partial\mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \times \mathcal{V}_{\mathcal{I}}(\mathfrak{p})/\partial\mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \to \mathcal{V}_{\mathcal{I}}(\mathfrak{p})/\partial\mathcal{V}_{\mathcal{I}}(\mathfrak{p}), \quad i = 1, 2 \tag{4.8}$$

such that $[\int\phi, \int\psi]_i := \int\{\phi,\psi\}_i$.

Lemma 4.8. *Let $q', r \in \mathcal{F}_{\mathfrak{g},f} = (\mathfrak{m}^\perp \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar 0}$. For $\epsilon \in \mathbb{C}$, we have*

$$\left.\frac{\int\phi(q'+\epsilon r) - \int\phi(q')}{\epsilon}\right|_{\epsilon=0} = \int\left(r^\sigma\,\middle|\,\frac{\delta\phi(q')}{\delta q}\right). \tag{4.9}$$

Proof. If $r \in (\mathfrak{m}^\perp)_{\bar 0} \otimes (\mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar 0}$ then the proof is similar to the non-super algebras cases in [1]. Suppose $r = q_i \otimes r^i$ for $q_i \in (\mathfrak{m}^\perp)_{\bar 1}$. Then

$$\left(r^\sigma\,\middle|\,\frac{\delta\phi(q')}{\delta q}\right) = \left(-q_i \otimes r^i\,\middle|\,\frac{\delta\phi(q')}{\delta q}\right) = r^i(q_i \otimes 1|q^i \otimes 1)\frac{\delta\phi(q')}{\delta q^i} = r^i\frac{\delta\phi(q')}{\delta q^i}. \tag{4.10}$$

Also, we can see the LHS of (4.9) is the same as $r^i\frac{\delta\phi(q')}{\delta q^i}$. In detail, for $\phi = \partial^{n_1}(q^{i_1})\,\partial^{n_2}(q^{i_2})\ldots\partial^{n_t}(q^{i_t})$, if we denote the first $k-1$ terms in ϕ by $\phi_k = \partial^{n_1}(q^{i_1})\,\partial^{n_2}(q^{i_2})\ldots\partial^{n_{k-1}}(q^{i_{k-1}})$ and the last $t-k$ terms in ϕ by $\psi_k = \partial^{n_{k+1}}(q^{i_{k+1}})\,\partial^{n_{k+2}}(q^{i_{k+2}})\ldots\partial^{n_t}(q^{i_t})$ for $k = 0, 1, \ldots, t$ then

$$\phi = \phi_k \cdot \partial^{n_k}q^{i_k} \cdot \psi_k$$

for any k. Now, for such ϕ, let $r = q_i \otimes r^i$ and $q' = \sum_{j\in I} q_j \otimes Q^j$. Then we have

$$\begin{aligned}\left.\frac{\phi(q'+\epsilon r)-\phi(q')}{\epsilon}\right|_{\epsilon=0} &= \textstyle\sum_{k=0}^t \phi_k(q')\cdot\partial^{n_k}q^{i_k}(r)\cdot\psi_k(q')\\ &= \textstyle\sum_{k=1}^t (-1)^{\mathrm{p}(q_i)\mathrm{p}(\phi_k)}\partial^{n_k}q^{i_k}(r)\cdot\phi_k(q')\psi_k(q')\\ &= \textstyle\sum_{k=1}^t \partial^{n_k}\left(r^i(q_i|q^{i_k})\right)(-1)^{\mathrm{p}(q_i)\mathrm{p}(\phi_k)}\phi_k(q')\psi_k(q')\\ &= \textstyle\sum_{n\geq 0}\partial^n r^i\frac{\partial\phi}{\partial q^{i\,(n)}}.\end{aligned} \tag{4.11}$$

The last equality holds since $(q_i|q^j) = \delta_{ij}$. Since $\int\partial(F)G = \int -F\partial(G)$, the lemma is proved by (4.10) and (4.11). □

Proposition 4.9. *The brackets* $[\ ,\ ]_i$ *for* $i = 1, 2$ *satisfy skew-symmetry.*

Proof. For $\phi = a \otimes F$ and $\psi = b \otimes G$ in $\mathfrak{g} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$, we have

$$\begin{aligned}&\textstyle\int(a \otimes F|[b \otimes G, \partial])\\ &\quad= \textstyle\int(a \otimes F| - b \otimes \partial G) = -\int(-1)^{\mathrm{p}(\phi)\mathrm{p}(\psi)}(b \otimes \partial G|a \otimes F)\\ &\quad= \textstyle\int(-1)^{\mathrm{p}(\phi)\mathrm{p}(\psi)}(b \otimes G|a \otimes \partial F) = -\int(-1)^{\mathrm{p}(\phi)\mathrm{p}(\psi)}(b \otimes G|[a \otimes F, \partial]).\end{aligned}$$

Hence, by invariance and skew-symmetry of the bilinear form on $\mathfrak{g} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$, we have

$$\begin{aligned}[\textstyle\int \phi, \int \psi]_i = \int \{\phi, \psi\}_i &= -(-1)^{\mathrm{p}(\phi)\mathrm{p}(\psi)} \int \{\psi, \phi\}_i\\ &= -(-1)^{\mathrm{p}(\phi)\mathrm{p}(\psi)}[\int \psi, \int \phi]_i, \quad i = 1, 2.\end{aligned}$$

□

Lemma 4.10. *For $\psi \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$, we have:*

$$\begin{aligned}&[\textstyle\int q^i, \int \psi]_1 = \sum_{j\in I} \int \left([q^i, q^j] - (q^i|q^j)k\partial\right) \frac{\delta}{\delta q^j}\psi,\\ &[\textstyle\int q^i, \int \psi]_2 = \sum_{j\in I} \int ([q^i, q^j]|s) \frac{\delta}{\delta q^j}\psi,\end{aligned}$$

where $\{q^j | j \in I\}$ is a basis of $\mathfrak{p}$.

Proof. By expanding the LHS of $\int\{q^i, \psi\}_1$, we obtain

$$\begin{aligned}[\textstyle\int q^i, \int \psi]_1 &= \textstyle\sum_{j,j'\in I} \int \left(q^i \otimes 1\middle| [q^j \otimes \frac{\delta}{\delta q^j}\psi, k\partial + \mathrm{p}(j')q_{j'} \otimes q^{j'} - f \otimes 1]\right)\\ &= \textstyle\sum_{j\in I} \int (q^i|q^j)(-k\partial)\frac{\delta}{\delta q^j}\psi + \sum_{j,j'\in I} \int q^{j'}(q_{j'}|[q^i, q^j])\frac{\delta}{\delta q^j}\psi\\ &\quad - \sum_{j\in I}([q^i, q^j]|f)\frac{\delta}{\delta q^j}\psi,\end{aligned} \tag{4.12}$$

where $\{q_j | j \in I\}$ is the basis of $\mathfrak{m}^\perp$ such that $(q_j|q^{j'}) = \delta_{j,j'}$ and $\mathrm{p}(j')$ is the parity of $q^{j'}$. In $\mathcal{V}_{\mathcal{I}}(\mathfrak{p})$, we have

$$\textstyle\sum_{j\in I}[q^i, q^j]\frac{\delta}{\delta q^j}\psi = \sum_{j,j'\in I} \int([q^i, q^j]|q_{j'})q^{j'} \frac{\delta}{\delta q^j}\psi - \sum_{j\in I}([q^i, q^j]|f)\frac{\delta}{\delta q^j}\psi.$$

Hence the first equality is proved. The second equality also can be proved similarly. □

Proposition 4.11. *We have the following equations:*

$$\begin{aligned}&[\textstyle\int \phi, \int \psi]_1 = \sum_{i,j\in I} \int \frac{\delta_R}{\delta_R q^i}\phi \left([q^i, q^j] - (q^i|q^j)k\partial\right) \frac{\delta}{\delta q^j}\psi,\\ &[\textstyle\int \phi, \int \psi]_2 = \sum_{i,j\in I} \int \frac{\delta_R}{\delta_R q^i}\phi([q^i, q^j]|s) \frac{\delta}{\delta q^j}\psi.\end{aligned} \tag{4.13}$$

Proof. Observe that

$$\left(\tfrac{\delta}{\delta q}\phi\middle| a \otimes F\right) = \textstyle\sum_{i\in I} \frac{\delta_R \phi}{\delta_R q^i}(q^i \otimes 1|a \otimes F)$$

for any $a\otimes F\in\mathfrak{g}\otimes\mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. Since $\frac{\delta}{\delta q}q^i=q^i\otimes 1$, we have

$$[\int\phi,\int\psi]_t=\int\sum_{i\in I}\frac{\delta_R\phi}{\delta_R q^i}\left(q^i\otimes 1\left|\left[\frac{\delta}{\delta q}\psi,\mathcal{L}^{\sigma}_{[t]}(\Lambda)\right]\right.\right)$$
$$=\int\sum_{i\in I}\frac{\delta_R\phi}{\delta_R q^i}\left(\frac{\delta}{\delta q}q^i\left|\left[\frac{\delta}{\delta q}\psi,\mathcal{L}^{\sigma}_{[t]}(\Lambda)\right]\right.\right)=\int\sum_{i\in I}\frac{\delta_R\phi}{\delta_R q^i}\{q^i,\psi\}_t \quad (4.14)$$

for $t=1,2$. Now, by the proof of Lemma 4.10, we can see (4.13). □

By previous propositions, we know how to compute the brackets $[\,,\,]_i$, $(i=1,2)$, defined on $\mathcal{V}_{\mathcal{I}}(\mathfrak{p})/\partial\mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. The next thing we want to show is that the brackets can be understood as brackets on $\mathcal{W}(\mathfrak{g},f,k)/\partial\mathcal{W}(\mathfrak{g},f,k)$:

$$[\;,\;]_i:\mathcal{W}(\mathfrak{g},f,k)/\partial\mathcal{W}(\mathfrak{g},f,k)\times\mathcal{W}(\mathfrak{g},f,k)/\partial\mathcal{W}(\mathfrak{g},f,k)\to\mathcal{W}(\mathfrak{g},f,k)/\partial\mathcal{W}(\mathfrak{g},f,k).$$

We provide proofs in two different ways. One (Proposition 4.13) is by the definition W-algebras in Sect. 3 and the other one (Proposition 4.14) is purely algebraic. Note that the first proof is inspired from [1] and the second one is inspired form [7].

