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Coherence on Fractals Versus Pointwise Convergence for the Schrödinger Equation

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Abstract: We consider Carleson's problem regarding convergence for the Schrödinger equation in dimensions $d \geq 2$. We show that if the solution converges almost everywhere with respect to α-Hausdorff measure to its initial datum as time tends to zero, for all data $H^s(\mathbb{R}^d)$, then $s \geq \frac{d}{2(d+2)}(d+1-\alpha)$. This strengthens and generalises results of Bourgain and Dahlberg–Kenig.

1. Introduction

Consider the Schrödinger equation, $i\partial_t u + \Delta u = 0$, in $\mathbb{R}^{d+1}$, with initial datum $u(\cdot, 0) = u_0$ in the Bessel potential/Sobolev space defined as usual by

$$H^s(\mathbb{R}^d) = (1-\Delta)^{-s/2} L^2(\mathbb{R}^d) := \Big\{ G_s * f \ : \ f \in L^2(\mathbb{R}^d) \Big\}.$$

The Bessel kernel G_s is defined via its Fourier transform; $\widehat{G_s} = (1+|\cdot|^2)^{-s/2}$. In [8], Carleson proposed the problem of identifying the exponents $s > 0$ for which

$$\lim_{t \to 0} u(x,t) = u_0(x), \qquad \text{a.e.} \quad x \in \mathbb{R}^d, \qquad \forall\, u_0 \in H^s(\mathbb{R}^d), \tag{1}$$

with respect to the Lebesgue measure, and proved that (1) holds as long as $s \geq 1/4$ and $d = 1$. Dahlberg and Kenig then showed that this condition is necessary in all dimensions, providing a complete solution for the one-dimensional case [11].

The problem in higher dimensions has since been studied by many authors; see for example [4,5,7,10,16,21,22,26,28–30]. The best known positive result in two dimensions is due to Lee [17], who proved that $s > 3/8$ is sufficient for (1) to hold via bilinear techniques. In higher dimensions, Bourgain [6] proved that $s > \frac{2d-1}{4d}$ is sufficient using multilinear arguments.

The solution is typically represented as $u(\cdot,t) := \lim_{N\to\infty} S_{\Psi_N}(t)u_0$, where

$$S_{\Psi_N}(t)u_0(x) := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} \Psi(N^{-1}\xi)\,\widehat{u}_0(\xi)\,e^{ix\cdot\xi - it|\xi|^2}\,d\xi, \tag{2}$$

and Ψ is a fixed function, equal to one near the origin, that decays in such a way that the integral is well-defined. For convenience we take $\Psi(\xi) = \prod_{j=1}^d \psi(\xi_j)$, where ψ is differentiable, supported in the interval $[-2,2]$ and equal to one on $[-1,1]$. The limit is usually taken with respect to the L^2-norm, but here we will take all limits pointwise, at each point that they exist. This coincides with the usual L^2-limit almost everywhere with respect to the Lebesgue measure, and will allow for a more refined analysis.

It was thought that the necessary condition of Dahlberg and Kenig could also be sufficient in higher dimensions; see for example [14]. However, Bourgain proved a stronger necessary condition in five and more dimensions. Combining their results we obtain the following theorem.

Theorem 1. [6,11] *Let $d \geq 2$. Then, for any*

$$s < \max\left\{\frac{1}{4}, \frac{d-2}{2d}\right\}, \tag{3}$$

there exists $u_0 \in H^s(\mathbb{R}^d)$ such that

$$\limsup_{t\to 0} |u(x,t)| = \infty$$

for all x in a set of positive Lebesgue measure.

This was recently strengthened in [19] to $s < 1/2 - 1/(d+2)$ when $d \geq 3$. Here we give a constructive proof of this result as well as generalising to α-Hausdorff measure (generalising also the $d = 2$ case of Theorem 1). The fractal measure version of the problem had been previously considered in [1,3,9,18,20,25]; however, no nontrivial counterexamples, beyond the Lebesgue measure case, had been given until now.

Theorem 2. *Let $d \geq 2$ and $d/2 \leq \alpha \leq d$. Then, for any*

$$s < \frac{d}{2(d+2)}\big(d+1-\alpha\big), \tag{4}$$

there exists $u_0 \in H^s(\mathbb{R}^d)$ such that

$$\limsup_{t\to 0} |u(x,t)| = \infty$$

for all x in a set of positive α-Hausdorff measure.

The range of α cannot be extended due to the positive results of [1]. There it was proven that divergence can only occur on sets which are null with respect α-Hausdorff measure when $s > (d-\alpha)/2$ and $\alpha \leq d/2$. This is optimal as the data may already be singular on sets of Hausdorff dimension α if $s \leq (d-\alpha)/2$; see [34], and here we see that it is also optimal with respect to the range of α. Indeed we see that $\alpha_d(s) = \sup_{u_0\in H^s} \dim\{x \in \mathbb{R}^d : \limsup_{t\to 0} |u(x,t)| = \infty\}$ is not differentiable at $s = d/4$.

The analogous convergence problem for the quantum harmonic oscillator, where the Laplacian is replaced by $-\Delta + |x|^2$, was introduced by Yajima [32,33]. Via an equivalence between maximal estimates with respect to fractal measures [18, Theorem 1.3], the problems turned out to be equivalent however, and so Theorem 2 also holds if the free Schrödinger equation is replaced by the Schrödinger equation associated to the harmonic oscillator.

In contrast to the earlier results, which relied on the Nikišin–Stein maximal principle, the proof of Theorem 2 will be explicit, identifying the datum u_0 for which the solution diverges. The necessary condition of Dahlberg and Kenig relied on the existence of concentrated solutions, or wave-packets, that travel over large areas. On the other hand, Bourgain considered sums of data, with different frequencies, carefully chosen to create regions of constructive interference, recalling Young's double slit experiment.

In the light of Bourgain's example, a physical interpretation of Carleson's problem could be to identify the lowest s (a measure of average frequency) at which an initial state (or configuration of slits) can generate interference patterns, thus obscuring their original state. Indeed, it seems reasonable to suppose that in the absence of interference the initial state should be recoverable by tracing back from the later states. Inspired by this interpretation, we take data similar to the Dirichlet kernels, see [2] and the forthcoming Fig. 1, for which the corresponding solutions interfere with themselves periodically in time. The constructive interference reoccurs in the same relatively small regions of space, however we can energise the data so that the whole solution travels in a single direction. We then use a quantitative ergodicity argument to show that this direction can be taken so that the regions of constructive interference reappear in different places sufficiently often.

The one-dimensional case seems qualitatively different from the higher dimensional cases. In part this is due to the fact that a travelling wave, from which the Dahlberg–Kenig example is derived, can fill the whole space in that case, beating our necessary condition derived from interference patterns. We can think of no good reason to suppose that our necessary condition should not also be sufficient in the higher-dimensional Lebesgue measure case. If that were true, then our necessary condition in the fractal cases would most likely also be sufficient, as (4) would then interpolate between two sharp results.

Finally, we remark that, by a standard density argument, Theorem 2 implies that a local maximal estimate can only hold for $s \geq 1/2 - 1/(d+2)$. Then by an equivalence between local and global estimates [24, Theorem 3], we obtain the following necessary condition. Note that the regularity is twice that required for almost everywhere convergence. These estimates can be viewed as mixed-norm Strichartz estimates, where the order of time and space has been reversed compared to those first considered in [15,31].

Corollary 1. *Let $d \geq 2$ and suppose that there is a constant C_s such that*

$$\Big\| \sup_{0<t<1} \big|u(\cdot, t)\big| \Big\|_{L^2(\mathbb{R}^d)} \leq C_s \|u_0\|_{H^s(\mathbb{R}^d)}$$

whenever u_0 is a Schwartz function. Then $s \geq \frac{d}{d+2}$.

