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Self-Similar 2d Euler Solutions with Mixed-Sign Vorticity

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Abstract: We construct a class of self-similar 2d incompressible Euler solutions that have initial vorticity of mixed sign. The boundaries between regions of positive and negative vorticity form algebraic spirals, similar to the Kaden spiral and as opposed to Prandtl's logarithmic vortex spirals. Also unlike the Prandtl case, spirals are not initially present.

1. Introduction

1.1. Main result. We seek solutions of the 2d incompressible Euler equations

$$\mathbf{v}_t + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) + \nabla \pi = 0, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } (\mathbf{x}, t) \in \mathbb{R}^2 \times]0, \infty[\tag{1}$$

with locally integrable *self-similar* initial data:

$$\omega(\mathbf{x}, t) \overset{t \searrow 0}{\rightarrow} r^{-\frac{1}{\mu}} \mathring{\omega}(\theta), \tag{2}$$

where $\omega = \nabla \times \mathbf{v}$ is vorticity and (r, θ) are polar coordinates centered in $\mathbf{x} = 0$.

Theorem 1. *Given $\epsilon > 0$ and $\mu \in]\frac{2}{3}, \infty[$, there is an $N_0 \in \mathbb{N}$ so that a weak solution of (1) and (2) exists for all initial data $\mathring{\omega}$ satisfying the following conditions:*

1. *Periodicity: $\mathring{\omega}$ is $\frac{2\pi}{N}$-periodic for $N \geq N_0$.*
2. *Dominant rotation: the Fourier coefficients satisfy*

$$|\mathring{\omega}^\wedge(0)| \geq \epsilon \sum_{n \neq 0} |\mathring{\omega}^\wedge(n)|. \tag{3}$$

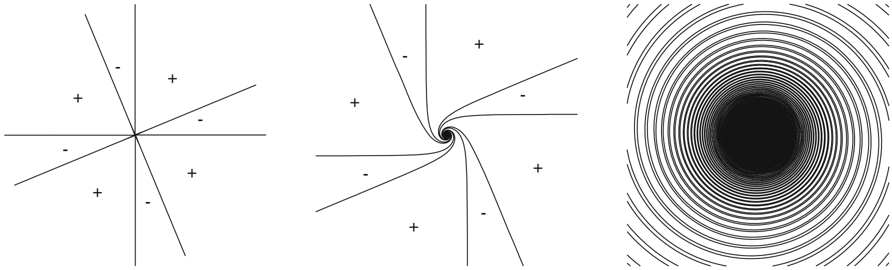

Fig. 1. *Left* sample initial data with $N = 4$ (the paper proves existence for N sufficiently large, not necessarily 4). *Center* for $t > 0$, positive and negative vorticity patches roll into an algebraic spiral. *Right* detail of inner spiral; note the densely packed almost circular turns

(The right-hand side and the integral in $\mathring{\omega}^\wedge(n) = \int_0^{2\pi} \mathring{\omega}(\theta)e^{-in\theta}\frac{d\theta}{2\pi}$ are assumed absolutely summable.) While standard theory [5,6,16,18] could possibly be adapted to provide mere existence in some cases, it appears unsuitable to obtain the detailed structural information below, especially because the known uniqueness results require $\omega \in \mathcal{L}^\infty$ [29] or slightly weaker assumptions [28,30] which are not satisfied by our initial data.

In contrast to our prior work [10], where $\mathring{\omega}$ had to be (in particular) $\mathcal{L}^\infty$-close to a nonzero constant, the present paper covers "large" initial data, allowing not only regions of zero vorticity (a necessary prerequisite for studying evolution of vortex patches with non-regular boundary [2–4]) but also flows with a mix of positive and negative vorticity (Fig. 1). In such initial data, vorticity of each sign would be in open cones with apex in the origin (Fig. 1 left).

If $\mathring{\omega}^\wedge(0) \neq 0$, then the initial velocity has a net rotation around the origin, so intuition suggests the cone tips will curl up into spirals (Fig. 1 center). Indeed we show:

Remark 1. In the setting of Theorem 1, for any $t > 0$ the boundaries between the regions of positive, zero and negative ω are algebraic spirals (Fig. 1 center and right), parametrized by

$$\mathbf{x}(\theta) = \underbrace{f(\theta)}_{\sim 1}\theta^{-\mu}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \quad \text{as } \theta \to \infty. \tag{4}$$

($f \lesssim g$ means there is a constant $C < \infty$ so that $|f| \leq Cg$ in some neighbourhood of the limit point; $f \sim g$ means $f \gtrsim g$ as well).

Flows with algebraic spiral rollup are important in applications since they are ubiquitous in physics [27, fig. 75, 84 and 92], in trailing vortices at aircraft wings, flow past a sharp corner, Mach reflections [1], turbulent eddies, detaching boundary layers, the Moore singularity of vortex sheets [23], etc. But prior to our work the mathematically rigorous construction of algebraic spiral flows was unsuccessful; see [12,19,22] for various attempts and insights. Even numerical approximation is notoriously difficult and unstable [13,20,21].

Since our initial data is self-similar, one would expect self-similar solutions as well, and indeed we show:

Remark 2. The solutions of Theorem 1 have the form

$$\omega(\mathbf{x}, t) = t^{-1}\check{\omega}(\check{\mathbf{x}}), \quad \mathbf{v}(\mathbf{x}, t) = t^{\mu-1}\check{\mathbf{v}}(\check{\mathbf{x}}), \quad \check{\mathbf{x}} = t^{-\mu}\mathbf{x} \tag{5}$$

(the accent does not represent Fourier transforms). To understand the t exponents, observe that $\omega = \nabla \times \mathbf{v}$ has dimensions of inverse time so that the initial condition (2) fixes $\mathbf{x} \sim t^{\mu}$.

Finally, we control the asymptotic behaviour of the solutions:

Remark 3. The solutions of Theorem 1 satisfy

$$\mathbf{v} \lesssim r^{1-\frac{1}{\mu}} \quad \text{and} \quad \omega \lesssim r^{-\frac{1}{\mu}} \quad \text{as } r \searrow 0 \textit{ and } \text{as } r \nearrow \infty, \ \text{uniformly in } t \in [0, \infty[; \tag{6}$$

$\mathbf{v}, \omega$ are continuous in $\mathbf{x} \neq 0$.

The periodicity and dominant-sign assumptions will be relaxed in later work, but cannot be expected to be fully removable: for $\mu \geq 1$, the initial vorticity $\omega(\mathbf{x}, 0) = r^{-1/\mu}\mathring{\omega}(\theta) \sim r^{-1/\mu}$ decays too slowly as $r \to \infty$ so that the Biot–Savart integral $\int_{\mathbb{R}^2} \frac{(\mathbf{x}-\mathbf{y})^{\perp}}{|\mathbf{x}-\mathbf{y}|^2}\omega(\mathbf{y})d\mathbf{y}$ (where $(x, y)^{\perp} = (-y, x)$) is not absolutely convergent at $|y| \to \infty$; definition in a principal-value sense requires a cancellation that occurs only for special $\mathring{\omega}$, such as $2\pi/N$-periodic $\mathring{\omega}$ for $N \geq 2$.

Moreover, when dominance changes from positive to negative vorticity, then the spirals we construct flip from counterclockwise to clockwise, with non-spiral borderline cases expected. Hence (3) cannot be omitted entirely.

Finally, the numerical work of Pullin [20,21] (see Fig. 2) on self-similar vortex sheet shows complicated bifurcation phenomena, in particular non-uniqueness (which is our main motivation [9]), a field of major recent activity [7,8,14,15,17,24–26].

1.2. Self-similarity for the Euler equations. To motivate our ansatz (2) we consider smooth solutions of the 2d vorticity equation

$$0 = \omega_t + \mathbf{v} \cdot \nabla\omega, \quad \mathbf{v} = \nabla_{\mathbf{x}}^{\perp} \Delta_{\mathbf{x}}^{-1}\omega$$

(where $\nabla^{\perp}\Delta^{-1}$ is the Biot–Savart operator on $\mathbb{R}^2$) and explore more general self-similar scalings of the form

$$\omega(\mathbf{x}, t) = f(t)\check{\omega}(\underbrace{g(t)\mathbf{x}}_{\check{\mathbf{x}}}),$$

where f, g are smooth and positive on some nonempty interval of times t. Then

$$\begin{aligned}
\nabla_{\mathbf{x}}\omega &= fg\nabla_{\check{\mathbf{x}}}\check{\omega}, \\
\partial_t\omega &= f'\check{\omega} + fg'\mathbf{x} \cdot \nabla_{\check{\mathbf{x}}}\check{\omega} = f'\check{\omega} + \frac{fg'}{g}\check{\mathbf{x}} \cdot \nabla_{\check{\mathbf{x}}}\check{\omega}, \\
\mathbf{v} &= \nabla_{\mathbf{x}}^{\perp}\Delta^{-1}\omega = \frac{f}{g}\underbrace{\nabla_{\check{\mathbf{x}}}^{\perp}\Delta_{\check{\mathbf{x}}}^{-1}\check{\omega}}_{=\check{\mathbf{v}}}.
\end{aligned}$$

The vorticity equation turns into

$$0 = f'(t)\check{\omega}(\check{\mathbf{x}}) + \frac{f(t)g'(t)}{g(t)}\check{\mathbf{x}} \cdot \nabla_{\check{\mathbf{x}}}\check{\omega}(\check{\mathbf{x}}) + f(t)^2\check{\mathbf{v}}(\check{\mathbf{x}}) \cdot \nabla_{\check{\mathbf{x}}}\check{\omega}(\check{\mathbf{x}}).$$

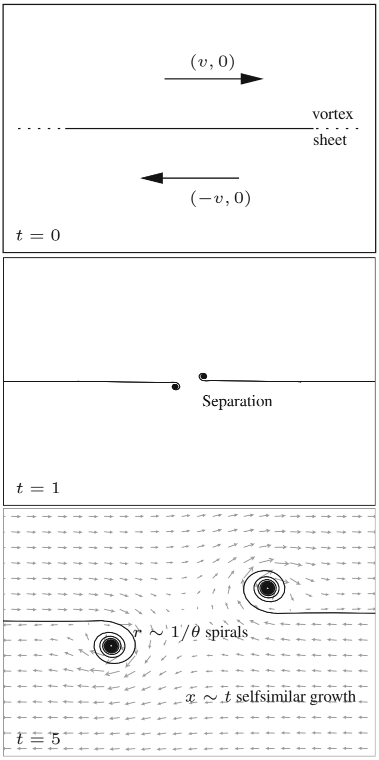

Fig. 2. Pullin's non-uniqueness example

We require the t-dependent factors to be equal up to multiplicative constants:

$$f' = \text{const}\cdot f^2, \quad \frac{fg'}{g} = \text{const}\cdot f^2.$$

The first equation has solutions

$$f(t) = \frac{1}{C_1 t + C_0}$$

for some constants C_1, C_0.

Case 1 ("logarithmic"): $C_1 = 0$. Then necessarily $C_0 > 0$. Dilation, reflection and translation in space and time are symmetries of the Euler equations, and if we consider two Euler solutions equivalent if they coincide after symmetry transformations, then we can eliminate C_0 and other constants: time dilation and time reversal yield

$$f(t) = 1.$$

Then the second equation $g'/g = \text{const}\cdot f$ has solutions $g(t) = C_2 e^{\alpha t}$ for constants $C_2 > 0$ and α. By space dilation and reflection we may take $C_2 = 1$:

$$g(t) = e^{\alpha t}.$$

Case 2 ("algebraic"): $C_1 \neq 0$. Using time dilation and reversal we obtain $C_1 = 1$, then a time shift yields $C_0 = 0$. Hence it is sufficient to consider

$$f(t) = \frac{1}{t}$$

for $t > 0$. Then the second equation $g'/g = \text{const} \cdot f$ has solutions $g(t) = C_2 t^{-\mu}$ for constants $C_2 > 0$ and μ. Again we may use invariance under space dilation and reflection to take $C_2 = 1$:

$$g = t^{-\mu}.$$

In summary, up to symmetries there are only two interesting families of f, g. (In fact with some changes $\mu \to \pm\infty$ formally yields the logarithmic case as a limit of the algebraic one.)

If we add dissipative terms, then the family of similarity laws narrows down considerably, for example for Navier–Stokes viscosity to $\mu = \frac{1}{2}$, for the unsteady *compressible* Euler equations to the *acoustic scaling* $\mu = 1$. For steady viscous flows we refer to the discussion in [11]. However, while the incompressible Euler equations feature only a single physical phenomenon (vortical motion), the compressible or viscous extensions mix other effects (acoustic waves, laminar viscosity, turbulence,...) with separation of time scales; physical observations show that in a large variety of physical flows either the viscous or the inviscid effects and either the acoustic or the incompressible features dominate the other in many regions of space-time. Hence imposing exact self-similarity, while leading to more tractable problems, excludes many interesting asymptotic behaviours for those models.

Our primary motivation [9] is to work towards a proof of the nonuniqueness examples of Pullin (Fig. 2, see [21]), which feature algebraic vortex spirals with μ in a range from slightly less than 1 to about 1.3. These examples are a special case of a more general problem: how do algebraic vortex spirals, which are ubiquitous in physical flow [27, fig. 83], arise from flows without spirals?

In the exponential case the flow is qualitatively the same for all real t, so if spirals are present at one time they are present at all times. But in the algebraic case

$$\omega(\mathbf{x}, t) = t^{-1}\check{\omega}(\check{\mathbf{x}}), \quad \check{\mathbf{x}} = t^{-\mu}\mathbf{x} \tag{7}$$

with $\mu > 0$ the limit $t \searrow 0$ is distinguished. To converge to nontrivial initial data $\omega(0, \mathbf{x})$ we need that at least in *some* fixed $\mathbf{x}$ the right-hand side $t^{-1}\check{\omega}(\check{\mathbf{x}})$ converges to a finite but nonzero limit, hence

$$|\check{\omega}(\check{\mathbf{x}})| \sim t = (|\mathbf{x}|/|\check{\mathbf{x}}|)^{1/\mu} \overset{\mathbf{x}\neq 0}{\sim} |\check{\mathbf{x}}|^{-1/\mu} \quad \text{as } \check{\mathbf{x}} \to \infty \text{ along the ray spanned by } \mathbf{x}.$$

Now we obtain on such a ray that

$$|\omega(\mathbf{x}, 0)| \leftarrow |\omega(\mathbf{x}, t)| = t^{-1}|\check{\omega}(t^{-\mu}\mathbf{x})| \sim t^{-1}|t^{-\mu}\mathbf{x}|^{-1/\mu} = |\mathbf{x}|^{-1/\mu}$$

Thus the initial values, more precisely their asymptotics at infinity, determine the value of μ.

Conversely, since smooth vorticity is known (under some additional assumptions) to persist uniquely for all time in 2d [18, ch. 3 and 4], we need to consider nonsmooth flows for formation of infinite spirals. We expect spirals to result from rollup of fluid by localized concentrations of vorticity; a natural ansatz for such concentrations are δ

functions (*point vortices*) inducing $\mathbf{v} \sim r^{-1}$, or slower algebraic blowup like $\mathbf{v} \sim r^{1-1/\mu}$ ($\mu > 0$) corresponding to $\omega(0, \mathbf{x}) \sim r^{-1/\mu}$ as $r = |\mathbf{x}| \to \infty$. We expect the behaviour of the solution for such initial data to be asymptotic (at $|\mathbf{x}| \to 0$ and $t \searrow 0$) to the solution of $\omega(0, \mathbf{x}) = r^{-1/\mu}\mathring{\omega}(\theta)$, which is precisely our choice (2). This initial data and the 2d incompressible Euler equations are invariant under the scaling

$$\omega(\mathbf{x}, t) \leftarrow s\omega(st, s^{\mu}\mathbf{x}), \quad \mathbf{v}(\mathbf{x}, t) \leftarrow s^{1-\mu}\mathbf{v}(st, s^{\mu}\mathbf{x}) \qquad (s > 0),$$

so it is natural (but, in absence of uniqueness, not forcing) to expect corresponding solutions to be invariant as well, which is precisely our choice (5).

1.3. Overview. The proof of Theorem 1 has three parts. In Sect. 2 we specialize the incompressible Euler equations (1) to the case of self-similar solutions. Then we change to a coordinate system $\mathbf{b} = (\beta, \phi)$ that is especially suitable for the construction of solutions with spiral stratification. The equations simplify in several crucial ways that are really the key to the overall solution. In the process we extract a 1d function $\Omega(\phi)$, a parameter of the problem, that corresponds closely to the initial data $\mathring{\omega}(\phi)$ (see (21) and Proposition 41). We linearize the equations around a special background solution and Fourier-transform the resulting operator in ϕ. The resulting system decomposes into infinitely many ordinary differential equations, one for each Fourier mode.

In Sect. 3 we define carefully designed function spaces and invert the ODEs on them; simultaneously we analyze the action of various constituent operators of the nonlinear PDE on these spaces. Our particular choice of function spaces is the other nontrivial key ingredient to the overall solution. We conclude that the system is invertible on certain spaces that render the nonlinear PDE $\mathcal{C}^1$, so that the implicit function theorem yields solutions near the background solution, for a range of Ω. Since some of the function spaces use negative norms, we are able to solve for "large" initial data. However, a smallness restriction remains, and indeed N_0 in Theorem 1 must be taken larger as ϵ approaches 0. (The existence of mixed-sign vorticity solutions only requires $\epsilon < 1$ though, hence a fixed N_0; although it is computable, we do not try to do so since we hope to weaken the restriction in future work.)

Finally in Sect. 4 we argue that the function obtained in new coordinates has sufficient regularity to be a weak solution of the original problem, prove that the variety of initial data generated by Ω is as large as claimed in Theorem 1, and show that the regions of negative and positive vorticity are bounded by algebraic spirals.

2. Equations

2.1. Coordinate changes. The solution of the problem begins with several changes of coordinates. Since we are looking for low-regularity solutions, some outer divergences will be treated as distributional derivatives, using the well-known formula

$$f = \nabla_{\mathbf{x}} \cdot \mathbf{w} \quad \Leftrightarrow \quad f \det \nabla_{\mathbf{y}}^T \mathbf{x} = \nabla_{\mathbf{y}} \cdot (\operatorname{adj} \nabla_{\mathbf{y}}^T \mathbf{x}\, \mathbf{w}) \tag{8}$$

where adj is the *classical adjoint* matrix, $\nabla_{\mathbf{y}}^T \mathbf{x}$ the Jacobian of the coordinate transform $\mathbf{y} \mapsto \mathbf{x}$. This formula is valid even if the divergence $\nabla_{\mathbf{x}} \cdot \mathbf{w}$ is interpreted in the distributional sense, assuming $\mathbf{x}$ and $\mathbf{w}$ are sufficiently regular to make all products well-defined in a distributional sense. Since some of the following coordinate changes are implicit (i.e. depend on the solution of the problem in new coordinates), we cannot comment on the validity of each manipulation until Sect. 4 where solutions have already been constructed; in the meantime the reader may simply assume all functions involved are sufficiently smooth.

