Commun. Math. Phys. 351, 201–247 (2017)
Digital Object Identifier (DOI) 10.1007/s00220-017-2843-8

Communications in Mathematical Physics

Global Well-Posedness of the Euler–Korteweg System for Small Irrotational Data

Corentin Audiard[1,2], **Boris Haspot**[3]

[1] Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, 75005 Paris, France
[2] CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, 75005 Paris, France.
E-mail: corentin.audiard@upmc.fr
[3] Ceremade, UMR CNRS 7534, Université Paris Dauphine, PSL Research University, Place du Maréchal De Lattre De Tassigny, 75775 Paris Cedex 16, France.
E-mail: haspot@ceremade.dauphine.fr

Received: 24 February 2016 / Accepted: 9 December 2016
Published online: 11 February 2017 –

Abstract: The Euler–Korteweg equations are a modification of the Euler equations that take into account capillary effects. In the general case they form a quasi-linear system that can be recast as a degenerate Schrödinger type equation. Local well-posedness (in subcritical Sobolev spaces) was obtained by Benzoni–Danchin–Descombes in any space dimension, however, except in some special case (semi-linear with particular pressure) no global well-posedness is known. We prove here that under a natural stability condition on the pressure, global well-posedness holds in dimension $d \geq 3$ for small irrotational initial data. The proof is based on a modified energy estimate, standard dispersive properties if $d \geq 5$, and a careful study of the structure of quadratic nonlinearities in dimension 3 and 4, involving the method of space time resonances.

Résumé Les équations d'Euler–Korteweg sont une modification des équations d'Euler prenant en compte l'effet de la capillarité. Dans le cas général elles forment un système quasi-linéaire qui peut se reformuler comme une équation de Schrödinger dégénérée. L'existence locale de solutions fortes a été obtenue par Benzoni–Danchin–Descombes en toute dimension, mais sauf cas très particuliers il n'existe pas de résultat d'existence globale. En dimension au moins 3, et sous une condition naturelle de stabilité sur la pression on prouve que pour toute donnée initiale irrotationnelle petite, la solution est globale. La preuve s'appuie sur une estimation d'énergie modifiée. En dimension au moins 5 les propriétés standard de dispersion suffisent pour conclure tandis que les dimensions 3 et 4 requièrent une étude précise de la structure des nonlinéarités quadratiques pour utiliser la méthode des résonances temps espaces.

Contents

1. Introduction

The compressible Euler–Korteweg equations read

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \ (x,t) \in \mathbb{R}^d \times I, \\ \partial_t u + u \cdot \nabla u + \nabla g(\rho) = \nabla\Big(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2 \Big), \ (x,t) \in \mathbb{R}^d \times I, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), \ x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

Here ρ is the density of the fluid, u the velocity, g the bulk chemical potential, related to the pressure by $p'(\rho) = \rho g'(\rho)$. $K(\rho) > 0$ corresponds to the capillary coefficient. On the left hand side we recover the Euler equations, while the right hand side of the second equation contains the so called Korteweg tensor, which is intended to take into account capillary effects and models, in particular the behavior at the interfaces of a liquid-vapor mixture. The system arises in various settings: the case $K(\rho) = \kappa/\rho$ corresponds to the so-called equations of quantum hydrodynamics (which are formally equivalent to the Gross–Pitaevskii equation through the Madelung transform; on this topic see the survey of Carles et al. [10]).

As we will see, in the irrotational case the system can be reformulated as a quasilinear Schrödinger equation; this is in sharp contrast with the non homogeneous incompressible case where the system is hyperbolic (see [9]). For a general $K(\rho)$, local well-posedness was proved in [6]. Moreover, (1.1) has a rich structure with special solutions such as planar traveling waves, namely solutions that only depend on $y = t - x \cdot \xi$, $\xi \in \mathbb{R}^d$, with possibly $\lim_\infty \rho(y) \neq \lim_{-\infty} \rho(y)$. The orbital stability and instability of such solutions has been largely studied over the last ten years (see [7] and the review article of Benzoni-Gavage [8]). The existence and non uniqueness of global non dissipative weak solutions,[1] in the spirit of De Lellis–Szekelehidi [12], was tackled by Donatelli et al. [13], while weak-strong uniqueness has been very recently studied by Giesselman et al. [18].

[1] These global weak solutions do not verify the energy inequality.

Our article deals with a complementary issue, namely the global well-posedness and asymptotically linear behaviour of small smooth solutions near the constant state $(\rho, u) = (\overline{\rho}, 0)$. To our knowledge, we obtain here the first global well-posedness result for (1.1) in the case of a general pressure and capillary coefficient. This is in strong contrast with the existence of infinitely many *weak* solutions from [13].

A precise statement of our results is provided in Theorems 2.1, 2.2 of Sect. 2, but first we will briefly discuss the state of well-posedness theory, the structure of the equation, and the tools available to tackle the problem. Let us start with the local well-posedness result from [6].

Theorem 1.1. *For $d \geq 1$, let $(\overline{\rho}, \overline{u})$ be a smooth solution whose derivatives decay rapidly at infinity, and $s > 1 + d/2$. Then for $(\rho_0, u_0) \in (\overline{\rho}, \overline{u}) + H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$, ρ_0 bounded away from 0, there exists $T > 0$ and a unique solution (ρ, u) of (1.1) such that $(\rho - \overline{\rho}, u - \overline{u})$ belongs to $C([0, T], H^{s+1} \times H^s) \cap C^1([0, T], H^{s-1} \times H^{s-2})$ and ρ remains bounded away from 0 on $[0, T] \times \mathbb{R}^d$.*

We point out that [6] includes local well-posedness results for initial data that are not perturbations of constants (see theorem 6.1 in [6]). The authors also obtained several blow-up criterion. In the irrotational case it reads:

Blow-up criterion: for $s > 1+d/2$, (ρ, u) solution on $[0, T) \times \mathbb{R}^d$ of (1.1), the solution can be continued beyond T provided

1. $\rho([0, T) \times \mathbb{R}^d) \subset J \subset \mathbb{R}^{+*}$, J compact and K is smooth on a neighbourhood of J.
2. $\int_0^T (\|\Delta\rho(t)\|_\infty + \|\mathrm{div} u(t)\|_\infty)dt < \infty$.

These results relied on energy estimates for an extended system that we write now. If $\mathcal{L}$ is a primitive of $\sqrt{K/\rho}$, setting $L = \mathcal{L}(\rho)$, $w = \sqrt{K/\rho}\nabla\rho = \nabla L$, $a = \sqrt{\rho K(\rho)}$, from basic computations we verify (see [6]) that the equations on (L, u, w) are

$$\begin{cases} \partial_t L + u \cdot \nabla L + a\mathrm{div} u = 0, \\ \partial_t u + u \cdot \nabla u - w \cdot \nabla w - \nabla(a\mathrm{div} w) = -\nabla g, \\ \partial_t w + \nabla(u \cdot w) + \nabla(a\mathrm{div} w) = 0, \end{cases}$$

or equivalently for $z = u + iw$

$$\begin{cases} \partial_t L + u \cdot \nabla L + a\mathrm{div} u = 0, \\ \partial_t z + u \cdot \nabla z + i(\nabla z) \cdot w + i\nabla(a\mathrm{div} z) = \nabla\widetilde{g}(L). \end{cases} \tag{1.2}$$

Here we set $\widetilde{a}(L) = a \circ \mathcal{L}^{-1}(L)$, $\widetilde{g}(L) = g \circ \mathcal{L}^{-1}(L)$ which are well-defined since $\sqrt{K/\rho} > 0$ thus $\mathcal{L}$ is invertible.

This change of unknown clarifies the underlying dispersive structure of the model as the second equation is a quasi-linear degenerate Schrödinger equation. It should be pointed out however that the local existence results of [6] relied on H^s energy estimates rather than dispersive estimates. On the other hand, we constructed recently in [4] global small solutions to (1.1) for $d \geq 3$ when the underlying system is *semilinear*, that is $K(\rho) = \kappa/\rho$ with κ a positive constant and for $g(\rho) = \rho - 1$. This case corresponds to the equations of quantum hydrodynamics. The construction relied on the so-called Madelung transform, which establishes a formal correspondence between these equations and the Gross–Pitaevskii equation, and recent results on scattering for the Gross–Pitaevskii equation [20,22]. Let us recall for completeness that $1 + \psi$ is a solution of the Gross–Pitaevskii equation if ψ satisfies

$$i\partial_t\psi + \Delta\psi - 2\mathrm{Re}(\psi) = \psi^2 + 2|\psi|^2 + |\psi|^2\psi. \tag{1.3}$$

For the construction of global weak solutions (no uniqueness, but no smallness assumptions) we refer also to the work of Antonelli–Marcati [1,2].

In this article we consider perturbations of the constant state $\rho = \rho_c,\ u = 0$ for a general capillary coefficient $K(\rho)$ that we only suppose smooth and positive on an interval containing ρ_c. In order to exploit the dispersive nature of the equation we need to work with irrotational data $u = \nabla\phi$ so that (1.2) reduces to the following system (where $L_c = \mathcal{L}(\rho_c)$ which has obviously similarities with (1.3) (more details are provided in Sects. 3 and 4):

$$\begin{cases} \partial_t\phi - \Delta(L - L_c) + \widetilde{g}'(L_c)(L - L_c) = \mathcal{N}_1(\phi, L), \\ \partial_t(L - L_c) + \Delta\phi = \mathcal{N}_2(\phi, L). \end{cases} \tag{1.4}$$

The system satisfies the dispersion relation $\tau^2 = |\xi|^2(\widetilde{g}'(L_c) + |\xi|^2)$, and the $\mathcal{N}_j$ are at least quadratic nonlinearities that depend on L, ϕ and their derivatives (the system is thus quasi-linear). We also point out that the stability condition $\widetilde{g}'(L_c) \geq 0$ is necessary in order to ensure that the solutions in τ of the dispersion relation are real.

The existence of global small solutions for nonlinear dispersive equations is a rather classical topic that is impossible by far to describe exhaustively in this introduction. We shall restrict the discussion to the main ideas that are important for our work here.

Dispersive estimates. For the Schrödinger equation, two key tools are the dispersive estimate

$$\|e^{it\Delta}\psi_0\|_{L^q(\mathbb{R}^d)} \lesssim \frac{\|\psi_0\|_{L^{q'}}}{t^{d(1/2-1/q)}}, \tag{1.5}$$

and the Strichartz estimates

$$\|e^{it\Delta}\psi_0\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))} \lesssim \|\psi_0\|_{L^2}, \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \tag{1.6}$$

$$\|\int_0^t e^{i(t-s)\Delta} f(s)ds\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))} \lesssim \|f\|_{L^{p_1'}(\mathbb{R},L^{q_1'}(\mathbb{R}^d))}, \quad \frac{2}{p_1} + \frac{d}{q_1} = \frac{d}{2}. \tag{1.7}$$

Both indicate decay of the solution for long time in $L^p(L^q)$ spaces, it is of course of interest when we wish to prove the existence of global strong solution since it generally requires some damping behavior for long time. Due to the pressure term the linear structure of our system is actually closer to the one of the Gross–Pitaevskii equation (see (1.3)), but the estimates are essentially the same as for the Schrödinger equation. Local smoothing is also an interesting feature of Schrödinger equations, in particular for the study of quasilinear systems. A result in this direction was obtained by the first author in [3] but we will not need it here. The main task of our proof will consist in proving dispersive estimates of the type (1.5) for long time, it is related to the notion of scattering for solutions of dispersive equations. Let us recall now some classical result on the theory of the scattering for the Schrödinger equations and the Gross Pitaevskii equation.

Scattering. Let us consider the following nonlinear Schrödinger equation

$$i\partial_t\psi + \Delta\psi = \mathcal{N}(\psi).$$

Due to the dispersion, when the nonlinearity vanishes at a sufficient order at 0 and the initial data is sufficiently small and localized, it is possible to prove that the solution is global and the integral $\int e^{-is\Delta}\mathcal{N}(\psi(s))ds$ converges in $L^2(\mathbb{R}^d)$, so that there exists $\psi_+ \in L^2(\mathbb{R}^d)$ such that

$$\|\psi(t) - e^{it\Delta}\psi_+\|_{L^2} \longrightarrow_{t\to\infty} 0.$$

In this case, it is said that the solution is asymptotically linear, or *scatters* to ψ_+.

In the case where $\mathcal{N}$ is a general power-like non-linearity, we can cite the seminal work of Strauss [27]. More precisely if $\mathcal{N}(a) = O_0(|a|^p)$, global well-posedness for small data in H^1 is merely a consequence of Strichartz estimates provided p is larger than the so-called Strauss exponent

$$p_S(d) = \frac{\sqrt{d^2+12d+4}+d+2}{2d}. \tag{1.8}$$

For example scattering for quadratic nonlinearities (independently of their structure ϕ^2, $\overline{\phi}^2$, $|\phi|^2$...) can be obtained for $d \geq 4$, indeed $p_S(3) = 2$. The case $p \leq p_S$ is much harder and is discussed later.

Mixing energy estimates and dispersive estimates. If $\mathcal{N}$ depends on derivatives of ϕ, due to the loss of derivatives the situation is quite different and it is important to take more precisely into account the structure of the system. In particular it is possible in some cases to exhibit energy estimates which often lead after a Gronwall lemma to the following situation:

$$\forall N \in \mathbb{N},\ \|\psi(t)\|_{H^N} \leq \|\psi_0\|_{H^N}\exp\bigg(C_N\int_0^t \|\psi(s)\|_{W^{k,\infty}}^{p-1}ds\bigg),$$
$$k \text{ ``small'' and independent on } N.$$

A natural idea consists in mixing energy estimates in the H^N norm, N "large", with dispersive estimates: if one obtains

$$\bigg\|\int_0^t e^{i(t-s)\Delta}\mathcal{N}ds\bigg\|_{W^{k,\infty}} \lesssim \frac{\sup_{[0,t]}(\|\psi(s)\|_{H^N}^p + s^\alpha\|\psi(s)\|_{W^{k,\infty}}^p)}{t^\alpha}, \quad \alpha(p-1) > 1,$$

then setting $\|\psi\|_{X_T} = \sup_{[0,T]}\big(\|\psi(t)\|_{H^N} + t^\alpha\|\psi(t)\|_{W^{k,\infty}}\big)$ and if $\|e^{it\Delta}\psi_0\|_{X_T} \leq \varepsilon << 1$ uniformly in T, the energy estimate and Duhamel formula yields

$$\|\psi\|_{X_T} \leq \|\psi_0\|_{H^N}\exp(C\|\psi\|_{X_T}^{p-1}) + C\|\psi\|_{X_T}^p + \varepsilon.$$

Therefore $\|\psi\|_{X_T}$ must remain small uniformly in T. This strategy seems to have been initiated independently by Klainerman and Ponce [24] and Shatah [25]. If the energy estimate is true, this method works "straightforwardly" and gives global well-posedness for small initial data (this is the approach from Sect. 4) if

$$p > \widetilde{p}(d) = \frac{\sqrt{2d+1}+d+1}{d} > p_S(d). \tag{1.9}$$

Again, there is a critical dimension: $\widetilde{p}(4) = 2$, thus any quadratic nonlinearity can be handled with this method if $d \geq 5$.

Normal forms, space-time resonances. When $p \le p_S$ (semi-linear case) or $\widetilde{p}$ (quasi-linear case), the strategies above cannot be directly applied, and one has to look more closely at the structure of the nonlinearity. For the Schrödinger equation, one of the earliest results in this direction was due to Cohn [11] who proved (extending Shatah's method of normal forms [26]) the global well-posedness in dimension 2 of

$$i\partial_t\psi + \Delta\psi = i\nabla\overline{\psi}\cdot\nabla\overline{\psi}. \tag{1.10}$$

The by now standard strategy of proof was to use a normal form that transformed the quadratic nonlinearity into a cubic one, and since $3 > \widetilde{p}(2) \simeq 2.6$ the new equation could be treated with the arguments from [24]. In dimension 3, similar results (with very different proofs using vector fields method and time non resonance) were then obtained for the nonlinearities ψ^2 and $\overline{\psi}^2$ by Hayashi, Nakao and Naumkin [23] (it is important to observe that the quadratic nonlinearity is critical in terms of Strauss exponent for the semi-linear case when $d = 3$). The existence of global solutions for the nonlinearity $|\psi|^2$ is however still open (indeed it corresponds to a nonlinearity where the set of time and space non resonance is not empty; we will give more explanations below on this phenomenon).

More recently, Germain–Masmoudi–Shatah [14–16] and Gustafson–Nakanishi–Tsai [21,22] shed a new light on such issues with the concept of space-time resonances. To describe it, let us rewrite the Duhamel formula for the profile of the solution $f = e^{-it\Delta}\psi$, in the case (1.10):

$$\begin{aligned} f &= \psi_0 + \int_0^t e^{-is\Delta}\frac{\mathcal{N}(e^{is\Delta}f}{i})ds \\ \Leftrightarrow \widehat{f} &= \widehat{\psi_0} - \int_0^t\int_{\mathbb{R}^d} e^{is(|\xi|^2+|\eta|^2+|\xi-\eta|^2)}\eta\cdot(\xi-\eta)\widehat{\overline{f}}(\eta)\widehat{\overline{f}}(\xi-\eta)d\eta ds. \end{aligned} \tag{1.11}$$

In order to take advantage of the non cancellation of $\Omega(\xi,\eta) = |\xi|^2 + |\eta|^2 + |\xi-\eta|^2$ one might integrate by part in time, and from the identity $\partial_t f = -ie^{-it\Delta}\mathcal{N}(\psi)$, we see that this procedure effectively replaces the quadratic nonlinearity by a cubic one, ie acts as a normal form.

On the other hand, if $\mathcal{N}(\psi) = \psi^2$ the phase becomes $\Omega(\xi,\eta) = |\xi|^2 - |\eta|^2 - |\xi-\eta|^2$, which cancels on a large set, namely the "time resonant set"

$$\mathcal{T} = \{(\xi,\eta):\ \Omega(\xi,\eta) = 0\} = \{\eta \perp \xi - \eta\}. \tag{1.12}$$

The remedy is to use an integration by part in the η variable using $e^{is\Omega} = \frac{\nabla_\eta\Omega}{is|\nabla_\eta\Omega|^2}\nabla_\eta(e^{is\Omega})$, it does not improve the nonlinearity, however the factor $1/s$ is a gain in time decay. This justifies to define the "space resonant set" as

$$\mathcal{S} = \{(\xi,\eta):\ \nabla_\eta\Omega(\xi,\eta) = 0\} = \{\eta = -\xi - \eta\}, \tag{1.13}$$

as well as the space-time resonant set

$$\mathcal{R} = \mathcal{S}\cap\mathcal{T} = \{(\xi,\eta):\ \Omega(\xi,\eta) = 0,\ \nabla_\eta\Omega(\xi,\eta) = 0\}. \tag{1.14}$$

For $\mathcal{N}(\psi) = \psi^2$, we simply have $\mathcal{R} = \{\xi = \eta = 0\}$; using the previous strategy Germain et al. [16] obtained global well-posedness for the quadratic Schrödinger equation.

Finally, for $\mathcal{N}(\psi) = |\psi|^2$ similar computations lead to $\mathcal{R} = \{\xi = 0\}$, the "large" size of this set might explain why this nonlinearity is particularly difficult to handle.

Smooth and non smooth multipliers. The method of space-time resonances in the case $(\nabla\overline{\phi})^2$ is particularly simple because after the time integration by part, the Fourier transform of the nonlinearity simply becomes

$$\frac{\eta\cdot(\xi-\eta)}{|\xi|^2+|\eta|^2+|\xi-\eta|^2}\partial_s\widehat{\overline{f}}(\eta)\widehat{\overline{f}}(\xi-\eta),$$

where the multiplier $\frac{\eta\cdot(\xi-\eta)}{|\xi|^2+|\eta|^2+|\xi-\eta|^2}$ is of Coifman-Meyer type, thus in term of product laws it is just a cubic nonlinearity. We might naively observe that this is due to the fact that $\eta\cdot(\xi-\eta)$ cancels on the resonant set $\xi=\eta=0$. Thus one might wonder what happens in the general case if the nonlinearity writes as a bilinear Fourier multiplier whose symbol cancels on $\mathcal{R}$. In [14], the authors treated the nonlinear Schrödinger equation for $d=2$ by assuming that the nonlinearity is of type $B[\psi,\psi]$ or $B[\overline{\psi},\overline{\psi}]$, with B a bilinear Fourier multiplier whose symbol is linear at $|(\xi,\eta)|\leq 1$ (and thus cancels on $\mathcal{R}$). Concerning the Gross–Pitaevskii equation (1.3), the nonlinear terms include the worst one $|\psi|^2$ but Gustafson et al. [22] managed to prove global existence and scattering in dimension 3; one of the important ideas of their proof was a change of unknown $\psi\mapsto Z$ (or normal form) that replaced the nonlinearity $|\psi|^2$ by $\sqrt{-\Delta/(2-\Delta)}|Z|^2$ which compensates the resonances at $\xi=0$. To some extent, this is also a strategy that we will follow here.

Finally, let us point out that the method of space-time resonances proved to be remarkably efficient for the water wave equation [15] partially because the group velocity $|\xi|^{-1/2}/2$ is large near $\xi=0$, while it might not be the most suited for the Schrödinger equation whose group velocity 2ξ cancels at $\xi=0$. The method of vector fields is an interesting alternative, and this approach was later chosen by Germain et al. [17] to study the capillary water waves (in this case the group velocity is $3|\xi|^{1/2}/2$). Nevertheless, in our case the term $\widetilde{g}(L_c)$ in (1.4) induces a lack of symmetry which seems to limit the effectiveness of this approach.

Plan of the article. In Sect. 2, we introduce the notations and state our main results. Section 3 is devoted to the reformulation of (1.1) as a non degenerate Schrödinger equation, and we derive the energy estimates in "high" Sobolev spaces. We use a modified energy compared with [6] in order to avoid some time growth of the norms. In Sect. 4 we prove our main result in dimension at least 5. Section 5 begins the analysis of dimensions 3 and 4, which is the heart of the paper. We only detail the case $d=3$ since $d=4$ follows the same ideas with simpler computations. We first introduce the functional settings, a normal form and check that it defines an invertible change of variable in these settings, then we bound the high order terms (at least cubic). In Sect. 6 we use the method of space-time resonances (similarly to [22]) to bound quadratic terms and close the proof of global well-posedness in dimension 3. The "Appendix" provides some technical multipliers estimates required for Sect. 6.

2. Main Results, Tools and Notations

The results. As pointed out in the introduction, we need a condition on the pressure.

Assumption 2.1. Throughout all the paper, we work near a constant state $\rho=\rho_c>0,\ u=0$, with $g'(\rho_c)>0$.

In the case of the Euler equation, this standard condition implies that the linearized system

$$\begin{cases} \partial_t \rho + \rho_c \mathrm{div} u = 0, \\ \partial_t u + g'(\rho_c)\nabla\rho = 0. \end{cases}$$

is hyperbolic, with eigenvalues (sound speed) $\pm\sqrt{\rho_c g'(\rho_c)}$.

Theorem 2.1. *Let $d \geq 5$, $\rho_c \in \mathbb{R}^{+*}$, $u_0 = \nabla\phi_0$ be irrotational. For $(n, k) \in \mathbb{N}$, $k > 2 + d/4$, $2n + 1 \geq k + 2 + d/2$, there exists $\delta > 0$, such that if*

$$\|u_0\|_{H^{2n}\cap W^{k-1,4/3}} + \|\rho_0 - \rho_c\|_{H^{2n+1}\cap W^{k,4/3}} \leq \delta,$$

then the unique local solution to (1.1) of Theorem 1.1 is global, and $\rho(t)$ is bounded away from 0 uniformly in t. Moreover we have

$$\sup_{t\geq 0}\Big(\|\rho(t)-\rho_0\|_{H^{2n+1}} + \|u(t)\|_{H^{2n}} + t^{d/4}\big(\|\rho(t)-\rho_0\|_{W^{k,4}} + \|u(t)\|_{W^{k-1,4}}\big)\Big) \lesssim \delta. \tag{2.1}$$

In the other main theorem, we denote $L^2/\langle x\rangle = \{u \in L^2 : \langle x\rangle u \in L^2\}$, $\langle x\rangle = \sqrt{x^2+1}$.

