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Ergodicity for the Stochastic Quantization Problems on the 2D-Torus

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Abstract: In this paper we study the stochastic quantization problem on the two dimensional torus and establish ergodicity for the solutions. Furthermore, we prove a characterization of the Φ_2^4 quantum field on the torus in terms of its density under translation. We also deduce that the Φ_2^4 quantum field on the torus is an extreme point in the set of all L-symmetrizing measures, where L is the corresponding generator.

1. Introduction

In this paper we consider stochastic quantization equations on $\mathbb{T}^2$: Let $H = L^2(\mathbb{T}^2)$:

$$\begin{aligned} dX &= (AX - a_1 : X^3 : + a_2 X)dt + dW(t), \\ X(0) &= x, \end{aligned} \tag{1.1}$$

where $A : D(A) \subset H \to H$ is the linear operator

$$Ax = \Delta x - x, \quad D(A) = H^2(\mathbb{T}^2).$$

$: x^3 :$, i.e., Wick power, whose definition will be given in Sect. 2, means the renormalization of x^3 $a_1 > 0$ and a_2 is a real parameter. W is the $L^2(\mathbb{T}^2)$-cylindrical $(\mathcal{F}_t)$-Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a normal filtration $(\mathcal{F}_t)$.

This equation arises in the stochastic quantization of Euclidean quantum field theory. For $a_2 > 0$, it is also called the Allen-Cahn equation in [BDW16] and the references therein. Consider the measure

$$\nu(d\phi) = ce^{-2\int :q(\phi):d\xi}\mu(d\phi),$$

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where $q(\phi) = \frac{a_1}{4}\phi^4 - \frac{a_2}{2}\phi^2$, c is a normalization constant and μ is the Gaussian free field. The latter will be defined in Sect. 2 as well as the renormalization $: \cdot :$. ν is called the Φ^4_2-quantum field in Euclidean quantum field theory. By heuristical calculations, ν is an invariant measure for the solution to (1.1), which has been made rigorous in [RZZ15]. There have been many approaches to the problem of giving a meaning to the above heuristic measure in the two dimensional case and the three dimensional case (see [GRS75, GlJ86, S74] and the references therein). In [PW81] Parisi and Wu proposed a program for Euclidean quantum field theory of getting Gibbs states of classical statistical mechanics as limiting distributions of stochastic processes, especially those which are solutions to non-linear stochastic differential equations. Then one can use the stochastic differential equations to study the properties of the Gibbs states. This procedure is called stochastic field quantization (see [JLM85]). The Φ^4_2 model is the simplest non-trivial Euclidean quantum field (see [GlJ86] and the reference therein). The issue of the stochastic quantization of the Φ^4_2 model is to solve the Eq. (1.1) and to prove that the invariant measure is the limit of the time marginals as $t \to \infty$. The marginals converge to the Euclidean quantum field.

In [JLM85] the existence of an ergodic, continuous, Markov process having ν as an invariant measure has been proved, where ν is constructed above with A changed to the Dirichlet Laplacian on a bounded domain. In fact, they consider the Markov process given by the solution to the following equation for $0 < \varepsilon < \frac{1}{10}$

$$
\begin{aligned}
&dX = [-(-A)^{\varepsilon}X - (-A)^{-1+\varepsilon}(a_1 : X^3 : + a_2 X)]dt + (-A)^{-\frac{1}{2}+\frac{\varepsilon}{2}}dW(t),\\
&X(0) = x,
\end{aligned}
$$

which is easier to solve than (1.1) (corresponding to the case $\varepsilon = 1$) because of the regularization of the operator A. Moreover, they prove that the associated semigroup converges to ν in the L^2-sense. In [AR91] weak solutions to (1.1) have been constructed by using the Dirichlet form approach in the finite and infinite volume case. In [MR99] the stationary solution to (1.1) has also been considered in their general theory of martingale solutions for stochastic partial differential equations. In [DD03] again in the case of the torus, i.e. in finite volume, Da Prato and Debussche define the Wick powers of solutions to the stochastic heat equation in the paths space and study a shifted equation instead of (1.1) in the finite volume case. They split the unknown X into two parts: $X = Y_1 + Z_1$, where $Z_1(t) = \int_{-\infty}^{t} e^{(t-s)A}dW(s)$. Observe that Y_1 is much smoother than X and that in the stationary case

$$
: X^k := \sum_{l=0}^{k} C_k^l Y_1^l : Z_1^{k-l} :, \tag{1.2}
$$

with $C_k^l = \frac{k!}{l!(k-l)!}$ being a binomial coefficient and $: Z_1^{k-l} :$ being the Wick product, which motivated them to consider the following shifted equation:

$$
\begin{aligned}
&\frac{dY_1}{dt} = AY_1 - a_1 \sum_{l=0}^{3} C_3^l Y_1^l : Z_1^{3-l} : + a_2(Y_1 + Z_1)\\
&Y_1(0) = x - Z_1(0).
\end{aligned} \tag{1.3}
$$

They obtain local existence and uniqueness of the solution Y_1 to (1.3) by a fixed point argument. By using the invariant measure ν they obtain a global solution to (1.1) by defining $X = Y_1 + Z$ starting from almost every starting point. In [MW15] the authors consider the following equivalent equation:

$$\begin{aligned}\frac{dY}{dt} &= AY - a_1 \sum_{l=0}^{3} C_3^l Y^l : \bar{Z}^{3-l} : + a_2(Y + \bar{Z}) \\ Y(0) &= 0,\end{aligned} \tag{1.4}$$

where $\bar{Z}(t) = e^{tA}x + \int_0^t e^{(t-s)A} dW(s)$ and $: \bar{Z}^{k-1-l} :$ will be defined later. We call (1.4) *the shifted equation* for short. They obtain global existence and uniqueness of the solution to (1.4) directly from every starting point both in the finite and infinite volume case. Actually, (1.3) is equivalent to (1.4). For the solution Y_1 to (1.3), defining $Y(t) := Y_1(t) + e^{tA} Z_1(0) - e^{tA}x$, we can easily check that Y is a solution to (1.4) by using the binomial formula (2.2) below.

In [RZZ15] we prove that $X - \bar{Z}$, where X is obtained by the Dirichlet form approach in [AR91] and $\bar{Z}(t) = \int_0^t e^{(t-s)A} dW(s) + e^{tA}x$, also satisfies the shifted equation (1.4). Moreover, we obtain that the Φ_2^4 quantum field ν is an invariant measure for the process $X_0 = Y + \bar{Z}$, where Y is the unique solution to the shifted equation (1.4). It is natural to ask whether this invariant measure ν is the unique invariant measure for X_0. If ν is the unique invariant measure for X_0, then ν is the limiting distribution of the stochastic processes X_0. This problem is the main point in the stochastic field quantization as we mentioned above in the Φ_2^4 model on the torus.

This problem has been studied in [AKR97] and the references therein. It is proved in [AKR97] that the stochastic quantization of a Guerra–Rosen–Simon Gibbs state on $\mathcal{S}'(\mathbb{R}^2)$ in infinite volume with polynomial interaction is ergodic if the Gibbs state is a pure phase, i.e. an extreme Gibbs state. This result also holds for the finite volume case if one takes Dirichlet boundary conditions. Moreover, by [R86] we know that ν constructed above with A changed to a Dirichlet Laplacian on a bounded domain is a pure phase, which implies that the stochastic quantization of the Gibbs state is ergodic. However, the idea in [R86] and the results in [AKR97] cannot be applied for the torus. In this case we don't know whether ν is a pure phase. We also emphasize that it is not obvious that ν is a pure phase even if ν is absolutely continuous with respect to μ. In this case, the zero set of $\frac{d\nu}{d\mu}$, i.e. $\{\frac{d\nu}{d\mu} = 0\}$, which is hard to analyze analytically, may divide the state space into different irreducible components, which immediately implies non-ergodicity, i.e. the existence of two invariant measures. In this paper we study this problem using the techniques from SPDE. We analyze the shifted equation directly and obtain that ν is the unique invariant measure of X_0.

We also emphasize that Dirichlet form theory is crucially used in [AKR97]. Hence for the Dirichlet boundary condition case, one can only obtain that the associated semigroup converges to the Gibbs state for quasi-every starting point. In our paper we analyze X_0 starting from every point in $\mathcal{C}^\alpha$ for some $\alpha < 0$, which will be defined in Sect. 2. As a result, we can conclude that the associated semigroup converges to ν for every starting point in $\mathcal{C}^\alpha$.

Theorem 1.1. *ν is the unique invariant probability measure for the process X_0. Moreover, the associated semigroup P_t converges to ν weakly in $\mathcal{C}^\alpha$, as t goes to ∞.*

Remark 1.2. As in [DD03,RZZ15], one can replace the term $- : X^3 :$ by any Wick polynomial of odd degree $2N - 1$ with negative leading coefficient and obtain the same results in an analogous way. Indeed, we can also obtain corresponding L^p estimates by similar calculations as in the proof of [RZZ15, Theorem 3.10] and Lemma 3.4. Moreover, we replace Y_1 in (4.5) by Y_1^{2N-2} and do similar calculations without changing the Besov space and can obtain the corresponding estimate required in the proof of Theorem 4.1 in an analogous way.

Remark 1.3. By [GlJ86] we know that for $q(\phi)$ given by a polynomial $\phi^4 - \lambda\phi^2$ with λ large enough, the quantum fields in the infinite volume case may have different phases. Indeed, the two different measures in [GJS75] have been obtained by taking approximation of two different convergent sequences of measures obtained from the underlying specification with two different boundary conditions (see [GRS75] and [R86]). The limits ν_1 and ν_2 of these two sequences are the same on locally measurable functions, but they differ on the tail field on which they are uniquely determined (see [P76] and [R86]). This will not happen in the finite volume case. We expect that ν_1 and ν_2 are two different invariant measures for X_0 in the infinite volume case if they have a similar property as in [GlJ86, Corollary 12.2.4]. However, so far one only knows one state in the infinite volume case obtained in [GlJ86, Chapter 11] satisfying the property in [GlJ86, Corollary 12.2.4].

For the proof of Theorem 1.1, we use an argument from an abstract framework developed for application to SDEs with delay [HMS11]. In general by applying a theorem in [HMS11] (see Theorem 4.1), we can reduce the problem of uniqueness of the invariant measure to the convergence of solutions of (1.1) to solutions of an auxiliary system when time tends to infinity. However, in our case we cannot consider the solution to (1.1) obtained by Dirichlet form theory directly since it does not start from every point in some Polish space and the regularity of the solution to (1.1) is too rough to be controlled. Formally : $X^3 := X^3 - \infty X$, which makes it more difficult to analyze the Eq. (1.1) directly. Instead we consider the shifted equation (1.4) and do the required a-priori estimates for the solutions to (1.4). Correspondingly, we also construct an auxiliary system for the shifted form [see (3.3)]. Moreover, to apply [HMS11] we have to construct a suitable set such that the generalized coupling has positive mass on it and the two solutions can converge to each other on this set when time tends to infinity.

As a consequence of Theorem 1.1 we can give a characterization of ν in terms of its density under translation:

Theorem 1.4. *ν is the unique probability measure such that the following hold*

(i) *ν is absolutely continuous with respect to μ with $\frac{d\nu}{d\mu} \in L^p(\mathcal{S}'(\mathbb{T}^2), \mu)$ for some $p > 1$;*

(ii) *("quasi-invariance") $\frac{d\nu(z+tk)}{d\nu(z)} = a_{tk}(z) = a_{tk}^0(z)a_{tk}^{a_1,a_2}(z)$ for $z \in H_2^{-1-\epsilon}, k \in C^\infty(\mathbb{T}^2)$, $t > 0$ with*

$$a_{tk}^0(z) = \exp[-t\langle(-\Delta+1)k, z\rangle - \frac{1}{2}t^2\langle(-\Delta+1)k, k\rangle]$$

and

$$a_{tk}^{a_1,a_2}(z) = \exp[-\frac{a_1}{2}\sum_{i=0}^{3} C_4^i t^{4-i} : z^i : (k^{4-i}) + a_2\sum_{i=0}^{1} C_2^i t^{2-i} : z^i : (k^{2-i})].$$

Here $H_2^{-1-\epsilon}$ for some $\epsilon > 0$ is defined in Sect. 2 and in the following $\langle \cdot, \cdot \rangle$ means the dualization between the elements in $C^\infty(\mathbb{T}^2)$ and $H_2^{-1-\epsilon}$, respectively. $: z^3 :$ is a fixed version of the Wick power we define in Sect. 2. By [AR91, Proposition 6.9] we can choose $z \to : z^3 :$ as a measurable map from $H_2^{-1-\epsilon}$ to $H_2^{-1-\epsilon}$.

Property (ii) is similar to the quasi-invariance (i.e. invariance up to a density) of the usual Gaussian measure. Here a_{tk}^0 is the translation density for the Gaussian part and $a_{tk}^{a_1,a_2}$ corresponds to the polynomial part.

Similarly we obtain the following uniqueness result for the L-symmetrizing measures.

Theorem 1.5. *ν is the unique probability measure such that the following hold:*

(i) *ν is absolutely continuous with respect to μ with $\frac{d\nu}{d\mu} \in L^p(\mathcal{S}'(\mathbb{T}^2), \mu)$ for some $p > 1$;*

(ii) *$\int Lu v d\nu = \int L v u d\nu$ for $u \in \mathcal{F}C_b^\infty$, where*

$$Lu(z) = \frac{1}{2} Tr(D^2 u)(z) + \langle z, ADu \rangle - \langle a_1 : z^3 : -a_2 z, Du(z) \rangle$$

for $z \in H_2^{-1-\epsilon}$ and $\mathcal{F}C_b^\infty$ is as defined in Sect. 4.

Remark 1.6. From the proof of Theorems 1.4 and 1.5 we know that assuming (i) in Theorem 1.4 to hold, it follows that (ii) in Theorems 1.4 and 1.5 are equivalent to the logarithmic derivative of ν along k being given by

$$\beta_k = 2\langle z, Ak \rangle - 2\langle a_1 : z^3 : -a_2 z, k \rangle,$$

for $z \in H_2^{-1-\epsilon}$, $k \in C^\infty(\mathbb{T}^2)$. Here the logarithmic derivative of a measure ν along k is a ν-integrable function β_k such that the following integration by parts formula holds:

$$\int \frac{\partial u}{\partial k} d\nu = - \int \beta_k u d\nu.$$

Moreover, we can prove that ν is an extreme point of the following convex set.