Lemma 4.12. *Let $\mathcal{V}$ be a superalgebra of differential polynomials with both even generators and odd generators. If $G,G'\in\mathcal{V}$ satisfy*

$$\int FG=\int FG' \quad or \quad \overline{FG}=\overline{FG'}\ in\ \mathcal{V}/\partial\mathcal{V}$$

for all $F\in\mathcal{V}_{\bar 0}$ or all $F\in\mathcal{V}_{\bar 1}$ then $G=G'$.

Proof. Let us denote $\mathcal{V}=\mathcal{V}_{\bar 0}\otimes\mathcal{V}_{\bar 1}$, where $\mathcal{V}_0=\mathbb{C}_{\text{diff}}[u_i|i=1,2,\ldots,k_0]$ and $\mathcal{V}_1=\mathbb{C}_{\text{diff}}[v_i|i=1,2,\ldots,k_1]$ for even variables u_i and odd variables v_i.

It is enough to show that if $\int u_i^{(n)}G=0$ (resp. $\int v_i^{(n)}G=0$) for any even (resp. odd) variable $u_i^{(n)}$ (resp. $v_i^{(n)}$) then $G=0$.

Let us first show that if $\int u_i^{(n)}G=0$ for any even variable $u_i^{(n)}$ then $G=0$. If $G\in\mathbb{C}^{\times}$ then $u_1G\notin\partial\mathcal{V}$. Suppose G is not a constant. Take an integer $m\in\mathbb{Z}_{\geq 0}$ such that no monomial in G has terms with $u_1^{(n)}$ for $n\geq m$. Then $u_1^{(m+1)}G\notin\partial\mathcal{V}$. That is because if $\partial G_1=u_1^{(m+1)}G$ then G_1 should have the term $u_1^{(m+1)}G_2$ for a nonconstant element G_2. Hence $\partial G_1=u_1^{(m+1)}G$ has the term with $u_1^{(m+2)}$. This is a contradiction to our assumption and G should be 0.

Suppose $\int v_i^{(n)}G=0$ for any odd variable $v_i^{(n)}$ then $G=0$. We have $G\notin\mathbb{C}^{\times}$ since otherwise $v_1G\notin\partial\mathcal{V}$. Similarly to the previous case, for a nonconstant element G, let us take an integer $m\in\mathbb{Z}_{\geq 0}$ such that no monomial in G has terms with $v_1^{(n)}$ for $n\geq m$. Then $v_1^{(m+2)}G$ is not in $\partial\mathcal{V}$. Here we note that $v_1^{(m+1)}G$ can be in $\partial\mathcal{V}$, for example $\partial(v_1^{(m)}v_1^{(m-1)})=v_1^{(m+1)}v_1^{(m-1)}$. Hence $G=0$. □

Proposition 4.13. *The brackets* (4.7) *induce brackets on $\mathcal{W}(\mathfrak{g},f,k)/\partial\mathcal{W}(\mathfrak{g},f,k)$.*

Proof. It is enough to show that for $\phi,\psi\in\mathcal{W}(\mathfrak{g},f,k)$, we have $\{\phi,\psi\}_1$ and $\{\phi,\psi\}_2$ are also in $\mathcal{W}(\mathfrak{g},f,k)$.

Let $X\in(\mathfrak{n}\otimes\mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar 0}$ and two elements $q,q'\in\mathcal{F}_{\mathfrak{g},f}$ be gauge equivalent by X, i.e.

$$k\partial+q'^{\sigma}-f\otimes 1=e^{\text{ad}X^{\sigma}}(k\partial+q^{\sigma}-f\otimes 1).$$

Note that, since $[s, \mathfrak{n}] = 0$, we have

$$k\partial + q'^{\sigma} - \Lambda \otimes 1 = e^{\mathrm{ad}X^{\sigma}}(k\partial + q^{\sigma} - \Lambda \otimes 1). \tag{4.15}$$

If $r^{\sigma} := e^{-\mathrm{ad}X^{\sigma}} r'^{\sigma} \in \mathcal{F}_{\mathfrak{g},f}$ for some $r' \in \mathcal{F}_{\mathfrak{g},f}$, we have

$$\phi(q) = \phi(q') \text{ and } \phi(q' + \epsilon r') = \phi(q + \epsilon r) \quad \text{for } \epsilon \in \mathbb{C}.$$

Hence, by Lemma 4.8, the equation $\phi(q + r) - \phi(q) = \phi(q' + r') - \phi(q')$ implies

$$\begin{aligned}\int \left(r'^{\sigma} \middle| \frac{\delta\phi(q')}{\delta q} \right) &= \int \frac{\phi(q'+\epsilon r')-\phi(q')}{\epsilon}\Big|_{\epsilon=0} = \int \frac{\phi(q+\epsilon r)-\phi(q)}{\epsilon}\Big|_{\epsilon=0} \\ &= \int \left(r^{\sigma} \middle| \frac{\delta\phi(q)}{\delta q} \right) = \int \left(e^{-\mathrm{ad}X^{\sigma}} r'^{\sigma} \middle| \frac{\delta\phi(q)}{\delta q} \right) = \int \left(r'^{\sigma} \middle| e^{\mathrm{ad}X^{\sigma}} \frac{\delta\phi(q)}{\delta q} \right).\end{aligned} \tag{4.16}$$

By Lemma 4.12 and (4.16), we have $\frac{\delta\phi(q')}{\delta q} = e^{\mathrm{ad}X^{\sigma}} \frac{\delta\phi(q)}{\delta q}$. It is obvious that ψ has the same property. Hence we have

$$\begin{aligned}\{\phi, \psi\}_1(q') &= \left(\frac{\delta\phi(q')}{\delta q} \middle| \left[\frac{\delta\psi(q')}{\delta q}, k\partial + q'^{\sigma} - 1 \otimes \Lambda \right] \right) \\ &= \left(e^{\mathrm{ad}X^{\sigma}} \frac{\delta\phi(q)}{\delta q} \middle| \left[e^{\mathrm{ad}X^{\sigma}} \frac{\delta\psi(q)}{\delta q}, e^{\mathrm{ad}X^{\sigma}}(k\partial + q^{\sigma} - 1 \otimes \Lambda) \right] \right) = \{\phi, \psi\}_1(q).\end{aligned} \tag{4.17}$$

On the other hand, since $e^{\mathrm{ad}X^{\sigma}}(s \otimes 1) = s \otimes 1$, we can prove that $\{\phi, \psi\}_2(q') = \{\phi, \psi\}_2(q)$ by the same arguement. □

The classical W-superalgebra $\mathcal{W}(\mathfrak{g}, f, k)$ is a PVA endowed with the λ-brackets induced from those on the affine PVA $S(\mathbb{C}[\partial] \otimes \mathfrak{g})$ such that

$$\{a_{\lambda} b\}_1 = [a, b] + k\lambda(a|b), \quad \{a_{\lambda} b\}_2 = (s|[a, b]) \quad \text{for } a, b \in \mathfrak{g}.$$

Hence there are Lie brackets $[\ ,\ ]'_i$, $i = 1, 2$, on $\mathcal{W}(\mathfrak{g}, f, k)/\partial\mathcal{W}(\mathfrak{g}, f, k)$ induced by the λ-brackets on $\mathcal{W}(\mathfrak{g}, f, k)$. More precisely,

$$[\textstyle\int W_1, \int W_2]'_i = \int \{W_{1\,\lambda} W_2\}_i|_{\lambda=0}. \tag{4.18}$$

Proposition 4.14. *Brackets* $[\ ,\ ]'_i$ *in* (4.18) *and* $[\ ,\ ]_i$ *in* (4.8) *for* $i = 1, 2$, *defined on* $\mathcal{W}(\mathfrak{g}, f, k)/\partial\mathcal{W}(g, f, k)$ *are same.*