Theorem 2.2. *Let $d = 3$ or 4, $u_0 = \nabla\phi_0$ irrotational, $k > 2 + d/4$, there exists $\delta > 0$, $n \in \mathbb{N}$ large, $\varepsilon > 0$ small, such that for $\frac{1}{p} = \frac{1}{2} - \frac{1}{d} - \varepsilon$, $p' = p/(p-1)$, if*

$$\begin{aligned} &\|u_0\|_{H^{2n}} + \|\rho_0 - \rho_c\|_{H^{2n+1}} + \|xu_0\|_{L^2} + \|x(\rho_0-\rho_c)\|_{L^2} \\ &+ \|u_0\|_{W^{k-1,p'}} + \|\rho_0 - \rho_c\|_{W^{k,p'}} \leq \delta, \end{aligned}$$

then the unique local solution to (1.1) from Theorem 1.1 is global and $\rho(t)$ is bounded away from 0 uniformly in t. Moreover, for $t \geq 0$, $(u, \rho - \rho_0)(t) \in (L^2/\langle x\rangle)^2$ and we have

$$\sup_{t\geq 0}\Big(\|\rho(t)-\rho_0\|_{H^{2n+1}} + \|u(t)\|_{H^{2n}} + t^{1+d\varepsilon}\big(\|\rho(t)-\rho_0\|_{W^{k,p}} + \|u(t)\|_{W^{k-1,p}}\big)\Big) \lesssim \delta.$$

Remark 2.1. Smallness in weighted spaces for the *profile* of the solution holds too, for simplicity in the statement we chose not to write it. As for the Schrödinger equation, the $W^{k,p'}$ regularity is not propagated, it is only used for the decay of the linear evolution $e^{itH}\psi_0$ (see formula (4.3)). On the other hand, our continuation argument requires *a priori* estimates for $(u, \rho - \rho_0)(t)$ in $W^{k,p} \cap L^2/\langle x\rangle)$, but it is not stated in [6] that the solution does belong to $W^{k,p}$ or in weighted spaces. The fact that (on the time of existence) $u(t) \in W^{k,p}$ is a consequence of the Sobolev embedding $H^{2n} \hookrightarrow W^{k,p}$ for n large enough, but the $\|xu\|_{L^2}$ bound requires to go back to the existence result in [6]. Let us sketch it shortly: as mentioned in the introduction, it is more convenient to solve

$$\begin{cases} \partial_t L + u\cdot\nabla L + a\mathrm{div} u = 0, \\ \partial_t z + u\cdot\nabla z + i(\nabla z)\cdot w + i\nabla(a\mathrm{div} z) = -\nabla\widetilde{g}(L). \end{cases} \tag{2.2}$$

In [6], the authors study the regularized equation

$$\partial_t z + u\cdot\nabla z + i(\nabla z)\cdot w + i\nabla(a\mathrm{div} z) + \tilde{g}'w + \varepsilon\Delta^2 z = f. \tag{2.3}$$

For $\varepsilon > 0$ fixed, the local well-posedness in H^s, s large enough, follows from a fixed point argument, the propagation of the property $\|xz\|_{L^2} < \infty$ can be done simply by including this norm in the fixed point procedure. The main issue is the existence of estimates uniform in ε. Denoting $A_\varepsilon(t) := \varepsilon\|\Delta L(t)\|_\infty + 1 + \|\nabla z(t)\|_\infty$, the authors prove the following estimate (corollary 4.2 in [6])

$$\|z\|_{L^\infty_T H^s} + \varepsilon\|\Delta z\|_{L^2_T H^s} \leq Ce^{\int_0^T A_\varepsilon dt}(1 + \|w\|^{\max(1,s)}_{L^\infty_T L^\infty_x})\big(\|z_0\|_{H^s} + \|f\|_{L^1_T H^s}\big). \quad (2.4)$$

Now for any $1 \leq i \leq d$, if z solves (2.2), $x_i z$ satisfies the equation

$$\partial_t(x_i z) + u \cdot \nabla(x_i z) + i\nabla(x_i z) \cdot w + i\nabla(a\mathrm{div}(x_i z)) + x_i \widetilde{g}' w + \varepsilon\Delta^2(x_i z) = R,$$

where R obviously does not contain any x_i factor. Estimate (2.4) with $s = 0$ gives

$$\|x_i z\|_{L^\infty_T L^2} + \varepsilon\|\Delta(x_i z)\|_{L^2_T L^2} \leq Ce^{\int_0^T A_\varepsilon dt}(1 + \|w\|_{L^\infty_T L^\infty_x})\big(\|x_i z_0\|_{L^2} + \|R\|_{L^1_T L^2}\big).$$

R contains third order derivatives of z, which is not an issue since the (non weighted) energy estimate ensures a priori bounds in H^s for s as large as needed. Similar computations are true for $x\rho$, and one can then follow the local existence procedure from [6] to construct local solutions such that $xu, x\rho \in L^\infty_T L^2$.

Finally, let us point out that rather than $xz \in L^\infty_T L^2$ our proof requires $xe^{-itH}z \in L^\infty_T L^2$, where $H = \sqrt{-\Delta(2 - \Delta)}$. This is also true, as can be seen from the commutation identity

$$x(e^{-itH}z) = e^{-itH}\left(xz - \frac{2it(1-\Delta)}{H}\nabla z\right),$$

and the boundedness of $\frac{(1-\Delta)\nabla}{H} : H^1 \to L^2$.

Remark 2.2. While the proof implies to work with the velocity potential, we only need assumptions on the physical variables velocity and density.

Remark 2.3. Actually the proof gives a stronger result: in the appropriate variables the solution scatters. The precise statement is different in dimension 3, 4 and ≥ 5. Let $\mathcal{L}$ be the primitive of $\sqrt{K/\rho}$ such that $\mathcal{L}(\rho_c) = 1$, $L = \mathcal{L}(\rho)$, $\mathcal{H} = \sqrt{-\Delta(\widetilde{g}'(1) - \Delta)}$, $\mathcal{U} = \sqrt{-\Delta/(\widetilde{g}'(1) - \Delta)}$, $f = e^{-it\mathcal{H}}(\mathcal{U}\phi + iL)$, then if $d = 3, 4$ there exists f_∞ such that

$$\forall s < 2n + 1, \quad \|f(t) - f_\infty\|_{H^s \cap (L^2/\langle x\rangle)} \longrightarrow_{t\to\infty} 0.$$

By $f \in L^2/\langle x\rangle$, we mean that $\langle x\rangle f \in L^2$, and $\|f\|_{L^2/\langle x\rangle} := \|\langle x\rangle f\|_{L^2}$. If $d \geq 5$ the convergence rate $\|f(t) - f_\infty\|_2 \lesssim t^{-d/2+1}$ holds. See Sect. 6.4 for a discussion in dimension 3.

By a careful inspection of the proof, it is also possible to quantify how large n should be in dimension 3 (at least of order 20, see Remark 6.5), and how small ε should be (at least smaller than $1/24$).

In both theorems, the size of k and n can be slightly decreased by working in fractional Sobolev spaces, but since it would remain quite large we chose to avoid these technicalities.

Some tools and notations. Most of our tools are standard analysis, except a singular multiplier estimate.

Functional spaces The usual Lebesgue spaces are L^p with norm $\|\cdot\|_p$, the Lorentz spaces are $L^{p,q}$. If $\mathbb{R}^+$ corresponds to the time variable, and for B a Banach space, we write for short $L^p(\mathbb{R}^+, B) = L^p_t B$, similarly $L^p([0,T], B) = L^p_T B$.

For $k \in \mathbb{N}$, the Sobolev spaces are $W^{k,p} = \{u \in L^p : \forall |\alpha| \leq k, \ D^\alpha u \in L^p\}$. For $kp < d$, the homogeneous spaces $\dot{W}^{k,p}$ is the closure of $\mathcal{S}(\mathbb{R}^d)$ for the norm $\sum_{|\alpha|=k} \|D^\alpha u\|_{L^p}$. We recall the Sobolev embeddings

$$\forall kp < d, \ \dot{W}^{k,p}(\mathbb{R}^d) \hookrightarrow L^{q,p} \hookrightarrow L^q, \ q = \frac{dp}{d-kq}, \ \forall kp > d, \ W^{k,p}(\mathbb{R}^d) \hookrightarrow L^\infty.$$

If $p = 2$, as usual $W^{k,2} = H^k$, for which we have equivalent norm $(\int_{\mathbb{R}^d} (1+|\xi|^2)^k |\widehat{u}|^2 d\xi)^{1/2}$, we define similarly H^s for $s \in \mathbb{R}$ and $\dot{H}^s$ for which the embeddings remain true. The Bessel potential spaces $H^{s,p}$ are defined by $\|u\|_{H^{s,p}} := \|(1-\Delta)^{s/2} u\|_{L^p} < \infty$. For s a positive integer they coincide with the usual Sobolev spaces. They satisfy the Sobolev embedding $H^{s,p} \hookrightarrow H^{s',q}$, $1/q = 1/p - (s-s')/d$. The following dual estimate will be of particular use

$$\forall d \geq 3, \ \|u\|_{\dot{H}^{-1}} \lesssim \|u\|_{L^{2d/(d+2)}}.$$

We will use the following Gagliardo-Nirenberg type inequality (see for example [28])

$$\forall l \leq p \leq k-1 \text{ integers}, \ \|D^l u\|_{L^{2k/p}} \lesssim \|u\|_{L^{2k/(p-l)}}^{(k-p)/(k+l-p)} \|D^{k+l-p} u\|_{L^2}^{l/(k+l-p)}, \quad (2.5)$$

and its consequence

$$\forall |\alpha| + |\beta| = k, \ \|D^\alpha f D^\beta g\|_{L^2} \lesssim \|f\|_\infty \|g\|_{\dot{H}^k} + \|f\|_{\dot{H}^k} \|g\|_\infty. \quad (2.6)$$

Finally, we have the basic composition estimate (see [5]): for F smooth, $F(0) = 0$, $u \in L^\infty \cap W^{k,p}$ then[2]

$$\|F(u)\|_{W^{k,p}} \lesssim C(k, \|u\|_\infty)) \|u\|_{W^{k,p}}. \quad (2.7)$$

Non standard notations Since we will often estimate indistinctly z or $\overline{z}$, we follow the notations introduced in [22]: $z^+ = z$, $z^- = \overline{z}$, and $z^\pm$ is a placeholder for z or $\overline{z}$. The Fourier transform of z is indistinctly $\widehat{z}$ or $\mathcal{F}(z)$, however we also need to consider the profile $e^{-itH} z$, whose Fourier transform will be denoted $\widetilde{z^\pm} := e^{\mp itH(\xi)} \widehat{z^\pm}$.

When there is no ambiguity, we write $W^{k,\frac{1}{p}}$ (or $L^{\frac{1}{p}}$) instead of $W^{k,p}$ (or L^p) since it is convenient to use Hölder's inequality.

Multiplier theorems The Riesz multiplier $\mathrm{Ri} := \nabla/|\nabla|$ is bounded on L^p, $1 < p < \infty$. A bilinear Fourier multiplier is defined by its symbol $B(\eta, \xi)$, it acts on $(f, g) \in \mathcal{S}(\mathbb{R}^d)$

$$\widehat{B[f,g]}(\xi) = \int_{\mathbb{R}^d} B(\eta, \xi - \eta) \widehat{f}(\eta) \widehat{g}(\xi - \eta) d\eta.$$

[2] $k \in \mathbb{R}^+$ is allowed, but not needed.

Theorem 2.3 (Coifman-Meyer). *If* $\partial_\xi^\alpha \partial_\eta^\beta B(\xi,\eta) \lesssim (|\xi|+|\eta|)^{-|\alpha|-|\beta|}$, *for sufficiently many* α,β *then for any* $1<r,p,q<\infty$, $1/r=1/p+1/q$,

$$\|B(f,g)\|_r \lesssim \|f\|_p \|g\|_q.$$

If moreover $supp(B(\eta,\xi-\eta)) \subset \{|\eta| \gtrsim |\xi-\eta|\}$, (p,q,r) *are finite and* $k\in\mathbb{N}$ *then*

$$\|\nabla^k B(f,g)\|_r \lesssim \|\nabla^k f\|_p \|g\|_q.$$

Mixing this result with the Sobolev embedding, we get for $2<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$

$$\|fg\|_{H^s} \lesssim \|f\|_{L^p}\|g\|_{H^{s,q}} + \|g\|_{L^p}\|f\|_{H^{s,q}} \lesssim \|f\|_{L^p}\|g\|_{H^{s+d/p}} + \|g\|_{L^p}\|f\|_{H^{s+d/p}}. \tag{2.8}$$

Due to the limited regularity of our multipliers, we will need a multiplier theorem with loss from [19] (inspired by corollary 10.3 from [22]). Let us first describe the norm on symbols: for χ_j a smooth dyadic partition of the space, $\text{supp}(\chi_j) \subset \{2^{j-2} \le |x| \le 2^{j+2}\}$, we set

$$\|B(\eta,\xi-\eta)\|_{\tilde{L}^\infty_\xi \dot{B}^s_{2,1,\eta}} := \|2^{js}\chi_j(\nabla_\eta)B(\eta,\xi-\eta)\|_{l^1(\mathbb{Z},L^\infty_\xi L^2_\eta)},$$

$$\|B(\xi-\zeta,\zeta)\|_{\tilde{L}^\infty_\xi \dot{B}^s_{2,1,\zeta}} := \|2^{js}\chi_j(\nabla_\zeta)B(\xi-\zeta,\zeta)\|_{l^1(\mathbb{Z},L^\infty_\xi L^2_\zeta)}.$$

The second norm is motivated by the equivalent formula for the action of a bilinear multiplier $\widehat{B[f,g]} = \int_{\mathbb{R}^d} B(\xi-\zeta,\zeta)\widehat{f}(\xi-\zeta)\widehat{g}(\zeta)d\zeta$. The relevant norm for rough multiplier estimates is

$$\|B\|_{[B^s]} = \min\Big(\|B(\eta,\xi-\eta)\|_{\tilde{L}^\infty_\xi \dot{B}^s_{2,1,\eta}},\ \|B(\xi-\zeta,\zeta)\|_{\tilde{L}^\infty_\xi \dot{B}^s_{2,1,\zeta}}\Big).$$

Theorem 2.4 [19]. *Let* $0\le s\le d/2$, q_1,q_2 *such that* $\dfrac{1}{q_2}+\dfrac{1}{2}=\dfrac{1}{q_1}+\left(\dfrac{1}{2}-\dfrac{s}{d}\right)$,[3] *and* $2\le q_1', q_2 \le \dfrac{2d}{d-2s}$, *then*

$$\|B(f,g)\|_{L^{q_1}} \lesssim \|B\|_{[B^s]}\|f\|_{L^{q_2}}\|g\|_{L^2}.$$

Furthermore for $\dfrac{1}{q_2}+\dfrac{1}{q_3}=\dfrac{1}{q_1}+\left(\dfrac{1}{2}-\dfrac{s}{d}\right)$, $2\le q_i\le \frac{2d}{d-2s}$ *with* $i=2,3$,

$$\|B(f,g)\|_{L^{q_1}} \lesssim \|B\|_{[B^s]}\|f\|_{L^{q_2}}\|g\|_{L^{q_3}}.$$

In practice, it is simpler to estimate $\|B\|_{L^\infty_\xi \dot{H}^s}$ and use the interpolation estimate (see [22])

$$\|B\|_{\tilde{L}^\infty_\xi \dot{B}^s_{2,1,\eta}} \lesssim \|B\|^\theta_{L^\infty_\xi \dot{H}^{s_1}}\|B\|^{1-\theta}_{L^\infty_\xi \dot{H}^{s_2}},\ s=\theta s_1+(1-\theta)s_2.$$

Dispersion for the group e^{-itH} According to (1.4), the linear part of the equation reads $\partial_t z - i\mathcal{H}z = 0$, with $\mathcal{H}=\sqrt{-\Delta(\widetilde{g}'(L_c)-\Delta)}$ (see also Sect. 4). With a change of variable it reduces to $\widetilde{g}'(L_c)=2$, set $H=\sqrt{-\Delta(2-\Delta)}$, and use the dispersive estimate from [20], the version in Lorentz spaces follows from real interpolation as pointed out in [22].

[3] We write the relation between (q_1,q_2) in a rather odd way in order to emphasize the similarity with the standard Hölder's inequality.

Theorem 2.5 [20,22]. *For $2 \leq p \leq \infty$, $s \in \mathbb{R}$, $U = \sqrt{-\Delta/(2-\Delta)}$, we have*

$$\|e^{itH}\varphi\|_{\dot{B}^s_{p,2}} \lesssim \frac{\|U^{(d-2)(1/2-1/p)}\varphi\|_{\dot{B}^s_{p',2}}}{t^{d(1/2-1/p)}},$$

and for $2 \leq p < \infty$

$$\|e^{itH}\varphi\|_{L^{p,2}} \lesssim \frac{\|U^{(d-2)(1/2-1/p)}\varphi\|_{L^{p',2}}}{t^{d(1/2-1/p)}}.$$

Remark 2.4. The slight low frequency gain $U^{(d-2)(1/2-1/p)}$ is due to the fact that $H(\xi) = |\xi|\sqrt{2+|\xi|^2}$ behaves like $|\xi|$ at low frequencies, which has a strong angular curvature and no radial curvature.

Remark 2.5. Combining the dispersion estimate and the celebrated TT^* argument, Strichartz estimates follow

$$\|e^{itH}\varphi\|_{L^pL^q} \lesssim \|U^{\frac{d-2}{2}(1/2-1/p)}\varphi\|_{L^2}, \quad \frac{2}{p}+\frac{d}{q}=\frac{d}{2}, \quad 2 \leq p \leq \infty,$$

however the dispersion estimates are sufficient for our purpose.

3. Reformulation of the Equations and Energy Estimate

As observed in [6], setting $w = \sqrt{K/\rho}\nabla\rho$, $\mathcal{L}$ the primitive of $\sqrt{K/\rho}$ such that $\mathcal{L}(\rho_c) = 1$, $L = \mathcal{L}(\rho)$, $z = u + iw$ the Euler–Korteweg system rewrites

$$\begin{aligned} \partial_t L + u\cdot\nabla L + a(L)\mathrm{div}u &= 0,\\ \partial_t u + u\cdot\nabla u - w\cdot\nabla w - \nabla(a(L)\mathrm{div}w) &= -\widetilde{g}'(L)w,\\ \partial_t w + \nabla(u\cdot w) + \nabla(a(L)\mathrm{div}u) &= 0, \end{aligned}$$

where the third equation is just the gradient of the first one. Setting $\ell = L-1$, in the potential case $u = \nabla\phi$, the system on ϕ, ℓ then reads

$$\begin{cases} \partial_t\phi + \frac{1}{2}\big(|\nabla\phi|^2 - |\nabla\ell|^2\big) - a(1+\ell)\Delta\ell = -\widetilde{g}(1+\ell),\\ \partial_t\ell + \nabla\phi\cdot\nabla\ell + a(1+\ell)\Delta\phi = 0, \end{cases} \tag{3.1}$$

with $\widetilde{g}(1) = 0$ since we look for integrable functions. As a consequence of the stability condition (2.1), up to a change of variables we can and will assume through the rest of the paper that

$$\widetilde{g}'(1) = 2. \tag{3.2}$$

The number 2 has no significance except that this choice gives the same linear part as for the Gross–Pitaevskii equation linearized near the constant state 1.

Proposition 3.1. *Let $n > d/4 + 1/2$, under the following assumptions*

- $(\nabla\phi_0, \ell_0) \in H^{2n} \times H^{2n+1}$,
- *Normalized (2.1): $\widetilde{g}'(1) = 2$,*
- $L(x,t) = 1 + \ell(x,t) \geq m > 0$ *for* $(x,t) \in \mathbb{R}^d \times [0,T]$,

there exists a continuous function C and a constant c depending only on m such that the solution of (3.1) satisfies the following estimate

$$\|\nabla\phi(t)\|_{H^{2n}} + \|\ell(t)\|_{H^{2n+1}} \leq c\big(\|\nabla\phi_0\|_{H^{2n}} + \|\ell_0\|_{H^{2n+1}}\big)\exp\Bigg(\int_0^t C\Big(\|\ell\|_{L^\infty}, \Big\|\frac{1}{\ell+1}\Big\|_{L^\infty}, \|z\|_{L^\infty}\Big) \times (\|\nabla\phi(s)\|_{W^{1,\infty}} + \|\ell(s)\|_{W^{2,\infty}})ds\Bigg),$$

where $z(s) = \nabla\phi(s) + i\nabla\ell(s)$.

This is almost the same estimate as in [6] but for an essential point: in the integrand of the right hand side there is no constant added to $\|\nabla\phi(s)\|_{W^{1,\infty}} + \|\ell(s)\|_{W^{2,\infty}}$, the price to pay is that we cannot control ϕ but its gradient (this is natural since the difficulty is related to the low frequencies). Before going into the detail of the computations, let us underline on a very simple example the idea behind it. We consider the linearized system

$$\partial_t\phi - \Delta\ell + 2\ell = 0, \tag{3.3}$$
$$\partial_t l + \Delta\phi = 0. \tag{3.4}$$

Multiplying (3.3) by ϕ, (3.4) by ℓ, integrating and using Young's inequality leads to the "bad" estimate

$$\frac{d}{dt}\big(\|\phi\|_{L^2}^2 + \|\ell\|_{L^2}^2\big) \lesssim 2(\|\phi\|_{L^2}^2 + \|\ell\|_{L^2}^2),$$

on the other hand if we multiply (3.3) by $-\Delta\phi$, (3.4) by $(-\Delta+2)\ell$ we get

$$\frac{d}{dt}\int_{\mathbb{R}^d}\Big(\frac{|\nabla\ell|^2 + |\nabla\phi|^2}{2} + \ell^2\Big)dx = 0,$$

the proof that follows simply mixes this observation with the gauge method from [6].