Corollary 1.7. *ν is an extreme point of the convex set $\mathcal{M}^a$, which denotes the set of all probability measures on $\mathcal{S}'(\mathbb{T}^2)$ satisfying (ii) in Theorem 1.4.*

Corollary 1.8. *ν is an extreme point of the convex set $\mathcal{G}$, which denotes the set of all probability measures on $\mathcal{S}'(\mathbb{T}^2)$ satisfying (ii) in Theorem 1.5.*

Remark 1.9. (i) By [AKR97, Theorem 3.3] we know that ν being an extreme point of the convex set $\mathcal{M}^a$ is equivalent to ν being $C^\infty(\mathbb{T}^2)$-ergodic, which is also equivalent to the maximal Dirichlet form $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ being irreducible. For the definition of the maximal Dirichlet form $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$, we refer to [AKR97, Section 3].

(ii) Since the irreducibility is so crucial we recall here some characterizations of it in terms of the semigroup $(T_t)_{t>0}$ and generator $(L, D(L))$ of $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$. The following are equivalent:

1. $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is irreducible.
2. $(T_t)_{t>0}$ is irreducible, i.e., if $g \in L^2(\nu)$ such that $T_t(gf) = gT_t f$ for all $t > 0$, $f \in L^2(\nu)$ then $g = const$.

3. If $g \in L^2(\nu)$ such that $T_t g = g$ for all $t > 0$ then $g = const$.
4. $\int (T_t g - \int g d\nu)^2 d\nu \rightarrow_{t\to\infty} 0$ for all $g \in L^2(\nu)$.
5. If $u \in D(L)$ with $Lu = 0$, then $u = const$.

Here we emphasize that we don't know whether the maximal Dirichlet form is the same as the minimal Dirichlet form defined in the proof of Theorem 1.4 below, which is the issue of the Markov uniqueness problem. If the maximal Dirichlet form is associated with a strong Markov process (i.e. is a quasi-regular Dirichlet form in the sense of [MR92]), then it is the same as the minimal Dirichlet form (see [RZZ15, Theorem 3.12]).

(iii) The fact that the maximal Dirichlet form $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is irreducible can also be proved by a similar argument as in the proof of [BR95, Theorem 6.15] and by using Theorem 1.4.

We also want to mention that recently there has arisen a renewed interest in SPDEs related to such problems, particularly in connection with Hairer's theory of regularity structures [Hai14] and related work by Gubinelli et al. [GIP13]. By using these theories one can obtain local existence and uniqueness of solutions to (1.1) in the three dimensional case (see [Hai14,CC13]). Furthermore, very recently in [MW16] Mourrat and Weber have obtained global well-posedness of the solution to (1.1) in the three dimensional case based on the paracontrolled distribution method.

This paper is organized as follows: In Sect. 2, we collect some results related to Besov spaces and we recall some basic facts on Wick powers. In Sect. 3, we prove the necessary a-priori estimates of solutions to (1.1). In Sect. 4, we prove Theorems 1.1, 1.4 and 1.5.

Note added in the revised version: After this paper had entered the refereeing process, two papers, by Tsatsolis and Weber [TW16] and Hairer and Mattingly [HM16], appeared on arXiv, addressing a similar problem as solved in this paper. However, their approaches are entirely different from ours and are based on first proving the strong Feller property for the process. In fact, the authors in [TW16] obtain the strong Feller property and exponential ergodicity for $p(\Phi)_2$ model whereas the authors in [HM16] present a general method to establish the strong Feller property for solutions of SPDE driven by singular noise in the framework of the theory of regularity structures.

2. Preliminaries

2.1. Notations and some useful estimates. In the following we recall the definitions of Besov spaces. For a general introduction to the theory we refer to [BCD11,Tri78,Tri06]. The space of real valued infinitely differentiable functions of compact support is denoted by $\mathcal{D}(\mathbb{R}^d)$ or $\mathcal{D}$. The space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^d)$. Its dual, the space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform and the inverse Fourier transform are denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$.

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions on $\mathbb{R}^d$, such that

(i) the support of χ is contained in a ball and the support of θ is contained in an annulus;

(ii) $\chi(\xi) + \sum_{j\geq 0} \theta(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^d$.

(iii) $\text{supp}(\chi) \cap \text{supp}(\theta(2^{-j}\cdot)) = \emptyset$ for $j \geq 1$ and $\text{supp}\theta(2^{-i}\cdot) \cap \text{supp}\theta(2^{-j}\cdot) = \emptyset$ for $|i - j| > 1$.

We call such a pair (χ, θ) dyadic partition of unity, and for the existence of dyadic partitions of unity see [BCD11, Proposition 2.10]. The Littlewood-Paley blocks are now defined as

$$\Delta_{-1}u = \mathcal{F}^{-1}(\chi\mathcal{F}u) \quad \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j}\cdot)\mathcal{F}u).$$

For $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, $u \in \mathcal{D}$ we define

$$\|u\|_{B^\alpha_{p,q}} := (\sum_{j\geq -1}(2^{j\alpha}\|\Delta_j u\|_{L^p})^q)^{1/q},$$

with the usual interpretation as l^∞ norm in case $q = \infty$. The Besov space $B^\alpha_{p,q}$ consists of the completion of $\mathcal{D}$ with respect to this norm and the Hölder-Besov space $\mathcal{C}^\alpha$ is given by $\mathcal{C}^\alpha(\mathbb{R}^d) = B^\alpha_{\infty,\infty}(\mathbb{R}^d)$. For $p, q \in [1, \infty)$,

$$B^\alpha_{p,q}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{B^\alpha_{p,q}} < \infty\}.$$
$$\mathcal{C}^\alpha(\mathbb{R}^d) \subsetneq \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{B^\alpha_{\infty,\infty}(\mathbb{R}^d)} < \infty\}.$$

The reason that $\mathcal{C}^\alpha$ is smaller is that some functions in $\{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{B^\alpha_{\infty,\infty}(\mathbb{R}^d)} < \infty\}$ cannot be approximated in $\mathcal{C}^\alpha$-norm by smooth functions. We point out that everything above and everything that follows can be applied to distributions on the torus (see [S85, SW71]). More precisely, let $\mathcal{S}'(\mathbb{T}^d)$ be the space of distributions on $\mathbb{T}^d$. Besov spaces on the torus with general indices $p, q \in [1, \infty]$ are defined as the completion of $C^\infty(\mathbb{T}^2)$ with respect to the norm

$$\|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)} := \left(\sum_{j\geq -1}(2^{j\alpha}\|\Delta_j u\|_{L^p(\mathbb{T}^d)})^q\right)^{1/q},$$

and the Hölder-Besov space $\mathcal{C}^\alpha$ is given by $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}(\mathbb{T}^d)$. We write $\|\cdot\|_\alpha$ instead of $\|\cdot\|_{B^\alpha_{\infty,\infty}(\mathbb{T}^d)}$ in the following for simplicity. For $p, q \in [1, \infty)$

$$B^\alpha_{p,q}(\mathbb{T}^d) = \{u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)} < \infty\}.$$
$$\mathcal{C}^\alpha \subsetneq \{u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_\alpha < \infty\}. \tag{2.1}$$

In this part we give estimates on the torus for later use. Set $\Lambda = (-A)^{\frac{1}{2}}$. For $s \geq 0$, $p \in [1, +\infty]$ we use H^s_p to denote the subspace of $L^p(\mathbb{T}^d)$, consisting of all f which can be written in the form $f = \Lambda^{-s}g$, $g \in L^p(\mathbb{T}^d)$ and the H^s_p norm of f is defined to be the L^p norm of g, i.e. $\|f\|_{H^s_p} := \|\Lambda^s f\|_{L^p(\mathbb{T}^d)}$.

To study (1.1) in the finite volume case, we will need several important properties of Besov spaces on the torus and we recall the following Besov embedding theorems on the torus first (c.f. [Tri78, Theorem 4.6.1], [GIP13, Lemma 41]):

Lemma 2.1. (i) *Let* $1 \leq p_1 \leq p_2 \leq \infty$ *and* $1 \leq q_1 \leq q_2 \leq \infty$, *and let* $\alpha \in \mathbb{R}$. *Then* $B^\alpha_{p_1,q_1}(\mathbb{T}^d)$ *is continuously embedded in* $B^{\alpha-d(1/p_1-1/p_2)}_{p_2,q_2}(\mathbb{T}^d)$.

(ii) *Let* $s \geq 0$, $1 < p < \infty$, $\epsilon > 0$. *Then* $H^{s+\epsilon}_p \subset B^s_{p,1}(\mathbb{T}^d) \subset B^s_{1,1}(\mathbb{T}^d)$.

(iii) *Let* $1 \leq p_1 \leq p_2 < \infty$ *and let* $\alpha \in \mathbb{R}$. *Then* $H^\alpha_{p_1}$ *is continuously embedded in* $H^{\alpha-d(1/p_1-1/p_2)}_{p_2}$.

Here $\subset$ *means that the embedding is continuous and dense.*

We recall the following Schauder estimates, i.e. the smoothing effect of the heat flow, for later use.

Lemma 2.2. [GIP13, Lemma 47]

(i) *Let $u \in B^\alpha_{p,q}(\mathbb{T}^d)$ for some $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$. Then for every $\delta \geq 0$*

$$\|e^{tA}u\|_{B^{\alpha+\delta}_{p,q}(\mathbb{T}^d)} \lesssim t^{-\delta/2}\|u\|_{B^\alpha_{p,q}(\mathbb{T}^d)},$$

where the constant we omit is independent of t.

(ii) *Let $\alpha \leq \beta \in \mathbb{R}$. Then*

$$\|(1 - e^{tA})u\|_\alpha \lesssim t^{\frac{\beta-\alpha}{2}}\|u\|_\beta.$$

One can extend the multiplication on suitable Besov spaces and also have the duality properties of Besov spaces from [Tri78, Chapter 4]:

Lemma 2.3. (i) *Let $\alpha, \beta \in \mathbb{R}$ and $p, p_1, p_2, q \in [1, \infty]$ be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

The bilinear map $(u; v) \mapsto uv$ extends to a continuous map from $B^\alpha_{p_1,q} \times B^\beta_{p_2,q}$ to $B^{\alpha\wedge\beta}_{p,q}$ if $\alpha + \beta > 0$.

(ii) *Let $\alpha \in (0, 1)$, $p, q \in [1, \infty]$, p' and q' be their conjugate exponents, respectively. Then the mapping $(u; v) \mapsto \int uvdx$ extends to a continuous bilinear form on $B^\alpha_{p,q}(\mathbb{T}^d) \times B^{-\alpha}_{p',q'}(\mathbb{T}^d)$.*

We recall the following interpolation inequality and multiplicative inequality for the elements in H^s_p (cf. [Tri78, Theorem 4.3.1], [Re95, Lemma A.4], [RZZ15a, Lemma 2.1]):

Lemma 2.4. (i) *Suppose that $s \in (0, 1)$ and $p \in (1, \infty)$. Then for $u \in H^1_p$*

$$\|u\|_{H^s_p} \lesssim \|u\|^{1-s}_{L^p(\mathbb{T}^d)}\|u\|^s_{H^1_p}.$$

(ii) *Suppose that $s > 0$ and $p \in (1, \infty)$. If $u, v \in C^\infty(\mathbb{T}^2)$ then*

$$\|\Lambda^s(uv)\|_{L^p(\mathbb{T}^d)} \lesssim \|u\|_{L^{p_1}(\mathbb{T}^d)}\|\Lambda^s v\|_{L^{p_2}(\mathbb{T}^d)} + \|v\|_{L^{p_3}(\mathbb{T}^d)}\|\Lambda^s u\|_{L^{p_4}(\mathbb{T}^d)},$$

with $p_i \in (1, \infty]$, $i = 1, \ldots, 4$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

2.2. Wick power. In the following we recall the definition of Wick powers. Let $\mu = N(0, \frac{1}{2}(-\Delta + 1)^{-1}) := N(0, C)$.

2.2.1. Wick power on $L^2(\mathcal{S}'(\mathbb{T}^2), \mu)$ In fact μ is a measure supported on $\mathcal{S}'(\mathbb{T}^2)$. We have the well-known (Wiener-Itô) chaos decomposition

$$L^2(\mathcal{S}'(\mathbb{T}^2), \mu) = \bigoplus_{n \geq 0} \mathcal{H}_n,$$

where $\mathcal{H}_n$ is the Wiener chaos of order n (cf. [Nua06, Section 1.1.1]). Now we define the Wick power by using approximations: for $\phi \in \mathcal{S}'(\mathbb{T}^2)$ define

$$\phi_\varepsilon := \rho_\varepsilon * \phi,$$

with ρ_ε an approximate delta function,

$$\rho_\varepsilon(\xi) = \varepsilon^{-2}\rho(\frac{\xi}{\varepsilon}) \in \mathcal{D}, \int \rho = 1.$$

Here the convolution means that we view ϕ as a periodic distribution in $\mathcal{S}'(\mathbb{R}^2)$. For every $n \in \mathbb{N}$ we set

$$: \phi_\varepsilon^n :_C := c_\varepsilon^{n/2} P_n(c_\varepsilon^{-1/2}\phi_\varepsilon),$$

where $P_n, n = 0, 1, \ldots,$ are the Hermite polynomials defined by the formula

$$P_n(x) = \sum_{j=0}^{[n/2]} (-1)^j \frac{n!}{(n-2j)!j!2^j} x^{n-2j},$$

and $c_\varepsilon := \int \phi_\varepsilon^2 \mu(d\phi) = \int\int \bar{G}(\xi_1 - \xi_2)\rho_\varepsilon(\xi_2)d\xi_2\rho_\varepsilon(\xi_1)d\xi_1 = \|\bar{K}_\varepsilon\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)}$. Then $: \phi_\varepsilon^n :_C \in \mathcal{H}_n$. Here and in the following $\bar{G}$ is the Green function associated with $-A$ on $\mathbb{T}^2$ and let $\bar{K}(t, \xi)$ be such that $\bar{K}(t, \xi_1 - \xi_2)$ is the heat kernel associated with A on $\mathbb{T}^2$ and $\bar{K}_\varepsilon = \bar{K} * \rho_\varepsilon$ with $*$ means convolution in space and we view $\bar{K}$ as a periodic function on $\mathbb{R}^2$.

