



The Kontsevich Matrix Integral: Convergence to the Painlevé Hierarchy and Stokes' Phenomenon

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Abstract: We show that the Kontsevich integral on $n \times n$ matrices ($n < \infty$) is the isomonodromic tau function associated to a 2×2 Riemann–Hilbert Problem. The approach allows us to gain control of the analysis of the convergence as $n \rightarrow \infty$. By an appropriate choice of the external source matrix in Kontsevich's integral, we show that the limit produces the isomonodromic tau function of a special tronquée solution of the first Painlevé hierarchy, and we identify the solution in terms of the Stokes' data of the associated linear problem. We also show that there are several tau functions that are analytic in appropriate sectors of the space of parameters and that the formal Witten–Kontsevich tau function is the asymptotic expansion of each of them in their respective sectors, thus providing an analytic tool to analyze its nonlinear Stokes' phenomenon.

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1. Introduction and Results

The Kontsevich matrix integral has been introduced in [18] as a tool to prove the Witten conjecture, which relates the intersection numbers of the Deligne–Mumford moduli space to a specific solution of the Korteweg–de Vries hierarchy. This integral is given by the following expression¹

$$Z_n(x; Y) := \frac{\int_{H_n} dM e^{\text{Tr} \left(i \frac{M^3}{3} - YM^2 + i x M \right)}}{\int_{H_n} dM e^{-\text{Tr} (YM^2)}}, \tag{1.1}$$

where the integral is over the space H_n of $(n \times n)$ Hermitian matrices and Y is a diagonal matrix whose entries $y_k, k = 1, \dots, n$ satisfy the condition $\text{Re } y_k > 0$ (to ensure convergence); the parameter x was absent in the original formulation and it is added here for later convenience. Kontsevich proved that the function $Z_n(0; Y)$ is a ratio of the Wronskian of Airy functions and the Vandermonde determinant of the eigenvalues of Y :

$$Z_n(x; Y) = 2^n \pi^{\frac{n}{2}} e^{\frac{2}{3} \text{Tr} Y^3 + x \text{Tr} Y} \frac{\det [\text{Ai}^{(j-1)}(y_k^2 + x)]_{k,j \leq n} \prod_{j=1}^n (y_j)^{\frac{1}{2}}}{\prod_{j < k} (y_j - y_k)}, \quad \text{Re } y_j > 0. \tag{1.2}$$

(See Appendix B for a simple proof). A closely related model is the *external source* matrix model, with a probability measure of the form

$$d\mu(M) \propto e^{\text{Tr} (V(M) + \Lambda M)} dM, \tag{1.3}$$

where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, where, in this context, the function $V(x)$ is a real-valued scalar function. If one considers it as a random matrix model for the eigenvalues of M then the usual approach of orthogonal polynomials [10] needs to be generalized to multi-orthogonal polynomials. Then the familiar 2×2 Riemann–Hilbert Problem for the orthogonal polynomials trades places with a different Riemann Hilbert Problem of size $r \times r$, where r is the number of distinct eigenvalues of the matrix Y and the orthogonality is replaced by multiple orthogonality [2]. In general (except for special cases [8]), the case with n distinct eigenvalues leads naturally to a Riemann–Hilbert Problem of size $n + 1$. Our goals are however different: we are interested in the integral (1.1) itself and to study rigorously its limit as $n \rightarrow \infty$ and its convergence to particular tau functions of the first Painlevé hierarchy.

The Eq. (1.2) is the key step to prove that the Kontsevich integral is a tau function (in the formal sense of Sato [22]) for the KdV hierarchy, where the eigenvalues y_k plays the role of Miwa variables (see Eq. (1.32)). The first goal of this paper is to identify the Kontsevich integral with another type of tau function, of the type introduced by Jimbo, Miwa and Ueno [16, 17] in the study of isomonodromic deformations of linear ODEs; the so called isomonodromic tau function.

¹ We normalize the variables of integration differently from [18]. See Remark 2.2 for the precise comparison.

Our approach is conceptually equivalent to the following: consider the “bare system”

$$\begin{aligned} \frac{d}{d\lambda} \Psi_0(\lambda; x) &= \begin{bmatrix} 0 & -i \\ i(\lambda + x) & 0 \end{bmatrix} \Psi_0(\lambda; x) \\ \frac{d}{dx} \Psi_0(\lambda; x) &= \begin{bmatrix} 0 & -i \\ i(\lambda + x) & 0 \end{bmatrix} \Psi_0(\lambda; x). \end{aligned} \tag{1.4}$$

A fundamental matrix joint solution of (1.4) (up to right multiplication by an invertible matrix) can be written explicitly in terms of Airy functions (Sect. 3.1). We then proceed with a “dressing”, namely, a sequence of n discrete Schlesinger transformations (in the sense of [16]) in which the monodromy data (Stokes’ matrices) are preserved but we allow Ψ_n to have n poles at the points $\{\lambda_1, \dots, \lambda_n\} := \vec{\lambda}$ with $\lambda_k = y_k^2$, $k = 1, \dots, n$. The result of this operation is a system of partial differential equations for the unknown matrix valued function Ψ_n of the form

$$\frac{\partial}{\partial \lambda} \Psi_n(\lambda; x, \vec{\lambda}) = A(\lambda; x, \vec{\lambda}) \Psi_n(\lambda; x, \vec{\lambda}) \tag{1.5}$$

$$\frac{\partial}{\partial x} \Psi_n(\lambda; x, \vec{\lambda}) = U(\lambda; x, \vec{\lambda}) \Psi_n(\lambda; x, \vec{\lambda}) \tag{1.6}$$

$$\frac{\partial}{\partial \lambda_k} \Psi_n(\lambda; x, \vec{\lambda}) = -\frac{A_k(x, \vec{\lambda})}{\lambda - \lambda_k} \Psi_n(\lambda; x, \vec{\lambda}) \tag{1.7}$$

where the matrices A, U have the form

$$A(\lambda; x, \vec{\lambda}) = i\sigma_+ - i \left(\lambda + \frac{x}{2} - \frac{da^{(n)}(x; \vec{\lambda})}{dx} \right) \sigma_- + \sum_{j=1}^n \frac{A_j(x; \vec{\lambda})}{\lambda - \lambda_j}, \tag{1.8}$$

$$U(\lambda; x, \vec{\lambda}) = i\sigma_+ - i \left(\lambda - 2 \frac{da^{(n)}(x; \vec{\lambda})}{dx} \right) \sigma_-. \tag{1.9}$$

The isomonodromic approach of [16, 17] proceeds as follows; one imposes the *compatibility* of the Eqs. (1.5) (1.6) (1.7), namely, that there exists a *simultaneous* solution $\Psi_n(\lambda; x, \vec{\lambda})$ of them. This requirement implies differential equations that determine the dependence on $x, \lambda_1, \dots, \lambda_n$ of the matrices A, U, A_k appearing in the equations. The ensuing equations are usually referred to as “zero curvature equations” and take the following form

$$\begin{aligned} \partial_x A - \partial_\lambda U + [A, U] &\equiv 0, & \frac{\partial_{\lambda_k} A_j}{\lambda - \lambda_j} - \frac{\partial_{\lambda_j} A_k}{\lambda - \lambda_j} + \left[\frac{A_j}{\lambda - \lambda_j}, \frac{A_k}{\lambda - \lambda_k} \right] &\equiv 0 \\ \partial_\lambda \frac{A_k}{\lambda - \lambda_k} - \partial_{\lambda_k} A + \left[\frac{A_k}{\lambda - \lambda_k}, A \right] &\equiv 0, \\ \frac{\partial_x A_k}{\lambda - \lambda_k} - \partial_{\lambda_k} U + \left[\frac{A_k}{\lambda - \lambda_k}, U \right] &\equiv 0. \end{aligned} \tag{1.10}$$

Vice versa, for any collection of matrices A, U, A_k satisfying (1.10) there exists a joint solution Ψ_n of Eqs. (1.5, 1.6, 1.7). Since the dependence on λ of A is rational, the fundamental solution Ψ_n of (1.5) (normalized in some way that is not essential to specify now) is not necessarily single-valued: the analytic continuation of Ψ_n along a non-contractible contour γ in $\mathbb{C} \setminus \{\lambda_1, \dots, \lambda_n\}$ yields a new matrix that solves the same ODE and hence

it is related as $\Psi_n \mapsto \Psi_n M_\gamma$. The matrix M_γ depends only on the homotopy class and is called “monodromy matrix” associated to γ : the collection of these matrices, for all homotopy classes, provides an (anti)-representation of the fundamental group of $\mathbb{C} \setminus \{\lambda_1, \dots, \lambda_n\}$. In addition to these matrices one needs to compute the matrix Stokes’ multipliers (we refer to the introduction of [17] for a recall of this notion for the interested reader) and the collection of monodromy matrices and Stokes’ multiplier is the “generalized” monodromy data: it is important to recall that these monodromy data are independent of $x, \lambda_1, \dots, \lambda_n$ precisely as a consequence of (1.10), so that they should be regarded as integrals of the motions.

In [17] the notion of isomonodromic tau function was then defined as follows; for any solution of (1.10) (and associated Ψ -function) we can define the “isomonodromic tau function” $\tau_n(x; \vec{\lambda})$ by means of

$$\partial_{\lambda_k} \ln \tau_n(x; \vec{\lambda}) = \operatorname{res}_{\lambda=\lambda_k} \operatorname{Tr} A^2 d\lambda; \quad \partial_x \ln \tau_n(x; \vec{\lambda}) = a^{(n)}(x; \vec{\lambda}). \quad (1.11)$$

The results of [17] showed (in a much more general setting) that the Eq. (1.11) form a compatible set of equations provided that the Eq. (1.10) hold, and hence they can be integrated to define $\tau_n(x; \vec{\lambda})$ (which is, however, defined only up to multiplication by a scalar independent of $x, \vec{\lambda}$). The τ function depends parametrically on the generalized monodromy data (Stokes matrices and monodromy matrices) which replace the initial value conditions: the case that shall be of interest for us is when the monodromy representation is *trivial* and there is only the Stokes’ phenomenon at $\lambda = \infty$.

1.1. Results. At this point we can advertise the gist of our first result in the form of the following Theorem.

Theorem 1.1. *Let $\tau_n(x; \vec{\lambda})$ be the isomonodromic tau function for the isomonodromic system (1.5, 1.6, 1.7). Then the Kontsevich integral (1.1) is equal to*

$$Z_n(x; Y) = e^{\frac{x^3}{12}} \tau_n(x; \vec{\lambda}). \quad (1.12)$$

after the identification $y_k = \sqrt{\lambda_k}$, $k = 1, \dots, n$.

The formulation of the result in terms of isomonodromic deformation may be more widely recognizable by the readership, but it is not the way we want to set up its proof; the keen reader may also observe that the isomonodromic problem that we have indicated is still largely ambiguous because we did not, for example, specify the precise generalized monodromy data. Moreover, the isomonodromic formulation makes it hard to analyze the situation when the size of the matrix integral in (1.1) (the number of poles in (1.5)) tends to infinity, which is our second main motivation to be discussed later on.

To remove all these ambiguities we will now reformulate the isomonodromic system (1.5, 1.6, 1.7) directly in terms of a suitable Riemann–Hilbert Problem, thus displaying explicitly its monodromy data. This reformulation allows us to handle rigorously a limit as $n \rightarrow \infty$. For technical reasons that should become clear later on, we shall formulate a slightly more general situation where the set λ of n points is partitioned in two $(\vec{\lambda}, \vec{\mu}) = (\lambda_1, \dots, \lambda_{n_1}, \mu_1, \dots, \mu_{n_2})$ ($n = n_1 + n_2$). Associated to this data we define the function

$$\mathbf{d}_n(\lambda) := \prod_{j=1}^{n_1} \frac{\sqrt{\mu_j} + \sqrt{\lambda}}{\sqrt{\mu_j} - \sqrt{\lambda}} \prod_{j=1}^{n_2} \frac{\sqrt{\lambda_j} - \sqrt{\lambda}}{\sqrt{\lambda_j} + \sqrt{\lambda}}. \quad (1.13)$$

Riemann-Hilbert Problem 1.2. Let Σ be the union of oriented rays shown in Fig. 3. Find a 2×2 matrix valued analytic function $\Gamma_n = \Gamma_n(\lambda; \vec{\lambda}, \vec{\mu})$ such that:

- Γ_n is locally bounded everywhere in \mathbb{C} , and analytic in $\mathbb{C} \setminus \Sigma$.
- It admits continuous boundary values $\Gamma_{n,\pm}$ on each ray and they satisfy the *jump conditions*

$$\Gamma_n(\lambda)_+ = \Gamma_n(\lambda)_- M_n(\lambda), \quad \lambda \in \Sigma, \tag{1.14}$$

where the matrix M_n is piecewise defined by

$$M_n(\lambda) = \begin{cases} \mathbf{1} + \mathbf{d}_n(\lambda) e^{-\frac{4}{3}\lambda^{\frac{3}{2}} - 2x\lambda^{\frac{1}{2}}} \sigma_+ & \lambda \in \varpi_0 := e^{i\theta_0} \mathbb{R}_+ \\ \mathbf{1} + \frac{1}{\mathbf{d}_n(\lambda)} e^{\frac{4}{3}\lambda^{\frac{3}{2}} + 2x\lambda^{\frac{1}{2}}} \sigma_- & \lambda \in \varpi_{\pm} := e^{i\theta_{\pm}} \mathbb{R}_+ \\ i\sigma_2 & \lambda \in \mathbb{R}_-. \end{cases} \tag{1.15}$$

- Near $\lambda = \infty$, in each sector, it satisfies the following asymptotic expansion

$$\Gamma_n(\lambda) = \lambda^{-\frac{\sigma_3}{4}} \frac{\mathbf{1} + i\sigma_1}{\sqrt{2}} \left(\mathbf{1} + \frac{a^{(n)}(x; \vec{\lambda}, \vec{\mu})}{\sqrt{\lambda}} \sigma_3 + \mathcal{O}(\lambda^{-1}) \right). \tag{1.16}$$

It is implied that the rays can be slightly deformed with respect to Fig. 3 so that none of the poles of $\mathbf{d}_n(\lambda)$ lie on ϖ_0 and none of the poles of $\mathbf{d}_n^{-1}(\lambda)$ lie on ϖ_{\pm} . As a matter of fact the problem can be posed on arbitrary (non-intersecting) contours issuing from the origin and extending to infinity as long as the asymptotic directions at infinity are the ones indicated.

Remark 1.3 (Gauge arbitrariness). The asymptotic condition (1.16) implies a gauge fixing; indeed we could multiply Γ_n on the left by a constant matrix of the form $\mathbf{1} + c\sigma_-$, and this would not change the jump conditions. However that coefficient matrix of $\lambda^{-\frac{1}{2}}$ in the expansion (1.16) would be changed by the addition of a term proportional to σ_1 . In other words the requirement that the $\mathcal{O}(\lambda^{-\frac{1}{2}})$ term is proportional to σ_3 is part of the normalization condition at infinity (otherwise there would be a one-parameter family of solutions). It is not hard to prove that the solution, if it exists, is unique under this normalization.

We now explain how the Riemann-Hilbert Problem 1.2 provides the precise (generalized) monodromy data for the isomonodromic approach. The matrix

$$\begin{aligned} \Psi_n &= \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) := \Gamma_n(\lambda) e^{-\vartheta(\lambda;x)\sigma_3} D^{-1}(\lambda), \\ \vartheta(\lambda; x) &:= \left(\frac{2}{3} \lambda^{\frac{3}{2}} + x\sqrt{\lambda} \right) \end{aligned} \tag{1.17}$$

$$D(\lambda) = D(\lambda; \vec{\lambda}, \vec{\mu}) := \begin{bmatrix} \prod_{j=1}^{n_2} (\sqrt{\lambda_j} + \sqrt{\lambda}) \prod_{j=1}^{n_1} (\sqrt{\mu_j} - \sqrt{\lambda}) & 0 \\ 0 & \prod_{j=1}^{n_2} (\sqrt{\lambda_j} - \sqrt{\lambda}) \prod_{j=1}^{n_1} (\sqrt{\mu_j} + \sqrt{\lambda}) \end{bmatrix} \tag{1.18}$$

satisfies a jump condition on Σ with matrices independent of $\lambda, x, \vec{\lambda}, \vec{\mu}$. It then follows by standard arguments that it satisfies an overdetermined system of PDEs generalizing Eqs. (1.5), (1.6), (1.7), namely

$$\frac{\partial}{\partial \lambda} \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) = A(\lambda; x, \vec{\lambda}, \vec{\mu}) \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) \tag{1.19}$$

$$\frac{\partial}{\partial x} \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) = U(\lambda; x, \vec{\lambda}, \vec{\mu}) \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) \tag{1.20}$$

$$\frac{\partial}{\partial \lambda_k} \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) = -\frac{A_k(x, \vec{\lambda}, \vec{\mu})}{\lambda - \lambda_k} \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}), \quad k = 1, \dots, n_1 \tag{1.21}$$

$$\frac{\partial}{\partial \mu_k} \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) = -\frac{B_k(x, \vec{\lambda}, \vec{\mu})}{\lambda - \mu_k} \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}), \quad k = 1, \dots, n_2, \tag{1.22}$$

where the matrices A, U now have the form

$$A(\lambda; x, \vec{\lambda}, \vec{\mu}) = i\sigma_+ - i \left(\lambda + \frac{x}{2} - \frac{da^{(n)}(x; \vec{\lambda}, \vec{\mu})}{dx} \right) \sigma_- + \sum_{j=1}^{n_1} \frac{A_j(x; \vec{\lambda}, \vec{\mu})}{\lambda - \lambda_j} + \sum_{j=1}^{n_2} \frac{B_j(x; \vec{\lambda}, \vec{\mu})}{\lambda - \mu_j}, \tag{1.23}$$

$$U(\lambda; x, \vec{\lambda}, \vec{\mu}) = i\sigma_+ - i \left(\lambda - 2 \frac{da^{(n)}(x; \vec{\lambda}, \vec{\mu})}{dx} \right) \sigma_-. \tag{1.24}$$

and the function $a^{(n)}(x; \vec{\lambda}, \vec{\mu})$ is defined (implicitly) above by the Eq. (1.16).

