Commun. Math. Phys. 351, 311–339 (2017)
Digital Object Identifier (DOI) 10.1007/s00220-016-2803-8

Communications in
**Mathematical
Physics**

Uniform Approximation of a Maxwellian Thermostat by Finite Reservoirs

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Received: 9 March 2016 / Accepted: 14 October 2016
Published online: 2 December 2016 – © Springer-Verlag Berlin Heidelberg 2016

Abstract: We study a system of M particles in contact with a large but finite reservoir of $N \gg M$ particles within the framework of the Kac master equation modeling random collisions. The reservoir is initially in equilibrium at temperature $T = \beta^{-1}$. We show that for large N, this evolution can be approximated by an effective equation in which the reservoir is described by a Maxwellian thermostat at temperature T. This approximation is proven for a suitable L^2 norm as well as for the Gabetta–Toscani–Wennberg (GTW) distance and is *uniform in time*.

1. Introduction

In [6], Kac studied a spatially homogeneous gas of M particles moving in one dimension and interacting through random collisions. After certain exponentially distributed time intervals, a pair of particles is randomly and uniformly selected and they undergo a random collision, i.e., their pre-collisional velocities are replaced by new velocities that are randomly and uniformly selected in such a way that the total energy is preserved. The intensity of the collision process is chosen so that the average time λ^{-1} between two successive collisions of a given particle, i.e., the *mean free time*, is independent of the number of particles. Thus, the $M \to \infty$ limit of the model can be thought of as a realization of the classical Grad–Boltzmann limit.

To keep the presentation simple we describe the Kac model first for the system of M particles only and deal with the full model afterwards. The sub- and superscript S refers to this system of M particles. For a spatially homogeneous gas the state of the system is given by a function $f(\vec{v})$, the probability density of finding the particles in the system with velocities $\vec{v} = (v_1, \ldots, v_M)$. The infinitesimal generator of this evolution is given by (see [2,6])

$$\mathscr{L}_S[f] = \frac{\lambda_S}{M-1} \sum_{i<j} (R^S_{i,j} - I)[f], \tag{1}$$

where I is the identity operator and $R^S_{i,j}$ describes the result of a collision between particle i and particle j, that is

$$R^S_{i,j}[f](\vec{v}) := \fint f(\vec{v}_{i,j}(\theta))d\theta \tag{2}$$

with

$$\begin{aligned} \vec{v}_{i,j}(\theta) &:= (v_1, \ldots, v_i^*(\theta), \ldots, v_j^*(\theta), \ldots, v_M) \\ v_i^*(\theta) &:= v_i \cos\theta + v_j \sin\theta \quad v_j^*(\theta) := -v_i \sin\theta + v_j \cos\theta, \end{aligned} \tag{3}$$

and

$$\fint f(\theta)d\theta := \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta.$$

The gain term $\frac{\lambda_S}{M-1} R^S_{i,j}$ in (1) implies that, in an interval of length dt, there is a probability $\frac{\lambda_S}{M-1}dt$ that particles i and j will collide with resulting velocities v_i and v_j. Because every particle label appears exactly $M-1$ times in (1), particle i has a probability $\lambda_S dt$ of being involved in a collision during the time interval dt. Thus, on average, the time between two collisions involving particle i is λ_S^{-1}. Since the above evolution is completely independent of the positions of the particles, and hence of their density, the mean free time is the only number of physical significance.

In [1] a Kac-type model was introduced with the additional feature that, besides the pair collisions, each particle in the system can interact with a thermostat. The interaction of particle j with the Maxwellian thermostat is given by

$$B_j[f](\vec{v}) := \int dw \fint d\theta \sqrt{\frac{\beta}{2\pi}} e^{-\frac{\beta}{2} w_j^{*2}(\theta)} f(\vec{v}_j(\theta, w)), \tag{4}$$

where

$$\vec{v}_j(\theta, w) = (v_1, \ldots, v_j \cos(\theta) + w \sin\theta, \ldots, v_M), \quad w_j^*(\theta) = -v_j \sin\theta + w\cos\theta. \tag{5}$$

As before, the interaction times with the thermostat are described by a Poisson process whose intensity μ is chosen so that the average time between two successive interactions of a given particle with the thermostat is independent of the number of particles in the system S. Thus, the time evolution for this model is given by

$$\dot{f} = \widetilde{\mathscr{L}}[f] = \mathscr{L}_S[f] + \widetilde{\mathscr{L}}_T[f], \tag{6}$$

where

$$\widetilde{\mathscr{L}}_T[f] = \mu \sum_{j=1}^{M} (B_j - I)[f]. \tag{7}$$

In order to facilitate the discussion, we will call this model the *Thermostated System* or T-system in short. The unique equilibrium distribution of this thermostated system is given by a Gaussian with inverse temperature β. In [1] it is shown that the evolution approaches this equilibrium exponentially fast in L^2 as well as in entropy *uniformly in* M. Moreover, propagation of chaos [7] holds for this system as well and, as $M \to \infty$, the evolution of the single particle marginal is given by a Boltzmann-type equation.

These results have been extended to a system where only a subgroup of the particles interact with the thermostat in [8].

The thermostat can be thought of as an infinite reservoir of particles at a fixed inverse temperature $T = \beta^{-1}$ in which every particle in the reservoir collides at most once with a particle in the system. Thus, $B_j[f](\vec{v})$ describes a collision between a system particle and a reservoir particle that is randomly drawn from a Maxwellian distribution with temperature β^{-1}. The reservoir is not affected by the collisions with the particles from the system S. If the system S interacts, instead, with a large but finite reservoir, the reservoir does not remain in equilibrium. Particles in the reservoir can re-collide with system particles and with other reservoir particles, pushing more reservoir particles out of equilibrium.

In the present paper we compare, in appropriate metrics, the evolution (6) with the evolution arising from the interaction of the system S with a large but finite reservoir R containing $N \gg M$ particles. This model is explained in Sect. 2. In Sect. 3 we state the main results of the paper, namely, that for N large this evolution stays close uniformly in time to the one with an infinite reservoir. Section 4 contains the proofs of our results. Section 5 further addresses the relevance of our results together with possible extensions. Finally, in the Appendices, we report some technical computations and discuss the optimality of our bounds.

2. A Model for a Finite Heat Reservoir

The evolution inside the reservoir R is also given by a standard Kac model. As above, we assume that the average time between two collisions between two particles in the reservoir R is fixed independently of N. We denote this time by λ_R^{-1}. Thus, the generator of the evolution of the reservoir is

$$\mathscr{L}_R[f] = \frac{\lambda_R}{N-1} \sum_{1 \le i < j \le N} (R^R_{i,j} - I)[f]. \tag{8}$$

Again, the quantities that refer to the reservoir have a sub- or superscript R. The evolution of the system S and the reservoir R *without interaction between the two* is determined by the generator

$$\mathscr{L}_K[f] = \mathscr{L}_S[f] + \mathscr{L}_R[f] \tag{9}$$

where $\mathscr{L}_S[f]$ is given by (1). The velocities of the particles in the system S are, as before, denoted by $v_1, \ldots, v_M$ and the velocities of the particles in the reservoir by $w_1, \ldots, w_N$. Similar to what we wrote before, $R^S_{i,j}$ describes a collision in the system S between particle i and j, and is given by (see (3))

$$R^S_{i,j}[f](\vec{v}, \vec{w}) := \fint f(\vec{v}_{i,j}(\theta), \vec{w}) d\theta$$

and $R^R_{i,j}$ describing a collision in the reservoir between particle i and j is written as

$$R^R_{i,j}[f](\vec{v}, \vec{w}) := \fint f(\vec{v}, \vec{w}_{i,j}(\theta)) d\theta$$

with $\vec{v}_{i,j}(\theta)$ defined in (3) and $\vec{w}_{i,j}(\theta)$ analogously defined.

Some thought has to be given to the modeling of the interaction between the system S and the reservoir R. Naturally, we want that the average time between two successive collisions of a *given particle in the system S with any particle in the reservoir R* to be fixed independently of N and M. This is achieved by defining the interaction generator as

$$\mathscr{L}_I[f] = \frac{\mu}{N} \sum_{i=1}^{M} \sum_{j=1}^{N} (R_{i,j}^I - I)[f] \tag{10}$$

where

$$R_{i,j}^I[f](\vec{v}, \vec{w}) := \fint f(\vec{v}_i(\theta), \vec{w}_j(\theta)) d\theta,$$

with

$$\begin{aligned} &\vec{v}_i(\theta) := (v_1, \ldots, v_i^*(\theta), \ldots, v_M) \quad \vec{w}_j(\theta) := (w_1, \ldots, w_j^*(\theta), \ldots, w_N) \\ &v_i^*(\theta) := v_i \cos\theta + w_j \sin\theta \quad w_j^*(\theta) := -v_i \sin\theta + w_j \cos\theta. \end{aligned} \tag{11}$$

Thus, the evolution equation for the combined system S and reservoir R is given by

$$\dot{f} = \mathscr{L}[f] = \mathscr{L}_K[f] + \mathscr{L}_I[f], \tag{12}$$

where f is a probability distribution in $L^1(\mathbb{R}^M \times \mathbb{R}^N)$. It is elementary so see that this property is preserved under the evolution (12). We will call this model the *Finite Reservoir System* or FR-system in short.

It is plain that for an arbitrary initial distribution $f_0(\vec{v}, \vec{w})$ the evolutions given by (12) and (6) need not be similar. The latter tends to an equilibrium given by a Gaussian at temperature β^{-1} whereas the former, as can be easily seen, tends to an equilibrium which is given by averaging $f_0(\vec{v}, \vec{w})$ over all rotations in $\mathbb{R}^{M+N}$. Clearly, there is no reason why these two equilibria are close in any sense. The choice of initial conditions plays a key role. We shall assume that initially the reservoir is in the canonical equilibrium at temperature $T = \beta^{-1}$, that is, the state of the reservoir is given by

$$\Gamma_{\beta,N}(\vec{w}) = \prod_{i=1}^{N} \Gamma_{\beta,1}(w_i) \quad \text{where} \quad \Gamma_{\beta,1}(w) = \sqrt{\frac{\beta}{2\pi}} e^{-\frac{\beta}{2} w^2}.$$

We assume that the system S is initially in a generic initial state $l_0(\vec{v})$ with $\int l_0(\vec{v}) d\vec{v} = 1$.

It is easy to see that if the total momentum is initially zero, it remains zero for all times. Hence, we set it equal to zero. Moreover, we assume that the average kinetic energy per particle in the system is finite. The particles are assumed to be indistinguishable so that $l_0(\vec{v})$ is invariant under permutation of its variables. This implies that

$$\int v_i l_0(\vec{v}) d\vec{v} = 0 \quad \int |v_i|^2 l_0(\vec{v}) d\vec{v} = E_2 < \infty \quad \forall i.$$

Finally, by a simple rescaling of the velocities, we can assume without loss of generality that $\beta = 2\pi$. Thus, the initial distribution of the *system plus reservoir*, is given by

$$f_0(\vec{v}, \vec{w}) = l_0(\vec{v}) \Gamma_N(\vec{w}). \tag{13}$$

where $\Gamma_N(\vec{w}) = \Gamma_{2\pi,N}(\vec{w})$.

The evolution given by $\widetilde{\mathscr{L}}$, defined in (6), does not act on the $\vec{w}$ variables and with a slight abuse of notation we will consider $\widetilde{\mathscr{L}}$ as an operator acting on functions $f(\vec{v}, \vec{w})$ of both $\vec{v}$ and $\vec{w}$, leaving the dependence on $\vec{w}$ unchanged. It will be sometimes convenient to replace the generator $\widetilde{\mathscr{L}}$ by $\widetilde{\mathscr{L}} + \mathscr{L}_R$. This substitute is legitimate, since the operator $\mathscr{L}_R$ leaves the reservoir at equilibrium.

The similarity of the two evolutions, the one given by (12) with the one in (6) acting on the same initial state (13), can be heuristically understood as follows. The form of the interaction term implies that, in contrast to the collisions between system particles, the mean time between two successive collisions of a *given particle in the reservoir R with any particle in the system S* is $\mu^{-1}N/M$ and thus it diverges with N. This implies that for a finite time t and for N very large, with respect to t, we can indeed assume that each particle in the reservoir collides at most once with a particle in the system. This idea is implemented through the choice of (4). Thus, it is not difficult to prove a convergence result for any fixed time t, as $N \to \infty$. The interesting point, however, is that over longer times re-collisions will occur. Moreover the interaction $\mathscr{L}_R$, the collisions among the particles in the reservoir, spreads the modification of the distribution of one particle to all the reservoir particles. Thus, after a time approaching N, we can no more think that a randomly selected particle from the reservoir has a Maxwellian distribution. Thus, the real issue is to understand these competing effects in order to obtain a result uniformly in time. From a physical point of view such a result can be expected, because the thermostat is introduced to drive the system as $t \to \infty$ to a particular equilibrium state.