2. Proof of Theorem 2

Let $0 < \sigma < \frac{d+1-\alpha}{d+2}$ and $\lambda := 2^{\frac{M}{1-\sigma}}$ with M a large integer to be fixed later. As we only consider $\alpha \geq d/2$, we have that $\sigma < 1/2$. For $j \in \mathbb{N}$, define

$$\Omega^j = \big\{\xi \in 2\pi\lambda^{j(1-\sigma)}\mathbb{Z}^d \ : \ \lambda^{j-1} \leq |\xi_m| < \lambda^j, \ m = 1, \ldots, d\big\} + Q\Big(0, \frac{\varepsilon_1}{\sqrt{d}}\Big),$$

where $\varepsilon_1 > 0$ is a small constant to be chosen later. Here $Q(0,\ell)$ denotes the closed cube centred at the origin with side-length ℓ, and we denote its interior by $\mathring{Q}(0,\ell)$. We consider initial data

$$u_0 := \sum_{j\in\mathbb{N}} f_{\theta_j}, \qquad \theta_j \in \mathbb{S}^{d-1}, \tag{5}$$

where, for $0 < \delta < \sigma/4$, we define f_{θ_j} by

$$f_{\theta_j}(x) := e^{i\pi\lambda^j\theta_j\cdot x} f_j(x), \qquad \widehat{f_j} := \lambda^{-j(d\sigma-\delta)}\chi_{\Omega^j}.$$

Note that[1] $|\Omega^j| \simeq \lambda^{jd\sigma}$, so that $\|f_j\|_{H^s} \simeq \lambda^{-j(\frac{d\sigma}{2}-\delta-s)}$, and so $u_0 \in H^s$ provided

$$s < \frac{d\sigma}{2} - \delta.$$

Eventually we will let σ tend to $\frac{d+1-\alpha}{d+2}$ and δ tend to zero, covering all the cases of the range (4).

Consider now the 'dual' sets

$$X_0^j := \{x \in \lambda^{j(\sigma-1)}\mathbb{Z}^d \,:\, |x| \le 2\} + \mathring{Q}(0, \varepsilon_2\lambda^{-j}),$$
$$T^j := \big\{t \in \lambda^{j(2\sigma-1)}\mathbb{Z} \,:\, 0 < t < 1\big\},$$

where $\varepsilon_2 > 0$ is a small constant to be chosen later. As shown by Barceló, Bennett, Carbery, Ruiz and Vilela in [2], we have

$$\left|S_{\Psi_N}\Big(\frac{t}{2\pi\lambda^j}\Big) f_j(x)\right| \gtrsim \lambda^{j\delta} \qquad \text{for all} \quad (x,t) \in X_0^j \times T^j, \tag{6}$$

as long as $N \ge \lambda^j$ and $\varepsilon_1, \varepsilon_2$ are taken sufficiently small. This is true because the phase in (2) never strays too far from zero modulo $2\pi i$, and so the different pieces of the integral, corresponding to different pieces of Ω^j, do not cancel each other out at these points (see Fig. 1 for a real-variable, one-dimensional and time-independent sketch).

Proof of (6) We first claim that

$$x\cdot\xi \in 2\pi\mathbb{Z} + (-\tfrac{1}{10}, \tfrac{1}{10}) \quad \text{if} \quad \xi \in \Omega^j \quad \text{and} \quad x \in X_0^j \tag{7}$$

and

$$\frac{t}{2\pi\lambda^j}|\xi|^2 \in \mathbb{Z} + \big(-\tfrac{1}{10}, \tfrac{1}{10}\big) \quad \text{if} \quad \xi \in \Omega^j \quad \text{and} \quad t \in T^j. \tag{8}$$

Indeed we can write

$$\xi = 2\pi\lambda^{j(1-\sigma)}\nu + v, \quad \text{where} \quad \nu \in \mathbb{Z}^d, \quad \frac{\lambda^{j\sigma-1}}{2\pi} \le |\nu_m| < \frac{\lambda^{j\sigma}}{2\pi}, \quad |v_m| \le \frac{\varepsilon_1}{2\sqrt{d}},$$

and

$$x = \lambda^{j(\sigma-1)}\omega + w, \quad \text{where} \quad \omega \in \mathbb{Z}^d, \quad |\omega| \le 2\lambda^{j(1-\sigma)}, \quad |w_m| < \frac{\varepsilon_2}{2}\lambda^{-j},$$

[1] We write $a \lesssim b$ ($a \gtrsim b$) whenever a and b are nonnegative quantities that satisfy $a \le Cb$ ($a \ge Cb$) for a constant $C > 0$. We write $a \simeq b$ when $a \lesssim b$ and $b \lesssim a$.

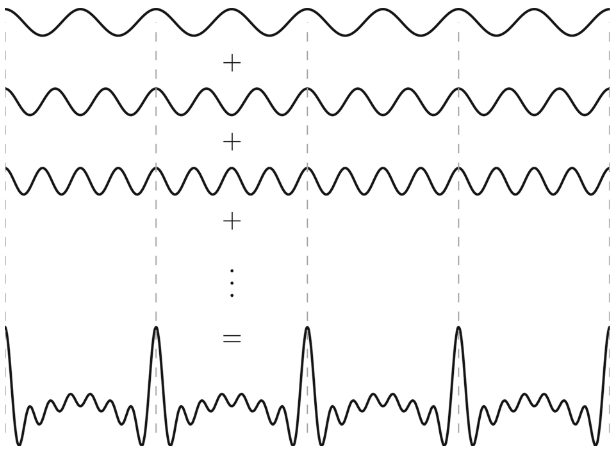

Fig. 1. Constructive interference

for $m = 1, \ldots, d$, so that

$$x \cdot \xi = 2\pi\omega \cdot \nu + \lambda^{j(\sigma-1)}\omega \cdot v + 2\pi\lambda^{j(1-\sigma)}\nu \cdot w + w \cdot v$$
$$=: I_1 + I_2 + I_3 + I_4.$$

Since $I_1 \in 2\pi\mathbb{Z}$ and

$$|I_2| \le \lambda^{j(\sigma-1)}|\omega||v| \le \lambda^{j(\sigma-1)}2\lambda^{j(1-\sigma)}\frac{\varepsilon_1}{2} = \varepsilon_1,$$
$$|I_3| \le 2\pi\lambda^{j(1-\sigma)}|\nu||w| \le \sqrt{d}\lambda^{j(1-\sigma)}\lambda^{j\sigma}\frac{\sqrt{d}\varepsilon_2}{2}\lambda^{-j} = \frac{d\varepsilon_2}{2},$$
$$|I_4| \le |w||v| \le \frac{\sqrt{d}}{4}\varepsilon_2\varepsilon_1\lambda^{-j},$$

we have that (7) holds by taking ε_1 and ε_2 sufficiently small. For (8) we write

$$t = \lambda^{j(2\sigma-1)}\tau, \qquad \text{where} \quad \tau \in \mathbb{N}, \;\; 0 < \tau \le \lambda^{j(1-2\sigma)},$$

so that

$$\frac{t}{2\pi\lambda^j}|\xi|^2 = \frac{1}{2\pi}\lambda^{2j(\sigma-1)}\tau(4\pi^2\lambda^{2j(1-\sigma)}|\nu|^2 + |v|^2 + 4\pi\lambda^{j(1-\sigma)}\nu \cdot v)$$
$$=: II_1 + II_2 + II_3.$$

Since $II_1 \in 2\pi\mathbb{Z}$, while

$$|II_2| \le \frac{1}{2\pi}\lambda^{2j(\sigma-1)}|\tau||v|^2 \le \frac{1}{2\pi}\lambda^{2j(\sigma-1)}\lambda^{j(1-2\sigma)}\frac{\varepsilon_1^2}{4} = \frac{\varepsilon_1^2\lambda^{-j}}{8\pi},$$
$$|II_3| \le 2\lambda^{j(\sigma-1)}|\tau||\nu||v| \le 2\lambda^{j(\sigma-1)}\lambda^{j(1-2\sigma)}\frac{\sqrt{d}\lambda^{j\sigma}}{2\pi}\frac{\varepsilon_1}{2} \le \frac{\sqrt{d}}{2\pi}\varepsilon_1,$$

we see that (8) is satisfied once ε_1 and ε_2 are taken sufficiently small.