These equations, together, represent a system of “monodromy preserving” deformation of the rational ODE (1.19), in the sense of [17].

Remark 1.4. While a general rational connection $\partial_\lambda - A$ with A as in (1.19) has nontrivial monodromy around the Fuchsian singularities of (1.5), our particular case corresponds to a situation where the monodromy is trivial; more specifically, the residue matrices $A_j(x; \vec{\lambda}, \vec{\mu})$ and $B_k(x; \vec{\lambda}, \vec{\mu})$ have all eigenvalues 0 and ± 1 , since they were produced adding zeros and poles of order one in the first or the second column of the jump matrix M_n in (1.15). Thus the Fuchsian ODE is “resonant” [23]. The no-monodromy condition is a special constraint that determines the particular solution relevant to our problem.

In this extended case the Jimbo–Miwa–Ueno definition of tau function translates to the following set of first order differential equations

$$\begin{aligned} \partial_{\lambda_k} \ln \tau_n(x; \vec{\lambda}, \vec{\mu}) &= \operatorname{res}_{\lambda=\lambda_k} \operatorname{Tr} A^2 d\lambda & \partial_{\mu_k} \ln \tau_n(x; \vec{\lambda}, \vec{\mu}) &= \operatorname{res}_{\lambda=\mu_k} \operatorname{Tr} A^2 d\lambda \\ \partial_x \ln \tau_n(x; \vec{\lambda}, \vec{\mu}) &= \operatorname{res}_{\lambda=\infty} \operatorname{Tr} (\Psi_n^{-1} \partial_\lambda \Psi_n \sqrt{\lambda} \sigma_3) = a^{(n)}(x; \vec{\lambda}, \vec{\mu}) \end{aligned} \tag{1.25}$$

generalizing straightforwardly the formulæ (1.11). Equation (1.25), as it is customary in the Jimbo–Miwa–Ueno setting, determine the τ function up to a multiplicative factor that may depend on the monodromy data of the problem. To address this ambiguity, the definition was generalized in [3] (with a correction in [5]) to one that applies also to general Riemann–Hilbert Problems:

$$\partial \ln \tau_{JMU} = \int_\Sigma \operatorname{Tr} \left(\Gamma_n^{-1} \Gamma'_n \partial M_n M_n^{-1} \right) \frac{d\lambda}{2i\pi}, \tag{1.26}$$

where $\Sigma = \mathbb{R}_- \cup \varpi_0 \cup \varpi_+ \cup \varpi_-$. Since the jump on \mathbb{R}_- in the Riemann–Hilbert Problem is independent of parameters, the integration in (1.26) extends only on the three rays $\varpi_{0,\pm}$.

In this case the two definitions are completely equivalent but we will continue using the second one. Note that, however, the function τ is only defined up to multiplicative constants. In the cases where explicit integration of the above equation is possible, the integration constant will be tacitly set to zero, without further comment.

Extension of the Kontsevich matrix integral to arbitrary sectors. The right side of (1.2), can be extended to an analytic function in the left planes of the variables because, up to the factor $\prod (y_j)^{\frac{1}{2}}$, the Airy functions are entire functions and the ratio in (1.2) is well defined on the “diagonal” sets $\{y_j = y_k, j, k = 1, \dots, n\}$; however we now contend that we need to define it differently. To explain the rationale we remind the reader that the interpretation of $Z_n(x; Y)$ as a generating function requires that it admits a *regular*² asymptotic expansion as $y_j \rightarrow \infty$. Using the well-known asymptotic expansion of the Airy function (Ai) in the sector $|\arg \lambda| < \pi$ we see that

$$\begin{aligned} \text{Ai}(\lambda) &= \frac{e^{-\frac{2}{3}\lambda^{\frac{3}{2}}}}{2\sqrt{\pi}\lambda^{\frac{1}{4}}}(1 + \mathcal{O}(\lambda^{-\frac{3}{2}})) \Rightarrow e^{\frac{2}{3}y^3+xy} \text{Ai}(y^2+x) \\ &= \begin{cases} \frac{e^{\frac{4}{3}y^3+2xy}}{2\sqrt{\pi}\sqrt{y}}(\mathbf{1} + \mathcal{O}(y^{-3})) & \arg y \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ \frac{1}{2\sqrt{\pi}\sqrt{y}}(\mathbf{1} + \mathcal{O}(y^{-3})) & \arg y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases} \end{aligned}$$




where we have used that $(y^2)^{\frac{3}{2}} = -y^3$ if $\text{Re } y \leq 0$ (and we use the principal roots). Therefore (1.2), as written, cannot possibly admit a regular asymptotic expansion if $y_j \rightarrow \infty$ in the sector $\text{Re } y_j \leq 0$ for some j .

The reader familiar with the Stokes’ phenomenon of the Airy function will see that the way to recover a regular expansion in the left half plane is to use either $\text{Ai}(\omega^{\pm 1}y^2)$ instead. To this end we introduce the following notation

$$\mathbf{Ai}_\nu(\lambda) := \text{Ai}(\omega^\nu \lambda), \quad \omega := e^{\frac{2i\pi}{3}}, \quad \nu = 0, 1, 2. \tag{1.27}$$

The functions \mathbf{Ai}_ν are solutions of the Airy equation and satisfy $\mathbf{Ai}_0 + \omega \mathbf{Ai}_1 + \omega^2 \mathbf{Ai}_2 \equiv 0$ and the functions $\sqrt{y}e^{\frac{2}{3}y^3} \mathbf{Ai}_\nu(y^2)$ admit a regular expansion in inverse integer powers (without exponential terms) as $|y| \rightarrow \infty$ within the following sectors:

$$\mathcal{S}_0 = \left\{ \arg(y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\}; \quad \mathcal{S}_1 = \left\{ \arg(y) \in \left(\frac{\pi}{6}, \frac{7\pi}{6}\right) \right\}; \quad \mathcal{S}_2 = \left\{ \arg(y) \in \left(\frac{5\pi}{6}, \frac{11\pi}{6}\right) \right\} \tag{1.28}$$

Definition 1.5. For any partition of the set \mathcal{Y} of the eigenvalues of Y into three disjoint sets $\mathcal{Y}^{(s)}$, $s = 0, 1, 2$ of respective cardinality n_0, n_1, n_2 ($n = n_0 + n_1 + n_2$), we consider the following determinant which we call *generalized Kontsevich integral*

² Here “regular” means that it is a (formal) series in inverse powers of the $y'_j s$, without exponential factors.

$$\begin{aligned}
 & Z_n(x; \mathcal{Y}^{(0)}, \mathcal{Y}^{(1)}, \mathcal{Y}^{(2)}) \\
 &= (-\omega)^{n_1-n_2} (2\sqrt{\pi})^n \frac{e^{\frac{2}{3}\text{Tr} Y^3 + x\text{Tr} Y} \prod_{j=1}^n (y_j)^{\frac{1}{2}}}{\prod_{j < k} (y_j - y_k)} \\
 & \det \begin{bmatrix} \left[\mathbf{Ai}_0^{(k-1)}(y_j^2 + x) \right]_{\substack{y_j \in \mathcal{Y}^{(0)} \\ 1 \leq k \leq n}} \\ \left[\mathbf{Ai}_1^{(k-1)}(y_j^2 + x) \right]_{\substack{y_j \in \mathcal{Y}^{(1)} \\ 1 \leq k \leq n}} \\ \left[\mathbf{Ai}_2^{(k-1)}(y_j^2 + x) \right]_{\substack{y_j \in \mathcal{Y}^{(2)} \\ 1 \leq k \leq n}} \end{bmatrix}. \tag{1.29}
 \end{aligned}$$

The generalized Kontsevich integrals (1.29) reduce to (1.2) if $\mathcal{Y}^{(0)} = \mathcal{Y}$, $\mathcal{Y}^{(1)} = \emptyset = \mathcal{Y}^{(2)}$ and hence Theorem 1.1 is a special case of the theorem below.

Theorem 1.6. [1] *The function $Z_n(x; \mathcal{Y}^{(0)}, \mathcal{Y}^{(1)}, \mathcal{Y}^{(2)})$ (1.29) and the isomonodromic tau function τ_n defined by (1.25) and associated to the Riemann–Hilbert Problem 1.2 are related by*

$$Z_n(x; \mathcal{Y}^{(0)}, \mathcal{Y}^{(1)}, \mathcal{Y}^{(2)}) = e^{\frac{x^3}{12}} \tau_n(x; \mathcal{Y}^{(0)}, \mathcal{Y}^{(1)}, \mathcal{Y}^{(2)}), \tag{1.30}$$

with the identification $y_i = \sqrt{\lambda_i}$ if $\text{Re}(y_i) > 0$ and $y_j = -\sqrt{\mu_j}$ if $\text{Re}(y_j) \leq 0$, all roots principal.

[2] *The expression (1.29) admits a regular asymptotic expansion if the variables y_j 's tend to infinity provided that $\mathcal{Y}^{(v)} \subset \mathcal{S}_v$ with the sectors \mathcal{S}_v defined in (1.28)*

[3] *This asymptotic expansion is independent of the assignment of the variables to the different groups $\mathcal{Y}^{(v)}$ or $\mathcal{Y}^{(v)}$ if they belong to the overlap of the sectors $\mathcal{S}_v \cap \mathcal{S}_{\bar{v}}$.*

The points [2], [3] of Theorem 1.6 follow simply from the fact that $e^{\frac{2}{3}y^3 + xy} \mathbf{Ai}_{v, \tilde{v}}(y^2 + x)$ have the same regular asymptotic expansion if $|y| \rightarrow \infty$ and $y \in \mathcal{S}_v \cap \mathcal{S}_{\tilde{v}}$. Indeed, the analysis of the asymptotic behaviour for $y \rightarrow \infty$ for the Airy function shows that $e^{\frac{2}{3}y^3} \mathbf{Ai}_v(y^2)$ admits the same regular (nontrivial) expansion in integer inverse powers of y 's if and only if y tends to infinity in the corresponding sectors \mathcal{S}_v (see [1], 10.4.59).

In particular, if all y_j 's tend to infinity in the right half-plane and we assign them all to \mathcal{S}_0 , then we get (1.1) and hence this expansion is the formal expansion that generates the intersection numbers of tautological classes as explained in [18].

Using the alternative but equivalent formula (1.26) we can restate the Theorem 1.6 in the form

Theorem 1.7. *The Kontsevich integral $Z_n(x; \mathcal{Y}^{(0)}, \mathcal{Y}^{(1)}, \mathcal{Y}^{(2)})$ in (1.29) satisfies*

$$\begin{aligned}
 \partial \ln Z_n(x; \vec{\lambda}, \vec{\mu}) &= \partial \frac{x^3}{12} + 2 \int_{\varpi_0} \left(\Gamma_n^{-1}(\lambda) \Gamma'_n(\lambda) \right)_{21} \partial \mathbf{d}_n(\lambda) e^{-\frac{4}{3}\lambda^{\frac{3}{2}} - 2x\lambda^{\frac{1}{2}}} \frac{d\lambda}{2i\pi} + \\
 &+ 2 \sum_{\pm} \int_{\varpi_{\pm}} \left(\Gamma_n^{-1}(\lambda) \Gamma'_n(\lambda) \right)_{12} \partial \mathbf{d}_n^{-1}(\lambda) e^{\frac{4}{3}\lambda^{\frac{3}{2}} + 2x\lambda^{\frac{1}{2}}} \frac{d\lambda}{2i\pi} \tag{1.31}
 \end{aligned}$$

where ∂ is the derivative with respect to any parameter $x, \vec{\lambda}, \vec{\mu}$. The relationship between the parameters $\{y_j\}$ and $\{\vec{\lambda}, \vec{\mu}\}$ is $y_i = \sqrt{\lambda_i}$ if $\text{Re}(y_i) > 0$ and $y_j = -\sqrt{\mu_j}$ if $\text{Re}(y_j) \leq 0$.

For the proof see Sec. 3.3.1.

The limit $n \rightarrow \infty$: first Painlevé hierarchy. It was one of the main points of Kontsevich’s original work [18] that the integral (1.1) is formally a KdV tau function in the Miwa variables³

$$T_k(Y) := -\frac{2^{-\frac{2k+1}{3}}}{(2k+1)!!} \text{Tr } Y^{-2k-1}. \tag{1.32}$$

More precisely, in these variables, the function $U(x; T) := \frac{\partial^2}{\partial T_0^2} \log Z_n(x; Y)$ satisfies the KdV hierarchy with the normalization adopted in [24]. This particular solution of the KdV hierarchy was known by physicists even before the formulation of Witten’s conjecture and its proof by Kontsevich [18]. In the physics literature, this is referred to as the *partition function of 2D topological gravity* (see the references in [12]). It can be defined as the solution satisfying the initial value condition $U(x, 0) = x$. As originally discovered by Douglas [13] using the so-called *string equation* (see Sect. 2 below, keeping in mind that there the normalization is different from Witten’s, see Remark 2.2), the Witten–Kontsevich solution of the KdV hierarchy satisfies an infinite number of ODE’s in T_0 (or x , which is the same) known as the Painlevé I hierarchy, and where the higher T_i ’s play the role of parameters. Indeed this is very close to the procedure of Flaschka and Newell [14] who deduce the Painlevé II hierarchy as a self-similar reduction of the modified KdV one. To see briefly how it works, recall that, in the Witten’s normalization, the equations of the KdV hierarchy are written as

$$\frac{\partial U}{\partial T_i} = \frac{\partial R_{i+1}}{\partial T_0}, \quad i \geq 0, \tag{1.33}$$

where the R_i are differential polynomials in $U(T_0)$ defined by the recursion

$$\frac{\partial R_{k+1}}{\partial T_0} = \frac{1}{2k+1} \left(\frac{\partial U}{\partial T_0} + 2U \frac{\partial}{\partial T_0} + \frac{1}{4} \frac{\partial^3}{\partial T_0^3} \right) R_k; \quad R_1(U) = U. \tag{1.34}$$

Besides these equations, the function $F(x; T) := \log Z_n(x; Y)$ satisfies also the first Virasoro constraint (which can be deduced as a consequence of the fact that $Z_n(x; Y)$ is a matrix integral)

$$\frac{\partial F}{\partial T_0} = \frac{T_0^2}{2} + \sum_{i=0}^{\infty} T_{i+1} \frac{\partial F}{\partial T_i}. \tag{1.35}$$

Differentiating once (1.35) with respect to T_0 and substituting the integrated version of (1.33) (integration constants are seen to be equal to zero) one obtains the set of ODE’s in T_0 , depending on the parameters $\{T_i\}$,

$$T_0 + (T_1 - 1)U + \sum_{i \geq 1} T_{i+1} R_{i+1} = 0. \tag{1.36}$$

³ The factor $2^{-\frac{2k+1}{3}}$ stems from our normalization, see Remark 2.2.

More precisely the N -th member of the Painlevé I hierarchy is obtained by putting $T_j = 0$ when $j \geq N + 1$ and it is an ODE in $x = T_0$ depending parametrically on T_1, \dots, T_N (more details are recalled in Sect. 2). Making sense of the formal statement “the Kontsevich matrix model $Z_n(x; Y)$ satisfies the Painlevé I hierarchy”, requires a limit $n \rightarrow \infty$ because, for fixed n , the variables T_j are not even independent. This leaves open the question as to what kind of convergence we should expect. Also, $Z_n(x; Y)$ is usually treated as a formal series, while it would be interesting to analyze the analytic properties of these solutions of the hierarchy. Thus the issue becomes:

Question 1.8. *How to choose a sequence $\mathcal{Y} = \mathcal{Y}_n$ of variables $\mathcal{Y}_n = \{y_1^{(n)}, \dots, y_n^{(n)}\}$ and an appropriate partition into three subsets in such a way that generalized Kontsevich integral (1.29) converges to a tau function of the N -th member of the first Painlevé hierarchy? Which particular solution does it converge to and in which domain of the parameters?*

To address the question, let $r \in \mathbb{N}$ and P_r be the r -th Padé approximant to e^{-z} (see (3.34)), which is a polynomial of degree r . Denote by a_1, \dots, a_r its zeroes. It is known [21] that they are all in the region $\text{Re } z > 0$ (see Fig. 7). Fix $N \in \mathbb{N}$ and set

$$\mathcal{Y} = \{y_1, \dots, y_n\} = \left\{ y : P_r(2ty^{2N+1}) = 0 \right\}, \quad n = r(2N + 1). \quad (1.37)$$

The set \mathcal{Y} is naturally partitioned into subsets of cardinality r as follows (see Figs. 1 and 2)

$$\mathcal{Y}_\kappa := \left\{ e^{\frac{i\pi\kappa}{2N+1}} \left(\frac{a_j}{2t} \right)^{\frac{1}{2N+1}}, \quad 1 \leq j \leq r \right\}; \quad \arg \mathcal{Y}_\kappa \subset \left(-\frac{\pi}{4N+2}, \frac{\pi}{4N+2} \right) - \frac{\arg t}{2N+1} \quad (1.38)$$

$$\kappa = -N, \dots, N - 1, N, \quad (1.39)$$

where $\arg(\mathcal{Y}_\kappa)$, denotes the set of the arguments of the elements in \mathcal{Y}_κ .