For Hermite polynomial P_n we have for $s, t \in \mathbb{R}$

$$P_n(s+t) = \sum_{m=0}^{n} C_n^m P_m(s)t^{n-m}, \tag{2.2}$$

where $C_n^m = \frac{n!}{m!(n-m)!}$.

A direct calculation yields the following:

Lemma 2.5. [RZZ15, Lemma 3.1] *Let $\alpha < 0$, $n \in \mathbb{N}$ and $p > 1$. $: \phi_\varepsilon^n :_C$ converges to some element in $L^p(\mathcal{S}'(\mathbb{T}^2), \mu; \mathcal{C}^\alpha)$ as $\varepsilon \to 0$. This limit is called n-th Wick power of ϕ with respect to the covariance C and denoted by $: \phi^n :_C$.*

Now we introduce the following probability measure. Let

$$\nu = c \exp(-\frac{1}{2}\int_{\mathbb{T}^2} (a_1 : \phi^4 :_C - 2a_2 : \phi^2 :_C)d\xi)\mu,$$

where c is a normalization constant. Then by [GlJ86, Sect. 8.6] for every $p \in [1, \infty)$, $\varphi(\phi) := \exp(-\frac{1}{2}\int_{\mathbb{T}^2}(a_1 : \phi^4 :_C - 2a_2 : \phi^2 :_C)d\xi) \in L^p(\mathcal{S}'(\mathbb{T}^2), \mu)$.

2.2.2. Wick power on a fixed probability space Now we fix a probability space $(\Omega, \mathcal{F}, P)$ equipped with a normal filtration $(\mathcal{F}_t)$ and W is an $L^2(\mathbb{T}^2)$-cylindrical $(\mathcal{F}_t)$-Wiener process. In the following we assume that $\mathcal{F}$ is the σ-field generated by $\{\langle W_t, h\rangle, h \in L^2(\mathbb{T}^2), t \in \mathbb{R}^+\}$. We also have the well-known (Wiener-Itô) chaos decomposition

$$L^2(\Omega, \mathcal{F}, P) = \bigoplus_{n\geq 0} \mathcal{H}'_n,$$

where $\mathcal{H}'_n$ is the Wiener chaos of order n (cf. [Nua06, Section 1.1.1]). In the following we set $Z(t) = \int_0^t e^{(t-s)A} dW(s)$, and we can also define Wick powers of $Z(t)$ by approximations: Let $Z_\varepsilon(t, \xi) = \int_0^t \langle \bar{K}_\varepsilon(t-s, \xi - \cdot), dW(s)\rangle$. Here $\langle \cdot, \cdot\rangle$ means inner product in $L^2(\mathbb{T}^2)$.

Lemma 2.6. [RZZ15, Lemma 3.4] *For $\alpha < 0$, $n \in \mathbb{N}$ and $t > 0$, $: Z^n_\varepsilon(t) ::= c_\varepsilon^{\frac{n}{2}} P_n(c_\varepsilon^{-\frac{1}{2}} Z_\varepsilon(t))$ converges in $L^p(\Omega, C((0, T]; \mathcal{C}^\alpha))$. Here the norm for $C((0, T]; \mathcal{C}^\alpha)$ is $\sup_{t\in[0,T]} t^\delta \|\cdot\|_\alpha$ for $\delta > 0$. The limit is called Wick power of $Z(t)$ with respect to the covariance C and denoted by $: Z^n(t) :$.*

Now following the technique in [MW15] we combine the initial value part with the Wick powers by using (2.2). We set $V(t) = e^{tA}x$, $x \in \mathcal{C}^\alpha$ for $\alpha < 0$ and

$$\bar{Z}_x(t) = Z(t) + V(t),$$

and for $n = 2, 3$,

$$: \bar{Z}^n_x(t) := \sum_{k=0}^{n} C^k_n V(t)^{n-k} : Z^k(t) : .$$

By Lemma 2.2 we know that $V \in C([0, T], \mathcal{C}^\alpha)$ and $V \in C((0, T], \mathcal{C}^\beta)$ for $\beta > -\alpha$ with the norm $\sup_{t\in[0,T]} t^{\frac{\beta-\alpha}{2}} \|\cdot\|_\beta$. Moreover,

$$t^{\frac{\beta-\alpha}{2}} \|V(t)\|_\beta \lesssim \|x\|_\alpha, \tag{2.3}$$

for $\beta > -\alpha$. Then by [RZZ15, Lemmas 3.5] we have $\bar{Z}_x \in L^p(C((0, T], \mathcal{C}^\alpha))$.

By (2.3) and Lemma 2.3 it is easy to obtain the following result:

Lemma 2.7. *Let $\alpha < 0$, $x \in \mathcal{C}^\alpha$. Then we have for $t > 0$ and any $\varepsilon > 0$*

$$\begin{aligned}
\|\bar{Z}_x(t)\|_\alpha &\leq \|Z(t)\|_\alpha + \|x\|_\alpha,\\
\| : \bar{Z}^2_x(t) : \|_\alpha &\leq C[\| : Z^2(t) : \|_\alpha + t^{\alpha-\varepsilon} \|x\|_\alpha \|Z(t)\|_\alpha + t^{\alpha-\varepsilon} \|x\|^2_\alpha],\\
\| : \bar{Z}^3_x(t) : \|_\alpha &\leq C[\| : Z^3(t) : \|_\alpha + t^{2\alpha-\varepsilon} \|x\|^2_\alpha \|Z(t)\|_\alpha\\
&\quad + t^{\alpha-\varepsilon} \|x\|_\alpha \| : Z^2(t) : \|_\alpha + t^{2\alpha-\varepsilon} \|x\|^3_\alpha].
\end{aligned}$$

3. The Necessary A-Priori Estimates

Now we follow [MW15,RZZ15] and give the existence and uniqueness of solutions to (1.1) by considering the shifted equation.

We fix $\alpha < 0$ with $-\alpha$ small enough, in the following. For $\underline{Z} = (Z, :Z^2:, :Z^3:)$, $0 < \delta < -\alpha$, define

$$\|\underline{Z}\|_{\mathfrak{L}_T} := \sup_{0\le t\le T} \left(\|Z(t)\|_\alpha, t^\delta\| :Z^2(t): \|_\alpha, t^\delta\| :Z^3(t): \|_\alpha\right)$$

and

$$\Omega_0 = \{Z \in C([0,T];\mathcal{C}^\alpha), :Z^n: \in C((0,T];\mathcal{C}^\alpha) \text{ for } n = 2,3, \|\underline{Z}\|_{\mathfrak{L}_T} < \infty \quad \forall T > 0\}.$$

Then

$$P[\Omega_0] = 1.$$

We also introduce the following notations: for $Y \in C([0,T],\mathcal{C}^\beta)$, $Z \in C([0,T];\mathcal{C}^\alpha)$ with $\beta > -\alpha$

$$\Psi(Y, Z) := -a_1(3Y^2 Z + 3Y :Z^2: + :Z^3:) + a_2(Y + Z).$$

By [RZZ15, Theorem 3.9] we know that the solution to (1.1) can be written as $X = Y + e^{tA}x + Z = Y + \bar{Z}_x$ for $x \in \mathcal{C}^\alpha$, where Y satisfies the following shifted equation:

$$dY = [AY - a_1 Y^3 + \Psi(Y, \bar{Z}_x)]dt, \quad Y(0) = 0. \tag{3.1}$$

Here and in the following (3.1) and other equations are interpreted in the mild sense:

$$Y(t) = \int_0^t e^{(t-s)A}[-a_1 Y^3 + \Psi(Y, \bar{Z}_x)](s)ds.$$

As a result, X is a mild solution to the following equation

$$dX = [AX - a_1(X - \bar{Z}_x)^3 + \Psi(X - \bar{Z}_x, \bar{Z}_x)]dt + dW, \quad X(0) = x. \tag{3.2}$$

That is to say:

Theorem 3.1. *For $\omega \in \Omega_0$, $x \in \mathcal{C}^\alpha$, there exists exactly one mild solution $X(\omega) \in C([0,\infty);\mathcal{C}^\alpha)$ to the Eq. (3.2) satisfying $(X - \bar{Z}_x)(\omega) \in C([0,\infty);\mathcal{C}^\beta)$ for some $\beta > -\alpha > 0$.*

Proof. For $\omega \in \Omega_0$, $x \in \mathcal{C}^\alpha$, by Lemma 2.7 we know that for $n = 2, 3$,

$$\bar{Z}_x \in C([0,\infty);\mathcal{C}^\alpha), \quad :\bar{Z}_x^n: \in C((0,\infty);\mathcal{C}^\alpha),$$

and for every $T > 0$

$$\sup_{0\le t\le T} \left[\|\bar{Z}_x(t)\|_\alpha, t^{-2\alpha+\delta}\| :\bar{Z}_x^2(t): \|_\alpha, t^{-2\alpha+\delta}\| :\bar{Z}_x^3(t): \|_\alpha\right] < \infty.$$

Then [MW15, Theorem 6.5] implies that for $\omega \in \Omega_0$ there exist exactly one solution $Y(\omega) \in C([0,\infty);\mathcal{C}^\beta)$ for some $\beta > -\alpha$ satisfying (3.2) in the mild sense. From this we can conclude the result easily. □

Remark 3.2. Here for $\omega \in \Omega_0$ X is an ω-wise mild solution to the Eq. (3.2), whose definition is stronger than the usual (probabilistically) mild solution to the stochastic differential equation (3.2).

Now we can define the semigroup associated with X and obtain an invariant measure for X: For $t \geq 0$, $f \in \mathcal{B}_b(\mathcal{C}^\alpha)$, define

$$P_t f(x) := Ef(X(t,x))$$

for the solution $X(t,x)$ to (3.2) with initial value $x \in \mathcal{C}^\alpha$. Here E denotes to take expectation under P. By Theorem 3.1 we obtain that X is a Markov process with $(P_t)_{t\geq 0}$ as the associated semigroup. By [RZZ15, Theorem 3.10] we know that ν is an invariant measure for X, where ν is defined after Lemma 2.5. In the next section we will prove that ν is the unique invariant measure for X. To prove this, we introduce the following equation, which has a new dissipation term compared to (3.1).

For given $x_0, x_1 \in \mathcal{C}^\alpha$ consider the following equation

$$\frac{d}{dt}\tilde{Y} = A\tilde{Y} - \lambda(\tilde{Y} - Y + e^{tA}(x_1 - x_0)) - a_1\tilde{Y}^3 + \Psi(\tilde{Y}, \bar{Z}_{x_1}), \quad \tilde{Y}(0) = 0, \tag{3.3}$$

where $\lambda > 1$ will be determined later.

By similar arguments as the proof of [MW15, Theorem 6.5] and [RZZ15, Theorem 3.10] we can easily derive the following result:

Theorem 3.3. *For $x_0, x_1 \in \mathcal{C}^\alpha$ and $\omega \in \Omega_0$, there exists a unique mild solution $\tilde{Y} \in C([0,\infty); \mathcal{C}^\beta)$ to the Eq. (3.3).*

Define $\tilde{X} := \tilde{Y} + e^{tA}x_1 + Z = \tilde{Y} + \bar{Z}_{x_1}$ with $\tilde{Y}$ obtained in Theorem 3.3. Then $\tilde{X}$ satisfies the following equation in the mild sense:

$$d\tilde{X} = [A\tilde{X} - \lambda(\tilde{X} - X) - a_1(\tilde{X} - \bar{Z}_{x_1})^3 + \Psi(\tilde{X} - \bar{Z}_{x_1}, \bar{Z}_{x_1})]dt + dW, \quad \tilde{X}(0) = x_1. \tag{3.4}$$

Now we give the necessary a-priori estimates for the solutions to (3.1) and (3.3) for later use. We will derive L^p-norm estimates for the solutions to (3.1) and (3.3) respectively. We can get the $\|\cdot\|_{L^p}$-norm estimate directly, but only with p depending on α, which is not enough for later use. Hence we first obtain the $\|\cdot\|_{L^p}$-norm estimates from 1 to t for every $p > 1$. Then we estimate $\|Y(1)\|_\beta$ for $\beta > 0$ by the following Steps 2 and 3.

Lemma 3.4. *Suppose that Y is the solution to (3.1) with $x = x_0$. For every even $p > 1$, there exist constants $C(p), C(\|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha) > 0, \gamma(p) > 1$ independent of ω, t such that for every $t \geq 1$ and $\omega \in \Omega_0$*

$$\|Y(t)\|_{L^p}^p + \int_1^t \|Y(s)\|_{L^p}^p ds + \int_1^t \|Y^{p-2}|\nabla Y|^2(s)\|_{L^1} ds$$
$$\leq C(\|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha) + C(p)\int_1^t (1 + \|x_0\|_\alpha^{\gamma(p)} + \|Z\|_\alpha^{\gamma(p)} + \sum_{n=2}^{3} \|:Z^n:\|_\alpha^{\gamma(p)})ds.$$

Here $C(\|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha)$ means a constant depending on $\|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha$.

Proof. We write the proof for Y and Z directly which is a bit informal, but it can be made rigorous by replacing Z by Z_ε and taking the limit as in the proof of [RZZ15, Theorem 6.5].