Proof. Let us start with the equation on $z = \nabla\phi + i\nabla\ell = u + iw$, we recall that $\widetilde{g}'(1) = 2$, so that we write it

$$\partial_t z + z\cdot\nabla z + i\nabla(a\text{div}z) = -2w + (2 - \widetilde{g}'(1+\ell))w. \tag{3.5}$$

We shortly recall the method from [6] that we will slightly simplify since we do not need to work in fractional Sobolev spaces. Due to the quasi-linear nature of the system (and in particular the bad "non transport term" $iw\cdot\nabla z$), it is not possible to directly estimate $\|z\|_{H^{2n}}$ by energy estimates, instead one uses a gauge function $\varphi_n(\rho)$ and control $\|\varphi_n\Delta^n z\|_{L^2}$. When we take the product of (3.5) with φ_n real, a number of commutators appear:

$$\varphi_n\Delta^n\partial_t z = \partial_t(\varphi_n\Delta^n z) - (\partial_t\varphi_n)\Delta^n z = \partial_t(\varphi_n\Delta^n z) + C_1, \tag{3.6}$$
$$\varphi_n\Delta^n(u\cdot\nabla z) = u\cdot\nabla(\varphi_n\Delta^n z) + [\varphi_n\Delta^n, u\cdot\nabla]z := u\cdot\nabla(\varphi_n\Delta^n z) + C_2, \tag{3.7}$$
$$i\varphi_n\Delta^n(w\cdot\nabla z) = iw\cdot\nabla(\varphi_n\Delta^n z) + [\varphi_n\Delta^n, w\cdot\nabla]z := iw\cdot\nabla(\varphi_n\Delta^n z) + C_3. \tag{3.8}$$

The term $\nabla(a\text{div}z)$ requires a bit more computations:

$$i\varphi_n\Delta^n\nabla(a\text{div}z) = i\nabla(\varphi_n\Delta^n(a\text{div}z)) - i(\nabla\varphi_n)\Delta^n(a\text{div}z),$$

then using recursively $\Delta(fg) = 2\nabla f \cdot \nabla g + f\Delta g + (\Delta f)g$ we get

$$\Delta^n(a\mathrm{div}z) = a\mathrm{div}\Delta^n z + 2n(\nabla a)\cdot\Delta^n z + C,$$

where C contains derivatives of z of order at most $2n-1$, so that

$$\begin{aligned} i\varphi_n\Delta^n\nabla(a\mathrm{div}z) &= i\nabla\Big(\varphi_n\big(a\mathrm{div}\Delta^n z + 2n(\nabla a)\cdot\Delta^n z\big)\Big) - i\nabla\varphi_n a\mathrm{div}\Delta^n z + i\nabla(\varphi_n C) \\ &= i\nabla\big(a\mathrm{div}(\varphi_n\Delta^n z)\big) + 2in\nabla a\cdot\varphi_n\nabla\Delta^n z - ia(\nabla + I_d\mathrm{div})\Delta^n z\cdot\nabla\varphi_n \\ &\quad + C_4, \end{aligned} \tag{3.9}$$

where C_4 contains derivatives of z of order at most $2n$ (in particular $\nabla(\varphi_n C)$ is included) and by notation $I_d\mathrm{div}\Delta^n z\cdot\nabla\varphi_n = \mathrm{div}(\Delta^n z)\,\nabla\varphi_n$. Finally, we define $C_5 = -\varphi_n\Delta^n\big((2-\widetilde{g}'(1+\ell))w\big)$. The equation on $\varphi_n\Delta^n z$ thus reads

$$\begin{aligned} &\partial_t(\varphi_n\Delta^n z) + u\cdot\nabla(\varphi_n\Delta^n z) + i\nabla\big(a\mathrm{div}(\varphi_n\Delta^n z)\big) + 2\varphi_n\Delta^n w + iw\cdot\nabla(\varphi_n\Delta^n z) \\ &= -\sum_1^5 C_k - 2in\varphi_n\nabla\Delta^n z\cdot\nabla a + ia(\nabla + I_d\mathrm{div})\Delta^n z\cdot\nabla\varphi_n. \end{aligned} \tag{3.10}$$

Taking the (complex) scalar product with $\varphi_n\Delta^n z$, integrating and taking the real part gives for the first four terms

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}(\varphi_n\Delta^n z)^2 dx - \frac{1}{2}\int_{\mathbb{R}^d}\mathrm{div}u|\varphi_n\Delta^n z|^2 dx + \int_{\mathbb{R}^d}2\varphi_n^2\Delta^n w\Delta^n u\,dx. \tag{3.11}$$

And we are left to control the remainder terms from (3.8, 3.9). Using $w = \frac{a}{\rho}\nabla\rho$, $\varphi_n = \varphi_n(\rho)$, we rewrite

$$\begin{aligned} &i\varphi_n w\cdot\nabla(\Delta^n z) + 2ni\varphi_n\nabla(\Delta^n z)\cdot\nabla a - ia\nabla(\Delta^n z)\cdot\nabla\varphi_n - ia\nabla\varphi_n\,\mathrm{div}\Delta^n z \\ &= i\varphi_n\left(w\cdot\nabla - \frac{a\nabla\varphi_n}{\varphi_n}\cdot\nabla - \frac{a\nabla\varphi_n}{\varphi_n}\mathrm{div} + 2n\nabla a\cdot\nabla\right)\Delta^n z \\ &= i\varphi_n\left[\left(\frac{a}{\rho} - a\frac{\varphi_n'}{\varphi_n}\right)\nabla\rho\cdot\nabla - \frac{a\varphi_n'}{\varphi_n}\nabla\rho\,\mathrm{div} + 2na'\nabla\rho\cdot\nabla\right]\Delta^n z. \end{aligned} \tag{3.12}$$

If the div operator was a gradient, the most natural choice for φ_n would be

$$\frac{a}{\rho} - \frac{2a\varphi_n'}{\varphi_n} + 2na' = 0 \Leftrightarrow \frac{\varphi_n'}{\varphi_n} = \frac{1}{2\rho} + \frac{na'}{a} \Leftarrow \varphi_n(\rho) = a^n(\rho)\sqrt{\rho}.$$

We make this choice, the remainder (3.12) rewrites

$$\begin{aligned} &\left[\left(\frac{a}{\rho} - a\frac{\varphi_n'}{\varphi_n}\right)\nabla\rho\cdot\nabla - \frac{a\varphi_n'}{\varphi_n}\nabla\rho\mathrm{div} + 2na'\nabla\rho\cdot\nabla\right]\Delta^n z \\ &= \left(\frac{a}{2\rho} + na'\right)\nabla\rho\cdot(\nabla - I_d\mathrm{div})\Delta^n z. \end{aligned}$$

Using the fact that $\varphi_n(a/(2\rho)+na')(\rho)\nabla\rho$ is a real valued gradient, say $\nabla G(\rho)$, the fact that z is irrotationnal and setting $z_n=\Delta^n z$, we have the following identity (with the Hessian Hess(G)):

$$\begin{aligned}\mathrm{Im}\int_{\mathbb{R}^d}\overline{z_n}\cdot(\nabla-I_d\mathrm{div})z_n\cdot\nabla G(\rho)dx &= \mathrm{Im}\sum_{i,j}\int_{\mathbb{R}^d}\overline{z_{i,n}}\partial_j z_{i,n}\partial_j G-\overline{z_{i,n}}\partial_j z_{j,n}\partial_i G\\ &= \mathrm{Im}\sum_{i,j}\int_{\mathbb{R}^d}\overline{z_n}\mathrm{Hess}(G)z_n-\Delta G|z_n|^2\\ &\quad -\partial_j G z_{i,n}(\overline{\partial_j z_{i,n}}-\overline{\partial_i z_{j,n}})dx\\ &=0,\end{aligned}$$

so that the contribution of (3.12) in the energy estimate is actually 0. Finally, we have obtained

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\|\varphi_n\Delta^n z\|_{L^2}^2dx-\frac{1}{2}\int_{\mathbb{R}^d}(\mathrm{div}u)|\varphi_n\Delta^n z|^2+2\int\varphi_n^2\Delta^n w\Delta^n u\,dx\\ &\quad=-\int\sum_{k=1}^{5}C_k\varphi_n\Delta^n\overline{z}dx.\end{aligned}\tag{3.13}$$

Note that the terms $C_k\varphi_n\Delta^n z$ are cubic while $\varphi_n\Delta^n w\Delta^n u$ is only quadratic, thus we will simply bound the first ones while we will need to cancel the later.

Control of the C_k: From their definition, it is easily seen that the $(C_k)_{2\le i\le 4}$ only contain terms of the kind $\partial^\alpha f\partial^\beta g$ with $f,g=u$ or w, $|\alpha|+|\beta|\le 2n$, thus

$$\forall\, 2\le k\le 4,\ \left|\int C_k\varphi_n\Delta^n z dx\right|\lesssim\sum_{|\alpha|+|\beta|\le 2n,\ f,g=u\text{ or }w}\|\partial^\alpha f\partial^\beta g\|_{L^2}\|z\|_{H^{2n}}.$$

When $|\alpha|=0,\ |\beta|=2n$, we have obviously $\|f\partial^\beta g\|_{L^2}\lesssim\|f\|_\infty\|g\|_{H^{2n}}$, while the general case $\|\partial^\alpha f\partial^\beta g\|_2\lesssim\|f\|_\infty\|g\|_{H^{2n}}+\|g\|_\infty\|f\|_{H^{2n}}$ is Gagliardo-Nirenberg's interpolation inequality (2.6). We deduce

$$\forall\, 2\le k\le 4,\ \left|\int C_k\varphi_n\Delta^n z dx\right|\lesssim\|z\|_\infty\|z\|_{H^{2n}}^2.$$

Let us deal now with $C_1=-\partial_t\varphi_n\Delta^n z$, since $\partial_t\varphi_n=-\varphi_n'\mathrm{div}(\rho u)$ we have

$$\left|\int_{\mathbb{R}^d}C_1\varphi_n\Delta^n\overline{z}dx\right|\lesssim F\big(\|\ell\|_{L^\infty},\|\frac{1}{\ell+1}\|_{L^\infty}\big)(\|u\|_{W^{1,\infty}}+\|z\|_{L^\infty}^2)\|z\|_{H^{2n}}^2,$$

with F a continuous function.

We now estimate the contribution of $C_5=-\varphi_n\Delta^n\big((2-\widetilde{g}'(1+\ell))w\big)$: since $\widetilde{g}'(1)=2$, from the composition rule (2.7) we have $\|\widetilde{g}'(1+\ell)-2\|_{H^{2n}}\lesssim F_1(\|\ell\|_{L^\infty},\|\frac{1}{\ell+1}\|_{L^\infty})\|\ell\|_{H^{2n}}$ with F_1 a continuous function so that

$$\begin{aligned}\left|\int_{\mathbb{R}^d}C_5\varphi_n\Delta^n\overline{z}dx\right| &\lesssim\|(2-\widetilde{g}')w\|_{H^{2n}}\|z\|_{H^{2n}}\\ &\lesssim(\|(2-\widetilde{g}'(1+\ell))\|_{L^\infty}\|z\|_{H^{2n}}\\ &\quad+F_1\big(\|\ell\|_{L^\infty},\|\frac{1}{\ell+1}\|_{L^\infty}\big)\|\ell\|_{H^{2n}}\|w\|_\infty\|z\|_{H^{2n}}.\end{aligned}$$

To summarize, for any $1 \leq k \leq 5$, we have

$$\left| \int_{\mathbb{R}^d} C_k \varphi_n \Delta^n z dx \right| \lesssim F_2(\|\ell\|_{L^\infty}, \|\frac{1}{\ell+1}\|_{L^\infty})(\|\ell\|_\infty + \|z\|_{W^{1,\infty}} + \|z\|_{L^\infty}^2)(\|\ell\|_{H^{2n}}^2 + \|z\|_{H^{2n}}^2), \quad (3.14)$$

with F_2 a continuous function.

Cancellation of the quadratic term. We start with the equation on ℓ to which we apply $\varphi_n \Delta^n$, multiply by $\varphi_n(\Delta^n \ell)/a$ and integrate in space

$$\int_{\mathbb{R}^d} \frac{\varphi_n^2}{a} \Delta^n l \partial_t \Delta^n l + \frac{\varphi_n^2}{a}(\Delta^n \ell)\Delta^n(\nabla\phi \cdot \nabla\ell) + \varphi_n^2 \Delta^n \ell \frac{\Delta^n(a\Delta\phi)}{a} = 0.$$

Commuting Δ^n and a, and using an integration by part, this rewrites

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} \frac{\varphi_n^2}{a}(\Delta^n \ell)^2 dx - \int_{\mathbb{R}^d} \partial_t\left(\frac{\varphi_n^2}{2a}\right)|\Delta^n \ell|^2 dx + \int_{\mathbb{R}^d} \frac{\varphi_n^2}{a}(\Delta^n \ell)\Delta^n(\nabla\phi \cdot \nabla\ell) \\
&\quad + \int_{\mathbb{R}^d} \varphi_n^2 \Delta^n \ell \, \Delta\Delta^n \phi dx + \frac{\varphi_n^2}{a}\Delta^n \ell[\Delta^n, a]\Delta\phi dx \\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} \frac{\varphi_n^2}{a}(\Delta^n \ell)^2 dx - \int_{\mathbb{R}^d} \partial_t\left(\frac{\varphi_n^2}{2a}\right)|\Delta^n \ell|^2 dx + \int_{\mathbb{R}^d} \frac{\varphi_n^2}{a}(\Delta^n \ell)\Delta^n(\nabla\phi \cdot \nabla\ell) \\
&\quad - \int_{\mathbb{R}^d} \varphi_n^2 \nabla\Delta^n \ell \cdot \nabla\Delta^n \phi \, dx - \int_{\mathbb{R}^d} \Delta^n \ell \, \nabla\varphi_n^2 \cdot \nabla\Delta^n \phi \, dx + \frac{\varphi_n^2}{a}\Delta^n \ell[\Delta^n, a]\Delta\phi dx.
\end{aligned}$$

We remark here that the integrand only depends on ℓ, $\nabla\phi$ and their derivatives, therefore using the same commutator arguments as previously, we get the bound

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} \frac{\varphi_n^2}{a}(\Delta^n \ell)^2 dx - \int_{\mathbb{R}^d} \varphi_n^2 (\Delta^n \nabla\phi) \cdot \Delta^n \nabla\ell dx \\
&\lesssim F_3(\|\ell\|_{L^\infty}, \|\frac{1}{\ell+1}\|_{L^\infty})(\|\ell\|_\infty + \|z\|_{W^{1,\infty}} + \|z\|_{L^\infty}^2)(\|\ell\|_{H^{2n}}^2 + \|z\|_{H^{2n}}^2), \quad (3.15)
\end{aligned}$$

with F_3 a continuous function. Now if we add (3.13) to $2 \times$ (3.15) and use the estimates on (C_k) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\varphi_n \Delta^n z\|_{L^2}^2 + 2\|\varphi_n \Delta^n \ell\|_{L^2}^2\right) \\
&\lesssim F_4(\|\ell\|_{L^\infty}, \|\frac{1}{\ell+1}\|_{L^\infty})(\|\ell\|_\infty + \|z\|_{W^{1,\infty}} + \|z\|_{L^\infty}^2)(\|\ell\|_{H^{2n}}^2 + \|z\|_{H^{2n}}^2),
\end{aligned}$$

with F_4 a continuous function. The conclusion then follows from Gronwall's lemma. □

4. Global Well-Posedness in Dimension Larger than 4

We first make a further reduction of the equations that will be also used for the cases $d = 3, 4$, namely we rewrite it as a linear Schrödinger equation with some remainder. In addition to $\widetilde{g}'(1) = 2$, we can also assume $a(1) = 1$, so that (3.1) rewrites[4]

$$\begin{cases} \partial_t \phi - \Delta \ell + 2\ell = (a(1+\ell) - 1)\Delta \ell - \frac{1}{2}\big(|\nabla \phi|^2 - |\nabla \ell|^2\big) + (2\ell - \widetilde{g}(1+\ell)), \\ \partial_t l + \Delta \phi = -\nabla \phi \cdot \nabla \ell + (1 - a(1+\ell))\Delta \phi. \end{cases} \tag{4.1}$$

The linear part precisely corresponds to the linear part of the Gross–Pitaevskii equation. In order to diagonalize it, following [20] we set

$$U = \sqrt{\frac{-\Delta}{2-\Delta}},\ H = \sqrt{-\Delta(2-\Delta)},\ \phi_1 = U\phi,\ \ell_1 = \ell.$$

The equation writes in the new variables

$$\begin{cases} \partial_t \phi_1 + H\ell_1 = U\Big((a(1+\ell_1) - 1)\Delta \ell_1 - \frac{1}{2}\big(|\nabla U^{-1}\phi_1|^2 - |\nabla \ell_1|^2\big) + (2\ell_1 - \widetilde{g}(1+\ell_1))\Big), \\ \partial_t \ell_1 - H\phi_1 = -\nabla U^{-1}\phi_1 \cdot \nabla \ell_1 - (1 - a(1+\ell_1))H\phi_1. \end{cases} \tag{4.2}$$

In a more compact notation if we set $\psi = \phi_1 + i\ell_1,\ \psi_0 = (U\phi + il)|_{t=0}$, the Duhamel formula gives

$$\psi(t) = e^{itH}\psi_0 + \int_0^t e^{i(t-s)H}\mathcal{N}(\psi(s))ds, \tag{4.3}$$

$$\text{with } \mathcal{N}(\psi) = U\big((a(1+\ell_1) - 1)\Delta \ell_1 - \frac{1}{2}\big(|\nabla U^{-1}\phi_1|^2 - |\nabla \ell_1|^2\big) + (2\ell_1 - \widetilde{g}(1+\ell_1))\big)$$
$$+ i\big(-\nabla U^{-1}\phi_1 \cdot \nabla \ell_1 - \big(1 - a(1+\ell_1)\big)H\phi_1\big). \tag{4.4}$$

We underline that for low frequencies the situation is more favorable than for the Gross–Pitaevskii equation, as all the terms where U^{-1} appears already contain derivatives that compensate this singular multiplier. Note however that the Gross–Pitaevskii equations are formally equivalent to this system in the special case $K(\rho) = \kappa/\rho$ via the Madelung transform, so our computations are a new way of seeing that these singularities can be removed in appropriate variables. Let us now state the key estimate:

Proposition 4.1. *Let $d \geq 5$, $T > 0$, $k \geq 2$, $N \geq k + 2 + d/2$, we set $\|\psi\|_{X_T} = \|\psi\|_{L^\infty([0,T],H^N)} + \sup_{t\in[0,T]} (1+t)^{d/4}\|\psi(t)\|_{W^{k,4}}$, then the solution of (4.3) satisfies*

$$\forall t \in [0,T],\ \|\psi(t)\|_{W^{k,4}} \lesssim \frac{\|\psi_0\|_{W^{k,4/3}} + \|\psi_0\|_{H^N} + G(\|\psi\|_{X_T}, \|\frac{1}{1+\ell_1}\|_{L^\infty_T(L^\infty)})\|\psi\|^2_{X_T}}{(1+t)^{d/4}},$$

with G a continuous function.

[4] The assumption $a(1) = 1$ should add some constants in factor of the nonlinear terms, we will neglect it as it will be clear in the proof that multiplicative constants do not matter.

Proof. We start with (4.3). From the dispersion estimate of Theorem 2.5 and the Sobolev embedding, we have for any $t \geq 0$

$$(1+t)^{d/4}\|e^{itH}\psi_0\|_{W^{k,4}} \lesssim (1+t)^{d/4}\min\left(\frac{\|U^{(d-2)/4}\psi_0\|_{W^{k,4/3}}}{t^{d/4}},\ \|\psi_0\|_{H^N}\right)$$
$$\lesssim \|\psi_0\|_{W^{k,4/3}} + \|\psi_0\|_{H^N}.$$

The only issue is thus to bound the nonlinear part. Let f, g be a placeholder for ℓ_1 or $U^{-1}\phi_1$, there are several kind of terms : $\nabla f \cdot \nabla g$, $(a(1+\ell_1)-1)\Delta f$, $2\ell_1 - \widetilde{g}(1+\ell_1)$, $|\nabla f|^2$, $\nabla f \cdot \nabla g$. The estimates for $0 \leq t \leq 1$ are easy (it corresponds to the existence of strong solution in finite time), so we assume $1 \leq t \leq T$ and we split the integral from (4.3) between $[0, t-1]$ and $[t-1, t]$. For the first integral we have from the dispersion estimate and (2.8):

$$\left\|\int_0^{t-1} e^{i(t-s)H}\nabla f \cdot \nabla g\, ds\right\|_{W^{k,4}} \lesssim \int_0^{t-1}\frac{\|\nabla f \cdot \nabla g\|_{W^{k,4/3}}}{(t-s)^{d/4}}ds$$
$$\lesssim \int_0^{t-1}\frac{\|\nabla f\|_{H^k}\|\nabla g\|_{W^{k-1,4}}}{(t-s)^{d/4}}ds,$$
$$\lesssim \|\psi\|_{X_T}^2\int_0^{t-1}\frac{1}{(t-s)^{d/4}(1+s)^{d/4}}ds$$
$$\lesssim \frac{\|\psi\|_{X_T}^2}{t^{d/4}}.$$

We have used the fact that ∇U^{-1} is bounded $W^{1,p} \to L^p$, $1 < p < \infty$ so that $\|\nabla f(s)\|_{H^k} \lesssim \|f\|_{X_T}$ for $s \in [0, t]$, $(1+s)^{d/4}\|\nabla g\|_{W^{k-1,4}} \lesssim \|g\|_{X_T}$.

For the second part on $[t-1, t]$ we use the Sobolev embedding $H^{d/4} \hookrightarrow L^4$ and (2.8):

$$\left\|\int_{t-1}^{t} e^{i(t-s)H}(\nabla f \cdot \nabla g)ds\right\|_{W^{k,4}} \lesssim \int_{t-1}^{t}\left\|\nabla f \cdot \nabla g\right\|_{H^{k+d/4}}ds$$
$$\lesssim \int_{t-1}^{t}\|\nabla f\|_{L^4}\|\nabla g\|_{H^{k+d/2}} + \|\nabla g\|_{L^4}\|\nabla f\|_{H^{k+d/2}}ds$$
$$\lesssim \|\psi\|_{X_T}^2\int_{t-1}^{t}\frac{1}{(1+s)^{\frac{d}{4}}}ds \lesssim \frac{\|\psi\|_{X_T}^2}{(1+t)^{d/4}}.$$

The terms of the kind $(a(1+\ell_1)-1)\Delta f$ are estimated similarly: splitting the integral over $[0, t-1]$ and $[t-1, t]$,

$$\left\|\int_0^{t-1} e^{i(t-s)H}(a(1+\ell_1)-1)\Delta f\, ds\right\|_{W^{k,4}} \lesssim \int_0^{t-1}\frac{\|a(1+\ell_1)-1\|_{W^{k,4}}\|\Delta f\|_{H^k}}{(t-s)^{d/4}}ds$$
$$\lesssim \int_0^{t-1}\frac{\|a(1+\ell_1)-1\|_{W^{k,4}}\|\nabla f\|_{H^{k+1}}}{(t-s)^{d/4}}ds.$$

As for the first kind terms, from the composition estimate we deduce that:

$$\|a(1+\ell_1)-1\|_{W^{k,4}} \lesssim F(\|\ell_1\|_{L^\infty_T(L^\infty)}, \|\frac{1}{1+\ell_1}\|_{L^\infty_T(L^\infty)})\|\ell_1\|_{W^{k,4}},$$

with F continuous, we can bound the integral above by $F(\|\psi\|_{X_T}, \|\frac{1}{1+\ell_1}\|_{L^\infty_T(L^\infty)})\|\psi\|^2_{X_T}$ $/t^{d/4}$. For the integral over $[t-1,t]$ we can do again the same computations using the composition estimates $\|a(1+\ell_1)-1\|_{H^{k+d/2}} \lesssim F_1(\|\ell_1\|_{L^\infty_T(L^\infty)}, \|\frac{1}{1+\ell_1}\|_{L^\infty_T(L^\infty)})$ $\|\ell_1\|_{H^{k+d/2}}$ with F_1 continuous. The restriction $N \geq k+2+d/2$ comes from the fact that we need $\|\Delta f\|_{H^{k+d/2}} \lesssim \|f\|_{X_T}$.