Proof. By Proposition 4.4, for $\phi, \psi \in \mathcal{W}(\mathfrak{g}, f, k)$, we have

$$\{\phi_{\lambda}\psi\}_1 = \sum_{\substack{i,j \in I \\ m,n \in \mathbb{Z}_{\geq 0}}} C^{\phi,\psi}_{i,j} \frac{\partial_R \psi}{\partial_R q^{j\,(n)}} (\lambda + \partial)^n \big([q^i, q^j] + (q^i|q^j)k(\lambda + \partial) \big) (-\lambda - \partial)^m \frac{\partial\phi}{\partial q^{i\,(m)}} \tag{4.19}$$

for the sign consideration $C^{\phi,\psi}_{i,j}$. If we apply $\lambda = 0$ to (4.19) then

$$\{\phi_{\lambda}\psi\}_1|_{\lambda=0} = \sum_{\substack{i,j \in I \\ m,n \in \mathbb{Z}_{\geq 0}}} C^{\phi,\psi}_{i,j} \frac{\partial_R \psi}{\partial_R q^{j\,(n)}} \partial^n \big([q^i, q^j] + (q^i|q^j)k\partial \big) (-\partial)^m \frac{\partial\phi}{\partial q^{i\,(m)}}$$

and

$$
\begin{aligned}
[\textstyle\int \phi, \int \psi]_1' &= \textstyle\int \{\phi_\lambda \psi\}_1|_{\lambda=0} \\
&= \sum_{\substack{i,j\in I\\ m,n\in\mathbb{Z}_{\geq 0}}} \int C_{i,j}^{\phi,\psi}\Big((-\partial)^n \tfrac{\partial_R \psi}{\partial_R q^{j\,(n)}}\Big)\Big([q^i, q^j] + (q^i|q^j)k\partial\Big)(-\partial)^m \tfrac{\partial \phi}{\partial q^{i\,(m)}} \\
&= \sum_{\substack{i,j\in I\\ m,n\in\mathbb{Z}_{\geq 0}}} \int \Big((-\partial)^m \tfrac{\partial_R \phi}{\partial_R q^{i\,(m)}}\Big)[q^i, q^j]\Big((-\partial)^n \tfrac{\partial \psi}{\partial q^{j\,(n)}}\Big) \\
&\quad + \sum_{\substack{i,j\in I\\ m,n\in\mathbb{Z}_{\geq 0}}} \int \Big((-\partial)^m \tfrac{\partial_R \phi}{\partial_R q^{i\,(m)}}\Big)(q^i|q^j)k(-\partial)\Big((-\partial)^n \tfrac{\partial \psi}{\partial q^{j\,(n)}}\Big) \\
&= \sum_{\substack{i,j\in I\\ m,n\in\mathbb{Z}_{\geq 0}}} \int \tfrac{\delta_R \phi}{\delta_R q^i}([q^i, q^j] - (q^i|q^j)k\partial)\tfrac{\delta \psi}{\delta q^j}.
\end{aligned}
\tag{4.20}
$$

Hence $[\int \phi, \int \psi]_1' = [\int \phi, \int \psi]_1$. By same arguments, we have $[\int \phi, \int \psi]_2' = [\int \phi, \int \psi]_2$. □

Theorem 4.15. *Brackets* $[\,,\,]_1$ *and* $[\,,\,]_2$ *are Lie brackets on* $\int \mathcal{W}(\mathfrak{g}, f, k) := \mathcal{W}(\mathfrak{g}, f, k)/\partial \mathcal{W}(\mathfrak{g}, f, k)$.

Proof. We know that if $\{\,_\lambda\,\}$ is a PVA bracket on $\mathcal{P}$ then $\{\,_\lambda\,\}|_{\lambda=0}$ is a Lie algebra bracket on $\mathcal{P}/\partial\mathcal{P}$. Hence the theorem directly follows from Proposition 4.14. □

Remark 4.16. If we consider $\mathcal{L}(\Lambda)$ instead of $\mathcal{L}^\sigma(\Lambda)$, we have

$$
\begin{aligned}
[\textstyle\int \phi, \int \psi]_{\mathcal{L},1} &:= \textstyle\int \Big(\frac{\delta\phi}{\delta q}\Big|\Big[\frac{\delta\psi}{\delta q}, \mathcal{L}_{[1]}(\Lambda)\Big]\Big) \\
&= \sum_{\substack{i,j\in I\\ m,n\in\mathbb{Z}_{\geq 0}}} (-1)^{\mathrm{p}(i)+\mathrm{p}(j)} \int \tfrac{\delta_R \phi}{\delta_R q^i}([q^i, q^j] - (q^i|q^j)k\partial)\tfrac{\delta \psi}{\delta q^j}
\end{aligned}
\tag{4.21}
$$

for bases $\{q_i\}_{i\in I}$ and $\{q^j\}_{j\in I}$ of $\mathfrak{p}$ and $\mathfrak{m}^\perp$ such that $(q_i|q^j) = \delta_{ij}$. In this article, we want to discuss integrable systems associated to a W-superalgebra whose PVA structure induces $[\,,\,]_1$ more than $[\,,\,]_{\mathcal{L},1}$. Hence we prefer to use $\mathcal{L}^\sigma(\Lambda)$ than $\mathcal{L}(\Lambda)$ (see also the Remark 5.15.)

5. super-Hamiltonian Equations and Poisson Vertex Algebras

Let us introduce super-Hamiltonian equations via Poisson vertex algebras. Recall that infinite dimensional Hamiltonian equation on the even differential algebra $\mathcal{P} = \mathbb{C}_{\mathrm{diff}}[\, u_i \mid i \in I\,]$ is an evolution equation of the form

$$
\frac{du}{dt} = H(\partial)\frac{\delta h}{\delta u} \tag{5.1}
$$

where

(1) the Poisson operator $H(\partial) = (H_{ij}(\partial))_{i,j\in I}$ is an $|I| \times |I|$ matrix operator such that
   (i) $H_{ij}(\partial) = \sum_{n=0}^N H_{ij;n}\partial^n$ for $H_{ij;n} \in \mathbb{C}[\,\partial^n u_i \mid i \in I, n \in \mathbb{Z}_{\geq 0}\,]$,
   (ii) if the λ-bracket on $\mathcal{P}$ is defined by $\{u_i\,_\lambda\, u_j\}_H = \sum_{n=0}^N H_{ij;n}\lambda^n$ then the differential algebra $\mathcal{P}$ with the induced λ-bracket $\{\,_\lambda\,\}_H$ is a Poisson vertex algebra,

(2) the Hamiltonian h is an element in $\mathcal{P}$.

Moreover, the Eq. (5.1) can be written with the λ-bracket $\{\ _\lambda\ \}_H$ as follows:

$$\frac{du_i}{dt} = \{h_\lambda u_i\}_H|_{\lambda=0} \quad i \in I.$$

For details, we refer to the paper [3].

Analogously, we define super-Hamiltonian systems and integrable systems when $\mathcal{P}$ is a differential superalgebra.

Definition 5.1. Let $\mathcal{P} = \mathbb{C}_{\text{diff}}[\, u_i \mid i \in I\,]$ be the superalgebra of differential polynomials and let $\mathcal{P}_{\bar{0}}$ and $\mathcal{P}_{\bar{1}}$ be the even and odd subspaces of P.

(1) A *super-Hamiltonian evolution equation* on the differential superalgebra on $\mathcal{P}$ is an evolution equation of the form

$$\frac{d\phi}{dt} = \{\, h\ {}_\lambda\ \phi\,\}|_{\lambda=0}, \quad \phi \in \mathcal{P} \tag{5.2}$$

for some $h \in \mathcal{P}_{\bar{0}}$.

(2) An *integral of motion* of (5.2) is the local functional $\int f \in \int \mathcal{P}$ such that

$$\int \frac{df}{dt} = 0.$$

(3) If (5.2) has infinitely many linearly independent integrals of motion then it is called an *integrable system.*

From now on, we let $\mathcal{P} = \mathbb{C}_{\text{diff}}[\, u_i \mid i \in I\,]$ be the differential superalgebra and u_i be homogeneous variables of $\mathcal{P}$, that is $I = I_{\bar{0}} \cup I_{\bar{1}}$ and u_i for $i \in I_{\bar{0}}$ (resp. $I_{\bar{1}}$) are even (resp. odd.)

Remark 5.2. For $f \in \mathcal{P}$, we let

$$\frac{df}{dt} = \sum_{n \in \mathbb{Z}_{\geq 0}, i \in I} \left(\partial^n \frac{du_i}{dt}\right) \frac{\partial f}{\partial u_i^{(n)}}, \tag{5.3}$$

inspired from chain rules. Then

$$\left[\, \frac{du_i}{dt} = \{\, h\ {}_\lambda\ u_i\,\}|_{\lambda=0} \ \ \text{for any } i \in I \,\right] \text{iff} \left[\, \frac{df}{dt} = \{\, h\ {}_\lambda\ f\,\}|_{\lambda=0} \ \ \text{for any } f \in \mathcal{P} \,\right]. \tag{5.4}$$

For the following proposition, recall if $\mathcal{P}$ is a PVA with a λ-bracket $\{\ _\lambda\ \}$ then the superspace $\mathcal{P}/\partial\mathcal{P}$ is a Lie superalgebra endowed with the bracket $[\int f, \int g] := \int \{f_\lambda g\}|_{\lambda=0}$.