We then address Question 1.8 for the subsequence $n = r(2N + 1)$ and let $r \rightarrow \infty$ while keeping N fixed. For a brief review of the first Painlevé hierarchy, its Riemann–Hilbert formulation and associated tau function we refer the reader to Sect. 2.

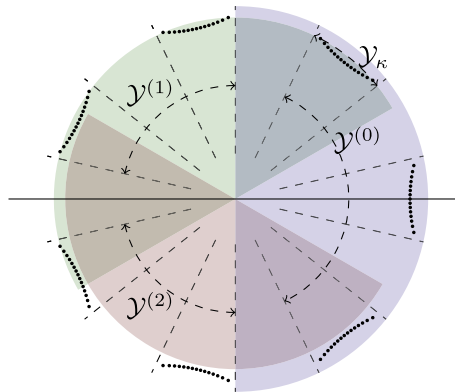


Fig. 1. Example with $N = 3$: the assignment of the \mathcal{Y}_κ 's to the disjoint subsets $\mathcal{Y}^{(0,1,2)}$ indicated in the figure corresponds to $k_+ = 1, k_0 = 0, k_- = -1$

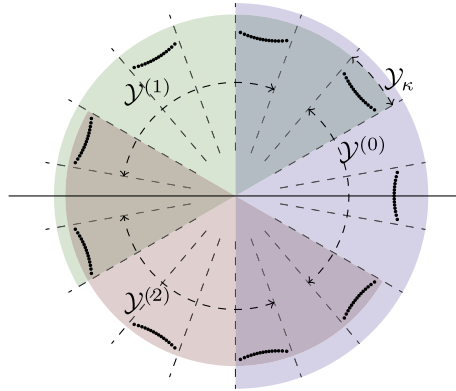


Fig. 2. Example with $N = 4$; the assignment of the \mathcal{Y}_κ 's to the disjoint subsets $\mathcal{Y}^{(0,1,2)}$ indicated in the figure corresponds to $k_+ = 1, k_0 = 0, k_- = -1$

Theorem 1.9. Fix $N \in \mathbb{N}$. Partition the set of indices κ (with $-N \leq \kappa \leq N$ modulo $2N + 1$) into three consecutive disjoint groups such that the corresponding subsets \mathcal{Y}_κ 's (1.38) belong to the sectors $\mathcal{S}_{0,1,2}$ (1.28); let $\mathcal{Y}^{(0,1,2)}$ be their respective unions, correspondingly. Define $k_+ \geq k_0 \geq k_-$, $k_+ > k_-$ as follows

- $k_0 = (\text{number of } \mathcal{Y}_\kappa \text{'s in the second quadrant that we assign to } \mathcal{Y}^{(2)}) - (\text{number of } \mathcal{Y}_\kappa \text{'s in the third quadrant that we assign to } \mathcal{Y}^{(1)})$;
- $k_- = -\lfloor \frac{N}{2} \rfloor + (\text{number of } \mathcal{Y}_\kappa \text{'s in the first quadrant that we assign to } \mathcal{Y}^{(1)})$;
- $k_+ = \lfloor \frac{N}{2} \rfloor - (\text{number of } \mathcal{Y}_\kappa \text{'s in the fourth quadrant that we assign to } \mathcal{Y}^{(2)})$;

Then

1. the formula (1.29) converges, with rate of convergence $\mathcal{O}(n^{-\infty})$ as $n \rightarrow \infty$, to the taufunction $\tau(x; t)$ of the special tronquée solution of the N -th member of the PI hierarchy defined by the formula (2.6) in terms of the solution of the RHP 3.5 with the chosen (k_+, k_0, k_-) . Then the function $u(x, t) := 2\partial_x^2 \ln \tau(x; t)$ satisfies the nonlinear ODE

$$(2N + 1)t\mathcal{L}_N[u(x; t)] + u(x; t) + x = 0. \tag{1.40}$$

2. All these particular solutions $u(x; t)$ have no poles for $|t|$ sufficiently small within an open sector of width at least π that contains $\arg(t) = 0$. Within the common sector where they have no poles they differ, as $|t| \rightarrow 0$, by $\mathcal{O}(|t|^\infty)$ terms. If $k_0 = 0$, then the width of this sector is at least π on either sides of $\arg t = 0$.
3. The limit at $t = 0$, from within this common sector, of the derivatives of arbitrary order equal those of the formal topological solution.

For the proof see Sec. 3.4. Here $\mathcal{L}_N[u]$ is the Lenard differential polynomial in $u(x)$ whose definition will be reviewed below in (2.9).

The idea behind the binning of the groups \mathcal{Y}_κ into the three disjoint subsets $\mathcal{Y}^{(0,1,2)}$ of Theorems 1.6 and 1.9 is as follows; referring to the Figs. 1, 2 we see that some groups

can only be assigned to one $\mathcal{Y}^{(\nu)}$ because they belong to only one \mathcal{S}_ν , while others fall in the intersection between two different sectors $\mathcal{S}_{0,1,2}$ and can be assigned to either. The different choices are reflected in the choices of the parameters k_+, k_0, k_- that characterize the particular tronquée solution in the Riemann–Hilbert Problem 3.5. The Theorem 1.9 is therefore a first foray in the study of the nonlinear Stokes’ phenomenon for the Witten–Kontsevich tau function.

Remark 1.10. From the recurrence relation (2.9) we can see that the Lenard polynomials $\mathcal{L}_N[u]$ are homogeneous of degree $2N$ under the rescaling $U(X) = \alpha^2 u(\alpha^{-1} X)$. Setting $U(X, t) := t^{\frac{2}{2N+1}} u(t^{-\frac{1}{2N+1}} X, t)$, we obtain the equation

$$(2N + 1)\mathcal{L}_N[U(X; t)] + t^{-\frac{3}{2N+1}} U(X; t) + X = 0 \tag{1.41}$$

and this shows that $u(x, t)$ is single-valued on the Riemann surface of $t^{\frac{1}{2N+1}}$. This explains how it is possible to have no poles in a sector of amplitude 2π or even bigger.

Example 1.11. For $N = 2, 3$ the Eq. (1.40) reads

$$N = 2; \frac{5}{8}t \left(u'' + 3u^2 \right) + u + x = 0 \tag{1.42}$$

$$N = 3; \frac{7}{32}t \left(u^{(4)} + 10uu'' + 5(u')^2 + 10u^3 \right) + u + x = 0. \tag{1.43}$$

The case $N = 2$ above is, up to the map $u(x) = \left(\frac{8}{5t}\right)^{\frac{2}{5}} U(X) - \frac{4}{15t}$, $x = -\left(\frac{t}{8}\right)^{\frac{1}{5}} X - \frac{2}{15t}$ the standard first Painlevé 1 equation $U'' + 3U^2 = X$; in this case the particular solution is precisely a tritronquée solution [15].

In Theorem 1.9 we restricted ourselves to the subsequence $n = r(2N+1)$ only because of the way we constructed the rational approximation of $e^{2t\lambda^{\frac{2N+1}{2}}}$; to extend the statement to the whole sequence one would have to consider the Padé approximants to $e^{2tz^{2N+1}}$ directly (of which, the polynomials $P_r(2tz^{2N+1})$ are a subsequence). More generally, the full fledged member of the Painlevé hierarchy as in (1.36) would require the analog of the estimate (3.39) for the location of the zeroes and the estimate of the remainder term of the general exponential $e^{\sum_{j=1}^N t_{2j+1} z^{2j+1}}$. We regard this issue as a technical one; we expect that the general phenomenon will be the same we observe in this restricted case.

2. The Riemann–Hilbert Problem for the First Painlevé Hierarchy and Associated τ function

In order to discuss the various solutions of the first Painlevé hierarchy, we need to review the relevant Riemann–Hilbert Problem ([7], page 365). The Riemann–Hilbert Problem of the N -th member of the first Painlevé hierarchy is constructed as follows; define the *phase function*

$$\vartheta(\lambda) := t_{2N+1} \lambda^{\frac{2N+1}{2}} + \sum_{j=0}^{N-1} t_{2j+1} \lambda^{\frac{2j+1}{2}}, \quad t_1 := x. \tag{2.1}$$

and let ϖ_ν be the rays $\arg(\lambda) - \frac{2 \arg(t)}{2N+1} = \frac{2\pi\nu}{2N+1}$, $-N \leq \nu \leq N$.

Riemann-Hilbert Problem 2.1. (*First Painlevé hierarchy*) Find a 2×2 matrix $\Gamma(\lambda)$, locally bounded everywhere in \mathbb{C} , analytic away from the rays ϖ_ν (oriented towards infinity) and \mathbb{R}_- (oriented towards the origin) and such that

- it admits non-tangential boundary values at the points of the rays and they satisfy

$$\Gamma_+(\lambda) = \Gamma_- e^{-\vartheta(\lambda)\sigma_3} S_\nu e^{\vartheta(\lambda)\sigma_3}, \quad \lambda \in \varpi_\nu, \quad \Gamma_+(\lambda) = \Gamma_-(\lambda) i \sigma_2, \quad \lambda \in \mathbb{R}_- \tag{2.2}$$

with

$$S_\nu = \begin{cases} S_{2j} = \begin{bmatrix} 1 & s_{2j} \\ 0 & 1 \end{bmatrix} & \nu = 2j \\ S_{2j+1} = \begin{bmatrix} 1 & 0 \\ s_{2j+1} & 1 \end{bmatrix} & \nu = 2j + 1. \end{cases}, \quad \nu = -N \dots, N, \tag{2.3}$$

and such that the $2N+2$ parameters s_{-N}, \dots, s_N are subject only to the *no monodromy condition*

$$S_{-N} \cdots S_0 \cdots S_N = i \sigma_2. \tag{2.4}$$

- Near $\lambda = \infty$ the solution has the same sectorial asymptotic expansion in each sector, normalized by

$$\Gamma(\lambda; \mathbf{t}) = \lambda^{-\frac{\sigma_3}{4}} \frac{\mathbf{1} + i \sigma_1}{\sqrt{2}} \left(\mathbf{1} + a(\mathbf{t}) \frac{\sigma_3}{\sqrt{\lambda}} + \mathcal{O}(\lambda^{-1}) \right). \tag{2.5}$$

The connection with the equations of the hierarchy arises as follows. The tau function of a solution corresponding to the above data is defined in [17] as

$$\partial_{t_j} \ln \tau_{P_{1N}}(\mathbf{t}) = - \operatorname{res}_{\lambda=\infty} \operatorname{Tr} \left(\lambda^{\frac{2j+1}{2}} \Gamma^{-1}(\lambda; \mathbf{t}) \Gamma'(\lambda; \mathbf{t}) \sigma_3 \right) d\lambda, \tag{2.6}$$

where the residue is to be intended as a formal one (the formal series in the residue turns out to have only integer powers of λ and the residue is the coefficient of the power λ^{-1} of the expression in the bracket). The function

$$\begin{aligned} u(x, t_3, t_5, \dots, t_{2N+1}) &:= 2\partial_x a(x, t_3, t_5, \dots, t_{2N+1}) \\ &= 2\partial_x^2 \ln \tau_{P_{1N}}(x, t_3, t_5, \dots, t_{2N+1}) \end{aligned} \tag{2.7}$$

satisfies the following ODE in $x = t_1$, depending parametrically on t_3, \dots, t_{2N+1} :

$$\sum_{k=1}^N (2k+1) t_{2k+1} \mathcal{L}_k[u] + x = 0. \tag{2.8}$$

Here $\mathcal{L}_k[u]$ are the Lenard-Magri differential polynomials defined [11] by the recursion relations:

$$\frac{\partial}{\partial x} \mathcal{L}_{n+1}[u] = \left(\frac{1}{4} \frac{\partial^3}{\partial x^3} + u(x) \frac{\partial}{\partial x} + \frac{1}{2} u_x(x) \right) \mathcal{L}_n[u], \quad \mathcal{L}_0[u] = 1, \quad \mathcal{L}_n[0] = 0 \tag{2.9}$$

The Stokes' parameters $\vec{s} = (s_{-N}, \dots, s_N)$ (subject to (2.4)) parametrize the solution space of (2.8).

In addition to the ODE (2.8) above, u satisfies also

$$\frac{\partial u}{\partial t_{2j+1}} = 2 \frac{\partial}{\partial x} \mathcal{L}_{j+1}[u], \quad j \leq N; \quad u = u(\mathbf{t}), \quad \mathbf{t} = (t_1, t_3, t_5, \dots). \tag{2.10}$$

Remark 2.2. The Eqs. (2.8), (2.9) and (2.10), up to a rescaling and a shift of $T_1 = t_3$, correspond to (1.36), (1.34), (1.33). The source of the difference comes from the normalization we have used for the matrix integration variable in (1.1); indeed Kontsevich writes the integrand as $\exp\left[\frac{i}{6}\text{Tr} X^3 - \frac{1}{2}\text{Tr} X \Lambda X\right]$, from which we conclude that the relationship between our M, Y and his \tilde{X}, Λ is

$$M = 2^{-\frac{1}{3}} X, \quad Y = 2^{-\frac{1}{3}} \Lambda. \tag{2.11}$$

This translates to the following scaling relationship for the times (comparing our phase function (2.1) with the phase function in [6], Eq. (1.9), which yields the correct normalizations)

$$t_{2j+1} = -\frac{2^{\frac{2j+1}{3}}(T_j - \delta_{j,1})}{(2j+1)!!}. \tag{2.12}$$

In particular our $t_1 = x$ corresponds to $-2^{\frac{1}{3}}T_0$.

Remark 2.3. The Eq. (2.8) is the statement that the tau function of the solution to RHP (2.1) is the reduction of a Korteweg–de Vries (KdV) tau function satisfying the string equation $[P, L] = 1$, where $L := \frac{\partial^2}{\partial x^2} + 2u(x, t_1, \dots, t_k)$ is the Lax operator for the KdV hierarchy and $P := \sum_{k=1}^N (2k+1)t_{2k+1}L_+^{\frac{2k+1}{2}}$, see for instance [19]. The formulation of the Painlevé I hierarchy in terms of the string equation is originally due to Douglas [13].

Example 2.4. The case $N = 2$ corresponds to the first Painlevé equation and the special solution is the famous *tri-tronquée* solution [15]. Also, for all even N these are the solutions (conjecturally) relevant to the study of the “higher order” critical behavior of the largest eigenvalue in certain random matrix models [7].

We shall need also the formula for the higher derivatives of $\ln \tau_{P_{1N}}(\mathbf{t})$; this formula is explained in [6] in the more general context of the KdV hierarchy (of which the Painlevé I hierarchy is a reduction).

$$\frac{\partial^k}{\partial t_{2j_1+1} \dots \partial t_{2j_k+1}} \ln \tau_{P_{1N}}(\mathbf{t}) = \prod_{j=1}^k \operatorname{res}_{\lambda_\ell=\infty} \lambda_\ell^{j_\ell} F_k(\lambda_1, \dots, \lambda_k) \tag{2.13}$$

$$F_k(\lambda_1, \dots, \lambda_k) = -\frac{1}{k} \sum_{\rho \in S_k} \frac{\operatorname{Tr}(\Theta(\lambda_{\rho_1}) \dots \Theta(\lambda_{\rho_k}))}{\prod_{j=1}^k (\lambda_{\rho_j} - \lambda_{\rho_{j+1}})} - \delta_{k,2} \frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_2)^2}, \quad k \geq 2. \tag{2.14}$$

$$\Theta(\lambda) = \Theta(\lambda; \mathbf{t}) = \Gamma(\lambda; \mathbf{t}) \sigma_3 \Gamma^{-1}(\lambda; \mathbf{t}). \tag{2.15}$$

where S_k is the permutation group of k elements and in the formula we convene that $\rho_{k+1} \equiv \rho_1$.

3. Kontsevich’s Integral as Isomonodromic Tau Function

This section contains the proof of the main Theorems 1.6 and 1.9, presented respectively in the Sects. 3.3 and 3.4.

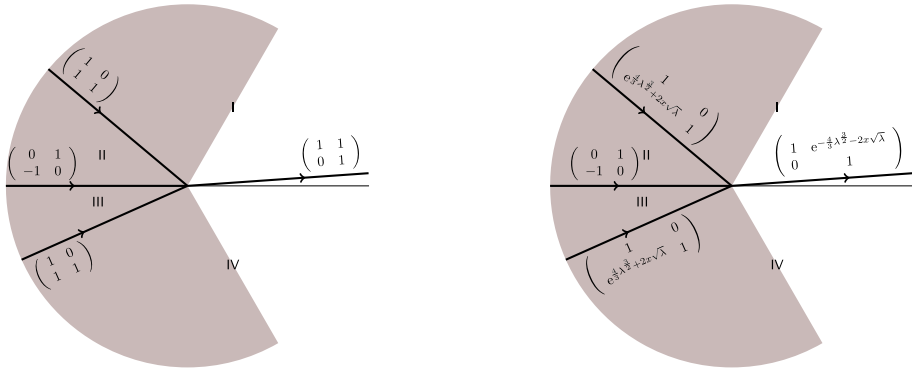


Fig. 3. The jumps of the Airy Riemann-Hilbert Problem

3.1. *The bare solution: Airy RHP.* The RHP (1.2) for $\mathbf{d}_0 \equiv 1$ corresponds to an explicitly solvable problem involving Airy functions: we call it the *bare* solution. This is also the solution of the Painlevé hierarchy (2.1) with $N = 1$ and $t_3 = \frac{2}{3}$.

