



Collision Index and Stability of Elliptic Relative Equilibria in Planar n -body Problem

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Abstract: It is well known that a planar central configuration of the n -body problem gives rise to solutions where each particle moves on a specific Keplerian orbit while the totality of the particles move on a homographic motion. When the eccentricity e of the Keplerian orbit belongs in $[0, 1)$, following Meyer and Schmidt, we call such solutions *elliptic relative equilibria* (shortly, ERE). In order to study the linear stability of ERE in the near-collision case, namely when $1 - e$ is small enough, we introduce the collision index for planar central configurations. The collision index is a Maslov-type index for heteroclinic orbits and orbits parametrised by half-lines that, according to the definition given by Hu and Portaluri (An index theory for unbounded motions of Hamiltonian systems, Hu and Portaluri (2015, preprint)), we shall refer to as half-clinic orbits and whose definition in this context, is essentially based on a blow up technique in the case $e = 1$. We get the fundamental properties of collision index and approximation theorems. As applications, we give some new hyperbolic criteria and prove that, generically, the ERE of minimal central configurations are hyperbolic in the near-collision case, and we give a detailed analysis of Euler collinear orbits in the near-collision case.

1. Introduction

For n particles of mass m_1, \dots, m_n , let $q_1, \dots, q_n \in \mathbb{R}^2$ be the position vectors, $p_1, \dots, p_n \in \mathbb{R}^2$ be the momentum vectors. Setting $d_{i,j} = \|q_i - q_j\|$, the Hamiltonian function has the form

$$H = \sum_{j=1}^n \frac{\|p_j\|^2}{2m_j} - U(q_1, \dots, q_n), \quad U = \sum_{1 \leq j < k \leq n} \frac{m_j m_k}{d_{jk}}. \quad (1.1)$$

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U will be defined on configuration space

$$\Lambda = \{x = (x_1, \dots, x_n) \in \mathbb{R}^{2n} \setminus \Delta : \sum_{i=1}^n m_i x_i = 0\},$$

where $\Delta = \{x \in \mathbb{R}^{2n} : \exists i \neq j, x_i = x_j\}$ is the collision set. A central configuration is a solution $(q_1, \dots, q_n) = (a_1, \dots, a_n)$ of

$$-\lambda m_j q_j = \frac{\partial U}{\partial q_j}(q_1, \dots, q_n) \tag{1.2}$$

for some constant λ . An easy computation shows that $\lambda = U(a)/\mathcal{I}(a) > 0$, where $\mathcal{I}(a) = \sum m_j \|a_j\|^2$ is the moment of inertia. Otherwise stated, a central configuration with $\mathcal{I}(a) = 1$ is a critical point of the function U constrained to the set $\mathcal{E} = \{x \in \Lambda \mid \mathcal{I}(x) = 1\}$. Please refer [35] for the properties of central configuration.

It is well known that a planar central configuration of the n -body problem gives rise to solutions where each particle moves on a specific Keplerian orbit while the totality of the particles move on a homographic motion. Following Meyer and Schmidt [31], we call these solutions as *elliptic relative equilibria* and in shorthand notation, simply ERE. Specifically, when the eccentricity $e = 0$, the Keplerian elliptic motion becomes circular and all the bodies move around the center of masses along circular orbits with the same frequency. Traditionally these orbits are called *relative equilibria*.

As pointed out in [31], there are two four-dimensional invariant symplectic subspaces, E_1 and E_2 , and they are associated to the translation symmetry, dilation and rotation symmetry of the system. In other words, there is a symplectic coordinate system in which the linearized system of the planar n -body problem decouples into three subsystems on E_1, E_2 and $E_3 = (E_1 \cup E_2)^\perp$, where \perp denotes the symplectic orthogonal complement. A symplectic matrix \mathcal{M} is called spectrally stable if all eigenvalues of \mathcal{M} belong to the unit circle \mathbb{U} of the complex plan, while \mathcal{M} is called hyperbolic if no eigenvalues of \mathcal{M} are on \mathbb{U} . The ERE is called hyperbolic (resp. stable) if the monodromy matrix \mathcal{M} is restricted to E_3 , that is, $\mathcal{M}|_{E_3}$ is hyperbolic (resp. stable).

There are many interesting results for the linear stability of ERE (cfr. [15,33,34,38,39,41] and references therein). Many of them investigated the relative equilibria for e small enough and as far as we know, only a few of them studied the linear stability of ERE with $e \in [0, 1)$. To our knowledge, the elliptic Lagrangian solution is the only case that is well studied. The Lagrangian solution, which was discovered by Lagrange in 1772 [22], is the ERE of the equilateral triangle central configuration in the planar three body problem.

It is well known that the stability of elliptic Lagrangian solutions depend on the eccentricity e and on

$$\beta = \frac{27(m_1 m_2 + m_1 m_3 + m_2 m_3)}{(m_1 + m_2 + m_3)^2}. \tag{1.3}$$

Long et al. used Maslov-type index and operator theory to study the stability problem, and gave out a full description of the bifurcation graph (cfr. [17,18]). Moreover, Wang et al. built up a trace formula for linear Hamiltonian systems and Sturm–Liouville systems, and used it to give an estimate of the stability region as well as of the hyperbolic region [20,21].

In the study of the near-collision case, that is, when $1 - e$ is small enough, a blow-up technique from Martínez et al. [28] is very useful to carry over our analysis. The authors

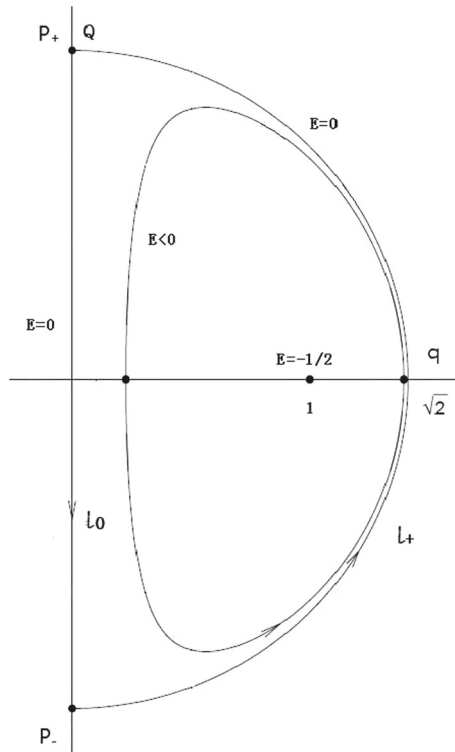


Fig. 1. Phase portrait of (2.7) from [29]

considered 4D linear system depending on a small parameter $\sigma > 0$ and the singular limit for $\sigma \rightarrow 0$. Based on it, they computed the trace $tr_1 = \lambda_1 + \lambda_1^{-1}$, $tr_2 = \lambda_2 + \lambda_2^{-1}$, where $\lambda_i, \lambda_i^{-1}, i = 1, 2$ are the eigenvalues of monodromy matrix. Under a “nondegenerate condition” they describe the asymptotic behaviour of $\log |tr_i|, i = 1, 2$ and tr_2 . Their study includes the ERE of Lagrangian equilateral triangle and Euler collinear central configurations.

Motivated by the these results, we will use blow-up technique and index theory to study the stability problem of ERE. The index theory we used will be based on the Maslov-type index. The Maslov index is associated to a given continuous path of Lagrangian subspaces and the Maslov-type index is assigned to a path of symplectic matrices. We briefly review the Maslov index theory in Sect. 2.2 and give its relation with the Maslov-type index. For the reader’s convenience, we now roughly describe the Maslov-type index theory. Let $Sp(2n)$ be the set of symplectic matrix in \mathbb{R}^{2n} equipped with the standard symplectic structure, and let I_{2n} be the identity matrix on \mathbb{R}^{2n} . Let $\gamma \in C([0, T], Sp(2n))$ with $\gamma(0) = I_{2n}$, for $\omega \in \mathbb{U}$, roughly speaking, the Maslov-type index $i_\omega(\gamma)$ is the intersection number (by a small perturbation) of γ and $D_\omega := \{M \in Sp(2n), \det(M - \omega I_{2n}) = 0\}$. Please refer to [24] for the details.

By the blow-up technique, the limit of ERE , as $e \rightarrow 1$, can be described by two heteroclinic orbits l_0, l_+ connected P_\pm (cfr. to Fig. 1). Throughout of the paper, we denote by γ_e the fundamental solution of the essential part of ERE, that is, $\dot{\gamma}_e(t) = JB(t)\gamma_e(t), t \in [0, 2\pi], \gamma_e(0) = id$, where $B(t)$ is defined in Eq. (2.2).

Given a symmetric matrix R we shall denote by $\lambda_1(R)$, the smallest eigenvalue of R . When the limit equilibrium P_{\pm} is non-hyperbolic, we have the following result.

Theorem 1.1. *Let $a_0 \in \mathcal{E}$ be a planar central configuration which satisfies*

$$\lambda_1(D^2U|_{\mathcal{E}}(a_0)) < -\frac{1}{8}U(a_0), \tag{1.4}$$

where $D^2U|_{\mathcal{E}}(a_0)$ is the Hessian of U restricted to \mathcal{E} at a_0 . Then $i_1(\gamma_e) \rightarrow \infty$ as $e \rightarrow 1$. For $\frac{1}{U(a_0)}\lambda_1(D^2U|_{\mathcal{E}}(a_0)) = -\frac{1}{8} - r_1$, let $\varepsilon = \frac{1}{2} \min\{\frac{r_1}{2r_1+5}, 1/8\}$, $\hat{e} = \frac{1-e^2}{2}$. If $\hat{e} < \varepsilon^3$, then we have

$$i_1(\gamma_e) \geq 2\frac{\sqrt{r_1}}{\pi} \ln\left(\frac{\varepsilon^2}{\sqrt{\hat{e}}}\right) - 6. \tag{1.5}$$

From [24], it is well-known that, for any $\omega \in \mathbb{U}$, $|i_{\omega} - i_1| \leq n$, then Theorem 1.1 shows also that $i_{\omega}(\gamma_e) \rightarrow \infty$ as $e \rightarrow 1$, which implies there exists a sequence $e_j(\omega)$ converging to 1, such that the system is ω -degenerate.

It is well known that any T -periodic solution is a critical point of the action functional

$$\mathcal{F}(q) = \int_0^T \left[\sum_{i=1}^n \frac{m_i \|\dot{q}_i(t)\|^2}{2} + U(q) \right] dt$$

defined on loop space $W^{1,2}(\mathbb{R}/T\mathbb{Z}, \Lambda)$. Let now x_e be the ERE corresponding to a_0 with eccentricity e , and let $\phi(x_e)$ be the Morse index of x_e (meaning that it is the total number of the negative eigenvalues of $\mathcal{F}''(x_e)$). Since the Morse index is equal to Maslov-type index (cfr. to Lemma 5.3). We have $\phi(x_e)$ is not less than the Maslov-type index $i_1(\gamma_e)$ coming from the essential part. Theorem 1.1 implies that, if a_0 satisfied (1.4), then $\phi(x_e) \rightarrow \infty$ as $e \rightarrow 1$ [19].

The above theorem is related to the result of the interesting paper of Barutello and Secchi [4]. They defined a collision Morse index for one-collision solution in n -body problem with α homogeneous potentials, and proved that the collision index is infinite under the condition (1.4) for the Newton potential. Their results show that a one-collision solution asymptotic to a_0 which satisfied (1.4) cannot be locally minimal for the action function. A Morse-type index theorem both for colliding and parabolic motions will be given in [5].

The most interesting case, however, is precisely when P_{\pm} is hyperbolic, in this case we can define the Maslov index for heteroclinic orbits and half-clinic orbits. (Cfr. Equations (3.2) and (3.4)). For *half-clinic orbits*, we mean a solution of Hamiltonian system $x(t)$ defined on \mathbb{R}^+ or \mathbb{R}^- , where \mathbb{R}^+ and \mathbb{R}^- stands for the non-negative and non-positive half-line, respectively and such that the initial condition $x(0)$ belongs to a Lagrangian subspace whilst $x(t)$ converge to an equilibrium point when $t \rightarrow \pm\infty$. Also in this last case, we assign a Maslov index to both l_0 and l_+ and we shall refer to it as the *collision index*. After defining the collision index, we shall prove Theorem 3.3 and we shall refer to it as the approximation theorem. Let us now show that, under a suitable non degenerate conditions for $e \rightarrow 1$, the Maslov index for γ_e is convergent to the sum of the collision index on l_0, l_+ . This is a main part (cfr. Sect. 3.1 for the details). In the study of the stability problem, the Dirichlet, Neumann, periodic, anti-periodic boundary conditions play an important role. Our key idea is to use the Maslov index corresponding to these 4 kinds of boundary conditions for determining the stability.

Throughout of the paper, we always let $V_d^j = \mathbb{R}^j \oplus 0$, $V_n^j = 0 \oplus \mathbb{R}^j$ be the Lagrangian subspace in $(\mathbb{R}^{2j}, \omega_0)$ which correspond to the Dirichlet and Neumann boundary conditions respectively, and we always omit the subscript if no confusion is possible. For ERE, by using the approximation theorem, we get the following result.

Theorem 1.2. *Let $a_0 \in \mathcal{E}$ be such that $\lambda_1(D^2U|_{\mathcal{E}}(a_0)) > -\frac{1}{8}U(a_0)$ and we assume that a_0 is nondegenerate and collision nondegenerate. Let $\phi(a_0)$ be the Morse index of a_0 which is the total number of negative eigenvalues of $D^2U|_{\mathcal{E}}(a_0)$. For $1 - e$ small enough, we have, $\gamma_e(2\pi)V_d \pitchfork V_d$,*

$$\mu(V_d, \gamma_e(t)V_d, t \in [0, 2\pi]) = k + i(V_d; l_+), \quad (1.6)$$

and $\gamma_e(2\pi)V_n \pitchfork V_n$,

$$\mu(V_n, \gamma_e(t)V_n, t \in [0, 2\pi]) = 2\phi(a_0) + i(V_d; l_+), \quad (1.7)$$

where \pitchfork means transversal, $k = 2n - 4$, $i(V_d; l_+)$ is the collision index on l_+ defined by (3.4) and μ is the Maslov index.

The definition of collision nondegenerate index is given in Definition 3.2. The degenerate problem on l_0 will be discussed in Sect. 3.2. We observe that, in contrast with respect to the nondegeneracy condition along l_0 , we didn't establish a useful criterion for detecting the nondegeneracy along l_+ .

If the central configurations have brake symmetry (cfr. Definition 4.1), the collision index of heteroclinic could be decomposed into the sum of index on half-clinic orbits and this will simplify the computation. To our knowledge, the Lagrangian and Euler central configurations both have brake symmetry. Another example is the $1 + n$ central configurations, that is regular polygon configurations with a central mass. It will be interesting to provide central configurations without this symmetry property.

As an application, we study the stability of ERE for minimizer central configurations. For a central configuration a_0 , it is obvious that $D^2U|_{\mathcal{E}}(a_0)$ has a trivial eigenvalue 0, which comes from the rotation invariant. The central configuration a_0 is called nondegenerate minimizer if all the nontrivial eigenvalues are bigger than 0, while a_0 is called strong minimizer if all the nontrivial eigenvalues are bigger than $U(a_0)$.

Theorem 1.3. *We assume that a_0 is a nondegenerate minimizer that satisfies the collision nondegenerate condition. If $1 - e$ is sufficiently small, then the ERE is hyperbolic.*

In the case $e = 0$, Moeckel conjectured [2] that a relative equilibrium is linearly stable only if it associated to a minimizing central configuration. Our results show that in the case that $1 - e$ small enough, it is generally hyperbolic. By the way, we conjecture that Theorem 1.3 is true also without the collision nondegenerate condition.

In the case a_0 strong minimizer the following result holds.

Theorem 1.4. *The ERE of a strong minimizer a_0 is hyperbolic for any $e \in [0, 1)$.*

A typical example of nondegenerate minimizer central configurations is the Lagrangian central configurations, which is strong minimizer if $\beta > 8$. For Lagrangian orbits, the conclusion of Theorem 1.3 was proved in [17] without the collision nondegenerate condition and the result of Theorem 1.4 was proved by Ou [36]. Another easy example is the $1 + 3$ -gon central configurations, that is, the regular triangular configurations with a central mass. The three unit masses with unit distance away from the mass m_c at the origin. As a direct application of Theorems 1.3 and 1.4, we get the next result.