3. Results

We will always assume that the initial state f_0 for the FR-system is of the form (13), that is, the system S is in a generic initial state while the reservoir R is in equilibrium at inverse temperature $\beta = 2\pi$. The state at time t of the FR-system is given by

$$f_t = e^{\mathscr{L}t} f_0.$$

As noted above, f_t reaches a *steady state* f_∞ when $t \to \infty$ and that we get:

$$f_\infty(\vec{v}, \vec{w}) = \lim_{t\to\infty} f_t(\vec{v}, \vec{w}) = \int_{\mathbb{S}^{M+N-1}(r)} l_0(\vec{v}')\Gamma_N(\vec{w}')d\sigma_r(\vec{v}', \vec{w}') \tag{14}$$

where $r = \sqrt{|\vec{v}|^2 + |\vec{w}|^2}$ and $\sigma_r(\vec{v}, \vec{w})$ is the normalized uniform measure on the sphere of radius r in $\mathbb{R}^{M+N}$.

We want to compare the evolution generated by $\mathscr{L}$ with the evolution generated by $\widetilde{\mathscr{L}}$, the generator for the T-system (see (6)). In order for them to be comparable, we think of $\widetilde{\mathscr{L}}$ as acting on functions of $M + N$ variables. Given an initial state f_0 of the form (13), let

$$\tilde{f}_t = e^{\widetilde{\mathscr{L}}t} f_0$$

be the state of the T-system at time t, where clearly we have $\tilde{f}_t(\vec{v}, \vec{w}) = l_t(\vec{v})\Gamma_N(\vec{w})$. Any comparison between f_t and $\tilde{f}_t$ will naturally yield an estimate on how much the reservoir deviates from its initial equilibrium state. Because $\mathscr{L}_R\Gamma_N = 0$, for an initial state f_0 of the form (13), we can write (see (9))

$$\widetilde{\mathscr{L}} = \widetilde{\mathscr{L}}_T + \mathscr{L}_K.$$

This modification clearly does not change the evolution of f_0, but simplifies some of the computations below. As $t \to \infty$, $\tilde{f}_t$ approaches a steady state $\tilde{f}_\infty$ given by

$$\tilde{f}_\infty(\vec{v}, \vec{w}) = \lim_{t\to\infty} \tilde{f}_t(\vec{v}, \vec{w}) = \Gamma_{M+N}(\vec{v}, \vec{w}). \tag{15}$$

It is worth observing that (14) and (15) remain valid even when $\lambda_R = \lambda_S = 0$.

As a first attempt given in Sect. 3.1, we will compare the above evolutions in the space $L^2(\mathbb{R}^M \times \mathbb{R}^N, \Gamma_{M+N})$. Since f_0 is a probability distribution, such an L^2 norm is not very natural, however, the computations are relatively simple. After discussing the limitations of the results in L^2 , we will, in Sect. 3.2, compare the evolutions in the Gabetta–Toscani–Wennberg (GTW) metric (see [5]). This metric is more natural but the computations are quite difficult.

3.1. Evolution in $L^2(\mathbb{R}^{M+N}, \Gamma_{M+N})$. As discussed in [1], it is natural to look at the evolution in the ground state representation by defining

$$f_t(\vec{v}, \vec{w}) = h_t(\vec{v}, \vec{w})\Gamma_{M+N}(\vec{v}, \vec{w})$$

where

$$f_0(\vec{v}, \vec{w}) = h_0(\vec{v})\Gamma_{M+N}(\vec{v}, \vec{w})$$

with $\int h_0(\vec{v})\Gamma_N(\vec{v})d\vec{v} = 1$ while $\int v_i h_0(\vec{v})\Gamma_N(\vec{v})d\vec{v} = 0$ and $\int |v_i|^2 h_0(\vec{v})\Gamma_N(\vec{v})d\vec{v} = E_2$, for every i.

Observe that $\mathscr{L}_K$ (see (9)) has the same form when acting on f or on h. More precisely we have that

$$\mathscr{L}_K[\Gamma_{M+N}h] = \Gamma_{M+N}\mathscr{L}_K[h].$$

This easily follows from the fact that Γ_{M+N} is a rotationally invariant function. On the other hand, in the case of the thermostat we have to note that

$$B_i[\Gamma_{M+N}h] = \Gamma_{M+N}T_i[h]$$

where B_i is given by (4) while

$$T_i[f] = \int dw e^{-\pi w^2} \fint f(\vec{v}_i(\theta, w))d\theta. \tag{16}$$

This means that the evolution of the initial state h_0 under the thermostated evolution can be written has

$$\tilde{h}_t = e^{\overline{\mathscr{L}}t}h_0$$

where

$$\overline{\mathscr{L}}[h] = \mathscr{L}_K[h] + \mathscr{L}_T[h]$$

with

$$\mathscr{L}_T[h] = \mu\sum_{i=1}^{M}(T_i - I)[h].$$

Recall that $\mathscr{L}_S + \mathscr{L}_T$ acts only on the $\vec{v}$ variables while $\mathscr{L}_R$ acts only on the $\vec{w}$ variables. Thus, if h_0 depends only on $\vec{v}$ then $e^{\overline{\mathscr{L}}t}h_0$ will depend only on $\vec{v}$ too. It follows that the term $\mathscr{L}_R$ is identically zero along the evolution of the chosen initial state. We keep it for future comparison with $\mathscr{L}$. Note that $\mathscr{L}[h\Gamma_{M+N}] = \mathscr{L}[h]\Gamma_{M+N}$ and hence the generator of the evolution for the FR-system requires no modifications.

It is easy to see that $\mathscr{L}$ and $\overline{\mathscr{L}}$ are bounded self-adjoint operators on $L^2(\mathbb{R}^{M+N}, \Gamma_{M+N})$ with the scalar product

$$\langle f, g\rangle = \int f(\vec{v}, \vec{w})g(\vec{v}, \vec{w})\Gamma_{M+N}(\vec{v}, \vec{w})d\vec{v}d\vec{w}. \tag{17}$$

Thus, it is natural to assume that $h_0 \in L^2(\mathbb{R}^{M+N}, \Gamma_{M+N}(\vec{v}, \vec{w}))$ and to study the evolution of $\|e^{\overline{\mathscr{L}}t}h_0 - e^{\mathscr{L}t}h_0\|_2$.

As a first step we estimate the behavior of the difference of the steady states. We clearly have

$$f_\infty(\vec{v}, \vec{w}) = \Gamma_{M+N}(\vec{v}, \vec{w})h_\infty(\vec{v})$$

with

$$h_\infty(\vec{v}, \vec{w}) = \int_{\mathbb{S}^{M+N-1}(r)} h(\vec{v})d\sigma_r(\vec{v}, \vec{w})$$

whereas $\tilde{h}_\infty \equiv 1$. In Appendix A.1, we show that

$$\|h_\infty - \tilde{h}_\infty\|_2^2 = \int_{\mathbb{R}^{M+N}} [h_\infty(\vec{v}, \vec{w}) - 1]^2\Gamma_{M+N}(\vec{v}, \vec{w})d\vec{v}d\vec{w} \leq \frac{M}{N-2}\|h_0 - 1\|_2^2 \tag{18}$$

Thus, the distance between the steady states is controlled by the distance between the initial state and the canonical equilibrium state and it vanishes as $1/\sqrt{N}$ as $N \to \infty$. This estimate, in a slightly weaker form, remains true for all t.

Theorem 1. *Let f_0 be the initial distribution for the system with reservoir and assume that it has the form*

$$f_0(\vec{v}, \vec{w}) = h_0(\vec{v})\Gamma_{M+N}(\vec{v}, \vec{w}) \tag{19}$$

with $h_0 \in L^2(\mathbb{R}^{M+N}, \Gamma(\vec{v}, \vec{w}))$. Then for every $t > 0$ we have

$$\|e^{\overline{\mathscr{L}}t}h_0 - e^{\mathscr{L}t}h_0\|_2 \leq \frac{M}{\sqrt{N}}(1 - e^{-\frac{\mu}{2}t})\|h_0 - 1\|_2. \tag{20}$$

This statement is proved in Sect. 4.1.

We close this section with some remarks about the meaning of Theorem 1. In view of the estimate on the steady states, we see that the dependence on N in (20) is optimal. Observe that the particles in the reservoir of the FR-model are at thermal equilibrium at time 0 and then evolve to a radially symmetric state for large time. Hence it is not surprising that the final state is close to a canonical distribution. Thus, the fact the their state remains close to a canonical distribution *uniformly in time* is the main point of the above theorem.

Observe that the dependence of the estimate on M during the evolution is not the same as in the steady state. It is not clear to us whether this is an artifact of our proof. The main ingredient in the proof is the estimate (32). In Appendix B, we show that this estimate is optimal in its M behavior. This implies that the time derivative at $t = 0$ of $\|e^{\overline{\mathscr{L}}t}h_0 - e^{\mathscr{L}t}h_0\|$ can actually be $M/\sqrt{N}$. But this may only be true for a very small time.

A disturbing aspect of the theorem is that it behaves very poorly when applied to some very reasonable initial distributions. Assume that the system is initially in equilibrium at a temperature $T_S = \beta_S^{-1} \neq \beta^{-1}$, that is $f_0(\vec{v}) = \Gamma_{\beta_S,M}(\vec{v})\Gamma_{\beta,M}(\vec{w})$. It follows that $h_0(\vec{v}) = \Gamma_{\beta_S,M}(\vec{v})/\Gamma_{\beta,M}(\vec{v})$. If $2\beta_S \geq \beta$ then $\|h_0\|_2 = C(\beta_S)^M$ where $C(\beta_S)^2 = \beta_S/\sqrt{\beta(2\beta_s - \beta)} > 1$. Thus, if the right hand side of (20) is to be small for such an initial state, we need a reservoir with a number of particles N exponentially large in M. In a sense, this makes the behavior in M discussed above rather unimportant. Also, if the initial temperature is sufficiently large, that is if $2\beta_S \leq \beta$, then $C(\beta_S) = \infty$, $h_0 \notin L^2(\mathbb{R}^M, \Gamma_M(\vec{v}))$ and our theorem does not apply in this situation. These are, perhaps, the main reasons why the Gabetta–Toscani–Wennberg metric is better suited for our purposes, although it is quite a bit more difficult to handle.

3.2. The Gabetta–Toscani–Wennberg metric. The Gabetta–Toscani–Wennberg (GTW) metric is a distance between probability densities. Let $f, g \in L^1(\mathbb{R}^{M+N})$ be two possible distributions for the FR-system where

$$\int v_i f(\vec{v},\vec{w})d\vec{v}d\vec{w} = \int w_j f(\vec{v},\vec{w})d\vec{v}d\vec{w} = 0$$
$$\int v_i^2 f(\vec{v},\vec{w})d\vec{v}d\vec{w}, \int w_j^2 f(\vec{v},\vec{w})d\vec{v}d\vec{w} < \infty \tag{21}$$

and analogously for g. We can define then

$$d_2(f,g) := \sup_{\vec{\xi}\neq 0, \vec{\eta}\neq 0} \frac{|\widehat{f}(\vec{\xi},\vec{\eta}) - \widehat{g}(\vec{\xi},\vec{\eta})|}{|\vec{\xi}|^2 + |\vec{\eta}|^2}. \tag{22}$$

Here, and in the following, we use the convention that $\widehat{f}$, the Fourier transform of f, is given by

$$\widehat{f}(\vec{\xi},\vec{\eta}) = \int_{\mathbb{R}^{M+N}} e^{-2\pi i(\vec{\xi},\vec{v})} e^{-2\pi i(\vec{\eta},\vec{w})} f(\vec{v},\vec{w})d\vec{v}d\vec{w},$$

where $\vec{\xi} = (\xi_1,\ldots,\xi_M)$ are the Fourier variables associated with the particles in the system S, while $\vec{\eta} = (\eta_1,\ldots,\eta_N)$ are the Fourier variables associated with the particles in the reservoir R. It is easily seen that under the stated conditions, $d_2(f,g)$ is defined. The metric d_2 in (22) is the more interesting member of a family of metrics $\{d_\alpha\}$ introduced in [5].