Definition 3.1. Let $\omega := e^{2i\pi/3}$ and $\mathcal{A}(\zeta)$ be the matrix satisfying the jumps indicated in Fig. 3 and such that

$$\mathcal{A}(\zeta) = \sqrt{2\pi} e^{-\frac{\pi i}{12}} \times \begin{cases} \begin{bmatrix} \text{Ai}(\zeta) & \text{Ai}(\omega^2 \zeta) \\ \text{Ai}'(\zeta) & \omega^2 \text{Ai}'(\omega^2 \zeta) \end{bmatrix} e^{-\frac{i\pi}{6} \sigma_3}, & \text{for } \zeta \in I, \\ \begin{bmatrix} -\omega \text{Ai}(\omega \zeta) & \text{Ai}(\omega^2 \zeta) \\ -\omega^2 \text{Ai}'(\omega \zeta) & \omega^2 \text{Ai}'(\omega^2 \zeta) \end{bmatrix} e^{-\frac{i\pi}{6} \sigma_3}, & \text{for } \zeta \in II, \\ \begin{bmatrix} -\omega^2 \text{Ai}(\omega^2 \zeta) & -\omega^2 \text{Ai}(\omega \zeta) \\ -\omega \text{Ai}'(\omega^2 \zeta) & -\text{Ai}'(\omega \zeta) \end{bmatrix} e^{-\frac{i\pi}{6} \sigma_3}, & \text{for } \zeta \in III, \\ \begin{bmatrix} \text{Ai}(\zeta) & -\omega^2 \text{Ai}(\omega \zeta) \\ \text{Ai}'(\zeta) & -\text{Ai}'(\omega \zeta) \end{bmatrix} e^{-\frac{i\pi}{6} \sigma_3}, & \text{for } \zeta \in IV, \end{cases} \quad (3.1)$$

where the four regions are separated by the rays $e^{i\theta_{0,\pm}} \mathbb{R}_+$ and \mathbb{R}_- with the angles $\theta_{0,\pm}$ in the ranges

$$\theta_0 \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right), \quad \theta_1 \in \left(\frac{\pi}{3}, \pi\right), \quad \theta_{-1} \in \left(-\pi, -\frac{\pi}{3}\right). \quad (3.2)$$

The matrix M has the same asymptotic expansion in each of the sectors I–IV (see, e.g. [9]);

$$\mathcal{A}(\zeta) \sim e^{-\frac{i\pi}{4} \sigma_3} \zeta^{-\frac{\sigma_3}{4}} \frac{\mathbf{1} + i\sigma_1}{\sqrt{2}} \left[\mathbf{1} + \sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{2}{3} \zeta^{3/2}\right)^{-k} \begin{bmatrix} (-1)^k (u_k + r_k) & i(u_k - r_k) \\ -i(-1)^k (u_k - r_k) & u_k + r_k \end{bmatrix} \right] e^{-\frac{2}{3} \zeta^{3/2} \sigma_3}, \quad (3.3)$$

$$u_k = \frac{\Gamma(3k + 1/2)}{54^k k! \Gamma(k + 1/2)}, \quad r_k = -\frac{6k + 1}{6k - 1} u_k, \quad \text{for } k \geq 1. \quad (3.4)$$

The matrix \mathcal{A} solves the Airy equation (in matrix form)

$$\frac{d}{d\zeta}\mathcal{A}(\zeta) = \begin{bmatrix} 0 & 1 \\ \zeta & 0 \end{bmatrix} \mathcal{A}(\zeta). \tag{3.5}$$

Now we define

$$\Gamma_0(\lambda; x) := e^{-\frac{i\pi}{4}\sigma_3} \mathcal{A}(\lambda + x) e^{\left(\frac{2}{3}\lambda^{\frac{3}{2}} + x\sqrt{\lambda}\right)\sigma_3}, \tag{3.6}$$

$$\Psi_0(\lambda; x) := e^{-\frac{i\pi}{4}\sigma_3} \mathcal{A}(\lambda + x) = \Gamma_0(\lambda; x) e^{-\left(\frac{2}{3}\lambda^{\frac{3}{2}} + x\sqrt{\lambda}\right)\sigma_3}. \tag{3.7}$$

The matrix Γ_0 provides the explicit solution of the Riemann–Hilbert Problem 1.2 for $n = 0$. Using the property (3.3) one can verify directly that

$$\Gamma_0(\lambda; x) = \lambda^{-\frac{\sigma_3}{4}} \frac{\mathbf{1} + i\sigma_1}{\sqrt{2}} \left(\mathbf{1} - \frac{x^2}{4} \frac{\sigma_3}{\sqrt{\lambda}} - \frac{x}{4\lambda} \sigma_2 + \mathcal{O}(\lambda^{-\frac{3}{2}}) \right) \quad \text{as } \lambda \rightarrow \infty. \tag{3.8}$$

By construction, $\mathcal{A}(\zeta)$ solves a Riemann–Hilbert Problem with jumps indicated in the left pane of Fig. 3. Consequently one can check that $\Gamma_0(\lambda; x)$ solves the Riemann–Hilbert Problem 1.2 for $n = 0$. The only point worth remarking is that the jump contours of \mathcal{A} should be preemptively translated by x so that the jump contours of Γ_0 coincide exactly with rays issuing from the origin.

The initial tau function. The isomonodromic tau function of the RHP for $n = 0$ (which is a function solely of x) is computed directly from the formula (1.11) (using (3.8)) which yields directly

$$\partial_x \ln \tau_0(x) = -\frac{x^2}{4} \Rightarrow \tau_0(x) = e^{-\frac{x^3}{12}}. \tag{3.9}$$

It is worth remarking that $\tau_0(x)$ is nowhere vanishing: this is a signal that the RHP (1.2) for $n = 0$ is always solvable.

3.2. The dressing: discrete Schlesinger transformations. The goal of this section is to determine the change of the following one form on the deformation space \mathbf{t} ;

$$\Omega(\partial; [M(\mathbf{t})]) := \int_{\Sigma} \text{Tr} \left(\Gamma_{0-}^{-1}(\lambda; \mathbf{t}) \Gamma'_{0-}(\lambda; \mathbf{t}) \partial M(\lambda; \mathbf{t}) M^{-1}(\lambda; \mathbf{t}) \right) \frac{d\lambda}{2i\pi} \tag{3.10}$$

when $M(\lambda; \mathbf{t})$ is replaced by $M_n(\lambda; \mathbf{t}, \vec{\lambda}, \vec{\mu}) := D_n^{-1}(\lambda) M(\lambda; \mathbf{t}) D_n(\lambda)$ and D given in (1.18). Now, in our setting the one form (3.10) is the total differential of the logarithm of the tau function (see also Eq. (1.26)).

Theorem 3.2. *The effect of the dressing of the jump matrices $M_n(\lambda; \mathbf{t}, \vec{\lambda}, \vec{\mu}) := D^{-1}(\lambda)M(\lambda; \mathbf{t})D(\lambda)$ on the one-form (3.10) is given by*

$$\Omega(\partial; [M_n(\mathbf{t})]) - \Omega(\partial; [M(\mathbf{t})]) = \partial \ln \left(\Delta(\vec{\lambda}, \vec{\mu}) \det \mathbb{G} \right) \tag{3.11}$$

where $\Delta(\vec{\lambda}, \vec{\mu})$ and the $n \times n$ matrix $\mathbb{G} (n = n_1 + n_2)$ are given by

$$\Delta(\vec{\lambda}, \vec{\mu}) := \frac{\prod_{j=1}^{n_2} \lambda_j^{\frac{1}{4}} \prod_{j=1}^{n_1} \mu_j^{\frac{1}{4}}}{\prod_{j < k \leq n_2} (\sqrt{\lambda_j} - \sqrt{\lambda_k}) \prod_{j < k \leq n_1} (\sqrt{\mu_j} - \sqrt{\mu_k}) \prod_{j \leq n_2, k \leq n_1} (\sqrt{\lambda_j} + \sqrt{\mu_k})} \tag{3.12}$$

$$\mathbb{G}_{k,\ell} = \begin{cases} \operatorname{res}_{\lambda=\infty} \frac{\lambda^{\lfloor \frac{\ell-1}{2} \rfloor} \mathbf{e}_2^T \Gamma_0^{-1}(\lambda_k) G_\infty(\lambda) \mathbf{e}_{((\ell-1) \bmod 2)+1}}{(\lambda - \lambda_k)} & 1 \leq k \leq n_2 \\ \operatorname{res}_{\lambda=\infty} \frac{\lambda^{\lfloor \frac{\ell-1}{2} \rfloor} \mathbf{e}_1^T \Gamma_0^{-1}(\mu_{k-n_2}) G_\infty(\lambda) \mathbf{e}_{((\ell-1) \bmod 2)+1}}{(\lambda - \mu_{k-n_2})} & n_2 + 1 \leq k \leq n_1 + n_2 \end{cases} \tag{3.13}$$

$$G_\infty(\lambda) := \Gamma_0(\lambda)D(\lambda) \frac{\mathbf{1} - i\sigma_1}{\sqrt{2}} \lambda^{\frac{\sigma_3}{4}} \begin{cases} \lambda^{-k} & n = 2k \\ \begin{bmatrix} \lambda^{-k-1} & 0 \\ 0 & \lambda^{-k} \end{bmatrix} & n = 2k + 1. \end{cases} \tag{3.14}$$

Here ∂ denotes any derivatives with respect to \mathbf{t} together with the variables $\vec{\lambda}, \vec{\mu}$.

A proof by induction can be extracted from [16] but we will provide a different one which relies upon prior work in [4] in Appendix A. We point out that the setting of Theorem 3.2 is precisely the one relevant to the Riemann–Hilbert Problem 1.2, where $M(\lambda; \mathbf{t})$ is the jump matrix of the Airy Riemann–Hilbert Problem for (3.6).

3.3. Proof of the main theorems. We now return to the original setting and Γ_0 as given in (3.6); in this case we can further simplify $\det \mathbb{G}$ and see that it provides the main ingredient for the Witten–Kontsevich tau integral (1.1) and extensions (1.29).

Proposition 3.3. *The following formula holds*

$$\det \mathbb{G} \propto e^{\frac{2}{3} \left(\sum \lambda_j^{\frac{3}{2}} - \sum \mu_k^{\frac{3}{2}} \right) + x \left(\sum \lambda_j^{\frac{1}{2}} - \sum \mu_k^{\frac{1}{2}} \right)} \det \left[\begin{array}{c} \left[\mathbf{A}_0^{(k-1)}(\lambda_j + x) \right]_{\substack{1 \leq k \leq n \\ \lambda_j \in I \cup IV}} \\ \left[\mathbf{A}_1^{(k-1)}(\lambda_j + x) \right]_{\substack{1 \leq k \leq n \\ \lambda_j \in II}} \\ \left[\mathbf{A}_1^{(k-1)}(\mu_j + x) \right]_{\substack{1 \leq k \leq n \\ \mu_j \in III \cup IV}} \\ \left[\mathbf{A}_2^{(k-1)}(\lambda_j + x) \right]_{\substack{1 \leq k \leq n \\ \lambda_j \in III}} \\ \left[\mathbf{A}_2^{(k-1)}(\mu_j + x) \right]_{\substack{1 \leq k \leq n \\ \mu_j \in I \cup II}} \end{array} \right] \tag{3.15}$$

where $\mathbf{Ai}_s(\lambda) = \text{Ai}(\omega^s \lambda)$, $\omega = e^{\frac{2i\pi}{3}}$, and I, II, III, IV denote the regions depicted in Fig. 3 and the proportionality is up to a constant independent of $\vec{\lambda}, \vec{\mu}$.

The proof is an elementary but somewhat lengthy manipulation using the form (3.13) of the entries of \mathbb{G} , the explicit form of the matrix Γ_0 (3.6), column operations and the differential equation satisfied by the Airy functions. We postpone it to the Appendix C.

3.3.1. *Proof of Theorems 1.6, 1.7, 1.1.* We denote $\lambda_k = y_k^2, \forall y_k \in \{\text{Re } y > 0\}$ and $\mu_\ell = y_\ell^2, \forall y_\ell \in \{\text{Re } y < 0\}$. Since the roots we use are all principal, we have

$$e^{\frac{2}{3}\lambda_k^{\frac{3}{2}} + x\lambda_k^{\frac{1}{2}}} = e^{\frac{2}{3}y_k^3 + xy_k}, \quad e^{-\frac{2}{3}\mu_\ell^{\frac{3}{2}} - x\mu_\ell^{\frac{1}{2}}} = e^{\frac{2}{3}y_\ell^3 + xy_\ell}. \tag{3.16}$$

Then the determinant in (3.15) becomes precisely the same as the determinant in (1.29) when written in terms of the y_j 's, while $\Delta(\vec{\lambda}, \vec{\mu})$ reduces to

$$\Delta(\vec{\lambda}, \vec{\mu}) \stackrel{(3.12)}{=} \pm \frac{\prod_{j=1}^n \sqrt{y_j}}{\prod_{j < k} (y_j - y_k)} \tag{3.17}$$

up to an inessential sign. We want to apply Theorem 3.2; in this case the jump matrix M_n depends on $\vec{\lambda}, \vec{\mu}$ and x only, and $\Omega(\partial; [M_n]) = \partial \ln \tau_n(x; \vec{\lambda}, \vec{\mu}), \Omega(\partial_x; [M]) = \partial_x \ln \tau_0(x) = -\frac{x^2}{4}$ (see (3.9)). From Theorem 3.2 we obtain

$$\partial \ln \frac{\tau_n(x; \vec{\lambda}, \vec{\mu})}{\tau_0(x)} = \partial \ln((\det \mathbb{G}) \Delta(\vec{\lambda}; \vec{\mu})) \stackrel{(1.29)}{=} \partial \ln Z_n(x; \vec{\lambda}, \vec{\mu}). \tag{3.18}$$

The isomonodromic tau function τ_n is defined up to a multiplicative constant and therefore we can claim (using the expression for $\tau_0(x)$ in (3.9))

$$\tau_n(x; \vec{\lambda}, \vec{\mu}) = e^{-\frac{x^3}{12}} Z_n(x; \vec{\lambda}, \vec{\mu}). \tag{3.19}$$

The proof is now complete. \square

Remark 3.4. The variables τ_n depend on, can be denoted as $\vec{\lambda}, \vec{\mu}$ or by $y_j = \sqrt{\lambda_j}, y_k = -\sqrt{\mu_k}$ (in the right/left half-planes of the y -plane). Furthermore, since the determinant in (3.15) is split into blocks depending on the index ν in Ai_ν , we can equivalently denote the dependence as $\tau_n(x; \mathcal{Y}^{(0)}, \mathcal{Y}^{(1)}, \mathcal{Y}^{(2)})$. This is the way it was presented in the statement of Theorem 1.6.

3.4. *Approximation of tau functions of the first Painlevé hierarchy: Proof of Theorem 1.9.* We start with the following specializations of Riemann–Hilbert Problem 2.1 as indicated below.