Corollary 1.5. *Let a_0 be a 1 + 3-gon central configuration with central mass m_c , for $m_c \in [0, \frac{81+64\sqrt{3}}{249})$ and we assume that a_0 is collision nondegenerate. If $1 - e$ is sufficiently small, then the ERE is hyperbolic. Furthermore, if $m_c \in [0, \frac{\sqrt{3}}{24})$ then the ERE is hyperbolic for any $e \in [0, 1)$.*

Another conjecture of Moeckel [2] states that a relative equilibrium is linearly stable only if it has a dominant mass. For example the Lagrangian orbits and ERE of 1 + 3-gon relative to a strong minimizer have no dominant mass. Thus Theorem 1.4 can be considered as a support of Moeckel's conjecture in the case of $e > 0$, so we guess Moeckel's conjecture is also true in the case of ERE.

The collision index plays an important role in the study of the stability problem. We shall give some conjectures for the collision index that are related with Long's conjecture for the Maslov-type index of ERE (cfr. Remark 5.6 for further details).

As a further application, we consider the ERE of Euler collinear central configurations [14], which we simply refer to as elliptic Euler orbits. The linear stability of this kind of orbit depends upon two parameters, e and δ , where the last one $\delta \in [0, 7]$ only depends on mass m_1, m_2, m_3 . (Cfr. Appendix A of [28] and [27] for further details). To our knowledge, the near collision case was first studied by Martínez et al. [28]. Long and Zhou used Maslov-type index theory in order to describe the ± 1 -degenerate curves; they also analysed the stability problem. It is worth noting that by their methods is not possible to explain the limit property of ± 1 -degenerate curves numerically proved by Martínez et al. [28]. Please refer to Figs. 7 and 8. Using the collision index, we explain the limit property. We show that $\delta > 1/8$ is equivalent to condition (1.4). Theorem 1.1 implies the ± 1 -degenerate curves don't intersect $[1/8, 7] \times 1$. In the case $\delta \in (0, 1/8)$, the collision index is well defined, we analyse the near-collision phenomena by the collision index. We can compute in detail for the collision index on l_0 , but unfortunately, we can't determine the collision index on l_+ by analytical method. Instead, we develop a numerical method to compute the collision index. Based on numerical results A, the collision index strictly proved the behaviour of the ± 1 in the near-collision case. Please refer to Sect. 5.2 for the details.

This paper is organized as follows. We review the Meyer-Schmidt reduction and Martínez, Samà, Simó blow up technique at Sect. 2.1. We give a brief introduction of the Maslov index theory and we prove Theorem 1.1 in Sect. 2.2. The definition of collision index is stated and the approximation theorem is proved in Sect. 3.1. The computation of the collision index along l_0 and some basic properties of the collision index are given in Sect. 3.2. In Sect. 4, we study the case of brake symmetric central configurations. We give some applications in Sect. 5. In Sect. 5.1, we study the minimizing central configurations and we prove Theorem 1.3 and Theorem 1.4. We use the collision index to analyse the Euler orbits in Sect. 5.2. At last, for the reader's convenience, we give the details of the numerical method used to compute the collision index in Sect. 6.

2. Blow Up and Limit Index for the Non-Hyperbolic Case

This section includes some basic preliminaries. We first briefly review the decomposition of ERE by following authors in [31] and the blow-up technique of Martínez, Samà and Simó [29] in Sect. 2.1, then we review the fundamental property of Maslov index in Sect. 2.2, and give the proof of Theorem 1.1.

2.1. *Reduction and blow up method.* In 2005, Meyer and Schmidt strongly used the structure of the central configuration for the elliptic Lagrangian orbits and symplectically decomposed the fundamental solution of the elliptic Lagrangian orbit into two parts, one of which corresponds to the Keplerian solution and the other is the essential part of the dynamics, needed for studying the stability. For the reader's convenience, we briefly review the central configuration coordinates, by following Meyer and Schmidt [31].

Suppose that $\mathcal{Q} = (q_1, \dots, q_n) \in \mathbb{R}^{2n}$ with mass m_1, \dots, m_n is a central configuration, and $\mathcal{P} = (p_1, \dots, p_n) \in \mathbb{R}^{2n}$. Let I_j be the identity matrix on \mathbb{R}^j , $J_{2j} = \begin{pmatrix} 0_j & -I_j \\ I_j & 0_j \end{pmatrix}$. We denote by $\mathbb{J}_n = \text{diag}(J_2, \dots, J_2)_{2n \times 2n}$ and $M = \text{diag}(m_1, m_1, m_2, m_2, \dots, m_n, m_n)_{2n \times 2n}$. We assume that $t \mapsto x(t)$ is a periodic solution of ERE, then the corresponding fundamental solution is

$$\dot{\gamma}(t) = J_{4n} H''(x(t)) \gamma(t), \quad \gamma(0) = I_{4n}. \tag{2.1}$$

As in [31, Corollary 2.1, pag. 266], Eq. (2.1) can be decomposed into 3 subsystems on E_1, E_2 and $E_3 = (E_1 \cup E_2)^\perp$ respectively. The basis of E_1 is $(0, u), (Mu, 0), (0, v), (Mv, 0)$, where $u = (1, 0, 1, 0, \dots), v = (0, 1, 0, 1, \dots)$, and E_2 is spanned by $(0, \mathcal{Q}), (M\mathcal{Q}, 0), (0, \mathbb{J}_n \mathcal{Q}), (\mathbb{J}_n M \mathcal{Q}, 0)$. For $X = (g, z, w) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{2n-4}$ and $Y = (G, Z, W) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{2n-4}$, we consider the linear symplectic transformation of the form $\mathcal{Q} = PX, \mathcal{P} = P^{-T} Y$, where P is such that $\mathbb{J}P = P\mathbb{J}, P^T M P = I_{2n}$ ([31], p263). Now $B(t) = H''(x(t))$ in this new coordinate system has the form $B(\mathcal{Q}) = B_1 \oplus B_2 \oplus B_3$, where $B_i = B|_{E_i}$. The essential part $B_3(t)$ is a path of $(4n - 8) \times (4n - 8)$ symmetric matrix.

In the rotating coordinate system and by using the true anomaly as the variable, Meyer and Schmidt [31] gave a very useful form of the essential part

$$B(t) = \begin{pmatrix} I_k & -\mathbb{J}_{k/2} \\ \mathbb{J}_{k/2} & I_k - \frac{I_k + \mathcal{D}}{1 + e \cos(t)} \end{pmatrix}, \quad t \in [0, 2\pi], \tag{2.2}$$

where $k = 2n - 4$ and e is the eccentricity, t is the true anomaly and

$$\mathcal{D} = \frac{1}{\lambda} P^T D^2 U(\mathcal{Q}) P|_{w \in \mathbb{R}^k}, \quad \text{with } \lambda = \frac{U(\mathcal{Q})}{\mathcal{I}(\mathcal{Q})}. \tag{2.3}$$

We denote by $R := I_k + \mathcal{D}$, which can be considered as the regularized Hessian of the central configurations. In fact, for $a_0 \in \mathcal{E}$ which is a central configurations, then $\mathcal{I}(a_0) = 1$. With respect to the mass matrix M inner product, the Hessian of the restriction of the potential to the inertia ellipsoid, is given by

$$D^2 U|_{\mathcal{E}}(a_0) = M^{-1} D^2 U(a_0) + U(a_0).$$

Then we have

$$P^{-1} D^2 U|_{\mathcal{E}}(a_0) P = P^T D^2 U(a_0) P + U(a_0), \tag{2.4}$$

and thus

$$R = \frac{1}{U(a_0)} P^{-1} D^2 U|_{\mathcal{E}}(a_0) P|_{w \in \mathbb{R}^k}. \tag{2.5}$$

Thus the corresponding Sturm–Liouville system is

$$-\ddot{y} - 2\mathbb{J}_{k/2}\dot{y} + \frac{R}{1 + e \cos(t)}y = 0. \tag{2.6}$$

In order to study the singular limit case $e \rightarrow 1$, we use a change of coordinates as in [29]. In fact, in the case of Newtonian potential, this can be interpreted as a McGehee change of coordinates (cfr. [30, 32]). Let $q = (1 + e \cos(t))^{1/2}$, $Q = -2\dot{q}$ and change the time variable to τ , where $dt = qd\tau$. Throughout the paper, we always use $x' = \frac{dx}{d\tau}$ and $\dot{x} = \frac{dx}{dt}$. Then we have

$$q' = -\frac{1}{2}qQ, \quad Q' = \frac{1}{2}Q^2 + q^2 - 1. \tag{2.7}$$

We observe that (2.7) is well defined for $q = 0$ and its first integral is $E = q^2(\frac{Q^2}{2} + \frac{q^2}{2} - 1)$. An easy computation shows that, for the orbits with eccentricity e , the first integral with $E = \frac{e^2-1}{2} = -\hat{e}$. The system has two equilibria $P_{\pm} = (0, \pm\sqrt{2})$ lying on the level set $E = 0$. We distinguish the level set $E = 0$ into two orbits

$$l_0 = \{(q, Q) \in \mathbb{R}^2 | q = 0, |Q| < \sqrt{2}\}, \tag{2.8}$$

and

$$l_+ = \{(q, Q) \in \mathbb{R}^2 | q > 0, Q^2 + q^2 = 2\}. \tag{2.9}$$

On l_0 , we have

$$q_{l_0}(\tau) = 0, \quad Q_{l_0}(\tau) = -\sqrt{2} \tanh\left(\frac{\sqrt{2}}{2}\tau\right), \tag{2.10}$$

and the system on l_+ is

$$q'_{l_+} = -\frac{1}{2}qQ, \quad Q'_{l_+} = -\frac{q^2}{2}. \tag{2.11}$$

The solution is given by

$$q_{l_+}(\tau) = \sqrt{2}/\cosh\left(\frac{\sqrt{2}\tau}{2}\right), \quad Q_{l_+}(\tau) = \sqrt{2} \tanh\left(\frac{\sqrt{2}\tau}{2}\right). \tag{2.12}$$

For convenience, we also let $l_0^- = \{(p, Q) \in l_0, Q \leq 0\}$ and $l_0^+ = \{(p, Q) \in l_0, Q \geq 0\}$, similarly, let $l_+^- = \{(p, Q) \in l_+, Q \leq 0\}$ and $l_+^+ = \{(p, Q) \in l_+, Q \geq 0\}$. Obviously, $l_0 = l_0^- \cup l_0^+$ and $l_+ = l_+^- \cup l_+^+$.

Throughout the paper, we always let γ_e ($e \in [0, 1)$) be the fundamental solution of (2.2). For simplicity, γ_e can also be considered as a function of τ . For $q \neq 0$ we consider the matrix $S = \text{diag}(q^{\frac{1}{2}}I_k, q^{-\frac{1}{2}}I_k) \in \text{Sp}(2k)$ and for $T = \tau(2\pi)$, we have $S(T) = S(0)$. We let $\hat{\gamma}_e(\tau) = S(\tau)\gamma_e(\tau)S^{-1}(0)$ and we observe that the associated monodromy matrix is similar to the one of γ_e . A direct computation shows that

$$\frac{d}{d\tau}\hat{\gamma}_e = J\hat{B}\hat{\gamma}_e, \quad \hat{\gamma}_e(0) = I_{2k}, \quad \tau \in [0, T], \tag{2.13}$$

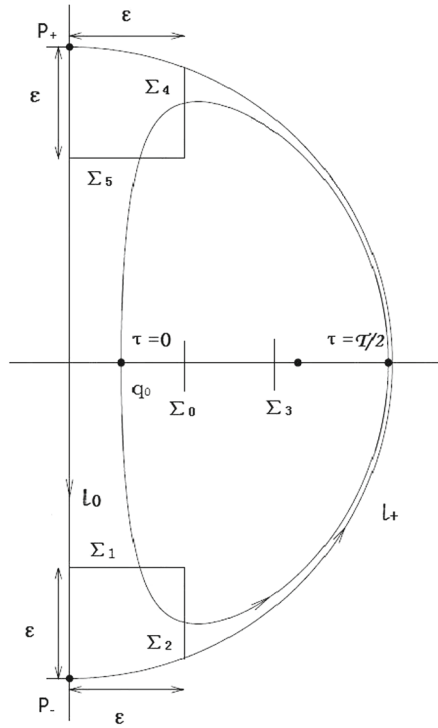


Fig. 2. An illustration of the sections used in the proof of Theorem 3.3

with

$$\hat{B} = \begin{pmatrix} I_k & \frac{Q}{4} I_k - q\mathbb{J}_{k/2} \\ \frac{Q}{4} I_k + q\mathbb{J}_{k/2} & q^2 I_k - R \end{pmatrix}. \tag{2.14}$$

The linear system (2.13) is well-defined also when $e = 1$. In this case, $E = 0$, the system has two equilibria corresponding to P_{\pm} , and the system can be considered as two heteroclinic orbits.

Proposition 2.1. P_{\pm} is hyperbolic if and only if $\lambda_1(R) > -\frac{1}{8}$.

Proof. We observe that at points P_{\pm} , the linear part with form $D_{\pm} = J_k \begin{pmatrix} I_k & \pm \frac{\sqrt{2}}{4} I_k \\ \pm \frac{\sqrt{2}}{4} I_k & -R \end{pmatrix}$. P_{\pm} is hyperbolic if and only if the eigenvalue of D_{\pm} is not on the imaginary line. Since R is diagonalizable by choose suitable bases, the results is from simple computations. \square

Given $\varepsilon < 1/8$, we define the following sections (see Fig. 2)

- $\Sigma_0 = \{(q, Q) | 0 < q < \varepsilon, Q = 0\}, \quad \Sigma_1 = \{(q, Q) | 0 < q < \varepsilon, Q = -\sqrt{2} + \varepsilon\},$
- $\Sigma_2 = \{(q, Q) | q = \varepsilon, -\sqrt{2 - \varepsilon^2} < Q < \varepsilon - \sqrt{2}\},$
- $\Sigma_3 = \{(q, Q) | 0 < \sqrt{2} - q < \varepsilon, Q = 0\}.$
- $\Sigma_4 = \{(q, Q) | q = \varepsilon, \sqrt{2} - \varepsilon < Q < \sqrt{2 - \varepsilon^2}\},$
- $\Sigma_5 = \{(q, Q) | 0 < q < \varepsilon, Q = \sqrt{2} - \varepsilon\}.$

If $\hat{e} = \frac{1-e^2}{2}$, $\tilde{E}(\hat{e}) := \{(q, Q), E = -\hat{e}\}$ denotes the energy level set. A direct computation show that $\tilde{E}(\hat{e})$ intersects $\Sigma_i, i = 0, \dots, 5$ simply, i.e. intersect at exactly one point when $\hat{e} < \varepsilon^3, \varepsilon < 1/8$. In this case, the Poincaré map $\mathcal{P}_i : \Sigma_{i-1} \mapsto \Sigma_i, i = 1, \dots, 5$ is well defined. In fact, $\hat{e} < \varepsilon^2$ ensures the intersection with Σ_0, Σ_3 . For the intersection of $\tilde{E}(\hat{e})$ and Σ_1 , we observe that, since $Q = -\sqrt{2} + \varepsilon$, are solutions of $q^2((-\sqrt{2}+\varepsilon)^2/2+q^2/2-1) = -\hat{e}$, we get $q_{\pm}^2 = \frac{1}{2}(\varepsilon(2\sqrt{2}-\varepsilon) \pm ((\varepsilon(2\sqrt{2}-\varepsilon))^2 - 8\hat{e})^{\frac{1}{2}})$. For $\varepsilon < 1/8, \hat{e} < \varepsilon^3$, we have $q_-^2 < \varepsilon^2 < q_+^2$, which guarantee the simple intersection. Since $\tilde{E}(\hat{e})$ is convex, this also ensure the simple intersection with Σ_2 , and we have the simple intersection of Σ_4, Σ_5 by symmetry.