Proposition 5.3. (Generalized Lenard–Magri scheme) *Suppose $\mathcal{P}$ is endowed with two compatible λ-brackets $\{\ _\lambda\ \}_H$ and $\{\ _\lambda\ \}_K$. If there are linearly independent even elements $\int h_i$, $i \in \mathbb{Z}_{\geq 0}$, in $\int \mathcal{P}$ such that*

$$[\textstyle\int h_m, \int u_i]_H = [\int h_{m+1}, \int u_i]_K \quad \text{for } m \in \mathbb{Z}_{\geq 0} \text{ and } i \in I, \tag{5.5}$$

then $\frac{d\phi}{dt} = \{h_m\, {}_\lambda \phi\}_H|_{\lambda=0}$ for $m \in \mathbb{Z}_{\geq 0}$ are integrable systems and $\int h_{m'}$, $m' \in \mathbb{Z}_{\geq 0}$ are integrals of motion.

Proof. If we assume $m > n$ then

$$[\int h_m, \int h_n]_H = [\int h_m, \int h_{n+1}]_K \text{ and } [\int h_m, \int h_n]_K = [\int h_{m-1}, \int h_n]_H.$$

Inductively, we can prove that if $m-n$ is odd then $[\int h_m, \int h_n]_H = [\int h_l, \int h_l]_K = 0$ (resp. $[\int h_m, \int h_n]_K = [\int h_l, \int h_l]_H = 0$) for some l with $m \geq l > n$ (resp. $m > l \geq n$). If $m-n$ is even then $[\int h_m, \int h_n]_H = [\int h_l, \int h_l]_H = 0$ (resp. $[\int h_m, \int h_n]_K = [\int h_l, \int h_l]_K = 0$) for some l with $m > l > n$. □

Remark 5.4. The Lenard–Magri scheme in Proposition 5.3 does not give any clue of finding *odd* integrals of motion.

Consider the adjoint map $\mathrm{ad}\Lambda : \mathfrak{g}((z^{-1})) \to \mathfrak{g}((z^{-1}))$ such that $\mathrm{ad}(\Lambda)(A) = [\Lambda, A]$ for $A \in \mathfrak{g}((z^{-1}))$. In the rest of this section, we assume that Λ is semisimple so that

$$\mathfrak{g}((z^{-1})) = \ker(\mathrm{ad}\Lambda) \oplus \mathrm{im}(\mathrm{ad}\Lambda). \tag{5.6}$$

Remark 5.5. The assumption that Λ is a semisimple element is quite a big constraint. Such Λ does not always exist for any nilpotent element f. However, when $\mathfrak{g}$ is $\mathrm{sl}(m|n)$ and the nilpotent element f corresponds to the pair of partitions λ and μ of m and n which has the form of one of followings:

$$\begin{aligned} &(1)\ \lambda = (r^{n_r}, 1^{n_1}), \quad \mu = (r^{m_r}, 1^{m_1}), \\ &(2)\ \lambda = ((r, r-1)^{p_{r,r-1}}, 1^{p_1}), \quad \mu = ((r, r-1)^{q_{r,r-1}}, 1^{q_1}), \end{aligned} \tag{5.7}$$

we can find an even element s such that Λ is semisimple. Here, $n_r, n_1, m_r, m_1, p_{r,r-1}, p_1, q_{r,r-1}, q_1$ are all nonnegative integers. Note that this remark directly follows from $\mathfrak{g} = \mathrm{sl}_n$ case (see [7] and Remark 2.30).

Take the gradation on $\mathfrak{g}((z^{-1}))$ defined by

$$\deg(z) = -d-1 \text{ and } \deg(g) = j/2 \text{ if } g \in \mathfrak{g} \text{ and } [h/2, g] = (j/2)g$$

and denote by $\mathfrak{g}((z^{-1}))_k$ the subspace of $\mathfrak{g}((z^{-1}))$ consisting of elements with degree k.

Proposition 5.6. *Let $L(\Lambda) = k\partial + q - \Lambda \otimes 1$ be a Lax operator. For a subspace $V \subset \mathfrak{g}((z^{-1}))$, denote $V_t = \mathfrak{g}((z^{-1}))_t \cap V$. There exist unique even element $S(q) \in \bigoplus_{t>0} im(ad\Lambda)_t \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ and unique even element $h(q) \in \bigoplus_{t>-1} ker(ad\Lambda)_t \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ such that*

$$L_0^\sigma(\Lambda) := e^{adS(q)^\sigma} L^\sigma(\Lambda) = k\partial + h(q)^\sigma - \Lambda \otimes 1. \tag{5.8}$$

Proof. We can decompose (5.8) via the gradation on $\mathfrak{g}((z^{-1}))$. Then degree $-\frac{1}{2}$ part of (5.8) is

$$q^\sigma_{-1/2} + [S^\sigma_{1/2}, -\Lambda \otimes 1] = h^\sigma_{-1/2},$$

where $q^\sigma_{-1/2}$, $S^\sigma_{1/2}$ and $h^\sigma_{-1/2}$ are degree $-1/2$, $1/2$ and $-1/2$ part of q^σ, $S(q)^\sigma$ and $h(q)^\sigma$. By (5.6), $S^\sigma_{1/2}$ and $q^\sigma_{-1/2}$ are uniquely determined. Also, since $q^\sigma_{-1/2}$ and Λ are even, both $S^\sigma_{1/2}$ and $q^\sigma_{-1/2}$ are even. Inductively, S^σ_n and h^σ_{n-1} are uniquely determined for $n \geq 1$. More precisely, if S^σ_m and h^σ_{m-1} are known for $m < n$ then degree $n-1$ part of (5.8) is

$$Q^\sigma_{n-1} + [S^\sigma_n, -\Lambda \otimes 1] = h^\sigma_{n-1},$$

where Q^σ_{n-1} is then degree $n-1$ part determined by S^σ_m's for $m < n$. Now, by (5.6), S^σ_n and q^σ_{n-1} are uniquely determined. □

Remark 5.7. By Proposition 5.6 and its proof, we can see that for an operator $L(\Lambda) = k\partial + q - \Lambda \otimes 1$, there is unique $h(q) \in \bigoplus_{t>-1} \ker(\mathrm{ad}\Lambda)_t \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ such that $L_0^\sigma(\Lambda) := e^{\mathrm{ad}S(q)^\sigma} L^\sigma(\Lambda) = k\partial + h(q)^\sigma - \Lambda \otimes 1$ for some $S(q) \in \bigoplus_{t>0} \mathfrak{g}((z^{-1}))_t \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ (which is not necessarily unique).

For $\mathcal{L}(\Lambda) = k\partial + q_{\mathrm{univ}} - \Lambda \otimes 1$, consider

$$H_n = (h(q_{\mathrm{univ}})^\sigma | z^n \Lambda \otimes 1) \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p}), \quad n \in \mathbb{Z}_{\geq 0}. \tag{5.9}$$

and fix

$$H_{-1} = ((h(q_{\mathrm{univ}})^\sigma - \Lambda \otimes 1 | z^{-1} \Lambda \otimes 1) = (-\Lambda \otimes 1 | z^{-1} \Lambda \otimes 1) \in \mathbb{C}. \tag{5.10}$$

Remark 5.8. Note that $(h(q_{\mathrm{univ}})^\sigma | z^n \Lambda \otimes 1) \neq 0$ only if $h(q_{\mathrm{univ}}) \in \mathfrak{g}_{\bar{0}} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})_{\bar{0}}$. Hence

$$H_n = (h(q_{\mathrm{univ}}) | z^n \Lambda \otimes 1) \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p}).$$

Lemma 5.9. *For $H_n \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$, $n \in \mathbb{Z}_{\geq -1}$, we have $H_n \in \mathcal{W}(\mathfrak{g}, f, k)$.*

Proof. It is enough to show that $h(q_{\mathrm{univ}})^\sigma \in \bigoplus_{t>-1} \ker(\mathrm{ad}\Lambda)_t \otimes \mathcal{W}(\mathfrak{g}, f, k)$.