Again we imagine that our system starts at time 0 in a state of the form

$$f_0(\vec{v},\vec{w}) = l_0(\vec{v})\Gamma_N(\vec{w})$$

and we want to estimate the d_2 distance between $f_t = e^{\mathscr{L}t} f_0$ and $\tilde{f}_t = e^{\widetilde{\mathscr{L}}t} f_0$. To see what kind of behavior to expect, we start from the distance between the steady states. Because the Fourier transform commutes with rotations we find

$$\widehat{f}_\infty(\vec{\xi},\vec{\eta}) = \int_{\mathbb{S}^{M+N-1}(r)} \widehat{l}_0(\vec{\xi})\Gamma_N(\vec{\eta})d\sigma_r(\vec{\xi},\vec{\eta})$$

and

$$\widehat{\tilde{f}}_\infty(\vec{\xi},\vec{\eta}) = \Gamma_{M+N}(\vec{\xi},\vec{\eta})$$

where we have used that Γ_1 is invariant under the Fourier transform. In Appendix A.2, we show that

$$d_2(f_\infty, \tilde{f}_\infty) \leq \frac{M}{M+N} d_2(l_0, \Gamma_M). \tag{23}$$

Again we want to obtain an estimate that remains true uniformly in time. In Sect. 4.2, we prove the following.

Theorem 2. *Let $f_0(\vec{v}, \vec{w})$ be the initial distribution for the system plus reservoir of the form*

$$f_0(\vec{v}, \vec{w}) = l_0(\vec{v})\Gamma_N(\vec{w}).$$

with l_0 symmetric and satisfying (21). Assume moreover that the fourth moment

$$\int v_i^4 l_0(\vec{v}) d\vec{v} = E_4 < \infty. \tag{24}$$

Then for every $t > 0$ we have

$$d_2\left(e^{\widetilde{\mathscr{L}}t} f_0, e^{\mathscr{L}t} f_0\right) \leq \frac{KM}{N}\left(1 - e^{-\frac{\mu}{4}t}\right)\sqrt{d_2(l_0, \Gamma_M)(F_4 + d_2(l_0, \Gamma_M))}\,. \tag{25}$$

with $F_4 = 48\pi^4(E_4 + 1)$ and $K = 16\sqrt{2}$.

The basic strategy of the proof of this theorem is similar to the one used for the proof of Theorem 1. Having said this, estimating the difference between $\widetilde{\mathscr{L}}_T$ and $\mathscr{L}_I$ in the d_2 metric turns out to be considerably more difficult than the one in the L^2 norm. Most of the work in the proof of Theorem 2 in Sect. 4.2 is devoted to carrying out these estimates which are summarized in Proposition 5. It is really in the proof of Proposition 5 that the extra condition (24) on the fourth order moment of the initial distribution is needed. In Appendix B we show that such a condition is indeed necessary for our proof.

We observe that $d_2(l_0, \Gamma_M)$ is well defined for any l_0 satisfying (21). Moreover, if l_0 is a product state, that is if

$$l_0(\vec{v}) = \prod_{i=1}^{M} \ell(v_i)$$

then, calling $\vec{\xi}^{<i} = (\xi_1, \ldots, \xi_{i-1})$, $\vec{\xi}^{>i} = (\xi_{i+1}, \ldots, \xi_M)$ and $\widehat{l_0^{>i}}(\vec{\xi}^{>i}) = \prod_{j>i} \widehat{\ell}(v_j)$, we get

$$\frac{|\Gamma_M(\vec{\xi}) - \widehat{l_0}(\vec{\xi})|}{|\vec{\xi}|^2} \leq \frac{\sum_i \Gamma_{i-1}(\vec{\xi}^{<i})\left|\Gamma_1(\xi_i) - \widehat{\ell}(\xi_i)\right| \widehat{l_0^{>i}}(\vec{\xi}^{>i})}{\sum_i \xi_i^2} \leq \sup_i \frac{|\Gamma_1(\xi_i) - \widehat{\ell}(\xi_i)|}{\xi_i^2}$$

so that

$$d_2(l_0, \Gamma_M) = d_2(\ell, \Gamma_1).$$

These observations address both problems found in the L^2 estimate.

4. Proof of Theorems 1 and 2

Both proofs are based on an expansion of the difference between two exponentials that we discuss here in the form needed for the L_2 estimates. A very similar expansion can be obtained for the d_2 case.

Observe that we can write

$$\begin{aligned}\mathscr{L} &= Q_S + Q_R + Q_I - \Lambda I \\ \overline{\mathscr{L}} &= Q_S + Q_R + Q_T - \Lambda I\end{aligned} \tag{26}$$

where

$$\Lambda = \frac{\lambda_S}{2} M + \frac{\lambda_R}{2} N + \mu M$$

while

$$Q_S = \frac{\lambda_S}{M-1} \sum_{1 \le i < j \le M} R^S_{i,j}, \quad Q_R = \frac{\lambda_R}{N-1} \sum_{1 \le i < j \le M} R^R_{i,j}.$$

Finally,

$$Q_I = \frac{\mu}{N} \sum_{i=1}^{M} \sum_{j=1}^{N} R^I_{i,j}, \quad Q_T = \mu \sum_{i=1}^{M} T_i.$$

We can thus write

$$e^{\mathscr{L}t} - e^{\overline{\mathscr{L}}t} = e^{-\Lambda t} \sum_{n=1}^{\infty} \frac{t^n}{n!} \left[(Q_S + Q_R + Q_I)^n - (Q_S + Q_R + Q_T)^n \right].$$

We further expand each term in the above sum as

$$\begin{aligned}&(Q_S + Q_R + Q_I)^n - (Q_S + Q_R + Q_T)^n \\ &\quad = \sum_{k=0}^{n-1} (Q_S + Q_R + Q_I)^{n-1-k} (Q_I - Q_T)(Q_S + Q_R + Q_T)^k\end{aligned}$$

so that we get

$$e^{\mathscr{L}t} - e^{\overline{\mathscr{L}}t} = e^{-\Lambda t} \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n-1} (Q_S + Q_R + Q_I)^{n-1-k} (Q_I - Q_T)(Q_S + Q_R + Q_T)^k. \tag{27}$$

The above expansion has three major advantages:

1. Isolating the factor $e^{-\Lambda t}$ avoids expanding a negative exponential as a power series.
2. As discussed in the previous section, we expect the difference between Q_I and Q_T to be small when they act on a function that depends only on $\vec{v}$. It is easy to see that $h_k(\vec{v}) := (Q_S + Q_R + Q_T)^k h_0(\vec{v})$ still depends only on $\vec{v}$ so that we expect to gain from the term $(Q_I - Q_T)h_k$.
3. Finally Λ is the largest eigenvalue of $Q_S + Q_R + Q_T$ corresponding to the eigenvector 1. But $(Q_I - Q_T)1 = 0$ so that, writing $h_k = 1 + u_k$, we expect that $\|u_k\|_2 < \Lambda^k$. A uniform version of this estimate, see (28) below, allows us to perform the sum over k in (27) without paying a factor of n. This is crucial in obtaining a bound uniform in t.

The following proofs consist, to a large extent, in a quantitative implementation of the above three observations.

4.1. Proof of Theorem 1. Observe that $(e^{\mathscr{L}t} - e^{\overline{\mathscr{L}}t})1 \equiv 0$ because the constant function 1 is a steady state for both evolutions. For this reason, we will write

$$h_0(\vec{v}) = 1 + u_0(\vec{v}) \quad \text{with} \quad \langle u_0, 1\rangle_M = 0$$

where $\langle \cdot, \cdot \rangle_M$ is the scalar product in $L^2(\mathbb{R}^M, \Gamma_M(\vec{v}))$, that is

$$\langle u, h\rangle_M = \int u(\vec{v})h(\vec{v})\Gamma_M(\vec{v})d\vec{v}.$$

From now on we will identify $L^2(\mathbb{R}^M, \Gamma_M(\vec{v}))$ with a subspace of $L^2(\mathbb{R}^{M+N}, \Gamma_{M+N}(\vec{v}, \vec{w}))$. We thus need to estimate the norm of

$$(Q_S + Q_R + Q_I)^{n-k-1}(Q_I - Q_T)(Q_S + Q_R + Q_T)^k u_0(\vec{v}).$$

To this end, observe that $R^S_{i,j}$ is the orthogonal projector on the subspace of functions that are invariant under rotations of v_i and v_j so that

$$\|R^\alpha_{i,j}\|_2 = 1 \quad \text{for} \quad \alpha = S,\ R \text{ or } I,$$

while

$$\|Q_T u\|_2 \le \mu\left(M - \frac{1}{2}\right)\|u\|_2 \quad \text{if} \quad \langle u, 1\rangle = 0.$$

Observe indeed that Q_T is a sum of operators acting independently on each variable v_i. Thus, its eigenvectors are tensor products of the eigenvectors of each of the T_i, while its eigenvalues are sums of their eigenvalues. It is possible to see that the Hermite polynomial $H_{2n}(v_i)$ of degree $2n$ and weight $e^{-\pi v_i^2}$ is an eigenvector of T_i with eigenvalue $a(n)$. The last inequality then follows from the fact that $a(0) = 1$ is the largest eigenvalue of T_i with eigenvector $H_0(v_i) = 1(v_i)$, while $a(n) \le 1/2$ for $n > 0$. It follows that $\|T_i l\|_2 \le (1/2)\|l\|$ when $\langle l, 1\rangle = 0$. With this, we get that

$$\langle (Q_S + Q_R + Q_T)u, 1\rangle = 0 \quad \text{if} \quad \langle u, 1\rangle = 0$$

and

$$\|u_k\|_2 \le \left(\Lambda - \frac{\mu}{2}\right)^k \|u_0\|_2, \tag{28}$$

where

$$u_k := (Q_S + Q_R + Q_T)^k u_0,$$

while

$$\|Q_S + Q_R + Q_I\|_2 \le \Lambda. \tag{29}$$

We thus have to estimate $\|(Q_I - Q_T)u\|_2$ where u depends only on $\vec{v}$.

Lemma 3. *Let $u(\vec{v})$ be any function in $L^2(\mathbb{R}^M, \Gamma_M(\vec{v}))$. Then*

$$\left\| \frac{1}{N}\sum_{j=1}^{N} R^I_{i,j}u - T_i u \right\|_2^2 = \frac{1}{N}\left(\langle T_i u, u\rangle - \langle T_i u, T_i u\rangle\right)$$

Proof. Consider for simplicity $i = 1$. We get

$$\left\| \frac{1}{N}\sum_{j=1}^{N} R^I_{1,j}u - T_1 u \right\|_2^2 = \frac{1}{N^2}\sum_{j,k=1}^{N}\int_{\mathbb{R}^{M+N}} R^I_{1,j}uR^I_{1,k}ud\mu(\vec{v},\vec{w})$$
$$-\frac{2}{N}\sum_{j=1}^{N}\int_{\mathbb{R}^{M+N}} R^I_{1,j}uT_1ud\mu(\vec{v},\vec{w})$$
$$+\int_{\mathbb{R}^{M+N}} |T_1u(v)|^2 d\mu(\vec{v},\vec{w}),$$

where $d\mu(\vec{v},\vec{w}) = \Gamma_{M+N}(\vec{v},\vec{w})d\vec{v}d\vec{w}$. Calling $\vec{v}^1 = (v_2,\ldots,v_M)$, we note that

$$\int_{\mathbb{R}^{M+N}} R^I_{1,1}uT_1ud\mu(\vec{v},\vec{w}) = \int_{\mathbb{R}^{M-1}}\int_{\mathbb{R}^2} ⨍ u(\sin\theta v_1 + \cos\theta w_1, \vec{v}^1)$$
$$\times\, d\theta T_1u(\vec{v})\Gamma_1(v_1)\Gamma_1(w_1)dv_1dw_1\Gamma_{M-1}(\vec{v}^1)d\vec{v}^1$$
$$= \int_{\mathbb{R}^M} |T_1u(\vec{v})|^2\Gamma_M(\vec{v})d\vec{v}. \tag{30}$$

Moreover,

$$\int_{\mathbb{R}^{M+N}} R^I_{1,1}uR^I_{1,2}ud\mu(\vec{v},\vec{w})$$
$$= \int_{\mathbb{R}^{M-1}}\int_{\mathbb{R}^3} ⨍ u(\sin\theta v_1 + \cos\theta w_1,\vec{v}^1)d\theta ⨍ u(\sin\theta v_1 + \cos\theta w_2,\vec{v}^1)d\theta$$
$$\cdot\, \Gamma_1(v_1)\Gamma_1(w_1)\Gamma_1(w_2)dv_1dw_1dw_2\Gamma_{M-1}(\vec{v}^1)d\vec{v}^1$$
$$= \int_{\mathbb{R}} |T_1(u)(\vec{v})|^2 d\mu(\vec{v},\vec{w}).$$