Step 1 We first prove that for every even $p > 1$, $t \geq 1$ there exists $\gamma(p) > 2$ such that

$$\|Y(t)\|_{L^p}^p + \int_1^t \|Y^{p-2}|\nabla Y|^2\|_{L^1}^p ds + \int_1^t \|Y(s)\|_{L^p}^p ds \leq \|Y(1)\|_{L^p}^p$$
$$+C(p)\int_1^t (1 + \|x_0\|_\alpha^{\gamma(p)} + \|Z\|_\alpha^{\gamma(p)} + \sum_{n=2}^3 \| : Z^n : \|_\alpha^{\gamma(p)})ds. \tag{3.5}$$

Testing against Y^{p-1}, we have that for $t \geq 1$, even $p > 1$,

$$\frac{1}{p}\|Y(t)\|_{L^p}^p + \int_1^t [(p-1)\langle \nabla Y(s), Y(s)^{p-2}\nabla Y(s)\rangle + a_1\|Y(s)^{p+2}\|_{L^1}]ds$$
$$= -\int_1^t [\|Y(s)\|_{L^p}^p + \langle \Psi(Y(s), \bar{Z}_{x_0}(s)), Y(s)^{p-1}\rangle]ds + \frac{1}{p}\|Y(1)\|_{L^p}^p.$$

Now we have $\langle \nabla Y(s), Y(s)^{p-2}\nabla Y(s)\rangle$ and $\|Y(s)^{p+2}\|_{L^1}$ on the left hand side of the equality, which can be used to control the right hand side of the above equation. By similar calculations as in the proof of [MW15, Theorem 6.4] and [RZZ15, Theorem 3.10] we deduce that there exists $\gamma_0 > 1$ such that

$$|\langle \Psi(Y(s), \bar{Z}_{x_0}(s)), Y(s)^{p-1}\rangle|$$
$$\leq C(p)(1 + \|\bar{Z}_{x_0}\|_\alpha^{\gamma_0} + \sum_{n=2}^3 \| : \bar{Z}_{x_0}^n : \|_\alpha^{p+2}) + \frac{1}{2}(a_1\|Y\|_{L^{p+2}}^{p+2} + \|Y^{p-2}|\nabla Y|^2\|_{L^1}),$$

which implies that

$$\frac{1}{p}\|Y(t)\|_{L^p}^p + \frac{1}{2}\int_1^t [(p-1)\langle \nabla Y(s), Y(s)^{p-2}\nabla Y(s)\rangle + a_1\|Y(s)^{p+2}\|_{L^1}]ds$$
$$\leq C(p)\int_1^t (1 + \|\bar{Z}_{x_0}\|_\alpha^{\gamma_0} + \sum_{n=2}^3 \| : \bar{Z}_{x_0}^n : \|_\alpha^{p+2})ds + \frac{1}{p}\|Y(1)\|_{L^p}^p \tag{3.6}$$
$$\leq C(p)\int_1^t (1 + \|x_0\|_\alpha^{\gamma(p)} + \|Z\|_\alpha^{\gamma(p)} + \sum_{n=2}^3 \| : Z^n : \|_\alpha^{\gamma(p)})ds + \frac{1}{p}\|Y(1)\|_{L^p}^p.$$

Here $\gamma(p) = 3(p+2) \vee \gamma_0$ and we used Lemma 2.7 in the last inequality. Now (3.5) follows.

Step 2 We prove that for even $p > 1$, with $-2\alpha(p+2) < 1$,

$$\sup_{0\leq t\leq 1} \|Y(t)\|_{L^p}^p + \int_0^1 \|Y^{p-2}|\nabla Y|^2\|_{L^1} ds \leq C(p, \|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha). \tag{3.7}$$

By similar arguments as in Step 1 we have for $0 \le t \le 1$

$$\begin{aligned}
&\frac{1}{p}\|Y(t)\|_{L^p}^p + \frac{1}{2}\int_0^t [(p-1)\langle \nabla Y(s), Y(s)^{p-2}\nabla Y(s)\rangle + a_1\|Y(s)^{p+2}\|_{L^1}]ds \\
&\le C(p)\int_0^t (1+\|\bar{Z}_{x_0}\|_\alpha^{\gamma_0} + \sum_{n=2}^3 \| :\bar{Z}_{x_0}^n: \|_\alpha^{p+2})ds \\
&\le C(p, \|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha).
\end{aligned}$$

Here we used Lemma 2.7 and $-2\alpha(p+2) < 1$ in the last inequality. Now (3.7) follows.

Step 3 We prove that for $0 < \beta < \frac{1}{2}+\alpha$

$$\|Y(1)\|_\beta \le C(\|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha). \tag{3.8}$$

Since Y satisfies the mild equation, we have

$$\begin{aligned}
\|Y(1)\|_\beta \le\, & C\int_0^1 (1-s)^{-\frac{\beta}{2}-\frac{1}{2}}\|Y^3\|_{L^2}ds \\
&+ C\int_0^1 (1-s)^{-\frac{\beta+1/2-\alpha}{2}}[\|Y^2\bar{Z}_{x_0}\|_{B^\alpha_{4,\infty}} + \|Y:\bar{Z}_{x_0}^2:\|_{B^\alpha_{4,\infty}}]ds \\
&+ C\int_0^1 (1-s)^{-(\beta-\alpha)/2}\|:\bar{Z}_{x_0}^3:\|_\alpha ds,
\end{aligned}$$

where we used Lemmas 2.1, 2.2 to deduce $\|e^{tA}x\|_\beta \le Ct^{-\frac{\beta}{2}-\frac{1}{2}}\|x\|_{L^2}$ and $\|\cdot\|_{\alpha-\frac{1}{2}} \le C\|\cdot\|_{B^\alpha_{4,\infty}}$. For the first term on the right hand side, we can use (3.7) for $p = 6$ to control it by $C(\|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha)$. Using Lemma 2.7 we can control the third term by $C(\|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha)$. Now we come to the second term:

$$\begin{aligned}
&\int_0^1 (1-s)^{-\frac{\beta+1/2-\alpha}{2}}[\|Y^2\bar{Z}_{x_0}\|_{B^\alpha_{4,\infty}} + \|Y:\bar{Z}_{x_0}^2:\|_{B^\alpha_{4,\infty}}]ds \\
&\le C\int_0^1 (1-s)^{-\frac{\beta+1/2-\alpha}{2}}[\|Y^2\|_{B^\beta_{4,\infty}}\|\bar{Z}_{x_0}\|_\alpha + \|Y\|_{B^\beta_{4,\infty}}\|:\bar{Z}_{x_0}^2:\|_\alpha]ds \\
&\le C\int_0^1 (1-s)^{-\frac{\beta+1/2-\alpha}{2}}[\|Y^2\|_{B^{\beta+1/2}_{2,\infty}}\|\bar{Z}_{x_0}\|_\alpha + \|Y\|_{B^{\beta+1/2}_{2,\infty}}\|:\bar{Z}_{x_0}^2:\|_\alpha]ds \\
&\le C(\|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha)\int_0^1 (1-s)^{-\frac{\beta+1/2-\alpha}{2}}[(\|\nabla Y^2\|_{L^2}^{\beta+1/2+\varepsilon}\|Y^2\|_{L^2}^{1/2-\beta-\varepsilon} + \|Y^2\|_{L^2}) \\
&\quad + s^{-\beta-\alpha}(\|\nabla Y\|_{L^2}^{\beta+1/2+\varepsilon}\|Y\|_{L^2}^{1/2-\beta-\varepsilon} + \|\nabla Y\|_{L^2})]ds \\
&\le C(\|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha) + C(\|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha)\int_0^1 (\|\nabla Y^2\|_{L^2}^2 + \|Y^2\|_{L^2})ds \\
&\le C(\|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha),
\end{aligned}$$

where $0 < \varepsilon < \frac{1}{2}-\beta$ and we used Lemma 2.3 in the first inequality, we used Lemma 2.1 in the second inequality, and the fact that

$$\|\cdot\|_{B^{\beta+\frac{1}{2}}_{2,\infty}} \le C\|\cdot\|_{B^{\beta+\frac{1}{2}}_{2,1}} \le C\|\cdot\|_{H_2^{\beta+\frac{1}{2}+\varepsilon}},$$

and Lemma 2.4 in the third inequality, and we used (3.7) for $p = 2$ and 4 in the last two inequalities. Combining the above estimates (3.8) follows.

Combining (3.5) and (3.8) and using $\|Y(1)\|_{L^p} \leq C\|Y(1)\|_\beta$, the result follows. □

The proof of Lemma 3.5 is similar to that of Lemma 3.4. But we should pay attention to how each term depends on λ, as $\tilde{Y}$ depends on λ.

Lemma 3.5. *For every even $p > 1$, $\lambda > 1$, there exist constants $C(p)$, $C(p, \lambda, \|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha, \|x_1\|_\alpha) > 0$, $\gamma(p) > 1$ independent of ω, t such that for every $t \geq 1$ and $\omega \in \Omega_0$*

$$\int_1^t \|\tilde{Y}(s)\|_{L^p}^p ds \leq C(p) \int_1^t (1 + \sum_{i=0}^1 \|x_i\|_\alpha^{\gamma(p)} + \|Z\|_\alpha^{\gamma(p)} + \sum_{n=2}^3 \| : Z^n : \|_\alpha^{\gamma(p)}) ds + C(p, \lambda, \|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha, \|x_1\|_\alpha).$$

$$\int_1^t \|\nabla\tilde{Y}(s)\|_{L^2}^2 ds + \|\tilde{Y}(t)\|_{L^p}^p \leq C(p)\lambda \int_1^t (1 + \sum_{i=0}^1 \|x_i\|_\alpha^{\gamma(p)} + \|Z\|_\alpha^{\gamma(p)} + \sum_{n=2}^3 \| : Z^n : \|_\alpha^{\gamma(p)}) ds + C(p, \lambda, \|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha, \|x_1\|_\alpha).$$

Proof. Step 1 We first prove that for every even $p > 1$ there exists $\gamma(p) > 1$ such that

$$\int_1^t \|\tilde{Y}(s)\|_{L^p}^p ds \leq \|\tilde{Y}(1)\|_{L^p}^p + C(p) \int_1^t (1 + \sum_{i=0}^1 \|x_i\|_\alpha^{\gamma(p)} + \|Z\|_\alpha^{\gamma(p)} + \sum_{n=2}^3 \| : Z^n : \|_\alpha^{\gamma(p)}) ds + C(\|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha) \tag{3.9}$$

Similarly as in the proof of Lemma 3.4 we have that for $t \geq 1$ and even $p > 1$

$$\begin{aligned} &\frac{1}{p}\|\tilde{Y}(t)\|_{L^p}^p + \lambda \int_1^t \|\tilde{Y}(s)\|_{L^p}^p ds \\ &\quad + \int_1^t [(p-1)\langle \nabla\tilde{Y}(s), \tilde{Y}(s)^{p-2}\nabla\tilde{Y}(s)\rangle + a_1\|\tilde{Y}(s)^{p+2}\|_{L^1}] ds \\ &= -\int_1^t [\|\tilde{Y}(s)\|_{L^p}^p + \langle \Psi(\tilde{Y}(s), \bar{Z}_{x_1}(s)), \tilde{Y}(s)^{p-1}\rangle] ds + \lambda \int_1^t \langle Y(s), \tilde{Y}(s)^{p-1}\rangle ds \\ &\quad -\lambda \int_1^t \langle e^{sA}(x_1 - x_0), \tilde{Y}(s)^{p-1}\rangle ds + \frac{1}{p}\|\tilde{Y}(1)\|_{L^p}^p. \end{aligned} \tag{3.10}$$

Now by similar calculations as in the proof of Lemma 3.4 and using (3.7) we deduce that there exist $\gamma(p) > 1$ such that

$$\begin{aligned}
&\|\tilde{Y}(t)\|_{L^p}^p + \lambda\int_1^t \|\tilde{Y}(s)\|_{L^p}^p ds + \frac{p-1}{2}\int_1^t \||\nabla\tilde{Y}(s)|^2\tilde{Y}^{p-2}(s)\|_{L^1}ds\\
&\le C(p)\int_1^t (1+\|\bar{Z}_{x_1}\|_\alpha^{\gamma_0} + \sum_{n=2}^3 \|:\bar{Z}_{x_1}^n:\|_\alpha^{p+2})ds + \lambda p\int_1^t \|Y(s)\|_{L^p}\|\tilde{Y}(s)\|_{L^p}^{p-1}ds\\
&\quad + C\lambda p\int_1^t s^{-\frac{\beta-\alpha}{2}}\|x_0-x_1\|_\alpha\|\tilde{Y}(s)\|_{L^p}^{p-1}ds + \|\tilde{Y}(1)\|_{L^p}^p\\
&\le C(p)\int_1^t (1+\|Z\|_\alpha^{\gamma(p)} + \sum_{n=2}^3 \|:Z^n:\|_\alpha^{\gamma(p)} + \|x_1\|_\alpha^{\gamma(p)})ds\\
&\quad + \lambda C(p)\int_1^t \|Y(s)\|_{L^p}^p ds + \frac{\lambda}{2}\int_1^t \|\tilde{Y}(s)\|_{L^p}^p ds\\
&\quad + C(p)\lambda\int_1^t \|x_0-x_1\|_\alpha^p ds + \|\tilde{Y}(1)\|_{L^p}^p,
\end{aligned}\tag{3.11}$$

where we used Hölder's inequality and Lemma 2.2 to control $\|e^{sA}(x_0-x_1)\|_{\mathcal{C}^\beta}$ by $Cs^{-\frac{\beta-\alpha}{2}}\|x_0-x_1\|_{\mathcal{C}^\alpha}$ for $\beta > -\alpha$ in the first inequality and we used Young's inequality in the second inequality. By Lemma 3.4 and the fact that $\lambda > 1$, (3.9) follows.

Step 2 We prove that for even $p > 1$ with $(-2\alpha+\delta)(p+2) < 1$

$$\sup_{0\le t\le 1}\|\tilde{Y}(t)\|_{L^p}^p + \int_0^1 \|\tilde{Y}^{p-2}|\nabla\tilde{Y}|^2\|_{L^1}ds \le \lambda C(p, \|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha, \|x_1\|_\alpha). \tag{3.12}$$

By similar arguments as in Step 1 we have for $0 \le t \le 1$, even $p > 1$ with $-2\alpha(p+2) < 1$ and $\varepsilon > 0$ small enough

$$\begin{aligned}
&\|\tilde{Y}(t)\|_{L^p}^p + \lambda\int_0^t \|\tilde{Y}(s)\|_{L^p}^p ds + \frac{p-1}{2}\int_0^t \||\nabla\tilde{Y}(s)|^2\tilde{Y}^{p-2}(s)\|_{L^1}ds\\
&\le C(p)\int_0^t (1+\|\bar{Z}_{x_1}\|_\alpha^{\gamma_0} + \sum_{n=2}^3 \|:\bar{Z}_{x_1}^n:\|_\alpha^{p+2})ds + \lambda\int_0^t \|Y(s)\|_{L^p}\|\tilde{Y}(s)\|_{L^p}^{p-1}ds\\
&\quad + C\lambda p\int_0^t s^{2\alpha}\|x_0-x_1\|_\alpha\|\tilde{Y}(s)\|_{L^p}^{p-1}ds\\
&\le C(p, \|\underline{Z}\|_{\mathfrak{L}_1})\int_0^t s^{(2\alpha-\varepsilon)(p+2)}(1+\|x_1\|_\alpha^{\gamma(p)})ds\\
&\quad + \lambda C(p)\int_0^t \|Y(s)\|_{L^p}^p ds + \frac{\lambda}{2}\int_0^t \|\tilde{Y}(s)\|_{L^p}^p ds\\
&\quad + C(p)\lambda\int_0^t s^{2\alpha p}\|x_0-x_1\|_\alpha^p ds,
\end{aligned}$$

where we used Lemma 2.7 in the last inequality. By Lemma 3.4, (3.7) and the fact that $\lambda > 1$, (3.12) follows.