Writing $2\ell_1 - \widetilde{g}(1+\ell_1) = \ell_1(2-\widetilde{g}(\ell_1)/\ell_1)$ we see that the estimate for the last term is the same as for $(a(1+\ell_1)-1)\Delta f$ but simpler so we omit it. □

End of the proof of theorem (2.1). We fix $k > 2+d/4$, n such that $2n+1 \geq k+2+d/2$, and use these values for $X_T = L^\infty([0,T], H^{2n+1} \cap (1+t)^{-d/4}W^{k,4})$. First note that since $\mathcal{L}$ is a smooth diffeomorphism near 1 and $u_0 = \nabla\phi_0$, we have

$$\begin{aligned}\|u_0\|_{H^{2n}\cap W^{k-1,4/3}} + \|\rho_0 - \rho_c\|_{H^{2n+1}\cap W^{k,4/3}} &\sim \|(U\phi_0, \mathcal{L}^{-1}(1+\ell_0)-1)\|_{(H^{2n+1}\cap W^{k,4/3})^2}\\ &\sim \|\psi_0\|_{H^{2n+1}\cap W^{k,4/3}},\end{aligned}$$

if $\|\ell_0\|_\infty$ is small enough. In particular we will simply write the smallness condition in term of ψ_0. Now using the embedding $W^{k,4} \hookrightarrow W^{2,\infty}$, the energy estimate of proposition (3.1) implies

$$\|\psi(t)\|_{H^{2n+1}} \lesssim \|\psi_0\|_{H^{2n+1}} \exp\left(C(\|\psi\|_{X_T})\int_0^t \|\psi\|_{W^{k,4}}ds\right),$$

with C continuous. Combining it with the decay estimate of proposition (4.1) we get with G continuous

$$\begin{aligned}\|\psi\|_{X_T} &\leq C_1\bigg(\|\psi_0\|_{W^{k,4/3}} + \|\psi_0\|_{H^{2n+1}} + \|\psi\|^2_{X_T} G(\|\psi\|_{X_T}, \|\frac{1}{1+\ell}\|_{L^\infty_T(L^\infty)})\\ &\quad + \|\psi_0\|_{H^{2n+1}}\exp\left(C(\|\psi\|_{X_T}, \|\frac{1}{1+\ell}\|_{L^\infty_T(L^\infty)})\int_0^T \|\psi\|_{W^{k,4}}ds\right)\\ &\leq C_1\bigg(\|\psi_0\|_{W^{k,4/3}} + \|\psi_0\|_{H^{2n+1}} + \|\psi\|^2_{X_T} G(\|\psi\|_{X_T}, \|\frac{1}{1+\ell}\|_{L^\infty_T(L^\infty)})\\ &\quad + \|\psi_0\|_{H^{2n+1}}\exp\Big(C\big(\|\psi\|_{X_T}, \|\frac{1}{\ell+1}\|_{L^\infty_T(L^\infty)})\|\psi\|_{X_T}\big)\bigg).\end{aligned}$$

For $\|\psi\|_{X_T}$ small enough, we have $\|1/(1+\ell_1)\|_\infty \lesssim 1 + \|\psi\|_{X_T}$, so that from the usual bootstrap argument we find that for $\|\psi_0\|_{W^{k,4/3}} + \|\psi_0\|_{H^N} \leq \varepsilon$ small enough then for any $T > 0$, $\|\psi\|_{X_T} \leq 3C_1\varepsilon$ (it suffices to note that for ε small enough, the application $m \mapsto C_1(\varepsilon + \varepsilon e^{C'm} + m^2)$ is smaller than m on some interval $[a,b] \subset]0,\infty[$ with $a \simeq 2C_1\varepsilon$).

In particular $\|\ell\|_\infty \lesssim \varepsilon$ and up to diminishing ε, we have

$$\|\rho - \rho_c\|_{L^\infty([0,T]\times\mathbb{R}^d)} = \|\mathcal{L}^{-1}(1+\ell) - \rho_c\|_\infty \leq \rho_c/2.$$

This estimate and the H^{2n+1} bound allow to apply the blow-up criterion (see p.3) of [6] to get global well-posedness.

5. The Case of Dimension d = 3, 4: Normal Form, Bounds for Cubic and Quartic Terms

In dimension $d = 4$ the approach of Sect. 4 fails, and $d = 3$ is even worse. Thus we need to study more carefully the structure of the nonlinearity. We recall $H = \sqrt{-\Delta(2-\Delta)}$, $U = \sqrt{-\Delta/(2-\Delta)}$, and start with (4.2), that we rewrite in complex form

$$\begin{aligned}\partial_t\psi - iH\psi &= U\big[(a(1+\ell)-1)\Delta\ell - \frac{1}{2}\big(|\nabla\phi|^2 - |\nabla\ell|^2\big) + (2l - \widetilde{g}(1+\ell))\big]\\ &\quad + i\big[-\nabla\phi\cdot\nabla\ell + \big(1-a(1+\ell)\big)\Delta\phi)\big]\\ &= U\mathcal{N}_1(\phi,\ell) + i\mathcal{N}_2(\phi,\ell) = \mathcal{N}(\psi).\end{aligned} \tag{5.1}$$

As explained in the introduction (see (1.11)), we can rewrite the Duhamel formula in term of the profile $e^{-itH}\psi$. In particular, (the Fourier transform of) quadratic terms read

$$I_{\text{quad}} = e^{itH(\xi)}\int_0^t e^{-is\left(H(\xi)\mp H(\eta)\mp H(\xi-\eta)\right)}\mathcal{B}(\eta,\xi-\eta)\widetilde{\psi^{\pm}}(\eta)\widetilde{\psi^{\pm}}(\xi-\eta)d\eta ds, \tag{5.2}$$

where we recall the notation $\widetilde{\psi^{\pm}} = e^{\mp itH}\widehat{\psi^{\pm}}$, and $\mathcal{B}$ is the symbol of a bilinear multiplier. For some $\varepsilon > 0$ to choose later, let $1/p = 1/6 - \varepsilon$, $T > 0$, $N = 2n+1$ and set:

$$\begin{cases}\|\psi\|_{Y_T} = \|xe^{-itH}\psi\|_{L^\infty_T L^2} + \|\langle t\rangle^{1+3\varepsilon}\psi\|_{L^\infty_T W^{k,p}},\\ \|\psi\|_{X(t)} = \|\psi(t)\|_{H^N} + \|xe^{-itH}\psi(t)\|_{L^2} + \|\langle t\rangle^{1+3\varepsilon}\psi(t)\|_{W^{k,p}},\\ \|\psi\|_{X_T} = \sup\limits_{[0,T]}\|\psi\|_{X(t)}.\end{cases} \tag{5.3}$$

The linear part of the evolution is controlled thanks to theorem (2.5): uniformly in T

$$\|e^{itH}\psi_0\|_{X_T} \lesssim \|\psi_0\|_{H^N\cap W^{k,p'}\cap(L^2/\langle x\rangle)}.$$

From the embedding $W^{3,p}\subset W^{2,\infty}$, Proposition 3.1 implies

$$\|\psi\|_{L^\infty_T H^{2n+1}} \lesssim \|\psi_0\|_{H^{2n+1}}\exp\big(C(\|\ell\|_{L^\infty_T L^\infty}, \|\frac{1}{\ell+1}\|_{L^\infty_T L^\infty}, \|\nabla\psi\|_{X_T})\|\psi\|_{X_T}\big). \tag{5.4}$$

with C a continuous function. Thus the main difficulty of this section will be to prove $\|I_{\text{quad}}\|_{Y_T} \lesssim \|\psi\|^2_{X_T}$, uniformly in T. Combined with the energy estimate (5.4) and similar (easier) bounds for higher order terms, this provides global bounds for ψ which imply global well-posedness.

In order to perform such estimates we can use integration by part in (5.2) either in s or η (for the relevance of this procedure, see the discussion on space time resonances in the introduction). It is thus essential to study where and at which order we have a cancellation of $\Omega_{\pm,\pm}(\xi,\eta) = H(\xi)\pm H(\eta)\pm H(\xi-\eta)$ or $\nabla_\eta\Omega_{\pm\pm}$. We will denote abusively $H'(\xi) = \frac{2+2|\xi|^2}{\sqrt{2+|\xi|^2}}$ the radial derivative of H and note that $\nabla H(\xi) = H'(\xi)\xi/|\xi|$, we also point out that $H'(r) = \frac{2+2r^2}{\sqrt{2+r^2}}$ is strictly increasing.

There are several cases that have some similarities with the situation for the Schrödinger equation, see (1.12), (1.13) and (1.14) for the definition of the resonant sets $\mathcal{T}$, $\mathcal{S}$, $\mathcal{R}$.

- $\Omega_{++} = H(\xi) + H(\eta) + H(\xi - \eta) \gtrsim (|\xi| + |\eta| + |\xi - \eta|)(1 + |\xi| + |\eta| + |\xi - \eta|)$, the time resonant set is reduced to $\mathcal{T} = \{\xi = \eta = 0\}$,
- $\Omega_{--} = H(\xi) - H(\eta) - H(\xi - \eta)$, we have $\nabla_\eta \Omega_{--} = H'(\eta)\frac{\eta}{|\eta|} + H'(\xi - \eta)\frac{\eta - \xi}{|\eta - \xi|}$. From basic computations

$$\nabla_\eta \Omega_{--} = 0 \Rightarrow \begin{cases} H'(\eta) = H'(\xi - \eta) \\ \frac{\xi - \eta}{|\eta - \xi|} = \frac{\eta}{|\eta|} \end{cases} \Rightarrow \begin{cases} |\eta| = |\xi - \eta| \\ \xi = 2\eta \end{cases}.$$

  On the other hand $\Omega_{--}(2\eta, \eta) = H(2\eta) - 2H(\eta) = 0 \Leftrightarrow \eta = 0$, thus $\mathcal{R} = \{\xi = \eta = 0\}$.
- $\Omega_{-+} = H(\xi) - H(\eta) + H(\xi - \eta)$, from similar computations we find that the space-time resonant set is $\mathcal{R} = \mathcal{S} = \{\xi = 0\}$. The case Ω_{+-} is symmetric.

The fact that the space-time resonant set for Ω_{+-} is not trivial explains why it is quite intricate to bound quadratic terms. An other issue pointed out in [22] for their study of the Gross–Pitaevskii equation is that the small frequency "parallel" resonances are worse than for the nonlinear Schrödinger equation. Namely near $\xi = \varepsilon\eta$, $\eta << 1$ we have

$$H(\varepsilon\eta) - H(\eta) + H((\varepsilon - 1)\eta) \sim \frac{-3\varepsilon|\eta|^3}{2\sqrt{2}} = \frac{-3|\xi|\,|\eta|^2}{2\sqrt{2}},$$
$$\text{while } |\varepsilon\eta|^2 - |\eta|^2 + |(1 - \varepsilon)\eta|^2 \sim -2|\eta|\,|\xi|,$$

we see that integrating by parts in time causes twice more loss of derivatives than prescribed by Coifman-Meyer's theorem, and there is no hope even for ξ/Ω to belong to any standard class of multipliers. Thus it seems unavoidable to use the rough multiplier Theorem 2.4.

5.1. Normal form. Let us recall that the nonlinearity reads as $U\mathcal{N}_1 + i\mathcal{N}_2$. After an integration by part in (5.2), it is necessary to divide the symbols of quadratic terms by Ω or $|\nabla_\eta \Omega|$, and we pointed out that both quantities cancel at $\xi = 0$. For the real quadratic terms, thanks to the factor $U(\xi) \sim_0 |\xi|$ there is some hope that the symbols $U/|\Omega|$ and $U/|\nabla_\eta \Omega|$ keep some boundedness properties, while on the imaginary part some terms appear that are unavoidably singular at $\xi = 0$.

In the spirit of [22] we will use a normal form such that the new variables satisfy a Schrödinger type equation where *all* quadratic terms have the same good structure as the real ones (essentially of the form $|\nabla| \circ \mathcal{B}[z, z]$ with $\mathcal{B}$ a smooth bilinear multipler).

Proposition 5.1. *For* $B(\eta, \xi - \eta) := \frac{(a'(1) - 1)\eta \cdot (\xi - \eta)}{2(2 + |\eta|^2 + |\xi - \eta|^2)}$, $\ell_1 = \ell - B[\phi, \phi] + B[\ell, \ell]$, *then* ℓ_1 *satisfies*

$$\partial_t \ell_1 + \Delta\phi = -\alpha div(\ell\nabla\phi) + R, \tag{5.5}$$

where R *contains cubic and higher order terms in* $\nabla\phi, \ell$.

Proof. From now on, we will use the notation R as a placeholder for remainder terms that should be at least cubic. In order to write the nonlinearity as essentially quadratic we set $a'(1) = \alpha$, and rewrite

$$\begin{aligned} \mathrm{Im}(\mathcal{N})(\psi) &= -\alpha\ell\Delta\phi - \nabla\phi \cdot \nabla\ell + \big[\big(1 + \alpha\ell - a(1 + \ell)\big)\Delta\phi\big] \\ &= -\alpha\ell\Delta\phi - \nabla\phi \cdot \nabla\ell + R. \end{aligned} \tag{5.6}$$

At the Fourier level, the quadratic terms $-\alpha\ell\Delta\phi - \nabla\phi\cdot\nabla\ell$ can be written as follows:

$$-\alpha\ell\Delta\phi - \nabla\phi\cdot\nabla\ell = -\alpha\mathrm{div}(\ell\nabla\phi) + (\alpha-1)\nabla\phi\cdot\nabla\ell. \tag{5.7}$$

We define the change of variables as $\ell_1 = \ell - B[\phi,\phi] + B[\ell,\ell]$, without assumption on B yet. We have

$$\begin{aligned}&\partial_t\big(-B[\phi,\phi]+B[\ell,\ell]\big)\\ &\quad= 2B[\phi,(-\Delta+2)\ell] + 2B[-\Delta\phi,\ell] + 2B\big[\phi,\mathcal{N}_1(\phi,\ell)\big] + 2B\big[\mathcal{N}_2(\phi,\ell),\ell\big]\\ &\quad= 2B[\phi,(-\Delta+2)\ell] + 2B[-\Delta\phi,\ell] + R,\end{aligned} \tag{5.8}$$

where the quadratic terms amount to a bilinear Fourier multiplier $B'[\phi,\ell]$, with symbol $B'(\eta,\xi-\eta) = 2B(\eta,\xi-\eta)\big(|\eta|^2+2+|\xi-\eta|^2\big)$. Using (5.7), (5.8) we see that the evolution equation on $\ell_1 = \ell - B[\phi,\phi] + B[\ell,\ell]$ is

$$\begin{gathered}\partial_t\ell_1 + \Delta\phi = B''(\phi,\ell) - \alpha\mathrm{div}(l\nabla\phi) + R,\\ B''(\eta,\xi-\eta) = 2B(\eta,\xi-\eta)(2+|\eta|^2+|\xi-\eta|^2) + (1-\alpha)\eta\cdot(\xi-\eta).\end{gathered}$$

and we see that in order to cancel B'' we should set

$$B(\eta,\xi-\eta) = \frac{(\alpha-1)\eta\cdot(\xi-\eta)}{2(2+|\eta|^2+|\xi-\eta|^2)}. \tag{5.9}$$

For this choice, we have indeed:

$$\partial_t\ell_1 + \Delta\phi = -\alpha\mathrm{div}(\ell\nabla\phi) + R, \tag{5.10}$$

which is the expected identity. □

In addition we get from (4.1):

$$\begin{aligned}\partial_t\phi - \Delta\ell_1 + 2\ell_1 = &-\Delta b(\phi,\ell) + 2b(\phi,\ell) + (a(1+\ell)-1)\Delta\ell - \frac{1}{2}\big(|\nabla\phi|^2 - |\nabla\ell|^2\big)\\ &+ (2\ell - \widetilde{g}(1+\ell)),\end{aligned} \tag{5.11}$$

with $\ell_1 = \ell - B[\phi,\phi] + B[\ell,\ell] = \ell + b(\phi,\ell)$. Setting $\phi_1 = U\phi$ the system becomes:

$$\begin{aligned}\partial_t\phi_1 + H\ell_1 &= U\bigg(\alpha\,\ell\Delta\ell - \frac{1}{2}\big(|\nabla U^{-1}\phi_1|^2 - |\nabla\ell|^2\big)\\ &\qquad + (-\Delta+2)b(\phi,\ell) - \widetilde{g}''(1)\ell^2\bigg) + R,\\ \partial_t\ell_1 - H\phi_1 &= -\alpha\mathrm{div}(\ell\nabla\phi) + R.\end{aligned}$$

Final form of the equation. Finally, if we replace in the quadratic terms $\ell = \ell_1 - b(\phi, \ell)$ and set $z = \phi_1 + i\ell_1$ we obtain

$$\begin{aligned}\partial_t z - iHz &= U\big[\alpha\,\ell_1\Delta\ell_1 - \frac{1}{2}\big(|\nabla U^{-1}\phi_1|^2 - |\nabla\ell_1|^2 - \widetilde{g}''(1)\ell_1^2\big) + (-\Delta+2)b(\phi,\ell_1)\big] - i\alpha\mathrm{div}(\ell_1\nabla\phi)\\ &\quad + U\big[\alpha\big(-b(\phi,\ell)\Delta\ell_1 - \ell_1\Delta b(\phi,\ell) + b(\phi,\ell)\Delta b(\phi,\ell)\big) - 2\nabla b(\phi,\ell)\cdot\nabla\ell + |\nabla b(\phi,\ell)|^2\\ &\quad + (-\Delta+2)(-2B[\ell_1, b(\phi,\ell)] + B[b(\phi,\ell), b(\phi,\ell)]) - \widetilde{g}''(1)(b(\phi,\ell))^2 + 2\widetilde{g}''(1)\ell_1 b(\phi,\ell)\big]\\ &\quad + i\alpha\mathrm{div}(b(\phi,\ell)\nabla\phi) + R\\ &:= Q(z) + R := \mathcal{N}_z, \end{aligned} \tag{5.12}$$

where $Q(z)$ contains the quadratic terms (the first line), R the cubic and quartic terms.

Remark 5.2. R contains a large amount of terms and we chose not to describe it in great detail here. Its detailed analysis is provided in Sect. 5.2.

Remark 5.3. It is noticeable that this change of unknown is not singular in term of the new variable $\phi_1 = U\phi$, indeed $B(\phi,\phi) = \widetilde{B}(\nabla\phi,\nabla\phi)$ where $\widetilde{B}(\eta,\xi-\eta) = \frac{\alpha-1}{(2+|\eta|^2+|\xi-\eta|^2)}$ is smooth, so that $B(\phi,\phi) = \widetilde{B}(\nabla U^{-1}\phi_1, \nabla U^{-1}\phi_1)$ acts on ϕ_1 as a composition of smooth bilinear and linear multipliers.

It remains to check that the normal form is well defined in our functional framework. We shall also prove that is cancels asymptotically.

Proposition 5.4. *We recall $b(\phi, l) = -B[\phi,\phi] + B[l,]$, B the bilinear multiplier given in (5.9). For $N > 4$, $k \geq 2$, the map $\phi_1 + il \mapsto z := \phi_1 + i(\ell + b(\phi,\ell))$ is bi-Lipschitz on the neighbourhood of 0 in X_∞, Moreover, $\psi = \phi_1 + i\ell$ and z have the same asymptotic as $t \to \infty$:*

$$\|\psi - z\|_{X(t)} = O(t^{-1/2}).$$

Proof. The terms $B[\phi,\phi]$ and $B[\ell,\ell]$ are handled in a similar way, we only treat the first case which is a bit more involved as we have the singular relation $\phi = U^{-1}\phi_1$. Note that $B[\phi,\phi] = \widetilde{B}(\nabla\phi,\nabla\phi)$, with $\widetilde{B}[\eta,\xi-\eta] = (\alpha-1)\frac{1}{2+|\eta|^2+|\xi-\eta|^2}$, and $\nabla U^{-1} = \langle\nabla\rangle \circ \mathrm{Ri}$ (we recall $\mathrm{Ri} = \nabla/|\nabla|$) so there is no real issue as long as we avoid the L^∞ space. Also, we split $B = B\chi_{|\eta|\gtrsim|\xi-\eta|} + B(1-\chi_{|\eta|\gtrsim|\xi-\eta|})$ where χ is smooth outside $\eta = \xi = 0$, homogeneous of degree 0, equal to 1 near $\{|\xi-\eta| = 0\} \cap \mathbb{S}^{2d-1}$ and 0 near $\{|\eta| = 0\} \cap \mathbb{S}^{2d-1}$. As can be seen from the change of variables $\zeta = \xi - \eta$, these terms are symmetric so we can simply consider the first case.

First note that by interpolation,

$$\forall\, 2 \leq q \leq p,\ \ \|\psi\|_{W^{k,q}} \lesssim \|\psi\|_{X(t)}/\langle t\rangle^{3(1/2-1/q)}. \tag{5.13}$$

For the H^N estimate we have from the Coifman-Meyer theorem (since the symbol $\widetilde{B}$ has the form $\frac{1}{2+|\eta|^2+|\xi-\eta|^2}$), the embedding $H^1 \mapsto L^3$ and the boundedness of the Riesz multiplier,

$$\|B[U^{-1}\phi_1, U^{-1}\phi_1]\|_{H^N} \lesssim \big\|\nabla U^{-1}\phi_1\big\|_{W^{N-2,3}}\big\|\nabla U^{-1}\phi_1\big\|_{L^6} \lesssim \|\phi_1\|_{X(t)}^2/\langle t\rangle.$$

For the weighted estimate $\|xe^{-itH}B[\phi,\phi]\|_{L^2}$, since $\phi = U^{-1}(\psi+\overline{\psi})/2$, we have a collection of terms that read in the Fourier variable:

$$\begin{aligned}
&\mathcal{F}\big(xe^{-itH}(\chi_{|\eta|\gtrsim|\xi-\eta|}B)[U^{-1}\psi^{\pm},U^{-1}\psi^{\pm}]\big)\\
&\quad=\nabla_\xi\int e^{-it\Omega_{\pm\pm}}B_1(\eta,\xi-\eta)\widetilde{\psi^{\pm}}(\eta)\widetilde{\psi^{\pm}}(\xi-\eta)d\eta,\\
&\qquad\text{where } B_1=\frac{\eta U^{-1}(\eta)\cdot(\xi-\eta)U^{-1}(\xi-\eta)}{2+|\eta|^2+|\xi-\eta|^2}\chi_{|\eta|\gtrsim|\xi-\eta|},\\
&\qquad\Omega_{\pm\pm}=H(\xi)\mp H(\eta)\mp H(\xi-\eta).
\end{aligned}$$

If the derivative hits B_1, in the worst case it adds a singular term $U^{-1}(\xi-\eta)$, so that from the embedding $\dot{H}^1\hookrightarrow L^6$

$$\begin{aligned}
&\left\|\int e^{-it\Omega_{\pm\pm}}(\nabla_\xi B_1)\widetilde{\psi^{\pm}}(\eta)\widetilde{\psi^{\pm}}(\xi-\eta)d\eta\right\|_{L^2}\\
&\quad=\big\|\nabla_\xi B_1[\psi^{\pm},\psi^{\pm}]\big\|_{L^2}\lesssim\|U^{-1}\psi\|_{W^{1,6}}\|\psi\|_{W^{1,3}}\\
&\qquad\lesssim\|\psi\|_{X(t)}^2/\langle t\rangle^{1/2}.
\end{aligned}$$

If the derivative hits $\widetilde{\psi^{\pm}}(\xi-\eta)$ we use the fact that the symbol $\frac{\langle\xi-\eta\rangle^2\chi_{|\eta|\gtrsim|\xi-\eta|}}{2+|\eta|^2+|\xi-\eta|^2}$ is of Coifman-Meyer type

$$\begin{aligned}
&\left\|\int e^{-it\Omega_{\pm\pm}}B_1(\eta,\xi-\eta)\widetilde{\psi^{\pm}}(\eta)\nabla_\xi\widetilde{\psi^{\pm}}(\xi-\eta)d\eta\right\|_{L^2}\\
&\quad\lesssim\|\langle\nabla\rangle\psi\|_{L^6}\|\langle\nabla\rangle^{-2}\langle\nabla\rangle e^{itH}xe^{-itH}\psi\|_{L^3}\\
&\quad\lesssim\|\psi\|_{X(t)}^2/\langle t\rangle.
\end{aligned}$$

Finally, if the derivative hits $e^{-it\Omega_{\pm\pm}}$ we note that $\nabla_\xi\Omega_{\pm\pm}=\nabla_\xi H(\xi)\mp\nabla_\xi H(\xi-\eta)$, where both term are multipliers of order 1 so

$$\begin{aligned}
\left\|\int e^{-it\Omega_{\pm\pm}}it(\nabla_\xi\Omega_{\pm\pm})B_1\widetilde{\psi^{\pm}}(\eta)\widetilde{\psi^{\pm}}(\xi-\eta)d\eta\right\|_{L^2}&\lesssim t\|\psi\|_{W^{1,3}}\|\psi\|_{W^{1,6}}\\
&\lesssim\|\psi\|_{X(t)}^2/\langle t\rangle^{1/2}.
\end{aligned}$$

The $W^{k,p}$ norm is also estimated using the Coifman-Meyer theorem and the boundedness of the Riesz multipliers:

$$\|B_1[\psi^{\pm}(t),\psi^{\pm}(t)]\|_{W^{k,p}}\lesssim\|\psi\|_{W^{k-1,1/12-\varepsilon/2}}^2\lesssim\|\psi\|_{W^{k,1/6-\varepsilon}}^2\lesssim\frac{\|\psi\|_{X(t)}^2}{\langle t\rangle^{2+6\varepsilon}}.$$

Gluing all the estimates we have proved

$$\|B[U^{-1}\psi,U^{-1}\psi]\|_{X(t)}^2\lesssim\|\psi\|_{X(t)}^2/\langle t\rangle^{1/2},\ \|B[U^{-1}\psi,U^{-1}\psi]\|_{X_\infty}^2\lesssim\|\psi\|_{X_\infty}^2,$$

thus using the second estimate we obtain from a fixed point argument that the map $\phi_1+il\mapsto\phi_1+i(\ell-B[\phi,\phi]+B[\ell,\ell])$ defines a diffeomorphism on a neighbourhood of 0 in X_∞. The first estimate proves the second part of the proposition. □

With similar arguments, we can also obtain the following:

Proposition 5.5. *Let* $z_0=U\phi_0+i(\ell_0-B[\phi_0,\phi_0]+B[\ell_0,\ell_0])$*, the smallness condition of theorem (2.2) is equivalent to the smallness of* $\|z_0\|_{H^{2n+1}}+\|xz_0\|_{L^2}+\|z_0\|_{W^{k,p'}}$.