Finally, we observe that $R^I_{i,j}$ is a projection, so that

$$\int_{\mathbb{R}^{M+N}} R^I_{1,1}uR^I_{1,1}ud\mu(\vec{v},\vec{w}) = \int_{\mathbb{R}^{M+N}} uR^I_{1,1}ud\mu(\vec{v},\vec{w}) = \int_{\mathbb{R}^{M+N}} uT_1ud\mu(\vec{v},\vec{w})$$

where the last equality follows as in (30). Collecting all terms proves the lemma. □

It thus follows that

$$\|(Q_I - Q_T)u_k\|_2^2 = \mu\left\|\sum_{i=1}^{M}\left(\frac{1}{N}\sum_{j=1}^{N}R^I_{i,j} - T_i\right)u_k\right\|_2^2$$
$$\le \mu M\sum_{i=1}^{M}\left\|\left(\frac{1}{N}\sum_{j=1}^{N}R^I_{i,j} - T_i\right)u_k\right\|_2^2$$
$$\le \frac{\mu M}{N}\sum_{i=1}^{M}(u_k, T_iu_k) - (T_iu_k, T_iu_k). \tag{31}$$

Observe that if $\langle u, 1\rangle = 0$, we can write $u = \bar{u} + \tilde{u}$ where $\bar{u}$ does not depend on v_1 while

$$\int \tilde{u}(\vec{v})\Gamma_1(v_1)dv_1 = 0 \quad \forall \vec{v}^1.$$

It follows that

$$\langle T_1 u, u\rangle - \langle T_1 u, T_1 u\rangle = \langle T_1 \tilde{u}, \tilde{u}\rangle - \langle T_1 \tilde{u}, T_1 \tilde{u}\rangle \le \sup_k(\rho_k - \rho_k^2)\|\tilde{u}\|_2$$

where ρ_k are the eigenvalues of T_i different from 1. Since $\rho_k \le 1/2$ (see [1]) and $x^2 - x$ is increasing on [0, 1/2], we get

$$\|(Q_I - Q_T)u_k\|_2 \le \frac{\mu}{2}\frac{M}{\sqrt{N}}\|u_k\|_2. \tag{32}$$

Combining (32), (28) and (29), we get

$$\begin{aligned}&\|(Q_S + Q_R + Q_I)^{n-k-1}(Q_I - Q_T)(Q_S + Q_R + Q_T)^k h_0(\vec{v})\|_2\\ &\quad\le \frac{\mu}{2}\frac{M}{\sqrt{N}}\Lambda^{n-k-1}\left(\Lambda - \frac{\mu}{2}\right)^k \|h_0 - 1\|_2.\end{aligned}$$

Adding up, we obtain

$$\begin{aligned}&\left\|(Q_S + Q_R + Q_I)^n h_0 - (Q_S + Q_R + Q_T)^n h_0\right\|_2\\ &\quad\le \frac{\mu}{2}\frac{M}{\sqrt{N}}\Lambda^{n-1}\|h_0 - 1\|_2 \sum_{k=0}^{n-1}\left(1 - \frac{\mu}{2\Lambda}\right)^k\\ &\quad= \frac{M}{\sqrt{N}}\Lambda^n\left[1 - \left(1 - \frac{\mu}{2\Lambda}\right)^n\right]\|h_0 - 1\|_2\end{aligned}$$

Thus, finally,

$$\begin{aligned}\|(e^{\mathscr{L}t} - e^{\overline{\mathscr{L}}t})h_0\|_2 &\le \|h_0 - 1\|_2\frac{M}{\sqrt{N}}e^{-\Lambda t}\sum_{n=0}^{\infty}\frac{t^n}{n!}\Lambda^n\left[1 - \left(1 - \frac{\mu}{2\Lambda}\right)^n\right]\\ &= \|h_0 - 1\|_2\frac{M}{\sqrt{N}}\left(1 - e^{-\frac{\mu}{2}t}\right).\end{aligned} \tag{33}$$

This concludes the proof of Theorem 1.

4.2. Proof of Theorem 2. We can proceed as in Eq. (27) to obtain

$$e^{\mathscr{L}t} - e^{\widetilde{\mathscr{L}}t} = e^{-\Lambda t}\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{n-1}(Q_S+Q_R+Q_I)^{n-1-k}(Q_I-Q_B)(Q_S+Q_R+Q_B)^k. \tag{34}$$

where we set as before

$$\widetilde{\mathscr{L}} = Q_S + Q_R + Q_B - \Lambda I$$

with

$$Q_B = \mu \sum_{i=1}^{M} B_i .$$

Using this expansion in the definition (22) we get

$$d_2\left(e^{\mathscr{L}t} f_0, e^{\widetilde{\mathscr{L}}t} f_0\right) \\ \leq e^{-\Lambda t} \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n-1} \Lambda^k d_2\left((Q_S + Q_R + Q_I)^{n-1-k} Q_I[l_k\Gamma_N], (Q_S + Q_R + Q_I)^{n-1-k} Q_B[l_k\Gamma_N]\right) \tag{35}$$

where

$$l_k\Gamma_N = \Lambda^{-k}(Q_S + Q_R + Q_B)^k[l_0\Gamma_N] \quad \text{that is} \quad l_k = \Lambda^{-k}\left(Q_S + Q_B + \frac{\lambda_R N}{2} I\right)^k [l_0] \tag{36}$$

because Q_R acts as a multiple of the identity on Γ_N and Q_B as well as Q_S act only on l_0. We have introduced the factor Λ^{-k} to maintain the normalization of l_k, that is $\int l_k(\vec{v})d\vec{v} = 1$.

We thus need estimates for d_2 that can play an analogous role as Eqs. (28), (29) and (32) played in the proof of Theorem 1 in Sect. 4.1.

As a first thing, we need representations of the Fourier transform of the collision and thermostat operators. Let $f(\vec{v}, \vec{w})$ be a function of $(\vec{v}, \vec{w})$. Since the Fourier transform commutes with rotations, we get

$$\widehat{R^S_{i,j}[f]}(\vec{\xi}, \vec{\eta}) = \fint d\theta \hat{f}(\xi_{i,j}(\theta), \vec{\eta}) := \widehat{R^S_{i,j}}[\hat{f}](\vec{\xi}, \vec{\eta})$$

where $\xi_{i,j}(\theta)$ is defined as in (3). An analogous formula holds for $R^I_{i,j}$ and $R^R_{i,j}$. Moreover, we get

$$\widehat{B_i[f]}(\vec{\xi}, \vec{\eta}) = \fint d\theta \hat{f}(\xi_i(\theta, 0), \vec{\eta}) := \widehat{B}_i[\hat{f}](\vec{\xi}, \vec{\eta}).$$

The behavior of these two operators under the d_2 metric is contained in the following Lemma.

Lemma 4. *Let $f(\vec{v}, \vec{w})$ and $g(\vec{v}, \vec{w})$ be two distributions, with 0 first moment and finite second moment. We have*

$$d_2\left(\Lambda^{-1}(Q_S + Q_R + Q_I)f, \Lambda^{-1}(Q_S + Q_R + Q_I)g\right) \leq d_2(f, g). \tag{37}$$

Assume moreover that $f(\vec{v}, \vec{w}) = l(\vec{v})\Gamma_N(\vec{w})$ then

$$d_2\left(\Lambda^{-1}(Q_S + Q_R + Q_B)f, \Gamma_{M+N}\right) \leq \left(1 - \frac{\mu}{2\Lambda}\right) d_2(f, \Gamma_{M+N}) \\ = \left(1 - \frac{\mu}{2\Lambda}\right) d_2(l, \Gamma_M) \tag{38}$$

Proof. It is easy to see that $d_2(f, g)$ is jointly convex in f and g, that is for every $\alpha, \beta > 0$ with $\alpha + \beta = 1$, we have

$$d_2(\alpha f_1 + \beta f_2, \alpha g_1 + \beta g_2) \leq \alpha d_2(f_1, g_1) + \beta d_2(f_2, g_2). \tag{39}$$

We have

$$\widehat{R^S_{i,j}}[f](\vec{\xi}, \vec{\eta}) - \widehat{R^S_{i,j}}[g](\vec{\xi}, \vec{\eta}) = \fint d\theta \left(\hat{f}(\vec{\xi}_{i,j}(\theta), \vec{\eta}) - \hat{g}(\vec{\xi}_{i,j}(\theta), \vec{\eta}) \right)$$

and, because $|\vec{\xi}_{i,j}(\theta)| = |\vec{\xi}|$, we get

$$\begin{aligned} d_2\left(R^S_{i,j} f, R^S_{i,j} g\right) &\leq \sup_{\vec{\xi},\vec{\eta}\neq 0} \frac{\fint d\theta \left| \hat{f}(\vec{\xi}_{i,j}(\theta), \vec{\eta}) - \hat{g}(\vec{\xi}_{i,j}(\theta), \vec{\eta}) \right|}{|\vec{\xi}_{i,j}(\theta)|^2 + |\vec{\eta}|^2} \leq \\ &\leq \sup_{\vec{\xi},\vec{\eta}\neq 0,\theta} \frac{\left| \hat{f}(\vec{\xi}_{i,j}(\theta), \vec{\eta}) - \hat{g}(\vec{\xi}_{i,j}(\theta), \vec{\eta}) \right|}{|\vec{\xi}_{i,j}(\theta)|^2 + |\vec{\eta}|^2} = d_2(f, g) \end{aligned} \tag{40}$$

Clearly, an identical argument holds for $R^I_{i,j}$ and $R^R_{i,j}$. Equation (37) follows from the convexity property (39).

Because $B_i \Gamma_M = \Gamma_M$ we get

$$\begin{aligned} d_2\left(\frac{1}{M}\sum_{i=1}^{M} B_i l_0, \Gamma_M\right) &\leq \frac{1}{M} \sup_{\vec{\xi}\neq 0} \sum_{i=1}^{M} \fint \frac{\left|\hat{l}(\vec{\xi}_i(\theta,0)) - \Gamma_M(\vec{\xi}_i(\theta,0))\right| \Gamma_1(\zeta_i \sin\theta)}{|\vec{\xi}_i(\theta,0)|^2} \frac{\left|\vec{\xi}_i(\theta,0)\right|^2}{|\vec{\xi}|^2} d\theta \\ &\leq d_2(l, \Gamma_M) \frac{1}{M} \fint d\theta \sum_{i=1}^{M} \frac{|\vec{\xi}|^2 - \xi_i^2 \sin^2\theta}{|\vec{\xi}^2|} = \left(1 - \frac{\fint d\theta \sin^2\theta}{M}\right) d_2(l, \Gamma_M). \end{aligned} \tag{41}$$

Again (38) follows from (39). □

Combining (35) and (37) we get

$$d_2\left(e^{\mathscr{L}t} f_0, e^{\widetilde{\mathscr{L}}t} f_0\right) \leq e^{-\Lambda t} \sum_{n=1}^{\infty} \frac{t^n \Lambda^{n-1}}{n!} \sum_{k=0}^{n-1} d_2(Q_I[l_k \Gamma_N], Q_B[l_k \Gamma_N]) \tag{42}$$

Thus we want to estimate

$$\begin{aligned} &\frac{1}{M} d_2(Q_I[l_k\Gamma_N], Q_B[l_k\Gamma_N]) \\ &\quad = \frac{\mu}{MN} \sup_{\vec{\xi},\vec{\eta}\neq 0} \frac{1}{|\vec{\xi}|^2 + |\vec{\eta}|^2} \left| \sum_{i=1}^{M} \sum_{j=1}^{N} \left(\widehat{R}^I_{i,j}[\widehat{l_k}\Gamma_N](\vec{\xi}, \vec{\eta}) - \widehat{B}_i[\widehat{l_k}\Gamma_N](\vec{\xi}, \vec{\eta}) \right) \right|, \end{aligned} \tag{43}$$

where l_k is defined in (36). Setting

$$\widehat{F}_{k,i}(\vec{\xi}, \eta_j) = \fint d\theta \hat{l}_k(\xi_1, \ldots, \xi_i \cos\theta + \eta_j \sin\theta, \ldots, \xi^M) \Gamma_1(-\xi_i \sin\theta + \eta_j \cos\theta)$$

we can write

$$\widehat{R}^I_{i,j}[\widehat{l_k}\Gamma_N] = \Gamma_{N-1}(\vec{\eta}^j)\widehat{F}_{k,i}(\vec{\xi},\eta_j)$$

where $\vec{\eta}^j = (\eta_1, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_N)$. Likewise,

$$\begin{aligned}\widehat{B}_i[\widehat{l_k}\Gamma_N] &= \Gamma_N(\eta)\fint d\theta \hat{l}_k(\xi_1, \ldots, \xi_i\cos\theta, \ldots, \xi^M)\Gamma_1(-\xi_i\sin\theta)\\ &= \widehat{F}_{k,i}(\vec{\xi},0)\Gamma_1(\eta_j)\Gamma_{N-1}(\eta^j).\end{aligned}$$