Riemann–Hilbert Problem 3.5. Choose three integers $k_+, k_0, k_- \in \{-\lfloor \frac{N-1}{2} \rfloor, \dots, \lfloor \frac{N-1}{2} \rfloor\}$ with $k_+ > k_-, k_+ \geq k_0 \geq k_-$ and specialize the Riemann–Hilbert Problem

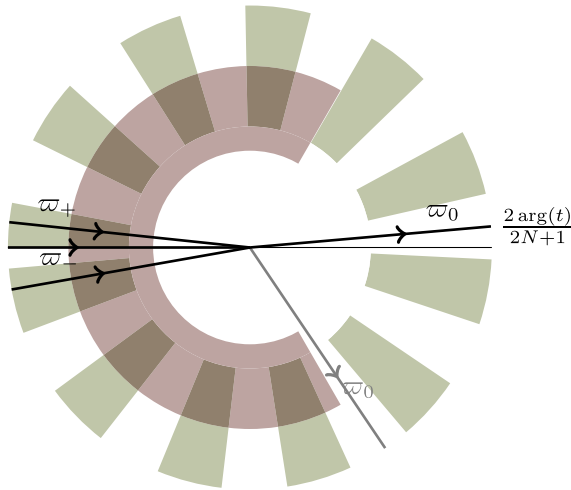


Fig. 4. The jumps of the Riemann–Hilbert Problem 3.5. In the example $N = 11$ and the width of each darker sector is $2\pi/23$. The rays ϖ_{\pm} must extend to infinity within the sector shaded in both hues, while ϖ_0 within the *white sectors*. In this example there are five choices for the ray ϖ_0 and four for each ϖ_{\pm} . Each choice determines a particular solution of the Eq. (1.40) of the P_1 hierarchy. In the example above (which is relevant to the setting of Theorem 1.9), shifting $\arg t$ by $\pm\pi$ one of the two dark sectors adjacent to \mathbb{R}_- disappears on the second sheet of $\sqrt{\lambda}$ and one of the rays ϖ_{\pm} is pinched. If we choose ϖ_0 as indicated by the lighter shade, then we can rotate $\arg t$ up to a smaller angle than $-\pi$ because the ray ϖ_0 will be forced to move in the sector S_1 , but we can still rotate up to π . In general, the reader can convince oneself that the minimum amplitude of rotation of $\arg t$ is indeed π in the positive and/or negative direction, and thus all these solutions of the hierarchy converge to the Airy parametrix exponentially fast as $|t| \rightarrow 0$ and $\arg(t)$ in a sector of width at least π that contains the positive real t -axis

2.1 to the case $s_{2k_0} = 1, s_{2k_{\pm} \pm 1} = -1$. Furthermore set $t_1 = x, t_3 = \frac{2}{3}, t_{2N+1} = t$ and all other $t_j = 0$. Explicitly, the jump matrices read (with the rays oriented as in Fig. 4)

$$M(\lambda) = \begin{cases} \mathbf{1} + e^{-2\vartheta(\lambda;t,x)}\sigma_+ & \lambda \in \varpi_0 := e^{i\theta_0}\mathbb{R}_+ \\ \mathbf{1} + e^{2\vartheta(\lambda;t,x)}\sigma_- & \lambda \in \varpi_{\pm} := e^{i\theta_{\pm}}\mathbb{R}_+ \\ i\sigma_2 & \lambda \in \mathbb{R}_- \end{cases} \quad (3.20)$$

$$\vartheta(\lambda; t, x) = t\lambda^{\frac{2N+1}{2}} + \frac{2}{3}\lambda^{\frac{3}{2}} + x\lambda^{\frac{1}{2}} \quad (3.21)$$

where the ray $\varpi_0 = e^{i\theta_0}\mathbb{R}_+$ is such that $\operatorname{Re} \lambda^{\frac{3}{2}} > 0 < \operatorname{Re} t\lambda^{\frac{2N+1}{2}}$, and the two rays $\varpi_{\pm} = e^{i\theta_{\pm}}\mathbb{R}_+$ are such that $\operatorname{Re} \lambda^{\frac{3}{2}} < 0 > \operatorname{Re} t\lambda^{\frac{2N+1}{2}}$ (Fig. 4 for example). Namely we must have (3.2) and

$$\begin{aligned} \theta_0 \in \mathcal{J}_0(k_0, t) &:= \left(-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}\right) + \frac{4k_0\pi}{2N+1} - \frac{2 \arg(t)}{2N+1}, \quad k_0 \in \mathbb{Z} \\ \theta_{\pm} \in \mathcal{J}_{\pm}(k_{\pm}, t) &:= \left(-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}\right) + \frac{(4k_{\pm} \pm 2)\pi}{2N+1} \\ &\quad - \frac{2 \arg(t)}{2N+1}, \quad k_{\pm} \in \mathbb{Z} \end{aligned} \quad (3.22)$$

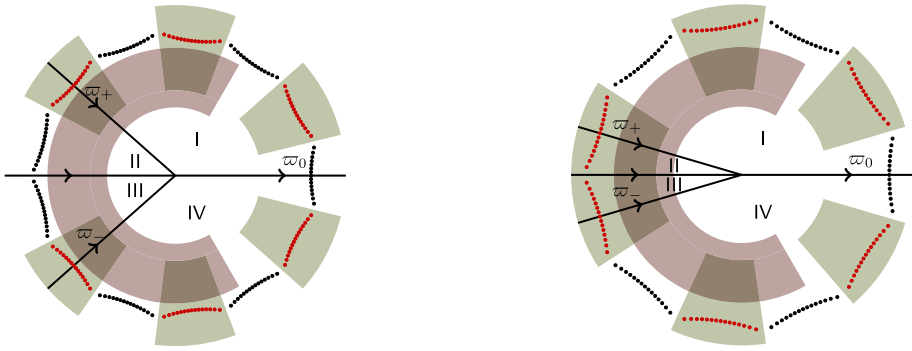


Fig. 5. The points μ_k (red) and λ_j (black). $N = 6$ (left) and $N = 5$ (right)

Proposition 3.6 [1]. Let $\Gamma(\lambda; t, x)$ be the solution of the Riemann–Hilbert Problem (3.5) with a choice of integers $\vec{k} = (k_0, k_+, k_-)$ such that

$$\begin{aligned} \mathcal{J}_0(k_0, 1) \cap \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) &\neq \emptyset & \mathcal{J}_+(k_+, 1) \cap \left(\frac{\pi}{3}, \pi\right) &\neq \emptyset \\ \mathcal{J}_-(k_-, 1) \cap \left(-\pi, -\frac{\pi}{3}\right) &\neq \emptyset. \end{aligned} \tag{3.23}$$

Then for x ranging in a compact set the solution Γ is analytic for t in a sector $\{|t| < r, \arg(t) \in (a, b)\}$ containing $\arg t = 0$, that depends on the choice of \vec{k} and has width at least π . For the case $k_0 = 0$ the sector contains the sector $\arg(t) \in (-\pi, \pi)$.

[2] This solution, for $t \rightarrow 0$, converges to the Airy parametrix (3.6) and also its tau function $\tau_{P1_N}(\mathbf{t})$ defined by (2.6) (with $t_1 = x, t_{2N+1} = t$ and all other $t_j = 0$), converges to $e^{\frac{x^3}{12}}$.

[3] The derivatives of arbitrary order with respect to $t_{2N+1} = t, t_1 = x$ of $\tau_{P1_N}(x, t)$ also converge as $|t| \rightarrow 0$ in the same sector to the derivatives of the topological solution.

Proof. [1] The matrix Γ_0 in Definition 3.1 solves the RHP (1.2) with $\mathbf{d}_0 \equiv 1$. The three rays $\omega \mathbb{R}_+$ can be rotated to rays $\varpi_j = e^{i\theta_j} \mathbb{R}_+, j = 0, \pm 1$ within the range (3.2).

Indeed, within these ranges the jump matrices in Fig. 3 are of the form $\mathbf{1} + \mathcal{O}(|\lambda|^{-\infty})$ as $|\lambda| \rightarrow \infty$ since the function $e^{-\frac{2}{3}\lambda^{\frac{3}{2}}}$ is decaying along ϖ_0 and $e^{\frac{2}{3}\lambda^{\frac{3}{2}}}$ is decaying along ϖ_{\pm} .

On the other hand the well-posedness of the Riemann–Hilbert Problem 1.2 for Γ requires that the rays satisfy (3.22) so that $e^{-t\lambda^{\frac{2N+1}{2}}}$ is decaying along ϖ_0 and $e^{t\lambda^{\frac{2N+1}{2}}}$ is decaying along ϖ_{\pm} , see Fig. 4.

In general there are several possible choices of k_0, k_{\pm} in (3.22) that satisfy both ranges (3.2), (3.22); for a given choice of rays, the conditions will remain satisfied within a certain maximal open sector in the t -plane; it should be noted that different choices of $k_{0,\pm}$ lead to inequivalent Riemann–Hilbert Problems and solutions of the first Painlevé hierarchy (see below); in particular the poles of the corresponding Painlevé transcendent depend on this choice.

Let us assume now that we make one such choice of sector and consider the RHP for the matrix \mathcal{E} with jumps on $\varpi_{0,\pm 1}$ as follows

$$\mathcal{E}_+(\lambda; t, x) = \mathcal{E}_-(\lambda; t, x) \left(\mathbf{1} + e^{-\frac{2}{3}\lambda^{\frac{3}{2}} - x\sqrt{\lambda}} \left(e^{-t\lambda^{\frac{2N+1}{2}}} - 1 \right) \Gamma_0(\lambda; x) \sigma_+ \Gamma_0^{-1}(\lambda; x) \right), \quad \lambda \in \varpi_0 \tag{3.24}$$

$$\mathcal{E}_+(\lambda; t, x) = \mathcal{E}_-(\lambda; t, x) \left(\mathbf{1} + e^{\frac{2}{3}\lambda^{\frac{3}{2}} + x\sqrt{\lambda}} \left(e^{t\lambda^{\frac{2N+1}{2}}} - 1 \right) \Gamma_0(\lambda; x) \sigma_- \Gamma_0^{-1}(\lambda; x) \right), \quad \lambda \in \varpi_{\pm} \tag{3.25}$$

$$\mathcal{E}(\lambda; t, x) = \mathbf{1} + \mathcal{O}(\lambda^{-1}), \quad \lambda \rightarrow \infty. \tag{3.26}$$

Along the rays in the chosen sector, the matrices Γ_0, Γ_0^{-1} (Airy parametrix) in (3.24), (3.25) remain bounded, uniformly with respect to x ranging in a compact set. Furthermore, within the chosen sector, we can send $|t| \rightarrow 0$ and the jump matrices will converge to the identity matrix in all L^p norms, $1 \leq p \leq \infty$, uniformly with respect to x in compact sets: for example on ϖ_+ the function $e^{\frac{2}{3}\lambda^{\frac{3}{2}} + x\sqrt{\lambda}} \left(e^{t\lambda^{\frac{2N+1}{2}}} - 1 \right)$ belongs to all $L^p(\varpi_+, |d\lambda|)$ because $e^{\frac{2}{3}\lambda^{\frac{3}{2}} + x\sqrt{\lambda}}$ does, and $\left(e^{t\lambda^{\frac{2N+1}{2}}} - 1 \right)$ is bounded. Consequently the matrix \mathcal{E} converges to the identity as $|t| \rightarrow 0$ in the given sector and has an expansion near $\lambda = \infty$ of the form

$$\mathcal{E}(\lambda; t, x) = \mathbf{1} + \frac{1}{\lambda} \mathcal{E}_1(t, x) + \mathcal{O}(\lambda^{-2}). \tag{3.27}$$

Most importantly, for $|t|$ sufficiently small, the existence of the solution \mathcal{E} (and its analyticity with respect to the parameters t, x) is guaranteed by standard arguments. By construction of the jump relations (3.24), (3.25), the matrix $\mathcal{E}(\lambda; t, x) \Gamma_0(\lambda; x)$ solves a Riemann–Hilbert Problem with the same jumps as $\Gamma(\lambda; t, x)$ but in a different gauge (see Remark 1.3). By a left multiplication with λ -independent matrix and by the uniqueness of the solution of the Riemann–Hilbert Problem 1.2, we deduce that

$$\Gamma(\lambda; t, x) = (\mathbf{1} - (\mathcal{E}_1(t, x))_{12} \sigma_-) \mathcal{E}(\lambda; t, x) \Gamma_0(\lambda; x). \tag{3.28}$$

The left multiplier is crafted so as to guarantee the same gauge as Γ at infinity (see Remark 1.3). We conclude that $\Gamma(\lambda; t, x)$ is analytic in the specified domain. The width of the sectors is explained by way of example in the caption of Fig. 4.

[2] Since $\mathcal{E} \rightarrow \mathbf{1}$, we deduce that the tau function for $\Gamma(\lambda; t, x)$ defined by (2.6) converges to that of Γ_0 as given in (3.9).

[3] By the same argument, using (2.13) we conclude that all derivatives of $\ln \tau_{P_{1N}}$ converge as $|t| \rightarrow 0$ within the common sector, to the same expression (2.13) evaluated using the Airy parametrix Γ_0 . These are [6] precisely the derivatives of the topological solution of KdV (note that we are using a different normalization of the time t from loc. cit. but this is inconsequential to our discussion). \square

3.4.1. *Equivalence to all orders of different solutions: proof of Theorem 1.9*_[3]. Proposition 3.6 has already established Theorem 1.9_[2] and part of Theorem 1.9_[3]. It remains to show that two solutions of the Riemann–Hilbert Problem 3.5 (and the corresponding tau functions) with different choices of $\vec{k} = (k_0, k_+, k_-)$ differ by exponentially small terms as $|t| \rightarrow 0$ as long as the corresponding sectors appearing in the Proposition 3.6 have non-empty overlap. See Fig. 6 illustrating a typical such setup.

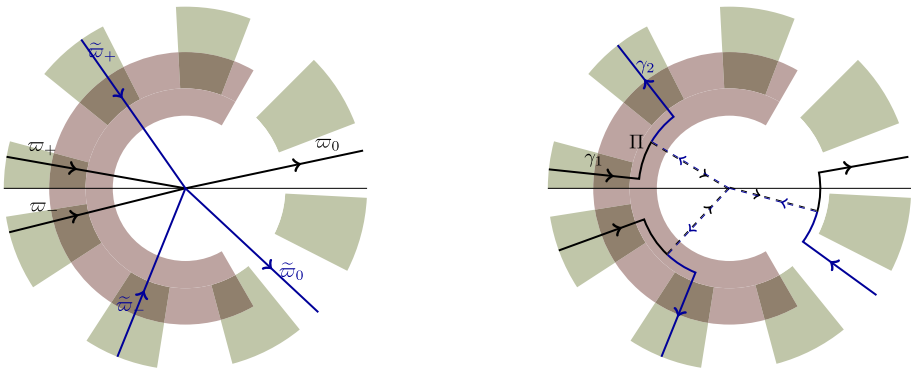


Fig. 6. *Left* the jump contours $\varpi_{0,\pm}$ and $\tilde{\varpi}_{0,\pm}$ (marked in different colors) of two solutions $\Gamma, \tilde{\Gamma}$ of the RHP 3.5 with different choices of the integers (k_0, k_+, k_-) (specifically, $N = 7$ and $(k_+ = 3, k_0 = 0, k_- = -3)$ while $(\tilde{k}_+ = 2, \tilde{k}_0 = -1, \tilde{k}_- = -2)$). *Right* the jumps of their ratio $\mathcal{E} = \Gamma \tilde{\Gamma}^{-1}$ after the contour deformation. On the *dashed arcs* the jumps cancel each other and hence \mathcal{E} is continuous across them

The proof is a simple application of perturbation analysis of Riemann–Hilbert Problems; the ratio of two solutions with different choices \vec{k}_1, \vec{k}_2 has a jump which approaches the identity in any L^p ($1 \leq p \leq \infty$) at an exponential rate in $\frac{1}{|t|^\mu}$, with a power law that we are going to compute.

We need to analyze the signs of the real part of the phase function ϑ (3.21) as $|t| \rightarrow 0$. Treating the term with t in ϑ as a perturbation, it is clear that for $|t|$ sufficiently small the signs of $\text{Re } \vartheta(\lambda; t, x)$ are dominated by those of $f_0 = \text{Re} \left(\frac{2}{3} \lambda^{\frac{2}{3}} + x \sqrt{\lambda} \right)$ in any fixed compact set in the λ -plane. Let us fix a bounded domain for x : $|x| < K$.

We are free to deform the jump contours of the RHP 3.5 as we wish as long as the asymptotic directions at infinity satisfy the appropriate conditions, we shall deform them in a way that we explain below.

Contour deformation. We consider $t \in \mathbb{R}_+$ for simplicity because the steps are valid in a small sector. For definiteness we only treat ϖ_+ , with similar considerations in the remaining cases. Let $\theta_1 := \frac{(2k_++1)2\pi}{2N+1}, \theta_2 := \frac{(2\tilde{k}_++1)2\pi}{2N+1}$ be the bisecants of the sectors visited by the jumps of the two solutions (refer to Fig. 6) and recall that by assumption we must have $\theta_{1,2} \in (\frac{\pi}{3}, \pi)$. Along the two rays $\arg \lambda = \theta_{1,2}$ we have (here $r = |\lambda|$)

$$\text{Re } \vartheta = -|t| r^{\frac{2N+1}{2}} + \frac{2}{3} r^{\frac{3}{2}} \cos\left(\frac{3}{2}\theta_j\right) + \text{Re } x \sqrt{\lambda} < \sqrt{r} \left[-|t| r^N - Q \frac{2}{3} r + K \right] \quad (3.29)$$

where $Q = \min |\cos(\frac{3}{2}\theta_j)|$ is a positive number because of the condition $\theta_j \in (\frac{\pi}{3}, \pi)$.

Consider the rays γ_j given by $\arg \lambda = \theta_j, r \geq r_0 = (Q/2)^{\frac{1}{N-1}} |t|^{\frac{-1}{N-1}}$; along these rays the function e^ϑ belongs to any L^p ($1 \leq p \leq \infty$) and the corresponding norm decays exponentially as $|t| \rightarrow 0$; in fact from (3.29)

$$\begin{aligned} \sup_{\lambda \in \gamma} \text{Re } \vartheta(\lambda) &\leq t^{\frac{-3}{2N-2}} \frac{Q^{\frac{N+1/2}{N-1}}}{2^{\frac{1}{N-1}}} \left(-\frac{1}{2} - \frac{2}{3} + \frac{K |t|^{\frac{1}{N-1}}}{Q^{\frac{N}{N-1}} 2^{\frac{3}{2N-2}}} \right) \\ &= t^{\frac{-3}{2N-2}} \frac{Q^{\frac{N+1/2}{N-1}}}{2^{\frac{1}{N-1}}} \left(-\frac{7}{6} + \frac{K |t|^{\frac{1}{N-1}}}{Q^{\frac{N}{N-1}} 2^{\frac{3}{2N-2}}} \right) \end{aligned} \quad (3.30)$$

which clearly tends to $-\infty$ as $|t| \rightarrow 0$. Consider now the arc of circle Π , joining the two points $r_0 e^{i\theta_1}$ to $r_0 e^{i\theta_2}$; along this arc we have (with $\varphi = \arg \lambda$);

$$\begin{aligned} \operatorname{Re} \vartheta &= -|t| r^{\frac{2N+1}{2}} \cos\left(\frac{2N+1}{2}\varphi\right) + \frac{2}{3} r^{\frac{3}{2}} \cos\left(\frac{3}{2}\varphi\right) + \operatorname{Re}(x\sqrt{\lambda}) \\ &\leq t^{\frac{-3}{2N-2}} \frac{Q^{\frac{N+1/2}{N-1}}}{2^{\frac{1}{N-1}}} \left(-\frac{1}{6} + \frac{K|t|^{\frac{1}{N-1}}}{Q^{\frac{N}{N-1}} 2^{\frac{3}{2N-2}}}\right) \end{aligned} \tag{3.31}$$

where we have used that $\cos(\frac{3}{2}\varphi) \leq -Q$ along the arc. Once more the L^p norm of e^ϑ along this arc is easily estimated to tend to zero exponentially.