Recall that there exists $X^\sigma \in (\mathfrak{n} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}}$ such that $e^{\mathrm{ad}X^\sigma} \mathcal{L}^\sigma(\Lambda) = k\partial + \sum_{i\in\mathcal{J}} (q_i \otimes w^i)^\sigma - \Lambda \otimes 1$ where w^i generate $\mathcal{W}(\mathfrak{g}, f, k)$. It is obvious that

$$h(\textstyle\sum_{i\in\mathcal{J}} q_i \otimes w^i) \in \bigoplus_{t>-1} \ker(\mathrm{ad}\Lambda)_t \otimes \mathcal{W}(\mathfrak{g}, f, k).$$

More precisely, there is $S \in \bigoplus_{t>0} \mathfrak{g}((z^{-1}))_t \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ such that

$$e^{\mathrm{ad}(S^\sigma + X^\sigma)} \mathcal{L}^\sigma(\Lambda) = e^{\mathrm{ad}S^\sigma} (e^{\mathrm{ad}X^\sigma} \mathcal{L}^\sigma(\Lambda)) = k\partial + h\big(\textstyle\sum_{i\in\mathcal{J}} q_i \otimes w^i\big)^\sigma - \Lambda \otimes 1.$$

Since $S + X \in \bigoplus_{k>0} \mathfrak{g}((z^{-1}))_k \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$, by Remark 5.7, we conclude that

$$h(q_{\mathrm{univ}}) = h(\textstyle\sum_{i\in\mathcal{J}} q_i \otimes w^i) \in \bigoplus_{t>-1} \ker(\mathrm{ad}\Lambda)_t \otimes \mathcal{W}(\mathfrak{g}, f, k).$$

Thus $H_n \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p}) = (h(q_{\mathrm{univ}}) | z^n \Lambda \otimes 1) \in \mathcal{W}(\mathfrak{g}, f, k)$. □

Lemma 5.10. *Let S be the same as in (5.8) and $r \in (\mathfrak{m}^\perp \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}}$. Then we have*

$$\int \left(r^\sigma \,\middle|\, \frac{\delta H_n}{\delta q} \right) = \int \left(r^\sigma \,\middle|\, e^{-adS^\sigma} (z^n \Lambda \otimes 1) \right), \quad n \in \mathbb{Z}_{\geq -1}, \tag{5.11}$$

where $q = (q^i)_{i\in I}$ for a basis q^i of $\mathfrak{p}$.

Proof. If $n = -1$, we have

$$e^{-\mathrm{ad}S^\sigma} (z^n \Lambda \otimes 1) \in \mathfrak{g}[z]z \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$$

so that $\left(r^\sigma \,\middle|\, e^{-\mathrm{ad}S^\sigma}(z^n \Lambda \otimes 1)\right) = 0$. Since $\frac{\delta H_{-1}}{\delta q} = 0$, the both sides of (5.11) are 0.

To show the proposition when $n \geq 0$, recall that

$$\int \left. \frac{H_n(q + \epsilon r) - H_n(q)}{\epsilon} \right|_{\epsilon=0} = \int \left(r^\sigma \,\middle|\, \frac{\delta H_n}{\delta q} \right)$$

for $r \in (\mathfrak{m}^\perp \otimes \mathcal{V}(\mathfrak{p}))_{\bar{0}}$ and $q = \sum_{i\in I} q_i \otimes Q^i$. Let us define $L_0(\epsilon)$ and $S(\epsilon)$ by

$$L_0^\sigma(\epsilon) = e^{\mathrm{ad}S(\epsilon)^\sigma} (\partial + (q^\sigma + \epsilon r^\sigma) - \Lambda \otimes 1) = \partial + h(q + \epsilon r)^\sigma - \Lambda \otimes 1.$$

Then

$$\left(\tfrac{d}{d\epsilon}L_0^\sigma(\epsilon)\big|\, z^n\Lambda\otimes 1\right)\Big|_{\epsilon=0} = \frac{H_n(q+\epsilon r)-H_n(q)}{\epsilon}\Big|_{\epsilon=0}.$$

By the process finding $S(\epsilon)$ in Proposition 5.6, we have

$$S(\epsilon) = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \cdots \in \mathrm{im}(\mathrm{ad}\Lambda)((z^{-1}))\otimes\mathcal{V}_{\mathcal{I}}(\mathfrak{p})[\epsilon],$$

for $S_0, S_1, \ldots \in \mathfrak{g}((z^{-1}))\otimes\mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. Also, we have

$$\tfrac{d}{d\epsilon}L_0^\sigma(\epsilon)\big|_{\epsilon=0} = e^{\mathrm{ad}S(\epsilon)^\sigma} r^\sigma\big|_{\epsilon=0} + \tfrac{d}{d\epsilon}\left(e^{\mathrm{ad}S(\epsilon)^\sigma}L^\sigma(0)\right)\big|_{\epsilon=0} \tag{5.12}$$

where $L^\sigma(0) = \partial + q^\sigma - \Lambda\otimes 1$ and $e^{\mathrm{ad}S_0^\sigma}L^\sigma(0) = L_0^\sigma(0) = \partial + h^\sigma(q) - \Lambda\otimes 1$. In order to investigate the last term in (5.12), we need the following facts:

(i) $\tfrac{d}{d\epsilon}\left(e^{\mathrm{ad}S(\epsilon)^\sigma}L^\sigma(0)\right)\big|_{\epsilon=0} = \sum_{n\in\mathbb{Z}_{\geq 1}}\tfrac{1}{n!}\sum_{m=0}^{n-1}\mathrm{ad}^m S_0^\sigma(\mathrm{ad}S_1^\sigma(\mathrm{ad}^{n-1-m}S_0^\sigma(L^\sigma(0))))$,

(ii) $\tfrac{1}{n!}\sum_{m=0}^{n-1}\mathrm{ad}^m S_0^\sigma(\mathrm{ad}S_1^\sigma(\mathrm{ad}^{n-1-m}S_0^\sigma(L^\sigma(0))))$
$= \sum_{m=0}^{n-1}\left[\tfrac{1}{(n-m)!}\mathrm{ad}^{n-m-1}S_0^\sigma(S_1^\sigma),\ \tfrac{1}{m!}\mathrm{ad}^m S_0^\sigma(L^\sigma(0))\right]$,

(iii) $\sum_{n\in\mathbb{Z}_{\geq 1}}\sum_{m=0}^{n-1}\left[\tfrac{1}{(n-m)!}\mathrm{ad}^{n-m-1}S_0^\sigma(S_1^\sigma),\ \tfrac{1}{m!}\mathrm{ad}^m S_0^\sigma(L^\sigma(0))\right]$
$= \sum_{l\in\mathbb{Z}_{\geq 0}}\left[\tfrac{1}{(l+1)!}\mathrm{ad}^l S_0^\sigma(S_1^\sigma),\, L_0^\sigma(0)\right].$

(5.13)

By (5.12) and (5.13), we have

$$\tfrac{d}{d\epsilon}\left(e^{\mathrm{ad}S(\epsilon)^\sigma}L^\sigma(0)\right)\big|_{\epsilon=0} = \sum_{l\in\mathbb{Z}_{\geq 0}}\left[\tfrac{1}{(l+1)!}\mathrm{ad}^l S_0^\sigma(S_1^\sigma),\, L_0^\sigma(0)\right].$$

Observe that

$$\int\left(r^\sigma\,\Big|\,\tfrac{\delta H_n}{\delta q}\right) = \int\left(\tfrac{d}{d\epsilon}L_0^\sigma(\epsilon)\big|\, z^n\Lambda\otimes 1\right)\big|_{\epsilon=0} \tag{5.14}$$

and

$$\begin{aligned}
&\left(\frac{d}{d\epsilon}L_0^\sigma(\epsilon)\Big|\, z^n\Lambda\otimes 1\right)\Big|_{\epsilon=0} = \left(e^{\mathrm{ad}S(\epsilon)^\sigma}r^\sigma\,\big|\, z^n\Lambda\otimes 1\right)\big|_{\epsilon=0}\\
&\quad + \sum_{l\in\mathbb{Z}_{\geq 0}}\frac{1}{(l+1)!}\left(\left[\mathrm{ad}^l S_0^\sigma(S_1^\sigma),\, L_0^\sigma(0)\right]\Big|\, z^n\Lambda\otimes 1\right)\Big|_{\epsilon=0}\\
&= \left(r^\sigma\,\big|e^{-\mathrm{ad}S(0)^\sigma}(z^n\Lambda\otimes 1)\right)\\
&\quad + \sum_{l\in\mathbb{Z}_{\geq 0}}\frac{1}{(l+1)!}\left(\mathrm{ad}^l S_0^\sigma(S_1^\sigma)\Big|\,[h^\sigma(q)-\Lambda\otimes 1, z^n\Lambda\otimes 1]\right)\Big|_{\epsilon=0}\\
&\quad + \sum_{l\in\mathbb{Z}_{\geq 0}}\tfrac{1}{(l+1)!}\left(-\partial(\mathrm{ad}^l S_0^\sigma(S_1^\sigma))\big|\, z^n\Lambda\otimes 1\right)\big|_{\epsilon=0}\\
&= \left(r^\sigma\,\big|e^{-\mathrm{ad}S(0)^\sigma}(z^n\Lambda\otimes 1)\right) + \sum_{l\in\mathbb{Z}_{\geq 0}}\tfrac{1}{(l+1)!}\left(-\partial(\mathrm{ad}^l S_0^\sigma(S_1^\sigma))\big|\, z^n\Lambda\otimes 1\right)\big|_{\epsilon=0}.
\end{aligned} \tag{5.15}$$

The last term in (5.15) is in $\partial\mathcal{V}_{\mathcal{I}}(\mathfrak{p})$. Hence, for $S(0)^\sigma = S^\sigma$, we have

$$\int\left(r^\sigma\,\Big|\,\tfrac{\delta H_n}{\delta q}\right) = \int\left(r^\sigma\,\big|e^{-\mathrm{ad}S^\sigma}(z^n\Lambda\otimes 1)\right).$$

