



SU(N) Transitions in M-Theory on Calabi–Yau Fourfolds and Background Fluxes

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Received: 8 April 2016 / Accepted: 29 June 2016

Published online: 23 September 2016 – © Springer-Verlag Berlin Heidelberg 2016

Abstract: We study M-theory on a Calabi–Yau fourfold with a smooth surface S of A_{N-1} singularities. The resulting three-dimensional theory has a $\mathcal{N} = 2$ $SU(N)$ gauge theory sector, which we obtain from a twisted dimensional reduction of a seven-dimensional $\mathcal{N} = 1$ $SU(N)$ gauge theory on the surface S . A variant of the Vafa–Witten equations governs the moduli space of the gauge theory, which—for a trivial $SU(N)$ principal bundle over S —admits a Coulomb and a Higgs branch. In M-theory these two gauge theory branches arise from a resolution and a deformation to smooth Calabi–Yau fourfolds, respectively. We find that the deformed Calabi–Yau fourfold associated to the Higgs branch requires for consistency a non-trivial four-form background flux in M-theory. The flat directions of the flux-induced superpotential are in agreement with the gauge theory prediction for the moduli space of the Higgs branch. We illustrate our findings with explicit examples that realize the Coulomb and Higgs phase transition in Calabi–Yau fourfolds embedded in weighted projective spaces. We generalize and enlarge this class of examples to Calabi–Yau fourfolds embedded in toric varieties with an A_{N-1} singularity in codimension two.

1. Introduction

The construction of gauge theories via dimensional reduction on Calabi–Yau varieties with singularities has become a powerful approach to study both supersymmetric gauge theories and moduli spaces of Calabi–Yau varieties in the vicinity of singularities. For gauge theories with eight supercharges—such as $\mathcal{N} = 2$ theories in four spacetime dimensions or $\mathcal{N} = 1$ theories in five spacetime dimensions—the interplay between Higgs and Coulomb branches of the gauge theory and the corresponding “extremal transitions” between geometric phases of singular Calabi–Yau threefolds has led to important insights into strongly coupled supersymmetric gauge theories and their moduli spaces [1–8].

While holomorphy strongly constrains supersymmetric theories with eight supercharges [9–13], it is a less powerful tool for supersymmetric theories with four supercharges [14–16], e.g., $\mathcal{N} = 1$ theories in four spacetime dimensions and $\mathcal{N} = 2$ theories in three spacetime dimensions. As a consequence the analysis of the gauge theory branches becomes more challenging but also richer. (For a proposal of a transition in a system with only two supercharges see Ref. [17].)

In this work gauge theories with four supercharges are constructed from M-theory on a Calabi–Yau fourfold. That is to say we want to make predictions regarding the relevant moduli spaces (and transitions among them) of the low-energy physics governing degrees of freedom localized near a complex codimension two singularity, which gives rise to a three-dimensional $\mathcal{N} = 2$ gauge theory, studied for instance in Refs. [18–27]. This should be contrasted with results obtained from codimension two singularities in type II string theories for theories with eight supercharges studied in Refs. [3, 4]. In both scenarios the essential idea is that such codimension two singularities are associated to non-Abelian enhanced gauge symmetry. This is understood from the duality between M-theory compactified on K3 to the heterotic string compactified on T^3 [28]. The charged degrees of freedom represent M2-branes wrapping the two-cycles whose volume vanishes in the singular limit. In the limit in which the K3 volume is large, bulk modes decouple from the low-energy dynamics of the seven-dimensional modes localized at the singular locus. The resulting gauge theory is described by a non-Abelian gauge theory with sixteen supercharges. Compactifying further on a circle shows that IIA theory near an ADE singularity exhibits enhanced gauge symmetry in six dimensions. Now the charged modes are associated to wrapped D2-branes.

In a Calabi–Yau n -fold— $n = 4$ for the M-theory compactifications to three dimensions or $n = 3$ for the IIA compactification to four dimensions—a codimension two singularity can be thought of as a face of suitable codimension in the Kähler cone, in which some number of divisors shrink to a $(n - 2)$ -dimensional locus S along which we find an ADE singularity. Deep in the interior of this face, the volume of S and those of any relevant submanifolds are large, which means we can study the low-energy dynamics by a suitably twisted dimensional reduction along S of the gauge theory from seven (respectively six) dimensions down to three (respectively four) dimensions. This will lead to a prediction for an $\mathcal{N} = 2$ gauge theory describing the low-energy dynamics near the singular locus. Kähler deformations away from the singularity will describe the Coulomb branch of this theory, while its Higgs branch, when present, will lead to a prediction for the complex structure moduli space of the related compactification on the Calabi–Yau space obtained via an extremal transition.

Compared to gauge theories with eight supercharges from singular Calabi–Yau threefolds [3, 4], constructing theories with four supercharges from Calabi–Yau fourfolds requires additional geometric data [29, 30]. Namely, in order to entirely describe the geometrically engineered gauge theory, it is necessary to specify the M-theory compactification on the Calabi–Yau fourfold together with a suitable four-form background flux. That is to say, the branches of the gauge theory are geometrically realized only if the four-form flux in the Calabi–Yau fourfold phases are specified correctly. The conditions which the four-form flux in M-theory must satisfy are rather stringent, but a large number of examples have been constructed in recent years as a by-product of the study of fluxes in F-theory [31–40], when those fluxes are being studied via M-theory. Note that the extremal transitions we are concerned with in this paper cannot be seen directly in F-theory because the Coulomb branch of the M-theory compactification does not lift

to F-theory. Nevertheless, they are an important feature of the corresponding M-theory compactification as was noticed in the above references.

For Calabi–Yau fourfolds with conifold singularities in codimension three—describing three-dimensional $\mathcal{N} = 2$ Abelian gauge theories at low energies—the role of background fluxes in the corresponding Calabi–Yau fourfold phases has been pioneered in the context of F-theory in Ref. [33] and studied in detail in Ref. [26]. In this work, we analyze the phase structure of $SU(N)$ gauge theories arising from A_{N-1} surface singularities in a Calabi–Yau fourfold. To obtain the anticipated branches of the three-dimensional $\mathcal{N} = 2$ $SU(N)$ gauge theory, we find that it is again essential to determine the correct four-form fluxes in the associated Calabi–Yau fourfold phases. In this work the focus is on $SU(N)$ gauge theories obtained from a twisted dimensional reduction on the surface S with a trivial principal $SU(N)$ bundle. Then the Coulomb branch of the gauge theory arises from the resolved Calabi–Yau fourfold phase in the absence of background flux, while the Higgs branch requires a specific non-trivial four-form flux. In this phase, the flux induces a superpotential which obstructs some of the Calabi–Yau fourfold moduli. These obstructions become essential because they allow us to identify the Higgs branch moduli space of the gauge theory with the unobstructed moduli space of the Calabi–Yau fourfold. Thus, while the four-form flux of the Higgs branch must fulfill the aforementioned consistency conditions, the flux is further constrained by the requirement that the Higgs branch moduli space arises from the flat directions of the flux-induced superpotential in the associated M-theory Calabi–Yau fourfold phase.

In order to explicitly check the anticipated interplay between phase transitions among gauge theory branches and their realizations as Calabi–Yau fourfolds, it is necessary to establish geometric tools to efficiently study the extremal transitions among the relevant Calabi–Yau geometries. The work of Mavlyutov [41,42] provides a mathematical framework to describe explicit examples, in which both the resolved and deformed Calabi–Yau fourfold phases are constructed as hypersurfaces and complete intersections in toric varieties, respectively. Analyzing this large class of examples, we demonstrate the anticipated agreement with the gauge theory predictions.

The organization of this work is as follows. In Sect. 2 we review the role of four-form background fluxes for M-theory compactified on Calabi–Yau fourfolds. In Sect. 3 we perform the twisted dimensional reduction of the $\mathcal{N} = 1$ seven-dimensional $SU(N)$ gauge theory on the surface S with trivial $SU(N)$ -principal bundle. For the resulting three-dimensional $\mathcal{N} = 2$ gauge theory, we deduce the spectrum and predict the geometry of the Coulomb and Higgs branch moduli spaces. In Sect. 4 the analysis of the gauge theory of the previous section is continued, emphasizing the M-theory compactification point of view and deducing some general geometric properties of the resolved and deformed Calabi–Yau fourfold phases. In Sect. 5 we construct two explicit examples based upon Calabi–Yau fourfold hypersurfaces with A_{N-1} surface singularities in weighted projective spaces. We construct both the resolved and the deformed Calabi–Yau fourfold phases in detail and verify the gauge theory predictions. In Sect. 6 we generalize these examples to hypersurfaces with A_{N-1} surface singularities embedded in toric varieties. For this large class of examples, we again find agreement with the gauge theory predictions. In Sect. 7 we present our conclusions.

2. M-Theory and G-Flux

The eleven-dimensional $\mathcal{N} = 1$ gravity multiplet in the supergravity limit of M-theory consists of the graviton and the anti-symmetric three-form tensor field as its bosonic

degrees of freedom. The expectation value of the field strength of the three-form tensor field is known as the four-form flux G . On a topologically non-trivial eleven-dimensional Lorentzian manifold M_{11} a consistently quantized four-form flux G fulfills the quantization condition [43]

$$\frac{G}{2\pi} + \frac{p_1(M_{11})}{4} = H^4(M_{11}, \mathbb{Z}), \tag{2.1}$$

where $p_1(M_{11})$ is the first Pontryagin class of the manifold M_{11} .¹ In this note we focus on M-theory compactifications on a compact Calabi–Yau fourfold X to three-dimensional Minkowski space $\mathbb{M}^{1,2}$. Then the quantization condition reduces to²

$$\frac{G}{2\pi} - \frac{c_2(X)}{2} \in H^4(X, \mathbb{Z}). \tag{2.2}$$

As a consequence, when the second Chern class $c_2(X)$ of the Calabi–Yau fourfold X is not divisible by 2, a consistent M-theory realization on the Calabi–Yau X requires a non-zero and half-integral background flux G . Furthermore, due to the compactness of X the Gauss law for the flux G demands the tadpole cancellation condition [29,30,43]

$$M = \frac{\chi(X)}{24} - \frac{1}{2} \int_X \frac{G}{2\pi} \wedge \frac{G}{2\pi}, \tag{2.3}$$

in terms of the Euler characteristic $\chi(X)$ of the fourfold X and an integer M which enumerates the net number of space-time filling (anti-)M2-branes. Note that the quantization condition (2.2) ensures that the right hand side of the tadpole condition is always integral [43]. In particular, a Calabi–Yau fourfold with an even second Chern class $c_2(X)$ admits an M-theory background with vanishing four-form flux G , because the evenness of $c_2(X)$ geometrically implies that the Euler characteristic $\chi(X)$ of such a Calabi–Yau fourfold X is divisible by 24, c.f., Ref. [43].

In this note we analyze the phase structure of M-theory arising from extremal transitions of Calabi–Yau fourfolds along A_{N-1} surface singularities. That is, we consider a singular Calabi–Yau fourfold X_0 with an A_{N-1} singularity along a smooth surface S , and we assume that X_0 admits a geometric transition to both a deformed Calabi–Yau fourfold X^b and a resolved Calabi–Yau fourfold X^\sharp .

In the context of M-theory compactifications extremal transitions among Calabi–Yau fourfolds are not automatically in accord with both the tadpole and the quantization condition [26,33]. Thus for a M-theory transition between X^b and X^\sharp we must specify the number of space-time filling M2-branes M^\sharp and M^b and the background four-form fluxes G^\sharp and G^b in the respective Calabi–Yau fourfold phases. As in Ref. [26], we consider phase transitions with a constant number of space-time filling M2-branes, i.e.,

$$M^\sharp = M^b, \tag{2.4}$$

which we assume are located far from the transition. With this assumption the physics of the transition is governed by the degrees of freedom arising in the vicinity of the surface

¹ For ease of notation we use the letter G for both the four-form flux and its cohomological representative.

² For complex manifolds X the first Pontryagin class is given by $p_1(X) = -c_2(TX \otimes \mathbb{C})$ in terms of the complexified tangent bundle $TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{\bar{1},0}X$ so that $p_1(X) = c_1(X)^2 - 2c_2(X)$ (with $c_1(X) = 0$ for Calabi–Yau fourfolds X).

singularity of the Calabi–Yau fourfold X_0 , and the tadpole cancellation condition yields the transition condition

$$\frac{\chi(X^b)}{24} - \frac{\chi(X^\sharp)}{24} = \frac{1}{2} \int_{X^b} \frac{G^b}{2\pi} \wedge \frac{G^b}{2\pi} - \frac{1}{2} \int_{X^\sharp} \frac{G^\sharp}{2\pi} \wedge \frac{G^\sharp}{2\pi}, \tag{2.5}$$

where the left-hand side is solely determined by the topological change between the transition fourfolds X^b and X^\sharp .

Choosing the right-hand side of (2.5)—combined with the quantization condition (2.2)—ensures that a M-theory phase transition is in accord with known anomaly cancellation conditions. It, however, does not guarantee that a transition can actually occur dynamically. As the M-theory background fluxes generate a flux-induced superpotential W and/or a twisted superpotentials \tilde{W} [44]

$$W = \int_X \Omega \wedge \frac{G}{2\pi}, \quad \tilde{W} = \int_X J \wedge J \wedge \frac{G}{2\pi}, \tag{2.6}$$

an unobstructed extremal transition in M-theory is only realized along a flat direction of the flux-induced scalar potential V , which is a function of these flux-induced superpotentials. Here Ω is the holomorphic four form and J is the Kähler form of the Calabi–Yau fourfold X .