Exponential rate of convergence. Now, referring to Fig. 6 we deform the rays $\overline{\omega}_+$, $\widetilde{\omega}_+$ as indicated and consider the RHP for the matrix \mathcal{E} with jumps only on the union of the rays $\gamma_{1,2}$ and the arc Π as shown in the figure and with the jump matrix given by $\mathbf{1} + e^{\vartheta(\lambda;t,x)} \widetilde{\Gamma}(\lambda; t, x) \sigma_- \widetilde{\Gamma}^{-1}(\lambda; t, x)$. We know from Proposition 3.6 that $\widetilde{\Gamma}(\lambda; t, x)$ tends to the Airy parametrix as $|t| \rightarrow 0$ in a small sector around \mathbb{R}_+ uniformly with respect to λ on the Riemann sphere and hence it remains bounded as $|t| \rightarrow 0$. The L^p norms of e^ϑ on the two rays are $\mathcal{O}\left(e^{-\widetilde{C}|t|^{-\frac{3}{2N-2}}}\right)$ while the L^p norms on the arc Π (whose length grows like $|t|^{-\frac{1}{N-1}}$), are all bounded by $\mathcal{O}\left(|t|^{-\frac{1}{N-1}} e^{-C|t|^{-\frac{3}{2N-2}}}\right)$, where C, \widetilde{C} are positive constants that follow from the estimates (3.29), (3.31). In total the L^p norm of e^ϑ along the whole contour are bounded by

$$\|e^\vartheta\|_{L^p(\gamma_1 \cup \gamma_2 \cup \Pi, |\mathrm{d}\lambda|)} = \mathcal{O}\left(t^{\frac{-1}{N-1}} e^{-C|t|^{-\frac{3}{2N-2}}}\right), \tag{3.32}$$

uniformly with respect to $1 \leq p \leq \infty$. By standard arguments [10] on small norm Riemann Hilbert Problems, we conclude that \mathcal{E} converges to the identity on the Riemann sphere, at the same exponential rate (3.32). Then, by the same argument used in Proposition 3.6, we conclude that

$$\Gamma(\lambda; t, x) = (\mathbf{1} - (\mathcal{E}_1(t, x))_{12} \sigma_-) \mathcal{E}(\lambda; t, x) \widetilde{\Gamma}(\lambda; x)$$

(with $\mathcal{E}_1(t, x)$ similar as in (3.27)) and hence $\widetilde{\Gamma}(\lambda; t, x)$, $\Gamma(\lambda, t, x)$ differ from each other by exponentially small terms as $|t| \rightarrow 0$ in a small sector around \mathbb{R}_+ . In particular, the ratio of the corresponding tau functions defined via (2.6) will also tend to unity exponentially fast as $|t| \rightarrow 0$; therefore, the asymptotic expansion of the logarithms of the two tau functions in powers of t will be identical in the overlapping sector. Since the estimates are uniform with respect to x in a compact set (we used $|x| < K$ in the estimates (3.29), (3.31)), the coefficients of these expansions must also be analytic in x at least in the same domain.

3.4.2. Padé approximation: Proof of Theorem 1.9_[1]. The exponential function admits a Padé approximation of the form

$$e^{-z} = \prod_{j=1}^r \frac{a_j - z}{a_j + z} + \mathcal{O}(z^{2r+1}) = \frac{P_r(z)}{P_r(-z)} + \mathcal{O}(z^{2r+1}). \tag{3.33}$$

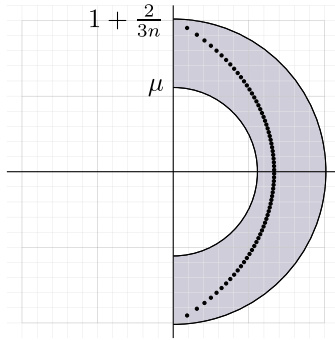


Fig. 7. The zeroes of $P_n(2nz)$ for $n = 70$

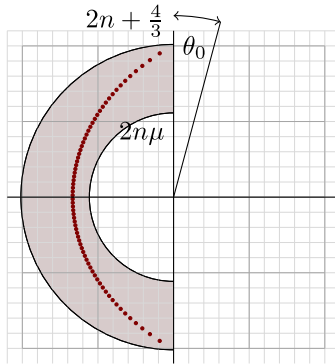


Fig. 8. The poles of the Padé approximation are within the shaded sectorial annulus

The polynomial P_n is explicitly known⁴ ([20], p. 433)

$$P_r(z) = \sum_{k=0}^r \frac{(2r - k)!(-z)^k}{k!(r - k)!}. \tag{3.34}$$

The location of the zeros $\mathfrak{Z}_r := \{a_j, j = 1, \dots, r\}$, of $P_r(z)$ (plotted in Fig. 7 by way of example) is known to belong to the annular sector [21]

$$2r\mu < |a_j| < 2r + \frac{4}{3}, \quad \mu > 0, \quad \mu e^{1+\mu} = 1 \quad (\mu \simeq 0.278465\dots) \tag{3.35}$$

$$|\arg(a_j)| \leq \cos^{-1}(1/r) \quad \operatorname{Re}(a_j) > 2\mu > \frac{1}{2}. \tag{3.36}$$

Estimate for the remainder. The remainder of the approximation is also known exactly

$$e^{-z} - \frac{P_r(z)}{P_r(-z)} = \frac{(-1)^{r+1} z^{2r+1}}{r! P_r(-z)} \int_0^1 e^{-tz} (1-t)^r t^r dt. \tag{3.37}$$

The estimate (3.35) on the position of zeroes implies that for $\operatorname{Re}(z) \geq 0$, the minimum distance from the poles $-\mathfrak{Z}_r$ for $|\arg(z)| \leq \frac{\pi}{2} - \theta_0$ for some small fixed θ_0 , is (see Fig. 8)

⁴ We have normalized the polynomial to be monic.

$$\text{dist}(z, -3r) \geq \frac{1}{\sqrt{(|z| - 2r\mu \cos \theta_0)^2 + (2r\mu \sin \theta_0)^2}} \geq \frac{2}{|z| + 2r\mu \sin \theta_0}. \tag{3.38}$$

Then we can estimate

$$\begin{aligned} \left| e^{-z} - \frac{P_r(z)}{P_r(-z)} \right| &\leq \frac{2|z|^r}{r!(\sin \theta_0)^r (2r\mu \sin \theta_0 + |z|)^r} \int_0^{\text{Re } z} e^{-t} t^r dt \\ &\leq \frac{2|z|^r}{(\sin \theta_0)^r (2n\mu \sin \theta_0 + |z|)^r}. \end{aligned} \tag{3.39}$$

Let $N \in \mathbb{N}$ be fixed. We want to obtain an approximation of $e^{-2t\lambda \frac{2N+1}{2}}$ in a suitable sector; of course we will use (3.33) replacing $z \mapsto 2t\lambda \frac{2N+1}{2}$.

*Proof of Theorem 1.9*_[1]. From (3.39) we have (we set $t = t_{2N+1}$, $n = r(2N + 1)$ for brevity)

$$\begin{aligned} \frac{P_r(2t\lambda \frac{2N+1}{2})}{P_r(-2t\lambda \frac{2N+1}{2})} - e^{-t\lambda \frac{2N+1}{2}} &= \mathcal{O} \left(\frac{|t|^r |\lambda|^{\frac{r(2N+1)}{2}}}{(\sin \theta_0)^r ((2n\mu \sin(\theta_0))^r + |t|^r |\lambda|^{\frac{r(2N+1)}{2}})} \right), \\ \text{arg } \lambda \in \mathcal{J}_0(k, t) &:= \left(-\frac{\pi}{2N + 1}, \frac{\pi}{2N + 1} \right) + \frac{4\pi k_0 - 2 \text{arg}(t)}{2N + 1}. \end{aligned} \tag{3.40}$$

The estimate above shows that inside the growing disk $|t\lambda \frac{2N+1}{2}| \leq Kr^{\frac{1}{2}}$ (with $K = (4N + 2)(\sin^2 \theta_0)$) the expression is bounded as follows:

$$\left| \frac{P_r(2t\lambda \frac{2N+1}{2})}{P_r(-2t\lambda \frac{2N+1}{2})} - e^{-t\lambda \frac{2N+1}{2}} \right| \leq \begin{cases} r^{-\frac{r}{2}} |\lambda| \leq K^{\frac{2}{2N+1}} |t|^{\frac{-2}{2N+1}} r^{\frac{1}{2N+1}}, & \text{arg}(\lambda) \in \mathcal{J}_0 \\ 1 & |\lambda| \geq K^{\frac{2}{2N+1}} |t|^{\frac{-2}{2N+1}} r^{\frac{1}{2N+1}}, \text{arg}(\lambda) \in \mathcal{J}_0. \end{cases} \tag{3.41}$$

Let $\Gamma_n(\lambda; t, x)$ ($n = r(2N + 1)$) be the solution of the RHP (1.2) with $\mathbf{d}_n = \frac{P_r(t\lambda \frac{2N+1}{2})}{P_r(-t\lambda \frac{2N+1}{2})}$ and let $\Gamma(\lambda; t, x)$ be the solution discussed in Proposition 3.6. Similar esti-

mates hold for $\frac{P_r(-2t\lambda \frac{2N+1}{2})}{P_r(2t\lambda \frac{2N+1}{2})} - e^{t\lambda \frac{2N+1}{2}}$ in the sectors \mathcal{J}_{\pm} (3.22). We now choose k_+ , k_0 , k_-

and the corresponding ways $\varpi_{0,\pm}$; the tau function $\tau_n(z; \vec{\lambda}, \vec{\mu})$ is then given by (1.29) according to Theorem 1.6_[1]. The relationship between the positioning of the jump rays $\varpi_{0,\pm}$ and the integers $k_{0,\pm}$ follows from the formula (3.15) and careful inspection.

It only remains to show that the solution of the Riemann–Hilbert Problem 1.2 converges to the solution of the Painlevé auxiliary Riemann–Hilbert Problem 2.1, which we now accomplish.

Following the same idea as in Proposition 3.6, let $\mathcal{E} = \mathcal{E}(\lambda; t, x, n)$ be the solution of the Riemann–Hilbert Problem with jumps on $\varpi_{0,\pm 1}$ of the form

$$\mathcal{E}_+ = \mathcal{E}_- \Gamma \left(\mathbf{1} + e^{-\frac{2}{3}\lambda^{\frac{3}{2}} - x\sqrt{\lambda}} \overbrace{\left(\frac{P_r(t\lambda \frac{2N+1}{2})}{P_r(-t\lambda \frac{2N+1}{2})} - e^{-t\lambda \frac{2N+1}{2}} \right)}^{:=\tilde{\mathfrak{F}}_0(\lambda)} \sigma_+ \right) \Gamma^{-1}, \quad \lambda \in \varpi_0 \tag{3.42}$$

$$\mathcal{E}_+ = \mathcal{E}_- \Gamma \left(\mathbf{1} + e^{\frac{2}{3}\lambda^{\frac{3}{2}+x}\sqrt{\lambda}} \underbrace{\left(\frac{P_r(-t\lambda^{\frac{2N+1}{2}})}{P_r(t\lambda^{\frac{2N+1}{2}})} - e^{t\lambda^{\frac{2N+1}{2}}} \right)}_{:=\mathfrak{F}_{\pm}(\lambda)} \right) \Gamma^{-1}, \quad \lambda \in \varpi_{\pm} \tag{3.43}$$

$$\mathcal{E}(\lambda; t, x, n) = \mathbf{1} + \frac{1}{\lambda} \mathcal{E}_1(t, x, n) + \mathcal{O}(\lambda^{-2}), \quad \lambda \rightarrow \infty. \tag{3.44}$$

Consider (3.42), with similar considerations for (3.43): given the estimate (3.41) we have that, uniformly for x in compact sets,

$$\left| e^{-\frac{2}{3}\lambda^{\frac{3}{2}-x}\sqrt{\lambda}} \mathfrak{F}_0(\lambda) \right| = \begin{cases} \mathcal{O}(r^{-\frac{r}{2}}) & |\lambda| \leq |t|^{\frac{-2}{2N+1}} r^{\frac{1}{2N+1}}, \lambda \in \varpi_0 \\ \mathcal{O} \left(\exp \left(-C|t|^{\frac{-3}{2N+1}} r^{\frac{3}{4N+2}} \right) \right) & |\lambda| \geq |t|^{\frac{-2}{2N+1}} r^{\frac{1}{2N+1}}, \lambda \in \varpi_0. \end{cases} \tag{3.45}$$

According to Proposition 3.6 for $|t|$ sufficiently small and x in a compact set, the function $\Gamma(\lambda; t, x)$ remains uniformly bounded and therefore it is easy to see, using (3.45), that the jump matrices (3.42), (3.43) converge to the identity in all L^p norms ($1 \leq p \leq \infty$) as $r \rightarrow \infty$, and hence so does \mathcal{E} ; the rate of convergence is the same as in (3.45) and it is faster than any inverse power of r and thus on n . Note that the angle of the ray ϖ_0 ranges in a sector where there are zeroes of the numerator in $\mathfrak{F}_0(\lambda)$ and since the zeroes and poles of the numerator/denominator are contained in disjoint sectors, the function $\mathfrak{F}_0(\lambda)$ is bounded and analytic along the ray ϖ_0 . Similarly for the other two rays.

Then, by the same token used in Proposition 3.6 we must have

$$\Gamma_n(\lambda; t, x) = \left(\mathbf{1} - \mathcal{E}_1(t, x, n)_{21} \sigma_- \right) \mathcal{E}(\lambda; t, x, n) \Gamma(\lambda; t, x) \tag{3.46}$$

and we conclude that the tau function for the problem Γ_n (i.e. Z_n) converges to the tau function of Γ . \square

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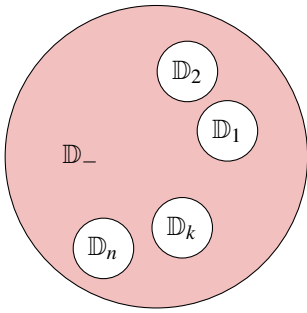
A. Proof of Theorem 3.2

Let $\Gamma_n = \Gamma_n(\lambda; \mathbf{t}, \vec{\lambda}, \vec{\mu})$ denote the solution of the Riemann–Hilbert Problem with the jump matrices $M_n(\lambda; \mathbf{t}, \vec{\lambda}, \vec{\mu})$. It can be written as $\Gamma_n(\lambda) = R(\lambda)\Gamma_0(\lambda)D(\lambda)$ where $R(\lambda)$ is a suitable rational matrix and $D(\lambda)$ as in (1.18). Indeed the matrix ratio

$$R(\lambda) := \Gamma_n(\lambda)D(\lambda)^{-1}\Gamma_0(\lambda)^{-1} \tag{A.1}$$

is seen to have no jumps on Σ . It clearly has at most simple poles at $\lambda = \lambda_k, \mu_k$ and decays algebraically at infinity. By Liouville’s theorem $R(\lambda)$ is a rational function.

We now seek a set of characterizing conditions for the matrix R as a solution of a given Riemann–Hilbert Problem. To this end let $r > 0$ be sufficiently small so that all the disks below are disjoint and define:



$$\begin{aligned}
 \mathbb{D}_+ &:= \mathbb{D}_\infty \cup \bigsqcup_{k=1}^n \mathbb{D}_k, \\
 \mathbb{D}_k &:= \{|\lambda - \lambda_k| < r\}, \quad k \leq n_2, \\
 \mathbb{D}_{k+n_2} &:= \{|\lambda - \mu_k| < r\}, \quad k \leq n_1 \\
 \mathbb{D}_\infty &:= \left\{ \left| \frac{1}{\lambda} \right| > r \right\} \\
 \mathbb{D}_- &:= \mathbb{C} \setminus \overline{\mathbb{D}_+}.
 \end{aligned} \tag{A.2}$$

Riemann-Hilbert Problem A.1. Find a 2×2 piecewise analytic function $\mathbf{R}(\lambda)$ on \mathbb{D}_+ and \mathbb{D}_- , admitting continuous boundary values and satisfying the following conditions

$$\mathbf{R}_+(\lambda) = \mathbf{R}_-(\lambda)\mathbf{J}(\lambda), \quad \mathbf{R}(\lambda) = \mathbf{1} + \mathcal{O}(\lambda^{-1}), \quad \lambda \rightarrow \infty \tag{A.3}$$

where $\mathbf{J}(\lambda)$ is the matrix that on each component of $\partial\mathbb{D}_+$ restricts to

$$\mathbf{J}(\lambda) = \begin{cases} J_k(\lambda) = \Gamma_0(\lambda)(\lambda_k - \lambda)^{E_{22}}, & |\lambda - \lambda_k| = r, \quad k \leq n_2 \\ J_{n_2+k}(\lambda) = \Gamma_0(\lambda)(\mu_k - \lambda)^{E_{11}}, & |\lambda - \mu_k| = r, \quad k \leq n_1 \\ J_\infty(\lambda) = \Gamma_0(\lambda)D(\lambda) \frac{\mathbf{1} - i\sigma_1}{\sqrt{2}} \lambda^{\frac{\sigma_3}{4}}, & |\lambda|^{-1} = r. \end{cases} \tag{A.4}$$

We shall use interchangeably $\mathbf{R}_\pm(\lambda)$ for the boundary values or the restriction of the solution $\mathbf{R}(\lambda)$ to \mathbb{D}_\pm , respectively. We prove the following:

Proposition A.2. *The matrix $R(\lambda)$ in (A.1) is a rational matrix-valued function with simple poles at $\lambda = \lambda_k, \mu_k$. Restricted to \mathbb{D}_- it coincides with $\mathbf{R}_-(\lambda)$ in the Riemann-Hilbert Problem A.1, up to a left constant multiplier of the form $\mathbf{1} - ia^{(n)}\sigma_-$. In particular, the matrix $\mathbf{R}_-(\lambda)$ extends to a rational function of λ .*

Proof. We already proved that $R(\lambda)$ is a rational function. Moreover near λ_k, μ_k we have, from (A.1),

$$R(\lambda) = \mathcal{O}(1)(\lambda - \lambda_k)^{-E_{22}}\Gamma_0^{-1}(\lambda), \quad R(\lambda) = \mathcal{O}(1)(\lambda - \mu_k)^{-E_{11}}\Gamma_0^{-1}(\lambda) \tag{A.5}$$

and thus $R(\lambda)$ must have simple poles at λ_k, μ_k (here $\mathcal{O}(1)$ stands for a locally analytic matrix function, with analytic inverse). Now in order to establish the connection between \mathbf{R} and R we have to study the behaviour at $\lambda = \infty$. We multiply both sides of $\Gamma_n = R\Gamma_0 D$ (D defined in (1.18)) as follows

$$\Gamma_n(\lambda) \left(\lambda^{-\frac{\sigma_3}{4}} \frac{\mathbf{1} + i\sigma_1}{\sqrt{2}} \right)^{-1} = R(\lambda) \underbrace{\Gamma_0(\lambda)D(\lambda) \frac{\mathbf{1} - i\sigma_1}{\sqrt{2}} \lambda^{\frac{\sigma_3}{4}}}_{J_\infty(\lambda)}. \tag{A.6}$$

Because of the asymptotic behavior (1.16) the left side admits a regular expansion at $\lambda = \infty$ with leading coefficient of the form $C_0 = \mathbf{1} + ia^{(n)}\sigma_-$.