Step 3 We prove that for $0 < \beta < \frac{1}{2} + \alpha$

$$\|\tilde{Y}(1)\|_\beta \le C(\lambda, \|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha, \|x_1\|_\alpha). \tag{3.13}$$

Since $\tilde{Y}$ satisfies the mild equation, by similar arguments as in Step 3 in the proof of Lemma 3.4 we have

$$\begin{aligned}\|\tilde{Y}(1)\|_\beta \le{}& C\int_0^1 (1-s)^{-\frac{\beta}{2}-\frac{1}{2}}[\|\tilde{Y}^3\|_{L^2} + \lambda(\|\tilde{Y}\|_{L^2} + \|Y\|_{L^2})]ds \\ &+ C\int_0^1 (1-s)^{-\frac{\beta+1/2-\alpha}{2}}[\|\tilde{Y}^2\bar{Z}_{x_1}\|_{B^\alpha_{4,\infty}} \\ &+ \|\tilde{Y} : \bar{Z}^2_{x_1} :\|_{B^\alpha_{4,\infty}}]ds \\ &+ C\int_0^1 (1-s)^{-(\beta-\alpha)/2}(\| : \bar{Z}^3_{x_1} : \|_\alpha + \lambda\|x_0\|_\alpha + \lambda\|x_1\|_\alpha)ds \\ \le{}& C(p, \lambda, \|\underline{Z}\|_{\mathfrak{L}_1}, \|x_0\|_\alpha),\end{aligned}$$

where for the term in the first integral we used (3.12) and for the term in the second and third integral we used similar arguments as in Step 3 in the proof of Lemma 3.4. Combining (3.9) and (3.13) the first result follows.

The second follows from (3.11), (3.12) and Lemma 3.4.

In the following we give an estimate of the Wick power $: Z^k :$, which is required in the proof of the main results.

For $\gamma > 0$ and $K > 0$ we introduce the following notations:

$$E_{K,\gamma} := \{\|\bar{Z}\|_{\mathfrak{L}_1} \le K, \int_1^t [\|Z\|^\gamma_\alpha + \| : Z^2 : \|^\gamma_\alpha + \| : Z^3 : \|^\gamma_\alpha]ds \le K(1+t), \forall t \ge 1\}. \tag{3.14}$$

Lemma 3.6. *For every $\gamma > 0$, $\varepsilon > 0$ there exists $K > 0$ such that $P(E_{K,\gamma}) \ge 1 - \varepsilon$.*

Proof. To prove this result, we first introduce the following stationary Markov process. Define $Z_1(t) = \int_{-\infty}^t e^{(t-s)A}dW(s)$. We also define

$$\begin{aligned}: Z_1^2 : &:= \lim_{\varepsilon\to 0}[(Z_1 * \rho_\epsilon)^2 - c_\epsilon] \text{ in } L^p(\Omega, C([0,\infty), \mathcal{C}^\alpha)), \\ : Z_1^3 : &:= \lim_{\epsilon\to 0}[(Z_1 * \rho_\epsilon)^3 - 3c_\epsilon Z_1 * \rho_\epsilon] \text{ in } L^p(\Omega, C([0,\infty), \mathcal{C}^\alpha)),\end{aligned}$$

for $p > 1$ as in [RZZ15, Lemma 3.3]. $(Z_1, : Z_1^2 :, : Z_1^3 :)$ are also stationary Markov processes. Here ρ_ϵ and c_ϵ are introduced in Sect. 2.2. By [DZ96, Theorem 3.3.1] we know that for every $q > 1$ there exists $\eta \in L^2(\Omega, P)$ such that

$$\mathfrak{Z}_T := \frac{1}{T}\int_0^T [\|Z_1\|^q_\alpha + \sum_{n=2}^3 \| : Z_1^n : \|^q_\alpha]ds \to \eta, \text{ as } T \to \infty, \quad P - a.s.,$$

which implies that for every $\varepsilon > 0$, there exists $\Omega_1 \subset \Omega$ such that $P(\Omega_1) < \varepsilon/4$ and

$$\sup_{\omega\in\Omega_1^c} |\mathfrak{Z}_T(\omega) - \eta(\omega)| \to 0, \text{ as } T \to \infty.$$

From this we can deduce that there exists T_0 independent of ω such that for $T \geq T_0$

$$\mathfrak{Z}_T(\omega) \leq \eta(\omega) + 1, \forall \omega \in \Omega_1^c,$$

which combined with $\eta \in L^2(\Omega; P)$ yields that there exists $K_1 > 0$ such that

$$P\{\int_0^T [\|Z_1\|_\alpha^{2\gamma} + \sum_{n=2}^3 \| : Z_1^n : \|_\alpha^{2\gamma}]ds \leq K_1 T, \forall T \geq T_0\} > 1 - \varepsilon/3.$$

Thus, there exists $K_2 > 0$ such that

$$P\{\int_0^T [\|Z_1\|_\alpha^{2\gamma} + \sum_{n=2}^3 \| : Z_1^n : \|_\alpha^{2\gamma}]ds \leq K_2(T+1), \forall T \geq 0\} > 1 - \varepsilon/3.$$

Now we give the relations of Z and Z_1. By (2.2) and similar arguments as in the proof of [RZZ15, Lemma 3.6] we have that

$$\begin{aligned}
Z(t) &= Z_1(t) - e^{tA} Z_1(0),\\
: Z^2(t) : &=: Z_1^2(t) : -2e^{tA} Z_1(0) Z_1(t) + (e^{tA} Z_1(0))^2,\\
: Z^3(t) : &=: Z_1^3(t) : +3(e^{tA} Z_1(0))^2 Z_1(t) - 3e^{tA} Z_1(0) : Z_1(t)^2 : -(e^{tA} Z_1(0))^3,
\end{aligned}$$

which combined with Lemma 2.3 implies that for $\beta > -\alpha > 0$

$$\begin{aligned}
\|Z(t)\|_\alpha &\leq \|Z_1(t)\|_\alpha + \|Z_1(0)\|_\alpha,\\
\| : Z^2(t) : \|_\alpha &\leq C[\| : Z_1^2(t) : \|_\alpha + \|e^{tA} Z_1(0)\|_\beta \|Z_1(t)\|_\alpha + \|e^{tA} Z_1(0)\|_\beta^2],\\
\| : Z^3(t) : \|_\alpha &\leq C[\| : Z_1^3(t) : \|_\alpha + \|e^{tA} Z_1(0)\|_\beta^2 \|Z_1(t)\|_\alpha\\
&\quad + \|e^{tA} Z_1(0)\|_\beta \| : Z_1^2(t) : \|_\alpha + \|e^{tA} Z_1(0)\|_\beta^3].
\end{aligned}$$

Now using Lemma 2.2 we have

$$\begin{aligned}
&\int_1^T [\|Z\|_\alpha^\gamma + \| : Z^2 : \|_\alpha^\gamma + \| : Z^3 : \|_\alpha^\gamma]ds\\
&\leq C \int_1^T [\|Z_1(0)\|_\alpha^{4\gamma} + 1 + \|Z_1(s)\|_\alpha^{2\gamma} + \| : Z_1^2(s) : \|_\alpha^{2\gamma} + \| : Z_1^3(s) : \|_\alpha^\gamma]ds,
\end{aligned}$$

which implies that there exists $K_3 > 0$ such that

$$P\{\int_1^T [\|Z\|_\alpha^\gamma + \sum_{n=2}^3 \| : Z^n : \|_\alpha^\gamma]ds \leq K_3(T+1), \forall T \geq 0\} > 1 - \varepsilon/2.$$

On the other hand, by Lemma 2.6 we have $E\|\underline{Z}\|_{\mathfrak{L}^1}^2 < \infty$, which implies that there exist $K_4 > 0$ such that

$$P(\|\underline{Z}\|_{\mathfrak{L}^1} \leq K_4) > 1 - \varepsilon/2.$$

Combining the above results we obtain that there exists $K > 0$ such that

$$P(E_{K,\gamma}) \geq 1 - \varepsilon.$$

□

4. Uniqueness of the Invariant Measure

In this section we will prove our main result: uniqueness of the invariant measure. We will use an asymptotic coupling argument to prove it. We first present an abstract result based on asymptotic coupling from [HMS11]: Let $\mathcal{P}$ be a Markov transition function on a Polish space $(\mathbb{X}, \rho)$ and let $\mathbb{X}_\infty = \mathbb{X}^{\mathbb{N}}$ be the space of all sequences in $\mathbb{X}$ with product topology. Denote the collection of all Borel probability measures on $\mathbb{X}$ by $\mathcal{M}(\mathbb{X})$ and $\mathcal{M}(\mathbb{X}_\infty)$ is defined correspondingly. Take $\mathcal{P}_\infty : \mathbb{X} \to \mathcal{M}(\mathbb{X}_\infty)$ to be the probability kernel defined by stepping with the Markov kernel $\mathcal{P}$. For $\mu_0 \in \mathcal{M}(\mathbb{X})$, let $\mu_0\mathcal{P}_\infty \in \mathcal{M}(\mathbb{X}_\infty)$ be the measure defined by $\int_{\mathbb{X}} \mathcal{P}_\infty(x, \cdot)d\mu_0(x)$. Given $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{X})$, consider the generalized coupling given by

$$\tilde{\mathcal{C}}(\mu_1\mathcal{P}_\infty, \mu_2\mathcal{P}_\infty) := \{\Gamma \in \mathcal{M}(\mathbb{X}_\infty \times \mathbb{X}_\infty) : \Gamma \circ \Pi_i^{-1} \ll \mu_i\mathcal{P}_\infty \text{ for each } i \in \{1, 2\}\},$$

where Π_i is the projection onto the ith coordinate. We also denote the diagonal at infinity by

$$D := \{(x, y) \in \mathbb{X}_\infty \times \mathbb{X}_\infty : \lim_{n\to\infty} \rho(x_n, y_n) = 0\}.$$

Now we recall the following asymptotic coupling argument from [HMS11].

Theorem 4.1. [HMS11, Corollary 2.2] *Suppose that there exists a Borel measurable set $B \subset \mathbb{X}$ such that*

(i) *for any $\mathcal{P}$-invariant Borel probability measure μ, $\mu(B) > 0$,*
(ii) *there exists a measurable map $B \times B \ni (x, y) \mapsto \Gamma_{x,y} \in \mathcal{M}(\mathbb{X}_\infty \times \mathbb{X}_\infty)$ such that, for all $x, y \in B$, $\Gamma_{x,y} \in \tilde{\mathcal{C}}(\delta_x\mathcal{P}_\infty, \delta_y\mathcal{P}_\infty)$ and $\Gamma_{x,y}(D) > 0$.*

Then there exists at most one invariant probability measure for $\mathcal{P}$.

Now we prove our main result by using Theorem 4.1.

Proof of Theorem 1.1. For a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and a cylindrical Wiener process W as in Sect. 3, we use $X(x_0)$ to denote the solution of Eq. (3.2) obtained in Sect. 3 starting from $x_0 \in \mathcal{C}^\alpha$. Now we apply Theorem 4.1 to $B = \mathbb{X} = \mathcal{C}^\alpha$ and $\mathcal{P} = P_1(x, dy)$ and $\mathcal{P}_\infty : \mathcal{C}^\alpha \mapsto \mathcal{M}(\mathcal{C}^\alpha_\infty)$ is defined as above, where $P_1(x, \cdot)$ denotes the marginal of $X(x)$ at time $t = 1$ for $x \in \mathcal{C}^\alpha$.

We also use $\tilde{X}(x_1)$ to denote the solutions of Eq. (3.4) obtained in Sect. 3 starting from $x_1 \in \mathcal{C}^\alpha$, which is used to construct the generalized coupling.

Girsanov transform

Set $v = \lambda(\tilde{X}(x_1) - X(x_0))$ and let $\tilde{W}(t) = W(t) - \int_0^{t\wedge\tau_R} v(s)ds$, where

$$\tau_R := \inf\{t > 0, \int_0^t \|X(x_0, s) - \tilde{X}(x_1, s)\|_{L^2}^2 ds \geq R\}.$$

Since

$$E \exp\left(\frac{1}{2}\int_0^{\tau_R} \|v(s)\|_{L^2}^2 ds\right) \leq e^{\frac{1}{2}R\lambda^2},$$

by the Girsanov theorem there is a probability measure Q on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ such that under Q, $\tilde{W}$ is a standard Wiener process. Moreover, it holds that $P \sim Q$ on $\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$.

Construction of the coupling

Let $\hat{Z}$ be the solution to the following linear equation

$$d\hat{Z}(t) = A\hat{Z}(t)dt + d\tilde{W}(t), \quad \hat{Z}(0) = 0,$$

and $:\hat{Z}^k:$ can be defined similarly as $:Z^k:$ as in Sect. 2. Moreover, we use similar notations as in Sect. 2: $\bar{\hat{Z}}_{x_1} := \hat{Z} + e^{tA}x_1$, and for $n = 2, 3$,

$$:\bar{\hat{Z}}^n_{x_1}(t) ::= \sum_{k=0}^{n} C_n^k (e^{tA}x_1)^{n-k} :\hat{Z}^k(t): .$$

Furthermore, we derive the relation between different Wick powers under P and Q respectively. Since $\hat{Z} = Z + a$ with $a(t) = -\int_0^t e^{(t-s)A} 1_{s\leq\tau_R} v(s)ds \in C([0,\infty); \mathcal{C}^\beta)$ for some $\beta > -\alpha$, we have that there exists $\Omega_0' \subset \Omega_0$ such that $P(\Omega_0') = 1$ and the following hold for $\omega \in \Omega_0'$ in $C((0,\infty); \mathcal{C}^\alpha)$

$$:\hat{Z}^2 :=: Z^2 : + 2Za + a^2, \tag{4.1}$$

and

$$:\hat{Z}^3 :=: Z^3 : + 3Za^2 + 3 : Z^2 : a + a^3. \tag{4.2}$$

We will prove (4.1) and (4.2) at the end of the proof.