A simple solution for a dynamical M-theory transition is realized by a vanishing flux G^\sharp and a non-vanishing primitive flux G^b , i.e.,

$$G^b \neq 0 \quad \text{with} \quad G^b \wedge J = 0, \quad G^\sharp = 0, \tag{2.7}$$

provided that the quantization condition (2.2) for both X^\sharp and X^b as well as the tadpole relation (2.5) are met. This solution is of particular importance to us, as it geometrically realizes the Coulomb–Higgs gauge theory transitions that we focus on in this work. On the one hand—due to $G^\sharp = 0$ —none of the geometric M-theory moduli are obstructed in the resolved Calabi–Yau fourfold X^\sharp . On the other hand, there is a flux-induced superpotential

$$W^b = \int_{X^b} \Omega \wedge \frac{G^b}{2\pi}, \tag{2.8}$$

which generates a potential for some of the complex structure moduli fields in the deformed Calabi–Yau fourfold X^b . At low energies the massive modes of the obstructed complex structure moduli are integrated out and a flux-restricted complex structure moduli space $\mathcal{M}_{\text{cs}}^b(G^b) \subset \mathcal{M}_{\text{cs}}^b$ remains. Geometrically, we can think of the unobstructed complex structure moduli as those complex structure deformation, under which the flux G^b remains of Hodge type $(2, 2)$, whereas the Hodge structure of G^b varies with respect to the obstructed complex structure moduli. Therefore, it is the flux-restricted moduli space $\mathcal{M}_{\text{cs}}^b(G^b)$ that yields the flat directions of the superpotential W^b and should thus be compared to the moduli space in the effective gauge theory description at low energies. Note that the Kähler moduli of the Calabi–Yau fourfold X^b nevertheless remain unobstructed because by the primitivity assumption of the flux G^b no twisted superpotential is generated.

3. Field Theory Analysis

In this section we discuss the predictions one obtains regarding the relevant moduli spaces from our understanding of the low-energy physics governing degrees of freedom localized near a (complex) codimension two singularity in M-theory. We start with a brief review of the results of Refs. [3,4] on codimension two singularities in type II string theory, and proceed to contrast this with the situation in M-theory.

3.1. Type IIA string theory on a Calabi–Yau threefold. The moduli space of type II compactifications on X^\sharp is identified with the moduli space of Calabi–Yau metrics and closed antisymmetric tensor fields on X^\sharp . Following Ref. [3] we discuss the IIA string in the vicinity of a face of the Kähler cone at which divisors are contracted to a smooth curve C of A_{N-1} singularities of genus g ; the superconformal field theory will be singular for suitably tuned B -field.

The transverse A_{N-1} singularity is resolved by blowing up along C . The vanishing cycles are described by a chain of $N - 1$ two-spheres Γ_i in this space, with their intersection matrix corresponding to the Dynkin diagram of A_{N-1} . As we move about C these spheres sweep out $N - 1$ divisors E_i on M . In homology there are then $(N - 1)$ shrinking two-cycles Γ_i , and $N - 1$ shrinking four-cycles E_i . The light soliton states are given by D2-branes wrapping chains of the form $\Gamma_i \cup \Gamma_{i+1} \cup \dots \cup \Gamma_j$ (with both orientations) and under the Ramond–Ramond gauge symmetry associated to E_i their charges fill out the roots of A_{N-1} .

In the limit in which C is large (deep in the associated face) we can approximately think of the low-energy theory as a twisted compactification on C of the six-dimensional theory obtained by including the massless solitons.

In flat space the six-dimensional theory contains a vector V_M , two complex scalars ϕ , and two fermions, all in the adjoint representation of $SU(N)$. The charged components are the soliton states; the neutral components are supplied by the moduli of the ALE space. The fields transform under a global $SU(2) \times SU(2)$ R -symmetry. The compactification breaks the local Lorentz group as $SO(6) \rightarrow SO(4) \times U(1) \sim SU(2) \times SU(2) \times U(1)$, and the requisite spin was determined in Ref. [3] to be the identification of

$$J' = J_L - J_3^{(1)} - J_3^{(2)}, \tag{3.1}$$

as the generator of rotations in tangent space TC , where J_L is the “standard” Lorentz generator, and the other two correspond to the Cartan elements of the $SU(2)$ factors. In four dimensions this leads to a theory with $\mathcal{N} = 2$ supersymmetry, $SU(N)$ gauge symmetry, and g massless hypermultiplets in the adjoint representation of the gauge group. For $g > 1$ the theory is IR free and a classical description is reliable in the vicinity of the singular locus; for a discussion of the special cases $g \leq 1$, c.f., Ref. [3].

The local structure of the moduli space near the singularity is modeled—deep in the cone—by the structure of the space of vacua of this gauge theory, which leads to the following predictions:

Kähler deformations away from the singular locus (and the associated B -fields) parameterize the Coulomb branch of the theory, along which the scalar ϕ in the vector multiplet acquires an expectation value, constrained by the potential to be diagonalizable by a gauge transformation. The eigenvalues of ϕ , subject to the tracelessness condition, form coordinates on a S_N cover of the Coulomb branch, on which the Weyl group $\mathcal{W}(SU(N)) \simeq S_N$ acts via the $N - 1$ dimensional representation. At generic

points—corresponding to smooth X^\sharp —the unbroken gauge symmetry is $U(1)^{N-1}$, and the massless matter comprises $g(N - 1)$ neutral hypermultiplets. The Weyl group acts on these as well via g copies of the same representation, so locally the moduli space of X^\sharp is a *quotient* of a product of special Kähler manifolds.

In addition, the gauge theory has a Higgs branch in which the hypermultiplets acquire nonzero expectation values. In terms of the compactification on X^\sharp this is the condensation of solitonic states [2]. This describes the deformations of X^b smoothing the singularity. At generic points on this branch the gauge group is completely broken and the (quaternionic) dimension of the Higgs branch is

$$\dim_{\mathbb{H}} \mathcal{H} = (g - 1)(N^2 - 1). \tag{3.2}$$

The Hodge numbers of the two spaces are thus related by

$$h^{1,1}(X^b) = h^{1,1}(X^\sharp) - (N - 1), \quad h^{2,1}(X^b) = h^{2,1}(X^\sharp) + (g - 1)(N^2 - 1) - g(N - 1), \tag{3.3}$$

where $g(N - 1)$ is subtracted in the last line of (3.3) since these moduli already appear on the Coulomb branch, as we recall below.

We can somewhat refine this prediction. There are special submanifolds on the Coulomb branch (meeting at the origin) along which non-Abelian subgroups of $SU(N)$ are unbroken. These correspond exactly to fixed point sets of the S_N action, where eigenvalues of ϕ coincide. In general an unbroken symmetry

$$SU(k_1) \times \cdots \times SU(k_p) \times U(1)^{p-1} \tag{3.4}$$

where $\sum k_i = N$, $k_i \geq 1$ (and factors of $SU(1)$ are simply to be ignored) will occur in codimension $N - p$, and there will be g massless hypermultiplets in the adjoint representation of the unbroken group. We can allow these to acquire expectation values, breaking the non-Abelian part completely and leading to a mixed branch $\mathcal{H}_{(k_1, \dots, k_p)}$ with the Higgs component having dimension

$$\dim_{\mathbb{H}} \mathcal{H}_{(k_1, \dots, k_p)} = (g - 1) \sum_{i=1}^p (k_i^2 - 1) + g(p - 1). \tag{3.5}$$

We can also see the transition to the Higgs branch along a different path which will prove more transparent in the geometrical analysis. At a generic point on the Coulomb branch, we can turn on expectation values for the $g(N - 1)$ neutral hypermultiplet scalars.³ Then, as we tune ϕ to zero, the non-Abelian symmetry is not restored, and the gauge symmetry remains $U(1)^{N-1}$. The theory is still IR free and we can use classical analysis. The hypermultiplet expectation values lead to masses for the off-diagonal components of rank $N(N - 1)$. Thus there are at the singular point an additional $(g - 1)N(N - 1)$ charged hypermultiplets with the $U(1)$ charges of $g - 1$ adjoints. When these acquire generic expectation values the gauge symmetry is Higgsed leading back to the Higgs branch dimension in Eq. (3.2). In other words, we can rewrite the second equation in (3.3) as

$$h^{2,1}(X^b) = h^{2,1}(X^\sharp) + (g - 1)(N^2 - N) - (N - 1), \tag{3.6}$$

understanding the additional moduli as arising from Higgsing the $U(1)^{N-1}$ under which the $(g - 1)N(N - 1)$ hypermultiplets are charged.

³ These are the moduli that we alluded to immediately after Eq. (3.3).

3.2. *M-theory on a Calabi–Yau fourfold.* The discrete choices determining a compactification of M-theory to three dimensions include, as discussed above, a choice of a topological type for the Calabi–Yau fourfold X as well as a choice of the four-form flux G and the number M of M2-branes, satisfying the conditions (2.2) and (2.3). Given such a choice, the moduli space is determined by the subspace of the space of Calabi–Yau metrics on X for which the chosen four-form flux G is of Hodge type $(2, 2)$ as well as primitive. There are additional moduli associated to periods of the three-form A_3 —i.e., $h^{2,1}(X)$ of these—as well as to the positions of the M2-branes.

As above we wish to consider a face of the Kähler cone of a Calabi–Yau fourfold X^\sharp at which a divisor contracts to a smooth surface S of transverse A_{N-1} singularities. As above, this is resolved by blowing up $N - 1$ times along S producing $N - 1$ exceptional divisors E_i and the vanishing cycles are the E_i as well as $N - 1$ two-cycles Γ_i . We will assume here that we can make a choice of flux on X^\sharp such that generic points in the vicinity of this face correspond to Kähler classes for which the flux is primitive. This means that there are M-theory vacua associated to a smooth X^\sharp in which the singularities have been resolved. This implies that the flux, if nonzero, is primitive for smooth X^\sharp , meaning that—if J^i are the $(1, 1)$ cohomology classes dual to E_i —then $G^\sharp \wedge J^i = 0$ for all i . In the cases we consider here this very restrictive condition will be met by setting $G^\sharp = 0$. Further, we assume that the positions of the M2-branes are all far from the contracting divisors, so that the worldvolume degrees of freedom decouple from the low-energy theory of the modes at the singularity.

In this situation, we can perform a calculation of the low-energy theory in the vicinity of the singular locus along similar lines to those followed above. In a suitable region (near a point deep in the face of the Kähler cone) bulk modes decouple from the dynamics near the singular locus and the low-energy dynamics is given by a $\mathcal{N} = 1$ seven-dimensional supersymmetric Yang–Mills theory on $M^{1,2} \times \mathbb{R}^4$, in which the charged modes are excitations of membranes wrapping vanishing cycles and the neutral modes describe the moduli. We then want to perform a suitably twisted dimensional reduction of this seven-dimensional theory, in which we replace \mathbb{R}^4 by a compact Kähler manifold S , such that we obtain an $\mathcal{N} = 2$ theory in three dimensions.

The seven-dimensional supersymmetric Yang–Mills theory is obtained by dimensional reduction from ten dimensions and is given in Ref. [45]. It has a global $SU(2)_R$ R-symmetry, and the fields are a gauge field A_M , a triplet S_i of scalars, and a doublet Ψ_α of gaugini satisfying the symplectic Majorana condition

$$\Psi_\alpha = \epsilon_{\alpha\beta} B \Psi^{\beta*}, \tag{3.7}$$

where the complex conjugation matrix B satisfies

$$B^{-1} \Gamma^M B = \Gamma^{M*} \quad B B^* = -\text{id}. \tag{3.8}$$

All the fields transform in the adjoint representation of the gauge group.

As noted above, the charged fields correspond to excitations of M2-branes wrapping the collapsing cycles, while the Cartan elements are associated to moduli of the compactification on X^\sharp . The gauge field A is associated to periods of the three-form field A_3 along the vanishing cycles; the scalars S_i are moduli of the metric resolving the singularity.

The supersymmetry variations are parameterized by a symplectic Majorana spinor doublet ϵ_I and the relevant one for us is [45]

$$\delta \Psi_\alpha = -\frac{1}{4} F_{MN} \Gamma^{MN} \epsilon_\alpha + \frac{i}{2} \Gamma^M D_M \left(S_i \sigma^i \right)_\alpha{}^\beta \epsilon_\beta + \frac{1}{4} \epsilon^{ijk} [S_i, S_j] (\sigma_k)_\alpha{}^\beta \epsilon_\beta. \tag{3.9}$$

3.2.1. *Twisted dimensional reduction.* When we dimensionally reduce, the Lorentz group is reduced $SO(1, 6) \times SU(2)_R \rightarrow SO(1, 2) \times SO(4) \times SU(2)_R$ with the $SO(4)$ eventually broken by the curvature of S . The representations in which the fields transform reduce as

$$\begin{aligned} A &: (\mathbf{7}, \mathbf{1}) \rightarrow (\mathbf{3}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4}_v, \mathbf{1}), \\ S &: (\mathbf{1}, \mathbf{3}) \rightarrow (\mathbf{1}, \mathbf{1}, \mathbf{3}), \\ \Psi &: (\mathbf{8}, \mathbf{2}) \rightarrow (\mathbf{2}, \mathbf{4}_s, \mathbf{2}). \end{aligned} \tag{3.10}$$

Since S is Kähler, the structure group is in fact $U(2) \sim SU(2)_L \times U(1)_I$, under which

$$\mathbf{4}_v \rightarrow \mathbf{2}_1 \oplus \mathbf{2}_{-1}, \quad \mathbf{4}_s \rightarrow \mathbf{2}_0 \oplus \mathbf{1}_1 \oplus \mathbf{1}_{-1}. \tag{3.11}$$

In a twisted reduction, we will replace $U(1)_I$ generated by J_I with $U(1)'$ generated by the linear combination $J' = J_I + 2J_R^3$. The curvature of S will then couple to the twisted $U(2)$ and the unbroken global symmetry will be $U(1)_R$ generated by $2J_I^3$. Under $SO(1, 2) \times SU(2)_L \times U(1)' \times U(1)_R$ we have the decompositions

$$\begin{aligned} (\mathbf{7}, \mathbf{1}) &\rightarrow (\mathbf{3}, \mathbf{1})_{0,0} \oplus (\mathbf{1}, \mathbf{2})_{1,1} \oplus (\mathbf{1}, \mathbf{2})_{-1,-1}, \\ (\mathbf{1}, \mathbf{3}) &\rightarrow (\mathbf{1}, \mathbf{1})_{2,2} \oplus (\mathbf{1}, \mathbf{1})_{-2,-2} \oplus (\mathbf{1}, \mathbf{1})_{0,0}, \\ (\mathbf{8}, \mathbf{2}) &\rightarrow (\mathbf{2}, \mathbf{2})_{1,1} \oplus (\mathbf{2}, \mathbf{2})_{-1,-1} \oplus (\mathbf{2}, \mathbf{1})_{2,1} \oplus (\mathbf{2}, \mathbf{1})_{0,-1} \oplus (\mathbf{2}, \mathbf{1})_{0,1} \oplus (\mathbf{2}, \mathbf{1})_{-2,-1}. \end{aligned} \tag{3.12}$$

We identify the corresponding modes of the fields by their transformation under $SU(2)_L \times U(1)'$ as

$$A_M \rightarrow A_\mu, \bar{A}_{\bar{m}}, A_m, \quad S_i \rightarrow \mathfrak{q}, \bar{\mathfrak{q}}, \quad \Psi \rightarrow \bar{\psi}_{\bar{m}}, \psi_m, \chi, \lambda_-, \lambda_+, \bar{\chi}. \tag{3.13}$$

Our model for the local moduli space will be the space of supersymmetric vacua of this theory. Following Ref. [46], we will construct this by evaluating the supersymmetry variation of Ψ_+ under the two unbroken supersymmetries. Setting this to zero yields a slight modification of the Vafa–Witten equations [47]

$$\begin{aligned} F^{(2,0)} = 0, \quad J \wedge F^{(1,1)} + [\mathfrak{q}, \bar{\mathfrak{q}}] = 0, \quad [\mathfrak{q}, \Phi] = 0, \\ D\Phi = \bar{D}\Phi = 0, \quad \bar{D}\mathfrak{q} = 0; \end{aligned} \tag{3.14}$$

setting the variation of Ψ_- to zero yields the complex conjugate equations by the symplectic Majorana condition.