Thus calling

$$\widehat{G}_k(\vec{\xi},\eta) = \frac{1}{M}\sum_{i=1}^{M}\left(\widehat{F}_{k,i}(\vec{\xi},\eta) - \widehat{F}_{k,i}(\vec{\xi},0)\Gamma_1(\eta)\right) \tag{44}$$

we can rewrite (43) in a more compact form

$$\frac{1}{M}d_2(Q_I[l_k\Gamma_N], Q_B[l_k\Gamma_N]) = \frac{\mu}{N}\sup_{\vec{\xi},\vec{\eta}\neq 0}\frac{1}{|\vec{\xi}|^2+|\vec{\eta}|^2}\sum_{j=1}^{N}\widehat{G}_k(\vec{\xi},\eta_j)\Gamma_{N-1}(\vec{\eta}^j). \tag{45}$$

Moreover, we have that

$$\begin{aligned}F_{k,i}(\vec{v},w) &= \fint d\theta \hat{l}_k(v_1, \ldots, v_i\cos\theta + w\sin\theta, \ldots, v^M)\Gamma_1(-v_i\sin\theta + w\cos\theta)\\ &= \fint d\theta \hat{l}_k(v_1, \ldots, v_i\cos(-\theta) - w\sin(-\theta), \ldots, v^M)\Gamma_1(v_i\sin(-\theta) - w\cos(-\theta))\\ &= F_{k,i}(\vec{v},-w)\end{aligned}$$

where we have used that Γ_1 is an even function. Thus $\widehat{F}_{k,i}(\vec{\xi},\eta)$ is even in η which makes $\widehat{G}_k(\vec{\xi},\eta)$ even in η. We also have $\widehat{G}_k(\vec{\xi},0) = 0$.

Our goal is to bound $d_2(Q_I[l_k\Gamma_N], Q_B[l_k\Gamma_N])$ in terms of $d_2(l_k,\Gamma_M)$. Thus, we focus on the supremum over the $\vec{\eta}$ variables of the reservoirs R, that is we look at

$$\mathscr{D}_N\left(\widehat{G}_k(\vec{\xi},\cdot), |\vec{\xi}|\right) = \sup_{\vec{\eta}\neq 0}\frac{1}{|\vec{\xi}|^2+|\vec{\eta}|^2}\sum_{j=1}^{N}\widehat{G}_k(\vec{\xi},\eta_j)\Gamma_{N-1}(\vec{\eta}^j). \tag{46}$$

In Proposition 5 we show that we can bound (46) in terms of $\mathscr{D}_1\left(\widehat{G}_k(\vec{\xi},\cdot), |\vec{\xi}|\right)$ and of $|\partial^p_\eta G_k(\vec{\xi},\eta)|$ for $p \le 4$, (see (47) and (48) below). Observe that $\mathscr{D}_1\left(\widehat{G}_k(\vec{\xi},\cdot), |\vec{\xi}|\right)$ refers to the situation where there is only one particle in the reservoir R, and thus, the supremum is over $\eta \in \mathbb{R}$ instead of $\vec{\eta} \in \mathbb{R}^N$.

Proposition 8 then shows that $|\partial^4_\eta G_k(\vec{\xi},\eta)|$ can be bounded in terms of the fourth moment E_4 of the initial distribution, (see (24)). We thus get a bound for $d_2(Q_I[l_k\Gamma_N], Q_B[l_k\Gamma_N])$ in terms of $d_2(Q_I[l_k\Gamma_1], Q_B[l_k\Gamma_1])$ and E_4. Together with (64) below, this will give us the desired estimate on $d_2(Q_I[l_k\Gamma_N], Q_B[l_k\Gamma_N])$ in terms of $d_2(l_k,\Gamma_M)$. The conclusion of the proof of Theorem 2 will then be very similar to the final steps of the proof of Theorem 1.

Proposition 5. *Let $H(\eta)$ be a bounded C^4 function of η. Assume that*

$$H(0) = 0 \quad H(\eta) = H(-\eta)$$

and

$$C_4 = \|H(\cdot)\|_{C^4} := \max_{p \le 4} \sup_{\eta} \left| \frac{d^p}{d\eta^p} H(\eta) \right| < \infty. \tag{47}$$

Calling

$$\mathscr{D}_N(H, a) = \sup_{\vec{\eta} \neq 0} \frac{1}{a^2 + |\vec{\eta}|^2} \left| \sum_{j=1}^{N} H(\eta_j) \Gamma_{N-1}(\vec{\eta}^j) \right| \tag{48}$$

we have

$$\mathscr{D}_N(H, a) \le [(8C_4 + \mathscr{D}_1(H, a))\mathscr{D}_1(H, a)]^{\frac{1}{2}} \tag{49}$$

One may hope that $\mathscr{D}_N(H, a) \le K \mathscr{D}_1(H, a)$ be true for some K independent of N. We will show in Appendix C that no such K exists. Observe that $\mathscr{D}_N(H, a)$ is of order 1 uniformly in N since we have

$$\mathscr{D}_N(H, a) \le \sup_{\vec{\eta} \neq 0} \frac{\sum_{j=1}^{N} \left| H(\eta_j) \right|}{\sum_{j=1}^{N} \eta_j^2} \le \sup_{\eta \neq 0} \frac{|H(\eta)|}{\eta^2} = \mathscr{D}_1(H, 0). \tag{50}$$

We were not able to use (50) directly. Indeed (50) and (45) give

$$\frac{1}{M} d_2(Q_I[l_k \Gamma_N], Q_B[l_k \Gamma_N]) = \frac{\mu}{N} \sup_{\vec{\xi}, \eta \neq 0} \frac{1}{|\vec{\eta}|^2} \sum_{j=1}^{N} \widehat{G}_k(\vec{\xi}, \eta)$$

and it is not clear how to relate the right side of the above equation to $d_2(l_k, \Gamma_M)$.

We can try to improve the above estimate observing that

$$|H(\eta)| \le \mathscr{D}_1(H, 0)\eta^2 \tag{51}$$

so that

$$\frac{\sum_{j=1}^{N} \left| H(\eta_j) \right| \Gamma_N(\vec{\eta}^j)}{a^2 + |\vec{\eta}|^2} \le \mathscr{D}_1(H, 0) \Gamma_N(\vec{\eta}) \frac{\sum_{j=1}^{N} \eta_j^2 e^{\pi \eta_j^2}}{a^2 + |\vec{\eta}|^2}.$$

Since $x e^{\pi x}$ is an increasing function for $x > 0$ we have that

$$\sum_{j=1}^{N} \eta_j^2 e^{\pi \eta_j^2} \le |\vec{\eta}|^2 e^{\pi |\vec{\eta}|^2}$$

that is, the supremum of $\sum_{j=1}^{N} \eta_j^2 e^{\pi \eta_j^2}$ on the set $|\vec{\eta}| = N$ is reached when $\eta_1 = N$ and $\vec{\eta}^1 = 0$. This observation will be useful in the following. Thus we get

$$\frac{\sum_{j=1}^{N} \left| H(\eta_j) \right| \Gamma_N(\vec{\eta}^j)}{a^2 + |\vec{\eta}|^2} \le \mathscr{D}_1(H, 0) \frac{|\vec{\eta}|^2}{a^2 + |\vec{\eta}|^2}. \tag{52}$$

Alas, this is not yet enough since after taking the supremum on $\vec{\eta}$ we are back to (50). Observe though that, if $\bar{\eta}$ is such that $|H(\bar{\eta})| = \sup_\eta |H(\eta)|$, then

$$\sup_{\vec{\eta}\neq 0} \frac{1}{a^2+|\vec{\eta}|^2}\left|\sum_{j=1}^{N} H(\eta_j)\Gamma_{N-1}(\vec{\eta}^j)\right| = \sup_{\vec{\eta}\neq 0, |\eta_i|\leq\bar{\eta}} \frac{1}{a^2+|\vec{\eta}|^2}\left|\sum_{j=1}^{N} H(\eta_j)\Gamma_{N-1}(\vec{\eta}^j)\right|$$

that is, we can limit the seprumum in (48) to the region where $\eta_i \leq \bar{\eta}$, for every i. But again we have no control on $\bar{\eta}$. In the first part of the proof of Proposition 5 we will use an improved version of the above argument to show that $\mathscr{D}_N(H,a)$ can be bounded in terms of $\mathscr{D}_1(H,0)/(1+a^2)$.

While it is obvious that $\mathscr{D}_1(H,a) \leq \mathscr{D}_1(H,0)$, the inverse inequality is generically far from true. In the second part of the proof, we find a lower bound on $\mathscr{D}_1(H,a)$ in terms of $\mathscr{D}_1(H,0)$ under the hypothesis that the fourth derivative of $H(\eta)$ is bounded. Observe indeed that, if $H(\eta)$ is of the form $H(\eta) = \frac{H''(0)}{2}\eta^2 - C\eta^4$ for some C, at least near $\eta = 0$, then $\mathscr{D}_1(H,a) \geq \frac{H''(0)^2}{2a^2C+H''(0)}$. In Lemma 7 we will show that a similar estimate holds for a generic H once we replace $H''(0)$ by $\mathscr{D}_1(H,0)$.

From these, Proposition 5 will easily follow.

Proof of Proposition 5. From (48) it follows that

$$|H(\eta)| \leq \mathscr{D}_1(H,a)(\eta^2+a^2).$$

Define

$$\begin{aligned}\widetilde{H}(\eta,a) &= \min\{\mathscr{D}_1(H,0)\eta^2, \mathscr{D}_1(H,a)(a^2+\eta^2)\} \\ &= \begin{cases}\mathscr{D}_1(H,0)\eta^2 & \text{if } \eta^2 \leq \eta_0^2(a) \\ \mathscr{D}_1(H,a)(a^2+\eta^2) & \text{if } \eta^2 \geq \eta_0^2(a)\end{cases},\end{aligned}$$

where

$$\eta_0^2(a) = \frac{\mathscr{D}_1(H,a)a^2}{\mathscr{D}_1(H,0)-\mathscr{D}_1(H,a)} \tag{53}$$

is chosen to make $\widetilde{H}$ continuous. We get $H(\eta) \leq \widetilde{H}(\eta,a)$ and thus $\mathscr{D}_N(H,a) \leq \mathscr{D}_N(\widetilde{H},a)$. The following Lemma contains our main improvement of (50) and (52).

Lemma 6. *Under the hypotheses of Proposition 5 we have*

$$\mathscr{D}_N(\widetilde{H},a) = \mathscr{D}_1(H,0) \sup_{k\leq N, |\eta|\leq\eta_0(a)} \frac{k\eta_0(a)^2 e^{-\pi((k-1)\eta_0(a)^2+\eta^2)} + \eta^2 e^{-\pi k\eta_0(a)^2}}{a^2+k\eta_0(a)^2+\eta^2} \tag{54}$$

that is, the supremum in (48) for $\widetilde{H}$ is attained for $\vec{\eta}$ of the form $\vec{\eta} = (\eta_0(a), \dots, \eta_0(a), \eta, 0, \dots, 0)$ for some η with $|\eta| \leq \eta_0(a)$.