First, the divergence constraint $\nabla_{\mathbf{x}} \cdot \mathbf{v} = 0$ implies

$$\mathbf{v} = \nabla_{\mathbf{x}}^{\perp} \psi \tag{9}$$

for a scalar *stream function* ψ; $(x, y)^{\perp} = (-y, x)$ is counterclockwise rotation by 90°. We focus on the *vorticity formulation*

$$0 = \partial_t \omega + \nabla_{\mathbf{x}} \cdot (\omega \mathbf{v}) \tag{10}$$

with

$$\omega = \nabla_{\mathbf{x}} \times \mathbf{v} = \Delta_{\mathbf{x}} \psi. \tag{11}$$

We seek *self-similar* solutions: with $\check{\mathbf{x}} = t^{-\mu}\mathbf{x}$,

$$\psi(t, \mathbf{x}) = t^{2\mu-1}\check{\psi}(\check{\mathbf{x}}), \quad \mathbf{v}(t, \mathbf{x}) = t^{\mu-1}\check{\mathbf{v}}(\check{\mathbf{x}}), \quad \omega(t, \mathbf{x}) = t^{-1}\check{\omega}(\check{\mathbf{x}}); \tag{12}$$

using (8) the problem reduces to

$$0 = (2\mu - 1)\check{\omega} + \nabla_{\check{\mathbf{x}}} \cdot \big(\check{\omega}(\check{\mathbf{v}} - \mu\check{\mathbf{x}})\big), \quad \check{\mathbf{v}} = \nabla_{\check{\mathbf{x}}}^{\perp}\check{\psi}, \quad \check{\omega} = \Delta_{\check{\mathbf{x}}}\check{\psi}. \tag{13}$$

To study spirals converging to a common origin it is convenient to use some form of polar coordinates, namely

$$\mathbf{a} = (a, \theta), \quad a = \log \check{r}, \quad \check{r} = |\check{\mathbf{x}}|, \quad \theta = \measuredangle \check{\mathbf{x}}, \tag{14}$$

where $\measuredangle$ is the counterclockwise angle from the positive horizontal axis (see Fig. 3), leading to

$$0 = (2\mu - 1)\check{r}^2\check{\omega} + (\check{\psi}_a\check{\omega})_\theta - ((\check{\psi}_\theta + \mu e^{2a})\check{\omega})_a, \quad e^{2a}\check{\omega} = \Delta_{\mathbf{a}}\check{\psi}. \tag{15}$$

Finally we change to coordinates $\mathbf{b} = (\beta, \phi)$ so that ∂_β is tangential to *pseudo-streamlines*, i.e. so that the transformed vorticity equation has a $(\partial_\beta, \partial_\phi)$ divergence with *zero* ∂_ϕ part. This scalar constraint does not determine the choice of (β, ϕ); we may impose another one: given the spiral behaviour it is natural to choose β and ϕ to be *angles*:

$$\theta = \beta + \phi. \tag{16}$$

After calculating with a general ansatz $a = a(\beta, \phi)$, the constraint implies

$$a = \frac{1}{2} \log \frac{\check{\psi}_\beta}{-\mu} \tag{17}$$

This solution-dependent change of coordinates is non-degenerate if and only if, using the convenient abbreviation

$$\partial_\varphi := \partial_\phi - \partial_\beta, \tag{18}$$

we have

$$\det \mathbf{a}_{\mathbf{b}} = -a_\varphi \neq 0 \tag{19}$$

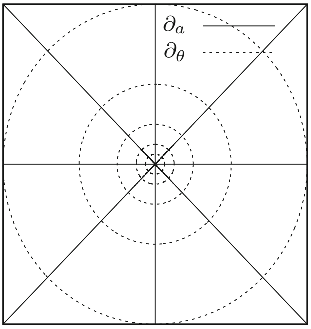

Fig. 3. $\mathbf{a} = (a, \theta)$ coordinates

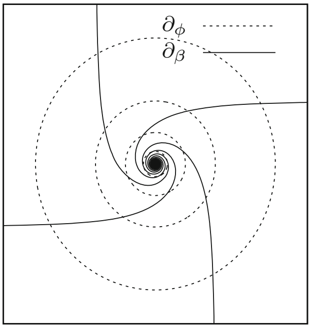

Fig. 4. $\mathbf{b} = (\beta, \phi)$ coordinates

everywhere. Now the vorticity equation reduces to

$$(1 - \frac{1}{2\mu})\check{\psi}_{\varphi\beta}\check{\omega} = \partial_\beta(\check{\omega}\check{\psi}_\varphi) \tag{20}$$

which has the surprisingly simple solution

$$\check{\omega} = \check{\psi}_\varphi^{-\frac{1}{2\mu}} \Omega(\phi) \tag{21}$$

where Ω is some function that can be chosen freely, as data. That is natural: we have a choice of initial data $\mathring{\omega}(\theta)$, another single-variable function; Ω will correspond closely, but not exactly, to $\mathring{\omega}$ (later investigated in Proposition 41).

The last remaining equation is the Poisson equation $\Delta\check{\psi} = \check{\omega}$ which transforms by a lengthy but elementary calculation using the adj formula for divergence (8) to

$$0 = \check{F} := \partial_\varphi \overbrace{\Big(2\check{\psi}_\beta\big(1 + (\frac{\check{\psi}_{\beta\phi}}{2\check{\psi}_\beta})^2\big)\frac{\check{\psi}_\varphi}{\check{\psi}_{\beta\varphi}} - \frac{\check{\psi}_\phi\check{\psi}_{\beta\phi}}{2\check{\psi}_\beta}\Big)}^{=:g^{(\varphi)}}$$
$$+\partial_\phi \underbrace{\Big(\frac{\check{\psi}_{\beta\varphi}\cdot\check{\psi}_\phi - \check{\psi}_{\beta\phi}\cdot\check{\psi}_\varphi}{2\check{\psi}_\beta}\Big)}_{=:g^{(\phi)}} + \underbrace{\frac{\check{\psi}_{\beta\varphi}\check{\psi}_\varphi^{-\frac{1}{2\mu}}}{2\mu}}_{=:g^{(0)}}\Omega \tag{22}$$

Although Eq. (22), a 3rd order nonlinear PDE for $\check{\psi}$, looks intimidating, we can identify some particular solutions that are inherited from the $\mathbf{x}$, t form. An important class are the stationary ones with $\omega(\mathbf{x}) = \omega(r)$ so that $\mathbf{v}$ is purely angular, hence orthogonal to $\nabla\omega$ so that the vorticity equation $0 = \mathbf{v} \cdot \nabla\omega$ is trivially satisfied. The initial condition (2) yields the special case $\omega(r) = \underline{\mathring{\omega}} r^{-\frac{1}{\mu}}$ with $\underline{\mathring{\omega}} = \text{const} \neq 0$ and $\psi = (2 - \frac{1}{\mu})^{-2} \underline{\mathring{\omega}} r^{2-\frac{1}{\mu}}$. These solutions are not only stationary, but also self-similar in the sense (5). (We emphasize that the solutions we are going to construct for nonconstant $\mathring{\omega}$ will *not* be stationary.) In our new coordinates the trivial solutions corresponds to multiples of

$$\check{\psi}_0 = \beta^{1-2\mu} \overline{\psi}_0, \quad \overline{\psi}_0 = \frac{1}{2\mu - 1} \tag{23}$$

(to see this, transform (23) back to $\mathbf{x}$ coordinates to compare to the given ω, ψ). $\check{\psi}_0$ solves (22) for data

$$\Omega_0 = \frac{2\mu - 1}{\mu} \tag{24}$$

(this can be checked quickly by substituting $\check{\psi}_0$, Ω_0 into (17) and using that all ∂_ϕ yield 0).

In Sect. 2.2 we scale the β-decay out of $\check{\psi}$ and $\check{F}$ in (22) to reach new variables $\overline{F}$, $\overline{\psi}$ (see (28)), then linearize $\overline{F}$ around $\overline{\psi}_0$. After defining function spaces Ψ for solution candidates $\overline{\psi}$ (Sect. 3.4), $\mathcal{W}$ for the data Ω (Sect. 3.8) and $\mathcal{F}$ for the values of $\overline{F}$ (Sect. 3.6), we prove in Sect. 3.9 that for some neighbourhood $B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0)$ of $\overline{\psi}_0$ in Ψ the map $\overline{F}$ is $\mathcal{C}^1$ on $B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0) \times \mathcal{W}$ into $\mathcal{F}$. In Sect. 3.7 we show that the Fréchet derivative $\partial \overline{F}/\partial \overline{\psi}(\overline{\psi}_0, \Omega_0)$ is a linear isomorphism on Ψ into $\mathcal{F}$. Hence the implicit function theorem shows there is a family of solutions for Ω_0 near Ω.

To achieve a large-data result we strongly exploit the divergence form of (22). In Sect. 3.2 we find a "near-maximal" multiplication algebra $\mathcal{G}$ for the nonlinear terms $g^{(\varphi)}$, $g^{(\phi)}$, $g^{(0)}$ in (22). This enables us to seek Ω in negative-norm spaces (namely $\langle D \rangle^{\frac{1}{2}} \mathcal{A}(\mathbb{T})$ where $\mathcal{A}(\mathbb{T})$ is the Wiener algebra, $\langle x \rangle = (1 + x^2)^{\frac{1}{2}}$ and $D = -i\partial_\phi$). In fact our techniques can solve the problem for some $\mathring{\omega}$ that are non-regular distributions, in fact non-*measure* distributions. However, physically interesting cases (such as δ functions) do not seem to be included among the non-function distributions, so to avoid technical bloat we state a result only for $\mathring{\omega}$ that are continuous functions.

A key difficulty of the problem is the asymptotic behaviour as $\beta \to \infty$. To solve the problem for nonconstant perturbations $\mathring{\omega}$ of the constant $\underline{\mathring{\omega}}$, we need to perturb the background solution $\check{\psi}$ and hence the coordinate transformation from β, ϕ to $\check{\mathbf{x}}$, whose curves of constant ϕ are algebraic spirals. A perturbation that does not decay sufficiently fast towards the spiral center will cause the curves to self-intersect (Fig. 5 right). This can easily happen since the spiral turns are very densely packed. Self-intersection corresponds to a degenerate coordinate transform $\mathbf{b} \mapsto \check{\mathbf{x}}$ which would manifest itself in the equations (22) as denominators $\check{\psi}_{\beta\varphi}$ crossing zero. The $\beta \to \infty$ problem will surface in many analytical details throughout the paper.

2.2. Scaling and linearization. It is convenient to remove the β decay from all quantities, leading to new versions with overbar, starting with

$$\check{\psi} = \beta^{1-2\mu} \overline{\psi}. \tag{25}$$

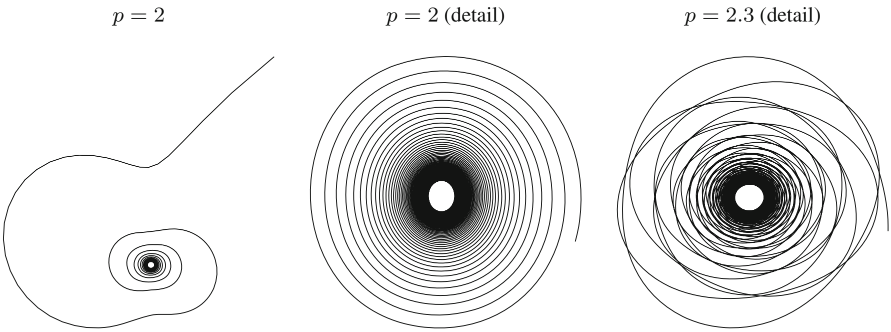

Fig. 5. The spiral $\mathbb{R}_+ \ni \beta \mapsto \beta^{-1}e^{i\beta}$ perturbed to $\beta^{-1}e^{i\beta}(1 + \alpha\beta^{-\delta}e^{ip\beta})$, with $\alpha = 0.5$, $\delta = 0.7$. (The spiral center is not drawn, leaving a white spot in the middle.) Integer frequency p: the perturbation is barely noticeable near the center. Non-integer frequency p: physically unreasonable self-intersection

We write $\acute{f}$ for multiplication-by-f operators. $a^\wedge = \mathcal{F}a$ and $a^\vee = \mathcal{F}^{-1}a$ represent the Fourier transform and inverse transform of $a = a(\beta, \phi)$ with respect to $\phi \in \mathbb{T}$, with dual variable n, leaving the first variable β untouched. We abbreviate

$$\overline{\partial}_\beta := \acute{\beta}^{2\mu}\partial_\beta\acute{\beta}^{1-2\mu} = \underbrace{\acute{\beta}\partial_\beta}_{=:B} + \acute{1} - \acute{2}\acute{\mu} \tag{26}$$

$$\overline{\partial}_\varphi := \acute{\beta}^{2\mu}\partial_\varphi\acute{\beta}^{1-2\mu} = \acute{\beta}\partial_\varphi + \acute{2}\acute{\mu} - \acute{1}, \quad \mathcal{F}^{-1}\overline{\partial}_\varphi\mathcal{F} = -\underbrace{\acute{\beta}(\partial_\beta - i\acute{n})}_{=:A} + \acute{2}\acute{\mu} - \acute{1} \tag{27}$$

Now (22) takes the form

$$0 = \overline{F} = 2\mu^2(-\overline{\partial}_\varphi\overline{g^{(\varphi)}} - \partial_\phi\overline{g^{(\phi)}} - \overline{g^{(0)}}\Omega) \tag{28}$$

$$\overline{g^{(\varphi)}} = 2\Big(1 + (\frac{\partial_\phi\overline{\partial}_\beta\overline{\psi}}{2\overline{\partial}_\beta\overline{\psi}})^2\Big)\overline{\partial}_\beta\overline{\psi}\frac{\overline{\partial}_\varphi\overline{\psi}}{(\overline{\partial}_\varphi + \acute{1})\overline{\partial}_\beta\overline{\psi}} - \frac{\partial_\phi\overline{\partial}_\beta\overline{\psi}\cdot\partial_\phi\overline{\psi}}{2\,\overline{\partial}_\beta\overline{\psi}} \tag{29}$$

$$\overline{g^{(\phi)}} = \frac{(\overline{\partial}_\varphi + \acute{1})\overline{\partial}_\beta\overline{\psi}\cdot\partial_\phi\overline{\psi} - \partial_\phi\overline{\partial}_\beta\overline{\psi}\cdot\overline{\partial}_\varphi\overline{\psi}}{2\,\overline{\partial}_\beta\overline{\psi}} \tag{30}$$

$$\overline{g^{(0)}} = \frac{1}{2\mu}(\overline{\partial}_\varphi + \acute{1})\overline{\partial}_\beta\overline{\psi}\cdot(\overline{\partial}_\varphi\overline{\psi})^{-\frac{1}{2\mu}} \tag{31}$$

At this point the reader should appreciate that no $\acute{\beta}$ remain except inside $\overline{\partial}_\varphi$ and $\overline{\partial}_\beta$. This is far from trivial: that the resulting derivatives of $\overline{\psi}$ can be estimated in spaces without β decay is crucial for solving the problem. Note that $\partial_\varphi = \partial_\phi - \partial_\beta$, so that one of $\partial_\phi, \partial_\beta, \partial_\varphi$ is redundant. However, $\overline{\partial}_\varphi, \overline{\partial}_\beta$ contain a $\acute{\beta}$ here while ∂_ϕ does not, so the choice of operators and the arrangement of the terms is a key step in overcoming the β decay problem.

We seek nontrivial solutions by linearizing the equation around the trivial $\overline{\psi} = \overline{\psi}_0$ and $\Omega = \Omega_0$ and taking the Fourier transform: another lengthy but elementary calculation yields

$$\mathcal{F}^{-1}\frac{\partial \overline{F}}{\partial \overline{\psi}}(\overline{\psi}_0,\Omega_0)\mathcal{F} = -\underbrace{\Big(\overbrace{A-\overbrace{((\acute{2}+\acute{n})\acute{\mu}-\acute{1})}^{=:\acute{m}_+}}^{=:A_+}\Big)\Big(\overbrace{A-\overbrace{((\acute{2}-\acute{n})\acute{\mu}-\acute{1})}^{=:\acute{m}_-}}^{=:A_-}\Big)\overbrace{(B+\acute{1})}^{=:B_1}}_{=:R}$$
$$+\underbrace{(2\mu-1)(A-B)}_{=:E} \tag{32}$$

We observe the essential fact that the operator has decomposed into infinitely many *ordinary* differential operators. Reason: we linearized around a function $\check{\psi}_0(\beta,\phi) = \text{const}\cdot\beta^{1-2\mu}$ that is constant in ϕ which therefore does not appear in the variable coefficients, so that Fourier transforms in that coordinate are effective.

We will invert this linearized operator by first inverting the 3rd order operator R and then absorbing E as a perturbation (see Sect. 3.7).

3. Function Spaces, Estimates and Implicit Function Theorem

In this section we define function spaces for the domain Ψ and codomain $\mathcal{F}$ of the nonlinear map $\overline{F}$ in (28) we are trying to invert, as well as several multiplication algebras $\mathcal{G}, \mathcal{G}_-, \mathcal{G}_0$ for the nonlinear parts of that map. Then on these spaces we carefully estimate the continuity of each differential operator constituting $\overline{F}$.

3.1. Wiener algebras with regularity. Since the linearization (32) decomposed into ordinary differential operators after taking Fourier transforms, it is convenient to analyze these operators separately for each frequency $n\in\mathbb{Z}$ on function spaces $\mathcal{X}_n$ and then to combine $\mathcal{X}_n$ into an overall space.

We utilize the following Wiener-type algebras, which are slightly nonstandard because we need $\mathcal{X}_n$ to depend on n. The reader may wish to note the definition and propositions, but skip over the standard proof techniques on first reading.

Definition 1. Let $(\mathcal{X}_n)_{n\in\mathbb{Z}}$ be a sequence of Banach spaces. $\ell^1(\mathcal{X}_n)$ is the space of sequences $u=(u_n)$ with $u_n\in\mathcal{X}_n$ for all $n\in\mathbb{Z}$ with finite norm

$$\|(u_n)\|_{\ell^1(\mathcal{X}_n)} = \sum_{n\in\mathbb{Z}}\|u_n\|_{\mathcal{X}_n}.$$

A straightforward adaptation of the standard proofs for n-independent $\mathcal{X}_n$ yields

Proposition 1. *$\ell^1(\mathcal{X}_n)$ is a Banach space.*

Let $\mathcal{X}$ be a Banach space, $\mathcal{Y}$ a linear space, $T:\mathcal{X}\to\mathcal{Y}$ linear injective. Then $T\mathcal{X}$ is a Banach space with the induced norm

$$\|Tx\|_{T\mathcal{X}} = \|x\|_{\mathcal{X}}$$

Alternatively, if T is a linear continuous operator on $\mathcal{D}'(\mathbb{R}_+)$ (but not necessarily injective; typically some differential operator), and $\mathcal{X}$ a Banach space continuously embedded in $\mathcal{D}'(\mathbb{R}_+)$, then $T\mathcal{X}$ is a Banach space with the standard induced norm

$$\|u\|_{T\mathcal{X}} = \inf\{\|x\|_{\mathcal{X}} \ :\ Tx=u\}.$$

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. $[\mathcal{X}, \mathcal{Y}]$ is the space of linear continuous operators on $\mathcal{X}$ into $\mathcal{Y}$, with operator norm

$$\|T\|_{[\mathcal{X},\mathcal{Y}]} := \inf\{C \in [0, \infty[\; : \; \forall x \in \mathcal{X} : \|Tx\|_{\mathcal{Y}} \leq C \|x\|_{\mathcal{X}}\}.$$

$[\mathcal{X}] = [\mathcal{X}, \mathcal{X}]$.

Henceforth, if a, b are expressions that may depend on j, k, n, we write

$$a \lesssim b$$

if $|a| \leq Cb$ for a constant $C < \infty$ independent of j, k, n. We write $a \sim b$ if $b \lesssim a$ as well. We write $\mathcal{X} \hookrightarrow \mathcal{Y}$ if a linear topological space $\mathcal{X}$ is continuously embedded in another linear topological space $\mathcal{Y}$. For Banach spaces $\mathcal{X}, \mathcal{Y}$ which vary with j, k, n, *n-uniform continuous embedding* means $\|\mathrm{id}\|_{[\mathcal{X},\mathcal{Y}]} \lesssim 1$, which we abbreviate as

$$\mathcal{X} \overset{\sim}{\hookrightarrow} \mathcal{Y}.$$

We use the standard notation $\langle n \rangle = (1 + n^2)^{\frac{1}{2}}$.