Following [29], we denote by $\tau_{l_0} > 0$ the time defined by $Q_{l_0}(\tau_{l_0}) = -\sqrt{2} + \varepsilon$ and $\tau_{l_+} > 0$ such that $q_{l_+}(-\tau_{l_+}) = \varepsilon$. It is obvious $Q_{l_0}(-\tau_{l_0}) = \sqrt{2} - \varepsilon$ and $q_{l_+}(\tau_{l_+}) = \varepsilon$. τ_{l_0} and τ_{l_+} are finite and independent of e once ε is fixed. Let $q_0 = q(0)$ and τ_1, τ_2 be the smallest positive time such that $(q(\tau_1), Q(\tau_1)) \in \Sigma_1$ and $(q(\tau_2), Q(\tau_2)) \in \Sigma_2$. It is clear that q_0, τ_1 and τ_2 depend on \hat{e} . Moreover $q_0 \rightarrow 0, \tau_1 \rightarrow \tau_{l_0}$ and $T/2 - \tau_2 \rightarrow \tau_{l_+}$ when $\hat{e} \rightarrow 0$. Similarly, let τ_4, τ_5 be the smallest positive time such that $(q(\tau_4), Q(\tau_4)) \in \Sigma_4$ and $(q(\tau_5), Q(\tau_5)) \in \Sigma_5$. We have $\tau_4 - T/2 \rightarrow \tau_{l_+}$ and $T - \tau_5 \rightarrow \tau_{l_0}$. The next lemma was proved in [29]; however, for the reader’s convenience, we proved it with a slight sharp estimates.

Lemma 2.2. For $\varepsilon < 1/8, \hat{e} < \varepsilon^3$, we have

$$(a) \quad \sqrt{2} \ln\left(\frac{\varepsilon}{q(\tau_1)}\right) \leq \tau_2 - \tau_1 \leq \frac{2}{\sqrt{2} - \varepsilon} \ln\left(\frac{\varepsilon}{q(\tau_1)}\right), \tag{2.15}$$

$$(b) \quad \int_{\tau_1}^{\tau_2} q(\tau) d\tau \leq \frac{2\varepsilon}{\sqrt{2} - \varepsilon} < 2\varepsilon, \tag{2.16}$$

$$(c) \quad \int_{\tau_1}^{\tau_2} |Q(\tau) + \sqrt{2}| d\tau < 2\varepsilon. \tag{2.17}$$

Proof. To prove (a), we observe that for $\tau \in [\tau_1, \tau_2], -\sqrt{2} \leq Q(\tau) \leq -\sqrt{2} + \varepsilon$. Multiplying by $-q(\tau)/2$ we get $\frac{1}{2}(\sqrt{2}-\varepsilon)q(\tau) \leq q'(\tau) \leq \frac{\sqrt{2}}{2}q(\tau)$ and hence Equation (2.15) by a direct integration. To prove (b), please note that $\int_{\tau_1}^{\tau_2} q(\tau) d\tau \leq \frac{2}{\sqrt{2}-\varepsilon} \int_{\tau_1}^{\tau_2} q'(\tau) d\tau \leq \frac{2}{\sqrt{2}-\varepsilon} q(\tau_2)$. Equation (2.16) follows by observing that $q(\tau_2) = \varepsilon$.

To prove (c), let $F(q) = \sqrt{2} - \sqrt{2 - q^2}$. Then $F(q) \leq cq$ for $q < \varepsilon$, with $c = \frac{\varepsilon}{\sqrt{2+\sqrt{2-\varepsilon^2}}}$. From (b), we have

$$\int_{\tau_1}^{\tau_2} F(q) d\tau \leq \frac{2c\varepsilon}{\sqrt{2} - \varepsilon}. \tag{2.18}$$

Please observe that, near P_-, l_+ is graph of $F(q) - \sqrt{2}$. Let $y = Q + \sqrt{2} - F(q)$, then $0 < y < \varepsilon$. By a direct computation it follows that

$$y' = \left(\frac{q^2}{2\sqrt{2-q^2}} - \sqrt{2-q^2}\right)y + y^2/2 = -\sqrt{2}y(1 + o_1),$$

where $|o_1| \leq \varepsilon$. Thus we have $-\sqrt{2}y(1 + \varepsilon) \leq y' \leq -\sqrt{2}y(1 - \varepsilon)$, then

$$\int_{\tau_1}^{\tau_2} y d\tau \leq \frac{1}{\sqrt{2}(1 - \varepsilon)} y(\tau_1) \leq \frac{\varepsilon}{\sqrt{2}(1 - \varepsilon)}. \tag{2.19}$$

From (2.18, 2.19), we have

$$\int_{\tau_1}^{\tau_2} |Q + \sqrt{2}|d\tau \leq \int_{\tau_1}^{\tau_2} (y + F(q))d\tau \leq \frac{\varepsilon}{\sqrt{2}(1 - \varepsilon)} + \frac{2c\varepsilon}{\sqrt{2} - \varepsilon} < 2\varepsilon. \quad (2.20)$$

□

2.2. *The index limit on the non-hyperbolic case* . We start by briefly reviewing the Maslov index theory [3, 6, 37] in this subsection. Let $(\mathbb{R}^{2n}, \omega)$ be the standard symplectic space, and $Lag(2n)$ the Lagrangian Grassmanian, i.e. the set of Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega)$. For two continuous paths $L_1(t), L_2(t), t \in [a, b]$ in $Lag(2n)$, the Maslov index $\mu(L_1, L_2)$ is an integer invariant. Here we use the definition from [6]. We list several properties of the Maslov index. The details could be found in [6].

Property I (Reparametrization invariance) Let $\varrho : [c, d] \rightarrow [a, b]$ be a continuous and piecewise smooth function with $\varrho(c) = a, \varrho(d) = b$, then

$$\mu(L_1(t), L_2(t)) = \mu(L_1(\varrho(\tau)), L_2(\varrho(\tau))). \quad (2.21)$$

Property II (Homotopy invariant with end points) For two continuous family of Lagrangian path $L_1(s, t), L_2(s, t), 0 \leq s \leq 1, a \leq t \leq b$, and satisfies $\dim L_1(s, a) \cap L_2(s, a)$ and $\dim L_1(s, b) \cap L_2(s, b)$ is constant, then

$$\mu(L_1(0, t), L_2(0, t)) = \mu(L_1(1, t), L_2(1, t)). \quad (2.22)$$

Property III (Path additivity) If $a < c < b$, then

$$\mu(L_1(t), L_2(t)) = \mu(L_1(t), L_2(t)|_{[a, c]}) + \mu(L_1(t), L_2(t)|_{[c, b]}). \quad (2.23)$$

Property IV (Symplectic invariance) Let $\gamma(t), t \in [a, b]$ is a continuous path in $Sp(2n)$, then

$$\mu(L_1(t), L_2(t)) = \mu(\gamma(t)L_1(t), \gamma(t)L_2(t)). \quad (2.24)$$

Property V (Symplectic additivity) Let $W_i, i = 1, 2$ be symplectic space, $L_1, L_2 \in C([a, b], Lag(W_1))$ and $\hat{L}_1, \hat{L}_2 \in C([a, b], Lag(W_2))$, then

$$\mu(L_1(t) \oplus \hat{L}_1(t), L_2(t) \oplus \hat{L}_2(t)) = \mu(L_1(t), L_2(t)) + \mu(\hat{L}_1(t), \hat{L}_2(t)). \quad (2.25)$$

In the case $L_1(t) \equiv V_0, L(t) = \gamma(t)V$, where γ is a path of symplectic matrix we have a monotonicity property (cfr. [21]).

Property VI (Monotone property) Suppose for $j = 1, 2, L_j(t) = \gamma_j(t)V$, where $\dot{\gamma}_j(t) = JB_j(t)\gamma_j(t)$ with $\gamma_j(t) = I_{2n}$. If $B_1(t) \geq B_2(t)$ in the sense that $B_1(t) - B_2(t)$ is non-negative matrix, then for any $V_0, V_1 \in Lag(2n)$, we have

$$\mu(V_0, \gamma_1 V_1) \geq \mu(V_0, \gamma_2 V_1). \quad (2.26)$$

One efficient way to study the Maslov index is via crossing form introduced by [37]. For simplicity and since it is enough for our purpose, we only review the case of the Maslov index for a path of Lagrangian subspace with respect to a fixed Lagrangian subspace. Let $\Lambda(t)$ be a C^1 -curve of Lagrangian subspaces with $\Lambda(0) = \Lambda$, and let V

be a fixed Lagrangian subspace which is transversal to Λ . For $v \in \Lambda$ and small t , define $w(t) \in V$ by $v + w(t) \in \Lambda(t)$. Then the form

$$Q(v) = \left. \frac{d}{dt} \right|_{t=0} \omega(v, w(t)) \tag{2.27}$$

is independent of the choice of V (cfr. [37]). A crossing for $\Lambda(t)$ is some t for which $\Lambda(t)$ intersects W nontrivially, i.e. for which $\Lambda(t) \in \overline{O_1(W)}$. The set of crossings is compact. At each crossing, the crossing form is defined to be

$$\Gamma(\Lambda(t), W, t) = Q|_{\Lambda(t) \cap W}. \tag{2.28}$$

A crossing is called *regular* if the crossing form is non-degenerate. If the path is given by $\Lambda(t) = \gamma(t)\Lambda$ with $\gamma(t) \in \text{Sp}(2n)$ and $\Lambda \in \text{Lag}(2n)$, then the crossing form is equal to $(-\gamma(t)^T J \dot{\gamma}(t)v, v)$, for $v \in \gamma(t)^{-1}(\Lambda(t) \cap W)$, where (\cdot, \cdot) is the standard inner product on \mathbb{R}^{2n} .

For $\Lambda(t)$ and W as before, if the path has only regular crossings, following [25], the Maslov index is equal to

$$\begin{aligned} \mu(W, \Lambda(t)) = & m^+(\Gamma(\Lambda(a), W, a)) + \sum_{a < t < b} \text{sign}(\Gamma(\Lambda(t), W, t)) \\ & - m^-(\Gamma(\Lambda(b), W, b)), \end{aligned} \tag{2.29}$$

where the sum runs all over the crossings $t \in (a, b)$ and m^+, m^- are the dimensions of positive and negative definite subspaces, $\text{sign} = m^+ - m^-$ is the signature. We note that for a C^1 -path $\Lambda(t)$ with fixed end points, and we can make it only have regular crossings by a small perturbation.

In contrast with the definition given in Eq. (2.29), the Maslov index defined in [37] has the following form

$$\begin{aligned} \mu_{RS}(\Lambda(t), W) = & \frac{1}{2} \text{sign}(\Gamma(\Lambda(a), W, a)) + \sum_{a < t < b} \text{sign}(\Gamma(\Lambda(t), W, t)) \\ & + \frac{1}{2} \text{sign}(\Gamma(\Lambda(b), W, b)). \end{aligned} \tag{2.30}$$

We observe that, for the non-degenerate path (i.e. $L(t) \cap W = 0$ for $t = a, b$),

$$\mu(W, L(t)) = \mu_{RS}(L(t), W).$$

Note that for $M \in \text{Sp}(2n)$, $Gr(M) := \{(x, Mx) \mid x \in \mathbb{R}^{2n}\}$ is a Lagrangian subspace of the symplectic vector space $(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, -\omega \oplus \omega)$. Let $\gamma(t)$ be a path of symplectic matrices, $\Lambda = \Lambda_1 \oplus \Lambda_2 \in \text{Lag}(4n)$, where $\Lambda_i \in \text{Lag}(2n)$, for $i = 1, 2$, then following [37] and by computing the crossing forms, we have

$$\mu(\Lambda_1 \oplus \Lambda_2, Gr(\gamma(t))) = \mu(\Lambda_2, \gamma(t)\Lambda_1). \tag{2.31}$$

For a continuous path $\gamma(t) \in \text{Sp}(2n)$ with $\gamma(0) = I_{2n}$, the Maslov-type index $i_\omega(\gamma) \in \mathbb{Z}$ is a very useful tool in studying the periodic orbits of Hamiltonian systems [24]. The next lemma [25, Corollary 2.1] gives its relation with the Maslov index.

Lemma 2.3. For any $\gamma(t)$, we have

$$i_1(\gamma) + n = \mu(\Delta, Gr(\gamma(t))), \quad (2.32)$$

and

$$i_\omega(\gamma) = \mu(Gr(\omega), Gr(\gamma(t))), \omega \in \mathbb{U} \setminus \{1\}, \quad (2.33)$$

where Δ is the diagonal $Gr(I_{2n})$, $Gr(\omega) = Gr(\omega I_{2n})$.

For $V_1, V_2 \in Lag(2n)$ and a Lagrangian path $t \mapsto \Lambda(t)$, the difference of the Maslov indexes with respect to the two Lagrangian subspaces is given in terms of the Hörmander index, i.e. [37, Theorem 3.5]

$$s(V_0, V_1; \Lambda(0), \Lambda(1)) = \mu(V_0, \Lambda) - \mu(V_1, \Lambda). \quad (2.34)$$

Obviously,

$$s(V_0, V_1; \Lambda(0), \Lambda(1)) = s(V_0, V_1; e^{-\varepsilon J} \Lambda(0), e^{-\varepsilon J} \Lambda(1)), \quad (2.35)$$

for $\varepsilon > 0$ small enough. The Hörmander index is independent of the choice of the path connecting $\Lambda(0)$ and $\Lambda(1)$. Under the non-degenerate condition, i.e. V_1, V_2 are transversal to $\Lambda(0), \Lambda(1)$ correspondingly. Two basic properties are given below

$$\begin{aligned} s(V_0, V_1; \Lambda(0), \Lambda(1)) &= -s(V_1, V_0; \Lambda(0), \Lambda(1)), \\ s(\Lambda(0), \Lambda(1); V_0, V_1) &= -s(V_0, V_1; \Lambda(0), \Lambda(1)), \end{aligned}$$

If $V_j = Gr(A_j)$, $\Lambda(j) = Gr(B_j)$ for symmetry matrices A_j and B_j , then

$$\begin{aligned} s(V_0, V_1; \Lambda(0), \Lambda(1)) &= \frac{1}{2} \text{sign}(B_0 - A_1) + \frac{1}{2} \text{sign}(B_1 - A_0) - \frac{1}{2} \text{sign}(B_1 - A_1) \\ &\quad - \frac{1}{2} \text{sign}(B_0 - A_0), \end{aligned} \quad (2.36)$$

where for a symmetric matrix A , $\text{sign}(A)$ is the signature of the symmetric form $\langle A \cdot, \cdot \rangle$. A direct corollary shows that

$$|s(V_0, V_1; \Lambda(0), \Lambda(1))| \leq 2n. \quad (2.37)$$

A sharp estimate for the difference of Neumann and Dirichlet boundary conditions has been given in [26].

Let γ be a fundamental solution of a periodic orbit, then $\gamma \in C([0, T], Sp(2n))$ with $\gamma(0) = I_{2n}$, as we have mentioned in the introduction, the Maslov index $\mu(V_n, \gamma V_n)$, $\mu(V_d, \gamma V_d)$ and Maslov-type index $i_1(\gamma), i_{-1}(\gamma)$ play an important role in the study of stability problem.

We come back to ERE, and we recall that $\hat{\gamma}_e(\tau) = S(\tau)\gamma_e(\tau)S^{-1}(0)$, with $S = \text{diag}\left(q^{\frac{1}{2}}I_k, q^{-\frac{1}{2}}I_k\right) \in Sp(2k)$. Please note that $S(T) = S(0)$, and the path $S(\tau)$ is contractible in $Sp(2n)$. In fact, if we set $S_\alpha = \text{diag}\left(q_\alpha^{\frac{1}{2}}I_k, q_\alpha^{-\frac{1}{2}}I_k\right)$ with $q_\alpha = (1 + \alpha \cos(t))^{1/2}$, then S_α is homotopy to the constant path $S_0(\tau) \equiv I_{2k}$ by S_α for $\alpha \in [0, e]$. We have

Lemma 2.4. For $e \in [0, 1)$, suppose $\dim Gr(S_\alpha(T)\gamma_e(T)S_\alpha^{-1}(0)) \cap \Lambda$ is constant for $\alpha \in [0, e]$, then $\mu(Gr(\gamma_e), \Lambda) = \mu(Gr(\hat{\gamma}_e), \Lambda)$.