5.2. Bounds for cubic and quartic nonlinearities. Let us first collect the list of terms in R (see (5.6), (5.8), (5.12) with $b = b(\phi, \ell)$):

$$(1 + \alpha\ell - (a(1+\ell))\Delta\phi,\ B[\phi, \mathcal{N}_1(\phi,\ell)],\ B[\mathcal{N}_2(\phi,\ell),\ell],\ i\alpha\text{div}(b\nabla\phi),$$
$$U\big(\alpha(-b\Delta\ell_1 - \ell_1\Delta b + b\Delta b - 2\nabla b\cdot\nabla\ell + |\nabla b|^2(-\Delta+2)b(\phi,-b)$$
$$- 2B[\ell_1, b] + B[b,b]\big).$$

We note that they are all either cubic (for example $B[\phi, |\nabla\phi|^2]$) or quartic (for example $B[b,b]$). B is a smooth bilinear multiplier and ϕ always appears with a gradient, so we can replace everywhere ϕ by $\phi_1 = U\phi$ up to the addition of Riesz multipliers.

Since the estimates are relatively straightforward, we only detail the case of the cubic term $B[\phi, |\nabla\phi|^2]$ which comes from $B[\phi, \mathcal{N}_1(\phi)]$ (quartic terms are simpler). Since $\phi = U^{-1}(\psi + \overline{\psi})/2$ we are reduced to bound in Y_T (see 5.3) terms of the form

$$I(t) = \int_0^t e^{i(t-s)H} B[U^{-1}\psi^{\pm}, |U^{-1}\nabla\psi^{\pm}|^2]ds.$$

Proposition 5.6. *For any $T > 0$, we have the a priori estimate*

$$\sup_{[0,T]} \|I(t)\|_{Y_T} \lesssim \|\psi\|_{X_T}^3.$$

Proof. **The weighted bound**

First let us write

$$\begin{aligned} xe^{-itH}I(t) = \int_0^t e^{-isH}\Big(& (-is\nabla_\xi H\, B[U^{-1}\psi^{\pm}, (U^{-1}\nabla\psi^{\pm})^2] \\ & + B[U^{-1}\psi^{\pm}, x(U^{-1}\nabla\psi^{\pm})^2] \\ & + \nabla_\xi B[U^{-1}\psi^{\pm}, (U^{-1}\nabla\psi^{\pm})^2]\Big)ds \\ = I_1(t) + I_2(t) + I_3(t). & \end{aligned}$$

Taking the L^2 norm and using the Strichartz estimate with $(p', q') = (2, 6/5)$ we get

$$\begin{aligned} \|I_1\|_{L_T^\infty L^2} &\lesssim \|(s\nabla_\xi H)B[U^{-1}\psi^{\pm}, (U^{-1}\nabla\psi^{\pm})^2]\|_{L^2(L^{6/5})} \\ &\lesssim \|sB[U^{-1}\psi^{\pm}, (U^{-1}\nabla\psi^{\pm})^2]\|_{L^2(W^{1,6/5})}, \\ \|I_2\|_{L_T^\infty L^2} &\lesssim \|B[U^{-1}\psi^{\pm}, x(U^{-1}\nabla\psi^{\pm})^2]\|_{L^2(L^{6/5})}. \end{aligned}$$

We have then from Coifman-Meyer's theorem, Hölder's inequality, continuity of the Riesz operator and (5.13)

$$\begin{aligned} \|sB[U^{-1}\psi^{\pm}, (U^{-1}\nabla\psi^{\pm})^2]\|_{L_T^2(W^{1,6/5})} &\lesssim \big\|s\|\psi\|_{W^{2,6}}^2\|\psi\|_{H^2}\big\|_{L_T^2} \lesssim \|\psi\|_{X_T}^3, \\ \|I_2\|_{L_T^\infty(L^2)} &\lesssim \Big\|\|\psi\|_{W^{1,6}}\|x(\nabla U^{-1}\psi^{\pm})^2\|_{L^{\frac{3}{2}}}\Big\|_{L_T^2}. \end{aligned} \tag{5.14}$$

The loss of derivatives in I_2 can be controlled thanks to a paraproduct: let $(\chi_j)_{j\geq 0}$ with $\sum \chi_j(\xi) = 1$, $\text{supp}(\chi_0) \subset B(0,2)$, $\text{supp}(\chi_j) \subset \{2^{j-1} \leq \xi \leq 2^{j+1}\}$, $j \geq 1$, and set $\widehat{\Delta_j \psi} := \chi_j \widehat{\psi}$, $S_j\psi = \sum_0^j \Delta_k \psi$. Then

$$(U^{-1}\nabla\psi^{\pm})^2 = \sum_{j\geq 0}(\nabla U^{-1}S_j\psi^{\pm})(\nabla U^{-1}\Delta_j\psi^{\pm}) + \sum_{j\geq 1}(\nabla U^{-1}S_{j-1}\psi^{\pm})(\nabla U^{-1}\Delta_j\psi^{\pm}).$$

For any term of the first scalar product we have

$$\begin{aligned} x\big((\partial_k U^{-1}S_j\psi^{\pm})(\partial_k U^{-1}\Delta_j\psi^{\pm})\big) &= (\partial_k U^{-1}S_j x\psi^{\pm})(\partial_k U^{-1}\Delta_j\psi^{\pm}) \\ &\quad + ([x, \partial_k U^{-1}S_j]\psi^{\pm})(\partial_k U^{-1}\Delta_j\psi^{\pm}). \end{aligned}$$

From Hölder's inequality, standard commutator estimates, the Besov embedding $W^{3,6} \hookrightarrow B^2_{6,1}$ and (6.1) we get

$$\begin{aligned} \sum_j \|(\partial_k U^{-1}S_j x\psi^{\pm})(\partial_k U^{-1}\Delta_j\psi^{\pm})\|_{L^{3/2}} &\lesssim \sum_j 2^j \|x\psi\|_{L^2} 2^j \|\Delta_j\psi\|_{L^6} \\ &\lesssim \|x\psi\|_{L^2}\|\psi\|_{W^{3,6}}, \quad (5.15) \\ \sum_j \|([x, \partial_k U^{-1}S_j]\psi^{\pm})(\partial_k U^{-1}\Delta_j\psi^{\pm})\|_{L3/2} &\lesssim \|U^{-1}\psi\|_{H^1}\|\psi\|_{W^{1,6}} \\ &\lesssim \|\psi\|^2_{X_T}/\langle t\rangle. \quad (5.16) \end{aligned}$$

Moreover, $x\psi = xe^{itH}e^{-itH}\psi = e^{itH}xe^{-itH}\psi + it\nabla_\xi H\psi$ so that :

$$\|x\psi(t)\|_{L^2} \lesssim \langle t\rangle \|\psi\|_{X_T}.$$

Similar computations can be done for $\sum_{j\geq 1}(\nabla U^{-1}S_{j-1}\psi^{\pm})(\nabla U^{-1}\Delta_j\psi^{\pm})$, finally (5.15), (5.16) and (5.13) imply

$$\|x(U^{-1}\nabla\psi^{\pm})^2\|_{L^{3/2}} \lesssim \|\psi\|^2_{X_T}.$$

Plugging the last inequality in (5.14) we can conclude

$$\|I_2\|_{L^\infty_T L^2} \lesssim \big\| \|\psi\|^3_{X_T}/\langle t\rangle \big\|_{L^2_T} \lesssim \|\psi\|^3_{X_T}.$$

The $W^{k,p}$ decay We split $[0,t] = [0,t-1] \cup [t-1,t]$. On $[0,t-1]$ we apply the dispersion estimate as in section 4:

$$\begin{aligned} &\left\| \int_0^{t-1} e^{i(t-s)H} B[U^{-1}\psi^{\pm}, (U^{-1}\nabla\psi^{\pm})^2] ds \right\|_{W^{k,p}} \\ &\lesssim \int_0^{t-1} \frac{\|B[U^{-1}\psi^{\pm}, (U^{-1}\nabla\psi^{\pm})^2]\|_{W^{k,p'}}}{(t-s)^{1+3\varepsilon}} ds \\ &\lesssim \int_0^{t-1} \frac{\|\nabla U^{-1}\psi\|^3_{W^{k,3p'}}}{(t-s)^{1+3\varepsilon}} ds \\ &\lesssim \int_0^{t-1} \frac{\|\psi\|^3_{W^{k+1,3p'}}}{(t-s)^{1+3\varepsilon}} ds. \quad (5.17) \end{aligned}$$

We then use interpolation and the estimate (5.13) with $q = 3p'$:

$$\|\psi\|_{W^{k+1,3p'}} \lesssim \|\psi\|_{W^{k,3p'}}^{(J-1)/J} \|\psi\|_{W^{k+J,3p'}}^{1/J}, \ \|\psi(t)\|_{W^{k,3p'}} \lesssim \frac{\|\psi(s)\|_{X_T}}{\langle s\rangle^{2/3-\varepsilon}}.$$

Since $3p' < 6$, we have $\|\psi\|_{W^{k+J,3p'}} \lesssim \|\psi\|_{H^{k+J+1}}$ by Sobolev embedding, so that for ε small enough, J large enough such that $(2-3\epsilon)(1-\frac{1}{J}) \geq 1+3\epsilon$ (but $J \leq N-k-1$) we observe that:

$$\|\psi(s)\|_{W^{k+1,3p'}}^3 \lesssim \frac{\|\psi\|_{X_T}^3}{\langle s\rangle^{1+3\epsilon}}$$

Plugging this inequality in (5.17) we conclude :

$$\int_0^{t-1} \frac{\|\psi\|_{W^{k+1,3p'}}^3}{(t-s)^{1+3\varepsilon}} \lesssim \frac{\|\psi\|_{X_T}^3}{\langle t\rangle^{1+3\varepsilon}}.$$

For the integral on $[t-1,t]$ it suffices to bound $\| \int_{t-1}^t e^{i(t-s)H} B[U^{-1}\psi^{\pm}, (U^{-1}\nabla\psi^{\pm})^2] ds\|_{W^{k,p}} \lesssim \| \int_{t-1}^t \|B[U^{-1}\psi^{\pm}, (U^{-1}\nabla\psi^{\pm})^2]ds\|_{H^{k+2}}$ and follow the argument of the proof of Proposition 4.1. □

6. Bounds for Quadratic Nonlinearities in Dimension 3, End of Proof

The following proposition will be repeatedly used (see proposition 4.6 [4] or [22]).

Proposition 6.1. *Let $\chi \in C_c^\infty(\mathbb{R}^+)$ such that supp$(\chi) \subset [0,2]$, $\chi|_{[0,1]} = 1$. We have the following estimates with $0 \leq \theta \leq 1$:*

$$\|\psi(t)\|_{\dot{H}^{-1}} \lesssim \|\psi(t)\|_{X(t)}, \tag{6.1}$$

$$\|U^{-2}\psi\|_{L^6} \lesssim \|\psi(t)\|_{X(t)}, \tag{6.2}$$

$$\begin{aligned} &\||\nabla|^{-2+\frac{5\theta}{3}}\chi(|\nabla|)\psi(t)\|_{L^6} \lesssim \min(1,t^{-\theta})\|\psi(t)\|_{X(t)}, \\ &\||\nabla|^{\theta}(1-\chi)(|\nabla|)\psi(t)\|_{L^6} \lesssim \min(t^{-\theta},t^{-1})\|\psi(t)\|_{X(t)}, \end{aligned} \tag{6.3}$$

$$\|U^{-1}\psi(t)\|_{L^6} \lesssim \langle t\rangle^{-\frac{3}{5}}\|\psi(t)\|_{X(t)}. \tag{6.4}$$

In this section, we will assume $\|\psi\|_{X_T} << 1$, in order to have that

$$\forall m \geq 2, \ \|\psi\|_{X_T}^2 + \|\psi\|_{X_T}^m \leq 2\|\psi\|_{X_T}^2.$$

All computations that follow can be done without any smallness assumption, but they would require to always add in the end some $\|\psi\|_{X_T}^m$, that we avoid for conciseness.

6.1. The L^p decay. We now prove decay for the quadratic terms in (5.12), namely

$$\langle t\rangle^{1+3\varepsilon}\Big\|\int_0^t e^{i(t-s)H}Q(z)(s)ds\Big\|_{W^{k,p}}\lesssim \|z\|_{X_T}^2.$$

For $t\leq 1$, the estimate is a simple consequence of the product estimate[5] $\|Q(z)\|_{H^{k+2}}\lesssim \|z\|_{H^N}^2$, and the boundedness of $e^{itH}: H^s\mapsto H^s$. Thus we focus on the case $t\geq 1$ and note that it is sufficient to bound $t^{1+3\varepsilon}\|\int_0^t e^{i(t-s)H}Q(z)(s)ds\|_{W^{k,p}}$.

We recall that the quadratic terms have the following structure (see (5.12))

$$\begin{aligned}Q(z)=U\Big(\alpha\,\ell_1\Delta\ell_1-\frac{1}{2}\big(|\nabla U^{-1}\phi_1|^2-|\nabla\ell_1|^2-\widetilde{g}''(1)\ell_1^2\big)+(-\Delta+2)b(\phi,\ell_1)\Big)\\ -i\alpha\mathrm{div}(\ell_1\nabla U^{-1}\phi_1),\end{aligned}\tag{6.5}$$

where $b=-B[\phi,\phi]+B[\ell_1,\ell_1],\ B(\eta,\xi-\eta)=\frac{(\alpha-1)\eta\cdot(\xi-\eta)}{2+|\eta|^2+|\xi-\eta|^2}$ so that any term in Q is of the form $(U\circ B_j)[z^{\pm},z^{\pm}],\ j=1\ldots 5$ where B_j satisfies $B_j(\eta,\xi-\eta)\lesssim 2+|\eta|^2+|\xi-\eta|^2$. From now on, we focus on the estimate

$$\sup_{0\leq t\leq T}\langle t\rangle^{1+3\varepsilon}\Big\|\int_0^t e^{i(t-s)H}U\circ B_j[z^{\pm},z^{\pm}](s)ds\Big\|_{W^{k,p}}\lesssim \|z\|_{X_T}^2.\tag{6.6}$$

6.1.1. Splitting of the phase space. Let $(\chi^a)_{a\in 2^{\mathbb{Z}}}$ be a standard dyadic partition of unity: $\chi^a\geq 0, \mathrm{supp}(\chi^a)\subset\{|\xi|\sim a\}, \forall\,\xi\in\mathbb{R}^3\backslash\{0\},\ \sum_a\chi^a(\xi)=1$. We define the frequency localized symbol

$$B_j^{a,b,c}=\chi^a(\xi)\chi^b(\eta)\chi^c(\zeta)B_j(\eta,\xi-\eta),\ \text{ where }\zeta=\xi-\eta.$$

While there are actually only two variables (η,ξ), in order to fully exploit Theorem 2.4 it is convenient to consider B both as a function of η,ξ and of ζ,ξ. Note that due to the relation $\xi=\eta+\zeta$, we have only to consider $B_j^{a,b,c}$ when $a\lesssim b\sim c,\ b\lesssim c\sim a$ or $c\lesssim a\sim b$.

Consider the Fourier transform of the frequency localized term

$$\begin{aligned}&\mathcal{F}\Big(\int_0^t e^{i(t-s)H}(U\circ B_j^{a,b,c})[z^{\pm},z^{\pm}](s)ds\Big)\\ &\quad=e^{itH(\xi)}\int_0^t\int_{\mathbb{R}^d}e^{-is\Omega}U(\xi)B_j^{a,b,c}(\eta,\xi-\eta)\widetilde{z^{\pm}}(s,\eta)\widetilde{z^{\pm}}(s,\xi-\eta)d\eta\,ds,\end{aligned}$$

where $\Omega=-i(H(\xi)\mp H(\eta)\mp H(\xi-\eta)),\ \widetilde{z^{\pm}}=\mathcal{F}\big((e^{-itH}z)^{\pm}\big)$. The main strategy to obtain estimate (6.6) follows the idea described in the introduction paragraph 206. Namely we perform an integration by part in the s or η variable, and use Theorem 2.4. In order to do so, we need estimates on the bilinear symbols that appear after integration by parts. The multiplier estimates are stated in Lemmas 6.1 and 6.2, however they require a further localization. In the "Appendix" we construct a function $\Phi(\xi,\eta)$ such that

[5] Q contains only derivatives of order at most 2, so $N\geq k+5$ suffices.

$\Phi + (1 - \Phi)$ splits the phase space in non time resonant and non space resonant parts in the following sense:

$$\forall a, b, c \begin{cases} \Phi U B_j^{a,b,c} := U B_j^{a,b,c,T} \text{ satisfies the estimates of Lemma 6.1,} \\ (1-\Phi) U B_j^{a,b,c} := U B_j^{a,b,c,X} \text{ satisfies the estimates of Lemma 6.2.} \end{cases} \tag{6.7}$$

Finally, we define

$$I^{a,b,c,T} = \int_0^t e^{i(t-s)H} U B_j^{a,b,c,T} ds, \quad I^{a,b,c,X} = \int_0^t e^{i(t-s)H} U B_j^{a,b,c,X} ds.$$

Using integration by parts in time (resp. in the "space" variable η), we will prove

$$\sup_{[0,T]} t^{1+3\varepsilon} \| \sum_{a,b,c} I^{a,b,c,T} \|_{W^{k,p}} \lesssim \|z\|_{X_T}^2, \text{ resp. } \sup_{[0,T]} t^{1+3\varepsilon} \| \sum_{a,b,c} I^{a,b,c,X} \|_{W^{k,p}} \lesssim \|z\|_{X_T}^2.$$

The estimates of $I^{a,b,c,T}$ are made in Sect. 6.1.2, the estimates of $I^{a,b,c,X}$ are made in Sect. 6.1.3.

Remark 6.2. The estimate $\sum_{a,b,c} \sup_{[0,T]} t^{1+3\varepsilon} \|I^{a,b,c,T}\|_{W^{k,p}} \lesssim \|z\|_{X_T}^2$ does not seem to be true. We will see later that it is the summation in a which causes an issue, but this can be overcome thanks to the fact that Theorem 2.4 only requires L_ξ^∞ bounds for the symbols, so that (crudely) we can replace the sum in a by an l_a^∞ bound.

Remark 6.3. While this is hidden by our notations, the function Φ depends on the various $\pm$ cases, the phase space partition to treat a z^2 type nonlinearity is not the same as for a $|z|^2$ type nonlinearity.

6.1.2. Control of non time resonant terms. The generic frequency localized quadratic term is

$$e^{itH(\xi)} \int_0^t \int_{\mathbb{R}^d} e^{-is(H(\xi) \mp H(\eta) \mp H(\xi-\eta))} U(\xi) B_j^{a,b,c,T} \\ \times (\eta, \xi - \eta) \widetilde{z^{\pm}}(s, \eta) \widetilde{z^{\pm}}(s, \xi - \eta) d\eta\, ds. \tag{6.8}$$

Regardless of the $\pm$, we set $\Omega = H(\xi) \mp H(\eta) \mp H(\xi - \eta)$. An integration by part in s gives using the fact that $e^{-is\Omega} = \frac{-1}{i\Omega} \partial_s (e^{is\Omega})$ and $\partial_s \widetilde{z^{\pm}}(\eta) = e^{\mp isH(\eta)} \widehat{(\mathcal{N}_z)}^{\pm}(\eta)$, $\partial_s \widetilde{z^{\pm}}(\xi - \eta) = e^{\mp isH(\xi-\eta)} \widehat{(\mathcal{N}_z)}^{\pm}(\xi - \eta)$:

$$\begin{aligned} I^{a,b,c,T} &= \mathcal{F}^{-1}\Big(e^{itH(\xi)} \int_0^t \int_{\mathbb{R}^N} \frac{1}{i\Omega} e^{-is\Omega} U(\xi) B_j^{a,b,c,T}(\eta, \xi-\eta) \partial_s \big(\widetilde{z^{\pm}}(\eta, s) \widetilde{z^{\pm}}(\xi - \eta, s) \big) d\eta ds \Big) \\ &\quad - \Big[\mathcal{F}^{-1} \Big(e^{itH(\xi)} \int_{\mathbb{R}^N} \frac{1}{i\Omega} e^{-is\Omega(\xi,\eta)} U(\xi) B_j^{a,b,c,T}(\eta, \xi - \eta) \big(\widetilde{z^{\pm}}(\eta, s) \widetilde{z^{\pm}}(\xi - \eta, s) \big) d\eta ds \Big) \Big]_0^t \\ &= \int_0^t e^{i(t-s)H} \Big(\mathcal{B}_3^{a,b,c,T}[(\mathcal{N}_z)^{\pm}(s), z^{\pm}(s)] + \mathcal{B}_3^{a,b,c,T}[z^{\pm}(s), (\mathcal{N}_z)^{\pm}(s)] \Big) ds \\ &\quad - \big[e^{i(t-s)H} \mathcal{B}_3^{a,b,c,T}[z^{\pm}(s), z^{\pm}(s)] \big]_0^t, \end{aligned} \tag{6.9}$$

with $\mathcal{B}_3^{a,b,c,T}(\eta,\xi-\eta)=\dfrac{U(\xi)}{i\Omega}B_j^{a,b,c,T}(\eta,\xi-\eta)$ (we drop the dependency in j as all estimates will not depend on it).

In order to use the rough multiplier estimate from Theorem 2.4, we need to control $\mathcal{B}_3^{a,b,c,T}$. The following lemma extends to our settings the crucial multiplier estimates from [22].

Lemma 6.1. *Let* $m=\min(a,b,c),\ M=\max(a,b,c),\ l=\min(b,c)$. *For* $0<s<2$, *we have*

$$\text{if } M\gtrsim 1,\ \|\mathcal{B}_3^{a,b,c,T}\|_{[B^s]}\lesssim\frac{\langle M\rangle l^{\frac{3}{2}-s}}{\langle a\rangle},\quad \text{if } M<<1,\quad \|\mathcal{B}_3^{a,b,c,T}\|_{[B^s]}\lesssim l^{1/2-s}M^{-s}. \tag{6.10}$$

We postpone the proof to the "Appendix".

Remark 6.4. We treat differently M small and M large since we have a loss of derivative on the symbol in low frequencies. Let us mention that the estimate (6.10) can be written simply as follows:

$$\|\mathcal{B}_3^{a,b,c,T}\|_{[B^s]}\lesssim\frac{\langle M\rangle\langle l\rangle l^{\frac{1}{2}-s}U(M)^{-s}}{\langle a\rangle}.$$

Lets us start by estimating the first term in (6.9): we split the time integral between $[0,t-1]$ and $[t-1,t]$. The sum over a,b,c involves three cases: $b\lesssim a\sim c,\ c\lesssim a\sim b$ and $a\lesssim b\sim c$.