Let $\mathcal{X}_n$ be a sequence of Banach spaces. Assume that for some $q < \infty$ the spaces $\langle \acute{n} \rangle^{-q} \mathcal{X}_n$ are n-uniformly continuously embedded in a Banach space $\mathcal{X}$ which is continuously embedded in $\mathcal{D}'(\mathbb{R}_+)$. (This is necessary because polynomial growth bounds on Fourier transforms are needed to render the inverse transform well-defined in the sense of temperate distributions; all needed results from their classical theory are adapted with obvious modifications.)

Definition 2.

$$\mathcal{A}^s(\mathcal{X}_n) := (\langle \acute{n} \rangle^{-s} \ell^1(\mathcal{X}_n))^\vee \tag{33}$$

($\mathcal{X}_n$ is permitted to vary with n, so s should be considered an index of smoothness only when $q = 0$ above.)

Remark 4. $\mathcal{A}^0(\mathcal{X}_n)$ with $\mathcal{X}_n = \mathbb{C}$ for all n is the classical Wiener algebra $\mathcal{A}(\mathbb{T})$.

Proposition 2. *$\mathcal{A}^s(\mathcal{X}_n)$ is a Banach space.*

Proof. $\mathcal{F}^{-1}$ and $\langle \acute{n} \rangle^{-s}$ are injective, and $\ell^1(\mathcal{X}_n)$ is a Banach space by Proposition 1, so $\mathcal{A}^s(\mathcal{X}_n)$ with the induced norm is a Banach space. □

Remark 5. If $T_n : \mathcal{X}_n \to \mathcal{Y}_n$ is linear with

$$\|T_n\|_{[\mathcal{X}_n,\mathcal{Y}_n]} \lesssim 1,$$

then $T(u_n) := (T_n u_n)$ defines a continuous linear map on $\mathcal{A}^s(\mathcal{X}_n)$ into $\mathcal{A}^s(\mathcal{Y}_n)$. In particular, if $\mathcal{X}_n \overset{\sim}{\hookrightarrow} \mathcal{Y}_n$, then

$$\mathcal{A}^s(\mathcal{X}_n) \hookrightarrow \mathcal{A}^s(\mathcal{Y}_n).$$

If additionally T_n is an isomorphism with

$$\left\| T_n^{-1} \right\|_{[\mathcal{X}_n,\mathcal{Y}_n]} \lesssim 1,$$

then T defines an isomorphism of $\mathcal{A}^s(\mathcal{X}_n)$ onto $\mathcal{A}^s(\mathcal{Y}_n)$. Finally,

$$\mathcal{A}^s(\mathcal{X}_n \oplus \mathcal{Y}_n) = \mathcal{A}^s(\mathcal{X}_n) \oplus \mathcal{A}^s(\mathcal{Y}_n). \tag{34}$$

Proposition 3. *Let* $(\mathcal{X}_n), (\mathcal{Y}_n), (\mathcal{Z}_n)$ *be sequences of Banach spaces embedded in* $\mathcal{D}'(\mathbb{R}_+)$. *Assume that for all* $j, k \in \mathbb{Z}$ *and* $x \in \mathcal{X}_j$, $y \in \mathcal{Y}_k$ *the product* $x \cdot y$ *is well-defined and in* $\mathcal{Z}_{j+k}$, *with*

$$\mathcal{X}_j \cdot \mathcal{Y}_k \overset{\sim}{\hookrightarrow} \mathcal{Z}_{j+k} \tag{35}$$

Finally, let $s \geq 0$.

Then for $x \in \mathcal{A}^s(\mathcal{X}_n)$ *and* $y \in \mathcal{A}^{-s}(\mathcal{Y}_n)$ *the right-hand side of*

$$(x \cdot y)^\wedge(n) := \sum_{k \in \mathbb{Z}} x^\wedge(n-k) \cdot y^\wedge(k)$$

is absolutely convergent, so we may define the product $x \cdot y$ *by it, and moreover*

$$\mathcal{A}^s(\mathcal{X}_n) \cdot \mathcal{A}^{-s}(\mathcal{Y}_n) \hookrightarrow \mathcal{A}^{-s}(\mathcal{Z}_n) \tag{36}$$

$$\mathcal{A}^s(\mathcal{X}_n) \cdot \mathcal{A}^{s}(\mathcal{Y}_n) \hookrightarrow \mathcal{A}^{s}(\mathcal{Z}_n) \tag{37}$$

Proof. In Appendix 5.1. □

Remark 6. The regularity exponents $\pm s$ cannot be improved, as simple examples show. In more familiar terms, the standard definition of distributions T of order s times functions u,

$$\forall \chi \in \mathcal{D} : \langle T \cdot u, \chi \rangle := \langle T, u \cdot \chi \rangle,$$

requires $u \cdot \chi \in \mathcal{C}^s$, hence $u \in \mathcal{C}^s$.

3.2. The multiplication algebras $\mathcal{G}$. $\mathcal{C}_b(\ldots)$ represents the continuous functions that are bounded (including up to infinity). If we omit the domain, then $\overline{\mathbb{R}_+}$ is implied.

For the remainder of the paper we choose some $\chi_0 \in \mathcal{C}^\infty(\overline{\mathbb{R}_+})$ so that $\chi_0 = 1$ near 0 and $\chi_0 = 0$ near ∞. Abbreviate

$$\chi_\infty = 1 - \chi_0$$

so that $\chi_\infty = 0$ near 0 and $\chi_\infty = 1$ near ∞. Let $\delta > 0$; we will restrict the value further later on. Abbreviate

$$\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b = \acute{\beta}^{\delta}\mathcal{C}_b \cap \acute{\beta}^{-\delta}\mathcal{C}_b. \tag{38}$$

If $\mathcal{X}, \mathcal{Y}$ are Banach spaces embedded in the same linear Hausdorff space, then $\mathcal{X} \cap \mathcal{Y}$ and $\mathcal{X} + \mathcal{Y}$ are also Banach spaces with the induced norms

$$\|u\|_{\mathcal{X}\cap\mathcal{Y}} := \max\{\|u\|_{\mathcal{X}}, \|u\|_{\mathcal{Y}}\},$$
$$\|u\|_{\mathcal{X}+\mathcal{Y}} := \inf\{\|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}} \ : \ u = x + y\};$$

$\acute{\beta}^{\delta}\mathcal{C}_b, \acute{\beta}^{-\delta}\mathcal{C}_b \hookrightarrow \mathcal{D}'(\mathbb{R}_+)$, so $\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b$ is a Banach space. Obviously

$$\mathcal{C}_b \cdot \bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b \hookrightarrow \bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b \tag{39}$$

The following Banach algebras $\mathcal{G}_{\ldots}$ are the spaces for the values of $\overline{g^{(\varphi)}}, \overline{g^{(\phi)}}, \overline{g^{(0)}}$ in (28). These three expressions are nonlinear in $\overline{\psi}$ and its derivatives, but $\overline{F}$ in (28) is linear in $\overline{g^{(\varphi)}}, \overline{g^{(\phi)}}, \overline{g^{(0)}}$. By multiplying before taking divergence, we are able to construct solutions of lower regularity.

We write $\mathbb{C}$ for subspaces of constant functions, usually functions of β (the domain will be clear from the context). For an element v of some vector space V, $\mathbb{C}v$ is the span of v.

Definition 3. For $n \in N\mathbb{Z}$ consider the following spaces of functions of β (with $\delta_{jk} = 0$ unless $j = k$ where $\delta_{jk} = 1$):

$$\mathcal{G}_0^n := \bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b \oplus \mathbb{C}\chi_0 \tag{40}$$

$$\mathcal{G}_-^n := \bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b \oplus \mathbb{C}\chi_0 \oplus \delta_{0n}\mathbb{C}\chi_\infty \overset{\chi_\infty=1-\chi_0}{=} \bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b \oplus \mathbb{C}\chi_0 \oplus \delta_{0n}\mathbb{C} \tag{41}$$

$$\mathcal{G}^n := \bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b \oplus \mathbb{C}\chi_0 \oplus \mathbb{C}\chi_\infty \overset{\chi_\infty=1-\chi_0}{=} \bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b \oplus \mathbb{C}\chi_0 \oplus \mathbb{C} \tag{42}$$

The $\oplus$ are direct sums because (consider the limits $\beta \searrow 0$ and $\beta \nearrow \infty$) χ_∞ and χ_0 are linearly independent and not contained in $\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b$. For $n \notin N\mathbb{Z}$ we set all these spaces to $\{0\}$. (After inverse transforms this corresponds to the $\frac{2\pi}{N}$-periodicity in Theorem 1.)

Functions in $\mathcal{G}^n$ have the asymptotic behaviours

$$\overbrace{c_0}^{\text{const}} + \beta^\delta \overbrace{r_0(\beta)}^{\text{bounded}} \quad \text{as } \beta \searrow 0,$$
$$\underbrace{c_\infty}_{\text{const}} + \beta^{-\delta}\underbrace{r_\infty(\beta)}_{\text{bounded}} \quad \text{as } \beta \nearrow \infty.$$

$\mathcal{G}_-^n$ allows a c_∞ term only in the $n = 0$ Fourier mode; $\mathcal{G}_0^n$ does not allow it at all.

It is natural to wonder why we would not allow $\overline{g^{(\varphi)}}, \overline{g^{(\phi)}}, \overline{g^{(0)}}$ to live in the larger space $\mathcal{C}_b$. That is because we will, in light of (32), represent $\overline{\psi}$ as $(B + \acute{1})^{-1}A_-^{-1}g$, and when applying $A + \acute{1}$ to $\overline{\psi}$ (as $\overline{g^{(\varphi)}}, \overline{g^{(\phi)}}, \overline{g^{(0)}}$ require us to be able to) we (formally) get

$$(A + \acute{1})(B + \acute{1})^{-1}A_-^{-1}g = B^{-1}AA_-^{-1}g$$

But $B^{-1} = (\beta\partial_\beta - 0)^{-1}$ is undefined on $\mathcal{C}_b$ because it has the characteristic exponent $s = 0$. Formally the Green function representation is

$$B^{-1}f(\beta) = \int_{\dots}^{\beta} \beta'^{-1} f(\beta')d\beta',$$

but for generic $f \in \mathcal{C}_b$ the β'^{-1} will cause logarithmic blowup either at $\beta = 0+$ or at $\beta = \infty$, no matter how we choose the boundary "...". B^{-1} is, however, well-defined on $\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b$ (albeit not into). (If our linearized operator happened to have $(B + \acute{s})^{-1}$ with a more convenient s instead of $(B + \acute{1})^{-1}$, then the paper could be shortened considerably.)

From the definitions the following products are obvious:

Proposition 4. *For all $j, k \in \mathbb{Z}$,*

$$\mathcal{G}_0^j \cdot \mathcal{G}^k \overset{\sim}{\hookrightarrow} \mathcal{G}_0^{j+k} \tag{43}$$

$$\mathcal{G}^j \cdot \mathcal{G}^k \overset{\sim}{\hookrightarrow} \mathcal{G}^{j+k} \tag{44}$$

$$\mathcal{G}_-^j \cdot \mathcal{G}_-^k \overset{\sim}{\hookrightarrow} \mathcal{G}_-^{j+k} \tag{45}$$

(All embeddings are uniform in n ($\tilde{\hookrightarrow}$) because $\mathcal{G}^n$, $\mathcal{G}^n_0$ are independent of n while $\mathcal{G}^n_-$ has only two different values.)

We also note the following obvious embeddings ($\mathcal{C}^\infty_c$ being $\mathcal{C}^\infty$ functions of compact support)

$$\mathcal{C}^\infty_c(\mathbb{R}_+) \hookrightarrow \bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b \tilde{\hookrightarrow} \mathcal{G}^n_0 \tilde{\hookrightarrow} \mathcal{G}^n_- \tilde{\hookrightarrow} \mathcal{G}^n \tilde{\hookrightarrow} \mathcal{C}_b \hookrightarrow \mathcal{D}'(\mathbb{R}_+) \tag{46}$$

The spaces $\mathcal{G}^n$, $\mathcal{G}^n_-$, $\mathcal{G}^n_0$ define the behaviour of $g^\wedge(\beta, n)$ for each n separately. The following spaces define the asymptotics of $g^\wedge$ as $|n| \to \infty$. We require regularity $\frac{1}{2}$, which will be motivated in Sect. 3.8.

Definition 4.

$$\mathcal{G} := \mathcal{A}^{\frac{1}{2}}(\mathcal{G}^n) \tag{47}$$

$$\mathcal{G}_- := \mathcal{A}^{\frac{1}{2}}(\mathcal{G}^n_-) \tag{48}$$

$$\mathcal{G}_0 := \mathcal{A}^{\frac{1}{2}}(\mathcal{G}^n_0) \tag{49}$$

Using (37) and (43), (44), (45) we immediately obtain

Proposition 5.

$$\mathcal{G}_0 \cdot \mathcal{G} \hookrightarrow \mathcal{G}_0 \tag{50}$$

$$\mathcal{G} \cdot \mathcal{G} \hookrightarrow \mathcal{G} \tag{51}$$

$$\mathcal{G}_- \cdot \mathcal{G}_- \hookrightarrow \mathcal{G}_- \tag{52}$$

Proposition 6.

$$1 \in \mathcal{G}_- \subset \mathcal{G} \tag{53}$$

Proof. The Fourier transform of 1 is δ_{0n} which is $= 1$ for $n = 0$ (in $\mathcal{G}^n_-$ by (41)), $= 0$ for $n \neq 0$ (trivially in $\mathcal{G}^n_-$). The $\mathcal{G}_-$ norm is obviously finite. □

3.3. L and L^{-1}. Now we start analyzing the differential operators in (32).

Definition 5. From now on consider

$$Z := \acute{\beta}(\partial_\beta - i\acute{z}) \tag{54}$$

$$L := Z - \acute{s} \tag{55}$$

where $s, z \in \mathbb{R}$.

Remark 7. Note $Z \overset{(26)}{=} B$ for $z = 0$, $Z \overset{(27)}{=} A$ for $z = n$.

If we regard these differential operators as defined on $\mathcal{D}'(\mathbb{R}_+)$, then

$$\ker L = \mathbb{C}\beta^s e^{iz\beta} \tag{56}$$

Proposition 7. *Let $s \neq 0$. Then L is injective on $\mathcal{C}_b$.*

Proof. If not, then by (56) $\beta^s e^{iz\beta} \in \mathcal{C}_b$. But for $s \neq 0$ that function is not bounded. □

Definition 6. Let $s \neq 0$ so that L is injective on $\mathcal{C}_b$ (Proposition 7). For the remainder of the paper, let $L^{-1} : L\mathcal{C}_b \to \mathcal{C}_b$ refer to the inverse of $L : \mathcal{C}_b \to L\mathcal{C}_b$:

$$L^{-1}L = \mathrm{id} \quad \text{on } \mathcal{C}_b, \tag{57}$$
$$LL^{-1} = \mathrm{id} \quad \text{on } L\mathcal{C}_b. \tag{58}$$

L is continuous on $\mathcal{D}'(\mathbb{R}_+)$, so $L\mathcal{C}_b$ is a Banach space with the induced norm, and obviously

$$L : \mathcal{C}_b \to L\mathcal{C}_b \text{ is an isometry,} \tag{59}$$
$$L^{-1} : L\mathcal{C}_b \to \mathcal{C}_b \text{ is an isometry.} \tag{60}$$

If we apply the inverse on $\mathcal{C}_b$ rather than $L\mathcal{C}_b$, we can use the following estimate.

Proposition 8. *Let $s \neq 0$. Then*

$$\mathcal{C}_b \hookrightarrow L\mathcal{C}_b, \tag{61}$$

so

$$LL^{-1} \overset{(58)}{=} \mathrm{id} \quad \text{on } \mathcal{C}_b, \tag{62}$$

and we have the estimate

$$\left\| L^{-1} \right\|_{[\mathcal{C}_b]} \leq |s|^{-1}. \tag{63}$$

Proof. Set

$$u(\beta) = \beta^s e^{iz\beta} \underbrace{\int_{\infty^{\operatorname{sign} s}}^{\beta} \frac{f(\beta')}{\beta'^s e^{iz\beta'}} \frac{d\beta'}{\beta'}}_{=:w(\beta)}$$

where

$$\infty^{\operatorname{sign} s} = \begin{cases} \infty, & s > 0 \\ 0, & s < 0 \end{cases}$$

so that (since f is bounded) the integrand is absolutely summable. It is easy to check that u solves $Lu = f$. Moreover

$$\begin{aligned} |u(\beta)| &= |\beta^s e^{iz\beta} \int_{\infty^{\operatorname{sign} s}}^{\beta} f(\beta')\beta'^{-s-1} e^{-iz\beta'} d\beta'| \\ &\overset{\beta'=x\beta}{\underset{x=\beta'/\beta}{=}} |\int_{\infty^{\operatorname{sign} s}}^{1} f(x\beta) x^{-s-1} e^{iz\beta(1-x)} dx| \\ &\leq \int_{\infty^{\operatorname{sign} s}}^{1} |f(x\beta)| x^{-s-1} dx \leq \|f\|_{\mathcal{C}_b} \int_{\infty^{\operatorname{sign} s}}^{1} x^{-s-1} dx = |s|^{-1} \|f\|_{\mathcal{C}_b} \end{aligned} \tag{64}$$

which yields (63), and $\|f\|_{L\mathcal{C}_b} = \left\| L^{-1} f \right\|_{\mathcal{C}_b} \leq \left\| L^{-1} \right\|_{[\mathcal{C}_b]} \|f\|_{\mathcal{C}_b}$ yields (61). □

Proposition 9.

$$(Z - \acute{s})\acute{\beta}^r = \acute{\beta}^r (Z - \acute{s} + \acute{r}) \quad \text{on } \mathcal{D}'(\mathbb{R}_+) \tag{65}$$

Proof. Trivial calculation. □

Proposition 10. *Let $r \in \mathbb{R}$ and $s \notin \{0, r\}$. If*

$$u \in (Z - \acute{s} + \acute{r})\mathcal{C}_b \quad (\textit{sufficient: } u \in \mathcal{C}_b), \ \textit{and} \tag{66}$$

$$u \in \acute{\beta}^{-r}(Z - \acute{s})\mathcal{C}_b \quad (\textit{sufficient: } u \in \acute{\beta}^{-r}\mathcal{C}_b), \tag{67}$$

then

$$\acute{\beta}^r (Z - \acute{s} + \acute{r})^{-1} u = (Z - \acute{s})^{-1} \acute{\beta}^r u + \beta^s e^{iz\beta} c \tag{68}$$

where the scalar c is zero if $\operatorname{sign} s = \operatorname{sign}(s - r)$ *for all n.*

Proof. By assumption $s \neq 0$ and $s - r \neq 0$, so (61) yields $\mathcal{C}_b \hookrightarrow (Z - \acute{s} + \acute{r})\mathcal{C}_b$ and $\acute{\beta}^{-r}\mathcal{C}_b \hookrightarrow \acute{\beta}^{-r}(Z - \acute{s})\mathcal{C}_b$ so that the right conditions in (66) and (67) imply the left ones.