□

Proposition 5.11. *Recall that* $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}$ *and suppose* $\{q^i | i \in I\}$ *is a basis of* $\mathfrak{p}$ *and* $\{q^i | i \in I'\}$ *is a basis of* $\mathfrak{m}$. *For* $Q = \sum_{i\in I} q^i \otimes Q_i + \sum_{i\in I'} q^i \otimes Q_i \in \mathfrak{g} \otimes \mathcal{V}(\mathfrak{p})$, *we denote* $Q|_{\mathfrak{p}} = \sum_{i\in I} q^i \otimes Q_i \in \mathfrak{p} \otimes \mathcal{V}(\mathfrak{p})$. *Then we have*

$$\frac{\delta H_n}{\delta q} = e^{-adS^\sigma}(z^n \Lambda \otimes 1)\Big|_{\mathfrak{p}}.$$

Proof. It directly follows from Lemma 5.10. □

Proposition 5.12. *Let* $\phi \in \mathcal{W}(\mathfrak{g}, f, k)$ *and* $a_{\mathfrak{m}} \otimes F \in (\mathfrak{m} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})_{\bar{0}}$. *Then we have*

$$\int \left(\frac{\delta\phi}{\delta q} \Big| [(a_{\mathfrak{m}} \otimes F)^\sigma, \mathcal{L}^\sigma_{[1]}(\Lambda)] \right) = \int \left(\frac{\delta\phi}{\delta q} \Big| z^{-1}[(a_{\mathfrak{m}} \otimes F)^\sigma, zs \otimes 1] \right) = 0.$$

Proof. Let us denote $S = a_{\mathfrak{m}} \otimes F \in (\mathfrak{m} \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p}))_{\bar{0}}$ and denote

$$\mathcal{L}^\sigma(\epsilon) := e^{\mathrm{ad}\epsilon S^\sigma}\mathcal{L}^\sigma(\Lambda).$$

Then $\mathcal{L}^\sigma(\epsilon) = \partial + q^\sigma(\epsilon) - \Lambda \otimes 1$ for $q^\sigma(\epsilon) = q^\sigma_{\mathrm{univ}} + \epsilon[S^\sigma, \mathcal{L}^\sigma(\Lambda)] + o(\epsilon^2)$. Since ϕ is gauge invariant, we have

$$0 = \int \frac{d\phi(q(\epsilon))}{d\epsilon}\Big|_{\epsilon=0} = \int \left(\frac{\delta\phi}{\delta q} \Big| [(a_{\mathfrak{m}} \otimes F)^\sigma, \mathcal{L}^\sigma(\Lambda)] \right)$$

for any F which has the same parity as $a_{\mathfrak{m}}$. Hence $\int \left(\frac{\delta\phi}{\delta q} \Big| [(a_{\mathfrak{m}} \otimes F)^\sigma, \mathcal{L}^\sigma_{[1]}(\Lambda)] \right) = 0.$

Also, the second equality follows from $[a_{\mathfrak{m}}, s] = 0$. □

Proposition 5.13. *Let* S *be the same as in* (5.8) *and let* $\phi \in \mathcal{W}(\mathfrak{g}, f, k)$. *Then we have*

$$\begin{aligned} &\int \left(\frac{\delta\phi}{\delta q} \Big| \left[\frac{\delta H_n}{\delta q}, \mathcal{L}^\sigma_{[1]}(\Lambda) \right] \right) = \int \left(\frac{\delta\phi}{\delta q} \Big| \left[e^{-adS^\sigma}(z^n \Lambda \otimes 1), \mathcal{L}^\sigma_{[1]}(\Lambda) \right] \right), \\ &\int \left(\frac{\delta\phi}{\delta q} \Big| \left[\frac{\delta H_n}{\delta q}, s \otimes 1 \right] \right) = \int \left(\frac{\delta\phi}{\delta q} \Big| \left[e^{-adS^\sigma}(z^n \Lambda \otimes 1), s \otimes 1 \right] \right). \end{aligned} \tag{5.16}$$

Proof. It follows from Propositions 5.11 and 5.12. □

Theorem 5.14. *Let us consider* $\mathcal{W}(\mathfrak{g}, f, k)$ *and* H_i *be defined as in* (5.9). *The equation*

$$\frac{du}{dt} = \{H_i\,{}_\lambda\, u\}_1|_{\lambda=0}, \quad u \in \mathcal{W}(\mathfrak{g}, f, k) \text{ and } i \in \mathbb{Z} \tag{5.17}$$

has linearly independent integrals of motion $\int H_j$ *for* $j \in \mathbb{Z}_{\geq 0}$. *Hence* (5.17) *is an integrable system.*

Proof. In order to use Lenard scheme, we aim to show that

$$[\textstyle\int H_i, \int u]_1 = [\int H_{i+1}, \int u]_2, \quad i \in \mathbb{Z}_{\geq -1}. \tag{5.18}$$

Recall that the brackets $[\int H_i, \int u]_1$ and $[\int H_{i-1}, \int u]_2$ are defined by

$$\begin{aligned} &[\textstyle\int H_i, \int u]_1 = \int \left(\frac{\delta H_i}{\delta q} \Big| \left[\frac{\delta u}{\delta q}, \mathcal{L}^\sigma_{[1]}(\Lambda) \right] \right) = -\int \left(\frac{\delta u}{\delta q} \Big| \left[\frac{\delta H_i}{\delta q}, \mathcal{L}^\sigma_{[1]}(\Lambda) \right] \right), \\ &[\textstyle\int H_{i+1}, \int u]_2 = \int \left(\frac{\delta H_{i+1}}{\delta q} \Big| \left[\frac{\delta u}{\delta q}, s \otimes 1 \right] \right) = -\int \left(\frac{\delta u}{\delta q} \Big| \left[\frac{\delta H_{i+1}}{\delta q}, s \otimes 1 \right] \right). \end{aligned} \tag{5.19}$$

Denote $\frac{\delta H(z)}{\delta q} := \sum_{i\in\mathbb{Z}} \frac{\delta H_i}{\delta q} z^{-i}$. Then

$$\left[\frac{\delta H(z)}{\delta q}, \mathcal{L}^\sigma(\Lambda)\right] = \sum_{i\in\mathbb{Z}} \left(\left[\frac{\delta H_i}{\delta q}, \mathcal{L}^\sigma_{[1]}(\Lambda)\right] - \left[\frac{\delta H_{i+1}}{\delta q}, s\otimes 1\right]\right) z^{-i}. \tag{5.20}$$

By (5.19) and (5.20), (5.18) is equivalent to

$$\begin{aligned}
&\int \left(\frac{\delta u}{\delta q}\middle| z^i \left[\frac{\delta H(z)}{\delta q}, \mathcal{L}^\sigma(\Lambda)\right]\right)\\
&= \int \left(\frac{\delta u}{\delta q}\middle| \left[e^{-\mathrm{ad} S^\sigma}(z^i\Lambda\otimes 1), \mathcal{L}^\sigma_{[1]}(\Lambda)\right]\right) + \int \left(\frac{\delta u}{\delta q}\middle| \left[z^{-1}e^{-\mathrm{ad} S^\sigma}(z^{i+1}\Lambda\otimes 1), zs\otimes 1\right]\right)\\
&= \int \left(\frac{\delta u}{\delta q}\middle| \left[e^{-\mathrm{ad} S^\sigma}(z^i\Lambda\otimes 1), \mathcal{L}^\sigma(\Lambda)\right]\right) = 0
\end{aligned} \tag{5.21}$$

for any $\phi \in \mathcal{W}(\mathfrak{g}, f, k)$ and $i \in \mathbb{Z}$. Here, we used Proposition 5.13 for the first equality. Hence we proved (5.19) by the following fact:

$$\left[e^{-\mathrm{ad} S^\sigma}(z^i\Lambda\otimes 1), \mathcal{L}^\sigma(\Lambda)\right] = e^{-\mathrm{ad} S^\sigma}\left[z^i\Lambda\otimes 1, e^{\mathrm{ad} S^\sigma}\mathcal{L}^\sigma(\Lambda)\right] = 0.$$

In particular, since H_{-1} is constant, we have $[\int H_{-1}, \int u]_1 = [\int H_0, \int u]_2 = 0$.

Now, the only thing to prove is that $\{\int H_j\}_{j\in\mathbb{Z}_{\geq 0}}$ is linearly independent. Since, for given $H, u \in \mathcal{W}(g, f, k)$ such that $\{H_\lambda u\}_1|_{\lambda=0} \neq 0$, the total degree of $\{H_\lambda u\}_1|_{\lambda=0}$ is greater than the total degree of $\{H_\lambda u\}_2|_{\lambda=0}$ in the algebra of polynomials $\mathbb{C}[(q^i)^{(n)}|i\in I, n\in\mathbb{Z}_{\geq 0}]$, where $\{q^i|i\in I\}$ is a basis of $\mathfrak{p}$. Hence, if we can show $\{H_{0\,\lambda}\mathcal{W}(g, f, k)\}|_{\lambda=0} \neq 0$ then the linearly independence of $\{\int H_j\}_{j\in\mathbb{Z}_{\geq 0}}$ follows. Indeed, this can be proved as below. Suppose $q^{\mathrm{can}}_{\mathrm{univ}}$ in Proposition 3.14 has the summand $f\otimes\phi_e$. Then $H_0 = \phi_e$ so that $\{H_{0\,\lambda}\mathcal{W}(g, f, k)\}|_{\lambda=0} \neq 0$. □

Remark 5.15. Recall that the formula (5.8) $e^{\mathrm{ad} S(q)^\sigma} L^\sigma(\Lambda)$ is same as $(e^{S(q)}L(\Lambda))^\sigma$. Also, since Λ is even, we have

$$(h(q_{\mathrm{univ}})^\sigma | z^{-n}\Lambda\otimes 1) = (h(q_{\mathrm{univ}})|z^{-n}\Lambda\otimes 1).$$

Hence we can use $L(\Lambda)$ instead of $L(\Lambda)^\sigma$ to compute H_n.