The case $b\lesssim a\sim c$: for $k_1\in[0,k]$ we have from Theorem 2.4 with $\sigma=1+3\varepsilon$:

$$\begin{aligned}
&\Big\|\nabla^{k_1}\int_0^{t-1}e^{i(t-s)H}\sum_{b\lesssim a\sim c}\mathcal{B}_3^{a,b,c,T}[\mathcal{N}_z^{\pm},z^{\pm}]ds\Big\|_{L^p}\\
&\lesssim\int_0^{t-1}\frac{1}{(t-s)^{1+3\varepsilon}}\sum_{b\lesssim a\sim c}\langle a\rangle^{k_1}\|\mathcal{B}_3^{a,b,c,T}[\mathcal{N}_z^{\pm},z^{\pm}]\|_{L^{p'}}ds,\\
&\lesssim\int_0^{t-1}\frac{1}{(t-s)^{1+3\varepsilon}}\Bigg(\sum_{b\lesssim a\sim c\lesssim 1}ab\|\mathcal{B}_3^{a,b,c,T}\|_{[B^\sigma]}\|U^{-1}Q(z)\|_{L^2}\|U^{-1}z\|_{L^2}\\
&\quad+\sum_{b\lesssim a\sim c,1\lesssim a}\langle c\rangle^{-N+k}U(b)\|\mathcal{B}_3^{a,b,c,T}\|_{[B^\sigma]}\|U^{-1}Q(z)\|_{L^2}\|\langle\nabla\rangle^N z\|_{L^2}\Bigg)ds+\mathcal{R},
\end{aligned}\tag{6.11}$$

where $\mathcal{R}=\displaystyle\int_0^{t-1}\frac{1}{(t-s)^{1+3\varepsilon}}\sum_{b\lesssim a\sim c}\langle a\rangle^{k_1}\|\mathcal{B}_3^{a,b,c,T}[R^{\pm},z^{\pm}]\|_{L^{p'}}ds$. Using Lemma 6.1 we have, provided $\epsilon<\frac{1}{12}$ and $N-k-\frac{1}{2}+3\epsilon>0$:

$$\sum_{b\lesssim a\sim c\lesssim 1} ab\|\mathcal{B}_3^{a,b,c,T}\|_{[B^\sigma]}\lesssim \sum_{a\lesssim 1}\sum_{b\lesssim a} abb^{1/2-1-3\varepsilon}a^{-1-3\varepsilon}\lesssim \sum_{a\lesssim 1} a^{1/2-6\varepsilon}\lesssim 1,$$

$$\sum_{b\lesssim a\sim c,\ a\gtrsim 1} U(b)\langle c\rangle^{-N+k}\|\mathcal{B}_3^{a,b,c,T}\|_{[B^\sigma]}\lesssim \sum_{a\gtrsim 1}\sum_{b\lesssim a} U(b)\frac{b^{\frac{1}{2}-3\epsilon}}{a^{N-k}}\lesssim \sum_{a\gtrsim 1}\frac{1}{a^{N-k-\frac{1}{2}+3\epsilon}}\lesssim 1.$$

Using the gradient structure of $Q(z)$ (see 5.12) and by interpolation :

$$\|U^{-1}Q(z)\|_{L^2}\lesssim \|z\|_{W^{2,4}}^2\lesssim \|z\|_{W^{2,6}}^{\frac{3}{2}}\|z\|_{H^2}^{\frac{1}{2}}, \tag{6.12}$$

so that if we combine these estimates with (6.1), we get

$$\|\nabla^{k_1}\int_0^{t-1} e^{i(s-t)H}\sum_{b\lesssim a\sim c}\mathcal{B}_3^{a,b,c,T}[Q(z)^{\pm},z]ds\|_{L^p}\lesssim \|z\|_{X_T}^3\int_0^{t-1}\frac{1}{(t-s)^{1+3\varepsilon}}\frac{1}{\langle s\rangle^{\frac{3}{2}}}ds$$
$$\lesssim \frac{\|z\|_{X_T}^3}{t^{1+3\varepsilon}}.$$

We bound now $\mathcal{R}$ from (6.11): contrary to the quadratic terms, cubic terms have no gradient structure, however the nonlinearity is so strong that we can simply use $\|1_{|\eta|\lesssim 1}U^{-1}R\|_2\lesssim \|R\|_{L^{6/5}}$. Using the same computations as for quadratic terms we get

$$\|\nabla^{k_1}\int_0^{t-1} e^{i(t-s)H}\sum_{b\lesssim a\sim c}\mathcal{B}_3^{a,b,c,T}[R,z^{\pm}]ds\|_{L^p}$$
$$\lesssim \int_0^{t-1}\frac{1}{(t-s)^{1+3\varepsilon}}\Big(\|1_{|\eta|\lesssim 1}U^{-1}R\|_{L^2}\|U^{-1}z\|_{L^2}+\|U^{-1}R\|_{L^2}\|\langle\nabla\rangle^N z\|_{L^2}\Big)ds.$$

According to (5.12) the cubic terms involve only smooth multipliers and do not contain derivatives of order larger than 2, thus we can generically treat them like $(\langle\nabla\rangle^2 z)^3$ using the Proposition 5.4; we have then:

$$\|R\|_{L^{6/5}}\lesssim \|z\|_{H^2}\|z\|_{W^{2,6}}^2\lesssim \frac{\|z\|_{X_T}^3}{\langle t\rangle^2},\ \ \|R\|_{L^2}\lesssim \|z\|_{W^{2,6}}^3\lesssim \frac{\|z\|_{X_T}^3}{\langle t\rangle^2}.$$

This closes the estimate as $\displaystyle\int_0^{t-1}\frac{1}{(t-s)^{1+3\varepsilon}\langle s\rangle^2}ds\lesssim \frac{1}{t^{1+3\varepsilon}}$. Similar computations can be done for the quartic terms.

It remains to deal with the term $\int_{t-1}^t$, using the Sobolev embedding we have:

$$\|\nabla^{k_1}\int_{t-1}^t e^{i(t-s)H}\sum_{b\lesssim a\sim c}\mathcal{B}_3^{a,b,c,T}[\mathcal{N}_z^{\pm},z^{\pm}]ds\|_{L^p}\lesssim \int_{t-1}^t \|(\cdots)\|_{H^{k_2}}ds,$$

with $k_2 = k + 1 + 3\varepsilon$. Again, with $\sigma = 1 + 3\varepsilon$ we get using Theorem 2.4 and Sobolev embedding:

$$\|\nabla^{k_1}\int_{t-1}^{t} e^{i(t-s)H}\sum_{b\lesssim a\sim c}\mathcal{B}_3^{a,b,c,T}[\mathcal{N}_z^{\pm},z^{\pm}]ds\|_{L^p}\lesssim\int_{t-1}^{t}\|\sum_{b\lesssim a\sim c}\mathcal{B}_3^{a,b,c,T}[\mathcal{N}_z^{\pm},z^{\pm}]\|_{H^{k_2}}ds$$
$$\lesssim\int_{t-1}^{t}\Big(\sum_{b\lesssim a\sim c\lesssim 1}ab\|\mathcal{B}_3^{a,b,c,T}\|_{[B^\sigma]}\|U^{-1}Q\|_{L^2}\|U^{-1}z\|_{L^p}$$
$$+\sum_{b\lesssim a\sim c,1\lesssim a}U(b)a^{k_2-(N-1-3\epsilon)}\|\mathcal{B}_3^{a,b,c,T}\|_{[B^\sigma]}\|U^{-1}Q\|_{L^2}\|\langle\nabla\rangle^N z\|_{L^2}\Big)ds+\mathcal{R},$$

where $\mathcal{R}$ contains higher order terms that are easily controlled. Using $\|U^{-1}z\|_{L^p}\lesssim\|z\|_{H^2}$ and the same estimates as previously, we can conclude for N sufficiently large:

$$\|\nabla^{k_1}\int_{t-1}^{t} e^{i(t-s)H}\sum_{b\lesssim a\sim c}\mathcal{B}_3^{a,b,c,T}[\mathcal{N}_z^{\pm},z^{\pm}]ds\|_{L^p}\lesssim\|u\|_{X_T}^3\int_{t-1}^{t}\frac{1}{\langle s\rangle^{3/2}}ds\lesssim\frac{\|z\|_{X_T}^3}{t^{1+3\varepsilon}}.$$

The case $c\lesssim a\sim b$ As for $b\lesssim a\sim c$ we use $\sigma = 1+3\varepsilon$ and start with

$$\Big\|\nabla^{k_1}\int_{1}^{t-1} e^{i(t-s)H}\sum_{c\lesssim a\sim b}\mathcal{B}_3^{a,b,c,T}[\mathcal{N}_z^{\pm},z^{\pm}]ds\Big\|_{L^p}$$
$$\lesssim\int_{1}^{t-1}\frac{1}{(t-s)^{1+3\varepsilon}}\Bigg(\sum_{c\lesssim a\sim b\lesssim 1}bc\|\mathcal{B}_3^{a,b,c,T}\|_{[B^\sigma]}\|U^{-1}Q(z)\|_{L^2}\|U^{-1}z\|_{L^2}$$
$$+\sum_{c\lesssim a\sim b,1\lesssim a}\langle b\rangle^{-1}\|\mathcal{B}_3^{a,b,c,T}\|_{[B^\sigma]}\|\langle\nabla\rangle^{k+1}Q(z)\|_{L^2}\|z\|_{L^2}\Bigg)ds+\mathcal{R}.$$

$\mathcal{R}$ contains the higher order nonlinear terms which, again, we will not detail. This case is symmetric from $b\lesssim a\sim c$ except for the term $\|\langle\nabla\rangle^{k+1}Q(z)\|_{L^2}$, which is estimated as follows. Let $1/q = 1/3+\varepsilon$, $k_3 = \frac{1}{2}-3\varepsilon$. If $k+2+k_3\le N$ then using the structure of Q (see (6.5)) and Gagliardo Nirenberg inequalities we get:

$$\|\langle\nabla\rangle^{k+1}Q(z)\|_{L^2}\lesssim\|z\|_{W^{2,p}}\|z\|_{W^{k+3,q}}\lesssim\|z\|_{W^{2,p}}\|z\|_{H^{k+3+k_3}}\lesssim\|z\|_X^2/\langle t\rangle^{1+3\varepsilon}.$$

Using the multiplier bounds as for the case $b\lesssim a\sim c$, we obtain via the Lemma 6.1:

$$\Big\|\nabla^{k_1}\int_{0}^{t-1} e^{i(t-s)H}\sum_{c\lesssim a\sim b}\mathcal{B}_3^{a,b,c,T}[\mathcal{N}_z^{\pm},z^{\pm}]ds\Big\|_{L^p}\lesssim\|z\|_X^3\int_0^{t-1}\frac{1}{(t-s)^{1+3\varepsilon}}\frac{1}{\langle s\rangle^{(1+3\varepsilon)}}ds$$
$$\lesssim\frac{\|z\|_X^3}{\langle t\rangle^{1+3\varepsilon}}.$$

The bound for the integral on $[t-1,t]$ is obtained by similar arguments.

The case $a \lesssim b \sim c$ We have using Theorem 2.4 and the fact that the support of $\mathcal{F}(\sum_{a\lesssim b} a^{k_1}\mathcal{B}_3^{a,b,c,T}[\mathcal{N}_z^{\pm}, z^{\pm}])$ is localized in a ball $B(0,b)$:

$$\|\nabla^{k_1}\int_0^{t-1} e^{i(t-s)H}\sum_{a\lesssim b\sim c}\mathcal{B}_3^{a,b,c,T}[\mathcal{N}_z^{\pm}, z^{\pm}]ds\|_{L^p}$$
$$\lesssim \int_0^{t-1}\frac{1}{(t-s)^{1+3\varepsilon}}\|\sum_{a\lesssim b\sim c} a^{k_1}\mathcal{B}_3^{a,b,c,T}[\mathcal{N}_z^{\pm}, z^{\pm}]\|_{L^{p'}}ds$$
$$\lesssim \int_0^{t-1}\frac{1}{(t-s)^{1+3\varepsilon}}\sum_{b\sim c}\frac{1}{\langle b\rangle^{N-2}}U(b)U(c)\left\|\sum_{a\lesssim b}\langle a\rangle^k\mathcal{B}_3^{a,b,c,T}\ \|_{[B^\sigma]}\|\ U^{-1}Q(z)\right\|_{L^2}$$
$$\left\|U^{-1}\langle\nabla\rangle^N z\right\|_{L^2}ds+\mathcal{R},$$

where as previously, $\mathcal{R}$ is a remainder of higher order terms that are not difficult to bound. We observe that for any symbols $(B^a(\xi,\eta))$ such that

$$\forall\,\eta,\ |a_1-a_2|\geq 2\Rightarrow \mathrm{supp}(B^{a_1}(\cdot,\eta))\cap\mathrm{supp}(B^{a_2}(\cdot,\eta))=\emptyset,$$

then

$$\|\sum_a B^a\|_{[B^\sigma]}\lesssim \sup_a\|B^a\|_{[B^\sigma]}. \tag{6.13}$$

This implies using Lemma 6.1 and provided that N is large enough:

$$\sum_{b\sim c}\frac{1}{\langle b\rangle^{N-2}}U(b)U(c)\|\sum_{a\lesssim b}\langle a\rangle^k\mathcal{B}_3^{a,b,c,T}\|_{[B^\sigma]}$$
$$\lesssim \sum_b\frac{1}{\langle b\rangle^{N-2}}U(b)^2\sup_{a\lesssim b}\langle a\rangle^k\frac{b^{\frac12-\sigma}U(M)^{-\sigma}\langle b\rangle\langle M\rangle}{\langle a\rangle}$$
$$\lesssim \sum_b\frac{U(b)^{5/2-2\sigma}}{\langle b\rangle^{N+\sigma-k-7/2}}\lesssim 1.$$

We have finally using (6.12):

$$\|\nabla^{k_1}\int_0^{t-1} e^{i(t-s)H}\sum_{a\lesssim b\sim c}\mathcal{B}_3^{a,b,c,T}[\mathcal{N}_z^{\pm}, z^{\pm}]ds\|_{L^p}\lesssim \|z\|_X^3\int_0^{t-1}\frac{1}{(t-s)^{1+3\varepsilon}}\frac{1}{\langle s\rangle^{3/2}}ds$$
$$\lesssim \frac{\|u\|_X^3}{t^{1+3\varepsilon}}.$$

We proceed in a similar way to deal with the integral on $[t-1,t]$. This end the estimate for the first term in (6.9).

The second term is symmetric from the first, it remains to deal with the boundary term: $\|\nabla^{k_1}\big[e^{i(t-s)H}\mathcal{B}_3^{a,b,c,T}[z^{\pm}, z^{\pm}]\big]_0^t\|_{L^p}$. We have:

$$\|\big[\nabla^{k_1}e^{i(t-s)H}\mathcal{B}_3^{a,b,c,T}[z^{\pm}, z^{\pm}]\big]_0^t\|_{L^p}\leq \|\nabla^{k_1}e^{-itH}\mathcal{B}_3^{a,b,c,T}[z_0^{\pm}, z_0^{\pm}]\|_{L^p}$$
$$+\|\nabla^{k_1}\mathcal{B}_3^{a,b,c,T}[z^{\pm}(t), z^{\pm}(t)]\|_{L^p}. \tag{6.14}$$

The first term on the right hand-side of (6.14) is easy to deal with using the dispersive estimates of the Theorem 2.5. For the second term we focus on the case $b \lesssim a \sim c$, the other areas can be treated in a similar way. Using Proposition 6.1, Sobolev embedding and the rough multiplier Theorem 2.4 with $\sigma = 1 + 3\epsilon$, $q_1 = q_2 = q_3 = p$ we have with χ as in prop 6.1:

$$\sum_{b \lesssim a \sim c \lesssim 1} \|\nabla^{k_1} \mathcal{B}_3^{a,b,c,T}[z^{\pm}(t), z^{\pm}(t)]\|_{L^p}$$

$$\lesssim \sum_{b \lesssim a \sim c} b^{-\frac{1}{2}-3\epsilon} a^{-1-3\epsilon} U(b)U(c) \|\chi(|\nabla|)U^{-1} z\|_{L^p}^2$$

$$\lesssim \sum_{b \lesssim a \sim c} b^{-\frac{1}{2}-3\epsilon} a^{-1-3\epsilon} U(b)U(c) \|U^{-1+3\epsilon} z\|_{L^6}^2 \lesssim \frac{\|z\|_{X_T}^2}{\langle t\rangle^{\frac{6}{5}+6\epsilon}},$$

$$\sum_{b \lesssim a \sim c,\ a \gtrsim 1} \|\nabla^{k_1} \mathcal{B}_3^{a,b,c,T}[z^{\pm}(t), z^{\pm}(t)]\|_{L^p} \lesssim \sum_{b \lesssim a \sim c,\ a \gtrsim 1} \frac{\langle a\rangle^{k_1} b^{1/2-3\epsilon}}{\langle a\rangle^{k_1+1}} \|z\|_{L^p} \|z\|_{W^{k+1,p}} \lesssim \frac{\|z\|_{X_T}^2}{\langle t\rangle^{\frac{3}{2}(1+3\varepsilon)}}.$$

where in the last inequality we also used $\|z\|_{W^{k+1,6}}^2 \lesssim \|z\|_{W^{k,p}} \|z\|_{W^{k+2,p}} \lesssim \|z\|_{W^{k,p}} \|z\|_{H^N}$.

6.1.3. Non space resonance. In this section we treat the term $\sum_{a,b,c} I^{a,b,c,X}$. Since control for t small just follows from the H^N bounds, we focus on $t \geq 1$, and first note that the integral over $[0, 1] \cup [t-1, t]$ is easy to estimate.

Bounds for $(\int_0^1 + \int_{t-1}^t) e^{i(t-s)H} Q(z) ds$

In order to estimate $\|\nabla^{k_1} \int_{t-1}^t e^{i(t-s)H} Q(z) ds\|_{L^p}$, with $k_1 \in [0, k]$ we can simply use Sobolev's embeddings $H^{k+2} \hookrightarrow W^{k,p}$, $H^N \hookrightarrow W^{k+4,q}$, $\frac{1}{2} = \frac{1}{q} + \frac{1}{p}$ and a Gagliardo-Nirenberg type inequality (2.8) :

$$\begin{aligned}\|\int_{t-1}^t \nabla^{k_1} e^{i(t-s)H} Q(z) ds\|_{L^p} &\lesssim \int_{t-1}^t \|Q(z)\|_{H^{k+2}} ds \\ &\lesssim \int_{t-1}^t \|z\|_{W^{k+4,q}} \|z\|_{W^{k,p}} ds \\ &\lesssim \|z\|_X^2 \int_{t-1}^t \frac{1}{\langle s\rangle^{1+3\varepsilon}} ds \lesssim \frac{\|z\|_X^2}{\langle t\rangle^{1+3\varepsilon}}.\end{aligned}$$

The estimate on $[0, 1]$ follows from similar computations using Minkowski's inequality and the dispersion estimate from Theorem 2.5.

Frequency splitting

Since we only control $xe^{-itH} z$ in $L^\infty L^2$, in order to handle the loss of derivatives we follow the idea from [15] which corresponds to distinguish low and high frequencies with a threshold frequency depending on t. Let $\theta \in C_c^\infty(\mathbb{R}^+)$, $\theta|_{[0,1]} = 1$, $\text{supp}(\theta) \subset [0, 2]$, $\Theta(t) = \theta(\frac{|\nabla|}{t^\delta})$, for any quadratic term $B_j[z^\pm, z^\pm]$, we write

$$B_j[z^\pm, z^\pm] = \overbrace{B_j[(1-\Theta(t))z^\pm, z^\pm] + B_j[\Theta(t)z^\pm, (1-\Theta)(t)z^\pm]}^{\text{high frequencies}} + \overbrace{B_j[\Theta(t)z^\pm, \Theta(t)z^\pm]}^{\text{low frequencies}}. \tag{6.15}$$

The main idea here is that thanks to the relation $|\xi|\theta(|\xi|/t^\delta) \lesssim t^\delta$, loss of derivatives is "paid" with some growth in t, but since the decay is slightly stronger than needed we can absorb this growth.

High frequencies. Using the dispersion Theorem 2.5, Gagliardo-Nirenberg estimate (2.8) and Sobolev embedding we have for $\frac{1}{p_1} = \frac{1}{3} + \varepsilon$ and for any quadratic term of Q written under the form $UB_j[z^\pm, z^\pm]$:

$$\begin{aligned}
&\left\| \int_1^{t-1} e^{i(t-s)H}\big(UB_j[(1-\Theta(s))z^\pm, z^\pm] + UB_j[\Theta(t)z, (1-\Theta)(s)z^\pm]\big)ds \right\|_{W^{k,p}} \\
&\quad \leq \int_1^{t-1} \frac{1}{(t-s)^{1+3\varepsilon}} \|z\|_{W^{k+2,p_1}} \|(1-\Theta(s))z\|_{H^{k+2}} ds \\
&\quad \leq \int_1^{t-1} \frac{1}{(t-s)^{1+3\varepsilon}} \|z\|_{H^N}^2 \frac{1}{s^{\delta(N-2-k)}} ds,
\end{aligned} \tag{6.16}$$

choosing N large enough so that $\delta(N-2-k) \geq 1+3\varepsilon$, we obtain the expected decay.

Low frequencies. The low frequency part of quadratic terms reads in the Duhamel formula

$$\mathcal{F}I_3^{a,b,c,X} = e^{itH(\xi)} \int_1^{t-1} \int_{\mathbb{R}^N} e^{-is\Omega} UB_j^{a,b,c,X}(\eta, \xi-\eta) \widetilde{\Theta z^\pm}(s,\eta) \widetilde{\Theta z^\pm}(s, \xi-\eta) d\eta ds,$$

with $\Omega = H(\xi) \mp H(\eta) \mp H(\xi-\eta)$. Using $e^{-is\Omega} = \dfrac{i\nabla_\eta \Omega}{s|\nabla_\eta\Omega|^2} \cdot \nabla_\eta e^{-is\Omega}$ and denoting $\text{Ri} = \frac{\nabla}{|\nabla|}$ the Riesz operator, $\Theta'(t) := \theta'(\frac{|\nabla|}{t^\delta})$, $J = e^{itH} x e^{-itH}$, an integration by part in η gives:

$$\begin{aligned}
I_3^{a,b,c,X} &= -\mathcal{F}^{-1}\Big(e^{itH(\xi)} \int_1^{t-1} \frac{1}{s} \int_{\mathbb{R}^N} e^{-is\Omega(\xi,\eta)} \big(\mathcal{B}_1^{a,b,c,X}(\eta, \xi-\eta) \cdot \nabla_\eta [\Theta \widetilde{z^\pm}(\eta) \Theta \widetilde{z^\pm}(\xi-\eta)] \\
&\quad + \mathcal{B}_2^{a,b,c,X}(\eta, \xi-\eta) \widetilde{\Theta z^\pm}(\eta) \widetilde{\Theta z^\pm}(\xi-\eta) d\eta \big) ds \Big) \\
&= -\int_1^{t-1} \frac{1}{s} e^{i(t-s)H} \Big(\mathcal{B}_1^{a,b,c,X}[\Theta(s)(Jz)^\pm, \Theta(s)z^\pm] - \mathcal{B}_1^{a,b,c,X}[\Theta(s)z^\pm, \Theta(s)(Jz)^\pm] \\
&\quad + \mathcal{B}_2^{a,b,c,X}[\Theta(s)z^\pm, \Theta(s)z^\pm] \Big) ds \\
&\quad - \int_1^{t-1} \frac{1}{s} e^{i(t-s)H} \Big(\mathcal{B}_1^{a,b,c,X}[\frac{1}{s^\delta} Ri\, \Theta'(s) z^\pm, \Theta(s)z^\pm] \\
&\quad - \mathcal{B}_1^{a,b,c,X}[\Theta(s)z^\pm, \frac{1}{s^\delta} Ri \Theta'(s) z^\pm] \Big) ds.
\end{aligned} \tag{6.17}$$

As previously, we drop the j index since all multipliers satisfy the same estimates:

$$\mathcal{B}_1^{a,b,c,X} = \frac{U(\xi)\nabla_\eta\Omega}{|\nabla_\eta\Omega|^2} B_j^{a,b,c,X},\ \mathcal{B}_2^{a,b,c,X} = \nabla_\eta B_j^{a,b,c,X}.$$

The following counterpart of Lemma 6.1 slightly improves the estimates from [22].