Let $u \in (Z - \acute{s} + \acute{r})\mathcal{C}_b$ and $\beta^r u \in (Z - \acute{s})\mathcal{C}_b$, then

$$(Z - \acute{s})(Z - \acute{s})^{-1} \underbrace{\acute{\beta}^r u}_{\in (Z - \acute{s})\mathcal{C}_b} \overset{(58)}{=} \acute{\beta}^r \underbrace{u}_{\in (Z - \acute{s} + \acute{r})\mathcal{C}_b} \overset{(58)}{=} \acute{\beta}^r (Z - \acute{s} + \acute{r})(Z - \acute{s} + \acute{r})^{-1} u$$

$$\overset{(65)}{=} (Z - \acute{s})\acute{\beta}^r (Z - \acute{s} + \acute{r})^{-1} u$$

We remove $Z - \acute{s}$ from the left of both sides, having to allow a multiple of its kernel element $\beta^s e^{iz\beta}$:

$$(Z - \acute{s})^{-1} \acute{\beta}^r u = \acute{\beta}^r (Z - \acute{s} + \acute{r})^{-1} u - c\beta^s e^{iz\beta}$$

which is (68). Solve for the constant c:

$$c = e^{-iz\acute{\beta}} \big(\acute{\beta}^{-(s-r)} \underbrace{(Z - \acute{s} + \acute{r})^{-1} u}_{\in \mathcal{C}_b} - \acute{\beta}^{-s} \underbrace{(Z - \acute{s})^{-1} \acute{\beta}^r u}_{\in \mathcal{C}_b} \big) \tag{69}$$

If $\operatorname{sign} s = \operatorname{sign}(s - r) \neq 0$, then either as $\beta \to 0$ or as $\beta \to +\infty$ *both* β^{-s} and $\beta^{-(s-r)}$ decay, whereas all other right-hand side factors are bounded, so $c = 0$. □

Proposition 11. *Let $s \notin [-\delta, \delta]$. Then*

$$\left\| (Z - \acute{s})^{-1} \right\|_{[\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b]} \leq \operatorname{dist}(s, [-\delta, \delta])^{-1} \tag{70}$$

Proof.

$$\begin{aligned}
\left\|(Z-\acute{s})^{-1}u\right\|_{\bigcap\acute{\beta}^{\pm\delta}\mathcal{C}_b} &= \max_{\sigma=\pm1}\left\|(Z-\acute{s})^{-1}u\right\|_{\acute{\beta}^{\sigma\delta}\mathcal{C}_b}\\
&= \max_{\sigma=\pm1}\left\|\acute{\beta}^{-\sigma\delta}(Z-\acute{s})^{-1}\acute{\beta}^{\sigma\delta}\acute{\beta}^{-\sigma\delta}u\right\|_{\mathcal{C}_b}\\
&\overset{(68),r=\sigma\delta}{\underset{\operatorname{sign}s=\operatorname{sign}(s-\sigma\delta)\neq0}{=}} \max_{\sigma=\pm1}\left\|(Z-\acute{s}+\sigma\acute{\delta})^{-1}\acute{\beta}^{-\sigma\delta}u\right\|_{\mathcal{C}_b}\\
&\le \max_{\sigma=\pm1}\underbrace{\left\|(Z-\acute{s}+\sigma\acute{\delta})^{-1}\right\|_{[\mathcal{C}_b]}}_{\overset{(63)}{\underset{s-\sigma\delta\neq0}{\le}}|s-\sigma\delta|^{-1}}\underbrace{\left\|\acute{\beta}^{-\sigma\delta}u\right\|_{\mathcal{C}_b}}_{=\|u\|_{\acute{\beta}^{\sigma\delta}\mathcal{C}_b}\le\|u\|_{\bigcap\acute{\beta}^{\pm\delta}\mathcal{C}_b}}\\
&\le\underbrace{\max_{\sigma=\pm1}|s-\sigma\delta|^{-1}}_{=\operatorname{dist}(s,[-\delta,\delta])^{-1}}\|u\|_{\bigcap\acute{\beta}^{\pm\delta}\mathcal{C}_b}.
\end{aligned}$$

□

For the rest of the paper we fix some constant δ with

$$0<\delta<\min\{1,2\mu-1\}. \tag{71}$$

Then $m_\pm \overset{(32)}{=} 2\mu-1\pm n\mu$ yields for N sufficiently large and $n\in N\mathbb{Z}$ that

$$\operatorname{dist}(m_\pm,[-\delta,\delta])\sim\langle n\rangle,\quad \text{in particular } m_\pm\neq0 \text{ and} \tag{72}$$
$$\operatorname{sign}m_+=\operatorname{sign}(m_++\delta)=\operatorname{sign}(m_+-\delta), \tag{73}$$
$$\operatorname{sign}m_-=\operatorname{sign}(m_-+\delta)=\operatorname{sign}(m_--\delta); \tag{74}$$
$$\operatorname{dist}(-1,[-\delta,\delta])\sim1,\quad \text{in particular} \tag{75}$$
$$\operatorname{sign}1=\operatorname{sign}(1+\delta)=\operatorname{sign}(1-\delta). \tag{76}$$

Lemma 1.

$$\left\|A_+^{-1}\right\|_{[\bigcap\acute{\beta}^{\pm\delta}\mathcal{C}_b]}\lesssim\langle n\rangle^{-1} \tag{77}$$
$$\left\|A_-^{-1}\right\|_{[\bigcap\acute{\beta}^{\pm\delta}\mathcal{C}_b]}\lesssim\langle n\rangle^{-1} \tag{78}$$
$$\left\|B_1^{-1}\right\|_{[\bigcap\acute{\beta}^{\pm\delta}\mathcal{C}_b]}\lesssim1 \tag{79}$$

Proof. Using $A_\pm^{-1}\overset{(32)}{=}(A-\acute{m}_\pm)^{-1}$, (77) and (78) follow from (70) with $Z=A$ and $s=m_\pm\overset{(72)}{\sim}\langle n\rangle$. Using $B_1^{-1}\overset{(32)}{=}(B-(-\acute{1}))^{-1}$, (79) follows from (70) with $Z=B$ and $s=-1$. □

3.4. The space Ψ for solution candidates. The space Ψ houses the solution candidates $\overline{\psi}$. Since our goal is to define the largest "convenient" multiplication algebra $\mathcal{G}$ for the nonlinear terms, it is convenient to define Ψ indirectly:

Definition 7.

$$\Psi^n := B_1^{-1} A_-^{-1} \mathcal{G}_-^n \tag{80}$$

We also use

$$\Psi_\delta^n := B_1^{-1} A_-^{-1} \bigcap \acute{\beta}^{\pm\delta} \mathcal{C}_b \tag{81}$$

$$\Psi_0^n := B_1^{-1} A_-^{-1} (\mathbb{C}\chi_0) \tag{82}$$

$$\Psi_-^n := B_1^{-1} A_-^{-1} (\delta_{0n}\mathbb{C}) \tag{83}$$

and note (see (41))

$$\Psi^n = \Psi_\delta^n \oplus \Psi_0^n \oplus \Psi_-^n \tag{84}$$

Proposition 12. *Ψ^n is a Banach space, and*

$$A_- B_1 : \Psi^n \to \mathcal{G}_-^n \text{ is an isometry.} \tag{85}$$

Proof. $A_- \overset{(32)}{=} A - \acute{m}_-$ and $B_1 \overset{(32)}{=} B - (-\acute{1})$. Note $m_- \overset{(72)}{\neq} 0 \neq -1$, so by Proposition 7 A_- and B_1 are injective on $\mathcal{C}_b$, and by (61) $A_-^{-1}\mathcal{C}_b \hookrightarrow \mathcal{C}_b$ and $B_1^{-1}\mathcal{C}_b \hookrightarrow \mathcal{C}_b$. Therefore $A_-^{-1}\mathcal{G}_-^n \overset{(46)}{\hookrightarrow} A_-^{-1}\mathcal{C}_b \hookrightarrow \mathcal{C}_b$ and $B_1^{-1}A_-^{-1}\mathcal{G}_-^n \overset{(46)}{\hookrightarrow} B_1^{-1}A_-^{-1}\mathcal{C}_b \hookrightarrow B_1^{-1}\mathcal{C}_b \hookrightarrow \mathcal{C}_b$. Hence A_- is also injective on the smaller space $A_-^{-1}\mathcal{G}_-^n$, and B_1 is also injective on $B_1^{-1}A_-^{-1}\mathcal{G}_-^n$. Thus $A_- : A_-^{-1}\mathcal{G}_-^n \to \mathcal{G}_-^n$ is bijective, hence an isometry with the induced norm that makes $A_-^{-1}\mathcal{G}_-^n$ a Banach space; in turn $\Psi^n = B_1^{-1}A_-^{-1}\mathcal{G}_-^n$ is a Banach space with the norm induced by B_1, and $B_1 : B_1^{-1}A_-^{-1}\mathcal{G}_-^n \to A_-^{-1}\mathcal{G}_-^n$ another isometry. Composition of two isometries yields (85). □

Remark 8. Note that inverses $(\acute{\beta}(\partial_\beta - i\acute{z}) - \acute{s})^{-1}$ increase β decay at $\beta \to \infty$ at all β-frequencies except near z. For example

$$(\acute{\beta}(\partial_\beta - i\acute{z}) - \acute{s})^{-1} e^{ip\beta} \overset{(64)}{=} \int_{\infty^{\operatorname{sign} s}}^{\beta} \beta^s e^{iz\beta} \beta'^{-s-1} e^{i(p-z)\beta'} d\beta'$$

(now integrate by parts a few times and estimate the remainder into O)

$$= \beta^{-1} e^{ip\beta} \Big(\frac{1}{i(p-z)} + O(\beta^{-1}) \Big)$$

So if $p \neq z$ we gain a β^{-1} over the input $e^{ip\beta}$. But clearly the coefficient blows up as $p \to z$, and for $p = z$ we have

$$(\acute{\beta}(\partial_\beta - i\acute{z}) - \acute{s})^{-1} e^{iz\beta} = \beta^s e^{iz\beta} \int_{\infty^{\operatorname{sign} s}}^{\beta} \beta'^{-s-1} d\beta'$$
$$= -\frac{1}{s} e^{iz\beta}$$

so we have not gained any decay.

Since $B_1^{-1} = \acute{\beta}\partial_\beta + \acute{1}$ and $A_-^{-1} = \acute{\beta}(\partial_\beta - i\acute{n}) - \acute{m}_-$ that means Ψ functions gain decay β^{-2} compared to $\mathcal{G}_-^n$ at frequencies separated from 0 and n, but only β^{-1} near them.

Such a function space with frequency-dependent weights at $\beta \to \infty$ could be defined more explicitly, but for our purposes that appears to be more cumbersome than the natural representation $B_1^{-1}A_-^{-1}\mathcal{G}_-^n$ suggested by the form of R in (32).

Definition 8.

$$\Psi := \mathcal{A}^{\frac{1}{2}}(\Psi^n) \tag{86}$$

Proposition 13. *Ψ is well-defined and a Banach space.*

Proof. By Proposition 12 Ψ^n are Banach spaces for all n, and $\Psi^n = B_1^{-1}A_-^{-1}\mathcal{G}_-^n \overset{\sim}{\hookrightarrow} B_1^{-1}A_-^{-1}\mathcal{C}_b \underset{(79)}{\overset{(78)}{\hookrightarrow}} \mathcal{C}_b \hookrightarrow \mathcal{D}'$ so that by Proposition 2 with $q = 0$ and $\mathcal{X} = \mathcal{C}_b$, Ψ is a Banach space as well. □

3.5. Continuity on Ψ. Now we come to the continuity of various operators on Ψ_δ^n. In the following result, observe that applying B_1 to $B_1^{-1}A_-^{-1}g$ first cancels B_1^{-1} before there is a need to commute A through it. Hence a β^0 term at $\beta \to 0$ or $\beta \to \infty$ is not produced.

Proposition 14.

$$\|\mathrm{id}\|_{[\Psi_\delta^n,\mathcal{G}_-^n]} \lesssim \langle n\rangle^{-1} \tag{87}$$

$$\|B\|_{[\Psi_\delta^n,\mathcal{G}_0^n]} \lesssim \langle n\rangle^{-1} \tag{88}$$

$$\|AB_1\|_{[\Psi_\delta^n,\mathcal{G}_0^n]} \lesssim 1 \tag{89}$$

Proof. $\Psi_\delta^n = B_1^{-1}A_-^{-1}\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b$ and $\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b \overset{(40)}{\hookrightarrow} \mathcal{G}_0^n \overset{(41)}{\hookrightarrow} \mathcal{G}_-^n$, so it is sufficient to show

$$\left\|\mathrm{id}\, B_1^{-1}A_-^{-1}\right\|_{[\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b,\mathcal{G}_-^n]} \lesssim \langle n\rangle^{-1}$$

$$\left\|BB_1^{-1}A_-^{-1}\right\|_{[\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b,\mathcal{G}_0^n]} \lesssim \langle n\rangle^{-1}$$

$$\left\|AB_1B_1^{-1}A_-^{-1}\right\|_{[\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b,\mathcal{G}_0^n]} \lesssim 1$$

These follow from (78) and (79) by linear combinations $B = B_1 - \acute{1}$ and $A = A_- + \acute{m}_-$ using $\acute{m}_- \overset{(72)}{\lesssim} \langle n\rangle$. □

In the next result we apply A without any helping B_1. As explained this turns the $\chi_\infty\beta^{-1}$ at $\beta \to \infty$ produced by B_1^{-1} into $\chi_\infty\beta^0$. The result can in general only fit into $\mathcal{G}^n$, not the smaller $\mathcal{G}_-^n$ or $\mathcal{G}_0^n$.

Proposition 15.

$$\|A\|_{[\Psi_\delta^n,\mathcal{G}^n]} \lesssim 1 \tag{90}$$

Proof. Consider $\overline{\psi} \in \Psi_\delta^n$, i.e.

$$\overline{\psi} = (B + \acute{1})^{-1}(A - \acute{m}_-)^{-1}g, \quad g \in \bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b \hookrightarrow \mathcal{C}_b. \tag{91}$$

We telescope $A = B - in\acute{\beta}$, so that the prior estimate (88) reduces the problem to an estimate for $in\acute{\beta}\overline{\psi}$. Only $n \neq 0$ is nontrivial. Consider

$$in\acute{\beta}\overline{\psi} = in\acute{\beta}(B+\acute{1})^{-1}\overbrace{(A-\acute{m}_-)^{-1}g}^{X}.$$

For $\sigma = \pm 1$,

$$X = \acute{\beta}^{\sigma\delta}\underbrace{\acute{\beta}^{-\sigma\delta}(A-\acute{m}_-)^{-1}}_{*}\underbrace{g}_{\in\mathcal{C}_b};$$

we apply (68) with $Z = A$, $s = m_- - \sigma\delta$ and $r = -\sigma\delta$ to $*$ to obtain

$$X = \acute{\beta}^{\sigma\delta}\overbrace{(A-\acute{m}_- + \sigma\acute{\delta})^{-1}}^{*}\overbrace{\acute{\beta}^{-\sigma\delta}g}^{\in\mathcal{C}_b};$$

to justify this we argue that $s = m_- - \sigma\delta \neq 0 \neq m_- = s - r$ by (72), and that $g, \acute{\beta}^{-\sigma\delta}g \in \mathcal{C}_b$ so that (66) and (67) are satisfied. Note $\operatorname{sign} s = \operatorname{sign}(m_- - \sigma\delta) = \operatorname{sign}(m_-) = \operatorname{sign}(s-r)$ (again by (72)) so that no c term appears. Now

$$in\acute{\beta}\overline{\psi} = in\acute{\beta}^{\sigma\delta}\underbrace{\acute{\beta}^{1-\sigma\delta}(B+\acute{1})^{-1}}_{**}\overbrace{\acute{\beta}^{\sigma\delta}(A-\acute{m}_- + \sigma\acute{\delta})^{-1}\acute{\beta}^{-\sigma\delta}g}^{X}$$

To $**$ we also apply (68), with $Z = B$, $s = -\sigma\delta$ and $r = 1 - \sigma\delta$, obtaining

$$in\acute{\beta}\overline{\psi} = \underbrace{\acute{\beta}^{\sigma\delta}(B+\sigma\acute{\delta})^{-1}\underbrace{in\acute{\beta}(A-\acute{m}_- + \sigma\acute{\delta})^{-1}\acute{\beta}^{-\sigma\delta}g}_{=:u}}_{=:r_\sigma} + c_\sigma \tag{92}$$

To justify the last step we argue that $s = -\sigma\delta \neq 0 \neq -1 = s - r$ by (71), and that $g \in \mathcal{C}_b$ and $m_- \overset{(72)}{\neq} 0$ imply $X = (A - \acute{m}_-)^{-1}g \in \mathcal{C}_b$, hence (66); justification of (67) in the form $u \in (B + \sigma\acute{\delta})\mathcal{C}_b$, is to follow below. Note c_- is a constant (in β), whereas $c_+ = 0$ since for $\sigma = +1$ we have $\operatorname{sign} s = \operatorname{sign}(-\sigma\delta) = \operatorname{sign}(-1) = \operatorname{sign}(s-r)$.

We telescope

$$in\acute{\beta} = B - A = (B + \sigma\acute{\delta}) - \acute{m}_- - (A - \acute{m}_- + \sigma\acute{\delta}) \quad \text{on } \mathcal{D}'(\mathbb{R}_+)$$

so that formally

$$\begin{aligned} u = {} & \overbrace{(B+\sigma\acute{\delta})(A-\acute{m}_- + \sigma\acute{\delta})^{-1}\acute{\beta}^{-\sigma\delta}g}^{R_1} - \overbrace{\acute{m}_-(A-\acute{m}_- + \sigma\acute{\delta})^{-1}\acute{\beta}^{-\sigma\delta}g}^{R_2} \\ & \underbrace{-(A-\acute{m}_- + \sigma\acute{\delta})(A-\acute{m}_- + \sigma\acute{\delta})^{-1}\acute{\beta}^{-\sigma\delta}g}_{R_3} \end{aligned} \tag{93}$$

We check each part: $\acute{\beta}^{-\sigma\delta}g \in \mathcal{C}_b$ and $m_- - \sigma\delta \overset{(72)}{\neq} 0$ mean by (62) that $R_3 = \acute{\beta}^{-\sigma\delta}g \in \mathcal{C}_b$ is well-defined, and by (63) that $(A - \acute{m}_- + \sigma\acute{\delta})^{-1}\acute{\beta}^{-\sigma\delta} \in \mathcal{C}_b$ is well-defined, so

R_2 is well-defined and in $\mathcal{C}_b$. Since $\delta \neq 0$ means $\mathcal{C}_b \overset{(61)}{\hookrightarrow} (B + \sigma\acute{\delta})\mathcal{C}_b$, we see that $u = R_1 + R_2 + R_3 \in (B + \sigma\acute{\delta})\mathcal{C}_b$ so that the steps to (92) are now fully justified.