Example 5.16. As in Example 3.15, the Lie superalgebra $\mathfrak{g} = \mathrm{spo}(2|1)$ is generated by e, e_{od}, h, f_{od} and f. For $\Lambda = f + ze$ and $K := -f + ze$, we can see that $\mathfrak{g}((z^{-1}))$ is the $\mathbb{C}((z^{-1}))$-module generated by $e_{od}, f_{od}, h = 2x, \Lambda, K$. The subspace $\mathrm{im}(\mathrm{ad}\Lambda)$ is generated by e_{od}, f_{od}, h, K and $ker(\mathrm{ad}\Lambda)$ is generated by Λ. Note that the Lie brackets between generators are

$$\begin{aligned}
&[x, \Lambda] = K, \quad [f_{od}, \Lambda] = -ze_{od},\\
&[e_{od}, \Lambda] = -f_{od}, \quad [K, \Lambda] = 2zh = 4zx, \quad [x, K] = \Lambda.
\end{aligned}$$

Consider the operator,

$$L(\Lambda) = k\partial + e_{od}\otimes\phi_{\mathrm{od}} + e\otimes\phi_{\mathrm{ev}} - \Lambda\otimes 1$$

for the generators $\phi_{\mathrm{od}} = -\frac{1}{2}f_{od} + \frac{k}{2}\partial e_{od} + \frac{1}{4}he_{od}$ and $\phi_{\mathrm{ev}} = f + \frac{1}{2}f_{od}e_{od} - \frac{1}{4}h^2 + \frac{k}{4}e_{od}\partial e_{od} - \frac{k}{2}\partial h$ of the algebra in Example 3.15. We want to find $h \in \bigoplus_{t>-1}\ker(\mathrm{ad}\Lambda)_t \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ such that

$$L_0(\Lambda) := e^{\mathrm{ad} S}L(\Lambda) = \partial + h - \Lambda\otimes 1. \tag{5.22}$$

for some $S \in \bigoplus_{t>0} \mathrm{im}(\mathrm{ad}\Lambda)_t \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$.

Let us denote $U = \sum_{t>0} U_t$ for $U_t \in \mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \otimes \mathfrak{g}((z^{-1}))_t$ and $h = \sum_{t>-1} h_t$ for $h_t = \mathcal{V}_{\mathcal{I}}(\mathfrak{p}) \otimes \big(\mathfrak{g}((z^{-1}))_t \cap ker(\mathrm{ad}\Lambda)\big)$. Since $z^i\Lambda$ has degree $-1-2i$, we have $h = \sum_{t \in \mathbb{Z}_{>-1}} h_{2t+1}$.

By comparing degree $\frac{1}{2}$ part of (5.22), we have $S_{1/2} = S_1 = 0$ and

$$e_{od} \otimes \phi_{od} - [S_{3/2}, \Lambda \otimes 1] = 0.$$

Hence $S_{3/2} = -z^{-1} f_{od} \otimes \phi_{od}$. By comparing degree 1 part of (5.22), we have

$$e \otimes \phi_{ev} - [S_2, \Lambda \otimes 1] = h_1$$

Hence $S_2 = \frac{1}{2} z^{-1} x \otimes \phi_{ev}$ and $h_1 = \frac{1}{2} z^{-1} \Lambda \otimes \phi_{ev}$ so that

$$H_0 = (h_1 | \Lambda \otimes 1) = \phi_{ev}.$$

By degree $\frac{3}{2}$ and 2 parts of (5.22), we have

$$\begin{aligned} &-[S_{5/2}, \Lambda \otimes 1] + [S_{3/2}, k\partial] = 0 \quad \text{and} \quad S_{5/2} = -z^{-1} e_{od} \otimes k\partial\phi_{od}; \\ &-[S_3, \Lambda \otimes 1] + [S_2, k\partial] + [S_{3/2}, e_{od} \otimes \phi_{od}] - [S_{3/2}, [S_{3/2}, \Lambda \otimes 1]] = 0 \end{aligned}$$

and $S_3 = -\frac{1}{8} z^{-2} K \otimes k\partial\phi_{ev}$. By degree 3 part of (5.22), we have

$$\begin{aligned} &-[S_4, \Lambda \otimes 1] + [S_3, k\partial] + [S_{5/2}, e_{od} \otimes \phi_{od}] + [S_2, e \otimes \phi_{ev}] \\ &\quad - \frac{1}{2}[S_2, [S_2, \Lambda \otimes 1]] - \frac{1}{2}[S_{3/2}, [S_{5/2}, \Lambda \otimes 1]] - \frac{1}{2}[S_{5/2}, [S_{3/2}, \Lambda \otimes 1]] \\ &\quad + \frac{1}{2}[S_{3/2}, [S_{3/2}, k\partial]] = h_3 \end{aligned} \tag{5.23}$$

Since the LHS of (5.23) is

$$\begin{aligned} &-[S_4, \Lambda \otimes 1] + z^{-2} \Lambda \otimes \left(\frac{1}{8}\phi_{ev}^2 + \frac{k}{2} \partial\phi_{od}\phi_{od} \right) \\ &\quad + z^{-2} K \otimes \left(\frac{k^2}{8} \partial^2 \phi_{ev} + \frac{1}{4}\phi_{ev}^2 + \frac{k}{2}\partial\phi_{od}\phi_{od} \right) \end{aligned}$$

and K is in the image of $\mathrm{ad}\Lambda$, we can conclude $h_3 = z^{-2}\Lambda \otimes \left(\frac{1}{8}\phi_{ev}^2 + \frac{k}{2}\partial\phi_{od}\phi_{od}\right)$ so that

$$H_1 = (h_3 | z\Lambda \otimes 1) = \frac{1}{4}\phi_{ev}^2 + k\partial\phi_{od}\phi_{od}.$$

Note that h_3 is the first integral of motion which gives rise to a nonlinear Hamiltonian equation. Now we get

$$\begin{cases} \dfrac{d\phi_{ev}}{dt} = \{H_1 {}_\lambda \phi_{ev}\}_H|_{\lambda=0} = -\dfrac{k^3}{4}\partial^3\phi_{ev} - \dfrac{3k}{2}\partial\phi_{ev}\phi_{ev} + 3k^2\phi_{od}\partial^2\phi_{od}, \\ \dfrac{d\phi_{od}}{dt} = \{H_1 {}_\lambda \phi_{od}\}_H|_{\lambda=0} = k^3\partial^3\phi_{od} - \dfrac{3k}{2}\partial\phi_{od}\phi_{ev} - \dfrac{3k}{4}\phi_{od}\partial\phi_{ev}, \end{cases}$$

which is same as super-KdV equation in [21] up to constant factors.

Example 5.17. Let $\mathfrak{g} = \mathrm{sl}(2|1)$ and $f = e_{21}$. With the notations in Example 3.16, we consider the operator

$$L(\Lambda) = k\partial + \tau \otimes \phi_\tau + e_{1\bar{1}} \otimes \phi_{1\bar{1}} + e_{\bar{1}2} \otimes \phi_{\bar{1}2} + e \otimes \phi_e - \Lambda \otimes 1.$$

Recall that $\phi_e = -kL - 2\phi_t^2$ for the energy momentum field L. It is not hard to check that

$$\{\phi_{\tau\,\lambda}\phi_\tau\} = -\frac{1}{2}k\lambda,\ \{\phi_{\tau\,\lambda}\phi_{1\bar{1}}\} = -\frac{1}{2}\phi_{1\bar{1}},\ \{\phi_{\tau\,\lambda}\phi_{\bar{1}2}\} = \frac{1}{2}\phi_{\bar{1}2},\ \{\phi_{1\bar{1}\,\lambda}\phi_{1\bar{1}}\} = \{\phi_{\bar{1}2\,\lambda}\phi_{\bar{1}2}\} = 0.$$