Now as long as $N \geq \lambda^j$ and $\xi \in \Omega^j$ we have $\Psi_N(\xi) = \Psi(N^{-1}\xi) = 1$ so that

$$S_{\Psi_N}\Big(\frac{t}{2\pi\lambda^j}\Big) f_j(x) = \frac{\lambda^{-j(d\sigma-\delta)}}{(2\pi)^{d/2}} \int_{\Omega^j} e^{ix\cdot\xi - i\frac{t}{2\pi\lambda^j}|\xi|^2} d\xi.$$

Then (7) and (8) imply that the phase is close enough to zero modulo $2\pi i$ as long as $(x,t) \in X_0^j \times T^j$ yielding (6). Here we have used that $|\Omega^j| \simeq \lambda^{jd\sigma}$. □

By a change of variables, the modulations that we introduced in the definition (5) of u_0 force the waves to travel in the following sense:

$$\Big|S_{\Psi_N}\Big(\frac{t}{2\pi\lambda^j}\Big) f_{\theta_k}(x)\Big| = \Big|S_{\Psi_N(\cdot+\pi\lambda^k\theta_k)}\Big(\frac{t}{2\pi\lambda^j}\Big) f_k(x - \lambda^{k-j}t\theta_k)\Big|. \tag{9}$$

As λ^{k-j} can be very large, we now look for an upper bound on the solutions associated to f_k, that is independent of the size of $|x|$. First we prove that when $k > 2j$, we have

$$\Big|S_{\Psi_N(\cdot+\pi\lambda^k\theta_k)}\Big(\frac{t}{2\pi\lambda^j}\Big) f_k(x)\Big| \lesssim \lambda^{-k\delta} \quad \text{for all} \quad (x,t) \in \mathbb{R}^d \times T^j. \tag{10}$$

When $j < k \leq 2j$, things are a little more difficult, and so we consider

$$X_0^{k,\delta} := \lambda^{k(\sigma-1)}\mathbb{Z}^d + Q(0, \varepsilon_2\lambda^{-k(1-2\delta)}).$$

This is a slightly fatter version of X_0^k, without the restriction that $|x| \leq 2$. In this case we are able to prove the same bound in a restricted region;

$$\Big|S_{\Psi_N(\cdot+\pi\lambda^k\theta_k)}\Big(\frac{t}{2\pi\lambda^j}\Big) f_k(x)\Big| \lesssim \lambda^{-k\delta} \quad \text{for all} \quad (x,t) \in (\mathbb{R}^d \backslash X_0^{k,\delta}) \times T^j \tag{11}$$

as long as $N \geq \lambda^j + \pi\lambda^{2j}$ with λ sufficiently large. Later we will see that this is sufficient.

Proof of (10) and (11) As before we write

$$\xi = 2\pi\lambda^{k(1-\sigma)}\nu + v \;\; \text{where} \;\; \nu \in \mathbb{Z}^d, \;\; \frac{\lambda^{k\sigma-1}}{2\pi} \leq |\nu_m| < \frac{\lambda^{k\sigma}}{2\pi}, \;\; |v_m| \leq \frac{\varepsilon_1}{2\sqrt{d}},$$
$$t = \lambda^{j(2\sigma-1)}\tau, \quad \text{where} \quad \tau \in \mathbb{N}, \;\; 0 < \tau \leq \lambda^{j(1-2\sigma)},$$

however for now we place no restriction on x;

$$x = \lambda^{k(\sigma-1)}\omega + w, \quad \text{where} \quad \omega \in \mathbb{Z}^d, \quad 0 \leq |w_m| < \lambda^{k(\sigma-1)}.$$

As before, we have that

$$\begin{aligned} x \cdot \xi &= (\lambda^{k(\sigma-1)}\omega + w) \cdot (2\pi\lambda^{k(1-\sigma)}\nu + v) \\ &= 2\pi\omega\cdot\nu + \lambda^{k(\sigma-1)}\omega\cdot v + 2\pi\lambda^{k(1-\sigma)}\nu\cdot w + w\cdot v. \end{aligned} \tag{12}$$

and the first term in (12) belongs to $2\pi\mathbb{Z}$. Similarly,

$$\begin{aligned} \frac{t}{2\pi\lambda^j}|\xi|^2 &= \frac{1}{2\pi}\lambda^{2j(\sigma-1)}\tau\big(4\pi^2\lambda^{2k(1-\sigma)}|\nu|^2 + |v|^2 + 4\pi\lambda^{k(1-\sigma)}\nu\cdot v\big) \\ &=: 2\pi\lambda^{2(k-j)(1-\sigma)}\tau|\nu|^2 + \frac{1}{2\pi}\lambda^{2j(\sigma-1)}\tau|v|^2 + 2\pi\lambda^{(k-2j)(1-\sigma)}\tau\nu\cdot v \end{aligned} \tag{13}$$

and again the first term belongs to $2\pi\mathbb{Z}$, since $\lambda = 2^{\frac{M}{1-\sigma}}$ with M integer. First we consider the case $k > 2j$ and prove (10). In fact, to help us with the proof of (11), we will suppose less, only that

$$\lambda^{(k-2j)(1-\sigma)}\tau > \frac{\varepsilon_2}{4}\lambda^{-k(\sigma-2\delta)}. \tag{14}$$

Using (12) and (13), we can write

$$S_{\Psi_N(\cdot+\pi\lambda^k\theta_k)}\Big(\frac{t}{2\pi\lambda^j}\Big) f_k = \frac{\lambda^{-k(d\sigma-\delta)}}{(2\pi)^{d/2}} \prod_{m=1}^{d} \sum_{\substack{\nu_m\in\mathbb{Z} \\ \lambda^{k\sigma-1}\le 2\pi|\nu_m|<\lambda^{k\sigma}}} e^{i\phi_{1,m}} \int_{-\frac{\varepsilon_1}{\sqrt{d}}}^{\frac{\varepsilon_1}{\sqrt{d}}} \psi_{N,m}\, e^{i\phi_{2,m}+i\phi_{3,m}} dv_m,$$

where

$$\begin{aligned} \psi_{N,m} &:= \psi\Big(\frac{2\pi\lambda^{j(1-\sigma)}\nu_m + v_m + \pi\lambda^k\theta_{k,m}}{N}\Big) \\ \phi_{1,m} &= 2\pi\lambda^{k(1-\sigma)}\nu_m w_m, \qquad \phi_{2,m} := w_m v_m - \frac{1}{2\pi}\lambda^{2j(\sigma-1)}\tau v_m^2, \\ \phi_{3,m} &= \lambda^{k(\sigma-1)}\omega_m v_m - 2\pi\lambda^{(k-2j)(1-\sigma)}\tau\nu_m v_m. \end{aligned}$$

We split each sum into two parts. For $m = 1, \dots, d$, we let $\mathcal{E}_m$ denote

$$\{\nu_m \in \mathbb{Z} : \lambda^{k\sigma-1} \le 2\pi|\nu_m| < \lambda^{k\sigma},\ |\lambda^{k(\sigma-1)}\omega_m - 2\pi\lambda^{(k-2j)(1-\sigma)}\tau\nu_m| < \lambda^{k\delta}\}.$$

As we are supposing (14), it is clear that $\#\mathcal{E}_m \lesssim \lambda^{k(\sigma-\delta)}$, and so trivially

$$\Big|\sum_{\nu_m\in\mathcal{E}_m} e^{i\phi_{1,m}} \int_{-\frac{\varepsilon_1}{\sqrt{d}}}^{\frac{\varepsilon_1}{\sqrt{d}}} \psi_{N,m} e^{i\phi_{2,m}+i\phi_{3,m}}\, dv_m\Big| \lesssim \lambda^{k(\sigma-\delta)}. \tag{15}$$

It remains to handle the contribution of the terms in the set $\mathcal{E}_m^c$ denoting

$$\{\nu_m \in \mathbb{Z} : \lambda^{k\sigma-1} \le 2\pi|\nu_m| < \lambda^{k\sigma},\ |\lambda^{k(\sigma-1)}\omega_m - 2\pi\lambda^{(k-2j)(1-\sigma)}\tau\nu_m| \ge \lambda^{k\delta}\}.$$

For this we use the cancelation in the integrals via van der Corput's lemma; see for example [27, pp. 344].

Lemma 1. *Let $\psi \in C^1(a,b)$ and $|\phi'(x)| \ge \gamma > 0$ with ϕ' monotone on (a,b). Then*

$$\Big|\int_a^b \psi(x)\, e^{i\phi(x)} dx\Big| \lesssim \gamma^{-1}(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}).$$

Since $|w_m| < \lambda^{k(\sigma-1)}$ and

$$\Big|\frac{1}{\pi}\lambda^{2j(\sigma-1)}\tau v_m\Big| \le \frac{1}{\pi}\lambda^{2j(\sigma-1)}\lambda^{j(1-2\sigma)}\frac{\varepsilon_1}{2\sqrt{d}} = \frac{\lambda^{-j}}{\pi}\frac{\varepsilon_1}{2\sqrt{d}},$$

we see that, for λ large enough, $|\partial_{v_m}\phi_{2,m}| \le 1$. On the other hand, by the definition of $\mathcal{E}_m^c$ we have $|\partial_{v_m}\phi_{3,m}| \ge \lambda^{k\delta}$, so that

$$|\partial_{v_m}(\phi_{2,m} + \phi_{3,m})| \ge \frac{1}{2}\lambda^{k\delta}.$$

Since $\|\psi_{N,m}\|_\infty$ and $\|\partial_{v_m}\psi_{N,m}\|_1$ are bounded by absolute constants, we can take $\gamma = \frac{1}{2}\lambda^{k\delta}$ in the lemma, so that

$$\Big|\sum_{\nu_m\in\mathcal{E}_m^c} e^{i\phi_{1,m}}\int_{-\frac{\varepsilon_1}{\sqrt{d}}}^{\frac{\varepsilon_1}{\sqrt{d}}} \psi_{N,m}\, e^{i\phi_{2,m}+i\phi_{3,m}} dv_m\Big| \lesssim \#\mathcal{E}_m^c\,\lambda^{-k\delta} \lesssim \lambda^{k(\sigma-\delta)}.$$