3.2.2. *Predictions for the moduli space.* Solutions to these equations provide our predictions for the local structure of the moduli space. Clearly the space of solutions breaks up into disjoint components labeled by the Chern classes of the flux F . In the situation we are describing, in which the generic point in the space of compactifications on X^\sharp near the singular locus is smooth, the bundle we obtain will be flat. The charged components of the curvature are certainly zero away from the singular locus since they are carried by wrapped branes which become massive; the neutral components can be described as the integrals of G^\sharp over the fibers Γ_i , which we are assuming vanish.

Three dimensional $\mathcal{N} = 2$ gauge theories contain additional discrete parameters, supersymmetric Chern–Simons couplings for the gauge fields. In general, the effective

gauge theory describing the low-energy physics near a singularity will have nonzero Chern–Simons couplings and these can have different values between the Coulomb and Higgs vacua [26]. In the cases we discuss here, these couplings vanish on the Coulomb branch and the fact that all of the chiral multiplets are in real representations means this will also be the case on the Higgs branch.

Linearizing about a trivial $SU(N)$ principal bundle, the modes of $(A_\mu, \lambda_\pm, \Phi)$ form vector multiplets with masses associated to eigenvalues of the Laplacian on S . The modes of (A_m, ψ_m) form chiral multiplets with masses associated to eigenvalues of the Laplacian on $(0, 1)$ -forms on S . The modes of (ϱ, χ) form chiral multiplets with masses associated to eigenvalues of the Laplacian on $(2, 0)$ -forms on S . The massless modes will thus be $h^0(S) = 1$ vector multiplet and $h^{0,1}(S) + h^{2,0}(S)$ chiral multiplets, all in the adjoint representation of the gauge group. For the multiplicities of the chiral multiplets, we will also use the irregularity q and the geometric genus p_g , which are respectively the conventional birational invariants for the dimensions $h^{1,0}(S) = h^{0,1}(S)$ and $h^{2,0}(S) = h^0(K_S)$ of the algebraic surface S .

We can then write the low-lying excitations in terms of a basis e_i for $H^{0,1}(S)$, a basis E_A for $H^0(K_S)$ and the dual basis E^B for $H^0(\overline{K}_S)$, i.e.,

$$\Phi = \phi, \quad A_m = \sum_i a^i e_i, \quad \varrho = \sum_A \varrho^A E_A, \quad \overline{\varrho} = \sum_B \overline{\varrho}_B \overline{E}^B, \quad (3.15)$$

with the three-dimensional fields, $\phi, a^i, \varrho_A, \overline{\varrho}^B$, taking values in the Lie algebra and ϕ is real. In terms of these the conditions for unbroken supersymmetry reduce to

$$\begin{aligned} [a^i, \phi] = 0, \quad \sum_{j,A} C_{ijA} [a^j, \varrho^A] = 0, \quad [\varrho^A, \phi] = 0, \\ [\overline{\varrho}_B, \phi] = 0, \quad \sum_A [\varrho^A, \overline{\varrho}_A] = 0, \end{aligned} \quad (3.16)$$

where

$$C_{ijA} = \int_S e_i \wedge e_j \wedge E. \quad (3.17)$$

The superpotential that leads to these equations has the form

$$W = \sum_{i,j,A} C_{ijA} \text{Tr} \left([a^i, a^j] \varrho^A \right). \quad (3.18)$$

In the following we restrict our analysis to surfaces S with $q = 0$ and $p_g \geq 1$. As we will see in the following, this assumption ensures that there are no non-perturbative corrections to the Coulomb branch of the gauge theory.

The model then has a Coulomb branch along which the real scalar ϕ acquires an expectation value. By a gauge transformation this can be taken to lie in the Cartan algebra. As in the previous subsection we can use the first $N - 1$ eigenvalues as coordinates, and the Weyl group $\mathcal{W}(SU(N)) \simeq S_N$ acts on these. At generic points the gauge symmetry is $U(1)^{N-1}$. In fact [27] the Coulomb branch is complex Kähler. The periods of the dual six-form A_6 along E_i are neutral scalars dual to the neutral gauge bosons, and they combine with ϕ_i to form holomorphic coordinates. At a generic point the massless modes of ϱ^A are those commuting with ϕ so we have $p_g(N - 1)$ neutral massless chiral multiplets. Again, the Weyl group S_N acts on these and the moduli space is a quotient.

There is another branch of the moduli space, along which \mathfrak{q}^A acquires a nonzero expectation value, completely breaking the gauge group and accordingly ϕ becomes massive. The last equation of Eqs. (3.14) is the moment map for the adjoint action of the gauge group leading as usual to a Higgs branch of complex dimension

$$\dim_{\mathbb{C}} \mathcal{H} = (p_g - 1)(N^2 - 1). \tag{3.19}$$

This (Higgs) branch is interpreted as a local model for the moduli space of the compactification on X^b , but we will in general find nonzero flux. Thus (3.19) does not lead directly to a prediction for the deformation space of the moduli of X^b .

As above, we have the more refined picture of the way these branches intersect at singular loci. At the codimension $N - p$ locus in the Coulomb branch along which the unbroken gauge group is given by Eq. (3.4) we will have p_g massless chiral multiplets in the adjoint representation. Turning on generic expectation values for \mathfrak{q}^A breaks the non-Abelian part completely and leads to a mixed branch $\mathcal{H}_{(k_1, \dots, k_p)}$ with the Higgs component having dimension

$$\dim_{\mathbb{C}} \mathcal{H}_{(k_1, \dots, k_p)} = (p_g - 1) \sum_{i=1}^p (k_i^2 - 1) + p_g(p - 1). \tag{3.20}$$

As in the previous subsection, there is another path in moduli space implementing the transition from the Coulomb branch to the Higgs branch. At a generic point on the Coulomb branch we can turn on expectation values for the $p_g(N - 1)$ neutral chiral fields. Then, as we tune ϕ to zero the gauge symmetry remains $U(1)^{N-1}$. The D-term condition—implementing the third equation of Eq. (3.14)—then leads to a mass term for the charged chiral fields leaving $(p_g - 1)N(N - 1)$ charged fields with the charges of $p_g - 1$ adjoints, leading as was found in Ref. [26] to a Higgs branch along which the gauge symmetry is completely broken and whose dimension agrees with our calculation above.

3.2.3. Quantum corrections and region of validity. Our discussion above has been entirely classical, and we need to address the degree to which quantum corrections might invalidate our conclusions. We are using the effective three dimensional field theory to make predictions about the moduli space of M-theory compactifications, and relating this to the moduli space of Kähler and complex structure deformations of the Calabi–Yau fourfolds X^a and X^b , respectively. We thus need to check separately the degree to which our description of the space of vacua is subject to corrections from non-trivial low-energy dynamics in the three-dimensional gauge theory, and the degree to which the geometric moduli space agrees with the space of M-theory compactifications. In the examples of Sect. 3.1 the four dimensional gauge theory was (for $g > 1$) IR free and semiclassical considerations provided an accurate description of the moduli space near the origin. The Coulomb branch suffered no string corrections, and α' corrections to the metric were computable using mirror symmetry or by taking advantage of special geometry to relate them to the holomorphic prepotential; the relevant parts of this, deep in the singular cone, were explicitly computed in Ref. [3]. On the Higgs branch, α' corrections were absent and non-perturbative string effects were suppressed deep in the cone.

In the case at hand, the first new phenomenon is that the encountered three-dimensional gauge theories are strongly coupled at low energy. The low-energy dynamics of $\mathcal{N} = 2$

gauge theories in three dimensions has been studied, for example in Refs. [18–27]. At the origin, we find non-trivial interacting superconformal field theories for $p_g > 1$. For $p_g = 1$ the low-energy supersymmetry is enhanced to $\mathcal{N} = 4$. The metric on the Coulomb branch is subject to both perturbative and non-perturbative corrections. For $p_g \geq 1$ the singularity at the origin of the Coulomb branch is unchanged by these, and the Higgs branch has the predicted singular structure [20].

The moduli space of M-theory compactifications on X is the space of Calabi–Yau metrics and A_3 periods subject to the conditions imposed by G -flux. The metric is subject to corrections but the superpotential is corrected only by five-brane instantons [48]. Deep in the cone we expect the contributions of these to be suppressed, with the exception of the contributions of five-branes wrapping the vanishing divisors E_i in the case of the Calabi–Yau fourfold X^\sharp . In the absence of background flux G^\sharp , these can contribute to the superpotential only if $\chi(E_i, \mathcal{O}_{E_i}) = 1$. But in the case at hand, the divisors E_i are \mathbb{P}^1 -fibrations over the surface S , and therefore we find $\chi(E_i, \mathcal{O}_{E_i}) = p_g - q + 1$. In the cases studied here, where $q = 0$ and $p_g > 0$, these instantons cannot contribute to the superpotential. Therefore, the classical description of the moduli space remains valid. The absence of a non-perturbative superpotential can be taken as another confirmed prediction of our identification of the gauge theory at the singularity.⁴

4. General Features of $SU(N)$ Models

In this section we describe some general geometric properties of the $SU(N)$ gauge theories studied in the previous section. In particular, we establish how M-theory compactified on Calabi–Yau fourfolds realizes the phase structure of the $SU(N)$ gauge theory.

4.1. Gauge theories from surface singularities in Calabi–Yau fourfolds. To realize geometrically the twisted dimensional reduction along the surface S , let us consider a Calabi–Yau fourfold X_0 with a smooth surface S of A_{N-1} singularities. We assume for simplicity of exposition that a tubular neighborhood of the surface S in X_0 is given by the hypersurface equation

$$xy = z^N, \tag{4.1}$$

in the total space of the bundle $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{K}_S$ with x, y and z sections of the bundles $\mathcal{L}_1, \mathcal{L}_2$ and the canonical line bundle \mathcal{K}_S , respectively. We further assume that the canonical line bundle \mathcal{K}_S is sufficiently ample.

M-theory compactified on the singular Calabi–Yau fourfold X_0 yields the twisted dimensional reduction along S studied in Sect. 3.2.1. That is to say that the eleven-dimensional supercharge Q_{11} dimensionally reduces on the surface S to the seven-dimensional supercharge Q_7 , which—due to the origin of S as a subvariety in the ambient space X_0 with trivial canonical class—becomes a section of the canonical spin^c bundle \mathcal{S}_S^c . This canonical spin^c bundle arises from a spin^c structure on S associated to a spin structure on $TS \oplus \mathcal{K}_S$. Therefore—assuming first that the surface S has a spin structure—the twisted dimensional reduction along S amounts to tensoring the spin bundle \mathcal{S}_S of the surface S with $\mathcal{K}_S^{1/2}$, namely

$$\mathcal{S}_S \xrightarrow[\text{twist}]{\otimes \mathcal{K}_S^{1/2}} \mathcal{S}_S \otimes \mathcal{K}_S^{1/2} = \mathcal{S}_S^c. \tag{4.2}$$

⁴ See Ref. [21], for a calculation of the non-perturbative superpotential in a situation where it is nonzero.

But even if the surface S is not spin—i.e., both the spin bundle \mathcal{S}_S and the square root of $K_S^{1/2}$ are simultaneously ill-defined—we can still formally perform the twist by tensoring with $K_S^{1/2}$, because there exists always a canonical spin^c structure on S such that the resulting spin^c bundle \mathcal{S}_S^c is well-defined.⁵

Since the twist acts on the supercharges, which generate the resulting spectrum of three-dimensional $\mathcal{N} = 2$ supermultiplets, tensoring with $K_S^{1/2}$ realizes the twist of J_I as in Sect. 3.2.1. This yields geometrically the previously determined three-dimensional $\mathcal{N} = 2$ supersymmetric Yang–Mills spectrum of a single vector multiplet and $p_g + q$ chiral matter multiplets in the adjoint representation of $SU(N)$.

4.2. The change in topology for the Coulomb–Higgs phase transition. In the M-theory compactification the Coulomb branch of the gauge theory realizes a crepant resolution of the singular Calabi–Yau fourfold X_0 to the resolved Calabi–Yau fourfold X^\sharp . This amounts to replacing the A_{N-1} surface singularity along S in X_0 by a chain of N \mathbb{P}^1 -bundles over S .

The Higgs branch of the gauge theory describes deformations of the singular Calabi–Yau fourfold X_0 to the Calabi–Yau fourfold X^b , locally given by the deformed hyper-surface equation

$$xy = z^N + \sum_{j=0}^N \omega_{N-j} z^j. \tag{4.3}$$

Here ω_j are sections of the pluri-canonical line bundles jK_S .