We proceed to estimate r_σ:

$$r_\sigma \overset{(92)}{=} (B + \sigma\acute{\delta})^{-1}u$$

(use $(B + \sigma\acute{\delta})^{-1}(B + \sigma\acute{\delta}) = \mathrm{id}$ on $\mathcal{C}_b$ by (57) with $\delta \overset{(71)}{\neq} 0$)

$$\begin{aligned} &\overset{(93)}{=} (A - \acute{m}_- + \sigma\acute{\delta})^{-1}\acute{\beta}^{-\sigma\delta}g \\ &- \acute{m}_-(B + \sigma\acute{\delta})^{-1}(A - \acute{m}_- + \sigma\acute{\delta})^{-1}\acute{\beta}^{-\sigma\delta}g \\ &- (B + \sigma\acute{\delta})^{-1}\acute{\beta}^{-\sigma\delta}g \end{aligned}$$

We estimate the three terms:

$$\left\| (A - \acute{m}_- + \sigma\acute{\delta})^{-1}\acute{\beta}^{-\sigma\delta}g \right\|_{\mathcal{C}_b} \underset{(72)}{\overset{(63)}{\lesssim}} \langle n\rangle^{-1} \left\| \acute{\beta}^{-\sigma\delta}g \right\|_{\mathcal{C}_b}$$

$$\left\| (B + \sigma\acute{\delta})^{-1}\acute{\beta}^{-\sigma\delta}g \right\|_{\mathcal{C}_b} \underset{(71)}{\overset{(63)}{\lesssim}} \left\| \acute{\beta}^{-\sigma\delta}g \right\|_{\mathcal{C}_b}$$

$$\left\| \acute{m}_-(B + \sigma\acute{\delta})^{-1}(A - \acute{m}_- + \sigma\acute{\delta})^{-1}\acute{\beta}^{-\sigma\delta}g \right\|_{\mathcal{C}_b} \underset{(72)}{\overset{(63)}{\lesssim}} \left\| \acute{\beta}^{-\sigma\delta}g \right\|_{\mathcal{C}_b}$$

so that altogether

$$\|r_\sigma\|_{\mathcal{C}_b} \lesssim \left\| \acute{\beta}^{-\sigma\delta}g \right\|_{\mathcal{C}_b} \lesssim \|\overline{\psi}\|_{\Psi^n_\delta}$$

Having estimated r_σ in (92), and c_+ being 0, it remains to estimate c_-: (92) with $\sigma = -1$ shows

$$c_- = \mathrm{const} = \beta\, \underbrace{in\, \overbrace{(B + \acute{1})^{-1}(A - \acute{m}_-)^{-1}g}^{=\overline{\psi}}}_{=:w} - \beta^{-\delta}r_-. \tag{94}$$

By now-familiar arguments w is well-defined and in $\mathcal{C}_b$, with

$$\|w\|_{\mathcal{C}_b} \lesssim \|\overline{\psi}\|_{\Psi^n_\delta}.$$

Since c_- is constant, it can be estimated by evaluating the right-hand side of (94) at any β, say $\beta = 1$, and we obtain

$$|c_-| \leq \|w\|_{\mathcal{C}_b} + \|r_-\|_{\mathcal{C}_b} \lesssim \|\overline{\psi}\|_{\Psi^n_\delta}$$

Altogether we have that

$$\begin{aligned} in\beta\overline{\psi} &= (\chi_0 + \chi_\infty)in\beta\overline{\psi} \overset{(92)}{=} \chi_0\beta^\delta r_+ + \chi_\infty(\beta^{-\delta}r_- + c_-) \\ &= \underbrace{\underbrace{\chi_0\beta^\delta r_+ + \chi_\infty\beta^{-\delta}r_-}_{\in\bigcap\acute{\beta}^{\pm\delta}\mathcal{C}_b} + \underbrace{\chi_\infty c_-}_{\in\mathbb{C}\chi_\infty}}_{\overset{(42)}{\in}\mathcal{G}^n} \end{aligned}$$

with $\mathcal{G}^n$ norm $\lesssim \|\overline{\psi}\|_{\Psi^n_\delta}$. □

Proposition 16.

$$\|AB\|_{[\Psi_\delta^n,\mathcal{G}^n]} \lesssim 1 \tag{95}$$

Proof.

$$\|AB\|_{[\Psi_\delta^n,\mathcal{G}^n]} \overset{AB=AB_1-A}{\leq} \underbrace{\|AB_1\|_{[\Psi_\delta^n,\mathcal{G}^n]}}_{\underset{\mathcal{G}_0^n\hookrightarrow\mathcal{G}^n}{\overset{(89)}{\lesssim}} 1} + \underbrace{\|A\|_{[\Psi_\delta^n,\mathcal{G}^n]}}_{\overset{(90)}{\lesssim} 1} \lesssim 1$$

□

Having dealt with the most difficult part Ψ_δ^n, we examine the fairly trivial action of the various differential operators id, B, A, AB on Ψ_-^n.

Proposition 17.

$$B_{|\Psi_-^n} = 0 \tag{96}$$
$$AB_{|\Psi_-^n} = 0 \tag{97}$$
$$A_{|\Psi_-^n} = 0 \tag{98}$$
$$AB_{1|\Psi_-^n} = 0 \tag{99}$$
$$\|\mathrm{id}\|_{[\Psi_-^n,\mathcal{G}_-^n]} \lesssim \langle n\rangle^{-1} \tag{100}$$

$$\overline{\psi}_0 \in \Psi \tag{101}$$

Proof. $\Psi_-^n \overset{(83)}{=} B_1^{-1}A_-^{-1}(\delta_{0n}\mathbb{C})$, so for $n \neq 0$ the statement is trivial. Consider $n = 0$:

$$B1 = \acute{\beta}\partial_\beta 1 = 0, \quad AB1 = A0 = 0, \quad A1 = \acute{\beta}(\partial_\beta - i\acute{n})1 \overset{n=0}{=} 0,$$
$$AoB_1 = A(B+\acute{1})1 = A1 = 0,$$

which yields the first four estimates.

$$A_-B_1\underbrace{\frac{1}{1-2\mu}}_{\in\mathcal{C}_b} \underset{n=0}{\overset{(32)}{=}} (A+\acute{1}-2\acute{\mu})(B+\acute{1})\frac{1}{1-2\mu} = 1$$

$$\overset{\text{Def. 6}}{\Rightarrow} \quad B_1^{-1}A_-^{-1}1 = \frac{1}{1-2\mu} \quad \Rightarrow \quad \mathrm{id}\,\Psi_-^n = B_1^{-1}A_-^{-1}(\delta_{0n}\mathbb{C}) = \delta_{0n}\mathbb{C}$$

Hence id maps Ψ_-^n into $\delta_{0n}\mathbb{C} \overset{(41)}{\hookrightarrow} \mathcal{G}_-^n$. (100) is trivial since Ψ_-^n is one-dimensional for $n = 0$ and zero for $n \neq 0$. Moreover

$$\frac{1}{2\mu-1} = B_1^{-1}A_-^{-1}(-1) \in B_1^{-1}A_-^{-1}(\delta_{0n}\mathbb{C}) \overset{(80)}{\in} \Psi^n$$

so that

$$\overline{\psi}_0 \overset{(23)}{=} (\delta_{0n}\frac{1}{2\mu-1})^\vee \in \Psi$$

(the $\Psi = \mathcal{A}^{\frac{1}{2}}(\Psi^n)$ norm is obviously finite). □

Finally we examine the action of $\acute{1}, B, A, AB$ on Ψ_0^n. $\mathbb{C}\chi_0$ introduces terms without decay at $\beta \searrow 0$. we remove the χ_0 terms at the cost of new terms in Ψ_δ^n on which we have estimates already.

Lemma 2.

$$\|B\chi_0\|_{\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b} \leq 1 \tag{102}$$

$$\|A\chi_0\|_{\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b} \leq \langle n\rangle \tag{103}$$

$$\|AB\chi_0\|_{\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b} \leq \langle n\rangle \tag{104}$$

Proof.

$$\begin{aligned} B\chi_0 &= \acute{\beta}\partial_\beta\chi_0 \\ A\chi_0 &= \acute{\beta}\partial_\beta\chi_0 - i\acute{n}\acute{\beta}\chi_0 \\ AB\chi_0 &= \acute{\beta}(\partial_\beta - i\acute{n})\acute{\beta}\partial_\beta\chi_0 = \acute{\beta}\partial_\beta\chi_0 + \acute{\beta}^2\partial_\beta^2\chi_0 - i\acute{n}\acute{\beta}^2\partial_\beta\chi_0 \end{aligned}$$

$\partial_\beta\chi_0, \partial_\beta^2\chi_0 \in \mathcal{C}_c^\infty(\mathbb{R}_+) \hookrightarrow \bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b$, and likewise $\beta\chi_0 \in \bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b$ (note $\delta \overset{(71)}{\leq} 1$). □

Lemma 3.

$$B_1^{-1}A_-^{-1}\chi_0 = -m_-^{-1}\chi_0 + r \quad \textit{where} \quad \|r\|_{\Psi_\delta^n} \lesssim 1 \tag{105}$$

Proof.

$$A_- \underbrace{\chi_0}_{\in\mathcal{C}_b} = -m_- \underbrace{\chi_0}_{\in\mathcal{C}_b \underset{m_-\neq 0}{\overset{(61)}{\hookrightarrow}} A_-\mathcal{C}_b} + \underbrace{A\chi_0}_{\overset{(103)}{\in} \mathcal{C}_b \hookrightarrow A_-\mathcal{C}_b}$$

$$\begin{aligned} \underset{m_-\neq 0}{\overset{(57)}{\Rightarrow}} \quad \chi_0 &= -m_-A_-^{-1}\chi_0 + A_-^{-1}A\chi_0 \\ \Leftrightarrow \quad A_-^{-1}\chi_0 &= -m_-^{-1}\chi_0 + A_-^{-1}(\underbrace{m_-^{-1}A\chi_0}_{=:g_1}) \end{aligned} \tag{106}$$

where

$$\|g_1\|_{\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b} \leq \underbrace{|m_-^{-1}|}_{\overset{(72)}{\lesssim}\langle n\rangle^{-1}} \underbrace{\|A\chi_0\|_{\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b}}_{\overset{(103)}{\lesssim}\langle n\rangle} \lesssim 1. \tag{107}$$

Similarly

$$\begin{aligned} B_1\chi_0 &= \chi_0 + B\chi_0 \\ \Rightarrow \quad \chi_0 &= B_1^{-1}\chi_0 + B_1^{-1}\underbrace{B\chi_0}_{\in\mathcal{C}_b} \\ \Leftrightarrow \quad B_1^{-1}\chi_0 &= \chi_0 + B_1^{-1}A_-^{-1}\underbrace{A_-B(-\chi_0)}_{=:g_2} \end{aligned} \tag{108}$$

$$\|g_2\|_{\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b} \leq \underbrace{\|AB\chi_0\|_{\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b}}_{\overset{(104)}{\lesssim}\langle n\rangle} + \underbrace{|m_-|}_{\overset{(72)}{\lesssim}\langle n\rangle}\underbrace{\|B\chi_0\|_{\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b}}_{\overset{(103)}{\lesssim}1} \lesssim \langle n\rangle \tag{109}$$

Combining (106) with (108) we get

$$B_1^{-1}A_-^{-1}\chi_0 \overset{(106)}{=} B_1^{-1}(-m_-^{-1}\chi_0 + A_-^{-1}g_1) \overset{(108)}{=} -m_-^{-1}\chi_0 + \underbrace{B_1^{-1}A_-^{-1}(-m_-^{-1}g_2 + g_1)}_{=:r}$$

Note

$$\|r\|_{\Psi_\delta^n} \overset{(81)}{=} \left\|g_1 - m_-^{-1}g_2\right\|_{\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b} \underset{(109)}{\overset{(107)}{\lesssim}} 1$$

□

Proposition 18.

$$\|\mathrm{id}\|_{[\Psi_0^n,\mathcal{G}_-^n]} \lesssim \langle n\rangle^{-1} \tag{110}$$

$$\|B\|_{[\Psi_0^n,\mathcal{G}_0^n]} \lesssim \langle n\rangle^{-1} \tag{111}$$

$$\|AB_1\|_{[\Psi_0^n,\mathcal{G}_0^n]} \lesssim 1 \tag{112}$$

$$\|A\|_{[\Psi_0^n,\mathcal{G}^n]} \lesssim 1 \tag{113}$$

$$\|AB\|_{[\Psi_0^n,\mathcal{G}^n]} \lesssim 1 \tag{114}$$

Proof.

$$\|\mathrm{id}\|_{[\Psi_0^n,\mathcal{G}_-^n]} \overset{(82)}{=} \left\|\mathrm{id}\, B_1^{-1}A_+^{-1}\chi_0\right\|_{\mathcal{G}_-^n}$$

$$\overset{(105)}{\lesssim} \underbrace{|m_-^{-1}|}_{\overset{(72)}{\lesssim}\langle n\rangle^{-1}}\underbrace{\|\chi_0\|_{\mathbb{C}\chi_0}}_{=1} + \underbrace{\|\mathrm{id}\|_{[\Psi_\delta^n,\mathcal{G}_-^n]}}_{\overset{(87)}{\lesssim}\langle n\rangle^{-1}}\underbrace{\|r\|_{\Psi_\delta^n}}_{\overset{(105)}{\lesssim}1} \lesssim \langle n\rangle^{-1}$$

$$\|B\|_{[\Psi_0^n,\mathcal{G}_0^n]} \overset{(82)}{=} \left\|BB_1^{-1}A_+^{-1}\chi_0\right\|_{\mathcal{G}_0^n}$$

$$\overset{(105)}{\lesssim} \underbrace{|m_-^{-1}|}_{\overset{(72)}{\lesssim}\langle n\rangle^{-1}}\underbrace{\|B\chi_0\|_{\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b}}_{\overset{(102)}{\lesssim}1} + \underbrace{\|B\|_{[\Psi_\delta^n,\mathcal{G}_0^n]}}_{\overset{(88)}{\lesssim}\langle n\rangle^{-1}}\underbrace{\|r\|_{\Psi_\delta^n}}_{\overset{(105)}{\lesssim}1} \lesssim \langle n\rangle^{-1}$$

$$\|AB_1\|_{[\Psi_0^n,\mathcal{G}_0^n]} \overset{(82)}{=} \left\|AB_1B_1^{-1}A_+^{-1}\chi_0\right\|_{\mathcal{G}_0^n}$$

$$\overset{(105)}{\lesssim} \underbrace{|m_-^{-1}|}_{\overset{(72)}{\lesssim}\langle n\rangle^{-1}}\underbrace{\|A(B+1)\chi_0\|_{\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b}}_{\underset{(103)}{\overset{(104)}{\lesssim}}\langle n\rangle} + \underbrace{\|AB_1\|_{[\Psi_\delta^n,\mathcal{G}_0^n]}}_{\overset{(89)}{\lesssim}1}\underbrace{\|r\|_{\Psi_\delta^n}}_{\overset{(105)}{\lesssim}1} \lesssim 1$$

$$\|A\|_{[\Psi_0^n,\mathcal{G}^n]} \overset{(82)}{=} \left\|AB_1^{-1}A_+^{-1}\chi_0\right\|_{\mathcal{G}^n}$$

$$\overset{(105)}{\lesssim} \underbrace{|m_-^{-1}|}_{\overset{(72)}{\lesssim}\langle n\rangle^{-1}} \underbrace{\|A\chi_0\|_{\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b}}_{\overset{(103)}{\lesssim}\langle n\rangle} + \underbrace{\|A\|_{[\Psi_\delta^n,\mathcal{G}^n]}}_{\overset{(90)}{\lesssim}1} \underbrace{\|r\|_{\Psi_\delta^n}}_{\overset{(105)}{\lesssim}1} \lesssim 1$$

$$\|AB\|_{[\Psi_0^n,\mathcal{G}^n]} \overset{(82)}{=} \left\|ABB_1^{-1}A_+^{-1}\chi_0\right\|_{\mathcal{G}^n}$$

$$\overset{(105)}{\lesssim} \underbrace{|m_-^{-1}|}_{\overset{(72)}{\lesssim}\langle n\rangle^{-1}} \underbrace{\|AB\chi_0\|_{\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b}}_{\overset{(104)}{\lesssim}\langle n\rangle} + \underbrace{\|AB\|_{[\Psi_\delta^n,\mathcal{G}^n]}}_{\overset{(95)}{\lesssim}1} \underbrace{\|r\|_{\Psi_\delta^n}}_{\overset{(105)}{\lesssim}1} \lesssim 1$$

□

Remark 9. We have treated each direct summand of Ψ^n; combining the estimates easily yields (remember (26),(27): the operator A is the Fourier transform of $-\acute{\beta}\partial_\varphi$, B the one of $\acute{\beta}\partial_\beta$ and in the transform of ∂_ϕ)

$$\|\mathrm{id}\|_{[\Psi^n,\mathcal{G}_-^n]} \overset{(84)}{\leq} \underbrace{\|\mathrm{id}\|_{[\Psi_\delta^n,\mathcal{G}_-^n]}}_{\overset{(87)}{\lesssim}\langle n\rangle^{-1}} + \underbrace{\|\mathrm{id}\|_{[\Psi_0^n,\mathcal{G}_-^n]}}_{\overset{(110)}{\lesssim}\langle n\rangle^{-1}} + \underbrace{\|\mathrm{id}\|_{[\Psi_-^n,\mathcal{G}_-^n]}}_{\overset{(100)}{\lesssim}\langle n\rangle^{-1}} \lesssim \langle n\rangle^{-1} \tag{115}$$

$$\Rightarrow \quad \|\mathrm{id}\|_{[\Psi,\mathcal{G}_-]} = \|\mathrm{id}\|_{[\mathcal{A}^{\frac{1}{2}}(\Psi^n),\mathcal{A}^{\frac{1}{2}}(\mathcal{G}_-^n)]} \overset{(115)}{<} \infty \tag{116}$$

$$\|B\|_{[\Psi^n,\mathcal{G}_0^n]} \overset{(84)}{\leq} \underbrace{\|B\|_{[\Psi_\delta^n,\mathcal{G}_0^n]}}_{\overset{(88)}{\lesssim}\langle n\rangle^{-1}} + \underbrace{\|B\|_{[\Psi_0^n,\mathcal{G}_0^n]}}_{\overset{(111)}{\lesssim}\langle n\rangle^{-1}} + \underbrace{\|B\|_{[\Psi_-^n,\mathcal{G}_0^n]}}_{\overset{(96)}{=}0} \lesssim \langle n\rangle^{-1} \tag{117}$$

$$\Rightarrow \quad \left\|\acute{\beta}\partial_\beta\right\|_{[\Psi,\mathcal{G}_0]} \overset{(117)}{<} \infty \tag{118}$$

$$\|AB_1\|_{[\Psi^n,\mathcal{G}_0^n]} \overset{(84)}{\leq} \underbrace{\|AB_1\|_{[\Psi_\delta^n,\mathcal{G}_0^n]}}_{\overset{(89)}{\lesssim}1} + \underbrace{\|AB_1\|_{[\Psi_0^n,\mathcal{G}_0^n]}}_{\overset{(112)}{\lesssim}1} + \underbrace{\|AB_1\|_{[\Psi_-^n,\mathcal{G}_0^n]}}_{\overset{(99)}{=}0} \lesssim 1 \tag{119}$$

$$\Rightarrow \quad \left\|\acute{\beta}\partial_\varphi(\acute{\beta}\partial_\beta + \acute{1})\right\|_{[\Psi,\mathcal{G}_0]} \overset{(119)}{<} \infty \tag{120}$$

$$\|A\|_{[\Psi^n,\mathcal{G}^n]} \overset{(84)}{\leq} \underbrace{\|A\|_{[\Psi_\delta^n,\mathcal{G}^n]}}_{\overset{(90)}{\lesssim}1} + \underbrace{\|A\|_{[\Psi_0^n,\mathcal{G}^n]}}_{\overset{(113)}{\lesssim}1} + \underbrace{\|A\|_{[\Psi_-^n,\mathcal{G}^n]}}_{\overset{(98)}{=}0} \lesssim 1 \tag{121}$$

$$\Rightarrow \quad \left\|\acute{\beta}\partial_\varphi\right\|_{[\Psi,\mathcal{G}]} \overset{(121)}{<} \infty \tag{122}$$

$$\|AB\|_{[\Psi^n,\mathcal{G}^n]} \overset{(84)}{\leq} \underbrace{\|AB\|_{[\Psi_\delta^n,\mathcal{G}^n]}}_{\overset{(95)}{\lesssim}1} + \underbrace{\|AB\|_{[\Psi_0^n,\mathcal{G}^n]}}_{\overset{(114)}{\lesssim}1} + \underbrace{\|AB\|_{[\Psi_-^n,\mathcal{G}^n]}}_{\overset{(97)}{=}0} \lesssim 1 \tag{123}$$

$$\Rightarrow \quad \left\|\acute{\beta}\partial_\varphi\acute{\beta}\partial_\beta\right\|_{[\Psi,\mathcal{G}]} \overset{(123)}{<} \infty \tag{124}$$

$$\left\|\partial_\phi\right\|_{[\Psi,\mathcal{G}_-]} \overset{(115)}{<} \infty \tag{125}$$

$$\left\|\overline{\partial}_\beta\right\|_{[\Psi,\mathcal{G}_-]} \overset{(26)}{\leq} \underbrace{\left\|\acute{\beta}\partial_\beta\right\|_{[\Psi,\mathcal{G}_-]}}_{\underset{\mathcal{G}_0\hookrightarrow\mathcal{G}_-}{\overset{(118)}{<}}\infty} + (2\mu-1)\underbrace{\|\mathrm{id}\|_{[\Psi,\mathcal{G}_-]}}_{\overset{(116)}{<}\infty} \underset{(116)}{\overset{(118)}{<}} \infty \tag{126}$$

$$\left\|\partial_\phi\overline{\partial}_\beta\right\|_{[\Psi,\mathcal{G}_-]} \overset{(26)}{\leq} \underbrace{\left\|\partial_\phi\acute{\beta}\partial_\beta\right\|_{[\Psi,\mathcal{G}_-]}}_{\underset{\mathcal{G}_0^n\hookrightarrow\mathcal{G}_-^n}{\overset{(117)}{<}}\infty} + (2\mu-1)\underbrace{\left\|\partial_\phi\right\|_{[\Psi,\mathcal{G}_-]}}_{\overset{(125)}{<}\infty} < \infty \tag{127}$$

$$\left\|\overline{\partial}_\varphi\right\|_{[\Psi,\mathcal{G}]} \overset{(27)}{\leq} \underbrace{\left\|\acute{\beta}\partial_\varphi\right\|_{[\Psi,\mathcal{G}]}}_{\overset{(122)}{<}\infty} + (2\mu-1)\underbrace{\|\mathrm{id}\|_{[\Psi,\mathcal{G}]}}_{\underset{\mathcal{G}_-\hookrightarrow\mathcal{G}}{\overset{(116)}{<}}\infty} < \infty \tag{128}$$

$$\begin{aligned}\left\|(\overline{\partial}_\varphi+\acute{1})\overline{\partial}_\beta\right\|_{[\Psi,\mathcal{G}]} &\underset{(26)}{\overset{(27)}{\leq}} \underbrace{\left\|\acute{\beta}\partial_\varphi\acute{\beta}\partial_\beta\right\|_{[\Psi,\mathcal{G}]}}_{\overset{(124)}{<}\infty} + (2\mu-1)\underbrace{\left\|\acute{\beta}\partial_\varphi\right\|_{[\Psi,\mathcal{G}]}}_{\overset{(122)}{<}\infty} \\ &\quad + 2\mu\underbrace{\left\|\acute{\beta}\partial_\beta\right\|_{[\Psi,\mathcal{G}]}}_{\underset{\mathcal{G}_0\hookrightarrow\mathcal{G}}{\overset{(118)}{<}}\infty} + 2\mu(2\mu-1)\underbrace{\|\mathrm{id}\|_{[\Psi,\mathcal{G}]}}_{\underset{\mathcal{G}_-\hookrightarrow\mathcal{G}}{\overset{(116)}{<}}\infty} < \infty\end{aligned} \tag{129}$$

3.6. The space $\mathcal{F}$ for values of $\overline{F}$. $\mathcal{F}$ is the space for the values of $\overline{F}$ in (28). These values arise from the outer divergence of $\overline{g^{(\varphi)}}$, $\overline{g^{(\phi)}}$, $\overline{g^{(0)}}$. No pointwise multiplications occur outside these three terms, so we can regard the outer divergence as distributional; $\mathcal{F}$ may contain non-regular distributions.