Combining this with (15), we obtain the desired bound

$$\Big|S_{\Psi_N(\cdot+\pi\lambda^k\theta_k)}\Big(\frac{t}{2\pi\lambda^j}\Big)f_k(x)\Big| \lesssim \lambda^{-k(d\sigma-\delta)}\lambda^{dk(\sigma-\delta)} = \lambda^{-(d-1)k\delta} \le \lambda^{-k\delta}.$$

Now we consider the case $k \le 2j$ and prove (11). In this case we are allowed to impose the extra condition

$$\frac{\varepsilon_2}{2}\lambda^{-k(1-2\delta)} < |w_m| < \lambda^{k(\sigma-1)} - \frac{\varepsilon_2}{2}\lambda^{-k(1-2\delta)}$$

for at least one m. Also we suppose that $N \ge \lambda^j + \pi\lambda^{2j}$ so that $\Psi_N(\cdot + \pi\lambda^k\theta_k) = 1$ in the support of Ω^k. Thus, using (12) and (13), we can write

$$S_{\Psi_N(\cdot+\pi\lambda^k\theta_k)}\Big(\frac{t}{2\pi\lambda^j}\Big)f_k(x) = \frac{\lambda^{-k(d\sigma-\delta)}}{(2\pi)^{d/2}}\prod_{m=1}^{d}\int_{-\frac{\varepsilon_1}{\sqrt{d}}}^{\frac{\varepsilon_1}{\sqrt{d}}} e^{i\phi_{4,m}} \sum_{\substack{\nu_m\in\mathbb{Z}\\ \lambda^{k\sigma-1}\le 2\pi|\nu_m|<\lambda^{k\sigma}}} e^{i2\pi\nu_m\phi_{5,m}}\, dv_m,$$

where

$$\phi_{4,m} = \lambda^{k(\sigma-1)}\omega_m v_m + w_m v_m - \frac{1}{2\pi}\lambda^{2j(\sigma-1)}\tau v_m^2$$

and

$$\phi_{5,m} = \lambda^{k(1-\sigma)}w_m - \lambda^{(k-2j)(1-\sigma)}\tau v_m.$$

Recognising the inner sum as part of the Dirichlet kernel, we note that

$$\Big|\sum_{\substack{\nu_m\in\mathbb{Z}\\ \lambda^{k\sigma-1}\le 2\pi|\nu_m|<\lambda^{k\sigma}}} e^{i2\pi\nu_m\phi_{5,m}}\Big| \lesssim \frac{1}{|\sin\pi\phi_{5,m}|}. \tag{16}$$

Recalling, that for at least one m we have that

$$\frac{\pi}{2}\varepsilon_2\lambda^{-k(\sigma-2\delta)} < |\pi\lambda^{k(1-\sigma)}w_m| < \pi - \frac{\pi}{2}\varepsilon_2\lambda^{-k(\sigma-2\delta)}, \tag{17}$$

we use (16) in that case and bound the other sums in a trivial fashion, so that

$$\Big|S_{\Psi_N(\cdot+\pi\lambda^k\theta_k)}\Big(\frac{t}{2\pi\lambda^j}\Big)f_k(x)\Big| \lesssim \lambda^{-k(d\sigma-\delta)}\lambda^{k(d-1)\sigma}\big|\sin\pi\phi_{5,m}\big|^{-1}.$$

Now given that we have already proven the desired bound (11) under the condition (14), we can suppose the opposite; that

$$\lambda^{(k-2j)(1-\sigma)}\tau \le \frac{\varepsilon_2}{4}\lambda^{-k(\sigma-2\delta)}.$$

Recalling that $\pi\phi_{5,m} = \pi\lambda^{k(1-\sigma)} w_m - \pi\lambda^{(k-2j)(1-\sigma)}\tau v_m$, by (17) we see that

$$\frac{\pi}{4}\varepsilon_2\lambda^{-k(\sigma-2\delta)} < |\pi\phi_{5,m}| < \pi - \frac{\pi}{4}\varepsilon_2\lambda^{-k(\sigma-2\delta)}.$$

Thus,

$$\left|S_{\Psi_N(\cdot+\pi\lambda^k\theta_k)}\left(\frac{t}{2\pi\lambda^j}\right) f_k(x)\right| \lesssim \lambda^{-k(d\sigma-\delta)}\lambda^{k(d-1)\sigma}\lambda^{k(\sigma-2\delta)} = \lambda^{-k\delta}$$

which completes the proof of (10). □

Now let $N \geq \lambda^j + \pi\lambda^{2j}$ and consider first

$$X^j_{t\theta_j} := X^j_0 + t\theta_j.$$

Noting that $\Psi_N = \Psi_N(\cdot + \pi\lambda^j\theta_j) = 1$ in Ω^j, and that

$$(x,t) \in X^j_{t\theta_j} \times T^j \quad \Rightarrow \quad (x - t\theta_j, t) \in X^j_0 \times T^j,$$

the combination of (6) and (9) tell us that

$$\left|S_{\Psi_N}\left(\frac{t}{2\pi\lambda^j}\right) f_{\theta_j}(x)\right| \gtrsim \lambda^{j\delta} \quad \text{for all} \quad (x,t) \in X^j_{t\theta_j} \times T^j. \tag{18}$$

Now let $j < k \leq 2j$ and consider

$$X^{k,\delta}_{\lambda^{k-j}t\theta_k} := X^{k,\delta}_0 + \lambda^{k-j}t\theta_k,$$

Noting that

$$(x,t) \in (\mathbb{R}^d \backslash X^{k,\delta}_{\lambda^{k-j}t\theta_k}) \times T^j \quad \Rightarrow \quad (x - \lambda^{k-j}t\theta_k, t) \in (\mathbb{R}^d \backslash X^{k,\delta}_0) \times T^j,$$

the combination of (9) and (11) tell us that

$$\left|S_{\Psi_N}\left(\frac{t}{2\pi\lambda^j}\right) f_{\theta_k}(x)\right| \lesssim \lambda^{-k\delta} \quad \text{for all} \quad (x,t) \in (\mathbb{R}^d \backslash X^{k,\delta}_{\lambda^{k-j}t\theta_k}) \times T^j. \tag{19}$$

Similarly for $k > 2j$, since (9) and (10), we have that

$$\left|S_{\Psi_N}\left(\frac{t}{2\pi\lambda^j}\right) f_{\theta_k}(x)\right| \lesssim \lambda^{-k\delta} \quad \text{for all} \quad (x,t) \in \mathbb{R}^d \times T^j. \tag{20}$$

Consider now

$$\Gamma^j_{t\theta_j} := X^j_{t\theta_j} \backslash \bigcup_{j<k\leq 2j} X^{k,\delta}_{\lambda^{k-j}t\theta_k} \quad \text{and} \quad \Gamma^j := \bigcup_{t\in T^j} \Gamma^j_{t\theta_j}.$$

An immediate consequence of (18), (19) and (20) is that if $x \in \Gamma^j$, there exists a time $t_j(x) \in T^j$ such that

(i) $\left|S_{\Psi_N}\left(\frac{t_j(x)}{2\pi\lambda^j}\right) f_{\theta_j}(x)\right| \gtrsim \lambda^{j\delta}$;

(ii) $\left|S_{\Psi_N}\left(\frac{t_j(x)}{2\pi\lambda^j}\right) f_{\theta_k}(x)\right| \lesssim \lambda^{-k\delta} \quad \text{for all} \quad k > j.$

Finally, we consider the set of x that belong to infinitely many Γ^j; that is

$$\Gamma := \bigcap_{n\in\mathbb{N}} \bigcup_{j>n} \Gamma^j.$$

Now for $x \in \Gamma$, there exists an infinite subset $J(x) \subset \mathbb{N}$ with an associated sequence of times $t_j(x) \in T^j$ for all $j \in J(x)$ such that both (i) and (ii) are satisfied. Recalling the definition (5) of u_0, by the triangle inequality

$$\Big|S_{\Psi_N}\Big(\frac{t_j(x)}{2\pi\lambda^j}\Big)f(x)\Big| \gtrsim \Big|S_{\Psi_N}\Big(\frac{t_j(x)}{2\pi\lambda^j}\Big)f_{\theta_j}(x)\Big| - |A_1| - |A_2|,$$

where

$$A_1 := \sum_{k<j} S_{\Psi_N}\Big(\frac{t_j(x)}{2\pi\lambda^j}\Big)f_{\theta_k}(x) \quad \text{and} \quad A_2 := \sum_{k>j} S_{\Psi_N}\Big(\frac{t_j(x)}{2\pi\lambda^j}\Big)f_{\theta_k}(x).$$