From Lemmas 2.3 and 2.4, we get

$$i_\omega(\gamma_e) = i_\omega(\hat{\gamma}_e), \quad \forall \omega \in \mathbb{U}, \quad e \in [0, 1). \tag{2.38}$$

We first consider the Maslov index on l_0^- . Let $\Psi_0(\tau)$ be the fundamental solution on l_0 , that is

$$\frac{d}{d\tau}\Psi_0(\tau) = J\hat{B}_0\Psi_0(\tau), \quad \Psi_0(0) = I_{2k}, \quad \tau \in [-\infty, +\infty), \tag{2.39}$$

with $\hat{B}_0 = \begin{pmatrix} I_k & \frac{Q_{l_0}}{4}I_k \\ \frac{Q_{l_0}}{4}I_k & -R \end{pmatrix}$.

Proposition 2.5. *Suppose $\lambda_1(R) = -(1/8 + r_1)$ with $r_1 > 0$, then, we have*

$$\mu(V_d, \Psi_0(\tau)V_d, \tau \in [0, \tau_0]) \geq \left\lceil \frac{\sqrt{r_1}}{\pi}\tau_0 \right\rceil, \tag{2.40}$$

where $\lceil Z \rceil$ denote the maximum integer which is not bigger than Z .

Proof. By changing the basis, we assume $R = \text{diag}(\lambda_1, \dots, \lambda_k)$ where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k$ and $\lambda_1 < -\frac{1}{8}$. Based on the property V of the Maslov index, we have the decomposition

$$\mu(V_d, \Psi_0(\tau)V_d) = \sum_{i=1}^k \mu(V_d^1, \Psi_0^i(\tau)V_d^1),$$

where $\Psi_0^i(\tau)$ satisfies the equation

$$\frac{d}{d\tau}\Psi_0^i(\tau) = J_2\hat{B}_i\Psi_0^i(\tau), \quad \Psi_0^i(0) = I_2, \quad \tau \in [0, +\infty), \tag{2.41}$$

with $\hat{B}_i = \begin{pmatrix} 1 & \frac{Q_{l_0}}{4} \\ \frac{Q_{l_0}}{4} & -\lambda_i(R) \end{pmatrix}$. Since $\hat{B}_i|_{V_d^1} > 0$, then $\Gamma(\Psi_0^i(\tau)V_d^1, V_d^1, \tau) > 0$, this implies that $\mu(V_d^1, \Psi_0^i(\tau)V_d^1, \tau \in [0, +\tau_0])$ is nondecreasing with respect to τ_0 . Moreover we have

$$\mu(V_d^1, \Psi_0^i(\tau)V_d^1, \tau \in [0, \tau_0]) = \sum_{0 < \tau_j < \tau_0} v^i(\tau_j),$$

where $v^i(\tau_j) = \dim V_d^1 \cap \Psi_0^i(\tau_j)V_d^1$. In order to compute the Maslov index $\mu(V_d^1, \Psi_0^i(\tau)V_d^1, \tau \in [0, \tau_0])$, we choose the basis $e_1 = (1, 0)^T$ of V_d^1 and we let $e_1^i(\tau) = \Psi_0^i(\tau)e_1$. Then $\mu(V_d^1, \Psi_0^i(\tau)V_d^1, \tau \in [0, \tau_0])$ is equals to the number of zeros of $f^i(\tau) = \det(M^i(\tau))$, for $M^i(\tau) = (e_1, e_1^i(\tau))$.

Let $\Psi_0^i(\tau) = \begin{pmatrix} a_i(\tau) & b_i(\tau) \\ c_i(\tau) & d_i(\tau) \end{pmatrix}$. Then $f^i(\tau) = c_i(\tau)$. From Eq. (2.41), we get that $c_i(\tau)$ satisfies the equation

$$\frac{d^2}{d\tau^2}c_i(\tau) = \left(\frac{3}{8} \tanh^2\left(\frac{\sqrt{2}}{2}\tau\right) - \frac{1}{4} + \lambda_i(R) \right) c_i(\tau),$$

$$c_i(0) = 0, \quad \dot{c}_i(0) = 1.$$

For $i = 1$, $\lambda_1(R) = -\frac{1}{8} - r_1$, then we have $\frac{3}{8} \tanh^2(\frac{\sqrt{2}}{2}\tau) - \frac{1}{4} + \lambda_1(R) \leq -r_1$. Using the Sturm comparison theorem, we know the number of zeros of $c_i(\tau)$ will be no less than $\lceil \frac{\sqrt{r_1}}{\pi} \tau_0 \rceil$. This is complete the proof. \square

In order to proof the Theorem 1.1, we need the lemma below. We will give an estimation of Maslov index on the period $[\tau_1, \tau_2]$. Let $\varepsilon_1 = \min\{\frac{r_1}{2r_1+5}, 1/8\}$, and $\hat{\gamma}(\tau, \tau_1)$ be the fundamental solution of (2.13) with $\hat{\gamma}(\tau_1, \tau_1) = I_{2k}$, we have

Lemma 2.6. For $\varepsilon \leq \frac{1}{2}\varepsilon_1$, $\hat{e} < \varepsilon^3$,

$$\mu(V_d, \hat{\gamma}(\tau, \tau_1)V_d; \tau \in [\tau_1, \tau_2]) \geq \frac{\sqrt{r_1}}{\pi} \ln\left(\frac{\varepsilon^2}{\sqrt{\hat{e}}}\right) - 3. \quad (2.42)$$

Proof. Let $\hat{B}_- = \begin{pmatrix} I_k & -\frac{\sqrt{2}}{4}I_k \\ -\frac{\sqrt{2}}{4}I_k & -R \end{pmatrix}$. Suppose $|\frac{Q+\sqrt{2}}{4} - q| < \varepsilon_1$ for some $\varepsilon_1 < 1$, then we have

$$\hat{B} - \hat{B}_- = \begin{pmatrix} 0_k & \frac{Q+\sqrt{2}}{4}I_k - q\mathbb{J}_{k/2} \\ \frac{Q+\sqrt{2}}{4}I_k + q\mathbb{J}_{k/2} & q^2 \end{pmatrix} > -\varepsilon_1 I_{2k}.$$

Let $\hat{B}_{\varepsilon_1} = \hat{B}_- - \varepsilon_1 I_{2k}$, by the monotonicity property of Maslov index, we have

$$\mu(V_d, \hat{\gamma}(\tau, \tau_1)V_d; \tau \in [\tau_1, \tau_2]) \geq \mu(V_d, \exp((\tau - \tau_1)J\hat{B}_{\varepsilon_1})V_d; \tau \in [\tau_1, \tau_2]). \quad (2.43)$$

Since R is diagonalizable, by using the \diamond -product, we can split \hat{B}_{ε_1} into the product of k two by two matrices, where the first factor (which is needed for computing the Maslov index) is given by $\hat{B}_1 := \begin{pmatrix} 1 - \varepsilon_1 & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & 1/8 + r_1 - \varepsilon_1 \end{pmatrix}$. By the direct sum property of Maslov index, we have

$$\begin{aligned} & \mu(V_d, \exp((\tau - \tau_1)J\hat{B}_{\varepsilon_1})V_d; \tau \in [\tau_1, \tau_2]) \\ & \geq \mu(V_d^1, \exp((\tau - \tau_1)J\hat{B}_1)V_d^1; \tau \in [\tau_1, \tau_2]). \end{aligned} \quad (2.44)$$

Let $f(\varepsilon_1, r_1) = \varepsilon_1^2 - (r + \frac{9}{8})\varepsilon_1 + r_1$, $\tilde{B} = \text{diag}(1, f(\varepsilon_1, r_1))$, $P = \begin{pmatrix} (1 - \varepsilon_1)^{-1/2} & 0 \\ 0 & (1 - \varepsilon_1)^{1/2} \end{pmatrix} \begin{pmatrix} 1 & \frac{\sqrt{2}}{4} \\ 0 & 1 \end{pmatrix}$, then

$$P^{-1} \exp((\tau - \tau_1)J\hat{B}_1)P = \exp((\tau - \tau_1)J\tilde{B}).$$

We have

$$\begin{aligned} \mu(V_d, \exp((\tau - \tau_1)J\hat{B}_1)V_d) &= \mu(P^{-1}V_d, \exp((\tau - \tau_1)J\tilde{B})P^{-1}V_d) \\ &\geq \mu(V_d, \exp((\tau - \tau_1)J\tilde{B})V_d) - 2, \end{aligned} \quad (2.45)$$

where the last inequality is from (2.37). In the next, we will estimate $\mu(V_d, \exp((\tau - \tau_1)J\tilde{B})V_d; \tau \in [\tau_1, \tau_2])$. A direct computation shows that $f(\varepsilon_1, r_1) > r_1/2$ if $\varepsilon_1 < \frac{r_1}{2r_1+5}$, and hence $\tilde{B} > \tilde{B}_{r_1/2} := \text{diag}(1, r_1/2)$. Moreover

$$\mu(V_d, \exp((\tau - \tau_1)J\tilde{B}_{r_1/2})V_d; \tau \in [\tau_1, \tau_2]) \geq \frac{\sqrt{r_1}(\tau_2 - \tau_1)}{\sqrt{2\pi}} - 1. \quad (2.46)$$

For $\varepsilon < \frac{\varepsilon_1}{2}$, then $|\frac{Q+\sqrt{2}}{4} - q| < \varepsilon_1$, From (2.45, 2.46) and (2.15), we have

$$\begin{aligned} \mu(V_d, \gamma(\tau, \tau_1)V_d; \tau \in [\tau_1, \tau_2]) &\geq \mu(V_d, \exp((\tau - \tau_1)J\hat{B}_1)V_d; \tau \in [\tau_1, \tau_2]) \\ &\geq \mu(V_d, \exp((\tau - \tau_1)J\tilde{B}_{r/2})V_d; \tau \in [\tau_1, \tau_2]) - 2 \\ &\geq \frac{\sqrt{r_1} \ln(\frac{\varepsilon}{q(\tau_1)})}{\pi} - 3. \end{aligned} \tag{2.47}$$

Direct compute show that $q^2(\tau_1) = \frac{1}{2}(\varepsilon(2\sqrt{2}-\varepsilon)-((\varepsilon(2\sqrt{2}-\varepsilon))^2-8\hat{\varepsilon})^{\frac{1}{2}}) \leq \frac{8\hat{\varepsilon}}{3\varepsilon(2\sqrt{2}-\varepsilon)}$, then

$$\ln\left(\frac{\varepsilon}{q(\tau_1)}\right) = \ln(\varepsilon) - \frac{1}{2} \ln(q^2(\tau_1)) \geq \ln\left(\frac{\varepsilon^2}{\sqrt{\hat{\varepsilon}}}\right). \tag{2.48}$$

The result is from (2.47–2.48). \square

Proof of Theorem 1.1. Under the assumption $\lambda_1(R) = -\frac{1}{8} - r_1$, from Lemma 2.6, we have For $\varepsilon \leq \frac{1}{2}\varepsilon_1, \hat{\varepsilon} < \varepsilon^3, \mu(V_d, \gamma(\tau, \tau_1)V_d; \tau \in [\tau_1, \tau_2]) \geq \frac{\sqrt{r_1}}{\pi} \ln\left(\frac{\varepsilon^2}{\sqrt{\hat{\varepsilon}}}\right) - 3$. Similarly

$$\mu(V_d, \gamma(\tau, \tau_4)V_d; \tau \in [\tau_4, \tau_5]) \geq \frac{\sqrt{r_1}}{\pi} \ln\left(\frac{\varepsilon^2}{\sqrt{\hat{\varepsilon}}}\right) - 3. \tag{2.49}$$

We have

$$\mu(V_d, \gamma(\tau, 0)V_d; \tau \in [0, T]) \geq 2\frac{\sqrt{r_1}}{\pi} \ln\left(\frac{\varepsilon^2}{\sqrt{\hat{\varepsilon}}}\right) - 6. \tag{2.50}$$

The results now follows from the fact that $i_1(\gamma) \geq \mu(V_d, \gamma V_d)$ (for Lagrangian system, we refer Sect. 5.1 for a detailed discussion). This complete the proof. \square

3. Collision Index for Planar Central Configurations

This section is the main part of our paper. We give the definition of the collision index in Sect. 3.1 and we prove the approximation theorem; we study the basic property of the collision index and we compute in detail the collision index on l_0 in Sect. 3.2.

3.1. Collision index. In this subsection, we will consider the Maslov index on the half line with a hyperbolic equilibrium. This is similar to the case of homoclinic orbit [10] and heteroclinic orbit [16], and a detailed study to the half-clinic orbits is given in [5, 16].

To define the Maslov index of the half line, we firstly review some basic fact of heteroclinic orbits. We consider the Hamiltonian flow induced by

$$\dot{z} = JB(t)z, \quad t \in \mathbb{R}. \tag{3.1}$$

We assume the limit is hyperbolic, meaning that

$$JB(\pm\infty) = \lim_{t \rightarrow \pm\infty} JB(\pm t)$$

is hyperbolic. It follows that $\mathbb{R}^{2n} = V_s^\pm \oplus V_u^\pm$, where $V_s^\pm (V_u^\pm)$ is the stable subspace (resp. unstable subspace) of the equilibria which is spanned by the generalized eigenvector

of eigenvalue with negative real part (positive real part) of $JB(\pm\infty)$. Moreover, both the stable subspace V_s^\pm and the unstable subspace V_u^\pm are Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$. The topology of Lagrangian Grassmannian $Lag(2n)$ is given by the metric

$$\rho(V, W) = \| \mathcal{P}_V - \mathcal{P}_W \|,$$

where $\mathcal{P}_V, \mathcal{P}_W$ is the orthogonal projection to V, W and $\| \cdot \|$ is the operator norm.

Let $\gamma(t, v)$ satisfy (3.1) with $\gamma(v, v) = I_{2n}$. In what follows we set $\gamma(t) := \gamma(t, 0)$. Clearly γ satisfies a semigroup property; that is, $\gamma(t, v)\gamma(v, \tau) = \gamma(t, \tau)$. For $v \in \mathbb{R}$, define

$$V_s(v) = \{ \xi | \xi \in \mathbb{R}^{2n} \text{ and } \lim_{t \rightarrow \infty} \gamma(t, v)\xi = 0 \},$$

and

$$V_u(v) = \{ \xi | \xi \in \mathbb{R}^{2n} \text{ and } \lim_{t \rightarrow -\infty} \gamma(t, v)\xi = 0 \}.$$

We remark that

$$\lim_{v \rightarrow \infty} V_s(v) = V_s^+ \text{ and } \lim_{v \rightarrow -\infty} V_u(v) = V_u^-.$$

It is well known that both $V_s(v)$ and $V_u(v)$ are Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$. An important property from [1] is the following: if V transversal to $V_s(0)$, then

$$\lim_{t \rightarrow \infty} \gamma(t, 0)V = V_u^+.$$

Similarly, if V transversal to $V_u(0)$, then

$$\lim_{t \rightarrow -\infty} \gamma(t, 0)V = V_s^-.$$

Let $\mathbb{R}^\pm := \{ \pm x \geq 0, x \in \mathbb{R} \}$. We will define the Maslov index of the half line \mathbb{R}^+ or \mathbb{R}^- . We notice that the discussions for heteroclinic orbit works for the half-clinic orbit. Firstly, we give the definition of nondegeneracy.

Definition 3.1. (i) The linear system (3.1) on \mathbb{R} is called nondegenerate if there is no bounded solution,

(ii) the linear system on \mathbb{R}^\pm is called nondegenerate with respect to V_0 , if there is no bounded solutions on \mathbb{R}^\pm which satisfies $z(0) \in V_0$.

We observe that for the system with hyperbolic limit, all the bounded solution must decay to 0 as $t \rightarrow \pm\infty$ [1].