Definition 9.

$$\mathcal{F}^n := A_+\mathcal{G}_-^n \tag{130}$$

Proposition 19. *$\mathcal{F}^n$ is a Banach space and*

$$A_+ : \mathcal{G}_-^n \to \mathcal{F}^n \textit{ is an isometry.} \tag{131}$$

Proof. $L = A_+ \overset{(32)}{=} A - m_+$ is injective on $\mathcal{C}_b$ by Proposition 7 (using $s = m_+ \overset{(72)}{\neq} 0$), hence also injective on $\mathcal{G}_-^n \overset{(46)}{\hookrightarrow} \mathcal{C}_b$, so $A_+ : \mathcal{G}_-^n \to A_+\mathcal{G}_-^n = \mathcal{F}^n$ is bijective, hence an isometry and $\mathcal{F}^n$ a Banach space with the induced norm. □

Definition 10.

$$\mathcal{F} := \mathcal{A}^{\frac{1}{2}}(\mathcal{F}^n) \tag{132}$$

This definition is valid because $\langle n\rangle^{-1}\mathcal{F}^n$, whose elements need not be functions, is continuously embedded into the Banach space $(\beta+\beta^{-1})(\mathcal{C}_b+\partial_\beta\mathcal{C}_b)$ (norm induced by $\partial_\beta, \beta+\beta^{-1}$) which in turn is continuously embedded in $\mathcal{D}'(\mathbb{R}_+)$; the embedding of $\langle n\rangle^{-1}\mathcal{F}^n$ is n-uniform because $A_+^{-1} = \acute{\beta}\partial_\beta - in\acute{\beta} - \acute{m}_+$ with $m_+ = 2\mu - 1 + n\mu$ can add at most one n growth which is absorbed by $\langle n\rangle^{-1}$. Hence $q = 1$ can be used in the runup to definition (33). We obtain

Proposition 20. *$\mathcal{F}$ is a Banach space.*

Proposition 21.

$$\|\mathrm{id}\|_{[\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b, \mathcal{F}^n]} \lesssim \langle n\rangle^{-1} \tag{133}$$

Proof.

$$\|\mathrm{id}\|_{[\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b, \mathcal{F}^n]} = \left\|A_+A_+^{-1}\right\|_{[\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b, A_+\mathcal{G}_-^n]} = \left\|A_+^{-1}\right\|_{[\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b, \mathcal{G}_-^n]}$$
$$\overset{(41)}{\leq} \left\|A_+^{-1}\right\|_{[\bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b, \bigcap \acute{\beta}^{\pm\delta}\mathcal{C}_b]} \overset{(77)}{\lesssim} \langle n\rangle^{-1}$$

□

Proposition 22.

$$\|\mathrm{id}\|_{[\mathbb{C}\chi_0, \mathcal{F}^n]} \lesssim \langle n\rangle^{-1} \tag{134}$$

Proof.

$$A_+\chi_0 \underset{(27)}{\overset{(32)}{=}} -\acute{m}_+\chi_0 - in\beta\chi_0 + \beta\partial_\beta\chi_0$$
$$\Leftrightarrow \quad \chi_0 = -m_+^{-1}\left(A_+\chi_0 + in\beta\chi_0 - \beta\partial_\beta\chi_0\right)$$
$$\Rightarrow \quad \|\chi_0\|_{\mathcal{F}^n} \leq \overbrace{|\acute{m}_+|^{-1}}^{\overset{(72)}{\lesssim}\langle n\rangle^{-1}} \Big(\overbrace{\|A_+\chi_0\|_{\mathcal{F}^n}}^{\overset{(130)}{=}\|\chi_0\|_{\mathcal{G}_-^n}\overset{(41)}{\leq}\|\chi_0\|_{\mathbb{C}\chi_0}=1} + \overbrace{\langle n\rangle\,\|\beta\chi_0\|_{\mathcal{F}^n}}^{\overset{(133)}{\lesssim}\|\beta\chi_0\|_{\bigcap\acute{\beta}^{\pm\delta}\mathcal{C}_b}\lesssim 1} + \underbrace{\left\|\beta\partial_\beta\chi_0\right\|_{\mathcal{F}^n}}_{\overset{(133)}{\lesssim}\langle n\rangle^{-1}\|\beta\partial_\beta\chi_0\|_{\bigcap\acute{\beta}^{\pm\delta}\mathcal{C}_b}\lesssim\langle n\rangle^{-1}} \Big)$$
$$\lesssim \langle n\rangle^{-1}$$

□

Lemma 4.

$$\|\mathrm{id}\|_{[\mathbb{C}\chi_\infty, \mathcal{F}^n]} \lesssim \langle n\rangle^{-1} \tag{135}$$

Proof. For $n = 0$:

$$A_+\chi_\infty = (\acute{\beta}(\partial_\beta - i\acute{n}) - \acute{m}_+)\chi_\infty$$
$$\underset{n=0}{\overset{(32)}{=}} (\acute{\beta}\partial_\beta - (2\mu - 1))\chi_\infty$$

$$\Leftrightarrow \quad \chi_\infty = (1-2\mu)^{-1}(A_+\chi_\infty - \beta\partial_\beta\chi_\infty)$$
$$\Leftrightarrow \quad \|\chi_\infty\|_{\mathcal{F}^n} \le |2\mu-1|^{-1}(\underbrace{\|A_+\chi_\infty\|_{\mathcal{F}^n}}_{\overset{(130)}{=}\|\chi_\infty\|_{\mathcal{G}^n_-}\overset{(41)}{\le}\|\chi_\infty\|_{\mathbb{C}\chi_\infty}=1} + \underbrace{\left\|\beta\partial_\beta\chi_\infty\right\|_{\mathcal{F}^n}}_{\overset{(133)}{\lesssim}\langle n\rangle^{-1}}) \lesssim 1$$

For $n \neq 0$:

$$\begin{aligned} A_+(\beta^{-1}\chi_\infty) &= (\acute{\beta}(\partial_\beta - i\acute{n}) - \acute{m}_+)(\beta^{-1}\chi_\infty) \\ &= -in\chi_\infty + (-m_+ - 1)\chi_\infty\beta^{-1} + \partial_\beta\chi_\infty \\ \Leftrightarrow \quad \chi_\infty &= (in)^{-1}\big((-m_+ - 1)\chi_\infty\beta^{-1} + \partial_\beta\chi_\infty - A_+(\chi_\infty\beta^{-1})\big) \\ \Rightarrow \quad \|\chi_\infty\|_{\mathcal{F}^n} &\le \langle n\rangle^{-1}\big(\underbrace{|m_+ + 1|}_{\lesssim\langle n\rangle}\underbrace{\left\|\chi_\infty\beta^{-1}\right\|_{\mathcal{F}^n}}_{\overset{(133)}{\lesssim}\langle n\rangle^{-1}} + \underbrace{\left\|\partial_\beta\chi_\infty\right\|_{\mathcal{F}^n}}_{\overset{(133)}{\lesssim}\langle n\rangle^{-1}} + \underbrace{\left\|A_+(\chi_\infty\beta^{-1})\right\|_{\mathcal{F}^n}}_{\overset{(130)}{=}\|\chi_\infty\beta^{-1}\|_{\mathcal{G}^n_-}\lesssim 1}\big) \\ &\lesssim \langle n\rangle^{-1} \end{aligned}$$

□

Proposition 23.

$$\|\mathrm{id}\|_{[\mathcal{G}^n,\mathcal{F}^n]} \lesssim \langle n\rangle^{-1} \tag{136}$$

Proof. Combine (133), (134) and (135). □

Proposition 24. *$\acute{n}$ maps $\mathcal{G}^n$ into $\mathcal{F}^n$, with*

$$\left\|\acute{n}\right\|_{[\mathcal{G}^n,\mathcal{F}^n]} \lesssim 1 \tag{137}$$

Proof. Immediate from (136). □

Proposition 25. *A maps $\mathcal{G}^n_-$ into $\mathcal{F}^n$, with*

$$\|A\|_{[\mathcal{G}^n_-,\mathcal{F}^n]} \lesssim 1 \tag{138}$$

Proof. $A = A_+ + m_+$, so

$$\|A\|_{[\mathcal{G}^n_-,\mathcal{F}^n]} \le \underbrace{\|A_+\|_{[\mathcal{G}^n_-,\mathcal{F}^n]}}_{\overset{(130)}{\le}1} + \underbrace{|m_+|}_{\overset{(72)}{\langle n\rangle}}\underbrace{\|\mathrm{id}\|_{[\mathcal{G}^n_-,\mathcal{F}^n]}}_{\underset{\mathcal{G}^n_-\hookrightarrow\mathcal{G}^n}{\overset{(136)}{\lesssim}}\langle n\rangle^{-1}} \lesssim 1$$

□

From (138), (137) and (136) we immediately get

Proposition 26.

$$\left\|\overline{\partial}_\varphi\right\|_{[\mathcal{G}_-,\mathcal{F}]} < \infty \tag{139}$$
$$\left\|\partial_\phi\right\|_{[\mathcal{G},\mathcal{F}]} < \infty \tag{140}$$
$$\|\mathrm{id}\|_{[\mathcal{G},\mathcal{F}]} < \infty \tag{141}$$

3.7. $\partial F/\partial\overline{\psi}$ is a linear isomorphism. Since we intend to apply the implicit function theorem it is essential to show that our linearized operator $R - E$ in (32) is invertible. This rests on two ideas: first, R is a product of the three first-order ordinary differential operators A_+, A_-, B_1 which are easy to invert. Second, $E = 0$ for $n = 0$, whereas it is small (relative to R) at large n. *Here* is one of the two points where the high periodicity we require in Theorem 1 is essential. (In future work we will discuss the technically more difficult inversion of $R - E$ for nonzero non-large n; then the value of μ will play a bigger role.)

Proposition 27. *For any $n \in N\mathbb{Z}$, R is an isometry on Ψ^n onto $\mathcal{F}^n$.*

Proof.

$$R \overset{(32)}{=} A_+ A_- B_1;$$

combine (85) with (131). □

Proposition 28.

$$\|E\|_{[\Psi^n,\mathcal{F}^n]} \lesssim N^{-1} \tag{142}$$

Proof.

$$E \overset{(32)}{=} (2\mu - 1)(A - B)$$

For $n = 0$ we have $A \underset{(26)}{\overset{(27)}{=}} B$, hence $E = 0$. For $n \in N\mathbb{Z}\backslash\{0\}$,

$$\|E\|_{[\Psi^n,\mathcal{F}^n]} \lesssim (\underbrace{\|A\|_{[\Psi^n,\mathcal{G}^n]}}_{\overset{(121)}{\lesssim} 1} + \underbrace{\|B\|_{[\Psi^n,\mathcal{G}^n]}}_{\underset{\mathcal{G}_0^n\subset\mathcal{G}^n}{\overset{(117)}{\lesssim}} \langle n\rangle^{-1}})\underbrace{\|\mathrm{id}\|_{[\mathcal{G}^n,\mathcal{F}^n]}}_{\overset{(136)}{\lesssim} \langle n\rangle^{-1}} \lesssim \langle n\rangle^{-1} \lesssim N^{-1}$$

□

Proposition 29. *For N $R - E$ is an n-uniform isomorphism on Ψ^n into $\mathcal{F}^n$.*

Proof. R is an isometry and $\|E\|_{[\Psi^n,\mathcal{F}^n]} \overset{(142)}{\leq} \frac{1}{2}$ for N sufficiently large, so that $R - E$ is an n-uniform isomorphism. □

Proposition 30. *For N sufficiently large, $\frac{\partial\overline{F}}{\partial\overline{\psi}}(\overline{\psi}_0, \Omega_0) : \Psi \to \mathcal{F}$ is an isomorphism.*

Proof. By Proposition 29, the Fourier transform $-(R-E)$ of $\frac{\partial\overline{F}}{\partial\overline{\psi}}(\overline{\psi}_0, \Omega_0)$ is n-uniformly an isomorphism on Ψ^n into $\mathcal{F}^n$, so $\frac{\partial\overline{F}}{\partial\overline{\psi}}(\overline{\psi}_0, \Omega_0)$ is an isomorphism on $\Psi = \mathcal{A}^{\frac{1}{2}}(\Psi^n)$ into $\mathcal{F} = \mathcal{A}^{\frac{1}{2}}(\mathcal{F}^n)$. □

3.8. The space $\mathcal{W}$ for data Ω. Now we define $\mathcal{W}$, the space for the data Ω. We choose *negative* regularity $-\frac{1}{2}$ in ϕ direction. This is crucial for obtaining large initial data that may change sign.

Definition 11. Let $\mathcal{W}^n = \mathbb{C}$ for $n \in N\mathbb{Z}$, $\mathcal{W} = 0$ for other $n \in \mathbb{Z}$.

$$\mathcal{W} := \mathcal{A}^{-\frac{1}{2}}(\mathcal{W}^n) \tag{143}$$

Motivation: say Ω was allowed to have regularity $-\sigma$. Then its product with $\overline{g^{(0)}}$, and hence F (see (28)), have regularity $-\sigma$ at most. From $\mathcal{F}$ to $\mathcal{G}$ we gain one order of regularity, to index $1-\sigma$ (put differently we gain two orders from $\mathcal{F}$ into Ψ, then lose one into $\mathcal{G}$ due to the various $\overline{\partial}_\varphi$, ∂_ϕ in $\overline{g^{(\varphi)}}$, $\overline{g^{(\phi)}}$, $\overline{g^{(0)}}$ which contain n). $\overline{g^{(0)}}$ is estimated in $\mathcal{G}$, so the two factors $\overline{g^{(0)}}$ and Ω have regularity $1-\sigma$ and $-\sigma$. To well-define the product we need (by Proposition 36) that $(1-\sigma)+(-\sigma) \geq 0$, hence $-\sigma \geq -\frac{1}{2}$.

(It is worth investigating whether a reformulation of the problem might overcome this limitation.)

Proposition 31.

$$\Omega_0 \in \mathcal{W} \tag{144}$$

Proof. $1^\wedge = \delta_{0n} \in \mathcal{W}^n$ for all n, and the $\mathcal{A}^{-\frac{1}{2}}(\mathcal{W}^n)$ norm is obviously finite. Hence multiples of 1, including Ω_0, are in $\mathcal{W}$. □

Lemma 5.

$$\mathcal{W}^j \cdot \mathcal{G}^k \overset{\sim}{\hookrightarrow} \mathcal{G}^{j+k} \tag{145}$$

Proof. Let $j,k \in \mathbb{Z}$. If $j \notin N\mathbb{Z}$, then $\mathcal{W}^j = 0$; if $k \notin N\mathbb{Z}$, then $\mathcal{G}^k = 0$; either way $\mathcal{W}^j \cdot \mathcal{G}^k = 0$. Let $j,k \in N\mathbb{Z}$ so that $j+k \in N\mathbb{Z}$. Then $\mathcal{W}^j \cdot \mathcal{G}^k = \mathbb{C} \cdot \mathcal{G}^k = \mathcal{G}^k \overset{(42)}{=} \mathcal{G}^{j+k}$. □

Proposition 32.

$$\mathcal{W} \cdot \mathcal{G} \hookrightarrow \mathcal{F} \tag{146}$$

In particular

$$\mathcal{W} \hookrightarrow \mathcal{F} \tag{147}$$

Proof. $1 \overset{(53)}{\in} \mathcal{G}$, so (147) is implied by (146) which we prove now:

$$\mathcal{W} \cdot \mathcal{G} = \mathcal{A}^{-\frac{1}{2}}(\mathcal{W}^n) \cdot \mathcal{A}^{\frac{1}{2}}(\mathcal{G}^n) \underset{(36)}{\overset{(145)}{\hookrightarrow}} \mathcal{A}^{-\frac{1}{2}}(\mathcal{G}^n) \overset{(137)}{\hookrightarrow} \mathcal{A}^{\frac{1}{2}}(\mathcal{F}^n) = \mathcal{F}$$

□

3.9. F is continuously differentiable. So far we have estimated linear maps: various derivatives applied to $\overline{\psi}$. Now we combine these into nonlinear terms by multiplication. We point out an essential problem: we have defined Ψ as $B_1^{-1}A_-^{-1}$ of $\mathcal{G}_-^n$, not $\mathcal{G}^n$. This is because $\mathcal{G}^n$ allows terms with asymptotics $c\beta^0$ ($c \neq 0$ constant) as $\beta \to \infty$ at *all* n; A_-^{-1} would map this into a term $c_2\beta^{-1} + o(\beta^{-1})$ and then $B_1^{-1} = (B - (-\acute{1}))^{-1}$, which has characteristic exponent -1, would produce $c_3\beta^{-1}\log\beta$, *not* $c_3\beta^{-1}$. After taking derivatives and multiplying (nonlinearity) the $\log\beta$ would cause havoc and prevent closure.