To determine the change in Euler characteristic for the transition, we compare the smooth Calabi–Yau fourfold X^\sharp to the smooth Calabi–Yau fourfold X^b . Let us specialize to the fourfold X^b arising from $\omega_j = 0$ for $j < N$ but with a generic section ω_N of NK_S , such that the Eq. (4.3) becomes $xy = z^N + \omega_N$. Then the A_{N-1} surface singularity in X^b is replaced by a bundle of a bouquet $S^2 \vee \dots \vee S^2$ of $N - 1$ two-spheres collapsing over the curve $\mathcal{C} \subset S$, where \mathcal{C} is the vanishing locus of ω_N . We assume that for the generic choice of ω_N the curve \mathcal{C} is smooth.⁶ Thus, as both smooth Calabi–Yau fourfold phases X^\sharp and X^b arise from topological fibrations of a bouquet of $N - 1$ two-spheres over $S \setminus \mathcal{C}$, the change in Euler characteristic between X^\sharp and X^b is determined by the difference in Euler characteristic of the fibrations along the curve \mathcal{C} . The Euler characteristic of the bouquet of N two-spheres fibered over \mathcal{C} in X^\sharp becomes $N \cdot \chi(\mathcal{C})$ and compares to the Euler characteristic $1 \cdot \chi(\mathcal{C})$ of the collapsed fibers over \mathcal{C} in X^b , so that

$$\chi(X^\sharp) - \chi(X^b) = (N - 1)\chi(\mathcal{C}). \tag{4.4}$$

The Euler characteristic $\chi(\mathcal{C})$ in turn is minus the degree of the canonical bundle $K_{\mathcal{C}}$ of \mathcal{C} , which by the adjunction formula is computed to be $K_{\mathcal{C}} = (N + 1)K_S|_{\mathcal{C}}$. Therefore, the canonical bundle $K_{\mathcal{C}}$ has degree $(N + 1)N K_S^2$, and we arrive with Eq. (4.4) at

$$\chi(X^b) - \chi(X^\sharp) = N(N - 1)(N + 1)K_S^2. \tag{4.5}$$

⁵ A non-vanishing second Stiefel–Whitney class $w_2(S) \in H^2(S, \mathbb{Z}_2)$ is the obstruction to the existence of a spin structure on S . As $w_2(S) = w_2(K_S)$, it is also the obstruction to the existence of a square root of the canonical line bundle. Therefore, we have $w_2(TS \oplus K_S) = w_2(S) + w_2(K_S) = 0$, which implies the existence of a canonical spin^c structure on S .

⁶ If the generic curve \mathcal{C} is not smooth, additional massless matter fields are present in the gauge theory spectrum.

This argument generalizes from the A_{N-1} case to Calabi–Yau fourfolds X_0 with a smooth surface S of ADE singularities. Then the curve $\mathcal{C} \subset S$ becomes the vanishing locus of a section of the line bundle $h_G K_S$ in terms of the dual Coxeter number h_G of the ADE group G , and we obtain

$$\chi(X^b) - \chi(X^\sharp) = r_G h_G (h_G + 1) K_S^2, \tag{4.6}$$

where r_G is the rank of the group G .

4.3. The gauge theory and geometric moduli space. Before we study explicit examples in the next section, we make some further general remarks about the relationship between the gauge theory and the geometric moduli spaces.

In the Coulomb branch of the gauge theory the adjoint-valued scalar field ϕ of the vector multiplet acquires an expectation value in the Cartan subalgebra

$$\langle \phi \rangle = \text{Diag}(\phi_1, \dots, \phi_N), \quad \phi_1 + \dots + \phi_N = 0, \tag{4.7}$$

which generically breaks the gauge group $SU(N)$ to its maximal Abelian subgroup $U(1)^{N-1}$. The Weyl group $\mathcal{W}(SU(N)) \simeq S_N$ of $SU(N)$ permutes the expectation values ϕ_j , $j = 1, \dots, N$. Hence, we can view the expectation values ϕ_j as coordinates on the S_N -covering space of the $N - 1$ -dimensional moduli space of the Coulomb branch. To describe the Coulomb moduli space itself—and not its $N!$ -fold cover—we pick in the Weyl orbit of diagonal expectation values (4.7) a representative obeying

$$\phi_1 \geq \phi_2 \geq \dots \geq \phi_N. \tag{4.8}$$

In the Calabi–Yau fourfold X^\sharp the non-negative differences $J^j = \phi_j - \phi_{j+1}$, $j = 1, \dots, N - 1$, become Kähler coordinates for the $N - 1$ exceptional divisors in the chain of $N - 1$ \mathbb{P}^1 fibrations over the surface S . If any two expectation values in $\langle \phi \rangle$ coincide, the representative (4.8) of the Weyl orbit ceases to be unique and the gauge group is not entirely broken to the maximal Abelian subgroup. Geometrically, some of the Kähler moduli J^j vanish, and hence we are on the boundary of the Kähler cone. This means that the A_{N-1} surface singularity is not entirely resolved in the Calabi–Yau fourfold X_0 , which geometrically reflects that the gauge group is only partially broken to a group properly containing its maximal Abelian subgroup.

In the Higgs branch of the gauge theory it is the adjoint-valued matter fields \mathfrak{q} that acquire an expectation value. The expectation values of the matter fields \mathfrak{q} are adjoint-valued sections of $H^0(S, K_S)$ and deform the ADE surface singularity (4.1). The deformations are governed by invariant theory of the $SU(N)$ gauge group and take the form

$$xy = \det(z \cdot I_N + M). \tag{4.9}$$

Here M is a traceless $N \times N$ matrix whose entries are sections of the canonical line bundle K_S . The deformed hypersurface Eq. (4.9) is manifestly Weyl invariant, as the Weyl group S_N acts on \mathfrak{q} by conjugation with permutation matrices

$$\sigma \in S_N : q \mapsto P_\sigma^\dagger q P_\sigma. \tag{4.10}$$

We observe that—in agreement with the gauge theory prediction for the Higgs branch in Sect. 3.2.2—the invariant deformations (4.9) parametrize a $(N^2 - 1) \cdot (p_g - 1)$ -dimensional subspace in the space of all hypersurface deformation (4.3). In the M-theory

compactification on the Calabi–Yau fourfold X^b , the gauge invariant deformations (4.9) become the flat directions of the superpotential (2.8) arising from a suitable four-form flux G^b . The flux G^b is required to be primitive, to fulfill the quantization condition (2.2), and to accommodate for the tadpole cancellation condition (2.5), which implies together with Eq. (4.5) that

$$\frac{1}{2} \int_{X^b} \frac{G^b}{2\pi} \wedge \frac{G^b}{2\pi} = \frac{1}{24} N(N-1)(N+1) K_S^2. \tag{4.11}$$

We now claim that in the neighborhood where the Calabi–Yau fourfold X^b is described in terms of the hypersurface Eq. (4.9) the flux G^b is locally given by

$$\frac{G^b}{2\pi} = \frac{N-1}{2} R - T. \tag{4.12}$$

Here R is the two-dimensional algebraic cycle arising from the intersection

$$R : x = z = 0, \tag{4.13}$$

while T is the two-dimensional algebraic cycle given by

$$T : x = 0, \text{ rank } S \leq N - 2, \tag{4.14}$$

in terms of the $N \times (N-1)$ submatrix S of the $N \times N$ matrix $z \cdot I_N + M$ obtained by deleting the last column. Note that the construction of the submatrix S is not gauge invariant, as gauge transformations act upon the matrix $z \cdot I_n + M$ by conjugation. As a matter of fact there is a whole Weyl orbit of algebraic cycles R obtained from conjugation by permutation matrices according to Eq. (4.10), which give rise to equivalent flux-restricted moduli space $\mathcal{M}_{\text{cs}}^b(G^b)$ in agreement with the Higgs branch moduli space (3.19).

The detailed local analysis of Calabi–Yau fourfolds with A_{N-1} singularities in codimension two together with the structure of local background fluxes is presented elsewhere [49]. Here we justify our proposal in the context of extremal transitions in global Calabi–Yau fourfolds. Namely, for a rather large class of toric example to be studied in the next two sections, we explicitly spell out a consistent background flux G^b , which in the vicinity of the deformed A_{N-1} surface singularity agrees with our local proposal (4.12) for the four-form flux G^b .

We observe that the Weyl group $\mathcal{W}(SU(N)) \simeq S_N$ acts on the matrix M according to Eq. (4.10), and hence induces a non-trivial action on the submatrix S of the flux component T , generating the Weyl orbit of fluxes G^b . While non-trivially acting on the flux G^b , the Weyl group S_N does not change any complex structure moduli because $\det(z \cdot I_N + M)$ remains invariant with respect to conjugation by permutation matrices. Therefore, the Weyl group action realizes a monodromy in the M-theory moduli space, which is fibered over the complex structure moduli space of X^b . Nevertheless, the restricted complex structure moduli space $\mathcal{M}_{\text{cs}}^b(G^b)$ —identified with the Higgs branch of the gauge theory—remains invariant under the monodromy action upon the flux G^b . It is rather surprising that we find a remnant of the Weyl group in the M-theory moduli space of the Calabi–Yau phase associated to the gauge theory Higgs branch. It would be interesting to further study the implications of this observation. We will give an explicit example of this phenomenon in Sect. 5.1.

5. Examples in Weighted Projective Spaces

In this section we provide two examples of hypersurfaces in weighted projective space to help fix ideas in a global setting: an $SU(2)$ example (which is similar to an example previously presented in Ref. [33]) and an $SU(6)$ example.

5.1. Calabi–Yau fourfolds from $\mathbb{P}^{(1,1,2,2,2,2)}$. We consider a generic weighted hypersurface X_0 of weight 10 in $\mathbb{P}^{(1,1,2,2,2,2)}$ defined by a weight 10 polynomial $f_{10}(x_1, \dots, x_6)$, its desingularization X^\sharp , and its smoothing X^b .

5.1.1. Geometric data. Since $\mathbb{P}^{(1,1,2,2,2,2)}$ is the quotient of \mathbb{P}^5 by the \mathbb{Z}_2 -action given by multiplication of the coordinates (x_1, \dots, x_6) by $(-1, -1, 1, 1, 1, 1)$, the codimension two locus Y defined by $x_1 = x_2 = 0$ has a transverse A_1 singularity. Then $Y \simeq \mathbb{P}^3$, with (x_3, x_4, x_5, x_6) serving as homogeneous coordinates. We can resolve the singularity of $\mathbb{P}^{(1,1,2,2,2,2)}$ by blowing up Y to get a smooth variety $\tilde{\mathbb{P}}$. Then the proper transform X^\sharp of X_0 in $\tilde{\mathbb{P}}$ is smooth.

Furthermore, X_0 is singular along $S = X_0 \cap Y$, which has equation $f_{10}(0, 0, x_3, x_4, x_5, x_6)$ in the homogeneous coordinates of $Y \simeq \mathbb{P}^3$. Since x_3, \dots, x_6 each have weight 2, then the weighted polynomial $f = f_{10}(0, 0, x_3, x_4, x_5, x_6)$ has degree 5 as an ordinary polynomial, and we have identified S as a quintic hypersurface in \mathbb{P}^3 , i.e. a quintic surface. By the Lefschetz hyperplane theorem, we have $q = h^{1,0}(S) = 0$.

By the adjunction formula, we have

$$K_S = \mathcal{O}_S(-4 + 5) = \mathcal{O}_S(1). \tag{5.1}$$

The number of adjoint chiral multiplets is then $p_g = h^0(K_S) = h^0(\mathcal{O}_S(1))$. We compute this space of sections using the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{r} \mathcal{O}_S(1) \rightarrow 0, \tag{5.2}$$

where α is multiplication by f and r is restriction to S . Taking cohomology of (5.2) and using $H^0(\mathcal{O}_{\mathbb{P}^3}(-4)) = H^1(\mathcal{O}_{\mathbb{P}^3}(-4)) = 0$, we get

$$p_g = h^0(\mathcal{O}_S(1)) = h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4. \tag{5.3}$$

Thus we have 4 adjoint chiral multiplets in our $SU(2)$ gauge theory.

To describe X^b , we first embed $\mathbb{P}^{(1,1,2,2,2,2)}$ as a singular quadric hypersurface in \mathbb{P}^6 by

$$\mathbb{P}^{(1,1,2,2,2,2)} \rightarrow \mathbb{P}^6, \quad (x_1, \dots, x_6) \mapsto (x_1^2, x_2^2, x_1x_2, x_3, x_4, x_5, x_6). \tag{5.4}$$

Letting (y_0, \dots, y_6) be homogeneous coordinates on \mathbb{P}^6 , we see that (5.4) embeds $\mathbb{P}^{(1,1,2,2,2,2)}$ isomorphically onto the singular quadric hypersurface with equation $q_0(y) = y_0y_1 - y_2^2 = 0$. Furthermore, after the substitution $y_0 = x_1^2, y_1 = x_2^2, y_2 = x_1x_2$, and $y_i = x_i$ for $3 \leq i \leq 6$ described by (5.4), we can find a homogenous degree five polynomial $g(y)$ with $g(y) = f_{10}(x)$, and $g(y)$ is unique up to multiples of $q_0(y)$. We conclude that X_0 is isomorphic to the complete intersection of $q_0(y)$ and $g(y)$, a (singular) complete intersection Calabi–Yau fourfold $\mathbb{P}^6[2, 5]$.

This description makes it clear how to smooth X_0 to obtain X^b : simply smooth the singular quadric $q_0(y)$ to a quadric $q^b(y)$ to obtain a more general complete intersection

Calabi–Yau $\mathbb{P}^6[2, 5]$. The generic $q^b = 0$ will intersect $g = 0$ transversely, so the resulting Calabi–Yau fourfold will be smooth. In fact, we can still get a smooth complete intersection Calabi–Yau if $q^b = 0$ has an isolated singularity at which g does not vanish. Such a q^b is a rank 6 quadric.

To count moduli for these deformations of $q_0(y)$, the space of first order deformations of q_0 is given by the degree 2 part of $\mathbb{C}[y_0, \dots, y_6]/J(q_0)$. Since the partial derivatives of q_0 are just y_0, y_1, y_2 up to multiple, the space of first order deformations is identified with homogeneous degree 2 polynomials in y_3, y_4, y_5, y_6 , a ten-dimensional space.

We make contact with the discussion in Sect. 4, where the deformation was described by (4.3), with $\omega_2 \in H^0(S, 2K_S)$. For the quintic surface, we have that $H^0(S, 2K_S) = H^0(S, \mathcal{O}_S(2))$. Tensoring (5.2) with $\mathcal{O}(1)$ and using the vanishing of the cohomologies of $\mathcal{O}_{\mathbb{P}^3}(-3)$, we conclude that $H^0(S, 2K_S)$ is identified with the space of degree 2 homogeneous polynomials in \mathbb{P}^3 . So the space of smoothings which we described explicitly above is canonically identified with $H^0(S, 2K_S)$.