For $\omega \in \Omega_0'$, by (4.1), (4.2) and $a \in C([0,\infty); \mathcal{C}^\beta)$ we know that for $n = 2, 3$, $T \in \mathbb{R}^+$

$$\hat{Z} \in C([0,T]; \mathcal{C}^\alpha), :\hat{Z}^n :\in C((0,T]; \mathcal{C}^\alpha), \|\underline{\hat{Z}}\|_{\mathfrak{L}_T} < \infty,$$

which by Theorem 3.1 implies that for $\omega \in \Omega_0'$ there exists a unique mild solution $\hat{Y}(\omega) \in C([0,\infty), \mathcal{C}^\beta)$ to the following equation

$$\frac{d\hat{Y}}{dt} = A\hat{Y} - a_1\hat{Y}^3 + \Psi(\hat{Y}, \bar{\hat{Z}}_{x_1}), \quad \hat{Y}(0) = 0. \tag{4.3}$$

Define

$$\hat{X}(x_1, \omega) = \begin{cases} [\hat{Y} + e^{tA}x_1 + \hat{Z}](\omega), & \text{if } \omega \in \Omega_0' \\ 0 & \text{otherwise.} \end{cases}$$

Then we conclude that under Q, $\hat{X}(x_1)$ is also a mild solution to the Eq. (3.2) with $x = x_1$ and with W replaced by $\tilde{W}$, which combined with Theorem 3.1 and the Yamada-Watanabe Theorem in [Kurz07] implies that under Q, $\hat{X}(x_1)$ has the same law as the solution $X(x_1)$ to the Eq. (3.2) starting from x_1. Since $P \sim Q$, we have that under P the law of the pair $(X(x_0), \hat{X}(x_1))$ has marginals which are equivalent to the marginals of the solutions to (3.2) starting respectively from x_0 and x_1. Set $\Gamma_{x_0,x_1} :=$ law of $(X(x_0), \hat{X}(x_1))$ for $x_0, x_1 \in \mathcal{C}^\alpha$. It follows that $\Gamma_{x_0,x_1} \in \tilde{\mathcal{C}}(\delta_{x_0}\mathcal{P}_\infty, \delta_{x_1}\mathcal{P}_\infty)$. It remains to show that $\Gamma_{x_0,x_1}(D) > 0$.

We have that $\hat{X}(x_1)$ satisfies the following equation in the mild sense P-a.s.:

$$d\hat{X} = [A\hat{X} - a_1(\hat{X} - \bar{\hat{Z}}_{x_1})^3 + \Psi(\hat{X} - \bar{\hat{Z}}_{x_1}, \bar{\hat{Z}}_{x_1})]dt + d\tilde{W}.$$

By (4.1), (4.2) we have that there exists $\Omega_2 \subset \Omega_0'$ such that $P(\Omega_2) = 1$ and for $\omega \in \Omega_2$, $\hat{X}(x_1, \omega)$ also satisfies the following equation in the mild sense :

$$d\hat{X} = [A\hat{X} - a_1(\hat{X} - \bar{Z}_{x_1})^3 + \Psi(\hat{X} - \bar{Z}_{x_1}, \bar{Z}_{x_1})]dt + dW - v1_{t \le \tau_R} dt.$$

Then on $\{\tau_R = \infty\} \cap \Omega_2$, $\hat{X} - \bar{Z}_{x_1}$ also satisfies (3.3). By Theorem 3.3 we obtain that $\hat{X} - \bar{Z}_{x_1} = \tilde{Y}$ on $\{\tau_R = \infty\} \cap \Omega_2$, which implies that $\hat{X} = \tilde{X}$ on $\{\tau_R = \infty\} \cap \Omega_2$. Here $\tilde{Y}$ is the solution to (3.3) and $\tilde{X}(x_1) = \tilde{Y} + e^{tA}x_1 + Z$. Now to prove $\Gamma_{x_0,x_1}(D) > 0$, it suffices to estimate $X(x_0) - \tilde{X}(x_1)$.

Estimate of $X(x_0) - \tilde{X}(x_1)$

In the following we estimate $X(x_0) - \tilde{X}(x_1)$ and we do all the calculations informally, but all the calculations below can be made rigorous by approximation as done in the proof of [RZZ15, Theorem 3.10]. Set $Y_1 = Y + e^{tA}x_0$, $\tilde{Y}_1 = \tilde{Y} + e^{tA}x_1$ and $u = X - \tilde{X} = Y_1 - \tilde{Y}_1$. By the binomial formula (2.2) we have that P-a.s. Y_1 and $\tilde{Y}_1$ are the mild solutions to the following equations

$$\frac{d}{dt}Y_1 = AY_1 - [a_1 Y_1^3 + \Psi(Y_1, Z)], \quad Y_1(0) = x_0,$$

and

$$\frac{d}{dt}\tilde{Y}_1 = A\tilde{Y}_1 + \lambda(Y_1 - \tilde{Y}_1) - [a_1\tilde{Y}_1^3 + \Psi(\tilde{Y}_1, Z)], \quad Y_1(0) = x_1$$

respectively. It is obvious that P-a.s. u is the mild solution to the following equation:

$$\frac{d}{dt}u = Au - \lambda u - [a_1 Y_1^3 - a_1\tilde{Y}_1^3 + \Psi(Y_1, Z) - \Psi(\tilde{Y}_1, Z)], \quad u(0) = x_0 - x_1.$$

Standard energy estimates yield

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \lambda\|u\|_{L^2}^2 \le -\langle(\Psi(Y_1, \underline{Z}) - \Psi(\tilde{Y}_1, Z)), u\rangle, \tag{4.4}$$

where we used that

$$-\langle Y_1^3 - \tilde{Y}_1^3, u\rangle \le 0.$$

Now we calculate each term in $\langle(\Psi(Y_1, Z) - \Psi(\tilde{Y}_1, Z)), u\rangle$: For the first term we have

$$\begin{aligned}
3a_1|\langle(Y_1^2 - \tilde{Y}_1^2)Z, u\rangle| &= 3a_1|\langle u^2, (Y_1 + \tilde{Y}_1)Z\rangle| \\
&\le C_S\|u^2\|_{B^{-\alpha}_{\frac{4}{3},1}}(\|Y_1 Z\|_{B^\alpha_{4,\infty}} + \|\tilde{Y}_1 Z\|_{B^\alpha_{4,\infty}}) \\
&\le C_S\|\Lambda^{\frac{1}{2}}u\|_{L^2}^2(\|Y_1 Z\|_{B^\alpha_{4,\infty}} + \|\tilde{Y}_1 Z\|_{B^\alpha_{4,\infty}}) \\
&\le C_S\|u\|_{L^2}(\|\nabla u\|_{L^2} + \|u\|_{L^2})(\|Y_1 Z\|_{B^\alpha_{4,\infty}} + \|\tilde{Y}_1 Z\|_{B^\alpha_{4,\infty}}) \\
&\le C_S\|u\|_{L^2}^2(\|Y_1 Z\|^2_{B^\alpha_{4,\infty}} + \|\tilde{Y}_1 Z\|^2_{B^\alpha_{4,\infty}} + 1) + \frac{1}{4}\|\nabla u\|_{L^2}^2,
\end{aligned} \tag{4.5}$$

where C_S is a constant changing from line to line and we used Lemma 2.3 in the first inequality and Lemmas 2.1 and 2.4 to deduce that

$$\|u^2\|_{B^{-\alpha}_{\frac{4}{3},1}} \le C_S\|\Lambda^s u^2\|_{L^{\frac{4}{3}}} \le C_S\|\Lambda^s u\|_{L^2}\|u\|_{L^4} \le C_S\|\Lambda^{\frac{1}{2}}u\|_{L^2}^2, \tag{4.6}$$

for $\frac{1}{2} > s > -\alpha$ in the second inequality and we used Lemma 2.4 in the third inequality and Young's inequality in the last inequality. For the second term we have

$$\begin{aligned} 3a_1|\langle Y_1 : Z^2 : -\tilde{Y}_1 : Z^2 :, u\rangle| &\le C_S\|u^2\|_{B^{-\alpha}_{1,1}}\| : Z^2 : \|_\alpha \\ &\le C_S\|u\|_{L^2}(\|\nabla u\|_{L^2} + \|u\|_{L^2})\| : Z^2 : \|_\alpha \\ &\le C_S\|u\|^2_{L^2}(\| : Z^2 : \|^2_\alpha + 1) + \frac{1}{4}\|\nabla u\|^2_{L^2}, \end{aligned} \tag{4.7}$$

where we used Lemma 2.4 in the first inequality, Lemma 2.1 and (4.6) to deduce that

$$\|u^2\|_{B^{-\alpha}_{1,1}} \le \|u^2\|_{B^{-\alpha}_{\frac{4}{3},1}} \le C_S\|\Lambda^{\frac{1}{2}}u\|^2_{L^2},$$

for $\frac{1}{2} > s > -\alpha, q > 1, \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, q_2 < 4, \frac{2}{q_1} - s > \frac{1}{2}$ in the second inequality and we used Young's inequality in the last inequality.

For the last term we have

$$|a_2\langle Y_1 - \tilde{Y}_1, u\rangle| \le C\|u\|^2_{L^2}. \tag{4.8}$$

Combining (4.4)-(4.8) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \lambda\|u\|^2_{L^2} \\ &\quad\le \|u\|^2_{L^2}C_S[\|Y_1Z\|^2_{B^\alpha_{4,\infty}} + \|\tilde{Y}_1Z\|^2_{B^\alpha_{4,\infty}} + \| : Z^2 : \|^2_\alpha + 1] := \|u\|^2_{L^2}L. \end{aligned}$$

Then Gronwall's inequality yields that

$$\|u(t)\|^2_{L^2} \le \|u(1)\|^2_{L^2}\exp\int_1^t 2(-\lambda + L(s))ds.$$

Here we use Gronwall's inequality starting from $t = 1$ instead of $t = 0$ since $u(0)$ is not in L^2.

Recall that for $\gamma, K > 0$, $E_{K,\gamma}$ has been defined in (3.14). By Lemma 3.6 we know that for every $\gamma > 0$ there exists $K > 0$, such that $P(E_{K,\gamma}) > 0$. In the following we estimate each term in L on $E_{K,\gamma}$ with $\gamma > 0$ to be determined later: We have that on $E_{K,\gamma}$

$$\begin{aligned} &\int_1^t \|Y_1Z\|^2_{B^\alpha_{4,\infty}}ds \\ &\le C_S\int_1^t \|Y_1\|^2_{B^\beta_{4,\infty}}\|Z\|^2_\alpha ds \\ &\le C_S\int_1^t (\|\nabla Y_1\|^{2\beta_0}_{L^2}\|Y_1\|^{2(1-\beta_0)}_{L^2} + \|Y_1\|^2_{L^2})\|Z\|^2_\alpha ds \\ &\le C_S[(\int_1^t \|\nabla Y_1\|^{2\beta_0p_1}_{L^2}ds)^{\frac{1}{p_1}}\left(\int_1^t \|Y_1\|^{2(1-\beta_0)p_2}_{L^2}ds\right)^{\frac{1}{p_2}}\left(\int_1^t \|Z\|^{2p_3}_\alpha ds\right)^{\frac{1}{p_3}} \\ &\quad + (\int_1^t \|Y_1\|^4_{L^2}ds)^{\frac{1}{2}}\left(\int_1^t \|Z\|^4_\alpha ds\right)^{\frac{1}{2}}], \end{aligned} \tag{4.9}$$

where $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$, $p_i>1, i=1,2,3, \beta>-\alpha, \beta_0=\beta+\frac{1}{2}+\varepsilon, \varepsilon>0, 2\beta_0 p_1\leq 2$, and we used Lemma 2.3 in the first inequality, Lemma 2.1 to deduce that

$$\|Y_1\|_{B^\beta_{4,\infty}}\leq C_S\|Y_1\|_{B^{\beta+1/2}_{2,\infty}}\leq C_S\|Y_1\|_{B^{\beta+1/2}_{2,1}}\leq C_S\|Y_1\|_{H^{\beta_0}_2}$$

in the second inequality and Hölder's inequality in the last inequality. In the following we estimate each term on the right hand side of (4.9): by Lemma 3.4 we know that for any even $p>1$ we have on $E_{K,\gamma}$ with $\gamma\geq\gamma(p)$

$$\begin{aligned}\int_1^t\|Y_1(s)\|^p_{L^p}ds&\leq C(p)[\int_1^t\|Y(s)\|^p_{L^p}ds+\int_1^t\|e^{sA}x_0\|^p_{L^p}ds]\\&\leq C(p)\int_1^t[1+\|Z\|^{\gamma(p)}_\alpha+\|:Z^2:\|^{\gamma(p)}_\alpha+\|:Z^3:\|^{\gamma(p)}_\alpha]ds\\&\quad+C(\|x_0\|_\alpha)(1+t)+C(\|\underline{Z}\|_{\mathfrak{L}_1},\|x_0\|_\alpha)\\&\leq C(p,\|x_0\|_\alpha,K)(1+t),\end{aligned}$$

where we used Lemma 2.2 to control $\|e^{sA}x_0\|_{L^p}\leq C_S s^\alpha\|x_0\|_\alpha$ in the second inequality. Similarly, by Lemma 3.4 we have that on $E_{K,\gamma}$ for $\gamma\geq\gamma(2)$

$$\begin{aligned}\int_1^t\|\nabla Y_1(s)\|^{2\beta_0p_1}_{L^2}ds&\leq C(p_1)[\int_1^t\|\nabla Y(s)\|^2_{L^2}ds+t+\int_1^t\|\nabla e^{sA}x_0\|^{2\beta_0p_1}_{L^2}ds]\\&\leq C(p_1)\int_1^t[1+\|x_0\|^{\gamma(2)}_\alpha+\|Z\|^{\gamma(2)}_\alpha+\|:Z^2:\|^{\gamma(2)}_\alpha\\&\quad+\|:Z^3:\|^{\gamma(2)}_\alpha]ds+t\\&\quad+C\int_1^t s^{-(1+\varepsilon-\alpha)\beta_0p_1}\|x_0\|^{2\beta_0p_1}_\alpha ds+C(\|\underline{Z}\|_{\mathfrak{L}_1},\|x_0\|_\alpha)\\&\leq C(p_1,\|x_0\|_\alpha,K)(1+t),\end{aligned}$$

where we used Young's inequality and $2\beta_0p_1\leq 2$ in the first inequality and Lemmas 2.1, 2.2 to deduce that $\|\nabla e^{sA}x_0\|_{L^2}\leq C_S s^{-(1+\varepsilon-\alpha)/2}\|x_0\|_\alpha$ in the second inequality. Choose

$$\gamma\geq\gamma(2(1-\beta_0)p_2)\vee 2p_3\vee\gamma(2)\vee\gamma(4)\vee 4\vee\gamma(2(p_0-1))$$

for some p_0 satisfying $p_0>-\frac{2}{\alpha}$, which will be used later. Combining the above estimates we obtain that on $E_{K,\gamma}$

$$\int_1^t\|Y_1Z\|^2_{B^\alpha_{4,\infty}}ds\leq C(p_1,p_2,\|x_0\|_\alpha,K)(1+t).$$