But now when we apply the inverse $R^{-1} = B_1^{-1}A_-^{-1}A_+^{-1}$ to defects f we need to ensure that $A_+^{-1}f$ is indeed in the smaller space $\mathcal{G}_-^n$. This required the choice $\mathcal{F} = A_+\mathcal{G}_-^n$ in (132). The price to pay is that the outer divergence of F (28) must map into $A_+\mathcal{G}_-^n$ as opposed to a larger space $A_+\mathcal{G}^n$. Since the factors that constitute $\overline{g^{(\varphi)}}$, $\overline{g^{(\phi)}}$, $\overline{g^{(0)}}$ involve $\overline{\partial}_\varphi\overline{\psi}$ which lives in $\mathcal{G}^n$ (as discussed in the context of (90)), this appears difficult.

However, we make a crucial observation: the key difficulty in (28) is $\overline{\partial}_\varphi\overline{g^{(\varphi)}}$ (since ∂_ϕ on $\overline{g^{(\phi)}}$ does not involve $\acute{\beta}$, hence cannot lose decay, and similarly $\overline{g^{(0)}}\Omega$ is harmless). Fortunately the values of $\overline{g^{(\varphi)}}$ are indeed in the smaller space $\mathcal{G}_-^n$: $\overline{\partial}_\varphi$ appears only in one particular fraction in $\overline{g^{(\varphi)}}$ (see (29)), which has a crucial cancellation that eliminates all $n \neq 0$ occurrences of $+c\chi_\infty(\beta)\beta^0$. This is proven in detail in the last part of the following proof.

Definition 12. Let $B_{\epsilon_\psi}^{(\Psi)}(\overline{\psi}_0)$ be the ball in Ψ of radius ϵ_ψ around $\overline{\psi}_0$ (which is $\in \Psi$ by (101), so the ball is well-defined). Let $B_{\epsilon_\Omega}^{(\mathcal{W})}(\Omega_0)$ be the ball in $\mathcal{W}$ of radius ϵ_Ω around Ω_0 (which is $\in \mathcal{W}$ by (144)).

Proposition 33. *Let $\alpha \in \{-1/(2\mu), -1\}$. For ϵ_ψ sufficiently small, $\overline{\psi} \in B_{\epsilon_\psi}^{(\Psi)}(\overline{\psi}_0)$:*

$$\inf \overline{\partial}_\varphi\overline{\psi} > 0, \quad \overline{\psi} \mapsto (\overline{\partial}_\varphi\overline{\psi})^\alpha \in \mathcal{C}^1(B_{\epsilon_\psi}^{(\Psi)}(\overline{\psi}_0); \mathcal{G}) \tag{148}$$

$$\inf(\overline{\partial}_\varphi + \acute{1})\overline{\partial}_\beta\overline{\psi} > 0, \quad \overline{\psi} \mapsto 1/(\overline{\partial}_\varphi + \acute{1})\overline{\partial}_\beta\overline{\psi} \in \mathcal{C}^1(B_{\epsilon_\psi}^{(\Psi)}(\overline{\psi}_0); \mathcal{G}) \tag{149}$$

$$\sup \overline{\partial}_\beta\overline{\psi} < 0, \quad \overline{\psi} \mapsto 1/\overline{\partial}_\beta\overline{\psi} \in \mathcal{C}^1(B_{\epsilon_\psi}^{(\Psi)}(\overline{\psi}_0); \mathcal{G}_-) \tag{150}$$

$$\overline{\psi} \mapsto \overline{\partial}_\varphi\overline{\psi}/(\overline{\partial}_\varphi + \acute{1})\overline{\partial}_\beta\overline{\psi} \in \mathcal{C}^1(B_{\epsilon_\psi}^{(\Psi)}(\overline{\psi}_0); \mathcal{G}_-) \tag{151}$$

Proof. (148): $\mathcal{G}$ is a unital Banach algebra ((53), (51)); Taylor expansion of $x \mapsto x^\alpha$ around $x = 1$ yields (for either α) an analytic function $f : U \to \mathcal{G}$ on a neighbourhood $U \subset \mathcal{G}$ of $1 \overset{(27)}{\underset{(23)}{=}} \overline{\partial}_\varphi\overline{\psi}_0$. By (128) $\overline{\partial}_\varphi$ is $\mathcal{C}^1$ (in fact linear continuous) on $B_{\epsilon_\psi}^{(\Psi)}(\overline{\psi}_0)$ into $\mathcal{G}$, so we can restrict ϵ_ψ so that $\overline{\partial}_\varphi$ maps $B_{\epsilon_\psi}^{(\Psi)}(\overline{\psi}_0)$ into U and thus $f \circ \overline{\partial}_\varphi$ is $\mathcal{C}^1$ on $B_{\epsilon_\psi}^{(\Psi)}(\overline{\psi}_0)$ into $\mathcal{G}$.

(149): analogous, expanding $x \mapsto x^{-1}$ around $-2\mu \overset{(27),(26)}{\underset{(23)}{=}} (\overline{\partial}_\varphi + \acute{1})\overline{\partial}_\beta\overline{\psi}_0$ and observing that $(\overline{\partial}_\varphi + \acute{1})\overline{\partial}_\beta \overset{(129)}{\in} \mathcal{C}^1(B_{\epsilon_\psi}^{(\Psi)}(\overline{\psi}_0); \mathcal{G})$.

(150): $\mathcal{G}_-$ is also a unital Banach algebra ((53), (52)), so we can expand around $\overline{\partial}_\beta\overline{\psi}_0 \overset{(26)}{\underset{(23)}{=}} -1$ and use $\overline{\partial}_\beta \overset{(126)}{\in} \mathcal{C}^1(B_{\epsilon_\psi}^{(\Psi)}(\overline{\psi}_0); \mathcal{G}_-)$;

(151):

$$(\overline{\partial}_\varphi + \acute{1})\overline{\partial}_\beta \overset{(27)}{\underset{(26)}{=}} \acute{2}\acute{\mu}\acute{\beta}\partial_\beta + \acute{\beta}\partial_\varphi(\acute{\beta}\partial_\beta + \acute{1}) - \acute{2}\acute{\mu}(\underbrace{\acute{\beta}\partial_\varphi + \acute{2}\acute{\mu} - \acute{1}}_{=\overline{\partial}_\varphi})$$

$$(\overline{\partial}_\varphi\overline{\psi})^{-1}\cdot(\overline{\partial}_\varphi+\acute{1})\overline{\partial}_\beta\overline{\psi} = \underbrace{\underbrace{(\overline{\partial}_\varphi\overline{\psi})^{-1}}_{\overset{(148)}{\underset{\alpha=-1}{\in}}\mathcal{G}} \cdot \underbrace{\big(\acute{2}\acute{\mu}\underbrace{\acute{\beta}\partial_\beta\overline{\psi}}_{\overset{(118)}{\in}\mathcal{G}_0} + \underbrace{\acute{\beta}\partial_\varphi(\acute{\beta}\partial_\beta+\acute{1})\overline{\psi}}_{\overset{(120)}{\in}\mathcal{G}_0}\big)}_{\in\mathcal{G}_0}}_{\overset{(50)}{\in}\mathcal{G}_0\hookrightarrow\mathcal{G}_-} - \underbrace{\acute{2}\acute{\mu}}_{\overset{(53)}{\in}\mathcal{G}_-}$$

Hence $\overline{\psi} \mapsto (\overline{\partial}_\varphi\overline{\psi})^{-1}(\overline{\partial}_\varphi+\acute{1})\overline{\partial}_\beta\overline{\psi}$ is $\mathcal{C}^1(B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0);\mathcal{G}_-)$. Moreover the expression is $= -2\mu$ at $\overline{\psi} = \overline{\psi}_0$ (see above), hence invertible in $\mathcal{G}_-$ on some neighbourhood U of -2μ. We restrict $\epsilon_\psi > 0$ further to stay in that neighbourhood. □

Note that the troublesome $\overline{\partial}_\varphi$ factors have been rewritten as a combination of $B\overline{\psi}$, $AB_1\overline{\psi}$ and $\overline{\psi}$, *without* $A\overline{\psi}$. This enables us to map into $\mathcal{G}^n_-$ and achieve closure of the estimates. If the fraction was replaced by $(\overline{\partial}_\varphi\overline{\psi})^{-1}$ alone or $(\overline{\partial}_\varphi+\acute{1})\overline{\partial}_\beta\overline{\psi}$ alone, this closure would not be possible.

Definition 13. Henceforth we restrict ϵ_ψ so that the preconditions of Proposition 33 hold.

Proposition 34. *For $\epsilon_\psi > 0$ sufficiently small,*

$$\overline{\psi} \mapsto \overline{g^{(\phi)}} \in \mathcal{C}^1(B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0);\mathcal{G}) \tag{152}$$

$$\overline{\psi} \mapsto \overline{g^{(0)}} \in \mathcal{C}^1(B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0);\mathcal{G}) \tag{153}$$

$$\overline{\psi} \mapsto \overline{g^{(\varphi)}} \in \mathcal{C}^1(B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0);\mathcal{G}_-) \tag{154}$$

Proof. (152): inspect $\overline{g^{(\phi)}}$ in (30) and observe that every term is $\mathcal{C}^1$ in $\overline{\psi} \in B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0)$ into $\mathcal{G}$ by (150) and (125) to (129).

(153): inspect $\overline{g^{(0)}}$ in (31): every term is $\mathcal{C}^1$ into $\mathcal{G}$, by (129) and (148) (using Definition 13).

(154): inspect $\overline{g^{(\varphi)}}$ in (29). The only factor using $\overline{\partial}_\varphi$ is the fraction in (151) which is $\mathcal{C}^1$ into $\mathcal{G}_-$. Every other factor uses only $\overline{\partial}_\beta$, ∂_ϕ, $\partial_\phi\overline{\partial}_\beta$, hence is *clearly* into $\mathcal{G}_-$: $1/\overline{\partial}_\beta\overline{\psi}$ by (150), the rest by (126), (125) and (127). $\mathcal{G}_-$ is an algebra by (52). □

Proposition 35. $\overline{\psi} \mapsto \overline{F}(\overline{\psi},\Omega)$ *is* $\mathcal{C}^1$ *on* $B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0)\times\mathcal{W}$ *into* $\mathcal{F}$.

Proof.

$$\overline{F} \overset{(28)}{=} 2\mu^2(-\overbrace{\overline{\partial}_\varphi}^{\overset{(139)}{\in}\mathcal{C}^1(\mathcal{G}_-;\mathcal{F})}\overbrace{\overline{g^{(\varphi)}}}^{\overset{(154)}{\in}\mathcal{C}^1(B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0);\mathcal{G}_-)} - \overbrace{\partial_\phi}^{\overset{(140)}{\in}\mathcal{C}^1(\mathcal{G};\mathcal{F})}\overbrace{\overline{g^{(\phi)}}}^{\overset{(152)}{\in}\mathcal{C}^1(B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0);\mathcal{G})}$$

$$- \underbrace{\underbrace{\Omega}_{\in\mathcal{W}} \cdot \underbrace{\overline{g^{(0)}}}_{\overset{(153)}{\in} \mathcal{C}^1(B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0);\mathcal{G})})}_{\overset{(146)}{\in} \mathcal{C}^1(B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0)\times\mathcal{W};\mathcal{F})}$$

□

3.10. Implicit function theorem.

Proposition 36. *For N sufficiently large and $\epsilon_\psi, \epsilon_\Omega > 0$ sufficiently small, there is a $\mathcal{C}^1$ map $H : B^{(\mathcal{W})}_{\epsilon_\Omega}(\Omega_0) \to B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0)$ so that*

$$\overline{F}(H(\Omega), \Omega) = 0 \tag{155}$$

for all $\Omega \in B^{(\mathcal{W})}_{\epsilon_\Omega}(\Omega_0)$. $H(\Omega)$ is real if Ω is real.

Proof. By Proposition 35 $\overline{F}$ is $\mathcal{C}^1$ on $B^{(\Psi)}_{\epsilon_\psi}(\overline{\psi}_0) \times \mathcal{W}$ into $\mathcal{F}$. Moreover $\overline{F}(\overline{\psi}_0, \Omega_0) = 0$. By Proposition 30 $\partial\overline{F}/\partial\overline{\psi}(\overline{\psi}_0, \Omega_0)$ is a linear isomorphism on Ψ into $\mathcal{F}$. Hence the implicit function theorem for Banach spaces yields existence and $\mathcal{C}^1$ regularity of H.

It is obvious from inspection of (28) that $\overline{F}$ maps real $\overline{\psi}$, Ω into real distributions, so the implicit function theorem yields a real-valued $H(\Omega)$ for real Ω. □

Definition 14. For the remainder of the paper we take

$$\overline{\psi} = H(\Omega). \tag{156}$$

4. Solution Properties

Having completed the hard step of constructing $\overline{\psi}$, it remains to show that it is a weak solution and has the properties claimed in the introduction.

$$\{\overline{\partial}_\varphi, \partial_\phi, \mathrm{id}\}\{\overline{\partial}_\beta, \mathrm{id}\}\Psi \overset{(125)\text{ to }(129)}{\hookrightarrow} \mathcal{G} \overset{(47)}{=} \mathcal{A}^{\frac{1}{2}}(\mathcal{G}^n) \overset{(46)}{\hookrightarrow} \mathcal{A}^{\frac{1}{2}}(\mathcal{C}_b) \hookrightarrow \mathcal{C}_b(\overline{\mathbb{R}}_+ \times \mathbb{T}). \tag{157}$$

Using $\partial_\beta \overset{(18)}{=} \partial_\phi - \partial_\varphi$ that means

$$\check{\psi}_{\beta\beta}, \check{\psi}_{\beta\phi}, \check{\psi}_\phi \in \mathcal{C}^0_{(\beta,\phi)}(\mathbb{R}_+ \times \mathbb{T}), \quad \check{\psi}_\beta, \check{\psi} \in \mathcal{C}^1_{(\beta,\phi)}(\mathbb{R}_+ \times \mathbb{T}) \tag{158}$$

(we do not control $\check{\psi}_{\phi\phi}$).

By Definition 13,

$$\check{\psi}_\varphi = \acute{\beta}^{-2\mu}\overline{\partial}_\varphi\overline{\psi} \overset{(148)}{>} 0, \tag{159}$$

$$\check{\psi}_{\beta\varphi} = \acute{\beta}^{-1-2\mu}(\overline{\partial}_\varphi + \acute{1})\overline{\partial}_\beta\overline{\psi} \overset{(149)}{>} 0, \tag{160}$$

$$\check{\psi}_\beta = \acute{\beta}^{-2\mu}\overline{\partial}_\beta\overline{\psi} \overset{(150)}{<} 0. \tag{161}$$

In particular for $x = \beta, \phi$ we have

$$a_x \overset{(17)}{=} \frac{\check{\psi}_{\beta x}}{2\check{\psi}_\beta} \in \mathcal{C}^0_{(\beta,\phi)}(\mathbb{R}_+ \times \mathbb{T}).$$

Hence $\mathbf{b} = (\beta, \phi) \mapsto \mathbf{a} = (a, \theta)$ is $\mathcal{C}^1$, and

$$\det \mathbf{a_b} \overset{(19)}{=} -a_\varphi \overset{(17)}{=} -\frac{\check{\psi}_{\beta\varphi}}{2\check{\psi}_\beta} \underset{(161)}{\overset{(160)}{>}} 0$$

so $\mathbf{b} \mapsto \mathbf{a}$ is a $\mathcal{C}^1$ diffeomorphism, as well as onto $\mathbb{R} \times \mathbb{T}$ (same proof as in [10, Section 7.4]).

$\mathbf{a} \mapsto \check{\mathbf{x}}$ is obviously a diffeomorphism and maps $\mathbb{R} \times \mathbb{T}$ onto $\mathbb{R}^2 \backslash \{0\}$.

Since $\check{\mathbf{x}} \mapsto \mathbf{b}$ and $\mathbf{b} \mapsto \check{\psi}$ are both $\mathcal{C}^1$, we obtain

$$\check{\mathbf{v}} = \nabla^\perp_{\check{\mathbf{x}}} \check{\psi} \in \mathcal{C}^0_{\check{\mathbf{x}}}(\mathbb{R}^2 \backslash \{0\})$$

Definition 15. Henceforth we restrict $\Omega \in B^{(\mathcal{W})}_{\epsilon_\Omega}(\Omega_0)$ to be an element of the Wiener algebra $\mathcal{A}^0(\mathbb{T})$ and hence of $\mathcal{C}^0(\mathbb{T})$ as well.

Thus

$$\check{\omega} \overset{(21)}{=} \check{\psi}_\varphi^{-\frac{1}{2\mu}} \Omega \underset{(159)}{\overset{(158)}{\in}} \mathcal{C}^0_{(\beta,\phi)}(\mathbb{R}_+ \times \mathbb{T}) = \mathcal{C}^0_{\check{\mathbf{x}}}(\mathbb{R}^2 \backslash \{0\})$$

and altogether

$$\mathbf{v}, \omega \in \mathcal{C}^0_{(\mathbf{x},t)}((\mathbb{R}^2 \backslash \{0\}) \times]0, \infty[).$$

Since $\mathbf{v} \in \mathcal{C}^0$, in contrast to our prior work where $\mathbf{v} \in \mathcal{C}^1$, some (elementary) care is needed with the nonlinear part of the vorticity equation.

Lemma 6. *Let $\Sigma \subset \mathbb{R}^2$ be open. With all derivatives in the $\mathcal{D}'(\Sigma)$ sense, let $\mathbf{v} = \nabla^\perp \psi$ with $\psi \in \mathcal{C}^1(\Sigma)$ and $\check{\omega} = \nabla \times \mathbf{v} = \Delta\psi \in \mathcal{C}^0(\Sigma)$. Then*

$$\nabla \times [\nabla \cdot (\mathbf{v} \otimes \mathbf{v})] = \nabla \cdot (\mathbf{v}\omega). \tag{162}$$

Proof. Let θ^δ be a standard mollifier. Consider $\psi^\delta := \psi * \theta^\delta$ which is well-defined outside $\Sigma \backslash B_\delta(\Sigma)$ which converges monotonically to Σ as $\delta \searrow 0$. Set $\varepsilon^{12} = 1 = -\varepsilon^{21}$, $\varepsilon^{11} = \varepsilon^{22} = 0$. We use the Einstein convention; subscripts are derivatives:

$$\begin{aligned}
&\psi \in \mathcal{C}^1(\Sigma) \Rightarrow \psi^\delta = \psi * \theta^\delta \to \psi \quad \text{in } \mathcal{C}^0(\Sigma) \text{ and} \\
&\psi^\delta_j = \psi_j * \theta^\delta \to \psi_j \quad \text{in } \mathcal{C}^0(\Sigma), \\
&\omega \in \mathcal{C}^0(\Sigma) \Rightarrow \psi^\delta_{kk} = \Delta\psi^\delta = \omega * \theta^\delta \to \omega = \psi_{kk} \quad \text{in } \mathcal{C}^0(\Sigma)
\end{aligned}$$

and therefore

$$\begin{aligned}
\psi^\delta_k \psi^\delta_p &\to \psi_k \psi_p \quad \text{and} \\
\psi^\delta_{kk} \psi^\delta_p &\to \psi_{kk} \psi_p \quad \text{in } \mathcal{C}^0(\Sigma)
\end{aligned}$$

so that (all remaining limits in $\mathcal{D}'(\Sigma)$)

$$
\begin{aligned}
(\psi_k^\delta \psi_p^\delta)_{km} &\to (\psi_k \psi_p)_{km} \\
(\psi_{kk}^\delta \psi_p^\delta)_m &\to (\psi_{kk} \psi_p)_m \\
\nabla \times [\nabla \cdot (\mathbf{v} \otimes \mathbf{v})] &= -\varepsilon^{jk}(v^j v^m)_{km} = -\varepsilon^{jk}(\varepsilon^{\ell j}\psi_\ell \varepsilon^{pm}\psi_p)_{km} = \varepsilon^{pm}(\psi_k \psi_p)_{km} \\
&\leftarrow \varepsilon^{pm}(\psi_k^\delta \psi_p^\delta)_{km} \\
&= \varepsilon^{pm}(\psi_{kk}^\delta \psi_p^\delta)_m + \varepsilon^{pm}(\psi_k^\delta \psi_{kp}^\delta)_m \\
&= \varepsilon^{pm}(\psi_{kk}^\delta \psi_p^\delta)_m + \underbrace{(\frac{\psi_k^\delta \psi_k^\delta}{2})_{pm}\varepsilon^{pm}}_{=0} \\
&\to \varepsilon^{pm}(\psi_{kk}\psi_p)_m = (\omega v^m)_m = \nabla \cdot (\mathbf{v}\omega)
\end{aligned}
$$

□

Proposition 37. *ψ defines a solution of*

$$0 = \nabla \times \big(\partial_t \mathbf{v} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} \otimes \mathbf{v})\big) \tag{163}$$

Proof. First we observe that

$$\nabla_{\mathbf{x}} \cdot \mathbf{v} = 0$$

is simply a consequence of our definition

$$\mathbf{v} = \nabla_{\mathbf{x}}^\perp \psi.$$

Moreover, it is easy to check that all our coordinate changes correctly treated the outer divergences as distributional, while all other derivatives are defined in the classical sense (with results in $\mathcal{C}^0$). Hence we may also trace from (28) back to the original curl constraint (11) which then shows

$$\nabla_{\mathbf{x}} \times \mathbf{v} = \omega \quad \text{on } (\mathbb{R}^2 \backslash \{0\}) \times \mathbb{R}_+ \tag{164}$$

Moreover we may trace from our definition

$$\overline{\omega} = (\overline{\partial}_\varphi \overline{\psi})^{-\frac{1}{2\mu}} \Omega,$$

back to (21), i.e.

$$\check{\omega} = \check{\psi}_\varphi^{-\frac{1}{2\mu}} \Omega,$$

which yields a weak solution of (20), i.e.

$$0 = (1 - \frac{1}{2\mu})\check{\psi}_{\varphi\beta}\check{\omega} - \partial_\beta(\check{\omega}\check{\psi}_\varphi), \quad \text{on } (\mathbb{R}^2 \backslash \{0\}) \times \mathbb{R}_+$$

which can then be traced back to the original divergence-form vorticity equation (10):

$$0 = \partial_t \omega + \nabla_{\mathbf{x}} \cdot (\omega \mathbf{v})$$
$$\overset{(164)}{=} \partial_t \nabla \times \mathbf{v} + \nabla_{\mathbf{x}} \cdot (\underbrace{(\nabla \times \mathbf{v})}_{\in \mathcal{C}^0} \underbrace{\mathbf{v}}_{\in \mathcal{C}^0})$$
$$\overset{(162)}{=} \nabla \times \big(\partial_t \mathbf{v} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} \otimes \mathbf{v})\big) \quad \text{on } (\mathbb{R}^2 \backslash \{0\}) \times \mathbb{R}_+.$$

□

Hence the solution we constructed does define a weak solution in the interior; now we discuss the initial data and asymptotics.