5.1.2. *Adding G-flux.* Letting $L \in H^2(X^\sharp)$ be the proper transform of the divisor $(x_1 = 0) \subset X_0$ and $M \in H^2(X^\sharp)$ be the proper transform of the divisor $(x_3 = 0) \subset X_0$, explicit computation gives⁷

$$c_2(X^\sharp) = 2LM + 10M^2, \tag{5.5}$$

which is visibly an even class. Therefore $G^\sharp = 0$ satisfies the quantization condition.

We now exhibit explicit smoothings X^b which satisfy a G-flux constraint, parallel to a construction previously presented in Ref. [33]. In the example under investigation, S is a quintic surface and $K_S = \mathcal{O}_S(1)$ so we have that $K_S^2 = 5$. It follows according to Eq. (4.4) that the Euler characteristic changes by $30 = (N + 1)N(N - 1)K_S^2$ with $N = 2$.

Starting the transition with $G^\sharp = 0$ as above, then for G^b we require $\frac{1}{2} \left(\frac{G^b}{2\pi}\right)^2 = \frac{30}{24} = \frac{5}{2}$. We also require G^b to satisfy the quantization condition (2.2) that $c_2(X^b) - 2 \cdot \frac{G^b}{2\pi}$ is even.

We can find a suitable G^b after constraining q^b to be a rank 6 quadric (whose singular point p is not contained in the quintic hypersurface $g = 0$). We have seen that we can parametrize the moduli of q^b as

$$q^b = y_0y_1 - y_2^2 + \hat{q}(y_3, y_4, y_5, y_6). \tag{5.6}$$

In this parametrization, q^b has rank 6 if and only if \hat{q} has rank 3. Writing

$$\hat{q} = {}^t(y_3, \dots, y_6)Q(y_3, \dots, y_6) \tag{5.7}$$

in terms of a 4×4 symmetric matrix Q , the condition for \hat{q} to have rank 3 is that $\det Q = 0$, a codimension one condition. This gives a $10 - 1 = 9$ dimensional moduli space, which we will identify with the Higgs branch of the gauge theory after exhibiting G^b . As a check, the dimension of the Higgs branch of an $SU(2)$ gauge theory with 4 adjoints is $3 \cdot 4 - 3 = 9$.

If \hat{q} has rank 3, then it can be put in the form $\hat{q} = y_3^2 + y_4y_5$ after a change of coordinates, leading to

$$q^b = y_0y_1 - y_2^2 + y_3^2 + y_4y_5. \tag{5.8}$$

⁷ Since this is a standard computation, we will content ourselves with explaining how to perform an equivalent computation in a more general context in Sect. 6.

This equation can be compared with (4.9) by rewriting it as

$$y_0 y_1 = \det \left(y_2 I_2 + \begin{pmatrix} y_3 & y_4 \\ y_5 & -y_3 \end{pmatrix} \right) \tag{5.9}$$

Then the quadric $q^b = 0$ contains the 3-planes P_1 and P_2 defined by $y_0 = y_3 - y_2 = y_4 = 0$ and $y_0 = y_3 - y_2 = y_5 = 0$ as codimension two subvarieties. The 3-planes described explicitly above are in different rulings. However, nonsingular (rank 7) quadrics have only one ruling.

We wish to emphasize this point, which encodes the key geometric property of our choice of flux. A homogeneous quadric of rank r in \mathbb{P}^{r-1} contains an irreducible family of linear subspaces when r is odd, but contains two distinct families of linear subspaces when r is even.⁸ This statement about linear subspaces depends only on the rank, not on the dimension in which the quadric has been embedded. Thus, a quadric in \mathbb{P}^{2k} of rank $2k$ has a unique singular point and two families of linear subspaces, but when we smooth this quadric to one of rank $2k + 1$ (the generic case) there is only one family of linear subspaces. In particular, the difference $P_1 - P_2$ of spaces from the two families exists as a cycle on the rank $2k$ quadric which cannot be extended to a cycle on the nonsingular quadric of rank $2k + 1$.

Restricting to X^b by intersection with $g = 0$, these P_i yield codimension two subvarieties of X^b with cohomology classes $T_1, T_2 \in H^4(X^b, \mathbb{Z})$. Since on q^b we have that $y_0 = y_3 - y_2 = 0$ is $P_1 \cup P_2$, we have in cohomology that $H^2 = T_1 + T_2$, where H is the hyperplane class of \mathbb{P}^6 restricted to X^b . As in Refs. [26,33], we then take

$$\frac{G^b}{2\pi} = \frac{1}{2} (T_1 - T_2). \tag{5.10}$$

Since the cycles T_1 and T_2 have the same degrees, and $H^6(X^b) = 0$ by Lefschetz, we see that $H \cdot G^b = 0$ and G^b is primitive. Furthermore, G^b is of Hodge type $(2, 2)$, as it is an algebraic cohomology class. Thus all of the directions in moduli corresponding to rank 6 q^b are flat directions relative to the superpotential generated by G^b .

However, if we try to deform further to a rank 7 (nonsingular) quadric q^b , then there is only one ruling on q^b so we do not have cycles T_1 and T_2 in that case. In the absence of cycles T_1 and T_2 , there is no reason for G^b to remain of type $(2, 2)$ and we expect that it is not of type $(2, 2)$. It would be interesting to verify this expectation.

Returning to the situation where q^b has rank 6, since each $T_i \subset \mathbb{P}^6$ is a complete intersection of three linear forms and a quintic, we compute the Chern class of its normal bundle in X^b as

$$c(N_{T_i, X^b}) = \frac{(1 + H)^3 (1 + 5H)}{(1 + 2H) (1 + 5H)}. \tag{5.11}$$

Expanding (5.11), we get $c_2(N_{T_i, X^b}) = H^2$, which is numerically 5, as T_i has degree 5, owing to the intersection with $g = 0$. Thus

$$T_1^2 = T_2^2 = 5. \tag{5.12}$$

We then compute $T_1 \cdot T_2 = T_1 \cdot (H^2 - T_1) = 5 - 5 = 0$. Thus $\left(\frac{G^b}{2\pi}\right)^2 = \frac{5}{2}$ as required.

⁸ This is closely related to the familiar fact that an orthogonal group in a space of odd dimension has a single irreducible spinor representation, but an orthogonal group in a space of even dimension has two distinct spinor representations.

For the quantization condition, explicit computation gives $c_2(X^b) = 11H^2$. Since we only need to compute mod 2, we can replace $2 \cdot \frac{G^b}{2\pi} = T_1 - T_2$ by $T_1 + T_2 = H^2$. We learn that $c_2(X^b) - 2 \cdot \frac{G^b}{2\pi}$ is congruent mod 2 to $10H^2$, which is even. So the quantization condition is satisfied.

Looking at Eq. (5.9), the action of the Weyl group $\mathcal{W}(SU(2)) \simeq \mathbb{Z}_2$ is realized by interchanging rows and columns, i.e.

$$\begin{pmatrix} y_3 & y_4 \\ y_5 & -y_3 \end{pmatrix} \mapsto \begin{pmatrix} -y_3 & y_5 \\ y_4 & y_3 \end{pmatrix}, \tag{5.13}$$

which has the effect of switching T_1 and T_2 . So the Weyl group sends G^b to $-G^b$.

Recall that G^b is determined by a choice of ruling of a quadric, or equivalently, by a choice of a matrix in the representation (5.9) of one of the equations for X^b . Then the Weyl group action can be explicitly realized as a monodromy in the M-theory moduli space over the complex structure moduli space. We realize this monodromy as follows. Consider the space \mathcal{M} of 2×2 matrices of linear forms in y_3, y_4, y_5 whose determinant is a rank 3 quadric. We choose a path in \mathcal{M} which starts at the matrix on the left of (5.13) and ends at the matrix on the right of (5.13). Explicitly, we can take

$$M(\theta) = \begin{pmatrix} e^{i\theta} y_3 & \frac{y_4(1+e^{i\theta})+y_5(1-e^{i\theta})}{2} \\ \frac{y_5(1+e^{i\theta})+y_4(1-e^{i\theta})}{2} & -e^{i\theta} y_3 \end{pmatrix}, \quad 0 \leq \theta \leq \pi, \tag{5.14}$$

so that θ parametrizes a path in the unobstructed complex structure moduli space $\mathcal{M}_{cs}^b(G^b)$. For $\theta = 0$ we get G^b as in Eq. (5.10), while for $\theta = \pi$ we get $-G^b$.

5.2. *Calabi–Yau fourfolds from $\mathbb{P}^{(1,5,6,6,6,6)}$.* We consider a generic weighted hypersurface X_0 of weight 30 in $\mathbb{P}^{(1,5,6,6,6,6)}$ defined by a weight 30 polynomial $f_{30}(x_1, \dots, x_6)$, its desingularization X^\sharp , and its smoothing X^b .

5.2.1. *Geometric data.* Since $\mathbb{P}^{(1,5,6,6,6,6)}$ is the quotient of \mathbb{P}^5 by the \mathbb{Z}_6 -action given by multiplication of the coordinates (x_1, \dots, x_6) by $(\omega, \omega^5, 1, 1, 1, 1)$ with $\omega^6 = 1$ and an additional \mathbb{Z}_5 -action, the codimension two locus Y defined by $x_1 = x_2 = 0$ has a transverse A_5 singularity, at least away from the point $p = (0, 1, 0, 0, 0, 0)$ which is the isolated fixed point of the additional \mathbb{Z}_5 . Then $Y \simeq \mathbb{P}^3$, with (x_3, x_4, x_5, x_6) serving as homogeneous coordinates. We can resolve the singularity of $\mathbb{P}^{(1,5,6,6,6,6)}$ (away from p) by blowing up Y to get a smooth variety $\tilde{\mathbb{P}}$. Since a generic f_{30} does not vanish at p , it follows that the proper transform X^\sharp of X_0 in $\tilde{\mathbb{P}}$ is smooth.

Furthermore, X_0 is singular along $S = X_0 \cap Y$, which has equation $f_{30}(0, 0, x_3, x_4, x_5, x_6)$ in the homogeneous coordinates of $Y \simeq \mathbb{P}^3$. Since x_3, \dots, x_6 each have weight 6, then the weighted polynomial $f = f_{10}(0, 0, x_3, x_4, x_5, x_6)$ has degree 5 as an ordinary polynomial, and we have identified S as a quintic hypersurface in \mathbb{P}^3 , i.e., a quintic surface. So again we have $q = h^{1,0}(S) = 0, K_S^2 = 5$ and there are 4 adjoints in our $SU(6)$ gauge theory.

To describe X^b , we first embed $\mathbb{P}^{(1,5,6,6,6,6)}$ as a singular weighted hypersurface in $\mathbb{P}^{(1,5,1,1,1,1,1)}$ by

$$\mathbb{P}^{(1,5,6,6,6,6)} \rightarrow \mathbb{P}^{(1,5,1,1,1,1,1)}, \quad (x_1, \dots, x_6) \mapsto (x_1^6, x_2^6, x_1x_2, x_3, x_4, x_5, x_6). \tag{5.15}$$

Letting (y_0, \dots, y_6) be homogeneous coordinates on $\mathbb{P}^{(1,5,1,1,1,1,1)}$, we see that (5.15) embeds $\mathbb{P}^{(1,5,6,6,6,6)}$ isomorphically onto the weight 6 hypersurface with equation $q_0(y) = y_0 y_1 - y_2^6 = 0$. Furthermore, after the substitution $y_0 = x_1^6, y_1 = x_2^6, y_2 = x_1 x_2$, and $y_i = x_i$ for $3 \leq i \leq 6$ described by (5.15), we can find a homogeneous degree five polynomial $g(y)$ with $g(y) = f_{30}(x)$. Note that g does not vanish at the unique singular point $p = (0, 1, 0, 0, 0, 0, 0)$ of $\mathbb{P}^{(1,5,1,1,1,1,1)}$. We conclude that X_0 is isomorphic to the complete intersection of $q_0(y)$ and $g(y)$, a (singular) complete intersection Calabi–Yau fourfold $\mathbb{P}^{(1,5,1,1,1,1,1)}$ [6, 5].

This description makes it clear how to smooth X_0 to obtain X^b : simply smooth the $q_0(y)$ to a general weight 6 hypersurface $q^b(y)$ to obtain a more general complete intersection Calabi–Yau $\mathbb{P}^{(1,5,1,1,1,1,1)}$ [6, 5]. The generic $q^b = 0$ will intersect $g = 0$ transversely, so the resulting complete intersection Calabi–Yau will be smooth.

To count moduli for these deformations of $q_0(y)$, the space of first order deformations of q_0 modulo g is given by the degree 6 part of $\mathbb{C}[y_0, \dots, y_6]/(J(q_0), g)$.⁹ Since the partial derivatives of q_0 are just y_0, y_1, y_2^5 up to multiple, the space of first order deformations is identified with homogeneous degree 6 polynomials in y_2, y_3, y_4, y_5, y_6 modulo g , where y_2 occurs with degree at most 4.

Write these deforming polynomials as

$$\sum_{j=0}^4 h_{6-j}(y_3, y_4, y_5, y_6) y_2^j. \tag{5.16}$$

Since h_{6-j} is to be taken modulo g , we view the coefficients of y_2^d as $h_{6-d} \in H^0(S, \mathcal{O}_S(6-j))$. So we see that the space of smoothings is identified with $\bigoplus_{j=2}^6 H^0(S, jK_S)$, in complete agreement with (4.3).

5.2.2. Adding G-flux. Resolving the A_5 singularity introduces five exceptional divisors, which we denote by E_1, \dots, E_5 . Letting $L \in H^2(X^\sharp)$ be the proper transform of the divisor $(x_1 = 0) \subset X_0$, $M \in H^2(X^\sharp)$ be the proper transform of the divisor $(x_2 = 0) \subset X_0$, and $N \in H^2(X^\sharp)$ be the proper transform of the divisor $(x_3 = 0) \subset X_0$, explicit computation gives¹⁰

$$c_2(X^\sharp) = (2E_3 + 6E_2 + 12E_1 + 20L + 2M + 10N)N, \tag{5.17}$$

which is visibly an even class. Therefore $G^\sharp = 0$ satisfies the quantization condition.