Hence we define $\mathbf{R}(\lambda)$ as

$$\mathbf{R}(\lambda) = \begin{cases} (\mathbf{1} - ia^{(n)}\sigma_-) \Gamma_n(\lambda) D^{-1}(\lambda) (\lambda - \lambda_k)^{E_{22}} & \lambda \in \mathbb{D}_k, k = 1, \dots, n_2, \\ (\mathbf{1} - ia^{(n)}\sigma_-) \Gamma_n(\lambda) D^{-1}(\lambda) (\lambda - \mu_k)^{E_{11}} & \lambda \in \mathbb{D}_{n_2+k}, k = 1, \dots, n_1, \\ (\mathbf{1} - ia^{(n)}\sigma_-) \Gamma_n(\lambda) \frac{1-i\sigma_1}{\sqrt{2}} \lambda^{\frac{\sigma_3}{4}} & \lambda \in \mathbb{D}_\infty, \\ (\mathbf{1} - ia^{(n)}\sigma_-) \Gamma_n(\lambda) D^{-1}(\lambda) \Gamma_0^{-1}(\lambda) & \lambda \in \mathbb{D}_-. \end{cases} \tag{A.7}$$

It is easy to check that \mathbf{R} , defined in this way, satisfies the jump conditions in (A.3) and, moreover, also the asymptotic condition at infinity. The identification (up to a normalization constant) between \mathbf{R}_- and R is read off directly from the last line of (A.7). \square

The jump matrix $\mathbf{J}(\lambda)$ admits (formal) meromorphic extension in the interior of \mathbb{D} (by construction) and moreover the total index of $\det \mathbf{J}(\lambda)$ around $\partial\mathbb{D}$ is zero. Under these conditions, the analysis in Appendix B of [4] applies. We briefly remind it with notation adapted to the current use.

Let \mathcal{H}_\pm denote the vector space of (formally) analytic valued row-vectors in \mathbb{D}_\pm (respectively) and C_\pm the Cauchy projection operator; consider the following two (finite dimensional, see B.14 in loc. cit.) subspaces of \mathcal{H}_-

$$V := C_-[\mathcal{H}_+\mathbf{J}^{-1}(\lambda)], \quad W := C_-[\mathcal{H}_+\mathbf{J}(\lambda)] \tag{A.8}$$

Proposition A.3 (Proposition B.3 in [4]). *The solution of the Riemann–Hilbert Problem A.1 exists, and is unique, if and only if the linear map*

$$\begin{aligned} \mathcal{G} : V &\longrightarrow W \\ v &\longmapsto C_-[v\mathbf{J}]. \end{aligned} \tag{A.9}$$

is invertible. In this case, the inverse is

$$\begin{aligned} \mathcal{G}^{-1} : W &\longrightarrow V \\ w &\longmapsto C_-[w\mathbf{J}^{-1}\mathbf{R}^{-1}]\mathbf{R}. \end{aligned} \tag{A.10}$$

Remark A.4. Even if R and \mathbf{R} differ by the multiplication of a constant left multiplier of the form $\mathbf{1} + \star\sigma_-$, the expression of the inverse (A.10) is unaffected by such multiplier.

Now, following [4], we choose properly two bases for V and W in such a way that the determinant \mathbb{G} of \mathcal{G} gives the variation of the one form Ω as in (3.11).

The jump on the large circle given by J_∞ (A.4) has an asymptotic expansion in integer powers of λ of the form (the \star symbol means a constant of no interest to us, which turns out to be $-\frac{x^2}{4}$)

$$\begin{aligned} \mathbf{n} = 2\mathbf{k} \equiv \mathbf{0} \pmod{2} \quad J_\infty(\lambda) &= (-1)^{n_1} \lambda^k \left(\mathbf{1} + \star\sigma_- + \mathcal{O}(\lambda^{-1}) \right) \\ &=: G_\infty(\lambda) \lambda^{k\mathbf{1}}, \end{aligned} \tag{A.11}$$

$$\begin{aligned} \mathbf{n} = 2\mathbf{k} + \mathbf{1} \equiv \mathbf{1} \pmod{2} \quad J_\infty(\lambda) &= (-1)^{n_1} \lambda^k \left(i\lambda\sigma_- - i\sigma_+ + \star\mathbf{1} + \mathcal{O}(\lambda^{-1}) \right) = \\ &= (-1)^{n_1} \left(\sigma_2 + \star E_{22} + \mathcal{O}(\lambda^{-1}) \right) \lambda^{(k+1)E_{11}+kE_{22}} \\ &=: G_\infty(\lambda) \lambda^{(k+1)E_{11}+kE_{22}}, \end{aligned} \tag{A.12}$$

where $G_\infty(\lambda)$ (as in (3.14)) has been introduced for convenience. Following [4], Appendix B, we choose the bases

$$V = \bigoplus_{k=1}^{n_2+n_1} \mathbb{C}\{v_k\}; \quad v_j = C_-[\mathbf{e}_2^T \mathbf{J}^{-1}(\lambda)] = \frac{\mathbf{e}_2^T \Gamma_0^{-1}(\lambda_j)}{\lambda - \lambda_j}, \quad 1 \leq j \leq n_2$$

$$v_{j+n_2} = C_-[\mathbf{e}_1^T \mathbf{J}^{-1}(\lambda)] = \frac{\mathbf{e}_1^T \Gamma_0^{-1}(\mu_j)}{\lambda - \mu_j}, \quad 1 \leq j \leq n_1, \tag{A.13}$$

$$W = \bigoplus_{\ell=1}^{n_2+n_1} \mathbb{C}\{w_\ell\}; \quad w_{n-\ell+1} = \mathbf{e}_{(\ell-1 \bmod 2)+1}^T \lambda^{\lfloor(\ell-1)/2\rfloor}, \quad 1 \leq \ell \leq n. \tag{A.14}$$

The basis of $W = C_-[\mathcal{H}_+ \mathbf{J}]$ is obtained by noticing the vector space is the same as $C_-[\mathcal{H}_+ G_\infty^{-1} \mathbf{J}]$ (because G_∞ is (formally) analytic at $\lambda = \infty$) and hence it is the same as $C_-[\mathcal{H}_+ \lambda^{\lfloor(n+1)/2\rfloor} E_{11} + \lambda^{\lfloor n/2\rfloor} E_{22}]$. The matrix $\mathbb{G}_{k,\ell}$ representing \mathcal{G} (A.9) for $k \leq n_2$ is then given by a direct computation as

$$\mathbb{G}_{k,\ell} = \operatorname{res}_{\lambda=\infty} \operatorname{res}_{\zeta=\lambda_k} \frac{\mathbf{e}_2^T \Gamma_0^{-1}(\lambda_k)}{(\zeta - \lambda_k)(\lambda - \zeta)} G_\infty(\lambda) \lambda^{\lfloor \frac{n+1}{2} \rfloor} E_{11} + \lambda^{\lfloor \frac{n}{2} \rfloor} E_{22} \frac{\mathbf{e}_{(\ell-1 \bmod 2)+1}}{\lambda^{\lfloor(\ell-1)/2\rfloor+1}} \tag{A.15}$$

$$= \operatorname{res}_{\lambda=\infty} \frac{\mathbf{e}_2^T \Gamma_0^{-1}(\lambda_k)}{(\lambda - \lambda_k)} G_\infty(\lambda) \lambda^{\lfloor \frac{n+1}{2} \rfloor} E_{11} + \lambda^{\lfloor \frac{n}{2} \rfloor} E_{22} \frac{\mathbf{e}_{(\ell-1 \bmod 2)+1}}{\lambda^{\lfloor(\ell-1)/2\rfloor+1}} \tag{A.16}$$

A similar computation yields the rest of formula (3.13). Since we are interested in the determinant of \mathbb{G} (up to multiplicative constants), we rearrange the basis in W ; then the matrix can be written more transparently as

$$\mathbb{G}_{k,\ell} = \operatorname{res}_{\lambda=\infty} \frac{\lambda^{\lfloor \frac{\ell-1}{2} \rfloor} \mathbf{e}_{\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}}^T \Gamma_0^{-1} \left(\begin{Bmatrix} \lambda_k \\ \mu_{k-n_2} \end{Bmatrix} \right) G_\infty(\lambda) \mathbf{e}_{(\ell-1 \bmod 2)+1}}{\left(\lambda - \begin{Bmatrix} \lambda_k \\ \mu_{k-n_2} \end{Bmatrix} \right)} \tag{A.17}$$

where for brevity the notation $\{ \}$ denotes two choices, according to the cases $k \leq n_2$ (top) or $k \geq n_2 + 1$ (bottom).

Variations of $\det \mathbb{G}$. It was shown in Appendix B of [4], Theorem B.1, that (translating to the current setting)

$$\begin{aligned} \partial \ln \det \mathbb{G} &= \oint_{\partial \mathbb{D}} \operatorname{Tr} \left(\mathbf{R}^{-1} \mathbf{R}' \partial \mathbf{J} \mathbf{J}^{-1} \right) \frac{d\lambda}{2i\pi} + \\ &+ \sum_{k=1}^{n_2} \oint_{\partial \mathbb{D}_k} \operatorname{Tr} \left(\Gamma_0^{-1} \Gamma_0' \partial (\lambda_k - \lambda)^{E_{22}} (\lambda_k - \lambda)^{-E_{22}} \right) \frac{d\lambda}{2i\pi} \\ &+ \sum_{k=1}^{n_1} \oint_{\partial \mathbb{D}_{n_2+k}} \operatorname{Tr} \left(\Gamma_0^{-1} \Gamma_0' \partial (\mu_k - \lambda)^{E_{11}} (\mu_k - \lambda)^{-E_{11}} \right) \frac{d\lambda}{2i\pi} \\ &= \oint_{\partial \mathbb{D}} \operatorname{Tr} \left(\mathbf{R}^{-1} \mathbf{R}' \partial \mathbf{J} \mathbf{J}^{-1} \right) \frac{d\lambda}{2i\pi} + \sum_{\substack{\zeta \in \bar{\lambda}, \bar{\mu} \\ \lambda = \zeta}} \operatorname{res} \operatorname{Tr} \left(\Gamma_0^{-1} \Gamma_0' \partial D D^{-1} \right), \end{aligned} \tag{A.18}$$

where ∂ means any derivative of the λ_k 's, μ_k 's or x and in the last step we have used the fact that

$$\frac{\partial \sqrt{\lambda_k}}{\sqrt{\lambda_k} - \sqrt{\lambda}} = \frac{\partial \lambda_k}{\lambda_k - \lambda} + \mathcal{O}(1), \quad \lambda \rightarrow \lambda_k, \tag{A.19}$$

and similarly for μ_k .

Proof of Theorem 3.2. We use the trivial algebra below with $\Gamma_n = R\Gamma_0 D$ and $M_n = D^{-1}MD$:

$$\begin{aligned} \Gamma_n^{-1}\Gamma'_n &= D^{-1}\Gamma_0^{-1}R^{-1}R'\Gamma_0 D + D^{-1}\Gamma_0^{-1}\Gamma'_0 D + D^{-1}D', \\ \partial(D^{-1}MD)D^{-1}M^{-1}D &= D^{-1}\partial M M^{-1}D - \partial D D^{-1} + D^{-1}M D^{-1}\partial D M^{-1}D, \\ \Gamma_{n,-}\partial M_n M_n^{-1}\Gamma_{n,-}^{-1} &= \partial\Gamma_{n,+}\Gamma_{n,+}^{-1} - \partial\Gamma_{n,-}\Gamma_{n,-}^{-1}. \end{aligned} \tag{A.20}$$

Plugging into the integrand of (3.10) and simplifying, using the cyclicity of the trace several times and (A.20), we find (below Δ denotes the jump operator $\Delta(f) = f_+ - f_-$):

$$\begin{aligned} &\text{Tr} \left[\Gamma_{n,-}^{-1}\Gamma'_{n,-}\partial(D^{-1}MD)D^{-1}M^{-1}D \right] = \\ &= \text{Tr} \left[\Gamma_{0,-}^{-1}\Gamma'_{0,-}\partial M M^{-1} + R^{-1}R'\Delta \left(\partial(\Gamma_0 D)D^{-1}\Gamma_0^{-1} \right) + \Delta \left(\Gamma_0^{-1}\Gamma'_0\partial D D^{-1} \right) + \right. \\ &\quad \left. -M^{-1}M'\partial D D^{-1} + D^{-1}D' \left(\partial M M^{-1} - \partial D D^{-1} + M D^{-1}\partial D M^{-1} \right) \right]. \end{aligned} \tag{A.21}$$

Given the particular triangularity of the jump matrices M , all terms on the last line of (A.21) are traceless and thus drop out.

Now, if we have $\int_{\Sigma} \Delta F \frac{dz}{2i\pi}$ and F has some poles outside of Σ then this reduces, by the Cauchy theorem, to the sum of the residues of F . We are thus left with

$$\begin{aligned} &\Omega(\partial; [M_n]) - \Omega(\partial; [M]) \\ &= \int_{\Sigma} \text{Tr} \left(\Gamma_n^{-1}\Gamma'_n\partial M_n M_n^{-1} \right) \frac{d\lambda}{2i\pi} \\ &\quad - \int_{\Sigma} \text{Tr} \left(\Gamma_0^{-1}\Gamma'_0\partial M M^{-1} \right) \frac{d\lambda}{2i\pi} \end{aligned} \tag{A.22}$$

$$\begin{aligned} &= \oint_{\partial\mathbb{D}} \text{Tr} \left(R^{-1}R'\partial(\Gamma_0 D)D^{-1}\Gamma_0^{-1} \right) \frac{d\lambda}{2i\pi} \\ &\quad + \sum_{\substack{\text{res} \\ \lambda=\zeta \\ \zeta \in \bar{\lambda}, \bar{\mu}}} \text{Tr} \left(\Gamma_0^{-1}\Gamma'_0\partial D D^{-1} \right) \end{aligned} \tag{A.23}$$

$$\begin{aligned} &= \oint_{\partial\mathbb{D}} \text{Tr} \left(\mathbf{R}^{-1}\mathbf{R}'\partial\tilde{\mathbf{J}}\tilde{\mathbf{J}}^{-1} \right) \frac{d\lambda}{2i\pi} \\ &\quad + \sum_{\substack{\text{res} \\ \lambda=\zeta \\ \zeta \in \bar{\lambda}, \bar{\mu}}} \text{Tr} \left(\Gamma_0^{-1}(\lambda)\Gamma'_0(\lambda)\partial D(\lambda)D^{-1}(\lambda) \right) \end{aligned} \tag{A.24}$$

$$\stackrel{(A.18)}{=} \partial \ln \det \mathbb{G} + \oint_{\partial\mathbb{D}} \text{Tr} \left(\mathbf{R}^{-1}\mathbf{R}'(\partial\tilde{\mathbf{J}}\tilde{\mathbf{J}}^{-1} - \partial\mathbf{J}\mathbf{J}^{-1}) \right) \frac{d\lambda}{2i\pi}. \tag{A.25}$$

Note that we can substitute R with \mathbf{R} in (A.24), because these two matrices differ by a left multiplication with a λ -independent matrix, and the expression (A.24) is invariant under this operation. The matrix $\mathbf{J}(\lambda)$ is read off the above formula and is given by

$$\tilde{\mathbf{J}}(\lambda) = \begin{cases} \Gamma_0(\lambda)D(\lambda) & \lambda \in \partial\mathbb{D}_k \\ \Gamma_0(\lambda)D(\lambda)\frac{1-i\sigma_1}{\sqrt{2}}\lambda^{\frac{\sigma_3}{4}} & \lambda \in \partial\mathbb{D}_{\infty}. \end{cases} \tag{A.26}$$

Therefore the matrix $\tilde{\mathbf{J}}$ differs from \mathbf{J} (A.4) only on the boundaries of the finite disks \mathbb{D}_k , $k = 1, \dots, n$ by the factor $T(\lambda)$ given by the diagonal matrix below

$$\tilde{\mathbf{J}}(\lambda) = \mathbf{J}(\lambda)T(\lambda), \quad T(\lambda) = D(\lambda) \prod_{k=1}^{n_2} (\lambda_k - \lambda)^{-E_{22}\chi_{\mathbb{D}_k}} \prod_{k=1}^{n_1} (\mu_k - \lambda)^{-E_{11}\chi_{\mathbb{D}_{n_2+k}}}, \tag{A.27}$$

where χ_X denotes the indicator function of the set X . Note that $T(\lambda)$ belongs to $\mathcal{H}_+(\mathbb{D}_k)$, $\forall k$.