Similarly by Lemma 3.5 we have for even $p>1$ with $\gamma\geq\gamma(p)$ that on $E_{K,\gamma}$

$$\begin{aligned}&\int_1^t\|\tilde{Y}_1(s)\|^p_{L^p}ds\\&\leq C(p)[\int_1^t\|\tilde{Y}(s)\|^p_{L^p}ds+\int_1^t\|e^{sA}x_1\|^p_{L^p}ds]\\&\leq C(p,\|x_0\|_\alpha,\|x_1\|_\alpha)\int_1^t[1+\|Z\|^{\gamma(p)}_\alpha+\|:Z^2:\|^{\gamma(p)}_\alpha+\|:Z^3:\|^{\gamma(p)}_\alpha]ds\\&\quad+C(\|x_1\|_\alpha)(1+t)+C(p,\lambda,\|\underline{Z}\|_{\mathfrak{L}_1},\|x_0\|_\alpha,\|x_1\|_\alpha)\\&\leq C(p,\|x_0\|_\alpha,\|x_1\|_\alpha,K)t+C(p,\lambda,K,\|x_0\|_\alpha,\|x_1\|_\alpha),\end{aligned}$$

and

$$\begin{aligned}&\int_1^t \|\nabla\tilde{Y}_1(s)\|_{L^2}^{2\beta_0 p_1}ds\\&\leq C(p_1,\|x_0\|_\alpha,\|x_1\|_\alpha)\lambda\int_1^t[1+\|Z\|_\alpha^{\gamma(2)}+\|:Z^2:\|_\alpha^{\gamma(2)}+\|:Z^3:\|_\alpha^{\gamma(2)}]ds\\&\quad+C(\|x_1\|_\alpha)(1+t)+C(p_1,\lambda,\|\underline{Z}\|_{\mathfrak{L}_1},\|x_0\|_\alpha,\|x_1\|_\alpha)\\&\leq C(p_1,\|x_0\|_\alpha,\|x_1\|_\alpha,K)t\lambda+C(p_1,\lambda,K,\|x_0\|_\alpha,\|x_1\|_\alpha).\end{aligned}$$

Then we have on $E_{K,\gamma}$,

$$\begin{aligned}\|u(t)\|_{L^2}^2&\leq\|u(1)\|_{L^2}^2\exp[\int_1^t 2(-\lambda+L(s))ds]\\&\leq\|u(1)\|_{L^2}^2\exp[-\lambda t+C(p_1,p_2,\|x_0\|_\alpha,\|x_1\|_\alpha,K)t\lambda^{\frac{1}{p_1}}\\&\quad+C(p_1,p_2,\lambda,\|x_0\|_\alpha,\|x_1\|_\alpha,K)].\end{aligned}$$

By (3.7) and (3.12) we have $\|u(1)\|_{L^2}^2\leq C(\lambda,\|x_0\|_\alpha,\|x_1\|_\alpha,K)$ on $E_{K,\gamma}$. Then we choose λ large enough so that there exist constants $C_0,C_1>0$ such that

$$\|u(t)\|_{L^2}^2\leq C_0e^{-C_1t}\to 0 \text{ on } E_{K,\gamma},\ \text{ as } t\to\infty.$$

On the other hand by Lemmas 3.4 and 3.5 we know that for $p_0>-\frac{2}{\alpha}$ on $E_{K,\gamma}$ and $t>1$

$$\|Y_1^{2p_0-2}(t)\|_{L^1}+\|\tilde{Y}_1^{2p_0-2}(t)\|_{L^1}\leq C(p_0,\|x_0\|_\alpha,\|x_1\|_\alpha,K,\lambda)(1+t),$$

which by Hölder's inequality implies that

$$\|u(t)\|_{L^{p_0}}\leq\|u(t)\|_{L^2}(\|Y_1^{2p_0-2}(t)\|_{L^1}^{\frac{1}{2}}+\|\tilde{Y}_1^{2p_0-2}(t)\|_{L^1}^{\frac{1}{2}})\to 0 \text{ on } E_{K,\gamma},\ \text{ as } t\to\infty,$$

Thus Lemma 2.1 yields that

$$\|u(t)\|_\alpha\to 0 \text{ on } E_{K,\gamma},\ \text{ as } t\to\infty.$$

From the above we also obtain that for fixed $K>0$, there exists $R>0$ such that $\tau_R=\infty$ on $E_{K,\gamma}$. It follows that

$$\Gamma_{x_0,x_1}(D)\geq P(E_{K,\gamma})>0.$$

Now by Theorem 4.1 the first result of Theorem 1.1 follows.

4.0.3. Proof of weak convergence In the following we prove that for fixed $x \in \mathcal{C}^\alpha$, $P_t(x, dy)$ converges to ν weakly, where $P_t(x, dy)$ denote the distribution of the $X(t)$ starting from x. We use similar arguments as the proof of [KS16, Theorem 2.7].

By similar argument as the proof of [RZZ15, Theorem 3.10] we have that the solution X to the Eq. (3.2) is continuous with respect to initial value in $\mathcal{C}^\alpha$, which implies the Feller property of the semigroup easily.

Now for $x \in \mathcal{C}^\alpha$ we prove the tightness of $\{P_n(x, dy), n \geq 1\}$. By Lemma 3.6 for every $\varepsilon > 0$, $y \in \mathcal{C}^\alpha$, $\alpha < \alpha' < -\frac{2}{p_0}$ for p_0 above, we can find a generalized coupling $\Gamma^\varepsilon_{x,y} \in \tilde{\mathcal{C}}(\delta_x \mathcal{P}_\infty, \delta_y \mathcal{P}_\infty)$ as above such that

$$\Gamma^\varepsilon_{x,y}(D) \geq 1 - \varepsilon/2, \quad \Gamma^\varepsilon_{x,y}(\lim_{n\to\infty} \|x_n - y_n\|_{\alpha'} = 0) \geq 1 - \varepsilon/2,$$

and $\Gamma^\varepsilon_{x,y} \circ \Pi_1^{-1} = \delta_x \mathcal{P}_\infty$, where $\delta_x \mathcal{P}_\infty$ denote the law of the sequence $\{X(n)\}$ on $\mathcal{C}^\alpha_\infty$ starting from x. In fact, $\Gamma^\varepsilon_{x,y}$ is the law of $(X(x), \hat{X}(y))$ as before and we choose $E_{\gamma,K(\varepsilon)}$ such that $P(E_{\gamma,K(\varepsilon)}) \geq 1 - \varepsilon/2$ and λ depends on $K(\varepsilon)$, which makes the coupling dependent on ε.

Define a measure on $\mathcal{C}^\alpha_\infty \times \mathcal{C}^\alpha_\infty$

$$\Gamma^\varepsilon(A) = \int \Gamma^\varepsilon_{x,y}(A)\nu(dy), \quad A \in \mathcal{M}(\mathcal{C}^\alpha_\infty) \times \mathcal{M}(\mathcal{C}^\alpha_\infty).$$

We have

$$\Gamma^\varepsilon(D) \geq 1 - \varepsilon/2, \quad \Gamma^\varepsilon(\lim_{n\to\infty} \|x_n - y_n\|_{\alpha'} = 0) \geq 1 - \varepsilon/2. \tag{4.10}$$

Since $\Gamma^\varepsilon \circ \Pi_1^{-1} = \delta_x \mathcal{P}_\infty$, $\Gamma^\varepsilon \circ \Pi_2^{-1} \ll \nu \mathcal{P}_\infty$, we deduce that $\Gamma^\varepsilon \in \tilde{\mathcal{C}}(\delta_x \mathcal{P}_\infty, \nu \mathcal{P}_\infty)$. Moreover we have for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Gamma^\varepsilon(y_n \in K_0^c) \leq \varepsilon/3, \quad n \geq 1, \tag{4.11}$$

if a compact set $K_0 \subset \mathcal{C}^\alpha$ is chosen such that $\nu(K_0) \geq 1 - \delta$.

Since the embedding $\mathcal{C}^{\alpha'} \subset \mathcal{C}^\alpha$ is compact and by [RZZ15, Lemma 3.1] we have that for every $k \in \mathbb{N}$, $\int \|\phi\|^k_{\mathcal{C}^{\alpha'}} \nu(d\phi) < \infty$, we can choose C large enough such that for compact sets $K_1 := \{\|\cdot\|_{\alpha'} \leq C\}$, $K_2 := \{\|\cdot\|_{\alpha'} \leq C + 1\}$

$$\nu(K_2) \geq \nu(K_1) \geq 1 - \delta.$$

By (4.10) we know that there exists D_1 such that $\Gamma^\varepsilon(D_1^c) \leq \frac{2\varepsilon}{3}$ and $\|x_n - y_n\|_{\alpha'}$ converges to 0 uniformly on D_1. For n large enough, we have

$$\Gamma^\varepsilon(x_n \in K_2^c) \leq \Gamma^\varepsilon(\{x_n \in K_2^c\} \cap D_1) + \Gamma^\varepsilon(D_1^c) \leq \Gamma^\varepsilon(y_n \in K_1^c) + \Gamma^\varepsilon(D_1^c) \leq \varepsilon,$$

where we used (4.10), (4.11) in the last inequality. Since $\Gamma^\varepsilon \circ \Pi_1^{-1} = \delta_x \mathcal{P}_\infty$ we deduce the tightness of $\{P_n(x, \cdot), n \geq 1\}$.

In the following we prove the weak convergence: If we assume that $P_n(x, \cdot)$ does not weakly converge to ν, there exists some probability measure $\nu_0 \neq \nu$ and a subsequence $P_{m_k}(x, \cdot)$ converges to ν_0 weakly. Fix a bounded Lipschitz continuous function $f : \mathcal{C}^\alpha \to \mathbb{R}$ such that $\int f d\nu_0 \neq \int f d\nu$ and set $U_n = \frac{1}{n}\sum_{k=1}^n f(x_{m_k})$. Now we want to prove that U_n converges to $\int f d\nu$ in probability with respect to $\delta_x \mathcal{P}_\infty$. For every $\varepsilon > 0$ as above and construct corresponding Γ^ε such that (4.10) holds.

By [KS16, Corollary 2.6] we have that U_n converges to $\int f d\nu$ in probability with respect to $\nu\mathcal{P}_\infty$, which implies that U_n converges to $\int f d\nu$ in probability with respect to $\Gamma^\varepsilon \circ \Pi_2^{-1}$. In fact, for every subsequence $\{n_r\}$ there exists another subsequence $\{n_{r_l}\}$ such that $U_{n_{r_l}}$ converges to $\int f d\nu$ $\nu\mathcal{P}_\infty$-a.s. Since $\Gamma^\varepsilon \circ \Pi_2^{-1} \ll \nu\mathcal{P}_\infty$, $U_{n_{r_l}}$ converges to $\int f d\nu$ $\Gamma^\varepsilon \circ \Pi_2^{-1}$-a.s.

Since f is bounded and Lipschitz continuous, by (4.10) we have

$$\Gamma^\varepsilon(\lim_{n\to\infty} |\frac{1}{n}\sum_{k=1}^n f(x_{m_k}) - \frac{1}{n}\sum_{k=1}^n f(y_{m_k})| = 0) \geq 1 - \varepsilon/2. \tag{4.12}$$

We have for every $\varepsilon_0 > 0$

$$\begin{aligned}
&\delta_x\mathcal{P}_\infty(|U_n - \int f d\nu| < \varepsilon_0)\\
&= \Gamma^\varepsilon(|U_n - \int f d\nu| < \varepsilon_0)\\
&\geq 1 - \Gamma^\varepsilon(|\frac{1}{n}\sum_{k=1}^n f(x_{m_k}) - \frac{1}{n}\sum_{k=1}^n f(y_{m_k})| + |\frac{1}{n}\sum_{k=1}^n f(y_{m_k}) - \int f d\nu| \geq \varepsilon_0)\\
&\geq 1 - \Gamma^\varepsilon(|\frac{1}{n}\sum_{k=1}^n f(x_{m_k}) - \frac{1}{n}\sum_{k=1}^n f(y_{m_k})| \geq \frac{\varepsilon_0}{2})\\
&\quad - \Gamma^\varepsilon(|\frac{1}{n}\sum_{k=1}^n f(y_{m_k}) - \int f d\nu| \geq \frac{\varepsilon_0}{2}),
\end{aligned}$$

which combined with (4.12) and the fact that U_n converges to $\int f d\nu$ in probability with respect to $\Gamma^\varepsilon \circ \Pi_2^{-1}$ implies that

$$\lim_{n\to\infty} \delta_x\mathcal{P}_\infty(|U_n - \int f d\nu| < \varepsilon_0) \geq 1 - \varepsilon.$$

Since ε is arbitrary we deduce that U_n converges to $\int f d\nu$ in probability with respect to $\delta_x\mathcal{P}_\infty$.

Moreover, f is bounded, we have that

$$\int U_n d\delta_x\mathcal{P}_\infty \to \int f d\nu.$$

On the other hand $P_{m_k}(x,\cdot)$ converges to ν_0 weakly, we have

$$\int U_n d\delta_x\mathcal{P}_\infty = \frac{1}{n}\sum_{k=1}^n \int f(y) P_{m_k}(x, dy) \to \int f d\nu_0 \neq \int f d\nu,$$

which is a contradiction finishing the proof of the second result.