Proof. Let

$$\widetilde{\mathscr{H}_N}(a,\vec{\eta}) = \frac{\sum_{i=1}^{N} \widetilde{H}(\eta_i)\Gamma_{N-1}(\vec{\eta}^i)}{a^2+|\vec{\eta}|^2}$$

and suppose $\vec{\eta}$ has $|\eta_i| > \eta_0(a)$ for some i. By differentiating we get

$$\partial_{\eta_i} \widetilde{\mathscr{H}_N}(a, \vec{\eta}) = \partial_{\eta_i} \left(\widetilde{H}(a, \eta_i) e^{\pi \eta_i^2} \right) \frac{\Gamma_N(\vec{\eta})}{a^2 + \vec{\eta}^2} - 2\eta_i \left(\pi + \frac{1}{a^2 + \vec{\eta}^2} \right) \widetilde{\mathscr{H}_N}(a, \vec{\eta})$$

where we used

$$\partial_\eta \left(\widetilde{H}(a, \eta) e^{\pi \eta^2} \right) = 2\eta \left(\pi \widetilde{H}(a, \eta) + \mathscr{D}_1(H, a) \right) e^{\pi \eta^2}$$

whenever $\eta \geq \eta_0(a)$. Because

$$\frac{\widetilde{H}(a, \eta_i) \Gamma_{N-1}(\vec{\eta}^i)}{a^2 + \vec{\eta}^2} \leq \widetilde{\mathscr{H}_N}(a, \vec{\eta}) \quad \text{and} \quad \frac{\mathscr{D}_1(\widetilde{H}, a) \Gamma_{N-1}(\vec{\eta}^i)}{a^2 + \vec{\eta}^2} \leq \frac{\mathscr{D}_N(\widetilde{H}, a)}{a^2 + \vec{\eta}^2},$$

with equality holding only if $\vec{\eta}^i = 0$, we have

$$\partial_{\eta_i} \widetilde{\mathscr{H}_N}(a, \vec{\eta}) < 0.$$

This implies that

$$\sup_{\vec{\eta} \neq 0} \widetilde{\mathscr{H}_N}(a, \vec{\eta}) = \sup_{\vec{\eta} \neq 0, |\eta_i| \leq \eta_0} \widetilde{\mathscr{H}_N}(a, \vec{\eta}) .$$

Now we show that there can be at most 1 coordinate i such that $0 < |\eta_i| < \eta_0(a)$. Consider

$$L(x, y) := x^2 e^{\pi x^2} + y^2 e^{\pi y^2}$$

and observe that $L(r \cos \theta, r \sin \theta)$ is maximal for $\theta = n \frac{\pi}{2}$ and minimal for $\theta = \frac{\pi}{4} + n \frac{\pi}{2}$. Moreover, it is strictly increasing for $\frac{\pi}{4} + n \frac{\pi}{2} < \theta < (n + 1) \frac{\pi}{2}$ and strictly decreasing for $n \frac{\pi}{2} < \theta < n \frac{\pi}{2} + \frac{\pi}{4}$. For $|\eta_i| \leq \eta_0(a)$ we have

$$\widetilde{\mathscr{H}_N}(a, \vec{\eta}) = \frac{\mathscr{D}_1(H, a) L(\eta_1, \eta_2) \Gamma_{N-2}(\eta_3 \ldots, \eta_N) + \sum_{i=3}^N \widetilde{H}(a, \eta_i) \Gamma_{N-1}(\vec{\eta}^i)}{a^2 + |\vec{\eta}^2|},$$

so that there can be no maximum for $\widetilde{\mathscr{H}_N}(a, \vec{\eta})$ for which both $0 < \eta_1 < \eta_0(a)$ and $0 < \eta_2 < \eta_0(a)$. Repeating this argument for each pair η_i, η_j with $1 \leq i, j \leq N$ we get that for all but possibly one i, we must have $\eta_i = 0$ or $\eta_i = \eta_0(a)$. □

To complete the proof of the first part of Proposition 5 we will simplify the right hand side of Eq. (54). Observe first that

$$\frac{k\eta_0(a)^2 e^{-\pi((k-1)\eta_0^2(a)+\eta^2)} + \eta^2 e^{-\pi k \eta_0(a)^2}}{a^2 + k\eta_0(a)^2 + \eta^2}$$
$$\leq \max \left\{ \frac{\eta_0^2(a)}{\frac{a^2}{2} + \eta_0(a)^2}, \frac{(k-1)\eta_0(a)^2 e^{-\pi((k-1)\eta_0^2(a)+\eta^2)} + \eta^2 e^{-\pi k \eta_0(a)^2}}{\frac{a^2}{2} + (k-1)\eta_0(a)^2 + \eta^2} \right\}.$$

From (53) we have

$$\frac{\eta_0^2(a)}{\frac{a^2}{2} + \eta_0(a)^2} \leq 2 \frac{\mathscr{D}_1(H, a)}{\mathscr{D}_1(H, 0)}$$

while

$$\sup_{k\le N,|\eta|\le\eta_0(a)} \frac{(k-1)\eta_0(a)^2 e^{-\pi((k-1)\eta_0^2(a)+\eta^2)} + \eta^2 e^{-\pi k\eta_0(a)^2}}{\frac{a^2}{2} + (k-1)\eta_0(a)^2 + \eta^2}$$
$$\le \sup_{k\le N,|\eta|\le\eta_0(a)} \frac{((k-1)\eta_0(a)^2 + \eta^2)e^{-\pi((k-1)\eta_0^2(a)+\eta^2)}}{\frac{a^2}{2} + (k-1)\eta_0(a)^2 + \eta^2} \le 2\sup_{y>0} \frac{ye^{-\pi y}}{\frac{a^2}{2}+y} \tag{55}$$

Clearly we have

$$\frac{ye^{-\pi y}}{\frac{a^2}{2}+y} \le \frac{y}{(\frac{a^2}{2}+y)(1+\pi y)} \le \frac{1}{\frac{\pi a^2}{2}+1}$$

so that

$$\mathscr{D}_N(H,a) \le \max\left\{\mathscr{D}_1(H,a), 2\frac{\mathscr{D}_1(H,0)}{1+\frac{\pi}{2}a^2}\right\}. \tag{56}$$

This concludes the first part of the proof. We start the second part with a couple of simple observations.

From the hypotheses of Proposition 5, it follows that

$$\frac{|H''(0)|\eta^2}{2} - \frac{C_4\eta^4}{4!} \le |H(\eta)| \le \frac{|H''(0)|\eta^2}{2} + \frac{C_4\eta^4}{4!}. \tag{57}$$

Let now $M = \sup_\eta |H(\eta)|$ and observe that there exists a finite $\tilde\eta$ such that $|H(\tilde\eta)| > M/2$. Moreover $\tilde\eta \ne 0$ since $H(0) = 0$. Thus $\mathscr{D}_1(H,0) \ge M/(2\tilde\eta^2)$ while

$$\frac{|H(\eta)|}{\eta^2} < \frac{M}{2\tilde\eta^2} \quad \text{if} \quad \eta^2 > 2\tilde\eta^2$$

Thus there exists η_m such that $\eta_m^2 \le \tilde\eta^2$ and $|H(\eta_m)| = \mathscr{D}_1(H,0)\eta_m^2$. We also know from (51) that

$$|H''(0)| \le 2\mathscr{D}_1(H,0),$$

with equality if and only if $\eta_m^2 = 0$.

Lemma 7. *Under the hypotheses of Proposition 5 we have*

$$\mathscr{D}_1(H,a) \ge \frac{\mathscr{D}_1(H,0)^2}{\frac{3}{2}C_4a^2 + 4\mathscr{D}_1(H,0)}$$

Proof. From (57) it follows that

$$\frac{|H(a,\eta)|}{a^2+\eta^2} \ge \frac{\frac{|H''(0)|\eta^2}{2} - \frac{C_4\eta^4}{4!}}{a^2+\eta^2}$$

and, choosing η^2 to be $\frac{6|H''(0)|}{C_4}$, we get that

$$\sup_\eta \frac{|H(a,\eta)|}{a^2+\eta^2} \ge \frac{|H''(0)|^2}{4|H''(0)| + \frac{3}{2}C_4a^2}. \tag{58}$$

Since, there is no positive lower bound for $|H''(0)|$, we complement this inequality using the second inequality in (57). We find that for all η

$$|H(\eta)| - \mathscr{D}_1(H,0)\eta^2 \le \frac{(|H''(0)| - 2\mathscr{D}_1(H,0))\eta^2}{2} + \frac{C_4\eta^4}{4!}$$

Since $|H''(0)| - 2\mathscr{D}_1(H,0) \le 0$ we get

$$\eta_m^2 \ge \frac{12(2\mathscr{D}_1(H,0) - |H''(0)|)}{C_4}.$$

This implies that

$$\begin{aligned}\sup_\eta \frac{|H(\eta)|}{a^2+\eta^2} \ge \frac{|H(\eta_m)|}{a^2+\eta_m^2} &\ge \liminf_{\epsilon\to 0} \frac{|H(\eta_m)|}{\eta_m^2+\epsilon}\frac{\eta_m^2}{a^2+\eta_m^2} \\ &\ge \frac{12\mathscr{D}_1(H,0)(2\mathscr{D}_1(H,0) - |H''(0)|)}{C_4a^2 + 12(2\mathscr{D}_1(H,0) - |H''(0)|)}. \end{aligned} \tag{59}$$

Observe now that the right hand side of (58) is an increasing function of $|H''(0)|$ while the right hand side of (59) is decreasing. Thus, we have

$$\mathscr{D}_1(H,a) \ge \min_{0\le h\le 2\mathscr{D}_1(H,0)} \max\left\{\frac{h^2}{4h+\frac{3}{2}C_4a^2}, \frac{12\mathscr{D}_1(H,0)(2\mathscr{D}_1(H,0)-h)}{C_4a^2+12(2\mathscr{D}_1(H,0)-h)}\right\}$$

Moreover

$$\begin{aligned}\frac{12\mathscr{D}_1(H,0)(2\mathscr{D}_1(H,0)-h)}{C_4a^2+12(2\mathscr{D}_1(H,0)-h)} &\ge \frac{12\mathscr{D}_1(H,0)^2}{12\mathscr{D}_1(H,0)^2+C_4a^2} \quad \text{for} \quad h \le \mathscr{D}_1(H,0) \\ \frac{h^2}{4h+\frac{3}{2}C_4a^2} &\ge \frac{\mathscr{D}_1(H,0)^2}{\frac{3}{2}C_4a^2+4\mathscr{D}_1(H,0)} \quad \text{for} \quad h \ge \mathscr{D}_1(H,0).\end{aligned}$$

The above, together with the observation

$$\frac{\mathscr{D}_1(H,0)^2}{\frac{3}{2}C_4a^2+4\mathscr{D}_1(H,0)} \le \frac{12\mathscr{D}_1(H,0)^2}{12\mathscr{D}_1(H,0)^2+C_4a^2}$$

concludes the proof. □

Observe finally that from $2|H(\eta)|/\eta^2 \le \sup_\eta |H''(\eta)|$ it follows that $2\mathscr{D}_1(H,0) \le \sup_\eta |H''(\eta)| \le C_4$. Thus we can write

$$\mathscr{D}_1(H,a) \ge \frac{2\mathscr{D}_1(H,0)^2}{C_4}\frac{1}{3a^2+4}. \tag{60}$$

Putting together (56) and (60) establishes the claim of Proposition 5. □

To apply Proposition 5 to (46), we need to estimate $\|\widehat{G}_k(\vec{\xi},\cdot)\|_{C^4}$, where $\widehat{G}_k(\vec{\xi},\eta)$ is defined in (44). Observe that for $p \le 4$ we have by Jensen's inequality

$$\begin{aligned}\left|\partial_{\eta_j}^p \widehat{R}^I_{i,j}[\widehat{l_k}\Gamma_N](\vec{\xi},\vec{\eta})\right| &\le (2\pi)^4 \int |w_j|^p R^I_{i,j}[l_k\Gamma_N](\vec{v},\vec{w})d\vec{v}d\vec{w} \\ &\le (2\pi)^4 \left(\int |w_j|^4 R^I_{i,j}[l_k\Gamma_N](\vec{v},\vec{w})d\vec{v}d\vec{w}\right)^{\frac{p}{4}} \\ &= (2\pi)^4 \left(\int (w_j^2+v_i^2)^2 l_k(\vec{v})\Gamma_N(\vec{w})d\vec{v}d\vec{w}\right)^{\frac{p}{4}} \\ &= (2\pi)^4 \left(E_{4,k} + 2\frac{E_{2,k}}{\sqrt{2\pi}} + \frac{3}{2\pi}\right)^{\frac{p}{4}} \le 32\pi^4(E_{4,k}+1)\end{aligned}$$

where

$$E_{n,k} = \int v_i^n l_k(\vec{v})d\vec{v} = \int v_i^n \left(Q_S + Q_B + \frac{\lambda_R N}{2} I\right)^k [l_0](\vec{v})d\vec{v}.$$

Using (44) we get

$$\|\widehat{G}_k(\vec{\xi},\cdot)\|_{C^4} \le 32\pi^4\left(E_{4,k}+1\right). \tag{61}$$

To estimate $E_{4,k}$ we need to study the action of Q_S and Q_B^* on v_i^4, where Q_B^* is the adjoint of Q_B. This is done in the following Lemma.

Proposition 8. *Given a symmetric distribution l_0 on $\mathbb{R}^M$ such that*

$$\int v_i^4 l_0(\vec{v})d\vec{v} = E_4 < \infty$$

we have

$$E_{4,k} = \int v_i^4 l_k(\vec{v})d\vec{v} \le 2(E_4+1)$$

where $l_k = \Lambda^{-k}\left(Q_S + Q_B + \frac{\lambda_R N}{2} I\right)^k l_0$.