Also,

$$\{\phi_{1\bar{1}\,\lambda}\phi_{\bar{1}2}\} = kL - k\partial\phi_\tau - 2k\lambda\phi_\tau - k^2\lambda^2,\ \ \{\phi_{\bar{1}2\,\lambda}\phi_{1\bar{1}}\} = kL + k\partial\phi_\tau + 2k\lambda\phi_\tau - k^2\lambda^2,$$

for the energy momentum field L in Proposition 2.15. Recall that $\phi_e = -kL - 2\phi_\tau^2$ and λ-brackets between ϕ_e and elements in $\mathcal{W}(\mathfrak{g}, f, k)$ can be computed using

$$\{L_\lambda\phi_\tau\} = (\partial + \lambda)\phi_\tau, \quad \{L_\lambda\phi_{1\bar{1}}\} = (\partial + \frac{3}{2}\lambda)\phi_{1\bar{1}} \quad \{L_\lambda\phi_{\bar{1}2}\} = (\partial + \frac{3}{2}\lambda)\phi_{\bar{1}2},$$

and

$$\{L_\lambda L\} = (\partial + 2\lambda)L - \frac{1}{2}k\lambda^3.$$

Now, let us consider $S \in \bigoplus_{t>0} \mathrm{im}(\mathrm{ad}\Lambda)_t \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ and $h \in \bigoplus_{t>-1} \ker(\mathrm{ad}\Lambda)_t \otimes \mathcal{V}_{\mathcal{I}}(\mathfrak{p})$ such that

$$L_0(\Lambda) = k\lambda + h - \Lambda \otimes 1 = e^{\mathrm{ad}S}L(\Lambda). \tag{5.24}$$

By equating degree $\leq 1/2$-parts of the both sides of (5.24), we get

$$S_{1/2} = S_1 = 0, \text{ and } S_{3/2} = -z^{-1}e_{2\bar{1}} \otimes \phi_{1\bar{1}} + z^{-1}e_{\bar{1}1} \otimes \phi_{\bar{1}2}.$$

By equating degree 1-parts of the both sides of (5.24):

$$[S_2, \Lambda \otimes 1] + h_1 = e \otimes \phi_e = \frac{1}{2}z^{-1}\Lambda \otimes \phi_e + \frac{1}{2}z^{-1}K \otimes \phi_e,$$

for $K = ze - f$, we get

$$S_2 = \frac{1}{2}z^{-1}x \otimes \phi_e, \quad h_1 = \frac{1}{2}z^{-1}\Lambda \otimes \phi_e, \quad H_0 = (h_1|\Lambda \otimes 1) = \phi_e.$$

Hence

$$\begin{cases} \dfrac{d\phi_\tau}{dt} = \{\phi_{e\,\lambda}\phi_\tau\}|_{\lambda=0} = k(\lambda + \partial)\phi_\tau|_{\lambda=0} = k\partial\phi_\tau, \\ \dfrac{d\phi_{1\bar{1}}}{dt} = \{\phi_{e\,\lambda}\phi_{1\bar{1}}\}|_{\lambda=0} = -k(\partial + \dfrac{3}{2}\lambda)\phi_{1\bar{1}} + 2\phi_{1\bar{1}}\phi_\tau\Big|_{\lambda=0} = -k\partial\phi_{1\bar{1}} + 2\phi_{1\bar{1}}\phi_\tau, \\ \dfrac{d\phi_{\bar{1}2}}{dt} = \{\phi_{e\,\lambda}\phi_{\bar{1}2}\}|_{\lambda=0} = -k(\partial + \dfrac{3}{2}\lambda)\phi_{\bar{1}2} - 2\phi_{\bar{1}2}\phi_\tau\Big|_{\lambda=0} = -k\partial\phi_{\bar{1}2} - 2\phi_{\bar{1}2}\phi_\tau, \\ \dfrac{d\phi_e}{dt} = \{\phi_{e\,\lambda}\phi_e\}|_{\lambda=0} = k^2(\partial + 2\lambda)L - \dfrac{k^3}{2}\lambda^3\Big|_{\lambda=0} = k^2\partial L = -k\partial(\phi_e + 2\phi_\tau^2), \end{cases} \tag{5.25}$$

is the simplest integrable system associated to sl(2|1).

The degree 2-part of (5.24) is

$$\begin{aligned}[S_{5/2}, \Lambda \otimes 1] &= [S_{3/2}, k\partial + \tau \otimes \phi_\tau] \\ &= z^{-1}e_{2\bar{1}} \otimes (k\partial\phi_{1\bar{1}} - \phi_\tau\phi_{1\bar{1}}) + z^{-1}e_{\bar{1}1} \otimes (-k\partial\phi_{\bar{1}2} - \phi_\tau\phi_{\bar{1}2}).\end{aligned} \tag{5.26}$$

Hence $S_{5/2} = z^{-1}e_{1\bar{1}} \otimes (-k\partial\phi_{1\bar{1}} + \phi_\tau\phi_{1\bar{1}}) + z^{-1}e_{\bar{1}2} \otimes (-k\partial\phi_{\bar{1}2} - \phi_\tau\phi_{\bar{1}2})$.

The degree 2-part of (5.24) is

$$\begin{aligned}&[S_3, \Lambda \otimes 1] + h_2 \\ &= [S_2, k\partial + \tau \otimes \phi_\tau] + [S_{3/2}, e_{1\bar{1}} \otimes \phi_{1\bar{1}} + e_{\bar{1}2} \otimes \phi_{\bar{1}2}] - \frac{1}{2}[S_{3/2}, [S_{3/2}, \Lambda \otimes 1]].\end{aligned} \tag{5.27}$$

Hence $S_3 = z^{-2}K \otimes -\frac{k}{8}\partial\phi_e$ and $h_2 = z^{-1}\tau \otimes \frac{1}{2}\phi_{1\bar{1}}\phi_{\bar{1}2}$.

The degree 3-part of (5.24) is

$$\begin{aligned}&[S_4, \Lambda \otimes 1] + h_3 \\ &= [S_3, k\partial + \tau \otimes \phi_\tau] + [S_{5/2}, e_{1\bar{1}} \otimes \phi_{1\bar{1}} + e_{\bar{1}2} \otimes \phi_{\bar{1}2}] + \frac{1}{2}[S_{3/2}, [S_{3/2}, k\partial + \tau \otimes \phi_\tau]] \\ &\quad - \frac{1}{2}[S_2, [S_2, \Lambda \otimes 1]] - \frac{1}{2}[S_{3/2}, [S_{5/2}, \Lambda \otimes 1]] - \frac{1}{2}[S_{5/2}, [S_{3/2}, \Lambda \otimes 1]].\end{aligned} \tag{5.28}$$

Hence $S_4 = z^2 h \otimes \frac{1}{2}\left(\frac{k^2}{8}\partial^2\phi_e + \frac{1}{2}\phi_\tau\phi_{\bar{1}2}\phi_{1\bar{1}} + \frac{k}{2}\partial\phi_{\bar{1}2}\phi_{1\bar{1}} - \frac{k}{2}\phi_{\bar{1}2}\partial\phi_{1\bar{1}}\right)$ and

$$h_3 = z^{-1}\Lambda \otimes \left(-\frac{1}{8}\phi_e^2 + \frac{1}{2}\phi_\tau\phi_{\bar{1}2}\phi_{1\bar{1}} + \frac{k}{2}\partial\phi_{\bar{1}2}\phi_{1\bar{1}} - \frac{k}{2}\phi_{\bar{1}2}\partial\phi_{1\bar{1}}\right)$$

so that

$$H_1 = (h_3|z\Lambda \otimes 1) = -\frac{1}{4}\phi_e^2 + \phi_\tau\phi_{\bar{1}2}\phi_{1\bar{1}} + k\partial\phi_{\bar{1}2}\phi_{1\bar{1}} - k\phi_{\bar{1}2}\partial\phi_{1\bar{1}}.$$

Hence the second integrable system associated to sl_2, getting by the Hamiltonian $4H_1$, is

$$\begin{cases} \dfrac{d\phi_\tau}{dt} = 6k(\partial\phi_{1\bar{1}}\phi_{\bar{1}2} + \phi_{1\bar{1}}\partial\phi_{\bar{1}2}), \\ \dfrac{d\phi_{1\bar{1}}}{dt} = \phi_{1\bar{1}}(8\phi_\tau^3 - 8\phi_e\phi_\tau - 12k\phi_\tau\partial\phi_\tau - 3k\partial\phi_3 + 4k^2\partial^2\phi_\tau), \\ \qquad + \partial\phi_{1\bar{1}}(6k\phi_e - 24k\phi_\tau^2 + 16k^2\partial\phi_\tau) + 20k^2\partial^2\phi_{1\bar{1}}\phi_\tau + 8k^3\partial^3\phi_{1\bar{1}} \\ \dfrac{d\phi_{\bar{1}2}}{dt} = \phi_{\bar{1}2}(-8\phi_\tau^3 + 8\phi_e\phi_\tau - 12\phi_\tau\partial\phi_\tau + 3k\partial\phi_e - 4k^2\partial^2\phi_\tau), \\ \qquad + \partial\phi_{\bar{1}2}(10k\phi_e - 24k\phi_\tau^2 - 16k^2\partial\phi_\tau) - 12k^2\partial^2\phi_{\bar{1}2}\phi_\tau - 8k^3\partial^3\phi_{\bar{1}2}, \\ \dfrac{d\phi_e}{dt} = -12k\partial H_1 + k^3\partial\phi_e + 24k\phi_\tau\phi_{\bar{1}2}\partial\phi_{1\bar{1}} - 24k\phi_\tau\phi_{1\bar{1}}\partial\phi_{\bar{1}2}. \end{cases} \tag{5.29}$$

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