We now exhibit explicit smoothings X^b which satisfy a G-flux constraint. In the example under investigation, S is a quintic surface and $K_S = \mathcal{O}_S(1)$ so we have that $K_S^2 = 5$. It follows that the Euler characteristic changes by $1050 = N(N-1)(N+1)K_S^2$ with $N = 6$ according to Eq. (4.5).

Starting the transition with $G^\sharp = 0$ as above, then for G^b we require $\frac{1}{2} \left(\frac{G^b}{2\pi}\right)^2 = \frac{1050}{24}$, or $\left(\frac{G^b}{2\pi}\right)^2 = \frac{175}{2}$. We also require G^b to satisfy the quantization condition (2.2) that $c_2(X^b) - 2 \cdot \frac{G^b}{2\pi}$ is even.

⁹ In the case of $\mathbb{P}^{(1,1,2,2,2,2)}$ [10], we did not have to consider the deforming polynomials modulo g since the degree of q_0 was less than the degree of g .

¹⁰ Since this is a standard computation, we again content ourselves with explaining how to perform an equivalent computation in a more general context in Sect. 6.

We can find a suitable G^b after constraining q^b to be of the form

$$y_0 y_1 = \det(y_2 I_6 + M(y)), \tag{5.18}$$

where $M(y)$ is a traceless 6×6 matrix of linear forms in y_3, \dots, y_6 and I_6 is the 6×6 identity matrix. There are 35×4 moduli for the entries of $M(y)$, which must be reduced by 35 since conjugation by an $SU(6)$ matrix does not alter q^b . These $35 \times 4 - 35$ moduli precisely match the moduli of the Higgs branch of an $SU(6)$ theory with 4 adjoints. Note that $M \equiv 0$ corresponds to $q^b = q_0$.

Let $S(y)$ be the 6×5 submatrix of $y_2 I_6 + M(y)$ obtained by deleting its last column. Let $R \subset X^b$ be the 4-cycle defined by $y_0 = y_2 = 0$ and let $T \subset X^b$ be the 4-cycle defined by

$$T = \{y \in X^b \mid y_0 = 0, \text{rank } S(y) \leq 4\}. \tag{5.19}$$

We put

$$\frac{G^b}{2\pi} = \frac{5}{2}R - T \in H^4(X^b), \tag{5.20}$$

which is of Hodge type $(2, 2)$ since it is an algebraic cohomology class.

We check that G^b is primitive by computing that its image in the cohomology of the fivefold F defined by $g = 0$ vanishes.

Let H be the restriction to F of hyperplane class of $\mathbb{P}^{(1,5,1,1,1,1)}$. Since F has weight 5 and the weighted projective space has a \mathbb{Z}_5 quotient, we have $\int_F H^5 = 5/5 = 1$.

Since R is defined in F by $q^b = y_0 = y_2 = 0$, its image in F is $6H^3$. By Porteous’s formula, T has image $15H^3$ in F . Thus the image of the class of $(5/2)R - T$ in F vanishes and we have verified primitivity.

We compute $\left(\frac{G^b}{2\pi}\right)^2$ by computing the intersections R^2, RT , and T^2 in X^b . Since X^b is a $(6, 5)$ complete intersection in $\mathbb{P}^{(1,5,1,1,1,1)}$ and R is a complete intersection of two linear forms, we have

$$R^2 = \frac{6 \cdot 5 \cdot 1^4}{5} = 6, \tag{5.21}$$

where the denominator of 5 arises from the \mathbb{Z}_5 quotient in the weighted projective space.

For RT , we can replace R by the algebraically equivalent cycle $y_2 = y_3 = 0$. Computing RT inside F we get

$$RT = \int_F 15H^3 \cdot H^2 = 15. \tag{5.22}$$

Finally, we compute T^2 as the degree of the second Chern class of the normal bundle N_{T, X^b} of T in X^b . First we define

$$\tilde{T} = \{(y, z) \in X^b \times \mathbb{P}^4 \mid S(y)z = 0\}. \tag{5.23}$$

The projection $\pi : X^b \times \mathbb{P}^4 \rightarrow X^b$ maps \tilde{T} to T . This projection fails to be an isomorphism only over points of T at which $S(y)$ has rank 3 or less. Since the rank 3 condition is codimension 6 in X^b , we see that $\tilde{T} \rightarrow T$ is an isomorphism.

We have

$$c(N_{T, X^b}) = \frac{c(X^b)}{c(T)}, \tag{5.24}$$

where we omit restrictions to T for brevity. Similarly,

$$c(N_{\tilde{T}, F \times \mathbb{P}^4}) = \frac{c(F \times \mathbb{P}^4)}{c(\tilde{T})} = \frac{c(F)c(\mathbb{P}^4)}{c(\tilde{T})} \tag{5.25}$$

with omitted restrictions to \tilde{T} .

Letting η be the hyperplane class \mathbb{P}^4 , we have

$$c(N_{\tilde{T}, F \times \mathbb{P}^4}) = (1 + H)(1 + H + \eta)^6, \tag{5.26}$$

since the six components of $S(y)z$ are bilinear and together with $y_0 = 0$ define \tilde{T} as a complete intersection. Since X^b is the hypersurface in F defined by $q^b = 0$, we have

$$c(F)|_{X^b} = c(X^b)(1 + 6H). \tag{5.27}$$

Identifying \tilde{T} with T via π , we get from (5.24)–(5.27)

$$c(N_{T, X^b}) = \frac{(1 + H + \eta)^6(1 + H)}{(1 + \eta)^5(1 + 6H)} \tag{5.28}$$

which gives

$$c_2(N_{T, X^b}) = 15H^2 - 5H\eta, \tag{5.29}$$

identified as a class on \tilde{T} . We can easily push (5.29) to $F \times \mathbb{P}^4$ since \tilde{T} is a complete intersection of 6 divisors in the class $H + \eta$ and the divisor $y_0 = 0$ of class H :

$$c_2(N_{T, X^b}) = (15H^2 - 5H\eta)H(H + \eta)^6 \in H^{18}(F \times \mathbb{P}^4) \tag{5.30}$$

We project (5.30) to F by extracting the coefficient of η^4 , which is $125H^5 \in H^{10}(F)$, which evaluates to 125. So finally

$$T^2 = 125. \tag{5.31}$$

Putting (5.21), (5.22), and (5.31) together, we get

$$\left(\frac{G^b}{2\pi}\right)^2 = \frac{25}{4}R^2 - 5RT + T^2 = \frac{75}{2} - 75 + 125 = \frac{175}{2} \tag{5.32}$$

as required.

For the quantization condition, explicit computation gives $c_2(X^b) = 15H^2$. Since we only need to compute mod 2, we can replace $2 \cdot \frac{G^b}{2\pi} = 5R - 2T$ by $5R = 5H^2$ because R is a complete intersection of two linear equations in X^b . We learn that $c_2(X^b) - 2 \cdot \frac{G^b}{2\pi}$ is congruent mod 2 to $10H^2$, which is even. So the quantization condition is satisfied.

Looking at (5.18), the action of the Weyl group $\mathcal{W}(SU(6)) \simeq S_6$ is realized by permuting the rows and columns of $M(y)$. As we did at the end of Sect. 5.1, we can realize this Weyl group action as a monodromy. We rewrite the submatrix $S(y)$ of $y_2I_6 + M(y)$ as

$$S(y) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} (y_2I_6 + M(y)). \tag{5.33}$$

The projection matrix P appearing in (5.33) has columns e_1, \dots, e_5 chosen from the standard basis of \mathbb{C}^6 . Choose a permutation σ in the Weyl group. We can then choose a path in the space of 6×5 matrices of maximal rank starting from P and ending at the matrix whose columns are $e_{\sigma(1)}, \dots, e_{\sigma(5)}$ to realize the Weyl action as a monodromy.

6. Toric Geometry and Further Examples

We begin by reviewing the setup for investigating Calabi–Yau hypersurfaces and complete intersections in toric varieties, specialized to Calabi–Yau fourfolds. See [50, 51].

Let N and M be a pair of dual lattices of rank 5. We consider a pair (Δ, Δ°) of 5-dimensional reflexive polytopes, with $\Delta \subset M_{\mathbb{R}}$ spanned by vertices in M , and $\Delta^\circ \subset N_{\mathbb{R}}$ spanned by vertices in N . The origin is the unique interior point of $\Delta \cap M$ and of $\Delta^\circ \cap N$. The polytopes are related by

$$\Delta^\circ = \{n \in N_{\mathbb{R}} \mid \langle m, n \rangle \geq 1 \text{ for all } m \in \Delta\}. \tag{6.1}$$

The toric variety \mathbb{P}_Δ can also be described as the toric variety associated to fan obtained by taking the cones over the faces of Δ° . Since this toric variety is typically highly singular, we choose a maximal projective crepant subdivision of that fan to obtain a toric variety with controllable singularities. The fan Σ^\sharp of this toric variety satisfies

- (i) $\Sigma^\sharp(1) = \Delta^\circ \cap N - \{0\}$
- (ii) X_{Σ^\sharp} is projective and simplicial

We let $X^\sharp \subset X_{\Sigma^\sharp}$ be a general anticanonical hypersurface, so that X^\sharp is a Calabi–Yau fourfold.

6.1. *Gauge group SU(N).* To achieve the situation of A_{N-1} singularities, we assume:

$$\Delta^\circ \text{ has a one-dimensional edge } \Gamma \text{ containing } N - 1 \text{ interior lattice points} \tag{6.2}$$

Let v_1 and v_2 be the endpoints of Γ . We can choose an $m_\Gamma \in M$ so that $\langle m_\Gamma, v_1 \rangle = N - 1$ and $\langle m_\Gamma, v_2 \rangle = -1$.

If we remove the cones containing the interior lattice points of Γ from the fan Σ^\sharp , we obtain a fan Σ_0 . The natural map $\pi : X_{\Sigma^\sharp} \rightarrow X_{\Sigma_0}$ blows down a divisor to a threefold with transverse A_{N-1} singularities. After intersecting with an anticanonical hypersurface in X_{Σ^\sharp} , we get a map $X^\sharp \rightarrow X_0$ of Calabi–Yau fourfolds, contracting a divisor to a surface S of A_{N-1} singularities.

To begin to understand S , we consider the dual face $\Gamma^\circ \subset \Delta$ defined by

$$\Gamma^\circ = \{m \in M_{\mathbb{R}} \mid \langle m, v_1 \rangle = -1, \langle m, v_2 \rangle = -1\}. \tag{6.3}$$

Then Γ° is a 3-dimensional polytope. We have for the geometric genus of S

$$p_g = |\text{int}(\Gamma^\circ) \cap M|, \tag{6.4}$$

as we will check later.

We denote the primitive integral generators of the other one-dimensional cones in Σ_0 by v_3, \dots, v_k . We also denote by $D_i \subset X_{\Sigma_0}$ the toric divisor associated with the edge v_i , $1 \leq i \leq k$. Similarly, we denote by $D_i^\sharp \subset X_{\Sigma^\sharp}$ the toric divisor associated with the

edge v_i . For D_i^\sharp , we can have $1 \leq i \leq k$ as above, or v_i can denote one of the $n - 1$ interior lattice points of Γ .

We form a new simplicial fan Σ^b in $(N \oplus \mathbb{Z})_{\mathbb{R}}$ with 1-dimensional cones w_0, \dots, w_n given by the vertices

$$\begin{aligned} w_0 &= \left(\frac{v_1 - v_2}{N}, -(N - 1) \right), \\ w_1 &= (0, 1), \\ w_2 &= (v_2, 0), \\ w_i &= (v_i, -N \langle m_\Gamma, v_i \rangle), \quad i \geq 3. \end{aligned} \tag{6.5}$$

The six-dimensional cones $\Sigma^b(6)$ of Σ^b can be described as follows. Let $\sigma \in \Sigma_0(5)$ be a 5-dimensional (simplicial) cone of Σ_0 . We partition the edges $\sigma(1)$ of σ into the set $\sigma(1)_1$ of edges spanned by v_1 or v_2 , and the set $\sigma(1)_2$ of edges spanned by v_i with $i \geq 3$. We have abused notation slightly by labeling the edges by their primitive integral generators v_i .

Then we form $\Sigma^b(6)$ as follows: for each $\sigma \in \Sigma(5)$ we form one or more 6-dimensional simplicial cones as the span of the vectors w_{i-1} for each $v_i \in \sigma(1)_1$, together with the vectors w_i for each $v_i \in \sigma(1)_2$, and exactly one more vector from among $\{w_0, w_1, w_2\}$. A distinct cone is included in $\Sigma^b(6)$ for each choice of this additional vector w_0, w_1 or w_2 . The fan Σ^b is the fan whose cones are the faces of one of the top-dimensional cones just described.

Using the fact that Σ_0 is a fan, it is straightforward to check that the intersection of any two cones of Σ^b is a face of each, so that Σ^b is indeed a fan. Furthermore, it is straightforward to check that Σ^b is complete since Σ_0 is. We let $D_i^b \subset X_{\Sigma^b}$ be the toric divisor associated with the edge w_i .

Example. Let Σ_0 be a fan for $\mathbb{P}^{(1,1,2,2,2,2)}$. A convenient choice is to take the complete simplicial fan with edges spanned by the rows of

$$\begin{pmatrix} -1 & -2 & -2 & -2 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.6}$$

We label the rows as v_1, \dots, v_6 in order. The edge Γ joining v_1 and v_2 has one interior lattice point $v_0 = (0, -1, -1, -1, -1)$ and we have an $SU(2)$ example. The fan for Σ^\sharp is obtained from the fan for Σ by replacing each cone containing both v_1 and v_2 by two cones: one cone in which v_1 and v_2 are replaced by v_1 and v_0 , and other cone in which v_1 and v_2 are replaced by v_0 and v_2 .

Choosing $m_\Gamma = (-1, 0, 0, 0, 0) \in M$ we have

$$\langle m_\Gamma, v_1 \rangle = 1, \quad \langle m_\Gamma, v_2 \rangle = -1, \quad \langle m_\Gamma, v_i \rangle = 0 \quad \text{for } i \geq 3. \tag{6.7}$$

Then (6.5) gives the edges of the fan Σ^b as the rows of

$$\begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{6.8}$$

with the labeling w_0, \dots, w_6 . The cones of Σ^b are immediately seen to consist of all cones spanned by any proper subset of $\{w_0, \dots, w_6\}$. We therefore obtain the fan for \mathbb{P}^6 , the space that we embedded $\mathbb{P}^{(1,1,2,2,2,2)}$ into in Sect. 5.1.1.