Proof. First we observe that, due to symmetry,

$$E_{4,k} = \int \frac{1}{M}\sum_{i=1}^{M} v_i^4 l_k(\vec{v})d\vec{v}.$$

Calling

$$\overline{Q}_S := \frac{1}{\binom{M}{2}}\sum_{i<j} R^S_{i,j} = \frac{2}{\lambda_S M} Q_S, \quad \overline{Q}_B := \frac{1}{M}\sum_{i=1}^{M} B_i = \frac{1}{\mu M} Q_B$$

we have that

$$\int v_i^4 \overline{Q}_S[l](\vec{v})d\vec{v} = \int \overline{Q}_S[v_i^4]l(\vec{v})d\vec{v} \quad \int v_i^4 \overline{Q}_B[l](\vec{v})d\vec{v} = \int \overline{Q}_T[v_i^4]l(\vec{v})d$$

where

$$\overline{Q}_T := \frac{1}{M}\sum_{i=1}^{M} T_i$$

with T_i defined in (16). It is easy to see that $\overline{Q}_S$ and $\overline{Q}_T$ leave the space V of even polynomials of degree at most 4 in the v_i invariant. Calling $H_n(v)$ the monic Hermite polynomial of degree n (with weight $\Gamma_1(v) = e^{-\pi v^2}$), a natural basis in V is given by

$$\mathscr{H}_4(\vec{v}) = \frac{1}{M}\sum_{i=1}^{M} H_4(v_i), \quad \mathscr{H}_3(\vec{v}) = \frac{2}{M(M-1)}\sum_{i<j} H_2(v_i)H_2(v_j)$$
$$\mathscr{H}_2(\vec{v}) = \frac{1}{M}\sum_{i=1}^{M} H_2(v_i), \quad \mathscr{H}_0(\vec{v}) = 1$$

and we have

$$\frac{1}{M}\sum_{i=1}^{M} v_i^4 = a_4\mathscr{H}_4(\vec{v}) + a_3\mathscr{H}_3(\vec{v}) + a_2\mathscr{H}_2(\vec{v}) + a_0\mathscr{H}_0(\vec{v}),$$

where $\vec{a} = (a_4, a_3, .a_2, a_0) = (1, 0, \frac{3}{\pi}, \frac{3}{4\pi^2})$ and $|\vec{a}| \le \sqrt{2}$. From [1] we know that the action of $\overline{Q}_S$ and $\overline{Q}_T$ on V with the basis $\mathscr{H}_i$ is given by two positive definite matrices L_S and L_T with spectral (and thus L^2) norm 1. Thus, also the action of $\Lambda^{-k}\left(Q_S + Q_T + \frac{\lambda_R N}{2} I\right)$ is given by a positive definite matrix L with norm 1. We get

$$\Lambda^{-k}\left(Q_S + Q_T + \frac{\lambda_R N}{2} I\right)\left(\frac{1}{M}\sum_{i=1}^{M} v_i^4\right)$$
$$= a_{4,k}\mathscr{H}_4(\vec{v}) + a_{3,k}\mathscr{H}_3(\vec{v}) + a_{2,k}\mathscr{H}_2(\vec{v}) + a_{0,k}\mathscr{H}_0(\vec{v})$$

where $\vec{a}_k = L^k\vec{a}$. Clearly we have $|\vec{a}_k| \le |\vec{a}| \le \sqrt{2}$. We integrate both sides against $l_0(\vec{v})$ to obtain

$$E_{4,k} = a_{4,k}\left(E_4 - \frac{3}{\pi}E_2 + \frac{3}{4\pi^2}\right) + a_{3,k}\left(E_3 - \frac{1}{\pi}E_2 + \frac{1}{4\pi^2}\right) + a_{2,k}\left(E_2 - \frac{1}{2\pi}\right) + a_{0,k}$$

where

$$E_2 = \int v_i^2 l_0(\vec{v})d\vec{v} \le \frac{1}{2}(1 + E_4) \quad E_3 = \int v_i^2 v_j^2 l_0(\vec{v})d\vec{v} \le E_4.$$

After some rearranging and neglecting terms with negative coefficients, we obtain

$$E_{4,k} \le E_4\left(\left(1 - \frac{3}{2\pi}\right)a_{4,k} + \left(1 - \frac{1}{2\pi}\right)a_{3,k} + \frac{1}{2}a_{2,k}\right) + \left(a_{0.k} + \left(\frac{1}{2} - \frac{1}{2\pi}\right)a_{2,k}\right)$$
$$\le |\vec{a}|\left(E_4\sqrt{\left(1 - \frac{3}{2\pi}\right)^2 + \left(1 - \frac{1}{2\pi}\right)^2 + \frac{1}{4}} + \sqrt{1 + \left(\frac{1}{2} - \frac{1}{2\pi}\right)^2}\right)$$

proving the result. Here we applied Cauchy–Schwarz inequality twice in the last step. □

It thus follows from (61) that

$$\|G_k(\vec{\xi}, \cdot)\|_{C^4} \le 96\pi^4(E_4 + 1) := 2F_4. \tag{62}$$

Applying Proposition 8 and Proposition 5 to (45), (46) and using (62) we get that

$$\begin{aligned} &d_2(Q_I[l_k\Gamma_N], Q_B[l_k\Gamma_N]) \\ &\quad\le \frac{\mu M}{N}\sqrt{\big(2KF_4 + (\mu M)^{-1}d_2(Q_B[l_k\Gamma_1], Q_I[l_k\Gamma_1])\big)(\mu M)^{-1}d_2(Q_B[l_k\Gamma_1], Q_I[l_k\Gamma_1])}, \end{aligned} \tag{63}$$

where K is defined in Theorem 2. It is easy to see that

$$\begin{aligned} \frac{1}{M}d_2(Q_I[l_k\Gamma_1], Q_B[l_k\Gamma_1]) &\le d_2(M^{-1}Q_I[l_k\Gamma_1], \mu\Gamma_{M+1}) + d_2(M^{-1}Q_B[l_k\Gamma_1], \mu\Gamma_{M+1}) \\ &\le 2\mu d_2(l_k, \Gamma_M). \end{aligned} \tag{64}$$

Combining (63) and (64) gives

$$d_2(Q_I[l_k\Gamma_N], Q_B[l_k\Gamma_N]) \le 2\frac{\mu M}{N}\sqrt{(8F_4 + d_2(l_k, \Gamma_M))d_2(l_k, \Gamma_M)}. \tag{65}$$

We can now conclude our proof. Indeed, going back to Eq. (42), we can write

$$\begin{aligned} &d_2\left(e^{\mathscr{L}t}f_0, e^{\widetilde{\mathscr{L}}t}f_0\right) \\ &\quad\le 2\frac{\mu M}{N}e^{-\Lambda t}\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{n-1}\Lambda^{n-1}\sqrt{(8F_4 + d_2(l_k, \Gamma_M))d_2(l_k, \Gamma_M)} \\ &\quad\le 2\frac{\mu M}{N}e^{-\Lambda t}\sum_{n=1}^{\infty}\frac{t^n\Lambda^{n-1}}{n!}\sum_{k=0}^{n-1}\left(1 - \frac{\mu}{2\Lambda}\right)^{\frac{k}{2}}\sqrt{(8F_4 + d_2(l_0, \Gamma_M))d_2(l_0, \Gamma_M)} \\ &\quad= 8\frac{M}{N}\left(1 - e^{-\frac{\mu}{4}t}\right)\sqrt{(8F_4 + d_2(l_0, \Gamma_M))d_2(l_0, \Gamma_M)} \end{aligned}$$

where we have used (38) in Lemma 4 together with $\left(1 - \frac{\mu}{2\Lambda}\right)^{\frac{1}{2}} \le 1 - \frac{\mu}{4\Lambda}$.

5. Conclusions and Outlooks

We have shown that a *small* system out of equilibrium interacting with a *large* system initially in equilibrium (the reservoir) can be well approximated in certain norms by the same small system interacting with a thermostat. This approximation moreover is uniform in time. Our proof is not based on a projection or conditioning method. Indeed, it is hard to see how one can apply such an argument to the d_2 metric. In particular, we obtain that also the reservoir remains uniformly close to the equilibrium state.

We can also think of our system as describing a local perturbation in a large system initially in equilibrium at a given temperature. In this spirit we see our results as an initial attempt to understand the return to equilibrium from an initial state that is locally close to equilibrium. We hope to come back to this problem in forthcoming research.

In the case of the L^2 norm introduced in Sect. 3.1, the derivation of the above approximation is rather direct. We believe that this is at least in part due to the fact that the generators $\mathscr{L}$ (see (12)) and $\overline{\mathscr{L}}$ (see (6)) both have a spectral gap uniform in N. This implies that both systems approach exponentially fast their respective steady states f_∞ and $\tilde{f}_\infty$, (14) and (15). Notwithstanding this, such a norm behaves poorly with the size of the system and it excludes altogether perfectly reasonable initial states.

Partly for this reason we have studied the d_2 metric defined in (22). Such a metric is well defined for all reasonable initial states and behaves much better as a function of the size of the system. The control of this norm is harder. The main ingredient is contained in Proposition 5 in Sect. 4.2. It requires an extra fourth moment assumption on the initial state and some substantial analysis of an associated functional inequality.

It is not hard to show that $e^{\widetilde{\mathscr{L}}t} f_0$ approaches $\tilde{f}_\infty$ exponentially fast in the d_2 metric (see [3,4]). On the other hand, it is an open question whether $e^{\mathscr{L}t} f_0$ approaches f_∞ exponentially fast in the d_2 metric at a rate uniform in N. Our result is not enough to give an answer but it makes such a question rather natural.

Finally in [3], the authors consider a system interacting with more than one thermostat. They start at the level of the Boltzmann equation but it would be interesting to see in which sense one can approximate such a system with a system interacting with several large but finite reservoirs at different temperatures. Observe that in such a case, if the reservoirs are kept finite, they will reach a steady state in which they all have the same temperature (or better, average kinetic energy). This will create a more complex and interesting interplay between the large N and large t limit, with more than one time scale involved.

Acknowledgements. Michael Loss and Hagop Tossounian acknowledge partial support from the NSF Grant DMS-1301555. Michael Loss also acknowledges partial support from the NSF Grant DMS- 1600560 and the Humboldt Foundation.

A. Estimates on the Steady States

In this Appendix we derive (18) and (23).

A.1. Derivation of (18). Because h_∞ depends only on $r = \sqrt{|\vec{v}|^2 + |\vec{w}|^2}$ we can set

$$H(r) = h_\infty(\vec{v}, \vec{w})$$

Moreover, setting

$$w_j = \tilde{w}_j \sqrt{r^2 - |\vec{v}|^2}$$

we get $r^2 - |\vec{w}|^2 = (r^2 - |\vec{v}|^2)(1 - |\tilde{\vec{w}}|^2)$ and

$$H(r) = \frac{2}{|\mathbb{S}^{M+N-1}| r^{M+N-1}} \int_{|\vec{v}|^2 \le r^2} h_0(\vec{v}) r \left(r^2 - |\vec{v}|^2\right)^{\frac{N-2}{2}} d\vec{v}$$
$$\times \int_{\sum_{i \le N-1} w_i^2 \le 1} \frac{1}{\sqrt{1 - \sum_{j=1}^{N-1} \tilde{w}_j^2}} d\tilde{w}_1 \cdots d\tilde{w}_{N-1}$$

so that we have

$$H(r) = \frac{|\mathbb{S}^{N-1}|}{|\mathbb{S}^{M+N-1}| r^M} \int_{\mathbb{R}^M} h_0(\vec{v}) \left(1 - \frac{|\vec{v}|^2}{r^2}\right)_+^{(N-2)/2} d\vec{v}$$

where $(x)_+ = x$ if $x \geq 0$ and $(x)_+ = 0$ otherwise. Because $\int \Gamma_N(\vec{v}) h_0(\vec{v}) d\vec{v} = 1$ and

$$\frac{|\mathbb{S}^{N-1}|}{|\mathbb{S}^{M+N-1}| r^M} \int_{\mathbb{R}^M} \left(1 - \frac{|\vec{v}|^2}{r^2}\right)_+^{(N-2)/2} d\vec{v} = 1$$

we may write

$$\begin{aligned} H(r) - 1 &= \int_{\mathbb{R}^M} \left[\frac{|\mathbb{S}^{N-1}|}{|\mathbb{S}^{M+N-1}| r^M} \left(1 - \frac{|\vec{v}|^2}{r^2}\right)_+^{(N-2)/2} - \Gamma_N(\vec{v}) \right] (h_0(\vec{v}) - 1) d\vec{v} \\ &= \int_{\mathbb{R}^M} \left[\frac{|\mathbb{S}^{N-1}|}{|\mathbb{S}^{M+N-1}| r^M} \left(1 - \frac{|\vec{v}|^2}{r^2}\right)_+^{(N-2)/2} e^{\pi|\vec{v}|^2/2} - e^{-\pi|\vec{v}|^2/2} \right] e^{-\pi|\vec{v}|^2/2} (h_0(\vec{v}) - 1) d\vec{v} \end{aligned}$$

and using Cauchy–Schwarz's inequality we find that

$$\begin{aligned} |H(r) - 1|^2 \leq& \int_{\mathbb{R}^M} \Gamma_N(\vec{v}) (h_0(\vec{v}) - 1)^2 d\vec{v} \int_{\mathbb{R}^M} \\ & \times \left[\frac{|\mathbb{S}^{N-1}|}{|\mathbb{S}^{M+N-1}| r^M} \left(1 - \frac{|\vec{v}|^2}{r^2}\right)_+^{(N-2)/2} e^{\pi|\vec{v}|^2/2} - e^{-\pi|\vec{v}|^2/2} \right]^2 d\vec{v}. \end{aligned}$$