We firstly give the definition of Maslov index on \mathbb{R}^+ . For, let $V_0, V_1 \in Lag(2n)$ and we suppose that the system is nondegenerate with respect to V_0 , that is $V_0 \pitchfork V_s(0)$. Then $\gamma(t, 0)V_0$ is a path of Lagrangian subspaces having limit V_u^+ and so, we define the Maslov index on \mathbb{R}^+ with V_0, V_1 by

$$i_+(V_1, V_0) := \mu(V_1, \gamma(t, 0)V_0, t \in \mathbb{R}^+). \tag{3.2}$$

In the case \mathbb{R}^- , recall that $V_u(t)$ is a path of Lagrangian subspace and $\lim_{t \rightarrow -\infty} V_u(t) = V_u^-$, then for $V \in Lag(2n)$, we define

$$i_-(V) := \mu(V, V_u(t), t \in \mathbb{R}^-). \tag{3.3}$$

We observe that the definition on \mathbb{R}^- does not need the nondegenerate condition. Finally, we will define the Maslov index on \mathbb{R} which is fully studied in [16]. Supposing that the linear system is nondegenerate on \mathbb{R} , then $\lim_{t \rightarrow -\infty} V_u(t) = V_u^-$ and $\lim_{t \rightarrow \infty} V_u(t) = V_u^+$. Thus we define

$$i(V) := \mu(V, V_u(t), t \in \mathbb{R}). \tag{3.4}$$

Under the nondegenerate condition, it is obvious that

$$i(V) = i_-(V) + i_+(V, V_u(0)).$$

We come back to ERE. By assuming that $\lambda_1(R) > -\frac{1}{8}$, we can identify l_0, l_+ with \mathbb{R} , and identify l_0^\mp, l_+^\pm with \mathbb{R}^\pm . For V_0, V satisfying the nondegenerate conditions, $i(V_1)$ on l_0 or l_+ , $i_+(V_1, V_0)$ on l_0^- or l_+^+ and $i_-(V_1)$ on l_0^+ and on l_+^- are well defined, and we shall refer to as *collision index*.

Definition 3.2. The planar central configuration is called collision nondegenerate if the corresponding system on l_+ is nondegenerate.

We identify \mathbb{R} with l_+ , and let $V_u(\tau)$ be the unstable subspace. Under the nondegenerate conditions,

$$\lim_{\tau \rightarrow \pm\infty} V_u(\tau) = V_u^\pm.$$

Let $V_{u,0}$ be the unstable subspace on l_0^- , then

$$\lim_{\tau \rightarrow \pm\infty} V_{u,0}(\tau) = V_u^\mp.$$

For $V_0, V_1 \in \text{Lag}(2n)$, satisfying $V_0 \pitchfork V_s(0)$, then the Maslov index $i_+(V_0, V_1)$ and $i_-(V_1)$ are well defined. As $e \rightarrow 1$, we have the next approximation theorem which plays a key role in our paper.

Theorem 3.3. Assuming $\lambda_1(R) > -\frac{1}{8}$, we have: (i) If $V_u^- \pitchfork V_1$, the system is nondegenerate with respect to V_0 on l_0^- , and nondegenerate with respect to V_1 on l_+^- , then, for $1 - e$ small enough, $V_1 \pitchfork \hat{\gamma}_e(\mathcal{T}/2)V_0$ and

$$\mu(V_1, \hat{\gamma}_e(\tau)V_0, \tau \in [0, \mathcal{T}/2]) = i_+(V_1, V_0; l_0^-) + i_-(V_1; l_+^-). \tag{3.5}$$

(ii) If $V_u^+ \pitchfork V_1$, the system is nondegenerate with respect to V_0 on l_0^+ , and nondegenerate with respect to V_1 on l_0^+ , then, for $1 - e$ small enough, $V_1 \pitchfork \gamma_e(\mathcal{T})\hat{\gamma}_e^{-1}(\mathcal{T}/2)V_0$ and

$$\mu(V_1, \hat{\gamma}_e(\tau)\hat{\gamma}_e^{-1}(\mathcal{T}/2)V_0, \tau \in [\mathcal{T}/2, \mathcal{T}]) = i_+(V_1, V_0; l_0^+) + i_-(V_1; l_0^+). \tag{3.6}$$

(iii) If $V_u^\pm \pitchfork V_1$, the system is collision nondegenerate, and nondegenerate with respect to V_0, V_1 on l_0^-, l_0^+ correspondingly, then, for $1 - e$ small enough, $V_1 \pitchfork \hat{\gamma}_e(\mathcal{T})V_0$ and

$$\mu(V_1, \hat{\gamma}_e(\tau)V_0, \tau \in [0, \mathcal{T}]) = i_+(V_1, V_0; l_0^-) + i_-(V_1; l_0^+) + i(V_1; l_+). \tag{3.7}$$

The proof is based on a series lemmas, we firstly give the next lemma which is from [29].

Lemma 3.4. *Let us consider the linear system*

$$x' = Dx + C(\tau)x, \tag{3.8}$$

where D is $k \times k$ diagonal matrix and $C(t)$ is a continuous matrix in $t \in [0, \hat{t}]$, such that $\int_0^{\hat{t}} \|C(s)\| ds < \hat{\varepsilon}$, for some constant $\hat{\varepsilon}$ which satisfies $\frac{6\sqrt{k}\hat{\varepsilon}}{1-3\hat{\varepsilon}} < 1$ and let $\gamma(\tau)$ be the fundamental solution of (3.8). Then for $t \in [0, \hat{t}]$, we have

$$\gamma(t) = (I + O(t)) \exp(Dt)(I + S), \tag{3.9}$$

where $\|O(t)\| \leq \frac{3\sqrt{k}\hat{\varepsilon}}{1-3\hat{\varepsilon}}$, $\|S\| \leq \frac{6\sqrt{k}\hat{\varepsilon}}{1-3\hat{\varepsilon}}$.

Proof. From lemma 6 of [29] and for $\hat{\varepsilon} < 1/4$, let λ be an eigenvalue of D and V be the corresponding eigenvector. Then there exists a solution $\varphi(t)$ of (3.8) such that

$$\|e^{-\lambda t} \varphi(t) - V\| \leq \frac{3\hat{\varepsilon}}{1-3\hat{\varepsilon}}.$$

Let $e_i, i = 1, \dots, k$ be the canonical basis, $Y(t)$ be the matrix defined by $\varphi_1, \dots, \varphi_k$ as column vectors. We define $O(t) := Y(t) \exp(-Dt) - I$, then $\|O(t)\| \leq \frac{3\sqrt{k}\hat{\varepsilon}}{1-3\hat{\varepsilon}}$ for $t \in [0, \hat{t}]$. Obviously, $\gamma(t) = Y(t)Y^{-1}(0) = (I + O(t)) \exp(Dt)(I + O(0))^{-1}$. Let $S := (I + O(0))^{-1} - I$, then for $\|O(0)\| < 1/2$, we have

$$\|S\| \leq \frac{\|O(0)\|}{1 - \|O(0)\|} < 2\|O(0)\| \leq \frac{6\sqrt{k}\hat{\varepsilon}}{1-3\hat{\varepsilon}},$$

which complete the proof. \square

Assume that $\lambda_1(R) > -\frac{1}{8}$, then P_{\pm} are hyperbolic. We recall that we set $D_{\pm} = J\hat{B}(P_{\pm})$ having the form $J_k \begin{pmatrix} I_k & \pm \frac{\sqrt{2}}{4} I_k \\ \pm \frac{\sqrt{2}}{4} I_k & -R \end{pmatrix}$. An easy computation shows that the eigenvalues of D_{\pm} are real if $\lambda_1(R) > -\frac{1}{8}$. Choose basis such that R is diagonalisable, that is $R = \text{diag}(\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \leq \dots \leq \lambda_k$. Let $P_1 = \begin{pmatrix} I_k & \frac{\sqrt{2}}{4} \\ 0_k & I_k \end{pmatrix}$, $P_2 = \begin{pmatrix} \sqrt{1/8 + R} I_k & \sqrt{1/8 + R} I_k \\ I_k & -I_k \end{pmatrix}$, $P = P_1 P_2$. By a direct computation we get that

$$P^{-1} D_- P = \text{diag}(\eta_1, \dots, \eta_k, -\eta_1, \dots, -\eta_k), \tag{3.10}$$

where $\eta_j = \sqrt{1/8 + \lambda_j}$. Let $\hat{\eta} = \max\{\eta_j, \eta_j^{-1}; j = 1, \dots, k\}$, then easy computation show that

$$\max\{\|P\|, \|P^{-1}\|\} \leq 2(1 + \hat{\eta}). \tag{3.11}$$

Based on Lemma 3.4, we firstly prove the important lemma below.

Lemma 3.5. *We assume that $\lambda_1(R) > -\frac{1}{8}$ and we let $\hat{\gamma}(\tau, \tau_1)$ be the fundamental solution of (2.13) with $\hat{\gamma}(\tau_1, \tau_1) = I_{2k}$. Then for $\varepsilon < \varepsilon_0$, $\hat{\varepsilon} < \varepsilon^3$, we have the following estimate below*

$$\hat{\gamma}(\tau, \tau_1) = P(I_{2k} + \Delta(\tau))D(\tau)(I_{2k} + S)P^{-1}, \quad \tau \in [\tau_1, \tau_2], \tag{3.12}$$

where the matrices $\Delta(\tau)$, S satisfy $\|\Delta\| \leq \frac{c_1}{2}\varepsilon$, $\|S\| \leq c_1\varepsilon$, for ε_0, c_1 is constant and dependent on R and $D(\tau) = \text{diag}(e^{\eta_1(\tau-\tau_1)}, \dots, e^{\eta_k(\tau-\tau_1)}, e^{-\eta_1(\tau-\tau_1)}, \dots, e^{-\eta_k(\tau-\tau_1)})$.

Proof. Simple computation shows that $J\hat{B}(\tau) = D_- + C(\tau)$ with

$$C(\tau) = \begin{pmatrix} -\frac{Q+\sqrt{2}}{4}I_k - q\mathbb{J}_{k/2} & -q^2I_k \\ 0_k & \frac{Q+\sqrt{2}}{4}I_k - q\mathbb{J}_{k/2} \end{pmatrix}.$$

Let $W(\tau) = P^{-1}\hat{\gamma}(\tau, \tau_1)P$, then

$$\dot{W}(\tau) = (P^{-1}D_-P + P^{-1}C(\tau)P)W(\tau).$$

Let $\varepsilon < \varepsilon_0$, where $\varepsilon_0 < 1/8$ will be fixed later. It is obvious $\int_{\tau_1}^{\tau_2} q^2(\tau) \leq \int_{\tau_1}^{\tau_2} q(\tau)d\tau$ for $0 < q < 1$. From Eqs. (2.16–2.17), we have $\int_{\tau_1}^{\tau_2} q(\tau) \leq 2\varepsilon$ as well as $\int_{\tau_1}^{\tau_2} |Q(\tau) + \sqrt{2}| \leq 2\varepsilon$. Then we have

$$\int_{\tau_1}^{\tau_2} \|C(\tau)\|d\tau \leq \int_{\tau_1}^{\tau_2} \left(\frac{1}{2}\|Q + \sqrt{2}\| + 2\|q\| + \|q^2\|\right)d\tau \leq 6\varepsilon, \tag{3.13}$$

and from (3.11) also that

$$\int_{\tau_1}^{\tau_2} \|P^{-1}C(\tau)P\|d\tau \leq 24(1 + \hat{\eta})^2\varepsilon. \tag{3.14}$$

Let $\varepsilon_0 = (24(3 + 6\sqrt{k})(1 + \hat{\eta})^2)^{-1}$ and for $\varepsilon < \varepsilon_0$, we denote $\hat{\varepsilon} = 24(1 + \hat{\eta})^2\varepsilon$. Then, $\varepsilon < \varepsilon_0$ implies $\frac{6\sqrt{k}\hat{\varepsilon}}{1-3\hat{\varepsilon}} < 1$. From Lemma 3.4, we have

$$\|\Delta\| \leq \frac{3\sqrt{k}\hat{\varepsilon}}{1-3\hat{\varepsilon}} \leq \frac{c_1}{2}\varepsilon, \quad \|S\| \leq \frac{6\sqrt{k}\hat{\varepsilon}}{1-3\hat{\varepsilon}} \leq c_1\varepsilon, \tag{3.15}$$

where $c_1 = 24^2\sqrt{k}(1 + \hat{\eta})^2$ only depend on R . This complete the proof. \square

Given two subspaces, graphs of two linear operators, it is possible to introduce a norm topology on the Lagrangian Grassmannian, equivalent to the gap topology, as below. More precisely, if $E = E^- \oplus E^+$, a sequence of operators $(L_n)_{n \in \mathbb{N}} \subset L(E^-, E^+)$ converges to L if and only if their graphs converge to the graph of L . Another important property is that, the image TV of a closed subspace V by an invertible linear operator T , is continuously depending on (T, V) . From Lemma 3.5, $P^{-1}(V_u) = V_d$ and $P^{-1}(V_s) = V_n$. For $V \in \text{Lag}(2n)$ with $V \pitchfork V_s^-$, then $\exists L_V$ such that

$$P^{-1}V = Gr(L_V).$$

We give a equivalent metric

$$\hat{\rho}(V, W) = \|L_V - L_W\|.$$

It is obvious that $L_{V_u^-} = 0_k$, since $V_u^- \pitchfork V_1$, so $\exists \sigma_1 > 0$ such that $V \pitchfork V_1$ if $\|L_V\| \leq \sigma_1$. For $\sigma > 0$, we always denote

$$B_{\hat{\rho}}(V, \sigma) = \{W \in \text{Lag}(2n), \hat{\rho}(V, W) < \sigma\}.$$

Lemma 3.6. *Let $\bar{V} = \text{Gr}(L_V)$, i.e. $\bar{V} = PV$. Then for any $\bar{V} \in U(\sigma) = \{\bar{V} : \|L_V\| < \sigma\}$ with $\sigma < 1$ and if $\|\Lambda\| < \sigma/6$ then $(I + \Lambda)\bar{V} \in U(2\sigma)$.*

Proof. For $\Lambda = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{pmatrix}$, we have $(I + \Lambda) \begin{pmatrix} x \\ L_V x \end{pmatrix} = \begin{pmatrix} (I + \Lambda_1 + \Lambda_2 L_V)x \\ (\Lambda_3 + (I + \Lambda_4)L_V)x \end{pmatrix}$. Let $y = (I + \Lambda_1 + \Lambda_2 L_V)x$ and choose $\|\Lambda\|$ small enough, then we have $(I + \Lambda)\bar{V} = \text{Gr}((\Lambda_3 + (I + \Lambda_4)L_V)(I + \Lambda_1 + \Lambda_2 L_V)^{-1})$. Since $\|\Lambda\| < \sigma/6$, an easy computation shows that

$$\begin{aligned} \|(\Lambda_3 + (I + \Lambda_4)L_V)(I + \Lambda_1 + \Lambda_2 L_V)^{-1}\| &\leq \frac{\|\Lambda_2\| + \|I + \Lambda_4\| \|L_V\|}{I - \|\Lambda_1\| - \|\Lambda_2\| \|L_V\|} \\ &< \frac{\|\Lambda\| + (1 + \|\Lambda\|) \|L_V\|}{I - \|\Lambda\| - \|\Lambda\| \|L_V\|} < 2\sigma. \end{aligned} \quad (3.16)$$

□

Lemma 3.7. *For any $0 < \sigma < 1$, we let $\varepsilon_\sigma := \min\{\varepsilon_0, \frac{\sigma}{24c_1}\}$. If $\varepsilon \leq \varepsilon_\sigma$, $V \in B_{\hat{\rho}}(V_u^-, \sigma/4)$ and $\hat{e} < \varepsilon^3$, then for every $\tau \in [\tau_1, \tau_2]$, we have $\hat{\gamma}_e(\tau, \tau_1)V \in B_{\hat{\rho}}(V_u^-, \sigma)$.*

Proof. From (3.12), $P^{-1}\hat{\gamma}_e(\tau, \tau_1)V = (I_{2k} + \Delta(\tau))D(\tau)(I_{2k} + S)P^{-1}V$ where $V \in B_{\hat{\rho}}(V_u^-, \sigma/4)$, $P^{-1}V \in U(\sigma/4)$. Since $\|S\| \leq C_1\varepsilon \leq \frac{\sigma}{24}$, then we have $(I_{2k} + S)P^{-1}V \in U(\sigma/2)$ by Lemma 3.6. Obviously $D(\tau)U(\sigma/2) \subset U(\sigma/2)$, using Lemma 3.6 again, we have $(I_{2k} + \Delta(\tau))D(\tau)(I_{2k} + S)P^{-1}V \in U(\sigma)$, which conclude the proof. □

Let Ψ_0 be the fundamental solution on l_0 as given in Equation (2.39) and $\Psi_+(\tau, \nu)$ be the fundamental solution on l_+ . Let $\sigma < \frac{1}{3} \min\{\rho(V_u^-, V_1), \rho(V_u^-, V_s^-)\}$ small enough such that $B_{\hat{\rho}}(V_u^-, 3\sigma) \pitchfork V_s^-, V_1$. For this σ , $\exists \sigma_1 > 0$ such that $B_{\hat{\rho}}(V_u^-, \sigma_1) \subset B_{\hat{\rho}}(V_u^-, \sigma)$, and let ε_{σ_1} be the number corresponding to σ_1 in Lemma 3.7.