Lemma 6.2. *Denoting* $M = \max(a,b,c)$, $m = \min(a,b,c)$ *and* $l = \min(b,c)$ *we have:*

- *If* $M << 1$ *then for* $0 \le s \le 2$*:*

$$\|\mathcal{B}_1^{a,b,c,X}\|_{[B^s]} \lesssim l^{\frac{3}{2}-s} M^{1-s},\quad \|\mathcal{B}_2^{a,b,c,X}\|_{[B^s]} \lesssim l^{\frac{1}{2}-s} M^{-s}, \tag{6.18}$$

- *If* $M \gtrsim 1$ *then for* $0 \le s \le 2$*:*

$$\|\mathcal{B}_1^{a,b,c,X}\|_{[B^s]} \lesssim \langle M\rangle^2 l^{3/2-s}\langle a\rangle^{-1},\quad \|\mathcal{B}_2^{a,b,c,X}\|_{[B^s]} \lesssim \langle M\rangle^2 l^{1/2-s}\langle a\rangle^{-1}. \tag{6.19}$$

We now use these estimates to bound the first term of (6.17). As in 6.1.2, there are three cases to consider: $b \lesssim c \sim a,\ c \lesssim c \lesssim a \sim b,\ a \lesssim b \sim c$.

Estimates for quadratic terms involving $\mathcal{B}_1^{a,b,c}$ In the case $c \lesssim a \sim b$, let $\varepsilon_1 > 0$ to be fixed later. Using Minkowski's inequality, dispersion and the rough multiplier Theorem 2.4 with $s = 1+\varepsilon_1$, $\frac{1}{q} = 1/2+\varepsilon-\frac{\varepsilon_1}{3}$ for $a \lesssim 1$, $s = 4/3$, $\frac{1}{q_1} = 7/18+\varepsilon$ for $a \gtrsim 1$ we obtain

$$\begin{aligned}
&\Big\|\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\sum_{c\lesssim a\sim b}\mathcal{B}_1^{a,b,c,X}[\Theta(s)(Jz)^{\pm},\Theta(s)z^{\pm}]ds\Big\|_{L^p}\\
&\lesssim \int_1^{t-1}\frac{1}{s(t-s)^{1+3\varepsilon}}\sum_{c\lesssim a\sim b\lesssim 1}\|\mathcal{B}_1^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]}\|\Theta(s)Jz\|_{L^2}\|\Theta(s)z]\|_{L^q}\\
&\quad+\sum_{c\lesssim a\sim b,\ 1\lesssim a\lesssim s^\delta} a^k\|\mathcal{B}_1^{a,b,c,X}\|_{[B^{4/3}]}\|\Theta(s)Jz\|_{L^2}\|\Theta(s)z]\|_{L^{q_1}}\Big)ds\\
&\lesssim \int_1^{t-1}\frac{1}{s(t-s)^{1+3\varepsilon}}\Bigg(\sum_{a\lesssim 1}\sum_{c\lesssim a\sim b}\|\mathcal{B}_1^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]}\|\Theta(s)Jz\|_{L^2}\|\Theta(s)z]\|_{L^q}\\
&\quad+\sum_{1\lesssim a\lesssim s^\delta}a^k\sum_{c\lesssim a\sim b}\|\mathcal{B}_1^{a,b,c,X}\|_{[B^{4/3}]}\|\Theta(s)Jz\|_{L^2}\|\Theta(s)z]\|_{L^{q_1}}\Bigg)ds.
\end{aligned}$$

Using Lemma 6.2 and interpolation we have for $\varepsilon_1 < 1/4$ and $\varepsilon_1 - 3\varepsilon > 0$,

$$\sum_{a\lesssim 1}\sum_{c\lesssim a\sim b}\|\mathcal{B}_1^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]} \lesssim \sum_{a\lesssim 1}a^{1-(1+\varepsilon_1)}\sum_{c\lesssim a}c^{\frac{3}{2}-(1+\varepsilon_1)} \lesssim 1,$$

$$\|\psi(s)\|_{L^q} \lesssim \|\psi(s)\|_{L^p}^{\frac{\varepsilon_1-3\varepsilon}{1+3\varepsilon}}\|\psi(s)\|_{L^2}^{1-\frac{\varepsilon_1-3\varepsilon}{1+3\varepsilon}} \lesssim \frac{\|\psi\|_X}{s^{\varepsilon_1-3\varepsilon}}.$$

In high frequencies we have:

$$\sum_{1\lesssim a\lesssim s^\delta} a^k \sum_{c\lesssim a\sim b} \frac{\langle M\rangle^2 c^{3/2-4/3}}{\langle a\rangle} \lesssim s^{\delta(k+7/6)},\ \ \|\psi(s)\|_{L^{q_1}} \lesssim \frac{\|\psi\|_{X_T}}{s^{1/3-3\varepsilon}}.$$

Finally we conclude that if $\min\big(\varepsilon_1-3\varepsilon, 1/3-3\varepsilon-\delta(k+7/6)\big)\geq 3\varepsilon$ (this choice is possible provided ε and δ are small enough):

$$\begin{aligned}&\|\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\sum_{a,b,c}\mathcal{B}_1^{a,b,c,X}[\Theta(s)(Jz)^\pm,\Theta(s)z^\pm]ds\|_{L^p}\\ &\lesssim \int_1^{t-1}\frac{\|z\|_X^2}{s^{1+3\varepsilon}(t-s)^{1+3\varepsilon}}ds\\ &\lesssim \frac{\|z\|_{X_T}^2}{t^{1+3\varepsilon}}.\end{aligned}$$

We do not detail the case $b\lesssim c\sim a$ which is very similar. The case $a\lesssim b\sim c$ involves an infinite sum over a which can be handled as in the non time resonant case with observation (6.13). The term $\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\mathcal{B}_1^{a,b,c,X}[\Theta(s)z^\pm,\Theta(s)(Jz)^\pm]ds$ is symmetric while the terms

$$\begin{aligned}&\Big\|\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\big(\mathcal{B}_1^{a,b,c,X}[\frac{1}{s^\delta}\mathrm{Ri}\Theta'(s)z^\pm,\Theta(s)z^\pm]\\ &\quad -\mathcal{B}_1^{a,b,c,X}[\Theta(s)z^\pm,\frac{1}{s^\delta}\mathrm{Ri}\Theta'(s)z^\pm]\big)ds\Big\|_{L^p},\end{aligned}$$

are simpler since there is no weighted term Jz involved.

Estimates for quadratic terms involving $\mathcal{B}_2^{a,b,c}$ The last term to consider is

$$\Big\|\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\sum_{a,b,c}\mathcal{B}_2^{a,b,c,X}[\Theta(s)z^\pm,\Theta(s)z^\pm]ds\Big\|_{L^p}.$$

Let us start with the zone $b\lesssim a\sim c$. We use the same indices as for $\mathcal{B}_1^{a,b,c}$: $s=1+\varepsilon_1$, $\frac{1}{q}=1/2+\varepsilon-\varepsilon_1/3$, $s_1=4/3$, $\frac{1}{q_1}=7/18+\varepsilon$,

$$\begin{aligned}&\Big\|\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\sum_{b\lesssim a}\mathcal{B}_2^{a,b,c,X}[\Theta(s)z^\pm,\Theta(s)z^\pm]ds\Big\|_{L^p}\\ &\lesssim\int_1^{t-1}\frac{1}{s(t-s)^{1+3\varepsilon}}\Big(\sum_{a\lesssim 1}\sum_{b\lesssim a\sim c}U(b)U(c)\|\mathcal{B}_2^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]}\|U^{-1}\Theta(s)z\|_{L^2}\|U^{-1}\Theta(s)z]\|_{L^q}\\ &\quad+\sum_{1\lesssim a\lesssim s^\delta}a^k\sum_{b\lesssim a\sim c}\frac{U(b)}{\langle c\rangle^k}\|\mathcal{B}_2^{a,b,c,X}\|_{[B^{4/3}]}\|U^{-1}\Theta(s)z\|_{L^2}\|\langle\nabla\rangle^k\Theta(s)z]\|_{L^{q_1}}\Big)ds\end{aligned}\tag{6.20}$$

For $M\lesssim 1$ we have if $\varepsilon_1<1/4$:

$$\sum_{a\lesssim 1}\sum_{b\lesssim c\sim a}U(b)U(c)\|\mathcal{B}_2^{a,b,c,X}\|_{[B^{1+\varepsilon_1}]}\lesssim\sum_{a\lesssim 1}\sum_{b\lesssim c\sim a}b^{1/2-\varepsilon_1}a^{-\varepsilon_1}\lesssim 1.$$

Furthermore we have from Proposition 6.1:

$$\|U^{-1}\psi(s)\|_{L^2} \lesssim \|\psi\|_X,\ \|U^{-1}\psi(s)\|_{L^q} \lesssim \|U^{-1}\psi\|_{L^2}^{1-\varepsilon_1+3\varepsilon}\|U^{-1}\psi\|_{L^6}^{\varepsilon_1-3\varepsilon} \lesssim \frac{\|\psi\|_{X_T}}{s^{\frac{3(\varepsilon_1-3\varepsilon)}{5}}}.$$

Now for $M \gtrsim 1$

$$\sum_{1\lesssim a\lesssim s^\delta} a^k \sum_{b\lesssim c\sim a} \frac{U(b)\langle M\rangle^2 b^{1/2-4/3}}{\langle a\rangle\langle c\rangle^k} \lesssim \sum_{1\lesssim a\lesssim s^\delta} a^{1/6} \lesssim s^{\delta/6},$$

$$\|\langle\nabla\rangle^k\Theta(s)z\|_{L^{q_1}} \lesssim \frac{\|z\|_{X_T}}{s^{1/3-3\varepsilon}}.$$

If $\min\big(3(\varepsilon_1-3\varepsilon)/5, 1/3-3\varepsilon-\delta/6\big) \gtrsim 3\varepsilon$, injecting these estimates in (6.20) gives

$$\Big\|\nabla^{k_1}\int_1^{t-1}\frac{1}{s}e^{i(t-s)H}\sum_{b\lesssim c\sim a}\mathcal{B}_2^{a,b,c,X}[\Theta(s)Jz,\Theta(s)z]ds\Big\|_{L^p}$$

$$\lesssim \int_1^{t-1}\frac{\|z\|_X^2}{(t-s)^{1+3\varepsilon}s^{1+3\varepsilon}}ds \lesssim \frac{\|z\|_{X_T}^2}{t^{1+3\varepsilon}}.$$

The two other cases $c \lesssim a \sim b$ and $a \lesssim b \sim c$ can be treated in a similar way, we refer again to the observation (6.13) in the case $a \lesssim b \sim c$.

Conclusion. The estimates from Sects. 6.1.2 and 6.1.3 imply

$$\forall t \in [0,T],\ \Big\|\int_0^t e^{i(t-s)H}Q(z(s))ds\Big\|_{W^{k,p}} \lesssim \frac{\|z\|_{X_T}^2+\|z\|_{X_T}^3}{\langle t\rangle^{1+3\varepsilon}}.$$

Remark 6.5. From the energy estimate, we recall that we need $k \geq 3$ (see (5.3)). The strongest condition on N seems to be $(N-2-k)\delta > 1$. In the limit $\varepsilon \to 0$, we must have at least $1/3-\delta(k+7/6) > 0$, so that $N \geq 18$. On the other hand, the strongest condition on ε seems to be $3(\varepsilon_1-3\varepsilon)/5 \geq 3\varepsilon$, with $\varepsilon_1 < 1/4$, so that $\varepsilon < 1/32$.

6.2. Bounds for the weighted norm. The estimate for $\|x\int_0^t e^{-isH}B_j[z,z]ds\|_{L^2}$ can be done with almost the same computations as in section 10 from [22]. The only difference is that Gustafson et al. deal with nonlinearities without loss of derivatives. As we have seen in Sect. 6.1.3, the remedy is to use an appropriate frequency truncation, so we will only give a sketch of proof for the bound in this paragraph.

First reduction. Applying xe^{-itH} to the generic bilinear term $U\circ B_j[z^\pm,z^\pm]$, we have for the Fourier transform:

$$\mathcal{F}\Big(xe^{-itH}\int_0^t e^{i(t-s)H}UB_j[z^\pm,z^\pm]\Big)$$

$$=\int_0^t\int_{\mathbb{R}^d}\nabla_\xi\Big(e^{-is\Omega}UB_j(\eta,\xi-\eta)\widetilde{z^\pm}(s,\eta)\widetilde{z^\pm}(s,\xi-\eta)\Big)d\eta\, ds. \quad (6.21)$$

As the X_T norm only controls $\|Jz\|_{L^2}$, we have to deal with the loss of derivative in the nonlinearities. It is then convenient that $|\xi-\eta| \lesssim |\eta|$ in order to absorb the loss of derivatives; to do this we use a cut-off function $\theta(\xi,\eta)$ which is valued in $[0,1]$, homogeneous of degree 0, smooth outside of $(0,0)$ and such that $\theta(\xi,\eta)=0$ in a neighborhood of $\{\eta=0\}$ and $\theta(\xi,\eta)=1$ in a neighborhood of $\{\xi-\eta=0\}$ on the sphere. Using this splitting we get two terms

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}\nabla_\xi\Big(e^{-is\Omega}UB_j(\eta,\xi-\eta)\theta(\xi,\eta)\widetilde{z^\pm}(s,\eta)\widetilde{z^\pm}(s,\xi-\eta)\Big)d\eta\,ds,\\&\int_0^t\int_{\mathbb{R}^d}\nabla_\xi\Big(e^{-is\Omega}(1-\theta(\xi,\eta))UB_j(\eta,\xi-\eta)\widetilde{z^\pm}(s,\eta)\widetilde{z^\pm}(s,\xi-\eta)\Big)d\eta\,ds.\end{aligned}\tag{6.22}$$

By symmetry it suffices to consider the first one which corresponds to a region where $|\eta|\gtrsim|\xi|,|\xi-\eta|$ so that we avoid loss of derivatives for $\nabla_\xi\widetilde{z^\pm}(s,\xi-\eta)$.

An estimate in a different space and high frequency losses. Depending on which term ∇_ξ lands on, the following integrals arise:

$$\begin{aligned}\mathcal{F}I_1&=\int_0^t\int_{\mathbb{R}^N}e^{-is\Omega}\nabla_\xi(\theta(\xi,\eta)UB_j(\eta,\xi-\eta))\widetilde{z^\pm}(s,\eta)\widetilde{z^\pm}(s,\xi-\eta)d\eta ds,\\\mathcal{F}I_2&=\int_0^t\int_{\mathbb{R}^N}e^{-is\Omega}\theta(\xi,\eta)UB_j(\eta,\xi-\eta)\widetilde{z^\pm}(s,\eta)\nabla_\xi^{(\eta)}\widetilde{z^\pm}(s,\xi-\eta)d\eta ds,\\\mathcal{F}I_3&=\int_0^t\int_{\mathbb{R}^N}e^{-is\Omega}(is\nabla_\xi\Omega)\theta(\xi,\eta)UB_j(\eta,\xi-\eta)\widetilde{z^\pm}(s,\eta)\widetilde{z^\pm}(s,\xi-\eta)d\eta ds\\&:=\mathcal{F}\Big(\int_0^t e^{-isH}s\mathcal{B}_j[z^\pm,z^\pm]ds\Big),\end{aligned}$$

with:

$$\mathcal{B}_j(\eta,\xi-\eta)=(i\nabla_\xi\Omega)\theta(\xi,\eta)UB_j(\eta,\xi-\eta).$$

The control of the L^2 norm of I_1 and I_2 is not a serious issue: basically we deal here with smooth multipliers, and from the estimate $\|z\,xe^{-itH}z\|_{L^1_TL^2}\lesssim\|z\|_{L^1_TL^\infty}\|xe^{-itH}z\|_{L^\infty_TL^2}\lesssim\|z\|^2_{X_T}$ it is apparent that we can conclude. The only point is that we can control the loss of derivative on Jz via the truncation function θ_1 and it suffices to absorb the loss of derivatives by z. Due to the s factor, the case of I_3 is much more intricate and requires to use again the method of space-time resonances.

Let us set

$$\begin{aligned}\|z\|_{S_T}&=\|z\|_{L^\infty_TH^1}+\|U^{-1/6}z\|_{L^2_TW^{1,6}},\\\|z\|_{W_T}&=\|xe^{-itH}z\|_{L^\infty_TH^1}.\end{aligned}$$

Gustafson et al. prove in [22] the key estimate

$$\Big\|\int_0^t e^{-isH}sB[z^\pm,z^\pm]ds\Big\|_{L^\infty_TL^2}\lesssim\|z\|^2_{S_T\cap W_T},$$

where B is a class of multipliers very similar to our $\mathcal{B}_j$, the only difference being that they are associated to semilinear nonlinearities, and thus cause no loss of derivatives at high frequencies. We point out that the S_T norm is weaker than the X_T norm, indeed $\|U^{-1/6}z\|_{L^2_TW^{1,6}}\lesssim\|z\|_{L^2_TW^{2,9/2}}\lesssim\|z\|_{X_T}\|1/\langle t\rangle^{5/6}\|_{L^2_T}\lesssim\|z\|_{X_T}$. Moreover we have already seen how to deal with high frequency loss of derivatives with the low/high frequency splitting (as for (6.15))

$$\mathcal{B}_j[z^\pm, z^\pm] = \mathcal{B}_j[1-\Theta(t)z^\pm, z^\pm] + \mathcal{B}_j[\Theta(t)z^\pm, z^\pm]. \tag{6.23}$$

Let $1/q = 1/3 + \varepsilon$, the first term is estimated using Sobolev embedding and the fact that N is large enough compared to δ:

$$\begin{aligned}\Big\| \int_0^t \int_{\mathbb{R}^N} e^{-isH} s\mathcal{B}_j[z^\pm, z^\pm] ds \Big\|_{L^2} &\lesssim \int_0^t s\|(1-\Theta(s))z\|_{W^{3,q}} \|z\|_{W^{3,p}} ds \\ &\lesssim \int_0^t \frac{\|z\|_{H^N}\|z\|_{X_T}}{\langle s\rangle^{(N-4)\delta}} ds \\ &\lesssim \|z\|_{X_T}^2.\end{aligned}$$

The estimate of the second term of (6.23) follows from the (non trivial) computations in [22], section 10. They are very similar to the analysis of the previous section (based on the method of space-time resonances), for the sake of completeness we reproduce hereafter a small excerpt from their computations.

As in Sect. 6.1, one starts by splitting the phase space

$$\int_0^t e^{i(t-s)H} s\mathcal{B}_j[\Theta(s)z^\pm, z^\pm] ds = \sum_{a,b,c} \int_0^t e^{i(t-s)H} s\big(\mathcal{B}_j^{a,b,c,T} + \mathcal{B}_j^{a,b,c,X}\big)[\Theta(s)z^\pm, z^\pm] ds.$$

For the time non-resonant terms, an integration by parts in s implies:

$$\begin{aligned}&\int_0^t e^{i(t-s)H} s\mathcal{B}_j^{a,b,c,T}[\Theta(s)z^\pm, z^\pm] ds \\ &= -\int_0^t e^{i(t-s)H} \Big((\mathcal{B}_j')^{a,b,c,T}[\Theta(s)z^\pm, z^\pm] ds + (\mathcal{B}_j')^{a,b,c,T}[s\Theta(s)\mathcal{N}_z^\pm, z^\pm] \\ &\quad + (\mathcal{B}_j')^{a,b,c,T}[\Theta(s)z^\pm, s\mathcal{N}_z^\pm] + (\mathcal{B}_j')^{a,b,c,T}[-\delta s^{-\delta}\Theta(s)|\nabla|z^\pm, z^\pm] \Big) ds \\ &\quad + \big[e^{isH}(\mathcal{B}_j')^{a,b,c,T}[s\Theta(s)z^\pm, z^\pm]\big]_0^t, \end{aligned} \tag{6.24}$$

with:

$$(\mathcal{B}_j')^{a,b,c,T} = \frac{1}{\Omega}\mathcal{B}_j^{a,b,c,T} = \frac{i\nabla_\xi \Omega}{\Omega} B_j^{a,b,c,T}\theta(\xi,\eta),$$

We only consider the second term in the right hand side of (6.24), in the case $c \lesssim b \sim a$. All the other terms can be treated in a similar way. The analog of Lemma 6.1 in these settings is the following:

Lemma 6.3. *Denoting* $M = \max(a,b,c)$*,* $m = \min(a,b,c)$ *and* $l = \min(b,c)$ *we have:*

$$\|(\mathcal{B}_j')^{a,b,c,T}\|_{[H^s]} \lesssim \langle M\rangle^2 \left(\frac{\langle M\rangle}{M}\right)^s l^{\frac{3}{2}-s}\langle a\rangle^{-1}. \tag{6.25}$$

We have then by applying Theorem 2.4:

$$\Big\| \int_0^T e^{-isH} \sum_{c\lesssim a\sim b} (\mathcal{B}'_j)^{a,b,c,T}[s\Theta(s)\mathcal{N}z^\pm, z^\pm]ds\Big\|_{L^2}$$
$$\lesssim \Big\| \sum_{c\lesssim a\sim b} \frac{U(c)}{\langle b\rangle^2} \|(\mathcal{B}'_j)^{a,b,c,T}\|_{[B^{1+\varepsilon}]} \|s\langle\nabla\rangle^2\mathcal{N}_z\|_{L^2}\|U^{-1}z\|_{L^6}\Big\|_{L^1_T}. \tag{6.26}$$

From Lemma 6.3 we find

$$\sum_{c\lesssim a\sim b} U(c)\|(\mathcal{B}'_3)^{a,b,c,T}\|_{[B^1]} \lesssim \sum_{c\lesssim a} \frac{U(c)}{\langle a\rangle^2}\langle a\rangle^2 a^{-1}c^{\frac12},$$
$$\lesssim \sum_{a\leq 1} a^{1/2} + \sum_{a\geq 1} a^{-1/2} \lesssim 1. \tag{6.27}$$

Next we have (as previously forgetting cubic and quartic nonlinearities)

$$\|\langle\nabla\rangle^2\mathcal{N}_z\|_{L^2} \lesssim \|z\|^2_{W^{4,4}} \lesssim \|z\|^2_{X_T}/\langle s\rangle^{3/4},$$

and from (6.4) $\|U^{-1}z(s)\|_{L^6} \lesssim \|z\|_{X_T}\langle s\rangle^{-3/5}$ so that

$$\Big\| \int_0^T e^{-isH} \sum_{c\lesssim a\sim b} (\mathcal{B}'_j)^{a,b,c,T}[s\mathcal{N}z^\pm, z^\pm]ds\Big\|_{L^2} \lesssim \Big\| \|z\|^3_{X_T}\langle s\rangle^{-27/20}\Big\|_{L^1_T} \lesssim \|z\|^3_{X_T}.$$

6.3. Existence and uniqueness. Combining the energy estimate (Proposition 3.1), the a priori estimates for cubic, quartic (Sect. 5.2) and quadratic nonlinearities (Sect. 6) and Proposition 5.4 we have uniformly in T

$$\|\psi\|_{X_T} \leq C_1\Big(\|\psi_0\|_{W^{k,4/3}} + \|\psi_0\|_{H^{2n+1}} + \|x\psi_0\|_{L^2} + \|\psi\|^2_{X_T} G\big(\|\psi\|_{X_T}, \big\|\frac{1}{1+\ell}\big\|_{L^\infty_T(L^\infty)}\big)$$
$$+ \|\psi_0\|_{H^{2n+1}}\exp\Big(C'\|\psi\|_{X_T} H\big(\|\psi\|_{X_T}, \big\|\frac{1}{\ell+1}\big\|_{L^\infty_T(L^\infty)}\big)\Big)\Big).$$

with G and H continuous functions. We can now use the same bootstrap argument as in Sect. 4 which ensures that $\|\psi\|_{X_T}$ remains small independently of T. Combined with the blow up criterion page 203 this ensures that the solution is global.