By (i) we have that

$$\Big|S_{\Psi_N}\Big(\frac{t_j(x)}{2\pi\lambda^j}\Big)f_{\theta_j}(x)\Big| \gtrsim \lambda^{j\delta}.$$

On the other hand, by bounding the terms trivially and taking λ sufficiently large, we can arrange that the following holds:

$$|A_1| \le \sum_{k<j} \lambda^{k\delta} \le \frac{1}{2}\Big|S_{\Psi_N}\Big(\frac{t_j(x)}{2\pi\lambda^j}\Big)f_{\theta_j}(x)\Big|.$$

Finally, by (ii) we have that

$$|A_2| \le \sum_{k>j} \lambda^{-k\delta} \lesssim 1,$$

so that

$$\Big|u\Big(x, \frac{t_j(x)}{2\pi\lambda^j}\Big)\Big| = \lim_{N\to\infty}\Big|S_{\Psi_N}\Big(\frac{t_j(x)}{2\pi\lambda^j}\Big)f(x)\Big| \gtrsim \lambda^{j\delta}.$$

We see that, for any $x \in \Gamma$, there is a sequence of times $\frac{t_j(x)}{2\pi\lambda^j}$ that satisfy

$$\Big|u\Big(x, \frac{t_j(x)}{2\pi\lambda^j}\Big)\Big| \to \infty \qquad \text{as} \qquad \frac{t_j(x)}{2\pi\lambda^j} \to 0.$$

Now, recalling that $s < \frac{d\sigma}{2} - \delta$, the proof would be complete if we could prove that the α-Hausdorff measure of Γ were positive, taking δ and σ sufficiently close to 0 and $\frac{d+1-\alpha}{d+2}$, respectively. For this we must choose the modulation directions θ_j appropriately, via the ergodic argument of the following section. In fact, in the final section, we do not conclude that the α-Hausdorff measure of Γ is positive, only that the β'-Hausdorff measure of Γ is positive for any $\beta' < \beta < \alpha$. However, this is sufficient as we could have started the proof with an $\alpha' > \alpha$ that also satisfies

$$s < \frac{d}{2(d+2)}\big(d - \alpha' + 1\big),$$

and performed all of the previous arguments for this $\alpha' > d/2$. The d-Hausdorff measure case is slightly different (we cannot choose an $\alpha' > d$), however in that case there is a slightly more direct argument that allows us to conclude that the measure of Γ is positive.

3. A Quantitative Ergodic Lemma

It is well-known that linear flow on the torus, in most directions, eventually passes arbitrarily close to every point. Here we show that this remains true when only considering equidistant points on the trajectory, and we quantify how long we must wait to get near to every point. To see that the quantification is sharp, we place a ball of radius $\varepsilon R^{(\kappa-1)/d}$ at each of the $R^{1-\delta}$ equidistant points. Even if these balls were disjoint their measure would be less than $\varepsilon^d R^{\kappa-\delta}$, and so we would not get close to all the points of the torus if $\delta \geq \kappa$.

Lemma 2. *Let $d \geq 2$, $0 < \varepsilon, \delta < 1$ and $\kappa > \frac{1}{d+1}$. Then, if $\delta < \kappa$ and R is sufficiently large, there is a $\theta \in \mathbb{S}^{d-1}$ for which, given any $y \in \mathbb{T}^d$, there is a $t_y \in R^\delta\mathbb{Z} \cap (0, R)$ such that*

$$|y - t_y\theta| \leq \varepsilon R^{(\kappa-1)/d}.$$

Proof. Let $\varphi : \mathbb{T}^d \to [0, (2/\varepsilon)^d)$ be smooth, supported in $B(0, \varepsilon/2)$, and such that $\int \varphi = 1$, and write

$$\varphi_R(y) := \varphi\big(R^{\frac{1-\kappa}{d}} y\big).$$

It will suffice to show that there exists $\theta \in \mathbb{S}^{d-1}$ such that, for all $y \in \mathbb{T}^d$, there is a $t \in (R^\delta\mathbb{Z} \cap (0, R)) + [-\frac{\varepsilon}{2}R^{-1}, \frac{\varepsilon}{2}R^{-1}]$ satisfying

$$\varphi_R(y - t\theta) > 0.$$

Let $\eta : \mathbb{R} \to [0, 2/\varepsilon)$ be a smooth function, supported in the interval $(-\varepsilon/2, \varepsilon/2)$, and such that $\int \eta = 1$. Defining

$$\eta_R(t) := R^\delta \sum_{\substack{j\in\mathbb{Z} \\ 0<j<R^{1-\delta}}} \eta\big(R(t - R^\delta j)\big),$$

and noting that η_R is supported in $(R^\delta\mathbb{Z} \cap (0, R)) + [-\frac{\varepsilon}{2}R^{-1}, \frac{\varepsilon}{2}R^{-1}]$, it will suffice to show that there exists $\theta \in \mathbb{S}^{d-1}$ such that, for all $y \in \mathbb{T}^d$,

$$\int_{\mathbb{R}} \varphi_R(y - t\theta)\eta_R(t)\, dt > 0.$$

Expanding in Fourier series;

$$\varphi_R(y - t\theta) = \widehat{\varphi_R}(0) + \sum_{\substack{k\in\mathbb{Z}^d \\ k\neq 0}} \widehat{\varphi_R}(k)e^{2\pi i y\cdot k}e^{-2\pi i t\theta\cdot k} =: \widehat{\varphi_R}(0) + \Phi(t, y, \theta),$$

and noting that $\int_{\mathbb{R}} \eta_R \simeq 1$ and $\widehat{\varphi_R}(0) = \int_{\mathbb{T}^d} \varphi_R = R^{\kappa-1}$, it would be sufficient to find $\theta \in \mathbb{S}^d$ such that

$$\Big|\int_{\mathbb{R}} \Phi(t, y, \theta)\eta_R(t)\, dt\Big| \lesssim R^{-\gamma}\log R, \qquad \gamma > 1 - \kappa, \tag{21}$$

whenever $y \in \mathbb{T}^d$.

For the proof of (21), we note that

$$\begin{aligned}\Big|\int_{\mathbb{R}} \Phi(t,y,\theta)\eta_R(t)\,dt\Big| &\leq \sum_{\substack{k\in\mathbb{Z}^d\\ k\neq 0}} \Big|\widehat{\varphi_R}(k)\Big|\,\Big|\int_{\mathbb{R}} e^{-2\pi i t\theta\cdot k}\eta_R(t)\,dt\Big| \\ &\lesssim \sum_{\substack{k\in\mathbb{Z}^d\\ k\neq 0}} \frac{R^{\kappa-1}}{\Big(1+R^{\frac{\kappa-1}{d}}|k|\Big)^{2d}}\Big|\widehat{\eta_R}(\theta\cdot k)\Big|,\end{aligned}$$

where the second inequality follows by integrating by parts in the formula for the Fourier coefficients. Noting that the right-hand side no longer depends on y, in order to find a $\theta\in\mathbb{S}^{d-1}$ such that (21) holds for all $y\in\mathbb{T}^d$, it will suffice to prove that the right-hand side is similarly bounded after averaging over the sphere; that is

$$\int_{\mathbb{S}^{n-1}} \sum_{\substack{k\in\mathbb{Z}^d\\ k\neq 0}} \frac{R^{\kappa-1}}{\Big(1+R^{\frac{\kappa-1}{d}}|k|\Big)^{2d}}\Big|\widehat{\eta_R}(\theta\cdot k)\Big|\,d\theta \lesssim R^{-\gamma}\log R,\quad \gamma>1-\kappa. \tag{22}$$