We now follow the exact same steps as in the proof of Theorem B.2 of [4] and have

$$\begin{aligned} & \oint_{\partial\mathbb{D}} \text{Tr} \left(\mathbf{R}^{-1}\mathbf{R}'(\partial\tilde{\mathbf{J}}\tilde{\mathbf{J}}^{-1} - \partial\mathbf{J}\mathbf{J}^{-1}) \right) \frac{d\lambda}{2i\pi} \\ &= - \sum_{k=1}^{n_2} \text{res}_{\lambda=\lambda_k} \text{Tr} \left(\frac{E_{22}}{\lambda - \lambda_k} \partial T T^{-1} \right) - \sum_{k=1}^{n_1} \text{res}_{\lambda=\mu_k} \text{Tr} \left(\frac{E_{11}}{\lambda - \mu_k} \partial T T^{-1} \right) \\ &= \sum_{k=1}^{n_2} \left(\frac{\partial\lambda_k}{4\lambda_k} - \sum_{j \neq k} \frac{\partial\sqrt{\lambda_j}}{\sqrt{\lambda_j} - \sqrt{\lambda_k}} \right) + \sum_{k=1}^{n_1} \left(\frac{\partial\mu_k}{4\mu_k} - \sum_{j \neq k} \frac{\partial\sqrt{\mu_j}}{\sqrt{\mu_j} - \sqrt{\mu_k}} \right) \\ &\quad - \sum_{k=1}^{n_2} \sum_{j=1}^{n_1} \frac{\partial(\sqrt{\mu_j} + \sqrt{\lambda_k})}{\sqrt{\mu_j} + \sqrt{\lambda_k}} \\ &= \partial \ln \frac{\prod_{j=1}^{n_2} \lambda_j^{\frac{1}{4}} \prod_{j=1}^{n_1} \mu_j^{\frac{1}{4}}}{\prod_{j < k \leq n_2} (\sqrt{\lambda_j} - \sqrt{\lambda_k}) \prod_{j < k \leq n_1} (\sqrt{\mu_j} - \sqrt{\mu_k}) \prod_{j=1}^{n_1} \prod_{k=1}^{n_2} (\sqrt{\mu_j} + \sqrt{\lambda_k})} \\ &=: \partial \ln \Delta(\vec{\lambda}, \vec{\mu}). \end{aligned} \tag{A.28}$$

Combining (A.28) with (A.25) we obtain

$$\begin{aligned} & \Omega(\partial; [M_n]) - \Omega(\partial; [M]) \stackrel{(A.25)}{=} \partial \ln \det \mathbb{G} \\ & \quad + \oint_{\partial\mathbb{D}} \text{Tr} \left(\mathbf{R}^{-1}\mathbf{R}'(\partial\tilde{\mathbf{J}}\tilde{\mathbf{J}}^{-1} - \partial\mathbf{J}\mathbf{J}^{-1}) \right) \frac{d\lambda}{2i\pi} \\ & \stackrel{(A.18)}{=} \partial \ln \left(\det \mathbb{G} \Delta(\vec{\lambda}, \vec{\mu}) \right) \end{aligned} \tag{A.29}$$

The proof is complete. \square

B. Explicit Computation of Z_n

For the benefit of the reader we derive (1.2) (in a way that is slightly different from [18]) as follows. Let dU be the Haar measure on $U(n)$, $S = \text{diag}(s_1, \dots, s_n)$ and the s_j 's the eigenvalues of M , and $\Delta(S) = \prod_{j < k} (s_j - s_k)$. Considering the numerator of (1.1), and setting $\Lambda = Y^2$ we have

$$\begin{aligned}
 & \int_{H_n} dM e^{\text{Tr} \left(i \frac{M^3}{3} - Y M^2 + i x M \right)} \stackrel{\star}{=} e^{\frac{2}{3} \text{Tr} Y^3 + x \text{Tr} Y} \int_{H_n} dM e^{i \text{Tr} \left(\frac{M^3}{3} + (Y^2 + x) M \right)} \\
 & \stackrel{\text{Weyl integration formula}}{=} C_n e^{\frac{2}{3} \text{Tr} Y^3 + x \text{Tr} Y} \int_{\mathbb{R}^n} \Delta^2(S) \prod_{j=1}^n e^{\frac{is^3}{3} + is_j x} ds_j \int_{U(n)} dU e^{i \text{Tr} (\Lambda U S U^\dagger)} \\
 & \stackrel{\text{(Harish-Chandra)}}{=} \tilde{C}_n e^{\frac{2}{3} \text{Tr} Y^3 + x \text{Tr} Y} \int_{\mathbb{R}^n} \frac{\Delta(S) \det [e^{is_j \lambda_k}]_{j,k \leq n}}{\Delta(\Lambda)} \prod_{j=1}^n e^{\frac{is^3}{3} + is_j x} ds_j \\
 & \stackrel{\text{Andreief}}{=} \frac{\tilde{C}_n n! e^{\frac{2}{3} \text{Tr} Y^3 + x \text{Tr} Y}}{\Delta(\Lambda)} \det \left[\int_{\mathbb{R}} s^{j-1} e^{\frac{is^3}{3} + is(\lambda_k + x)} ds \right]_{j,k \leq n} \\
 & = \tilde{C}_n n! (2\pi)^n e^{\frac{2}{3} \text{Tr} Y^3 + x \text{Tr} Y} \frac{\det [\text{Ai}^{(j-1)}(\lambda_k + x)]_{j,k \leq n}}{\Delta(\Lambda)} \tag{B.1}
 \end{aligned}$$

where C_n, \tilde{C}_n are proportionality constants (depending only on n) of no present interest (it turns out that $\tilde{C}_n = \frac{\pi^{\frac{n}{2}(n-1)}}{n!}$). In the step marked with \star we have performed a shift $M \mapsto M - iY$ and an analytic continuation; the integral is now only conditionally convergent and it can be understood as absolutely convergent integration on $H_n + i\epsilon \mathbf{1}$, $\epsilon > 0$. Recall now that

$$\int_{H_n} dM e^{-\text{Tr} (Y M^2)} = \frac{\pi^{\frac{n}{2} + \frac{n}{2}(n-1)}}{\prod_{j=1}^n \sqrt{y_j} \prod_{j < k} (y_j + y_k)} = \frac{\pi^{\frac{n^2}{2}}}{\prod_{j=1}^n \lambda_j^{\frac{1}{4}} \prod_{j < k=1}^n \sqrt{\lambda_j} + \sqrt{\lambda_k}}. \tag{B.2}$$

Thus, in total

$$Z_n(Y) = 2^n \pi^{\frac{n}{2}} e^{\frac{2}{3} \text{Tr} \Lambda^{\frac{3}{2} + x \text{Tr} \sqrt{\Lambda}}} \frac{\det [\text{Ai}^{(j-1)}(\lambda_k + x)]_{j,k \leq n} \prod_{j=1}^n (\lambda_j)^{\frac{1}{4}} \prod_{j < k} (\sqrt{\lambda_j} + \sqrt{\lambda_k})}{\Delta(\Lambda)}. \tag{B.3}$$

The overall proportionality constant is determined by observing that Z_n as defined in (1.1) tends to 1 as the eigenvalues of Y tend all to $+\infty$. The Airy function has the asymptotic behavior

$$\text{Ai}(\lambda) = \frac{e^{-\frac{2}{3} \lambda^{\frac{3}{2}}}}{2\sqrt{\pi} \lambda^{\frac{1}{4}}} (1 + \mathcal{O}(\lambda^{-\frac{3}{2}})), \quad |\arg(\lambda)| < \pi, \tag{B.4}$$

and hence

$$\det [\text{Ai}^{(j-1)}(y_k^2 + x)]_{j,k \leq n} \simeq \frac{e^{-\sum_{j=1}^n (\frac{2}{3} y_j^3 + x y_j)}}{2^n \pi^{\frac{n}{2}} \prod_{j=1}^n y_j^{\frac{1}{2}}} \prod_{j < k} (y_j - y_k) \quad \text{as } x \rightarrow \infty \tag{B.5}$$

from which the proportionality constant is deduced. The expression (1.2) follows from (B.1) by substituting $\Lambda = Y^2$ and simplifying.

Remark B.1. The above computation is carried out under the assumption that $\text{Re } y_j > 0$; to see what happens when $\text{Re } y < 0$, consider the simplest case $n = 1$ and $Y = y < 0$; then (set $x = 0$ for simplicity)

$$\frac{\int_{\ell} dM e^{i \frac{M^3}{3} - YM^2}}{\int_{\ell} dM e^{-YM^2}} = e^{\frac{2}{3}y^3} \int_{\ell} e^{i \frac{s^3}{3} + isy^2} ds \tag{B.6}$$

In order to be able to interpret (B.6) as an average of $e^{i \frac{M^3}{3}}$ with respect to a Gaussian measure, we must choose the path of integration ℓ in both numerator/denominator so as to have a convergent integral and also so that the term $e^{i \frac{M^3}{3}}$ is oscillatory. The choice $\arg(s) = e^{\pm i \frac{\pi}{3}}$ is possible. But this means that the integral gives now $\text{Ai}(\omega^{\pm 1} y^2)$ rather than $\text{Ai}(y^2)$. This is the underlying reason for the definition (1.29)

C. Proof of Propotion 3.3

Denote the column vectors $G_{\infty}(\lambda) \mathbf{e}_{1,2}$ (with G_{∞} as in (3.14)) by $H_{1,2}(\lambda)$ (respectively) and the row vectors

$$\mathbf{A}_k := \begin{cases} \mathbf{e}_2^T \Gamma_0^{-1}(\lambda_k) & k \leq n_2 \\ \mathbf{e}_1^T \Gamma_0^{-1}(\mu_{k-n_2}) & n_2 + 1 \leq k \leq n_1 + n_2. \end{cases} \tag{C.1}$$

The explicit expression depends on which sector λ_k 's, μ_j 's belong to, and can be read off from (3.1). Consider the wedge of the first two columns of \mathbb{G} ;

$$\left[\mathbf{A}_k \underset{\lambda=\infty}{\text{res}} \frac{H_1}{(\lambda - \lambda_k)} \right]_{k=1}^n \wedge \left[\mathbf{A}_k \underset{\lambda=\infty}{\text{res}} \frac{H_2}{(\lambda - \lambda_k)} \right]_{k=1}^n. \tag{C.2}$$

Here and below, $[\dots]_{k=1}^n$ denotes column vectors indexed by k . Depending on the parity of n and using (A.11), (A.12), we have (the symbol \star denotes a constant that eventually drops out of the computation, hence irrelevant)

$$\begin{aligned} \underset{\lambda=\infty}{\text{res}} \frac{H_1}{\lambda - \lambda_k} &= \underset{\lambda=\infty}{\text{res}} \frac{H_1}{\lambda} = \begin{cases} \mathbf{e}_1 + \star \mathbf{e}_2 & n \equiv 0 \pmod{2}, \\ -i \mathbf{e}_2 & n \equiv 1 \pmod{2}, \end{cases} \\ \underset{\lambda=\infty}{\text{res}} \frac{H_2}{\lambda - \lambda_k} &= \underset{\lambda=\infty}{\text{res}} \frac{H_2}{\lambda} = \begin{cases} \mathbf{e}_2 & n \equiv 0 \pmod{2}, \\ i \mathbf{e}_1 + \star \mathbf{e}_2 & n \equiv 1 \pmod{2}. \end{cases} \end{aligned} \tag{C.3}$$

Therefore, for any parity of n (up to an inessential sign), we have

$$\left[\mathbf{A}_k \underset{\lambda=\infty}{\text{res}} \frac{H_1}{(\lambda - \lambda_k)} \right]_{k=1}^n \wedge \left[\mathbf{A}_k \underset{\lambda=\infty}{\text{res}} \frac{H_2}{(\lambda - \lambda_k)} \right]_{k=1}^n = \pm [(\mathbf{A}_k)_1]_{k=1}^n \wedge [(\mathbf{A}_k)_2]_{k=1}^n. \tag{C.4}$$

Consider now the third and fourth columns: for the third one we have

$$\begin{aligned} \left[\mathbf{A}_k \underset{\lambda=\infty}{\text{res}} \frac{\lambda H_1}{(\lambda - \lambda_k)} \right]_{k=1}^n &= \left[\lambda_k \mathbf{A}_k \underset{\lambda=\infty}{\text{res}} \frac{H_1}{(\lambda - \lambda_k)} \right]_{k=1}^n + \left[\mathbf{A}_k \underset{\lambda=\infty}{\text{res}} \frac{(\lambda - \lambda_k) H_1}{(\lambda - \lambda_k)} \right]_{k=1}^n \\ &= \left[\lambda_k \mathbf{A}_k \underset{\lambda=\infty}{\text{res}} \frac{H_1}{\lambda} \right]_{k=1}^n + \left[\mathbf{A}_k \underset{\lambda=\infty}{\text{res}} H_1 \right]_{k=1}^n \end{aligned} \tag{C.5}$$

and similarly for the fourth

$$\left[\mathbf{A}_k \operatorname{res}_{\lambda=\infty} \frac{\lambda H_2}{(\lambda - \lambda_k)} \right]_{k=1}^n = \left[\lambda_k \mathbf{A}_k \operatorname{res}_{\lambda=\infty} \frac{H_2}{\lambda} \right]_{k=1}^n + \left[\mathbf{A}_k \operatorname{res}_{\lambda=\infty} H_2 \right]_{k=1}^n. \tag{C.6}$$

The last terms in (C.5), (C.6), respectively, are in the span on the first two columns appearing in (C.4) and hence can be dropped. The first two term are in the span of

$$[\lambda_k(\mathbf{A}_k)_1]_{k=1}^n \wedge [\lambda_k(\mathbf{A}_k)_2]_{k=1}^n. \tag{C.7}$$

Proceeding this way by induction we arrive at

$$\det \mathbb{G} = \bigwedge_{r=1}^n \left[\lambda_k^{\lfloor \frac{r-1}{2} \rfloor} \mathbf{A}_k \operatorname{res}_{\lambda=\infty} \frac{H_{2-(r \bmod 2)}}{\lambda} \right]_{k=1}^n. \tag{C.8}$$

To have more compact formulae we denote $f_k := \mathbf{Ai}_{s_k}(\lambda_k + x)$ and $g_k := \mathbf{Ai}_{s_k}(\mu_k + x)$, where s_k is the sector to which λ_k (or μ_k) belongs as indicated in the statement of the proposition and which follows from (3.1) and (C.1). Then, depending on the parity of n , we have explicitly, for even n :

$$\det \mathbb{G} = (-2i\pi)^{\frac{n}{2}} e^{\left(\sum_{j=1}^{n/2} \left(\frac{2}{3} \lambda_j^{\frac{3}{2}} + x \lambda_j^{\frac{1}{2}} \right) - \sum_{\ell=1}^{n/2} \left(\frac{2}{3} \mu_\ell^{\frac{3}{2}} + x \mu_\ell^{\frac{1}{2}} \right) \right)}$$

$$\det \begin{bmatrix} \left[\begin{array}{c|c|c|c} f_k & f'_k & \cdots & \lambda_k^{\frac{n}{2}} f_k \end{array} \right]_{k=1}^{n/2} \\ \left[\begin{array}{c|c|c|c} g_k & g'_k & \cdots & \mu_k^{\frac{n}{2}} g_k \end{array} \right]_{k=1}^{n/2} \end{bmatrix}$$

and, for odd n ,

$$\det \mathbb{G} = (-2i\pi)^{\frac{n}{2}} e^{\left(\sum_{j=1}^{n/2} \left(\frac{2}{3} \lambda_j^{\frac{3}{2}} + x \lambda_j^{\frac{1}{2}} \right) - \sum_{\ell=1}^{n/2} \left(\frac{2}{3} \mu_\ell^{\frac{3}{2}} + x \mu_\ell^{\frac{1}{2}} \right) \right)}$$

$$\det \begin{bmatrix} \left[\begin{array}{c|c|c|c} f_k & f'_k & \cdots & \lambda_k^{\frac{n-1}{2}} f_k \end{array} \right]_{k=1}^{n/2} \\ \left[\begin{array}{c|c|c|c} g_k & g'_k & \cdots & \mu_k^{\frac{n-1}{2}} g_k \end{array} \right]_{k=1}^{n/2} \end{bmatrix}.$$

Using the differential equation for the Airy functions repeatedly (all f_k, g_k 's solve the same Airy ODE), and further elementary (triangular) column operations, we see easily that in all cases (up to an inessential sign) we obtain the formula (3.15). \square

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