Choose $\varepsilon < \sigma_1$ small enough such that

$$\max\{\rho(\Psi_0(\tau_0)V_0, V_u^-,), \rho(V_u(-\tau_{l_+}), V_u^-,), \rho(\Psi_+(-\tau_{l_+}, 0)V_1, V_s^-,)\} < \sigma. \quad (3.17)$$

From Lemma 3.7, We have

Lemma 3.8. *For this fixed ε , $\hat{e} < \varepsilon^3$, $\rho(\hat{\gamma}_e(\tau)V_0, V_u^-) < \sigma$ for $\tau \in (\tau_1, \tau_2)$.*

Similarly, by the symmetric Fig. 2, for Σ_4, Σ_5 , we have the following result.

Lemma 3.9. *For this fixed ε , $\hat{e} < \varepsilon^3$, $\rho(\hat{\gamma}_e(\tau)\hat{\gamma}_e^{-1}(T/2)V_1, V_u^+) < \sigma$ for $\tau \in (\tau_4^+, \tau_5^+)$.*

Proof of Theorem 3.3. To prove (i), we will compute the Maslov index $\mu(V_1, \hat{\gamma}_e(\tau)V_0; \tau \in [0, T/2])$ on the three time interval $[0, \tau_1]$, $[\tau_1, \tau_2]$ and $[\tau_2, T/2]$. Form Lemma 3.8, for $\hat{e} < \varepsilon^3$,

$$\mu(V_1, \hat{\gamma}_e(\tau)V_0, \tau \in [0, \tau_1]) = \mu(V_1, \Psi_0(\tau)V_0, \tau \in [0, \tau_0]) = i(V_1, V_0; l_0^-). \quad (3.18)$$

Obviously

$$\mu(V_1, \hat{\gamma}_e(\tau)V_0, \tau \in [\tau_1, \tau_2]) = 0. \quad (3.19)$$

Now we consider the path on $[\tau_2, T/2]$. Please note that for $\hat{\varepsilon} < \varepsilon^3$, $\rho(\hat{\gamma}_\varepsilon(\tau_2)V_0, V_u^-) < \sigma$ by Lemma 3.8 and $\rho(\Psi_+(-\tau_{l_+}, 0)V_1, V_s^-,) < \sigma$ by (3.17), then $\hat{\gamma}_\varepsilon(\tau_2)V_0 \pitchfork \Psi_+(-\tau_{l_+}, 0)V_1$, which implies

$$\Psi_+(0, -\tau_{l_+})\hat{\gamma}_\varepsilon(\tau_2)V_0 \pitchfork V_1.$$

Since $\hat{\gamma}_\varepsilon(\tau - \tau_2)$ uniformly converges to $\Psi_+(\tau, -\tau_{l_+})$, we have for $\hat{\varepsilon} < \varepsilon^3$ small enough,

$$\mu(V_1, \hat{\gamma}_\varepsilon(\tau)V_0, \tau \in [\tau_2, T/2]) = \mu(V_1, \Psi_+(\tau, -\tau_{l_+})V_0, \tau \in [-\tau_{l_+}, 0]) = i_-(V_1; l_+^-). \tag{3.20}$$

The result of (i) is from (3.18), (3.19) and (3.20).

The proof of (ii) is based Lemma 3.9 and is totally analogous.

To prove (iii), we compute Maslov index $\mu(V_1, \hat{\gamma}_\varepsilon(\tau)V_0; \tau \in [0, T])$ on the five time intervals $[0, \tau_1]$, $[\tau_1, \tau_2]$, $[\tau_2, \tau_4]$, $[\tau_4, \tau_5]$ and $[\tau_5, T]$. By assumption of collision nondegenerate, the system is nondegenerate on l_+ , that is $\lim_{\tau \rightarrow +\infty} V_u(\tau) = V_u^+$. So for σ small enough, $V \in B_\sigma(V_u^-)$, we have $\lim_{\tau \rightarrow +\infty} \Psi(\tau, -\tau_{l_+})V = V_u^+$. If we consider $-\tau_{l_+}$ as the starting point, this means the system is nondegenerate with respect to V . By arguing as in step (i), we have for $\hat{\varepsilon} < \varepsilon^3$ small enough.

$$\begin{aligned} \mu(V_1, \hat{\gamma}_\varepsilon(\tau)V_0, \tau \in [\tau_2, \tau_4]) &= \mu(V_1, \Psi_+(\tau, -\tau_{l_+})(\hat{\gamma}_\varepsilon(\tau_2)V_0), \\ \tau \in [-\tau_{l_+}, \tau_{l_+}] &= i(V_1; l_+), \\ \mu(V_1, \hat{\gamma}_\varepsilon(\tau)V_0, \tau \in [\tau_4, \tau_5]) &= 0, \end{aligned}$$

and

$$\mu(V_1, \hat{\gamma}_\varepsilon(\tau)V_0, \tau \in [\tau_5, T]) = i_-(V_1; l_0^-),$$

with (3.18–3.19), we get the result. \square

3.2. *Some fundamental property of collision index.* We first compute the collision index

on l_0 . Recall that on line l_0 , $\hat{B} = \begin{pmatrix} I_k & \frac{Q}{4}I_k \\ \frac{Q}{4}I_k & -R \end{pmatrix}$. We can choose bases such that $R = \text{diag}(r_1, \dots, r_k)$, and set $\hat{B}_r = \begin{pmatrix} 1 & \frac{Q}{4} \\ \frac{Q}{4} & -r \end{pmatrix}$. Given any two $2m_k \times 2m_k$ square block matrices $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ with $k = 1, 2$, the symplectic sum of M_1 and M_2 is defined by

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

It is clear that, $\hat{B} = \hat{B}_{r_1} \diamond \dots \diamond \hat{B}_{r_k}$. We start by considering the two dimensional case. The linear systems $\dot{z} = J\hat{B}_r z$ with the form

$$\dot{y} = -\frac{Q}{4}y + rx, \tag{3.21}$$

$$\dot{x} = y + \frac{Q}{4}x, \tag{3.22}$$

where $z = (y, x)^T$. Assuming that $r \neq 0$, $r > -\frac{1}{8}$ and by taking derivative with respect to τ on both sides of (3.21), we have

$$\begin{aligned}\ddot{y} &= -\frac{\dot{Q}}{4}y - \frac{Q}{4}\dot{y} + r\dot{x} \\ &= \left(\frac{Q^2}{16} - \frac{\dot{Q}}{4} + r\right)y \\ &= \left(\frac{1}{4} - \frac{1}{8}\tanh^2\left(\frac{\sqrt{2}\tau}{2}\right) + r\right)y,\end{aligned}\tag{3.23}$$

where the second equality is from the fact that $x = \frac{1}{r}(\dot{y} + \frac{Q}{4}y)$ by (3.21). Let

$$f := \frac{1}{4} - \frac{1}{8}\tanh^2\left(\frac{\sqrt{2}\tau}{2}\right) + r = \frac{1}{8}(1 - \tanh^2\left(\frac{\sqrt{2}\tau}{2}\right)) + \left(r + \frac{1}{8}\right) > 0.\tag{3.24}$$

Lemma 3.10. *If $r > -\frac{1}{8}$, $r \neq 0$, then (i) for any $t_2 > t_1$, there is no nontrivial solution of (3.23) which satisfies boundary condition $y(t_1)\dot{y}(t_1) \geq y(t_2)\dot{y}(t_2)$; (ii) there is no nontrivial solution satisfying $y(0)\dot{y}(0) = 0$ and $y \rightarrow 0$, $\dot{y} \rightarrow 0$ as $\tau \rightarrow \pm\infty$; (iii) there is no nontrivial bounded solution on \mathbb{R} .*

Proof. Suppose y is solution of (3.23), then multiply by y and by integrating over the time interval $[t_1, t_2]$ we have

$$y(t_1)\dot{y}(t_1) - y(t_2)\dot{y}(t_2) + \int_{t_1}^{t_2} (\dot{y}^2 + fy^2)d\tau = 0.\tag{3.25}$$

The first conclusion now readily follows. By taking limit of (3.25) could get the second conclusion. The third conclusion follows from the fact that any bounded solution must decay exponential fast. \square

Recall that V_d, V_n are Lagrangian subspaces corresponding to the Dirichlet and Neumann boundary conditions. In the two dimensional case, let $e_1 = (1, 0)^T$, $e_2 = (0, 1)^T$. Then it is obvious that V_d, V_n are the linear spaces spanned by e_1, e_2 . An easy computation shows that $z(0) = e_1$ is equivalent to $y(0) = 1$ and $\dot{y}(0) = 0$. In the same way, $z(0) = e_2$ is equivalent to $y(0) = 0$ and $\dot{y}(0) = r$. The second conclusion of Lemma 3.10. implies that for $r \neq 0$, the system is nondegenerate. However this is not true for $r = 0$. In fact, let $r = 0$ in Eqs. (3.21–3.22). Then there is a nontrivial solution satisfies $Z(0) = e_2$ and $Z(t) \rightarrow 0$. Thus we have

Lemma 3.11. *Supposing $r > -1/8$, the system on l_0^- is nondegenerate with V_d , and it is nondegenerate with V_n if and only if $r \neq 0$.*

We firstly consider the Maslov index on l_0^+ , so we have the following result.

Lemma 3.12. *Suppose $r > -1/8$ and $r \neq 0$. Then*

$$i_-(V_d; l_0^+) = i_-(V_n; l_0^+) = 0.\tag{3.26}$$

Proof. By the definition of Maslov index, we only need to show that there is no nontrivial solution. If not, we assume that $Z(\tau)$ is the solution of the Eqs. (3.21–3.22). Then $Z(\tau) \rightarrow 0$ as $\tau \rightarrow -\infty$ and then $\dot{y}(\tau)y(\tau) \rightarrow 0$. Let $t_1 \rightarrow -\infty$ in (3.25), we have

$$-y(t_2)\dot{y}(t_2) + \int_{-\infty}^{t_2} (\dot{y}^2 + fy^2)d\tau = 0. \tag{3.27}$$

Please note that $y(t_2)\dot{y}(t_2) = -\frac{Q}{4}y^2(t_2) + rx(t_2)y(t_2)$. Then for $t_2 \in \mathbb{R}^-$, $Z(t_2) \in V_d$ or V_n implies $y(t_2)\dot{y}(t_2) \leq 0$, so we get the result. \square

We continuous to compute the Maslov index on l_0^- .

Corollary 3.13. *Suppose $r > -1/8$ and $r \neq 0$, then*

$$i_+(V_n, V_d; l_0^-) = 0, \tag{3.28}$$

and

$$i_+(V_n, V_n; l_0^-) = \begin{cases} 1 & \text{if } r \in (-\frac{1}{8}, 0), \\ 0 & \text{if } r > 0. \end{cases} \tag{3.29}$$

Proof. The proof follows from Eq. (2.29). Please note that Lemma 3.10. implies that there are no nontrivial solutions which satisfy $Z(0) = e_1, e_2$ and $Z(T) \in V_n$ for some $T > 0$. There is a crossing for $i_+(V_n, V_n; l_0^-)$ at $T = 0$. An easy computation shows that the crossing form $\Gamma(V_n, V_n, 0)$ is positive for $r \in (-\frac{1}{8}, 0)$ and negative for $r > 0$, which implies the results. \square

To compute the collision index $i_+(V_d, V_d; l_0^-)$ and $i_+(V_d, V_n; l_0^-)$, we will use the Hörmander index. We observe that, in the point $(0, Q_-)$, $J\hat{B}_r(-\infty) = \begin{pmatrix} \frac{\sqrt{2}}{4} & r \\ 1 & -\frac{\sqrt{2}}{4} \end{pmatrix}$ and hence the eigenvalues $\lambda_{\pm} = \pm(\frac{1}{8} + r)^{\frac{1}{2}}$ with eigenvector $e_{\pm}^- = (\frac{\sqrt{2}}{4} \pm (\frac{1}{8} + r)^{\frac{1}{2}}, 1)^T$. The unstable subspace V_u^- is spanned by e_+^- and stable subspace V_s^- is spanned by e_-^- . Similarly, at $(0, Q_+)$, $J\hat{B}_r(+\infty) = \begin{pmatrix} -\frac{\sqrt{2}}{4} & r \\ 1 & \frac{\sqrt{2}}{4} \end{pmatrix}$, the eigenvalues $\lambda_{\pm} = \pm(\frac{1}{8} + r)^{\frac{1}{2}}$ with eigenvector $e_{\pm}^+ = (-\frac{\sqrt{2}}{4} \pm (\frac{1}{8} + r)^{\frac{1}{2}}, 1)^T$. The unstable subspace V_u^+ is spanned by e_+^+ and stable subspace V_s^+ is spanned by e_-^+ . The Hörmander index could be computed by (2.35), (2.36), or we just choose simple Lagrangian paths connected $\Lambda(0), \Lambda(1)$ and compute the difference. For reader’s convenience, we list the result below.

$$s(V_d, V_n, V_d, V_u^-) = 1, \tag{3.30}$$

$$s(V_d, V_n, V_n, V_u^-) = 0, \tag{3.31}$$

$$s(V_n, V_d, V_u^-, V_u^+) = \begin{cases} 1 & \text{if } r \in (-\frac{1}{8}, 0), \\ 0, & \text{if } r > 0, \end{cases} \tag{3.32}$$

$$s(V_d, V_u^-, V_n, V_s^-) = s(V_d, V_u^-, V_d, V_s^-) = 0. \tag{3.33}$$

Figures 3, 4, 5, 6 illustrate the Hörmander index (3.30–3.33), where y is the horizontal coordinate, x is the vertical coordinate and the anticlockwise rotation is positive rotation.