In order to prove this, we note that

$$\begin{aligned}\widehat{\eta_R}(t) &= R^{\delta-1}\widehat{\eta}(R^{-1}t)\sum_{\substack{j\in\mathbb{Z}\\ 0<j<R^{1-\delta}}} e^{-2\pi i R^{\delta} j t} \\ &= R^{\delta-1}\widehat{\eta}(R^{-1}t)\frac{e^{-2\pi i R^{\delta}(N+1)t}-e^{-2\pi i R^{\delta}t}}{e^{-2\pi i R^{\delta}t}-1},\end{aligned} \tag{23}$$

where[2] $N:=\lfloor R^{1-\delta}\rfloor$. Now as $|\widehat{\eta}|\lesssim 1$ and $1-\delta>1-\kappa$, (22) would follow from

$$\int_{\mathbb{S}^{d-1}} \sum_{\substack{k\in\mathbb{Z}^d\\ k\neq 0}} \frac{R^{\kappa-1}}{\Big(1+R^{\frac{\kappa-1}{d}}|k|\Big)^{2d}}\Big|\frac{\sin(\pi N R^{\delta}\theta\cdot k)}{\sin(\pi R^{\delta}\theta\cdot k)}\Big|\,d\theta \lesssim \log R. \tag{24}$$

To see this, we parametrise the sphere, so that

$$\int_{\mathbb{S}^{d-1}} \Big|\frac{\sin(\pi N R^{\delta}\theta\cdot k)}{\sin(\pi R^{\delta}\theta\cdot k)}\Big|\,d\theta = 2|\mathbb{S}^{d-2}|\int_0^1 \Big|\frac{\sin(\pi N R^{\delta}|k|t)}{\sin(\pi R^{\delta}|k|t)}\Big|\,(1-t^2)^{\frac{d-3}{2}}\,dt. \tag{25}$$

Changing variables and using a well-known property of the Dirichlet kernel (see for example [23, pp. 3]), we get

$$\int_0^1 \Big|\frac{\sin(\pi N R^{\delta}|k|t)}{\sin(\pi R^{\delta}|k|t)}\Big|\,dt \leq \frac{1}{R^{\delta}|k|}\int_0^{R^{\delta}|k|} \Big|\frac{\sin(\pi N t)}{\sin(\pi t)}\Big|\,dt \lesssim \log N \lesssim \log R,$$

which is sufficient to conclude the proof of (24) when $d\geq 3$.

[2] $\lfloor x\rfloor$ denotes the largest integer strictly smaller than $x\in\mathbb{R}$.

When $d = 2$, (25) continues to hold, with $|\mathbb{S}^0| = 2$, but if $R^\delta |k|$ is close to an integer, the singularity in the integral will pick up one of the peaks of the Dirichlet kernel. However, we can still estimate (25) by a constant multiple of

$$\log N + \int_{1-(R^\delta|k|)^{-1}}^{1-(NR^\delta|k|)^{-1}} \Big|\frac{\sin(\pi N R^\delta |k| t)}{\sin(\pi R^\delta |k| t)}\Big| (NR^\delta|k|)^{1/2}\, dt + \int_{1-(NR^\delta|k|)^{-1}}^{1} N(1-t)^{-\frac{1}{2}}\, dt$$
$$\lesssim \log N + \frac{\log N}{R^\delta |k|}(NR^\delta|k|)^{1/2} + \frac{N}{(NR^\delta|k|)^{1/2}} \lesssim \log R + \frac{R^{1/2-\delta}\log R}{|k|^{1/2}}.$$

On the other hand, by an appropriate dyadic decomposition, we have that

$$\sum_{\substack{k\in\mathbb{Z}^2\\ k\neq 0}} \frac{R^{\kappa-1}}{\Big(1+R^{\frac{\kappa-1}{2}}|k|\Big)^4}\frac{1}{|k|^{1/2}} \lesssim R^{\frac{\kappa-1}{4}},$$

so that altogether we obtain (22) with $\gamma = 1/2+(1-\kappa)/4$. Given that we have restricted to $\kappa > 1/3$, we have that $\gamma > 1-\kappa$ and so we are done. $\square$

Remark 1. It might be possible to relax the restriction $\kappa > 1/(d+1)$. Indeed, $\kappa > 0$ suffices in higher dimensions and perhaps this could also be shown to be the case in two dimensions by using a slightly less trivial bound which estimated the number of points $k \in \mathbb{Z}^2$ such that $R^\delta|k|$ is within an $(R^\delta|k|)^{-1}$ neighbourhood of an integer—a relative of the Gauss circle problem. The restriction on κ is harmless for our purposes and so we do not pursue this further.

Definition 1. A set E is ϵ-dense in F if for every point $x \in F$ there is a point $y \in E$ such that $|x-y| < \epsilon$.

The following corollary allows us to choose directions θ_j so that the sets $\cup_{t\in T^j} X^j_{t\theta_j}$ have large measure. The corollary is also optimal, in the sense that the statement fails for larger σ. To see this, we can place balls of radius $\varepsilon R^{-\frac{\alpha}{d}}$ at the points of the following set and assume that the balls are disjoint. Then the volume of such a set would be of the order $R^{d+1-\alpha-(d+2)\sigma}$, a quantity that tends to zero as R tends to infinity when $\sigma > \frac{d+1-\alpha}{d+2}$.

Corollary 2. *Let $d \geq 2$, $d/2 \leq \alpha \leq d$ and $0 < \sigma < \frac{d+1-\alpha}{d+2}$. Then, for any $\varepsilon > 0$ and sufficiently large $R > 1$, there exists $\theta \in \mathbb{S}^{d-1}$ such that*

$$\bigcup_{t\in R^{2\sigma-1}\mathbb{Z}\cap(0,1)} \Big\{x \in R^{\sigma-1}\mathbb{Z}^d \,:\, |x| \leq 2\Big\} + t\theta$$

is $\varepsilon R^{-\frac{\alpha}{d}}$-dense in $B(0, 1/2)$.

Proof. By rescaling, this is equivalent to showing that

$$\bigcup_{t\in R^{\sigma}\mathbb{Z}\cap(0,R^{1-\sigma})} \Big\{x \in \mathbb{Z}^d \,:\, |x| \leq 2R^{1-\sigma}\Big\} + t\theta$$

is $\varepsilon R^{1-\sigma-\frac{\alpha}{d}}$-dense in $B(0, R^{1-\sigma}/2)$ for a certain $\theta \in \mathbb{S}^{d-1}$. By changing $R^{1-\sigma} \to R$, that is to say, for any $y \in B(0, R/2)$ there exists a $x_y \in \mathbb{Z}^d \cap B(0, 2R)$ and $t_y \in R^{\frac{\sigma}{1-\sigma}}\mathbb{Z} \cap (0, R)$ such that

$$|y - (x_y + t_y\theta)| < \varepsilon R^{1-\frac{\alpha}{d(1-\sigma)}},$$

for a fixed $\theta \in \mathbb{S}^{d-1}$, independent of y. By taking the quotient $\mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d$, this would follow if for any $[y] \in \mathbb{T}^d$ there exists $t_y \in R^{\frac{\sigma}{1-\sigma}}\mathbb{Z} \cap (0, R)$ such that

$$|[y] - [t_y\theta]| \; < \; \varepsilon R^{1-\frac{\alpha}{d(1-\sigma)}}, \tag{26}$$

and this is a direct consequence of Lemma 2, by taking $\kappa = d + 1 - \alpha/(1-\sigma)$ and $\delta = \sigma/(1-\sigma)$.

To see that it is enough to prove (26), we assume it is true and cover $B(0, R/2)$ with a family of disjoint copies of axis-parallel $\mathbb{T}^d$. Denote the copy that contains y by $\mathbb{T}^d_y$, and let z_y be the point in $\mathbb{T}^d_y$ such that $[z_y] = [t_y\theta]$. Then $x_y := z_y - t_y\theta \in \mathbb{Z}^d$ and by construction

$$|y - (x_y + t_y\theta)| = |[y] - [t_y\theta]| < \varepsilon R^{1-\frac{\alpha}{d(1-\sigma)}}.$$

Note that we also automatically have that

$$|x_y| \le |y| + |t_y| + \varepsilon R^{1-\frac{\alpha}{d(1-\sigma)}} < \frac{1}{2}R + R + \varepsilon R^{1-\frac{\alpha}{d(1-\sigma)}} < 2R,$$

and so we recover all of the required properties. □

4. The Measure of Γ

We start with the Lebesgue measure case, or equivalently $\alpha = d$. Note that $X^j_{t\theta_j}$ is a union of disjoint open cubes of side-length $\varepsilon_2\lambda^{-j}$, while $X^{k,\delta}_{\lambda^{k-j}t\theta_k}$ is a union of disjoint closed cubes of side-length $\varepsilon_2\lambda^{-(1-2\delta)k}$. The distance between the cubes is approximately $\lambda^{(\sigma-1)j}$ in the case of the former and $\lambda^{(\sigma-1)k}$ in the case of the latter. Thus we see that $\Gamma^j_{t\theta_j}$ is a union of disjoint open sets whose Lebesgue measure $|\cdot|$ is bounded from below by

$$\begin{aligned} &|Q(x, \varepsilon_2\lambda^{-j})| - \Big|Q(x, \varepsilon_2\lambda^{-j}) \cap \bigcup_{j<k\le 2j} X^{k,\delta}_{\lambda^{k-j}t\theta_k}\Big| \\ &\simeq \;\; \varepsilon_2^d\lambda^{-dj} - \varepsilon_2^d\lambda^{-dj}\sum_{k=j+1}^{2j}\lambda^{-d(1-2\delta)k}\lambda^{d(1-\sigma)k} \;\gtrsim\; \varepsilon_2^d\lambda^{-dj}, \end{aligned} \tag{27}$$

provided λ is large enough (recalling $\sigma > 4\delta$). We call these sets pseudo-cubes and denote them by $\mathcal{Q}(x, \varepsilon_2\lambda^{-j})$.