Proposition 38. *As $t \searrow 0$, $\omega(t, \cdot)$ satisfies our initial condition (2), with convergence in $\mathcal{C}^0(\mathbb{R}^2 \backslash \{0\})$ (i.e. locally uniformly), and*

$$\mathring{\omega} = \overline{h}_{|\beta=0, \phi=\theta} \tag{165}$$

where

$$\overline{h} := \overline{w} \Omega \tag{166}$$

with

$$\overline{w} := \Big(\frac{\overline{\partial}_\beta \overline{\psi}}{-\mu \overline{\partial}_\varphi \overline{\psi}}\Big)^{\frac{1}{2\mu}} \tag{167}$$

Proof.

$$\omega \overset{(5)}{=} t^{-1} \check{\omega} \overset{(21)}{=} t^{-1} (\partial_\varphi \check{\psi})^{-\frac{1}{2\mu}} \Omega \underset{(25)}{\overset{(27)}{=}} \Big(\frac{t}{\beta}\Big)^{-1} (\overline{\partial}_\varphi \overline{\psi})^{-\frac{1}{2\mu}} \Omega$$

and

$$r \overset{(5)}{=} t^\mu \check{r} \underset{(14)}{\overset{(17)}{=}} t^\mu \Big(\frac{\partial_\beta \check{\psi}}{-\mu}\Big)^{\frac{1}{2}} \underset{(25)}{\overset{(26)}{=}} \Big(\frac{t}{\beta}\Big)^\mu \underbrace{\Big(\frac{\overline{\partial}_\beta \overline{\psi}}{-\mu}\Big)^{\frac{1}{2}}}_{\overset{(150)}{\sim} 1} \tag{168}$$

so

$$r^{\frac{1}{\mu}} \omega = \underbrace{\Big(\frac{\overline{\partial}_\beta \overline{\psi}}{-\mu}\Big)^{\frac{1}{2\mu}} (\overline{\partial}_\varphi \overline{\psi})^{-\frac{1}{2\mu}}}_{=\overline{w}} \Omega = \overline{h} \underset{\text{Def. 15}}{\overset{(157)}{\in}} \mathcal{C}_b(\overline{\mathbb{R}}_+ \times \mathbb{T}) \tag{169}$$

(168) shows that holding $\mathbf{x}$ and hence $r = |\mathbf{x}|$ fixed while taking $t \searrow 0$ corresponds to $\beta \searrow 0$, and then $\mathbf{x}$ fixed also means $\measuredangle \mathbf{x} = \theta \overset{(16)}{=} \beta + \phi \to \phi$. So we find our initial data on the ray with angle θ by considering $(\beta, \phi) = (0, \theta)$. Since $\overline{h} \in \mathcal{C}_b(\overline{\mathbb{R}}_+ \times \mathbb{T})$ is ϕ-uniformly continuous at $\beta = 0+$ that means (with (165))

$$\omega(t, \mathbf{x}) \overset{t \searrow 0}{\to} r^{-\frac{1}{\mu}} \mathring{\omega}(\theta) \quad \text{in } \mathcal{C}^0(\mathbb{R}^2 \backslash \{0\}).$$

□

Proposition 39. *Our solutions satisfy* (6).

Proof. For $\mathbf{v}$ this is as in [10, Section 8.3]; for ω we observe

$$\omega \overset{(169)}{=} r^{-\frac{1}{\mu}} \underbrace{\overline{h}}_{\in \mathcal{C}_b} \lesssim r^{-\frac{1}{\mu}}.$$

□

Proposition 40. *If* $\frac{2}{3} < \mu$, *then our* $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ *is a weak solution on* $\mathbb{R}^2 \times [0, \infty[$ *(including origin); if* $\frac{1}{2} < \mu \leq \frac{2}{3}$, *then it is a weak solution on* $(\mathbb{R}^2 \backslash \{0\}) \times [0, \infty[$.

Proof. As in [10, Section 8.3]: integrate (163) against a test function supported in $(\mathbb{R}^2 \backslash \{0\}) \times [0, \infty[$ and use Proposition 38 for the boundary integral over $t = 0$. For $\mu > \frac{2}{3}$ observe that the asymptotics (6) are strong enough to remove $\mathbf{x} = 0$ from of the support of any test function. □

Having shown that our solution satisfies the initial condition as well, we turn to discuss the variety of initial data.

Proposition 41. *For* N *sufficiently large there is an* $\epsilon_{\mathring{\omega}} > 0$ *so that*

(i) $\Omega \mapsto \mathring{\omega}$ *maps* $B^{(\mathcal{W})}_{\epsilon_\Omega}(\Omega_0) \cap \mathcal{A}^0$ *onto a superset of* $B^{(\mathcal{W})}_{\epsilon_{\mathring{\omega}}}(\mathring{\omega}_0)$, *and*
(ii) $\mathring{\omega} \in \mathcal{A}^0(\mathcal{W}^n)$ *if and only if* $\Omega \in \mathcal{A}^0(\mathcal{W}^n)$.

Proof. (i) We want to show that at $\Omega = \Omega_0$

$$\Omega \mapsto \mathring{\omega} \underset{(166)}{\overset{(165)}{=}} \overline{w}_{|\beta=0} \cdot \Omega \tag{170}$$

is a local diffeomorphism in the norm of $\mathcal{W}$, which is regarded as a space of functions of ϕ alone. We have already shown $\mathcal{C}^1$ prior to our application of the implicit function theorem, so it is sufficient to show study the Fréchet derivative is an isomorphism.

$$\overline{\partial}_\beta \overline{\psi} \underset{\Omega=\Omega_0}{\overset{(26),(23)}{=}} (\beta\partial_\beta + \acute{1} - \acute{2}\acute{\mu})\frac{1}{2\mu - 1} = -1, \tag{171}$$

$$\overline{\partial}_\varphi \overline{\psi} \underset{\Omega=\Omega_0}{\overset{(27),(23)}{=}} (\beta\partial_\varphi - \acute{1} + \acute{2}\acute{\mu})\frac{1}{2\mu - 1} = 1 \quad \text{and} \tag{172}$$

$$(\overline{\partial}_\varphi + \acute{1})\overline{\partial}_\varphi \overline{\psi} \underset{\Omega=\Omega_0}{\overset{(27),(23)}{=}} (\beta\partial_\varphi + \acute{2}\acute{\mu})(\beta\partial_\beta + \acute{1} - \acute{2}\acute{\mu})\frac{1}{2\mu - 1} = -2\mu, \tag{173}$$

so

$$\overline{w} \underset{\Omega=\Omega_0}{\overset{(167)}{=}} \Big(\frac{-\overline{\partial}_\beta \overline{\psi}_0}{\mu \overline{\partial}_\varphi \overline{\psi}_0}\Big)^{\frac{1}{2\mu}} \overset{(23)}{=} \mu^{-\frac{1}{2\mu}}. \tag{174}$$

Write $d/d\Omega$ for a derivative with $\overline{\psi}$ as function of Ω, as per (156), but $\partial/\partial\Omega$ and $\partial/\partial\overline{\psi}$ for $\overline{\psi}$, Ω varied separately. Then

$$\frac{d\mathring{\omega}}{d\Omega} \underset{\Omega=\Omega_0}{\overset{(170),(174)}{=}} \underbrace{\mu^{-\frac{1}{2\mu}}}_{>0} \mathrm{id} + \Big(\frac{\partial \overline{w}}{\partial \overline{\psi}} \frac{d\overline{\psi}}{d\Omega}\Big)_{|\Omega=\Omega_0, \beta=0} \cdot \Omega_0.$$

The first term is clearly an isomorphism on $\mathcal{W}$, so it is sufficient to show the second term can be made small: Ω_0 is a constant, which multiplies all spaces used; we inspect the operators in the other factor. We have already shown in the context of the implicit function theorem argument (Proposition 36) that $d\overline{\psi}/d\Omega$ at $\Omega = \Omega_0, \overline{\psi} = \overline{\psi}_0$ is bounded on $\mathcal{W}$ into Ψ. Next,

$$\frac{\partial\overline{w}}{\partial\overline{\psi}} \overset{(174)}{=} \frac{1}{2\mu}\Big(\frac{-\overline{\partial}_\beta\overline{\psi}}{\mu\overline{\partial}_\varphi\overline{\psi}}\Big)^{\frac{1}{2\mu}}\Big(-\frac{1}{-\overline{\partial}_\beta\overline{\psi}}\overline{\partial}_\beta - \frac{1}{\overline{\partial}_\varphi\overline{\psi}}\overline{\partial}_\varphi\Big) \underset{\Omega=\Omega_0}{\overset{(171),(172)}{=}} -\frac{1}{2}\mu^{-\frac{1}{2\mu}-1}(\overline{\partial}_\beta + \overline{\partial}_\varphi);$$

this operator is bounded on Ψ into $\mathcal{G}$ by (126) and (128). Moreover

$$\mathcal{F}(\overline{\partial}_\beta + \overline{\partial}_\varphi)\mathcal{F}^{-1} \underset{(27)}{\overset{(26)}{=}} B - A = in\acute{\beta},$$

so the operator is bounded into the closed subspace of $\mathcal{G}$ functions g with $g^\wedge(0) = 0$. Evaluation at $\beta = 0$ on that subspace (regularity $\frac{1}{2}$) into $\mathcal{W}$ (regularity $-\frac{1}{2}$) has arbitrarily small operator norm for N sufficiently large. Thus the second term in (170) can be made arbitrarily small, which completes the proof of (i).

(ii) On one hand

$$\mathring{\omega} = \overline{w}_{|\beta=0}\cdot\Omega;$$

on the other hand at $\Omega = \Omega_0$ we have $\overline{w}_{|\beta=0} = \mu^{-\frac{1}{2\mu}} = \text{const} > 0$, and since $\overline{w}_{|\beta=0} \in \mathcal{A}^{\frac{1}{2}}$ depends continuously on Ω, it remains uniformly positive for Ω near Ω_0, and then

$$\Omega = \frac{1}{\overline{w}_{|\beta=0}}\cdot\mathring{\omega}.$$

Since $\overline{w}_{|\beta=0}$ and its inverse are in $\mathcal{A}^{\frac{1}{2}}$, $\mathring{\omega}$ is in $\mathcal{A}^0$ if and only if Ω is. $\square$

Proof of Theorem 1 According to Proposition 41 there is an $\epsilon_{\mathring{\omega}} > 0$ so that we can obtain a weak solution for any initial data $\mathring{\omega}$ with

$$\epsilon_{\mathring{\omega}} \ge \|\mathring{\omega} - \mathring{\omega}_0\|_{\mathcal{W}} = |\mathring{\omega}^\wedge(0) - \mathring{\omega}_0| + \sum_{n\in N\mathbb{Z}\setminus\{0\}} \underbrace{\langle n\rangle^{-\frac{1}{2}}}_{\le\langle N\rangle^{-\frac{1}{2}}} |\mathring{\omega}^\wedge(n)|$$

Focus on the ones with $\mathring{\omega}^\wedge(0) = \mathring{\omega}_0$. Then it is sufficient to satisfy

$$\sum_{n\in N\mathbb{Z}\setminus\{0\}} |\mathring{\omega}^\wedge(n)| \le \langle N\rangle^{\frac{1}{2}}\epsilon_{\mathring{\omega}} = \frac{\langle N\rangle^{\frac{1}{2}}\epsilon_{\mathring{\omega}}}{|\mathring{\omega}_0|}|\mathring{\omega}^\wedge(0)|$$

Given $\epsilon > 0$ as in Theorem 1 take N_0 so large that $N \ge N_0$ implies $\langle N\rangle^{\frac{1}{2}}\epsilon_{\mathring{\omega}}/|\mathring{\omega}_0| \ge \epsilon^{-1}$. Then we have solved the problem for all $\frac{2\pi}{N}$-periodic $\mathring{\omega}$ with

$$|\mathring{\omega}^\wedge(0)| \ge \epsilon \sum_{n\in\mathbb{Z}\setminus\{0\}} |\mathring{\omega}^\wedge(n)|$$

and $\mathring{\omega}^\wedge(0) = \mathring{\omega}_0 \neq 0$.

This is extended to arbitrary $\mathring{\omega}^\wedge(0) \neq 0$ by applying to the solutions the well-known Euler scaling symmetry

$$\omega(\mathbf{x}, t) \leftarrow s\omega(\mathbf{x}, st)$$

Hence we have constructed solutions for all initial data satisfying (3). □

To justify (4), observe that

$$\omega = \underbrace{(\frac{t}{\beta})^{-1}}_{>0} \underbrace{(\overline{\partial}_\varphi \overline{\psi})^{\frac{1}{2\mu}}}_{\overset{(159)}{>0}} \Omega.$$

Hence $\operatorname{sign}\omega = \operatorname{sign}\Omega$ in any $(\mathbf{x}, t) \in (\mathbb{R}^2 \backslash \{0\}) \times \mathbb{R}_+$. $\Omega = \Omega(\phi)$ has constant sign on curves of constant ϕ and varying β; for each fixed $t > 0$

$$r = (\frac{t}{\beta})^\mu \underbrace{(\frac{\overline{\partial}_\beta \overline{\psi}}{-\mu})^{\frac{1}{2}}}_{\sim 1} \sim \beta^{-\mu}$$

while $\theta = \beta + \phi$. Hence these curves are parametrized (for fixed ϕ and $\theta \to \infty$) by

$$\theta \mapsto \mathbf{x}(\theta) = \theta^{-\mu} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \underbrace{f(\theta)}_{\sim 1}$$

as claimed.

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5. Appendix

5.1. Wiener algebras.

Lemma 7.

$$\langle n \rangle \lesssim \langle k \rangle + \langle n - k \rangle \tag{175}$$

Proof.

$$\begin{aligned} \langle n \rangle &\lesssim \max\{1, |n|\} = \max\{1, |k + (n - k)|\} \leq \max\{1, |k| + |n - k|\} \\ &\leq \langle k \rangle + \langle n - k \rangle. \end{aligned}$$

□

Lemma 8.

$$\langle n \rangle \lesssim \langle n - k \rangle \langle k \rangle \tag{176}$$

and

$$\langle n \rangle^{-1} \lesssim \langle n - k \rangle \langle k \rangle^{-1}. \tag{177}$$

Proof.

$$\begin{aligned}\langle n\rangle &\overset{(175)}{\lesssim} \langle k\rangle + \langle n-k\rangle \\ &= \langle k\rangle \cdot 1 + 1 \cdot \langle n-k\rangle \\ &\leq \langle k\rangle \cdot \langle n-k\rangle + \langle k\rangle \cdot \langle n-k\rangle \sim \langle k\rangle\langle n-k\rangle\end{aligned}$$

This is (176); exchange k with n to obtain

$$\langle k\rangle \lesssim \langle n-k\rangle\langle n\rangle$$

and divide by $\langle n\rangle\langle k\rangle$ to obtain (177). □

Proof of Proposition 3 Let $x \in \mathcal{A}^s(\mathcal{X}_n)$ and $y \in \mathcal{A}^{-s}(\mathcal{Y}_n)$.

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} \langle n\rangle^{-s} \overbrace{\sum_{k\in\mathbb{Z}} \left\| x^\wedge(n-k) \cdot y^\wedge(k)\right\|_{\mathcal{Z}_n}}^{S} \\ &\overset{(35)}{\leq} \sum_{n\in\mathbb{Z}} \langle n\rangle^{-s} \sum_{k\in\mathbb{Z}} \left\| x^\wedge(n-k)\right\|_{\mathcal{X}_{n-k}} \left\| y^\wedge(k)\right\|_{\mathcal{Y}_k} \\ &\overset{(177)}{\lesssim} \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle n-k\rangle^s \left\| x^\wedge(n-k)\right\|_{\mathcal{X}_{n-k}} \langle k\rangle^{-s} \left\| y^\wedge(k)\right\|_{\mathcal{Y}_k} \\ &\leq \|x\|_{\mathcal{A}^s(\mathcal{X}_n)} \|y\|_{\mathcal{A}^{-s}(\mathcal{X}_n)} < \infty \end{aligned} \tag{178}$$

In particular the series S on the left-hand side is convergent, hence $x \cdot y$ is well-defined, and

$$\begin{aligned}\|x \cdot y\|_{\mathcal{A}^{-s}(\mathcal{Z}_n)} &= \left\| \sum_{n\in\mathbb{Z}} \langle n\rangle^{-s} (x\cdot y)^\wedge(n)\right\|_{\mathcal{Z}_n} \lesssim \sum_{n\in\mathbb{Z}} \langle n\rangle^{-s} \left\| \sum_{k\in\mathbb{Z}} x^\wedge(n-k)\cdot y^\wedge(k)\right\|_{\mathcal{Z}_n} \\ &\leq \sum_{n\in\mathbb{Z}} \langle n\rangle^{-s} \sum_{k\in\mathbb{Z}} \left\| x^\wedge(n-k)\cdot y^\wedge(k)\right\|_{\mathcal{Z}_n} \overset{(178)}{\lesssim} \|x\|_{\mathcal{A}^s(\mathcal{X}_n)} \|y\|_{\mathcal{A}^{-s}(\mathcal{X}_n)}\end{aligned}$$

which is (36). Using (176) instead of (177) we get (37). □

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