Proof of (4.1) and (4.2)

In the following we only have to prove (4.1), (4.2). Let $a_\varepsilon = a * \rho_\varepsilon$. By a similar argument as in the proof of [RZZ15, Lemma 3.4] we have for $p > 1$

$$: \hat{Z}^2 := \lim_{\varepsilon\to 0}(\hat{Z}_\varepsilon^2 - c_\varepsilon) \text{ in } L^p(\Omega, C((0,\infty);\mathcal{C}^\alpha), Q),$$
$$: Z^2 : +2Za + a^2 = \lim_{\varepsilon\to 0}(Z_\varepsilon^2 + 2Z_\varepsilon a_\varepsilon + a_\varepsilon^2 - c_\varepsilon) \text{ in } L^p(\Omega, C((0,\infty);\mathcal{C}^\alpha), P).$$

Since

$$Z_\varepsilon^2 + 2Z_\varepsilon a_\varepsilon + a_\varepsilon^2 - c_\varepsilon = \hat{Z}_\varepsilon^2 - c_\varepsilon,$$

and $P \sim Q$, we obtain that (4.1) holds P-a.s. (4.2) can also be proved by taking the limit as $\varepsilon \to 0$ for the following equation

$$\hat{Z}_\varepsilon^3 - 3c_\varepsilon \hat{Z}_\varepsilon = Z_\varepsilon^3 + 3Z_\varepsilon a_\varepsilon^2 + 3(Z_\varepsilon^2 - c_\varepsilon)a_\varepsilon + a_\varepsilon^3 - 3c_\varepsilon Z_\varepsilon.$$

□

In the following we will prove Theorem 1.4. First we introduce a space $\mathcal{F}C_b^\infty$, which will be used in the proof of Theorem 1.4.

Let $E = H_2^{-1-\epsilon}$, $E^* = H_2^{1+\epsilon}$ for some $\epsilon > 0$. We denote their Borel σ-algebras by $\mathcal{B}(E), \mathcal{B}(E^*)$ respectively. Define

$$\mathcal{F}C_b^\infty = \{u : u(z) = f({}_{E^*}\langle l_1, z\rangle_E, {}_{E^*}\langle l_2, z\rangle_E, \ldots, {}_{E^*}\langle l_m, z\rangle_E),$$
$$z \in E, l_1, l_2, \ldots, l_m \in E^*, m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m)\},$$

and for $u \in \mathcal{F}C_b^\infty$ and $l \in L^2(\mathbb{T}^2)$,

$$\frac{\partial u}{\partial l}(z) := \frac{d}{ds}u(z+sl)|_{s=0}, z \in E,$$

Let Du denote the L^2-derivative of $u \in \mathcal{F}C_b^\infty$, i.e. the map from E to $L^2(\mathbb{T}^2)$ such that

$$\langle Du(z), l\rangle = \frac{\partial u}{\partial l}(z) \text{ for all } l \in L^2(\mathbb{T}^2), z \in E.$$

Proof of Theorem 1.4. First we prove that ν satisfies (i) and (ii) in Theorem 1.4. (i) is obvious from [GlJ86, Sect. 8.6]. By [AR91, Theorem 7.11] the logarithmic derivative of ν along k is

$$\beta_k = 2\langle z, Ak\rangle - 2\langle a_1 : z^3 : -a_2 z, k\rangle,$$

for $z \in E$, $k \in C^\infty(\mathbb{T}^2)$, which implies (ii) by using [AR90, Corollary 4.8].

Let ν_0 be the measure satisfying (i), (ii) in Theorem 1.4. From (ii) we calculate the logarithmic derivative of ν_0: For $u \in \mathcal{F}C_b^\infty$, $k \in C^\infty(\mathbb{T}^2)$

$$\begin{aligned}\int \frac{\partial u}{\partial k} d\nu_0 &= \int \lim_{t\to 0}\frac{u(z+tk)-u(z)}{t} d\nu_0\\ &= \lim_{t\to 0}\int \frac{u(z+tk)-u(z)}{t} d\nu_0\\ &= \lim_{t\to 0}\int \frac{(a_{-tk}(z)-1)u(z)}{t} d\nu_0\\ &= \int \lim_{t\to 0}\frac{a_{-tk}(z)-1}{t} u(z) d\nu_0,\end{aligned}$$

where in the second equality we used that $u \in \mathcal{F}C_b^\infty$ and the dominated convergence theorem, and in the last equality we used (i) and [GlJ86, Section 8.6] to deduce the uniform integrability of a_{tk}. This implies the logarithmic derivative of ν_0 is the same as that of ν. Hence by [AR91] the diffusion process X^{ν_0} obtained from the Dirichlet form $\mathcal{E}_{\nu_0}^0$ also satisfies (1.1) and ν_0 is an invariant measure for X^{ν_0}. Here $\mathcal{E}_{\nu_0}^0$ is the closure of the pre-Dirichlet form

$$\mathcal{E}_{\nu_0}(u, v) := \frac{1}{2}\int_E \langle Du, Dv\rangle_{L^2} d\nu_0,$$

defined for $u, v \in \mathcal{F}C_b^\infty$ (see [AR91]). Moreover, by (i) we know that Lemma 3.6 in [RZZ15] also holds for ν_0. Furthermore, the same argument as in the proof of [RZZ15, Theorems 3.9] implies that X^{ν_0} also satisfies the shifted equation (3.2). By the uniqueness of the solution to (3.2) (see Theorem 3.1), we know that ν_0 is also an invariant measure for the solution to (3.2). By Theorem 1.1 the result follows. □

Proof of Theorem 1.5.. First we prove that ν satisfies (i) and (ii) in Theorem 1.5. As mentioned in the proof of Theorem 1.4, (i) is obvious and the logarithmic derivative of ν along k is

$$\beta_k = 2\langle z, Ak\rangle - 2\langle a_1 : z^3 : -a_2 z, k\rangle,$$

for $z \in E$, $k \in C^\infty(\mathbb{T}^2)$, which implies (ii) by direct calculations.

Let ν_0 be the measure satisfying (i), (ii) in Theorem 1.5. From (ii) we calculate the logarithmic derivative of ν_0: We follow the proof of [BR95, Theorem 3.10]: By (ii) we have $\int Lud\nu_0 = 0$ for $u \in \mathcal{F}C_b^\infty$. Hence for all $u, v \in \mathcal{F}C_b^\infty$

$$0 = \int L(uv)d\nu_0 = 2\int uLvd\nu_0 + \int \langle Du, Dv\rangle_{L^2} d\nu_0,$$

i.e.,

$$-\int uLvd\nu_0 = \frac{1}{2}\int \langle Du, Dv\rangle_{L^2} d\nu_0. \tag{4.13}$$

Let $g_n \in C_b^\infty(\mathbb{R})$, $n \in \mathbb{N}$, such that $g_n(t) = t$ on $[-n, n]$ and $\sup\{|g_n'(t)| + |g_n''(t)| : n \in \mathbb{N}, t \in \mathbb{R}\} < \infty$. Let $k \in C^\infty(\mathbb{T}^2)$. Applying (4.10) to $v := g_n(k)$ we can take $n \to \infty$ according to the dominated convergence theorem, and since

$$L(g_n(k)) = g_n''(k)\|k\|_{L^2}^2 + g_n'(k)(\langle z, Ak\rangle - \langle a_1 : z^3 : -a_2 z, k\rangle),$$

we obtain that

$$\int \frac{\partial u}{\partial k} d\nu_0 = -\int \beta_k u d\nu_0.$$

Then we can conclude that the logarithmic derivative of ν_0 along k is the same as that of ν. Hence by the same proof as that for Theorem 1.4, the result follows. □

In the following we only prove Corollary 1.7. Corollary 1.8 can be obtained similarly.

Proof of Corollary 1.7. Assume that ν can be written as a convex combination of two probability measures μ_1 and μ_2 in $\mathcal{M}^a$. Then μ_1 and μ_2 are absolutely continuous w.r.t. to ν with bounded densities and hence are also absolutely continuous w.r.t. the Gaussian measure μ with p-integrable densities for some $p > 1$. By Theorem 1.4 $\mu_1 = \mu_2 = \nu$. So, ν is extreme in the set $\mathcal{M}^a$. □

References

[AKR97] Albeverio, S., Kondratiev, Y.G., Röckner, M.: Ergodicity for the stochastic dynamics of quasi-invariant measures with applications to gibbs states. J. Funct. Anal. **149**(2), 415–469 (1997)

[AR90] Albeverio, S., Röckner, M.: Classical dirichlet forms on topological vector spaces closability and a cameron–martin formula. J. Funct. Anal. **88**, 395–436 (1990)

[AR91] Albeverio, S., Röckner, M.: Stochastic differential equations in infinite dimensions: solutions via dirichlet forms. Probab Theory Relat Field **89**, 347–386 (1991)

[BCD11] Bahouri, H., Chemin, J.-Y., Danchin, R.: Fourier Analysis and Nonlinear Partial Differential Equations, Vol. 343 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg (2011)

[BDW16] Berglund, N., Di Gesù, G., Weber, H.: An Eyring–Kramers law for the stochastic AllenCahn equation in dimension two. arXiv:1604.05742v1 (2016)

[BR95] Bogachev, V.I., Röckner, M.: Regularity of invariant measures on finite and infinite dimensional spaces and applications. J. Funct. Anal. **133**(1), 168–223 (1995)

[CC13] Catellier, R., Chouk, K.: Paracontrolled Distributions and the 3-dimensional Stochastic Quantization Equation. arXiv:1310.6869

[DD03] Da Prato, G., Debussche, A.: Strong solutions to the stochastic quantization equations. Ann. Probab. **31**(4), 1900–1916 (2003)

[DZ96] Da Prato, G., Zabczyk, J.: Ergodicity for Infinite Dimensional Systems, London Mathematical Society Lecture Notes, vol. 229. Cambridge University Press, Cambridge (1996)

[GIP13] Gubinelli, M., Imkeller, P., Perkowski, N.: Paracontrolled distributions and singular PDEs. Forum Math. Pi 3 no. 6 (2015)

[GlJ86] Glimm, J., Jaffe, A.: Quantum Physics: A Functional Integral Point of View. Springer, New York (1986)

[GRS75] Guerra, F., Rosen, J., Simon, B.: The $p(\Phi)_2$ euclidean quantum field theory as classical statistical mechanics. Ann. Math. **101**, 111–259 (1975)

[Hai14] Hairer, M.: A theory of regularity structures. Invent. Math. **198**(2), 269–504 (2014)

[HMS11] Hairer, M., Mattingly, J.C., Scheutzow, M.: Asymptotic coupling and a general form of harris theorem with applications to stochastic delay equations. Probab. Theory Relat. Fields **149**(1-2), 223–259 (2011)

[HM16] Hairer, M., Mattingly, J.C.: The strong Feller property for singular stochastic PDEs. arXiv:1610.03415v1 (2016)

[JLM85] Jona-Lasinio, G., Mitter, P.K.: On the stochastic quantization of field theory. Commun. Math. Phys. **101**(3), 409–436 (1985)

[KS16] Kulik, A., Scheutzow, M.: Generalized couplings and convergence of transition probabilities. arXiv:1512.06359v2

[Kurz07] Kurtz, T.G.: The yamada–watanabe–engelbert theorem for general stochastic equations and inequalities. Electron. J. Probab. **12**, 951–965 (2007)

[MR92] Ma, Z.M., Röckner, M.: Introduction to the Theory of (Non-symmetric) Dirichlet Forms. Springer, Berlin (1992)

[MR99] Mikulevicius, R., Rozovskii, B.: Martingale problems for stochasic PDE's. In: Carmona, R., Rozovskii, B.L. (eds.) Stochastic Partial Differential Equations: Six Perspectives, Volume 64 of Mathematical Surveys Monographs, pp. 243–325. American Mathematical Society, Providence (1999)

[MW15] Mourrat, J., Weber, H.: Global well-posedness of the dynamic Φ^4 model in the plane. arXiv:1501.06191v1 to appear in the Annals of Probability

[MW16] Mourrat, J.-C., Weber, H.: Global well-posedness of the dynamic Φ^4_3 model on the torus. arXiv:1601.01234

[Nua06] Nualart, D.: The Malliavin Calculus and Related Topics. Probability and Its Applications (New York). 2nd ed. Springer, Berlin (2006)

[P76] Preston, C.: Random Fields. Lecture Notes in Mathematics, Vol. 534. Springer, Berlin (1976)

[PW81] Parisi, G., Wu, Y.S.: Perturbation theory without gauge fixing. Sci. Sin. **24**(4), 483–496 (1981)

[Re95] Resnick, S.: Dynamical Problems in Non-linear Advective Partial Differential Equations. Ph.D. thesis, University of Chicago, Chicago (1995)

[R86] Röckner, M.: Specifications and martin boundaries for $p(\phi)_2$-random fields. Commun. Math. Phys. **106**, 105–135 (1986)

[RZZ15] Röckner, M., Zhu, R., Zhu, X.: Restricted Markov unqiueness for the stochastic quantization of $P(\phi)_2$ and its applications. arXiv:1511.08030 (2015), to appear in Journal of Functional Analysis

[RZZ15a] Röckner, M., Zhu, R., Zhu, X.: Sub- and supercritical stochastic quasi-geostrophic equation. Ann. Probab. **43**(3), 1202–1273 (2015)
[S74] Simon, B.: The $P(\phi)_2$ Euclidean (Quantum) Field Theory. Princeton University Press, Princeton (1974)
[S85] Sickel, W.: Periodic spaces and relations to strong summability of multiple fourier series. Math. Nachr. **124**, 15–44 (1985)
[SW71] Stein, E.M., Weiss, G.L.: Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton (1971)
[Tri78] Triebel, H.: Interpolation Theory, Function Spaces, Differential Operators. North-Holland Mathematical Library, vol. 18. North-Holland, Amsterdam (1978)
[Tri06] Triebel, H.: Theory of Function Spaces III. Birkhäuser, Basel (2006)
[TW16] Tsatsoulis, P., Weber, H.: Spectral Gap for the Stochastic Quantization Equation on the 2-dimensional Torus (2016). arXiv:1609.08447

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