Now using the $\alpha = d$-dimensional version of Corollary 2 with $R = \lambda^j$ and $d \ge 3$, we can choose θ_j in such a way that $B(0, 1/2) \subset \cup_{j\in T^j} X^j_{t\theta_j}$ for all j sufficiently large. Similarly, $\Gamma^j := \cup_{j\in T^j}\Gamma^j_{t\theta_j}$ is a union of pseudo-cubes whose centres are $\varepsilon_2\lambda^{-j}$-dense in $B(0, 1/2)$. This fact, and the bound (27), easily imply $|\Gamma^j| \gtrsim 1$. Thus we see that $\lim_{n\to\infty}|A_n| \gtrsim 1$, where $A_n = \bigcup_{j>n}\Gamma^j$. On the other hand, as everything is contained in the unit cube, it is clear that $|A_1| \lesssim 1$. Therefore, as $\{A_n\}_{n\ge 1}$ is a decreasing sequence of sets ($A_n \supseteq A_{n+1}$), we can conclude that

$$|\Gamma| = \Big|\bigcap_{n\in\mathbb{N}} A_n\Big| \gtrsim 1.$$

When $d/2 \leq \alpha < d$ we no longer have the upper bound on the measure of A_1 and so we will require a different argument.

We recall now some properties of Hausdorff measures and outer measures. For any $E \subset \mathbb{R}^d$ and for any $\delta \in (0, \infty]$ we define

$$\mathcal{H}^\alpha_\delta := \inf \Big\{ \sum_i \delta_i^\alpha \ : \ E \subset \bigcup_i Q(x_i, \delta_i), \ \delta_i \leq \delta \Big\}.$$

The α-Hausdorff measure of the set E is $\mathcal{H}^\alpha(E) := \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(E)$ and the Hausdorff dimension of E is $\dim(E) := \inf\{\alpha : \mathcal{H}^\alpha(E) = 0\}$. Thus if $\mathcal{H}^\alpha(E) > 0$ then $\dim(E) \geq \alpha$. We will also make use of the outer measure $\mathcal{H}^\alpha_\infty(E)$, which is at least sub-additive;

$$\mathcal{H}^\alpha_\infty(E) \leq \sum_j \mathcal{H}^\alpha_\infty(E_j), \quad \text{if} \quad E \subseteq \bigcup_j E_j$$

and satisfies $\mathcal{H}^\alpha_\infty(Q(\cdot, \delta)) = \delta^\alpha$. Combining Theorem 3.2 and Corollary 4.2 of [12] (see also [13, Proposition 8.5] for a simpler version), we obtain the following:

Lemma 3 [12]. *Suppose that there exists a constant* $c > 0$ *such that, for all* $\delta > 0$ *and all cubes* $Q(x, \delta) \subset B(0, 1/2)$*, we have the density condition*

$$\liminf_{j \to \infty} \mathcal{H}^\beta_\infty(E_j \cap Q(x, \delta)) \geq c\delta^\beta,$$

where $\{E_j\}_{j \geq 1}$ *is a sequence of open subsets of* $B(0, 1/2)$*. Then, for all* $\beta' < \beta$*,*

$$\mathcal{H}^{\beta'}\big(\textstyle\bigcap_{n \in \mathbb{N}} \bigcup_{j > n} E_j\big) > 0.$$

We will also require an elementary lemma regarding the following definition.

Definition 2. $A^* \subseteq B(0, 1/2)$ is a quasi-lattice with separation $\rho \ll 1$ if, for any $x \in \rho\mathbb{Z}^d \cap B(0, 1/2)$, there exists a unique $a \in A^*$ such that $|x - a| \leq \rho$.

Lemma 4. *Let* $0 < \epsilon \leq \rho/4$*. If* A *is* ϵ*-dense in* $B(0, 1/2)$*, as defined in Definition 1, then* A *contains a quasi-lattice* A^* *with separation* ρ*.*

Proof. Let $x \in \rho\mathbb{Z}^d \cap B(0, 1/2)$. Since A is ϵ-dense in $B(0, 1/2)$, there exists $a \in A$ such that

$$|x - a| \leq \epsilon. \tag{28}$$

The set A^* then consists of these $a \in A$ satisfying (28), after discarding redundant members in order to satisfy the uniqueness requirement. Note that this is possible since, given two different $x, y \in \rho\mathbb{Z}^d$, it cannot be that the same a satisfies both $|a - x| \leq \epsilon$ and $|a - y| \leq \epsilon$, as this would imply $|x - y| \leq 2\epsilon < \rho$, which would contradict our hypothesis. □

We again use Corollary 2, this time with a generic $d/2 < \alpha < d$, to choose θ_j such that $\Gamma^j := \cup_{j \in T^j} \Gamma^j_{t\theta_j}$ is the union of pseudo-cubes whose centres are $\varepsilon_2 \lambda^{-j\frac{\alpha}{d}}$-dense in $B(0, 1/2)$. Then, using Lemma 4 with $\epsilon = \varepsilon_2 \lambda^{-j\frac{\alpha}{d}}$ and $\rho = \lambda^{-j\frac{\alpha}{d}}$, we can suppose that Γ^j consists of the union of pseudo-cubes whose centres are a quasi-lattice with separation $\lambda^{-j\frac{\alpha}{d}}$.

Take $\beta \in (\frac{d}{d+2}(\alpha+1), \alpha)$. Note that the interval is not empty as we can assume that $\alpha > d/2$, and this also implies $\beta > d/2 \geq 1$. We will prove that $\mathcal{H}^{\beta'}(\Gamma) > 0$, for all $\beta' < \beta$, provided that we take δ and σ close enough to 0 and $\frac{d+1-\alpha}{d+2}$, respectively. By Lemma 3 it will suffice to prove the density condition

$$\liminf_{j\to\infty} \mathcal{H}^{\beta}_{\infty}(\Gamma^j \cap Q(x,\delta)) \geq c\delta^{\beta} \tag{29}$$

for all $\delta > 0$ and cubes $Q(x,\delta) \subset B(0, 1/2)$. That is to say, for sufficiently large j, it is essentially more efficient to cover $\Gamma^j \cap Q(x,\delta)$ with a single cube of side δ, rather than to cover it with the union of smaller cubes centred at the points of the quasi-lattice. Of course the only real competitor is the cover that consists of the disjoint union of cubes of side-length $\varepsilon_2\lambda^{-j}$ placed on top of the pseudo-cubes. However this cover is costed at

$$\sum_i \delta_i^{\beta} \simeq \left(\frac{\delta}{\lambda^{-j\frac{\alpha}{d}}}\right)^{d} (\varepsilon_2\lambda^{-j})^{\beta} = \varepsilon_2^{\beta}\delta^{d}\lambda^{j(\alpha-\beta)}$$

which diverges as $j \to \infty$. It remains to rule out all the other coverings.