6.4. Scattering. It remains to prove that $e^{-itH}\psi(t)$ converges in $H^s(\mathbb{R}^3)$, $s < 2n+1$. This is a consequence of the following lemma:

Lemma 6.4. *For any* $0 \leq t_1 \leq t_2$*, we have*

$$\Big\| \int_{t_1}^{t_2} e^{-isH}\mathcal{N}\psi\, ds\Big\|_{L^2} \lesssim \frac{\|\psi\|^2_{X_\infty}}{(t_1+1)^{1/2}}. \tag{6.28}$$

Proof. We focus on the quadratic terms since the cubic and quartic terms give even stronger decay. From Minkowski and Hölder's inequality and the dispersion $\|\psi(t)\|_{L^p} \leq \frac{\|\psi\|_X}{\langle t\rangle^{3(1/2-1/p)}}$:

$$\|\int_{t_1}^{t_2} e^{-isH}\mathcal{N}\psi ds\|_{L^2} \lesssim \int_{t_1}^{t_2} \|\langle\nabla\rangle^2\psi\langle\nabla\rangle^2\psi\|_{L^2}ds, \lesssim \int_{t_1}^{t_2} \|\langle\nabla\rangle^2\psi\|_{L^4}^2 ds,$$
$$\lesssim \|\psi\|_{X_\infty}^2 \int_{t_1}^{t_2} \frac{1}{\langle s\rangle^{3/2}}ds.$$

□

Interpolating between the uniform bound in H^{2n+1} and the decay in L^2 we get

$$\|e^{-it_1H}\psi(t_1) - e^{-it_2H}\psi(t_2)\|_{H^s} \lesssim 1/\langle t_1\rangle^{(2n+1-s)/(4n+2)},$$

thus $e^{-itH}\psi(t)$ converges in H^s for any $s < 2n+1$. For $d = 3$, the convergence of $xe^{-itH}\psi$ in L^2 follows from an elementary but cumbersome inspection of the proof of boundedness of $xe^{-itH}\psi$. If one replaces everywhere $\displaystyle\int_0^t xe^{-isH}\mathcal{N}_z ds$ by $\displaystyle\int_{t_1}^{t_2} xe^{-isH}\mathcal{N}_z ds$, every estimates ends up with $\|\psi\|_X^2 \int_{t_1}^{t_2} /(1+s)^{1+\varepsilon'}ds$, $k = 2, 3, 4$, for some small $\varepsilon' > 0$, so that $xe^{-itH}\psi(t)$ is a Cauchy sequence in L^2.

Acknowledgements. The work of Corentin Audiard was partially funded by the ANR Project BoND ANR-13-BS01-0009-01. The work of Boris Haspot was partially funded by the ANR Project INFAMIE ANR-15-CE40-0011.

A. The Multiplier Estimates

The aim of this section is to provide a brief sketch of proof of Lemmas 6.2 and 6.1, let us recall that $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ depend on the phase $\Omega = H(\xi) \mp H(\eta) \mp H(\xi-\eta)$ in the following way

$$\begin{aligned}
\mathcal{B}_3^{a,b,c,T} &= \frac{B_j}{\Omega}U(\xi)\chi^a(\xi)\chi^b(\eta)\chi^c(\xi-\eta),\\
\mathcal{B}_1^{a,b,c,X} &= \frac{B_j\nabla_\eta\Omega}{|\nabla_\eta\Omega|^2}U(\xi)\chi^a(\xi)\chi^b(\eta)\chi^c(\xi-\eta),\\
\mathcal{B}_2^{a,b,c,X} &= \nabla_\eta\mathcal{B}_1^{a,b,c,X}.
\end{aligned} \tag{A.1}$$

Recall the notations:

$$\begin{aligned}
&|\xi| \sim a,\ |\eta| \sim b,\ |\zeta| \sim c,\\
&M = \max(a,b,c),\ m = \min(a,b,c),\ l = \min(b,c).
\end{aligned} \tag{A.2}$$

The function χ_a, resp. χ_b, χ_c, are smooth cut-off functions that localize near $|\xi| \sim a \in 2^{\mathbb{Z}}$ (resp $|\eta| \sim b$, $|\zeta| \sim c$). We set as in [22] for any $\xi \in \mathbb{R}^3 \setminus \{0\}$, $\hat{\xi} = \xi/|\xi|$, and :

$$\alpha = |\hat{\zeta} - \hat{\xi}|,\ \beta = |\hat{\zeta} + \hat{\eta}|,\ \eta^\perp = \hat{\xi} \times \eta. \tag{A.3}$$

As a first reduction, we point out that the B_j's satisfy the pointwise estimate

$$|\nabla^k B_j(\eta, \xi - \eta)| \lesssim \langle M\rangle^2 l^{-k}. \tag{A.4}$$

with ∇ the derivative with respect to η or ζ. We will see that the term l^{-k} causes less loss of derivatives than if ∇_η hits $1/\Omega$ and $|\nabla_\eta\Omega|$, so that it will be sufficient to derive pointwise estimates for $\nabla^k(U/\Omega)$, $\nabla^k(U\nabla_\eta\Omega/|\nabla_\eta\Omega|^2$, and then multiply them by $\langle M^2\rangle$ to obtain pointwise estimates for the full multiplier.

A.1. The case $\Omega = H(\xi) + H(\eta) - H(\xi - \eta)$. Gustafson et al. [22] decompose the (ξ, η, ζ) region (with $\zeta = \xi - \eta$) into the following five cases where each later case excludes the previous ones:

1. $\{c << b \sim a\}$ defines a non time resonant set $\mathcal{S}_1$
2. $\mathcal{S}_1^c \cap \{\alpha > \sqrt{3}\}$ defines a non time resonant set $\mathcal{S}_2$.
3. $(\mathcal{S}_2 \cup \mathcal{S}_1)^c \cap \{c \gtrsim 1\}$ defines a non space resonant set $\mathcal{S}_3$.
4. $(\mathcal{S}_3 \cup \mathcal{S}_2 \cup \mathcal{S}_1)^c \cap \{|\eta^\perp| << M|\eta|\}$ defines a non time resonant set $\mathcal{S}_4$.
5. The rest defines $\mathcal{S}_5$, a non space resonant set.

The non time resonant set is thus $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_4$. The function Φ of (6.7) will be constructed as a partition of unity associated to $(\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_4) \sqcup (\mathcal{S}_3 \cup \mathcal{S}_5)$. The estimates of Lemmas 6.2 and 6.1 are essentially a consequence of the pointwise estimates[6] in [22], section 11, except in the fifth case where the pointwise estimate on $\nabla_\eta\Omega$ must be modified. We sketch all five cases for completeness.

1. If $a \sim b >> c$, we have

$$|\Omega| = \Omega = H(\xi) + H(\eta) - H(\zeta) \geq H(M) \sim M\langle M\rangle, \tag{A.5}$$

$$|\nabla_\zeta\Omega| \lesssim |\nabla H(\eta)| \lesssim \langle M\rangle, |\nabla_\zeta^2\Omega| \lesssim \frac{\langle m\rangle}{m}. \tag{A.6}$$

   From these estimates, the B_j estimate (A.4), the volume bound $|\{|\zeta| \sim m\}| \sim m^3$ and an interpolation argument we obtain $\|\frac{U(\xi)B_j}{\Omega}\chi^a\chi^b\chi^c\|_{L^\infty_\xi(\dot{H}^s_\zeta)} \lesssim m^{\frac{3}{2}-s}$, which is better than (6.10).
2. In the second case $\alpha > \sqrt{3}$ so that $|\zeta| \sim |\eta| \gtrsim |\xi|$.
   We cut-off the multipliers by: $\chi_{[\alpha]} = \Gamma(\hat{\xi} - \hat{\zeta})$, for a fixed $\Gamma \in C^\infty(\mathbb{R}^3)$ satisfying $\Gamma(x) = 1$ for $|x| \geq \sqrt{3}$ and $\Gamma(x) = 0$ for $|x| \leq \frac{3}{2}$, morover $|\nabla_\eta^k\Gamma| \lesssim M^{-k}$. In this region,

$$|\Omega| \geq \langle M\rangle|\xi| \sim \langle M\rangle m, |\nabla_\eta\Omega| \lesssim \frac{Mm}{\langle M\rangle} + \frac{\langle M\rangle m}{M} \lesssim \frac{|\Omega|}{M}, \tag{A.7}$$

$$|\nabla_\eta^2\Omega| = |\nabla^2 H(\eta) - \nabla^2 H(\zeta)| = |\nabla^2 H(\eta) - \nabla^2 H(-\zeta)| \lesssim \frac{\langle M\rangle m}{M^2} \lesssim \frac{|\Omega|}{M^2}. \tag{A.8}$$

[6] Note that these estimates must also take into account the partition function Φ, which turns out to be quite singular in some areas.

As a consequence:

$$\|\frac{U(\xi)}{\Omega}\chi_{[\alpha|}\chi^a\chi^b\chi^c\|_{L^\infty_\xi(\dot H^s_\eta)}\lesssim\frac{\langle M\rangle^2}{m\langle M\rangle}\frac{M^{\frac32}}{M^s}\frac{m}{\langle m\rangle}=\frac{\langle M\rangle M^{\frac32-s}}{\langle m\rangle}\sim\frac{\langle M\rangle l^{\frac32-s}}{\langle a\rangle}. \tag{A.9}$$

Remark A.1. The use of the normal form is essential here as for general $B_j^{a,b,c}$ we would obtain in equation (A.9):

$$\|\frac{U(\xi)}{\Omega}\chi_{[\alpha|}\chi^a\chi^b\chi^c\|_{L^\infty_\xi(\dot H^s_\eta)}\lesssim\frac{b^{3/2}}{m\langle M\rangle M^s\langle m\rangle}, \tag{A.10}$$

and the term $\frac{1}{m}$ could not be controlled. The same observation applies for the next areas.

3. The case $M\sim c\gtrsim 1$ and $\alpha<\sqrt3$. We are in a non space resonant area, the symbols to estimate are $\mathcal B_1^{a,b,c,X}$, $\mathcal B_2^{a,b,c,X}$ defined in (A.1). According to [22], the pointwise estimates in this region are

$$|\nabla_\eta\Omega|\sim||\zeta|-|\eta||+\langle\eta\rangle\beta\gtrsim|\xi|,\ |\nabla_\eta^k\Omega|\lesssim\frac{\langle\zeta\rangle}{|\zeta|}|\xi|\,|\eta|^{1-k}\lesssim|\xi|\,|\eta|^{1-k}. \tag{A.11}$$

Differentiating causes the same growth near $|\eta|=0$ as in (A.4), we deduce for $s\in[0,2]$, using the rough volume bound $b^{3/2}$

$$\begin{aligned}&\Big\|B_j\frac{\nabla_\eta\Omega}{|\nabla_\eta\Omega|^2}\chi^C_{[\alpha]}U(\xi)\chi^a(\xi)\chi^b(\eta)\chi^c(\xi-\eta)\Big\|_{\dot H^s_\eta}\lesssim\frac{\langle M\rangle^2b^{\frac32}}{ab^s}U(a)=\langle M\rangle^2l^{\frac32-s}\langle a\rangle^{-1},\\&\Big\|\nabla_\eta\cdot\Big(\frac{\nabla_\eta\Omega}{|\nabla_\eta\Omega|^2}\cdot B_j\chi^C_{[\alpha]}U(\xi)\chi^a(\xi)\chi^b(\eta)\chi^c(\xi-\eta)\Big)\Big\|_{\dot H^s_\eta}\lesssim l^{\frac12-s}\langle a\rangle^{-1}.\end{aligned} \tag{A.12}$$

4. The case $|\eta^\perp|<<M|\eta|$: it corresponds to a low frequency region, where the symbol has the bad "wave-like" behaviour. In this region

$$1>>M\sim|\zeta|,\ \alpha<\sqrt3,\ |\eta^\perp|=|\eta||\sin(\widehat{(\eta,\xi)})|<<M|\eta|, \tag{A.13}$$

The localization uses the (singular) cut-off multiplier $\chi_{[\perp]}=\chi\left(\frac{|\eta^\perp|}{100Mb}\right)$ with $\chi\in C_0^\infty(\mathbb R)$ satisfying $\chi(u)=1$ for $|u|\le1$ and $\chi(u)=0$ for $|u|\ge2$. In particular $|\nabla_\eta^k\chi_{[\perp]}|\lesssim\big(\frac{1}{Mb}\big)^k$, for all $k\ge1$. The worst case is $M=|\zeta|$, in this case Ω does not cancel thanks to the slight radial convexity of H:

$$\Omega=H(\xi+\eta)-H(\xi)-H(\eta)\sim\frac{|\xi||\eta|(|\xi|+|\eta|)}{\langle\xi\rangle+\langle\eta\rangle}\sim M^2m,\ |\nabla_\eta\Omega|<<|\xi|. \tag{A.14}$$

For higher derivatives we have:

$$|\nabla_\eta^{1+k}\Omega|=|\nabla^{k+1}H(\eta)-\nabla^{k+1}H(\zeta)|\lesssim\frac{|\xi|}{M|\eta|^k},\ |\nabla_\eta^kB_j|\lesssim l^{-k}. \tag{A.15}$$

For $|\eta| \sim b,\ |\eta^\perp| << Mb$, the region has for volume bound $b(Mb)^2 = M^2b^3$, we get by integration (for s integer) and interpolation

$$\left\| \frac{U(\xi)}{\Omega} \chi_{[\perp]} \chi^C_{[\alpha]} \chi^a \chi^b \chi^c \right\|_{L^2_\eta} \lesssim \frac{U(a)(M^2b^3)^{1/2}}{M^2 m(Mb)^s} \lesssim l^{\frac{1}{2}-s} M^{-s}. \tag{A.16}$$

5. In the last case we need a slight refinement of the symbol estimates from [22]: in the fifth area, $|\eta^\perp| \gtrsim Mb \sim |\zeta||\eta|$, $M \sim |\zeta| << 1$, $\alpha = |\widehat{\zeta} - \widehat{\xi}| \leq \sqrt{3}$. We have

$$|\nabla_\eta \Omega| = |H'(|\eta|)\widehat{\eta} + H'(|\zeta|)\widehat{\zeta}| \sim H'(|\eta|) - H'(|\zeta|) + |\widehat{\eta} + \widehat{\zeta}| \geq |\widehat{\eta} + \widehat{\zeta}|,$$

and for $\wedge$ the vector product

$$|\widehat{\eta} + \widehat{\zeta}| \geq \frac{|\eta \wedge \zeta|}{|\eta||\zeta|} = \frac{|\eta \wedge (\xi - \eta)|}{|\eta||\zeta|} = \frac{|\eta \wedge \xi|}{|\eta||\zeta|} = \frac{|\eta^\perp||\xi|}{|\eta||\zeta|}.$$

indeed, if η, ζ form an angle θ, $|\eta \wedge \zeta| = |\eta||\zeta||\sin\theta|$ and $|\widehat{\eta} + \widehat{\zeta}| \geq |\sin\theta)|$. Thus $|\nabla_\eta \Omega| \gtrsim |\xi||\eta^\perp|/(|\eta||\zeta|) \gtrsim |\xi|$.
For the higher derivatives, we combine (A.15) with $|\nabla_\eta \Omega| \gtrsim |\xi||\eta^\perp|/|\eta||\zeta|$ to get

$$\forall k \geq 2, \quad \frac{|\nabla^k_\eta \Omega|}{|\nabla_\eta \Omega|} \lesssim \frac{|\xi|}{M|\eta|^{k-1}\beta} \lesssim \frac{1}{|\eta|^{k-2}|\eta^\perp|}. \tag{A.17}$$

so that we have the pointwise estimate

$$\left| \nabla^k_\eta \frac{\nabla_\eta \Omega}{|\nabla_\eta \Omega|^2} \right| \sim \frac{1}{|\nabla \Omega|} \left(\frac{|\nabla^2_\eta \Omega|}{|\nabla_\eta \Omega|} \right)^k \lesssim \frac{1}{|\xi||\eta^\perp|^k}.$$

Following [22], we then use an angular dyadic decomposition $|\eta^\perp| \sim \mu \in 2^j\mathbb{Z}$, $Mb \lesssim \mu \lesssim b$. For each μ integrating gives a volume bound $\mu b^{1/2}$ and using interpolation we get for $s > 1$

$$\|U(\xi)/|\nabla_\eta \Omega|\|_{\dot{H}^s_\eta} \lesssim \sum_{Mb \lesssim \mu \lesssim b} \frac{U(a)\mu b^{1/2}}{a\mu^s} \sim l^{3/2-s} M^{1-s}.$$

A.2. The other cases.

The case $\Omega = H(\xi) - H(\eta) + H(\xi - \eta)$. This case is clearly symmetric from the $+-$ case.

The $--$ *case.* The decomposition follows the same line as in [22]. Note however that the analysis is simpler at least for $M \geq 1$. Indeed in this area $|\nabla_\eta \Omega| \sim |H'(\eta) - H'(\zeta)| + |\widehat{\eta} - \widehat{\zeta}| \gtrsim \big||\eta| - |\zeta|\big| + |\widehat{\eta} - \widehat{\zeta}| \sim |\eta - \zeta|$ so that we might split it as $\{|\eta - \zeta| \gtrsim \max(|\eta|, |\zeta|)\}$ and $\{|\eta - \zeta| << \max(|\eta|, |\zeta|)\}$. The first region is obviously space non resonant. The second region is time non resonant, indeed since $M \gtrsim 1$ we have in this region $|\xi| \sim |\eta| \sim |\zeta| \gtrsim 1$. Using a Taylor development gives

$$\begin{aligned} H(\xi) - H(\eta) - H(\zeta) &= H(2\eta + \zeta - \eta) - H(\eta) - H(\eta + \zeta - \eta) \\ &= H(2\eta) - 2H(\eta) + O(\langle a \rangle |\zeta - \eta|), \end{aligned}$$

this last quantity is bounded from below by $|\eta|^2$ for $|\eta| \gtrsim 1$, $|\zeta - \eta|$ small enough.

For $M < 1$, we can follow the same line as for $Z\overline{Z}$ by inverting the role of ξ and ζ. Note that the improved estimate in the last area relied on $|\nabla_\eta \Omega_{+-}| \gtrsim |\widehat{\eta} + \widehat{\zeta}| \geq |\eta^\perp|\xi|/(|\eta||\zeta|)$ and can just be replaced by $|\nabla_\eta \Omega_{--}| \gtrsim |\widehat{\eta} - \widehat{\zeta}| \geq |\eta^\perp|\xi|/(|\eta||\zeta|)$.

The ++ *case.* We have $\Omega = H(\xi) + H(\eta) + H(\zeta) \gtrsim (|\xi| + |\eta| + |\zeta|)(1 + |\xi| + |\eta| + |\zeta|)$, the area is time non resonant.

References

1. Antonelli, P., Marcati, P.: On the finite energy weak solutions to a system in quantum fluid dynamics. Commun. Math. Phys. **287**(2), 657–686 (2009)
2. Antonelli, P., Marcati, P.: The quantum hydrodynamics system in two space dimensions. Arch. Ration. Mech. Anal. **203**(2), 499–527 (2012)
3. Audiard, C.: Dispersive smoothing for the Euler–Korteweg model. SIAM J. Math. Anal. **44**(4), 3018–3040 (2012)
4. Audiard, C., Haspot, B.: From Gross–Pitaevskii equation to Euler–Korteweg system, existence of global strong solutions with small irrotational initial data. Preprint. https://hal.archives-ouvertes.fr/hal-01077281
5. Bahouri, H., Chemin, J.-Y., Danchin, R.: Fourier Analysis and Nonlinear Partial Differential Equations, Volume 343 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg (2011)
6. Benzoni-Gavage, S., Danchin, R., Descombes, S.: On the well-posedness for the euler-korteweg model in several space dimensions. Indiana Univ. Math. J. **56**, 1499–1579 (2007)
7. Benzoni-Gavage, S., Danchin, R., Descombes, S., Jamet, D.: Structure of Korteweg models and stability of diffuse interfaces. Interfaces Free Bound. **7**(4), 371–414 (2005)
8. Benzoni-Gavage, Sylvie: Planar traveling waves in capillary fluids. Differ. Integral Equ. **26**(3–4), 439–485 (2013)
9. Bulíček, M., Feireisl, E., Málek, J., Shvydkoy, R.: On the motion of incompressible inhomogeneous Euler–Korteweg fluids. Discrete Contin. Dyn. Syst. Ser. S **3**(3), 497–515 (2010)
10. Carles, R., Danchin, R., Saut, J.-C.: Madelung, Gross–Pitaevskii and Korteweg. Nonlinearity **25**(10), 2843–2873 (2012)
11. Cohn, S.: Global existence for the nonresonant Schrödinger equation in two space dimensions. Can. Appl. Math. Q. **2**(3), 257–282 (1994)
12. De Lellis, C., Székelyhidi, L. Jr.: On admissibility criteria for weak solutions of the Euler equations. Arch. Ration. Mech. Anal. **195**(1), 225–260 (2010)
13. Donatelli, D., Feireisl, E., Marcati, P.: Well/ill posedness for the Euler–Korteweg–Poisson system and related problems. Commun. Partial Differ. Equ. **40**(7), 1314–1335 (2015)
14. Germain, P., Masmoudi, N., Shatah, J.: Global solutions for 2D quadratic Schrödinger equations. J. Math. Pures Appl. (9) **97**(5), 505–543 (2012)
15. Germain, P., Masmoudi, N., Shatah, J.: Global solutions for the gravity water waves equation in dimension 3. Ann. Math. (2) **175**(2), 691–754 (2012)
16. Germain, P., Masmoudi, N., Shatah, J.: Global solutions for 3D quadratic Schrödinger equations. Int. Math. Res. Not. **2009**(3), 414–432 (2009). doi:10.1093/imrn/rnn135
17. Germain, P., Masmoudi, N., Shatah, J.: Global existence for capillary water waves. Commun. Pure Appl. Math. **68**(4), 625–687 (2015)

18. Giesselman, J., Lattanzio, C., Tzavaras, A.: Relative energy for the Korteweg theory and related hamiltonian flows in gas dynamics. Arch. Ration. Mech. Anal. **223**, 1427 (2017)
19. Guo, Y., Pausader, B.: Global smooth ion dynamics in the Euler–Poisson system. Commun. Math. Phys. **303**(1), 89–125 (2011)
20. Gustafson, S., Nakanishi, K., Tsai, T.-P.: Scattering for the Gross–Pitaevskii equation. Math. Res. Lett. **13**(2–3), 273–285 (2006)
21. Gustafson, S., Nakanishi, K., Tsai, T.-P.: Global dispersive solutions for the Gross–Pitaevskii equation in two and three dimensions. Ann. Henri Poincaré **8**(7), 1303–1331 (2007)
22. Gustafson, S., Nakanishi, K., Tsai, T.-P.: Scattering theory for the Gross–Pitaevskii equation in three dimensions. Commun. Contemp. Math. **11**(4), 657–707 (2009)
23. Hayashi, N., Naumkin, P.I.: On the quadratic nonlinear Schrödinger equation in three space dimensions. Int. Math. Res. Not. **2000**(3), 115–132 (2000). doi:10.1155/S1073792800000088
24. Klainerman, S., Ponce, G.: Global, small amplitude solutions to nonlinear evolution equations. Commun. Pure Appl. Math. **36**(1), 133–141 (1983)
25. Shatah, J.: Global existence of small solutions to nonlinear evolution equations. J. Differ. Equ. **46**(3), 409–425 (1982)
26. Shatah, J.: Normal forms and quadratic nonlinear Klein–Gordon equations. Commun. Pure Appl. Math. **38**(5), 685–696 (1985)
27. Strauss, W.: Nonlinear scattering theory at low energy. J. Funct. Anal. **41**, 110–133 (1981)
28. Taylor, M.E.: Partial Differential Equations III. Nonlinear Equations, Volume 117 of Applied Mathematical Sciences, 2nd edn. Springer, New York (2011)

Communicated by C. Mouhot