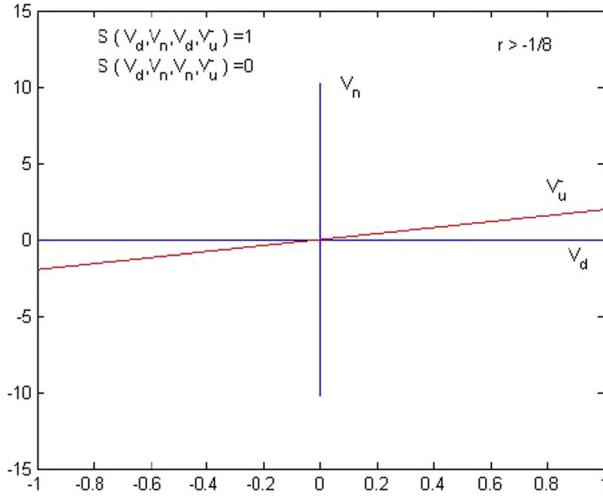


Fig. 3. $s(V_d, V_n, V_d, V_u^-)$ and $s(V_d, V_n, V_n, V_u^-)$

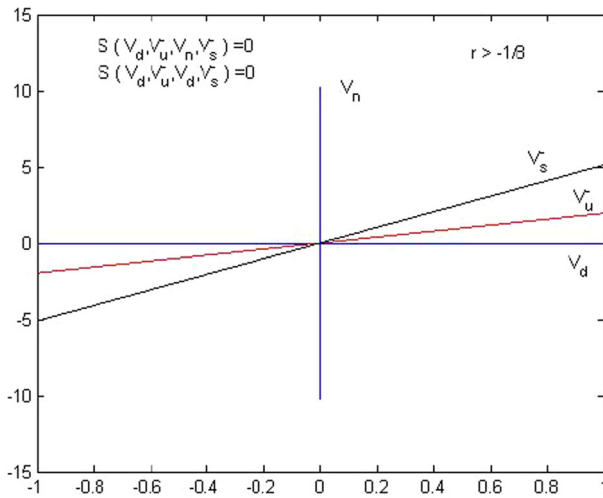


Fig. 4. $s(V_d, V_u^-, V_n, V_s^-)$ and $s(V_d, V_u^-, V_d, V_s^-)$

From Lemma 3.11., the system is nondegenerate for $r \neq 0$, so $\gamma(\tau)V_d \rightarrow V_u^-$ and $\gamma(\tau)V_n \rightarrow V_u^-$ as $\tau \rightarrow +\infty$. We have

$$i_+(V_d, V_d; l_0^-) = i(V_n, V_d; l_0^-) + s(V_d, V_n, V_d, V_u^-), \tag{3.34}$$

$$i_+(V_d, V_n; l_0^-) = i(V_n, V_n; l_0^-) + s(V_d, V_n, V_n, V_u^-). \tag{3.35}$$

Thus from Eqs. (3.30–3.31) we have

Corollary 3.14. *Suppose $r > -1/8$ and $r \neq 0$,*

$$i_+(V_d, V_d; l_0^-) = 1, \tag{3.36}$$

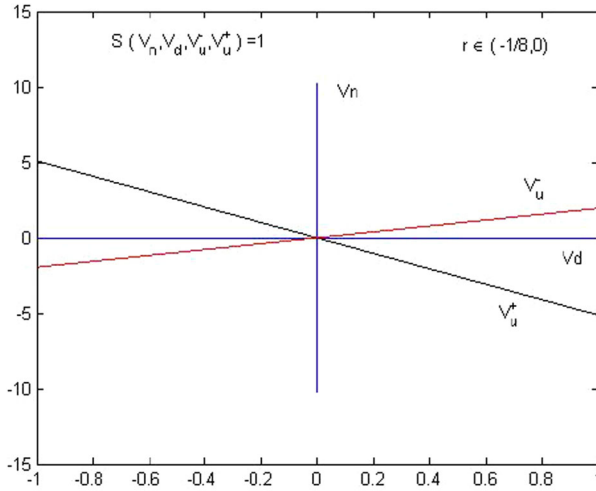


Fig. 5. $s(V_n, V_d, V_u^-, V_u^+)$ for $r \in (-\frac{1}{8}, 0)$

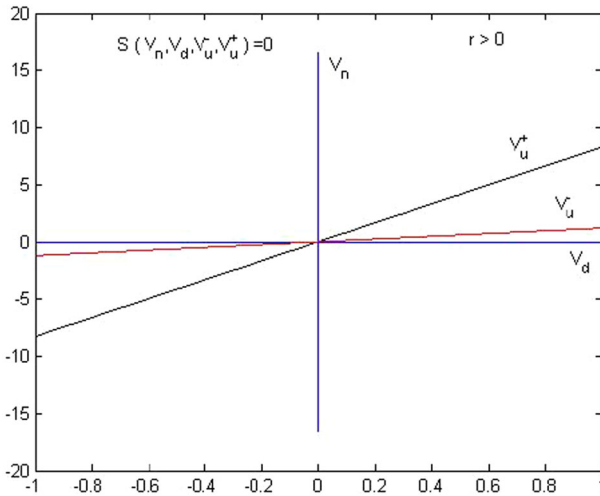


Fig. 6. $s(V_n, V_d, V_u^-, V_u^+)$ for $r > 0$

and

$$i_+(V_d, V_n; l_0^-) = \begin{cases} 1 & \text{if } r \in (-\frac{1}{8}, 0), \\ 0, & \text{if } r > 0. \end{cases} \tag{3.37}$$

We come back to the higher dimension. We denote by $\phi(R)$ the number of total negative eigenvalues of R . It is obvious that, $\phi(R) = \phi(a_0)$ which is the Morse index of the central configuration a_0 . By property V of Maslov index, we have

Corollary 3.15. *Supposing $\lambda_1(R) > -1/8$, R is nondegenerate, then we have*

$$i_-(V_d; l_0^+) = 0, \quad i_-(V_n; l_0^+) = 0, \quad (3.38)$$

$$i_+(V_d, V_d; l_0^-) = k, \quad i_+(V_d, V_n; l_0^-) = \phi(R), \quad (3.39)$$

and

$$i_+(V_n, V_d; l_0^-) = 0, \quad i_+(V_n, V_n; l_0^-) = \phi(R). \quad (3.40)$$

We then consider the linear system on l_+ , and recall that under the collision nondegenerate condition $i(V)$ is well defined.

Proposition 3.16. *Under the collision nondegenerate condition, we have*

$$i(V_n; l_+) = i(V_d; l_+) + \phi(R) \quad (3.41)$$

Proof. Please note that $i(V_n; l_+) = i(V_d; l_+) + s(V_n, V_d, V_u^-, V_u^+)$, we only need to show

$$s(V_n, V_d, V_u^-, V_u^+) = \phi(R). \quad (3.42)$$

We choose basis such that $R = \text{diag}(r_1, \dots, r_n)$ and consider the 2-dimension linear system $\dot{z} = J_2 \hat{B}_2(r)z$; (3.42) is from (3.32). \square

Now we give the proof of Theorem 1.2.

Proof. Since R is nondegenerate, $V_u, V_d \pitchfork V_u^\pm$, from (3.7),

$$\lim_{e \rightarrow 1} \mu(V_d, \hat{\gamma}_e(\tau)V_d, \tau \in [0, T]) = i_-(V_d; l_0^+) + i_+(V_d, V_d; l_0^-) + i(V_d; l_+),$$

from (3.38–3.39), we have

$$\lim_{e \rightarrow 1} \mu(V_d, \hat{\gamma}_e(\tau)V_d, \tau \in [0, T]) = k + i(V_d; l_+).$$

Then, (1.6) follows from Lemma 2.4 and the fact that $\gamma_e(2\pi) = \hat{\gamma}_e(T)$.

Similar,

$$\lim_{e \rightarrow 1} \mu(V_n, \hat{\gamma}_e(\tau)V_n, \tau \in [0, T]) = i_-(V_n; l_0^+) + i_+(V_n, V_n; l_0^-) + i(V_n; l_+), \quad (3.43)$$

(1.7) is from (3.38), (3.40–3.41) and Lemma 2.4. \square

It is worth noticing that in several cases we can't analytically compute the collision index and for this reason, we now introduce a numerical method which can be useful in this situation.

We first consider the case for \mathbb{R}^+ . We choose $V \in \text{Lag}(2n)$, such that $\hat{B}(\tau)|_V > 0$ for $\tau \in \mathbb{R}^\pm$. Then the crossing form $\Gamma(\Lambda(\tau), V, \tau) > 0$ and we have

$$\mu(V, \gamma(\tau)V_0) = \sum_{0 < \tau_j < \infty} \nu(\tau_j),$$

where $\nu(\tau_j) = \dim V \cap \gamma(\tau_j)V_0$. For the Lagrangian system, we can always choose $V_d = V$ and then $\hat{B}(t)|_{V_d} = I_n > 0$. We can get the Maslov index from the Hörmander index, in fact

$$\mu(V_1, \gamma(\tau)V_0, \tau \in [0, T]) = \mu(V_d, \gamma(\tau)V_0, [0, T]) + s(V_1, V_d, V_0, \gamma(T)V_0).$$

Under the nondegenerate conditions, $\lim_{T \rightarrow \infty} \gamma(T)V_0 = V_u^+$, then we have

$$\mu(V_1, \gamma(\tau)V_0, \tau \in \mathbb{R}^+) = \mu(V_d, \gamma(\tau)V_0, \tau \in \mathbb{R}^+) + s(V_1, V_d, V_0, V_u^+).$$

The cases of \mathbb{R}^- and \mathbb{R} are similar, so we just make $-T$ be the starting time, where $T > 0$ is large enough.

Remark 3.17. For computing the Maslov index $\mu(V_d, \gamma(\tau)V_0)$, we start by choosing a basis $\{\xi_1(0), \dots, \xi_n(0)\}$ of V_0 and, by using a numerical integrator, we get $\xi_k(t) := \gamma(t)\xi_k(0)$. For $j = 1, \dots, n$, let e_j the basis of V_d , $t \mapsto M(t)$ be the path of $2n \times 2n$ matrices defined by $M(t) := (e_1, \dots, e_n, \xi_1(t), \dots, \xi_n(t))$ and we set $f(t) = \det(M(t))$. Then $\mu(V_d, \gamma(t)V_0)$ is equal to the total number of zeros of $f(t)$. Since $t \mapsto |f(t)|$ is exponentially increasing, this method works well for not so large time. Instead, we use the robust numerical algorithm based on exterior algebra representation. We refer the interested reader to the following papers [7–9] in which the authors compute the Maslov index for homoclinic orbits (cfr. Sect. 6).

4. Collision Index for Brake Symmetry Central Configurations

In the next, we will consider the case that the central configuration with brake symmetry. We start with the following Definition.

Definition 4.1. The central configuration with normalized Hessian R is said with brake symmetry if there exists a $k \times k$ symmetry matrix N which satisfies $N^2 = I_k, N\mathbb{J} = -\mathbb{J}N, RN = NR$.

To our knowledge, the Lagrangian configuration [17], Euler central configuration [27] and the $1 + n$ central configurations have the brake symmetry property [34,40]. It could be interesting to find an example without having this symmetry property.

In the brake symmetry case, let $\hat{N} = \text{diag}(N, -N)$, and denote

$$g : x(\tau) \rightarrow \hat{N}x(T - \tau). \tag{4.1}$$

Obviously, $g^2 = id$ and $g \cdot -J \frac{d}{d\tau} = -J \frac{d}{d\tau} \cdot g$. From the fact of $q(\tau) = q(T - \tau), Q(\tau) = -Q(T - \tau)$, easy computations show that $\hat{N}\hat{B}(T - \tau) = \hat{B}(\tau)\hat{N}$, and consequently

$$g\hat{B} = \hat{B}g. \tag{4.2}$$

Let $E^\pm = \ker(g \mp I)$, then

$$\ker(-J \frac{d}{d\tau} - \hat{B}) = \ker((-J \frac{d}{d\tau} - \hat{B})|_{E^+}) \oplus \ker((-J \frac{d}{d\tau} - \hat{B})|_{E^-}). \tag{4.3}$$

Moreover, by the generalized Bott-type iteration formula for Maslov index [18, (Th 1.1)] or [26], we have

$$i_1(\hat{\gamma}) + k = \mu(V^+(\hat{N}), \hat{\gamma}(\tau)V^+(\hat{N}), \tau \in [0, T/2]) + \mu(V^-(\hat{N}), \hat{\gamma}(\tau)V^-(\hat{N}), \tau \in [0, T/2]), \tag{4.4}$$

$$i_{-1}(\hat{\gamma}) = \mu(V^+(\hat{N}), \hat{\gamma}(\tau)V^-(\hat{N}), \tau \in [0, T/2]) + \mu(V^-(\hat{N}), \hat{\gamma}(\tau)V^+(\hat{N}), \tau \in [0, T/2]), \tag{4.5}$$

where $V^\pm(\hat{N}) = \ker(\hat{N} \mp I_{2n})$, $k = 2n - 4$. Similarly, we can decompose of Dirichlet and Neumann boundary condition as follows

$$\mu(V_d, \hat{\gamma}(\tau)V_d, \tau \in [0, T]) = \mu(V^+(\hat{N}), \hat{\gamma}(\tau)V_d, \tau \in [0, T/2]) + \mu(V^-(\hat{N}), \hat{\gamma}(\tau)V_d, \tau \in [0, T/2]), \tag{4.6}$$

$$\mu(V_n, \hat{\gamma}(\tau)V_n, \tau \in [0, T]) = \mu(V^+(\hat{N}), \hat{\gamma}(\tau)V_n, \tau \in [0, T/2]) + \mu(V^-(\hat{N}), \hat{\gamma}(\tau)V_n, \tau \in [0, T/2]). \tag{4.7}$$

For the iteration formula of brake orbits we refer the interested reader to [23]. Then we consider the collision orbit, and let \mathcal{K} be the space of bounded solution of $\dot{z} = J\hat{B}(\tau)z$ on I_+ , and \mathcal{K}^\pm be the space of bounded solution on I_+ which satisfies $z(0) \in V^\pm(\hat{N})$. Similar to (4.3), we have

Lemma 4.2. *For the brake symmetry central configurations, on I_+ , we have*

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-. \tag{4.8}$$

Proof. Please note that on I_+ , $\hat{N}\hat{B}(-\tau) = \hat{B}(\tau)\hat{N}$, if $z(\tau)$ is one solution then $\hat{N}z(-\tau)$ is another solution. Let $z_\pm(\tau) = \frac{1}{2}(z(\tau) \pm \hat{N}z(-\tau))$, then $z_\pm \in \mathcal{K}_\mp$, which implies the result. \square

Obviously, there is standard brake symmetry on l_0 , that is $\hat{N}_0 = \text{diag}(I_k, -I_k)$, then $V^+(\hat{N}_0) = V_d$ and $V^-(\hat{N}_0) = V_n$. Let \mathcal{K}_0 be the space of bounded solution of $\dot{z} = J\hat{B}(\tau)z$ on l_0 , and \mathcal{K}_0^\pm be the space of bounded solution on l_0^- which satisfies $z(0) \in V^\pm(\hat{N}_0)$. We have on l_0 ,

$$\mathcal{K}_0 = \mathcal{K}_+^0 \oplus \mathcal{K}_-^0. \tag{4.9}$$

On l_0 , the nondegenerate condition is clear. Now, from (4.9) and Lemma 3.11, we have

Proposition 4.3. *The system on l_0 is nondegenerate if and only if R is nondegenerate.*

Since R satisfies brake symmetry, let $V^\pm = \ker(N \mp I_k)$, so it is obvious $\dim V^\pm = \frac{k}{2}$. Let $\hat{V}^\pm = JV^\pm \oplus V^\pm$ be symplectic subspace thus $\mathbb{R}^{2k} = \hat{V}^+ \oplus \hat{V}^-$. Set $R^\pm = R|_{V^\pm}$, and denote $\hat{B}_\pm = \begin{pmatrix} I_{\frac{k}{2}} & \frac{Q}{4}I_{\frac{k}{2}} \\ \frac{Q}{4}I_{\frac{k}{2}} & -R^\pm \end{pmatrix}$, we get $\hat{B} = \hat{B}_+ \diamond \hat{B}_-$, and the fundamental solution satisfies $\Psi_0 = \Psi_0|_{\hat{V}^+} \diamond \Psi_0|_{\hat{V}^-}$. Furthermore we have

$$i(V, W; l_0^-) = i(V|_{V_+}, W|_{V_+}; l_0^-) + i(V|_{V_-}, W|_{V_-}; l_0^-), \tag{4.10}$$

for $V, W = V^\pm(\hat{N})$ or V_d, V_n . Please note that

$$V^+(\hat{N})|_{\hat{V}^+} = V_d, \quad V^+(\hat{N})|_{\hat{V}^-} = V_n, \quad V^-(\hat{N})|_{\hat{V}^+} = V_n, \quad V^-(\hat{N})|_{\hat{V}^-} = V_d. \tag{4.11}$$

From (3.39)–(3.40), we have

$$i_+(V^+(\hat{N}), V_d; l_0^-) + i_+(V^-(\hat{N}), V_d; l_0^-) = k, \tag{4.12}$$

$$i_+(V^+(\hat{N}), V_n; l_0^-) + i_+(V^-(\hat{N}), V_n; l_0^-) = 2\phi(R), \tag{4.13}$$

$$i_+(V^+(\hat{N}), V^+(\hat{N}); l_0^-) + i_+(V^-(\hat{N}), V^-(\hat{N}); l_0^-) = k + \phi(R). \tag{4.14}$$

$$i_+(V^-(\hat{N}), V^+(\hat{N}); l_0^-) + i_+(V^+(\hat{N}), V^-(\hat{N}); l_0^-) = \phi(R). \tag{4.15}$$

For the brake symmetry central configurations, we get the following approximation theorem.