Using the sub-additivity of $\mathcal{H}^{\beta}_{\infty}$ and the fact that $\mathcal{H}^{\beta}_{\infty}(Q(x,\varepsilon_2\lambda^{-j}) = \varepsilon_2^{\beta}\lambda^{-\beta j}$, the $\mathcal{H}^{\beta}_{\infty}$-measure of any pseudo-cube $\mathcal{Q}(x,\varepsilon_2\lambda^{-j})$ is bounded from below by

$$\begin{aligned} &\mathcal{H}^{\beta}_{\infty}(Q(x,\varepsilon_2\lambda^{-j})) - \mathcal{H}^{\beta}_{\infty}\Big(Q(x,\varepsilon_2\lambda^{-j}) \cap \bigcup_{j<k\leq 2k} X^{k,\delta}_{\lambda^{k-j}t\theta_k}\Big) \\ &\gtrsim \quad \varepsilon_2^{\beta}\lambda^{-\beta j} - \varepsilon_2^{d}\lambda^{-dj}\sum_{k=j+1}^{2j} \lambda^{-\beta(1-2\delta)k}\lambda^{d(1-\sigma)k} \gtrsim \varepsilon_2^{\beta}\lambda^{-\beta j}, \end{aligned} \tag{30}$$

as long as λ is large enough and $(1-2\delta)\beta - (1-\sigma)d > 0$. This is satisfied, provided that we take δ and σ close enough to 0 and $\frac{d+1-\alpha}{d+2}$, respectively, due to the fact that we took $\beta > \frac{d}{d+2}(\alpha+1)$.

Let $\{Q(x_m,\delta_m)\}$ be a covering of $\Gamma^j \cap Q(x,\delta)$. We partition this in subsets

$$\mathcal{B}_k := \{Q(x_m,\delta_m) : Q(x_m,\delta_m) \text{ intersects } k \text{ pseudo-cubes}\}, \quad k = 1,\ldots,\mathcal{N}_{\delta},$$

where $\mathcal{N}_{\delta}$ is the number of all the pseudo-cubes that intersect $Q(x,\delta)$. Note that

$$\lim_{j\to\infty} \frac{\mathcal{N}_{\delta}}{\delta^{d}\lambda^{\alpha j}} \to 1. \tag{31}$$

Now using the fact that the separation of the pseudo-cubes is much greater than their side-lengths, by slightly enlarging some of the covering cubes if necessary, we can suppose that if a cube $Q(x_m,\delta_m)$ is involved in covering two different pseudo-cubes, then it covers them both completely. More specifically, by enlarging the side-lengths by a factor of at most $(1+2\lambda^{-j(1-\frac{\alpha}{d})})$, a cube that just touches two pseudo-cubes then covers them completely. Then if the original cover were so efficient as to not satisfy (29), then, given that the multiplication factor converges to one as j tends to infinity, the slightly enlarged cover would also fail to satisfy (29). After this enlarging process, we then discard the redundant members of $\mathcal{B}_1$, keeping only those which are required for the cover, i.e. those which are involved in covering a single pseudo-cube without the aid of a much larger covering cube. Finally, we partition further these smaller members of the partition $\mathcal{B}_1$ using the equivalence relation $Q(x_m,\delta_m) \equiv_1 Q(x_\ell,\delta_\ell)$ if $Q(x_m,\delta_m)$ and $Q(x_\ell,\delta_\ell)$ intersect the same pseudo-cube.

It is clear, by the geometry of Γ^j, that

$$\delta_m \geq (k^{\frac{1}{d}} - 1 - 2\varepsilon_2)\lambda^{-j\frac{\alpha}{d}} \quad \text{if } Q(x_m,\delta_m) \in \mathcal{B}_k \text{ with } k > 1. \tag{32}$$

Moreover we note

$$\sum_{k\geq 1} kN_k \geq \mathcal{N}_\delta, \quad \text{where } N_k := \begin{cases} \#\mathcal{B}_1/\equiv_1 & \text{for } k = 1 \\ \#\mathcal{B}_k & \text{for } k > 1, \end{cases} \tag{33}$$

where $\mathcal{B}_1/\equiv_1$ is the quotient set $\{[Q(x_m,\delta_m)]_{\equiv_1} : Q(x_m,\delta_m) \in \mathcal{B}_1\}$.

Given $Q(x_\ell,\delta_\ell) \in \mathcal{B}_1$ all the $Q(x_m,\delta_m) \in [Q(x_\ell,\delta_\ell)]_{\equiv_1}$ are a covering of the pseudo-cube associated to the class $[Q(x_\ell,\delta_\ell)]_{\equiv_1}$. Thus, recalling the lower bound (30) for the $\mathcal{H}^\beta_\infty$-measure of the pseudo-cubes, we have

$$\sum_{Q(x_m,\delta_m)\in[Q(x_\ell,\delta_\ell)]_{\equiv_1}} \delta_m^\beta \;\geq\; \mathcal{H}^\beta_\infty(\mathcal{Q}(\cdot,\varepsilon_2\lambda^{-j})) \;\geq\; C_1\varepsilon_2^\beta\lambda^{-\beta j}, \tag{34}$$

so that

$$\sum_{Q(x_m,\delta_m)\in\mathcal{B}_1} \delta_m^\beta \;\geq\; C_1 N_1 \varepsilon_2^\beta \lambda^{-\beta j}. \tag{35}$$

Thus, using (32), we obtain

$$\sum_m \delta_m^\beta \;=\; \sum_{\substack{k\geq 1\\ Q(x_m,\delta_m)\in\mathcal{B}_k}} \delta_m^\beta \;\geq\; C_1 N_1 \varepsilon_2^\beta \lambda^{-\beta j} \;+\; \sum_{k>1} N_k (k^{\frac{1}{d}} - 1 - 2\varepsilon_2)^\beta \lambda^{-\frac{\alpha\beta}{d}j}.$$

Now using the elementary inequality $k^{\frac{1}{d}} - 1 - 2\varepsilon_2 \geq C_2 k^{\frac{1}{d}}$, valid for $k > 1$ and small ε_2, and

$$\sum_{k>1} N_k k^{\frac{\beta}{d}} \;\geq\; \Big(\sum_{k>1} N_k k\Big)^{\frac{\beta}{d}} \;\geq\; (\mathcal{N}_\delta - N_1)^{\frac{\beta}{d}},$$

which follows by (33), this yields $\sum_m \delta_m^\beta \geq \mathcal{G}(N_1)$, where

$$\mathcal{G}(N) := C_1 N \varepsilon_2^\beta \lambda^{-\beta j} + C_2 \lambda^{-\frac{\alpha\beta}{d}j} (\mathcal{N}_\delta - N)^{\frac{\beta}{d}}.$$

Thus the proof would be complete if we could show that

$$\lim_{j\to\infty} \mathcal{G}(N) \geq C_2 \delta^\beta \quad \text{for all } N \in [0, \mathcal{N}_\delta]. \tag{36}$$

Proof of (36) By (31) we see that

$$\lim_{j\to\infty} \mathcal{G}(0) = C_2\lambda^{-\frac{\alpha\beta}{d}j}(\delta^d\lambda^{\alpha j})^{\frac{\beta}{d}} = C_2\delta^\beta \tag{37}$$

and on the other hand, using that $\alpha > \beta$, we have

$$\lim_{j\to\infty} \mathcal{G}(\mathcal{N}_\delta) = C_1\delta^d\lambda^{\alpha j}\varepsilon_2^\beta\lambda^{-\beta j} = C_1\varepsilon_2^\beta\delta^d\lambda^{(\alpha-\beta)j} = \infty. \tag{38}$$

In order to understand how $\mathcal{G}$ behaves in the interior interval $(0, \mathcal{N}_\delta)$, we calculate

$$\partial_N\mathcal{G}(N) = C_1\varepsilon_2^\beta\lambda^{-\beta j} - \frac{C_2\frac{\beta}{d}\lambda^{-\frac{\alpha\beta}{d}j}}{(\mathcal{N}_\delta - N)^{1-\frac{\beta}{d}}},$$

and note that

$$\partial_N\mathcal{G}(0) = C_1\varepsilon_2^\beta\lambda^{-\beta j} - C_2\tfrac{\beta}{d}\lambda^{-\frac{\alpha\beta}{d}j}\mathcal{N}_\delta^{\frac{\beta}{d}-1}.$$

Using the asymptotic expression (31), we see that the second term of the right-hand-side behaves, for large j, as $-C_2\frac{\beta}{d}\delta^{\beta-d}\lambda^{-\alpha j}$. Comparing this with the first term on the right-hand side (recalling that $\alpha > \beta$), we see that $\partial_N\mathcal{G}(0) > 0$ for sufficiently large j. Then, since there is only a single stationary point in $(0, \mathcal{N}_\delta)$, when $N = \mathcal{N}_\delta - (C_1^{-1}\varepsilon_2^{-\beta}C_2\frac{\beta}{d})^{\frac{d}{d-\beta}}\lambda^{\beta\frac{d-\alpha}{d-\beta}j}$, the function $\mathcal{G}$ increases until this point, and then decreases until it reaches the value $\mathcal{G}(\mathcal{N}_\delta)$. However we have already seen in (38) that this quantity diverges and so, for large j, the minimum is attained at $N = 0$. Thus (36) follows from (37), and the proof is complete. □

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