Thus, we get

$$\|h_\infty - 1\|^2 = |\mathbb{S}^{M+N-1}| \int r^{M+N-1} e^{-\pi r^2} |H(r) - 1| dr \leq C \|h\|_2^2$$

where

$$\begin{aligned} C = |\mathbb{S}^{M+N-1}| &\int_0^\infty dr r^{M+N-1} e^{-\pi r^2} \int_{\mathbb{R}^M} \\ & \times \left[\frac{|\mathbb{S}^{N-1}|}{|\mathbb{S}^{M+N-1}| r^M} \left(1 - \frac{|\vec{v}|^2}{r^2}\right)_+^{(N-2)/2} e^{\pi|\vec{v}|^2/2} - e^{-\pi|\vec{v}|^2/2} \right]^2 d\vec{v} \end{aligned}$$

By expanding the square, we can write the above integral as a sum of three integrals that can be computed explicitly as

$$\begin{aligned} &\int_0^\infty dr r^{M+N-1} e^{-\pi r^2} \int_{\mathbb{R}^M} \frac{|\mathbb{S}^{N-1}|^2}{|\mathbb{S}^{M+N-1}| r^{2M}} \left(1 - \frac{|\vec{v}|^2}{r^2}\right)_+^{(N-2)} e^{\pi|\vec{v}|^2} d\vec{v} \\ &\quad = \frac{\Gamma(\frac{M+N}{2})}{\Gamma(\frac{N}{2})\Gamma(\frac{M}{2})} \frac{\Gamma(\frac{N-2}{2})\Gamma(\frac{M}{2})}{\Gamma(\frac{M+N-2}{2})} = \frac{M+N-2}{N-2}, \\ &\int_0^\infty dr r^{M+N-1} e^{-\pi r^2} \int_{\mathbb{R}^M} \frac{|\mathbb{S}^{N-1}|}{r^M} \left(1 - \frac{|\vec{v}|^2}{r^2}\right)_+^{(N-2)/2} d\vec{v} = 1, \\ &|\mathbb{S}^{M+N-1}| \int_0^\infty dr r^{M+N-1} e^{-\pi r^2} \int_{\mathbb{R}^M} e^{-\pi|\vec{v}|^2} d\vec{v} = 1. \end{aligned} \tag{66}$$

We thus get

$$C = \frac{M}{N-2}.$$

A.2. Derivation of (23). Calling $r^2 = |\vec{\xi}|^2 + |\vec{\eta}|^2$, we have

$$d_2(f_\infty, \Gamma_{M+N}) = \sup_{r\neq 0} \int_{\mathbb{S}^{M+N-1}(r)} \frac{[\widehat{l_0}(\vec{\xi}) - \Gamma_M(\vec{\xi})]}{r^2} \Gamma_N(\vec{\eta}) d\sigma_r(\vec{\xi}, \vec{\eta})$$
$$\leq \left(\sup_{r\neq 0} \int_{\mathbb{S}^{M+N-1}(r)} \frac{|\vec{\xi}|^2}{r^2} \Gamma_N(\vec{\eta}) d\sigma_r(\vec{\xi}, \vec{\eta}) \right) d_2(l_0, \Gamma_M)$$

Observe now that

$$\int_{\mathbb{S}^{M+N-1}(r)} \frac{|\vec{\xi}|^2}{r^2} \Gamma_N(\vec{\eta}) d\sigma_r(\vec{\xi}, \vec{\eta}) = \int_{\mathbb{S}^{M+N-1}(1)} |\vec{\xi}|^2 \gamma\left(r^2(1 - |\vec{\xi}|^2)\right) d\sigma_1(\vec{\xi}, \vec{\eta})$$
$$\leq \frac{|\mathbb{S}^{N-1}|}{|\mathbb{S}^{M+N-1}|} \int_{|\vec{\xi}|^2 \leq 1} |\vec{\xi}|^2 \left(1 - |\vec{\xi}|^2\right)^{\frac{N-2}{2}} d\vec{\xi}$$
$$\leq \frac{|\mathbb{S}^{N-1}||\mathbb{S}^{M-1}|}{|\mathbb{S}^{M+N-1}|} \int_0^1 \rho^{M+1} \left(1 - \rho^2\right)^{\frac{N-2}{2}} d\rho$$
$$= \frac{1}{2} \frac{|\mathbb{S}^{N-1}||\mathbb{S}^{M-1}|}{|\mathbb{S}^{M+N-1}|} \int_0^1 s^{\frac{M}{2}} (1-s)^{\frac{N}{2}-1} ds$$
$$= \frac{1}{2} \frac{2\pi^{\frac{M}{2}} 2\pi^{\frac{N}{2}} \Gamma\left(\frac{M+N}{2}\right)}{\Gamma\left(\frac{M}{2}\right)\Gamma\left(\frac{N}{2}\right) 2\pi^{\frac{M+N}{2}}} \frac{\Gamma\left(\frac{M}{2}+1\right)\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{M+N}{2}+1\right)} = \frac{M}{M+N}.$$

B. Optimality of the Estimate (32)

In this appendix we show that there exists an initial state u_0 for which we have

$$\|(Q_I - Q_T)u_0\|_2 \geq C \frac{M}{\sqrt{N}} \|u_0\|_2.$$

thus saturating the bound in Lemma 3. We first observe that, by a similar analysis as Lemma 3, we get

$$\left\| \sum_{i=1}^M \left(\frac{1}{N} \sum_{j=1}^N R^I_{i,j} u - T_i u \right) \right\|_2^2 = \frac{M}{N} \left(\langle T_1 u, u\rangle - \langle T_1 u, T_1 u\rangle \right)$$
$$+ \frac{M(M-1)}{N} \left(\langle R^I_{1,1} u, R^I_{2,1} u\rangle - \langle T_1 u, T_2 u\rangle \right).$$

We thus need symmetric initial states such that $\langle R^I_{1,1} u, R^I_{2,1} u\rangle - \langle T_1 u, T_2 u\rangle = O(1)$ in M and N. To this end we set

$$u_{M,P}(\vec{v}) = \sum_{p_1+p_2+\cdots+p_M = P} \prod_{i=1}^M H_{2p_i}(v_i)$$

where $H_p(v)$ is the normalized Hermite polynomial of degree p with weight $\gamma(v) = e^{-\pi v^2}$. We get

$$R^I_{1,1} u_{M,P}(\vec{v}) = \sum_{p_1+p_2 \leq P} \widetilde{H}_{2p_1}(v_1, w_1) H_{2p_2}(v_2) u_{M-2, P-p_1-p_2}(\vec{v}^{1,2}).$$

where $\widetilde{H}_{2p}(v, w)$ is the only radially symmetric Hermite polynomial of degree $2p$. It follows that

$$\begin{aligned}&\langle R^I_{1,1}u_{M,P}, R^I_{2,1}u_{M,P}\rangle - \langle T_1 u_{M,P}, T_2 u_{M,P}\rangle \\ &\quad \geq \left(\langle R^I_{1,1}\bar{u}, R^I_{2,1}\bar{u}\rangle - \langle T_1\bar{u}, T_2\bar{u}\rangle\right)\|u_{P-2,M-2}\|_2\end{aligned}$$

where $\bar{u}(v_1, v_2) = H_4(v_1) + H_2(v_1)H_2(v_2) + H_4(v_2)$. Observe now that $\|u_{P,M}\|_2 = \binom{M+P}{P-1}$ while $\langle R^I_{1,1}\bar{u}, R^I_{2,1}\bar{u}\rangle - \langle T_1\bar{u}, T_2\bar{u}\rangle = \frac{11}{8}$ so that

$$\begin{aligned}&\langle R^I_{1,1}u_{M,P}, R^I_{2,1}u_{M,P}\rangle - \langle T_1 u_{M,P}, T_2 u_{M,P}\rangle \\ &\quad \geq \frac{11}{8}\frac{(P-1)(P-2)(M+1)M}{(M+P)(M+P-1)(M+P-2)(M+P-3)}\|u_{M,P}\|_2.\end{aligned}$$

By choosing $P = M$ we get

$$\langle R^I_{1,1}u_{M,M}, R^I_{2,1}u_{M,M}\rangle - \langle T_1 u_{M,M}, T_2 u_{M,M}\rangle \geq C\|u_{M,M}\|_2$$

with $C = 3/128$.
We can thus consider an initial state given by

$$h_0(\vec{v}) = 1 + a u_{M,M}(\vec{v}).$$

Observe that $u_{M,M}$ is an even polynomial in all its variables with positive coefficients for the terms of maximal degree. Thus $\inf_{\mathbb{R}^n} u_{M,M}(\vec{v}) > -\infty$ and choosing a small enough we get $h_0 \geq 0$.
Going back to (33) we can write

$$\begin{aligned}\|(e^{\mathscr{L}t} - e^{\overline{\mathscr{L}}t})h_0\|_2 &\geq \|h_0 - 1\|_2\frac{M}{\sqrt{N}}e^{-\Lambda t}\left(Ct - \sum_{n=2}^{\infty}\frac{t^n}{n!}\Lambda^n\left[1 - \left(1 - \frac{\mu}{2\Lambda}\right)^n\right]\right) \\ &\geq \|h_0 - 1\|_2\frac{M}{\sqrt{N}}t\left((C+1)e^{-\Lambda t} - 1\right)\end{aligned}$$

where we have used that $\left[1 - (1-x)^n\right] \leq nx$. Thus for this particular h_0 our estimate is saturated at least for a time order Λ^{-1}. Since $\Lambda > (\lambda_S/2 + \mu)M$ we cannot claim that for this example $\|(e^{\mathscr{L}t} - e^{\overline{\mathscr{L}}t})h_0\|_2$ actually grows to order $M/\sqrt{N}$.

C. Violation of $\mathscr{D}_N(H, a) \leq K\mathscr{D}_1(H, a)$

In this appendix we show that there cannot be a constant $K < N$ for which $\mathscr{D}_N(H, a) \leq K\mathscr{D}_1(H, a)$ holds for every H and a. Consider the family of function, parametrized by r, given by

$$H_r(x) = \eta^4\exp(-r\eta^2).$$

Then

$$\mathscr{D}_1(H_r, a) = \sup\frac{H_r(\eta)}{a^2 + \eta^2} = \frac{H_r(\eta(r))}{a^2 + \eta(r)^2}$$

for some $\eta(r)$ with $\eta(r)^2 \leq \frac{2}{r}$, since $H_r(\eta)/(a^2+\eta^2)$ is decreasing on $\eta^2 > \frac{2}{r}$. On the other hand, we get

$$\mathscr{D}_N(H_r, a) \geq \frac{N\eta(r)^4 \exp(-r\eta(r)^2)\exp(-\pi(N-1)\eta(r)^2)}{a^2 + N\eta(r)^2}$$

so that

$$\liminf_{r\to\infty} \frac{\mathscr{D}_N(H_r, a)}{\mathscr{D}_1(H_r, a)} \geq \liminf_{r\to\infty} N\frac{a^2+\eta(r)}{a^2+N\eta(r)^2}\exp(-\pi(N-1)\eta(r)^2) = N.$$

This bound is optimal since for any H and a we have

$$\mathscr{D}_N(H, a) \leq \sup_\eta \frac{\sum_{i=1}^N \mathscr{D}_1(H, a)(a^2+\eta^2)}{a^2+N\eta^2} \leq N\mathscr{D}_1(H, a). \tag{67}$$

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Communicated by C. Mouhot