Example. Let Σ_0 be a fan for $\mathbb{P}^{(1,5,6,6,6,6)}$. A convenient choice is to take the complete simplicial fan with edges spanned by the rows of

$$\begin{pmatrix} -5 & -6 & -6 & -6 & -6 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.9}$$

We label the rows as v_1, \dots, v_6 in order. The edge Γ joining v_1 and v_2 has 5 interior lattice points and we have an $SU(6)$ example. The interior lattice points are

$$\begin{aligned} v_0 &= (0, -1, -1, -1, -1), & v_{-1} &= (-1, -2, -2, -2, -2), \\ v_{-2} &= (-2, -3, -3, -3, -3), & & \\ v_{-3} &= (-3, -4, -4, -4, -4), & v_{-4} &= (-4, -5, -5, -5, -5). \end{aligned} \tag{6.10}$$

Choosing $m_\Gamma = (-1, 0, 0, 0, 0) \in M$ we have

$$\langle m_\Gamma, v_1 \rangle = 5, \quad \langle m_\Gamma, v_2 \rangle = -1, \quad \langle m_\Gamma, v_i \rangle = 0 \quad \text{for } i \geq 3. \tag{6.11}$$

Then (6.5) gives the edges of the fan Σ^b as the rows of

$$\begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{6.12}$$

with the labeling w_0, \dots, w_6 . The cones of Σ^b are immediately seen to consist of all cones spanned by any proper subset of $\{w_0, \dots, w_6\}$. We therefore obtain the fan for $\mathbb{P}^{(1,1,1,1,1,1,5)}$, the space that we embedded $\mathbb{P}^{(1,5,6,6,6,6)}$ into in Sect. 5.2.1.¹¹

Consider the map $\iota : X_{\Sigma_0} \rightarrow X_{\Sigma^b}$ given by

$$(y_0, y_1, y_2, y_3, \dots) = (x_1^N, x_2^N, x_1 x_2, x_3 \dots), \tag{6.13}$$

where (y_0, \dots, y_k) are the homogeneous coordinates of X_{Σ^b} . We check that the map is well-defined. If $g = (t_j) \in G(\Sigma_0)$, then $\prod t_j^{(m, v_j)} = 1$ for all $m \in M$. Then $g \cdot x$ maps to

$$(t_1^N x_1^N, t_2^N x_2^N, t_1 t_2 x_1 x_2, t_3 x_3 \dots), \tag{6.14}$$

¹¹ The general toric procedure requires us to add more edges from additional points of $\Delta^\circ \cap N$. However, for simplicity we can safely exclude them from discussion since the weighted hypersurface f considered in Sect. 5.2.1 does not contain the fixed point of the \mathbb{P}^5 action.

which we have to show is equivalent to $(x_1^N, x_2^N, x_1 x_2, \dots)$ up to an element of $G(\Sigma^b)$. In other words, we have to check that for all $(m, n) \in M \oplus \mathbb{Z}$ we have

$$(t_1^N)^{\langle(m,n),w_0\rangle} (t_2^N)^{\langle(m,n),w_1\rangle} (t_1 t_2)^{\langle(m,n),w_2\rangle} \prod_{j=3}^k (t_j)^{\langle(m,n),w_j\rangle} = 1. \tag{6.15}$$

But the left hand side of (6.15) simplifies to

$$t_1^{(m,v_1)-N(N-1)n} t_2^{(m,v_2)+Nn} \prod_{j=3}^k (t_j)^{\langle(m-Nnm_\Gamma,v_j)\rangle} = \prod_{j=1}^k (t_j)^{\langle(m-Nnm_\Gamma,v_j)\rangle}, \tag{6.16}$$

which is 1 because $m - Nnm_\Gamma \in M$ and $g \in G(\Gamma)$.

It is straightforward to check that the image of ι in coordinates lands in $\mathbb{C}^{k+1} - Z(\Sigma^b)$ and is an embedding after modding out by $G(\Sigma_0)$ and $G(\Sigma^b)$.

Example. For $\mathbb{P}^{(1,1,2,2,2,2)}$ the embedding into \mathbb{P}^6 is

$$(y_1, \dots, y_6) \mapsto (y_1^2, y_2^2, y_1 y_2, y_3, \dots, y_6) \tag{6.17}$$

in complete agreement with Sect. 5.1.1.

Example. For $\mathbb{P}^{(1,5,6,6,6,6)}$ the embedding into $\mathbb{P}^{(1,5,1,1,1,1)}$ is

$$(y_1, \dots, y_6) \mapsto (y_1^6, y_2^6, y_1 y_2, y_3, \dots, y_6) \tag{6.18}$$

in complete agreement with Sect. 5.2.1.

Clearly, $\iota(X_{\Sigma_0})$ is contained in the hypersurface $q_0(y) = y_0 y_1 - y_2^N = 0$. The linear equivalence $D_0^b + D_1^b \sim N D_2^b$ is realized by $(Nm_\Gamma, 1) \in M \oplus \mathbb{Z}$.

Since $\iota^*(\mathcal{O}_{X_{\Sigma^b}}(D_2^b)) \simeq \mathcal{O}_{X_{\Sigma_0}}(D_0 + D_1)$ and $\iota^*(\mathcal{O}_{X_{\Sigma^b}}(D_j^b)) \simeq \mathcal{O}_{X_{\Sigma_0}}(D_j)$ for $j \geq 3$, we see that

$$\iota^*(\mathcal{O}_{X_{\Sigma^b}}(\sum_{j=2}^k D_j^b)) \simeq \mathcal{O}_{X_{\Sigma_0}}(\sum_{j=1}^k D_j) = \mathcal{O}_{X_{\Sigma_0}}(-K_{X_\Sigma}), \tag{6.19}$$

and it is easy to see that X_0 pulls back from a section of $\mathcal{O}_{X_{\Sigma^b}}(\sum_{j=2}^k D_j^b) =: \mathcal{O}_{X_{\Sigma^b}}(D')$ which we denote by $f(y)$. Thus X_0 is identified with a complete intersection of $q_0(y)$ and $f(y)$ in X_{Σ^b} . The singular locus S is defined by $y_0 = y_1 = y_2 = f(y) = 0$. Adjunction again says that the canonical bundle of S is the restriction of D_2^b .

We can describe this complete intersection using a nef partition [51] if desired, partitioning the edges ρ_j spanned by the w_j into two sets:

$$\{\rho_0, \rho_1\}, \{\rho_2, \rho_3, \dots, \rho_k\} \tag{6.20}$$

and we are led to view $q_0(y)$ as a section of $\mathcal{O}_{X_{\Sigma^b}}(D_0 + D_1)$.

We can identify a basis of sections of $\mathcal{O}_{X_{\Sigma^b}}(D_2^b)$ with monomials $\chi^{(m,n)}$, $(m, n) \in M \oplus \mathbb{Z}$, satisfying

$$\begin{aligned} \langle(m, n), w_0\rangle &= \langle m, \frac{v_1-v_2}{2} \rangle - n && \geq 0 \\ \langle(m, n), w_1\rangle &= n && \geq 0 \\ \langle(m, n), w_2\rangle &= \langle m, v_2 \rangle && \geq -1 \\ \langle(m, n), w_j\rangle &= \langle m, v_j \rangle && \geq 0 \quad (j \geq 3) \end{aligned} \tag{6.21}$$

via the correspondence

$$\chi^{(m,n)} \leftrightarrow z^{(m,n)} := y_2 \prod_{j=0}^k y_j^{\langle(m,n),w_k\rangle} \tag{6.22}$$

If $m \in \text{int}(\Gamma^\circ) \cap M$, (6.21) is satisfied for $(m, 0) \in M \oplus \mathbb{Z}$, since $m \in \text{int}(\Gamma^\circ) \cap M$ is equivalent to

$$\langle m, v_1 \rangle = -1, \quad \langle m, v_2 \rangle = -1, \quad \langle m, v_j \rangle \geq 0 \quad (j \geq 3). \tag{6.23}$$

Thus $y_0 y_1, y_2^N$, and $y_2^r \prod_{j=1}^{N-r} z^{(m_{i_j}, 0)}$ ($m_{i_j} \in \text{int}(\Gamma^\circ) \cap M$) are identified with sections of $\mathcal{O}_{X_{\Sigma^b}}(ND_2^b)$. By the now-familiar argument, we will only need to consider $r \leq N - 2$.

We change notation and rewrite the p_g sections $z^{(m_i, 0)}$, $m_i \in \text{int}(\Gamma^\circ) \cap M$ as z_1, \dots, z_{p_g} . Then

$$q^b(y) = y_0 y_1 + y_2^N + \sum_{r=0}^{N-2} y_2^r \sum_{\substack{J \subset \{1, \dots, p_g\} \\ |J|=N-r}} a_{j_1 \dots j_{N-r}} \prod z_{j_i} \tag{6.24}$$

is a deformation of $q_0(y)$.

Suppose that $c_2(X^\sharp)$ is the restriction of an even toric class in $H^4(X_{\Sigma^\sharp}, \mathbb{Z})$, which implies that $G^\sharp = 0$ satisfies quantization. Then we will exhibit explicit smoothings X^b which satisfy a G-flux constraint.

The strengthened hypothesis on the evenness of $c_2(X^\sharp)$ is needed so that the exhibited G^b satisfies the quantization condition. The examples from Sect. 5 both satisfy this hypothesis. The tadpole condition is always satisfied, as we will see.

We can find a suitable G^b after constraining q^b to be of the form

$$y_0 y_1 = \det(y_2 I_N + M(y)), \tag{6.25}$$

where $M(y)$ is a traceless $N \times N$ matrix of linear forms in y_3, \dots, y_{p_g+2} and I_N is the $N \times N$ identity matrix. There are $p_g(N^2 - 1)$ moduli for the entries of $M(y)$, which must be reduced by $N^2 - 1$ since conjugation by an $SU(N)$ matrix does not alter q^b . These $(p_g - 1)(N^2 - 1)$ moduli precisely match the moduli of the Higgs branch of an $SU(N)$ theory with p_g adjoints. Note that $M \equiv 0$ corresponds to $q^b = q_0$.

Let $S(y)$ be the $N \times (N - 1)$ submatrix of $y_2 I_N + M$ obtained by deleting its last column. Let $R \subset X^b$ be the 4-cycle defined by $y_0 = y_2 = 0$ and let $T \subset X^b$ be the 4-cycle defined by

$$T = \{y \in X^b \mid y_0 = 0, \text{ rank } S(y) \leq N - 2\}. \tag{6.26}$$

We put

$$\frac{G^b}{2\pi} = \frac{N - 1}{2} R - T \in H^4(X^b), \tag{6.27}$$

which is of type (2, 2) since it is an algebraic cohomology class.

We check that G^b is primitive by computing that its image in the cohomology of the fivefold F defined by $g = 0$ vanishes.

Since R is defined in F by $q^b = y_0 = y_2 = 0$, its class in F is $ND_0^b(D_2^b)^2$. By Porteous’s formula, T has class $\frac{N(N-1)}{2}D_0^b(D_2^b)^2$. Thus the class of $\frac{N-1}{2}R - T$ vanishes in F and we have verified primitivity.

We compute $\left(\frac{G^b}{2\pi}\right)^2$ by computing R^2 , RT , and T^2 . Since X^b is a complete intersection of divisors in the classes ND_2^b and D' , R is a complete intersection of y_0 and y_2 , we have

$$R^2 = N(D_0^b)^2(D_2^b)^3 D' \in H^{12}(X_{\Sigma^b}). \tag{6.28}$$

Computing RT inside F we get

$$RT = \frac{N(N-1)}{2}(D_0^b)^2(D_2^b)^3 D' \in H^{10}(F). \tag{6.29}$$

Finally, we compute T^2 as the degree of the second Chern class of the normal bundle N_{T, X^b} of T in X^b . First we define

$$\tilde{T} = \left\{ (y, z) \in X^b \times \mathbb{P}^{N-2} \mid S(y)z = 0 \right\}. \tag{6.30}$$

The projection $\pi : X^b \times \mathbb{P}^{N-2} \rightarrow X^b$ maps \tilde{T} to T . This projection fails to be an isomorphism only over points of T at which $S(y)$ has rank $N - 3$ or less. Since the rank 3 condition is codimension 6 in X^b , we see that $\tilde{T} \rightarrow T$ is an isomorphism.

We have

$$c(N_{T, X^b}) = \frac{c(X^b)}{c(T)}, \tag{6.31}$$

where we omit restrictions to T for brevity. Also,

$$c(N_{\tilde{T}, F \times \mathbb{P}^{N-2}}) = \frac{c(F \times \mathbb{P}^{N-2})}{c(\tilde{T})} = \frac{c(F)c(\mathbb{P}^{N-2})}{c(\tilde{T})} = \frac{c(X^b)(1 + ND_2^b)c(\mathbb{P}^{N-2})}{c(\tilde{T})}. \tag{6.32}$$

We get

$$c(N_{\tilde{T}, F \times \mathbb{P}^{N-2}}) = (1 + D_0^b)(1 + D_2^b + \eta)^N, \tag{6.33}$$

since the N components of $S(y)z$ are bilinear in \mathbb{P}^{N-2} and sections of $\mathcal{O}(D_2^b)$, which define \tilde{T} as a complete intersection together with y_0 . Identifying \tilde{T} with T via π , we get

$$c(N_{T, X^b}) = \frac{(1 + D_2^b + \eta)^N (1 + D_0^b)}{(1 + ND_2^b)(1 + \eta)^{N-1}} \tag{6.34}$$

which gives

$$c_2(N_{T, X^b}) = (D_0^b - ND_1^b)\eta + \frac{N(N-1)}{2}(D_2^b)^2 = -D_1^b\eta + \frac{N(N-1)}{2}(D_2^b)^2, \tag{6.35}$$

where we have used $D_0^b + D_1^b \sim ND_2^b$. Computing the intersection on $F \times \mathbb{P}^{N-2}$, this is just

$$(-D_1^b\eta + \frac{N(N-1)}{2}(D_2^b)^2)D_0^b(D_2^b + \eta)^N \in H^{2N+6}(F \times \mathbb{P}^{N-2}). \tag{6.36}$$

We project down to T by extracting the coefficient of η^{N-2} , which is

$$\binom{N}{2}^2 D_0^b (D_2^b)^4 - \binom{N}{3} D_0^b D_1^b (D_2^b)^3 \in H^{10}(F). \tag{6.37}$$

Expressing this as a class on the toric variety finally gives

$$T^2 = \left(\binom{N}{2}^2 D_0^b (D_2^b)^4 - \binom{N}{3} D_0^b D_1^b (D_2^b)^3 \right) D' \in H^{12}(X_{\Sigma^b}). \tag{6.38}$$

Finally

$$\left(\frac{G^b}{2\pi} \right)^2 = \frac{(N-1)^2}{4} R^2 - (N-1)RT + T^2, \tag{6.39}$$

which evaluates on X_{Σ^b} to

$$\begin{aligned} & \frac{N(N-1)^2}{4} (D_0^b)^2 (D_2^b)^3 D' - \frac{N(N-1)^2}{2} (D_0^b)^2 (D_2^b)^3 D' + \binom{N}{2}^2 D_0^b (D_2^b)^4 D' \\ & - \binom{N}{3} D_0^b D_1^b (D_2^b)^3 D', \end{aligned} \tag{6.40}$$

which simplifies to

$$\left(\frac{G^b}{2\pi} \right)^2 = \frac{(N+1)N(N-1)}{12} D_0^b D_1^b (D_2^b)^3 D' \in H^{12}(X_{\Sigma^b}), \tag{6.41}$$

where we have again used $D_0^b + D_1^b \sim ND_2^b$.

Since S is a complete intersection of y_0, y_1, y_2 , and g , the class of S is $D_0^b D_1^b D_2^b D'$. By adjunction, we find

$$K_S = - \sum_{i=0}^k D_i^b + D_0^b + D_1^b + D_2^b + D' = D_2^b. \tag{6.42}$$

Thus we get

$$\frac{1}{2} \int_{X^b} \frac{G^b}{2\pi} \wedge \frac{G^b}{2\pi} = \frac{(N+1)N(N-1)}{24} K_S^2 = \frac{\chi(X^b) - \chi(X^\sharp)}{24}, \tag{6.43}$$

as required according to Eq. (4.11).

It remains to check the quantization condition. We first recall the computation of the Chern classes of a toric variety. Let X_Σ be a smooth projective toric variety of dimension n with ℓ edges in the fan Σ . Let $D_1, \dots, D_\ell \subset X_\Sigma$ be the corresponding toric divisors. Then we have a short exact sequence [52]

$$0 \rightarrow \mathcal{O}_{X_\Sigma}^{\ell-n} \rightarrow \bigoplus_{i=1}^{\ell} \mathcal{O}_{X_\Sigma}(D_i) \rightarrow T_{X_\Sigma} \rightarrow 0, \tag{6.44}$$

which gives

$$c(X_\Sigma) = \prod_{i=1}^{\ell} (1 + D_i). \tag{6.45}$$

So if $X \subset X_\Sigma$ is an anticanonical hypersurface we have

$$c(X) = \frac{\prod_{i=1}^\ell (1 + D_i)}{1 + \sum_{i=1}^\ell D_i}, \tag{6.46}$$

which gives

$$c_2(X) = \sum_{i < j} D_i D_j. \tag{6.47}$$

In particular, for $X^\sharp \subset X_{\Sigma^\sharp}$ we have

$$c_2(X^\sharp) = \sum_{i < j} D_i^\sharp D_j^\sharp. \tag{6.48}$$

Similarly, for X^b we get

$$c(X^b) = \frac{\prod_{i=0}^k (1 + D_i^b)}{(1 + D_0^b + D_1^b)(1 + \sum_{i=2}^k D_i^b)}, \tag{6.49}$$

which gives

$$c_2(X^b) = D_0^b D_1^b + \sum_{2 \leq i < j \leq k} D_i^b D_j^b. \tag{6.50}$$

We now let $f = \iota \circ \pi : X_{\Sigma^\sharp} \rightarrow X_{\Sigma^b}$ be the composition. Since (6.48) and (6.50) show that $c_2(X^\sharp)$ is the restriction of a cohomology class on X_{Σ^\sharp} and $c_2(X^b)$ is the restriction of a cohomology class on X_{Σ^b} , we are able to compare c_2 on both sides of the transition using f^* .

We continue our labeling conventions, so that the vertices in Γ are labeled, in order

$$v_1, v_0, v_{-1}, \dots, v_{2-N}, v_2. \tag{6.51}$$

We compute

$$\begin{aligned} f^*(D_0^b) &= N D_1^\sharp + (N - 1) D_0^\sharp + \dots + D_{2-N}^\sharp, \\ f^*(D_1^b) &= D_0^\sharp + \dots + (N - 1) D_{2-N}^\sharp + N D_2^\sharp, \\ f^*(D_2^b) &= D_1^\sharp + D_0^\sharp + \dots + D_{2-N}^\sharp + D_2^\sharp, \\ f^*(D_i^b) &= D_i^\sharp, \quad i \geq 3. \end{aligned} \tag{6.52}$$

As a check, note that $f^*(D_0^b + D_1^b - N D_2^b) = 0$, as it had to be owing to the linear equivalence $D_0^b + D_1^b \sim N D_2^b$.

Since (6.52) is a standard toric calculation, we content ourselves with just a few words of explanation. We have $f^* = \pi^* \circ \iota^*$, and ι^* is calculated below in (6.56). Since π is a blowdown, all that π^* can do is introduce the exceptional divisors with multiplicities. It then follows from the fact that the vertices (6.51) are on the edge Γ , in order, that for a pullback the coefficients of $D_1^\sharp, D_0^\sharp, \dots, D_{2-N}^\sharp, D_2^\sharp$ must be in arithmetic progression. Since the coefficients of D_1^\sharp and D_2^\sharp are fixed by (6.56), these observations are enough to completely determine (6.52).

For the quantization condition, we only need to compute mod 2. Since T is an integral class, we can replace $\frac{N-1}{2}R - T$ with $\frac{N-1}{2}R$ in checking quantization. We see that quantization follows immediately from two claims:

- $f^*(c_2(X^b)) - (N - 1)R \equiv c_2(X^\sharp) \pmod{2}$
- $f^* : H^*(X_{\Sigma^b}, \mathbb{Z}_2) \rightarrow H^*(X_{\Sigma^\sharp}, \mathbb{Z}_2)$ is injective

The first claim is checked by direct calculation, which can be separated into the cases where N is odd and N is even.

If N is odd, then (6.52) simplifies to

$$\begin{aligned} f^*(D_0^b) &= D_1^\sharp + D_{-1}^\sharp + \cdots + D_{4-N}^\sharp + D_{2-N}^\sharp, \\ f^*(D_1^b) &= D_0^\sharp + D_{-2}^\sharp + \cdots + D_{3-N}^\sharp + D_2^\sharp, \\ f^*(D_2^b) &= D_1^\sharp + D_0^\sharp + \cdots + D_{2-N}^\sharp + D_2^\sharp, \\ f^*(D_i^b) &= D_i^\sharp, \quad i \geq 3. \end{aligned} \tag{6.53}$$

Also, since $N - 1$ is even, we only have to show that $f^*(c_2(X^b)) \equiv c_2(X^\sharp) \pmod{2}$. This follows immediately from (6.48), (6.50), (6.53), and the Stanley–Reisner vanishings:

$$D_i \cdot D_j = 0 \text{ for } 2 - N \leq i < j \leq 2 \text{ unless } v_i, v_j \text{ are adjacent in the ordering} \tag{5.51}$$

(6.54)

Similarly, if N is even we have

$$\begin{aligned} f^*(D_0^b) &= D_0^\sharp + D_{-2}^\sharp + \cdots + D_{2-N}^\sharp, \\ f^*(D_1^b) &= D_0^\sharp + D_{-2}^\sharp + \cdots + D_{2-N}^\sharp, \\ f^*(D_2^b) &= D_1^\sharp + D_0^\sharp + \cdots + D_{2-N}^\sharp + D_2^\sharp, \\ f^*(D_i^b) &= D_i^\sharp, \quad i \geq 3. \end{aligned} \tag{6.55}$$

Then $(N - 1)R \equiv R \pmod{2}$. Since $R = D_0^b D_2^b$, the claim follows from $f^*(c_2(X^b) + D_0^b D_2^b) \equiv c_2(X^\sharp) \pmod{2}$, which is again checked by direct calculation as above.

The injectivity of f^* can be broken down into the injectivity of l^* and π^* separately. The injectivity of π^* follows since π is a blowup. The injectivity of l^* follows from the simpler computation analogous to (6.52)

$$\begin{aligned} l^*(D_0^b) &= N D_1, \\ l^*(D_1^b) &= N D_2, \\ l^*(D_2^b) &= D_1 + D_2, \\ l^*(D_i^b) &= D_i, \quad i \geq 3, \end{aligned} \tag{6.56}$$

which follows immediately from (6.13). The form of the fan Σ^\sharp shows that all linear equivalences and Stanley–Reisner relations among the D_i pull back from corresponding relations in the D_i^b . So we only have to look at (6.56) as a linear transformation to deduce that the kernel of l^* is generated by $D_0^b + D_1^b - N D_2^b$, which is zero.

7. Conclusions

In this work we studied the three-dimensional $\mathcal{N} = 2$ low energy theory of M-theory on Calabi–Yau fourfolds X_0 with a smooth surface S of A_{N-1} singularities. We found that—due to massless M2-brane degrees of freedom from the A_{N-1} singularity at codimension two—the three-dimensional effective theory resulted in a $\mathcal{N} = 2$ $SU(N)$ gauge theory with adjoint matter multiplets at low energies. Alternatively, we obtained the same gauge theory from a twisted dimensional reduction of the seven-dimensional $\mathcal{N} = 1$ $SU(N)$ gauge theory on the surface S .

From the twisted dimensional reduction, we derived for the three-dimensional $\mathcal{N} = 2$ $SU(N)$ gauge theory its matter spectrum, consisting of adjoint-valued $\mathcal{N} = 2$ chiral multiplets. Furthermore, we established that a variant of the Vafa–Witten equations [47] governed the supersymmetric ground states of the low energy theory. These equations allowed us to determine the moduli spaces of the Higgs and Coulomb branches of the gauge theory, where we in particular focus on the twisted dimensional reduction on S with a trivial $SU(N)$ principal bundle.

From the results of the performed gauge theory analysis, we predicted geometric properties of the M-theory compactification on the singular Calabi–Yau fourfold X_0 . First of all, we matched Coulomb and Higgs branches of the gauge theory with the crepant resolution to the (smooth) Calabi–Yau fourfolds X^\sharp and with the deformation to the (smooth) Calabi–Yau fourfold X^b , respectively. That is to say, a transition from the Coulomb to the Higgs branch in the gauge theory corresponded to an extremal transition between the resolved Calabi–Yau fourfold X^\sharp and the deformed Calabi–Yau fourfold X^b in M-theory. Furthermore, we argued that in order to arrive at the anticipated $SU(N)$ gauge theory branches—arising from a trivial $SU(N)$ principal bundle over S —the Coulomb–Higgs phase transition starting from a Calabi–Yau fourfold X^\sharp with no background flux ends at a Calabi–Yau fourfold X^b with non-trivial background four-form flux G^b .

The proposed flux G^b was required for consistency reasons so as to match the tadpole cancellation condition—due to the change of Euler characteristic along the extremal transition [26]—and to fulfill the flux quantization condition of M-theory [43]. But maybe even more importantly, the correct choice of the flux G^b was essential to be in accord with the moduli space of the Higgs branch of the $SU(N)$ gauge theory. Namely, we showed that the background flux G^b was primitive and generated a non-trivial M-theory superpotential. The flux G^b was of Hodge type $(2, 2)$ along the flat directions of the flux-induced superpotential, which in turn comprised the unobstructed complex structure moduli deformations associated to the Higgs branch of the described $SU(N)$ gauge theory. Furthermore, we observed that as we moved about the M-theory moduli space in the Calabi–Yau phase associated to the gauge theory Higgs branch, the flux G^b exhibited non-trivial monodromy behavior given in terms of the described Weyl group action $\mathcal{W}(SU(N))$ on the flux G^b .

In order to demonstrate our general arguments—inspired by Refs. [41,42]—using the framework of toric geometry we explicitly gave examples for extremal transitions between the Calabi–Yau fourfolds X^\sharp and X^b . Namely, starting from a five-dimensional toric varieties with A_{N-1} singularities in codimension two, we realized the Calabi–Yau fourfold X^\sharp as a hypersurface in the resolved toric variety, whereas we constructed the deformed Calabi–Yau fourfold X^b as a complete intersection in a six-dimensional toric variety. With the toric computational tools at hand, these examples allowed us to explicitly verify the general predictions concerning the interplay between the gauge theory moduli spaces and the M-theory background fluxes.

In this work we mainly focused on a particular gauge theory scenario arising from the twisted dimensional reduction of a trivial $SU(N)$ principal bundle over surface S . Firstly, extending the analysis to non-trivial $SU(N)$ principal bundles over the surface S would correspond to extremal transitions in M-theory with non-trivial background fluxes on both Calabi–Yau fourfolds X^{\sharp} and X^{\flat} —in analogy to the findings for M-theory four-form fluxes associated to phases of three-dimensional $\mathcal{N} = 2$ Abelian gauge theories [26]. Secondly, it would be interesting to extend the analysis to general ADE or even non-simply laced gauge groups. Note also that since the obtained results depended only on the local geometry in the vicinity of a codimension two singularity in the singular Calabi–Yau fourfold X_0 , the gauge theory branches are already captured in M-theory in terms of extremal transitions among suitable local Calabi–Yau fourfolds. Thus the relevant local Calabi–Yau fourfolds deserve further study as well. We plan to return to these issues in the future [49].

Acknowledgments. We would like to thank Lara Anderson, Paul Aspinwall, Sergei Gukov, Jonathan Heckman, Ken Intriligator, and Peter Mayr for discussions and correspondences. S.K., D.R.M., and M.R.P. thank the Aspen Center for Physics, which is supported by NSF Grant PHY-1066293, for hospitality during the initial stages of this work. The work of S.K. is supported by NSF Grants DMS-12-01089 and DMS-15-02170, the work of D.R.M. is supported by NSF Grant PHY-1307513, and the work of M.R.P. is supported by NSF Grant PHY-1521053.

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Communicated by N. A. Nekrasov