Theorem 4.4. *Let $\lambda_1(R) > -\frac{1}{8}$ be nondegenerate with brake symmetry property, and satisfying the collision nondegenerate conditions. Thus we have*

$$\lim_{e \rightarrow 1} \mu(V_d, \hat{\gamma}_e(\tau)V_d, \tau \in [0, T]) = k + i_-(V^-(\hat{N}); l_+^-) + i_-(V^+(\hat{N}); l_+^-), \tag{4.16}$$

$$\lim_{e \rightarrow 1} \mu(V_n, \hat{\gamma}_e(\tau)V_n, \tau \in [0, T]) = 2\phi(R) + i_-(V^-(\hat{N}); l_+^-) + i_-(V^+(\hat{N}); l_+^-), \tag{4.17}$$

$$\lim_{e \rightarrow 1} i_{-1}(\hat{\gamma}_e) = \lim_{e \rightarrow 1} i_1(\hat{\gamma}_e) = \phi(R) + i_-(V^-(\hat{N}); l_+^-) + i_-(V^+(\hat{N}); l_+^-). \tag{4.18}$$

Proof. Since R is collision nondegenerate, from (4.8), the system is nondegenerate with respect to $V^\pm(\hat{N})$ on l_+^- , and also from the condition that R is nondegenerate, then the system is nondegenerate with respect to $V^\pm(\hat{N}), V_n, V_d$ on l_0^- . From (i) of Theorem 3.3, we have

$$\lim_{e \rightarrow 1} \mu(V, \hat{\gamma}_e(\tau)W, \tau \in [0, T/2]) = i(V, W; l_0^-) + i(V; l_+^-),$$

for V, W is $V^\pm(\hat{N}), V_n, V_d$. From (4.4–4.7), we have

$$\begin{aligned} \lim_{e \rightarrow 1} \mu(V_d, \hat{\gamma}_e(\tau)V_d, \tau \in [0, T]) &= i_+(V^+(\hat{N}), V_d; l_0^-) + i_+(V^-(\hat{N}), V_d; l_0^-) \\ &\quad + i_-(V^+(\hat{N}); l_+^-) + i_-(V^-(\hat{N}); l_+^-), \end{aligned}$$

$$\begin{aligned} \lim_{e \rightarrow 1} \mu(V_n, \hat{\gamma}_e(\tau)V_n, \tau \in [0, T]) &= i_+(V^+(\hat{N}), V_n; l_0^-) + i_+(V^-(\hat{N}), V_n; l_0^-) \\ &\quad + i_-(V^+(\hat{N}); l_+^-) + i_-(V^-(\hat{N}); l_+^-), \end{aligned}$$

$$\begin{aligned} \lim_{e \rightarrow 1} i_1(\hat{\gamma}_e) &= i_+(V^+(\hat{N}), V^+(\hat{N}); l_0^-) + i_-(V^+(\hat{N}); l_+^-) \\ &\quad + i_+(V^-(\hat{N}), V^-(\hat{N}); l_0^-) + i_-(V^-(\hat{N}); l_+^-) - k, \end{aligned}$$

$$\begin{aligned} \lim_{e \rightarrow 1} i_{-1}(\hat{\gamma}_e) &= i_+(V^-(\hat{N}), V^+(\hat{N}); l_0^-) + i_-(V^+(\hat{N}); l_+^-) \\ &\quad + i_+(V^+(\hat{N}), V^-(\hat{N}); l_0^-) + i_-(V^-(\hat{N}); l_+^-). \end{aligned}$$

Then (4.16–4.18) is from (4.4–4.7) and (4.12–4.15). \square

Comparing of (1.6) in Theorem 1.2 and (4.16), we have

Corollary 4.5. *Let $\lambda_1(R) > -1/8$. If R is nondegenerate, has the brake symmetry property and if the collision nondegenerate condition is fulfilled, then we have*

$$i_-(V^-(\hat{N}); l_+^-) + i_-(V^+(\hat{N}); l_+^-) = i(V_d; l_+). \tag{4.19}$$

It is clear that $i(V_d; l_+) \geq 0$. We shall now prove that $i_-(V^\pm(\hat{N}); l_+^-)$ is also nonnegative. We consider the Maslov index on \mathbb{R}^- , supposing the system is nondegenerate with respect to V_1 , and $V_1 \pitchfork V_u^-$, then for $-\tau_0$ large enough,

$$\mu(V_1, V_u(\tau), \tau \in (-\infty, 0]) = \mu(V_1, \gamma(\tau, \tau_0)V_u(\tau_0), \tau \in [\tau_0, 0]).$$

From the property (III), (IV), for $\gamma(-s, 0), s \in [0, \infty)$

$$\mu(V_1, \gamma(\tau, \tau_0)V_u(\tau_0), \tau \in (\tau_0, 0)) = \mu(\gamma(-s, 0)V_1, V_u^-, s \in [0, \infty)),$$

we have

$$\mu(V_1, V_u(\tau), \tau \in (-\infty, 0]) = -\mu(V_u^-, \gamma(-\tau)V_1, \tau \in [0, \infty)).$$

By the nondegenerate condition, we have $\lim_{T \rightarrow \infty} \gamma(-T)V_1 = V_s$, then we have

$$\mu(V_1, V_u(\tau), \tau \in (-\infty, 0]) = s(V_d, V_u^-, V_1, V_s^-) - \mu(V_d, \gamma(-\tau)V_1, \tau \in [0, +\infty)).$$

In the case of ERE, an easy computation shows that

$$\frac{d}{d\tau} \Psi_+(-\tau) = -J\hat{B}(-\tau)\Psi_+(-\tau), \tag{4.20}$$

where $\Psi_+(\tau) = \Psi_+(\tau, 0)$ is the fundamental solution on l_+ . If the central configuration satisfies the brake symmetry, that is $\hat{N}\hat{B}(-\tau) = \hat{B}(\tau)\hat{N}$, then, direct computation shows that

$$\Psi_+(-\tau) = \hat{N}\Psi_+(\tau)\hat{N}, \tau \in [0, \infty).$$

So we have

$$\mu(V_d, \Psi_+(-\tau)V_1) = \mu(V_d, \hat{N}\Psi_+(\tau)\hat{N}V_1, \tau \in [0, \infty)).$$

Please note that if $-\hat{B}(\tau)|_V < 0$ for $t \in \mathbb{R}^+$, then the crossing form $\Gamma(\Lambda(t), V_d, t) < 0$, we have

$$\mu(V_d, \Psi_+(-\tau)V_1) = - \sum_{0 < \tau_j < \infty} v(\tau_j) \leq 0,$$

where $v(\tau_j) = \dim V_d \cap \Psi_+(-\tau_j)V_1$.

Please note that, in the case $V_1 = V_d^j \oplus V_n^{(k-j)}$, where $V_d^j \in V_d, V_n^{(k-j)} \in V_n$, from (3.33), we have

$$s(V_d, V_u^-, V_1, V_s^-) = 0.$$

Since $V^\pm(\hat{N})$ is a direct sum of Dirichlet Lagrangian subspace and Neumann Lagrangian subspace, by (4.11), we have

Lemma 4.6. *On l_+^- , we have*

$$i_-(V^\pm(\hat{N})) = \sum_{0 < \tau_j^\pm < \infty} v(\tau_j^\pm) \geq 0, \tag{4.21}$$

where $v(\tau_j^\pm) = \dim V_d \cap \Psi_+(-\tau_j^\pm)V^\pm(\hat{N})$.

5. Applications

We give applications for the collision index. In Sect. 5.1 we consider the ERE of minimal central configurations and prove some hyperbolicity results. At Sect. 5.2, we study the stability of Euler orbits.

5.1. *Minimal central configurations.* In order to give a hyperbolic criteria, we first review some results on Morse index. Consider the linear Sturm systems

$$-\frac{d}{dt}(P(t)\dot{y} + Q(t)y) + Q^T(t)\dot{y} + R(t)y = 0, \tag{5.1}$$

as P, R, Q are continuous path of matrices in \mathbb{R}^{2n} and satisfy $P(t) > 0, R(t) = R(t)^T$. This linear Sturm system (5.1) corresponds to the linear Hamiltonian system

$$\dot{z} = JB(t)z, z \in \mathbb{R}^{2n}, \tag{5.2}$$

where

$$B(t) = \begin{pmatrix} P^{-1}(t) & -P^{-1}(t)Q(t) \\ -Q^T P^{-1}(t) & Q^T(t)P^{-1}(t)Q(t) - R(t) \end{pmatrix}. \tag{5.3}$$

Let

$$L(t, x(t), \dot{x}(t)) = \frac{1}{2}((P\dot{x} + Qx) \cdot \dot{x} + Q^T \dot{x} \cdot x + Rx \cdot x), \tag{5.4}$$

and $\mathcal{F}(x) = \int_0^T \{L(t, x(t), \dot{x}(t))\}dt$ on $W^{1,2}([0, T], \mathbb{C}^n)$. We denote

$$D(\omega, T) = \{x \in W^{1,2}([0, T], \mathbb{C}^n), x(0) = \omega x(T)\}, \omega \in \mathbb{U}.$$

Obviously,

$$W_0^{1,2}([0, T], \mathbb{R}^n) \subset D(\omega, T) \subset W^{1,2}([0, T], \mathbb{C}^n).$$

Let $\mathcal{L} = \mathcal{F}''(0) := -\frac{d}{dt}(P(t)\frac{d}{dt} + Q(t)) + Q^T(t)\frac{d}{dt} + R(t)$, and more precisely, set $\mathcal{L}_n, \mathcal{L}_\omega, \mathcal{L}_d$ to be the operator with form \mathcal{L} under the Neumann, ω and Dirichlet boundary conditions separately. Let $\lambda_k(\mathcal{L})$ be the k -th eigenvalue of \mathcal{L} . From the monotonicity property of the eigenvalues [12], we have

$$\lambda_k(\mathcal{L}_n) \leq \lambda_k(\mathcal{L}_\omega) \leq \lambda_k(\mathcal{L}_d). \tag{5.5}$$

Let ϕ be the Morse index of \mathcal{L} which is defined to be the total number of negative eigenvalues, which is equal to the dimension of maximum negative definite subspace of \mathcal{F} . Let $\phi_d, \phi_\omega, \phi_n$ be the Morse index of $\mathcal{L}_d, \mathcal{L}_\omega, \mathcal{L}_n$ separately. From (5.5), we have

$$\phi_d \leq \phi_\omega \leq \phi_n.$$

Proposition 5.1. *The system is hyperbolic if $\phi_n = \phi_d$ and \mathcal{L}_n is nondegenerate.*

Proof. Please note that the system is hyperbolic, that is, $\sigma(\gamma(T)) \cap \mathbb{U} = \emptyset$ is equivalent to \mathcal{L}_ω which is nondegenerate for $\forall \omega \in \mathbb{U}$. Supposing $k_0 = \phi_n = \phi_d$, we have $\lambda_{k_0}(\mathcal{L}_\omega) < 0$ by (5.5). On the other hand, \mathcal{L}_n is nondegenerate which implies $\lambda_{k_0+1}(\mathcal{L}_n) > 0$ and hence $\lambda_{k_0+1}(\mathcal{L}_\omega) > 0$, which implies the result. \square

Please note that $\phi_n = 0$ implies $\phi_d = 0$, so we have

Corollary 5.2. *The system is hyperbolic if $\mathcal{L}_n > 0$.*

From Theorem 1.2 of [18] or P172 of [24], we list the relation of Morse index and Maslov index below.

Lemma 5.3. *Let γ be the fundamental solution of (5.2), then we have*

$$\phi_\omega(\mathcal{L}) = i_\omega(\gamma), \quad v_\omega(\mathcal{L}) = v_\omega(\gamma), \quad \forall \omega \in \mathbb{U}, \tag{5.6}$$

$$\phi_d(\mathcal{L}) + n = \mu(V_d, \gamma V_d), \quad \phi_n(\mathcal{L}) = \mu(V_n, \gamma V_n). \tag{5.7}$$

Proof of Theorem 1.3. Please note that the central configuration a_0 is non degenerate minimizer which implies $\lambda_1(R) > 0$, i.e. $\phi(R) = 0$ and R is nonsingular. Under the collision nondegenerate condition, from (1.6)–(1.7), for $1 - e$ small enough

$$\mu(V_d, \gamma_e(t)V_d, t \in [0, 2\pi]) - k = \lim_{e \rightarrow 1} \mu(V_n, \gamma_e(t)V_n, t \in [0, 2\pi]). \tag{5.8}$$

From (5.7), we have $\phi_d = \phi_n$. The nondegenerate of \mathcal{L}_n is from Theorem 1.2, so the result is from Proposition 5.1. \square

A typical example is the Lagrangian equilateral triangle central configuration. It is obvious that $R = \text{diag}((3 + \sqrt{9 - \beta})/2, (3 - \sqrt{9 - \beta})/2)$ satisfies the brake symmetry with $N = \text{diag}(1, -1)$. This fact had been used to decompose the -1 -degenerate curves in [17]. It is proved in [17] that for any $\beta \in (0, 9]$, $1 - e$ small enough, \mathcal{L}_n is positive, and consequently hyperbolic. By the approximation formula (4.17) and the nonnegative property (4.21), we have

Proposition 5.4. *If the Lagrangian central configurations is collision nondegenerate, then $i_-(V^-(\hat{N}); l_+) = i_-(V^+(\hat{N}); l_+) = 0$, and hence $i(V_d; l_+) = 0$.*

We continue by studying the case of strong minimizer, so please note that a central configuration is strong minimizer if it satisfies $\lambda_1(R) > 1$. The next lemma is important in the proof of Theorem 1.4.

Lemma 5.5 (see [20, Proposition 2]). *If $\delta > 1, \omega \in \mathbb{U}$, then $\mathcal{A}(e, \delta) = -\frac{d^2}{dt^2} - 1 + \frac{\delta}{1+e \cos(t)}$ is positive operator for all $e \in [0, 1)$ on its domain $\bar{D}_1(\omega, 2\pi)$, where $\bar{D}_n(\omega, 2\pi) = \{y \in W^{2,2}([0, 2\pi], \mathbb{C}^n) \mid y(2\pi) = \omega y(0), \dot{y}(2\pi) = \omega \dot{y}(0)\}$.*

Now we can proof Theorem 1.4.

Proof. For the ERE, we have

$$\mathcal{L} = -\frac{d^2}{dt^2} I_k - 2\mathbb{J}_{k/2} \frac{d}{dt} + \frac{R}{1 + e \cos(t)}.$$

then $\mathcal{L} > \hat{\mathcal{L}} := -\frac{d^2}{dt^2} I_k - 2\mathbb{J}_{k/2} \frac{d}{dt} + \frac{\lambda_1(R)I_k}{1+e \cos(t)}$. We only need to show $\hat{\mathcal{L}} > 0$ with domain $\bar{D}_k(\omega, 2\pi)$ for any $\omega \in \mathbb{U}$. Let $\mathcal{R}(t) = \begin{pmatrix} \cos(t)I_k & -\sin(t)I_k \\ \sin(t)I_k & \cos(t)I_k \end{pmatrix}$, then

$$\mathcal{R}\hat{\mathcal{L}}\mathcal{R}^T = -\frac{d^2}{dt^2} I_k - I_k + \frac{\lambda_1(R)I_k}{1 + e \cos(t)}.$$

Since $\lambda_1(R) > 1$, we get the result from Lemma 5.5. \square

It is obvious that for the strong minimizer if it is collision nondegenerate, the approximation theorem implies $i(V_d, l_+) = 0$.

In the special case, the ERE of Lagrangian central configurations is hyperbolic for $\beta > 8, e \in [0, 1)$, which had proved directly in [36]. As another example, we consider the 1 + 3 central configurations, and let $m_1 = m_2 = m_3 = 1$ and $m_0 = m_c$, the essential part $R = I_4 + \mathcal{D}$ with

$$\mathcal{D} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{3\sqrt{3u(3+m_c)}}{2(1+\sqrt{3m_c})} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{3\sqrt{3m_c(3+m_c)}}{2(1+\sqrt{3m_c})} \\ -\frac{3\sqrt{3m_c(3+m_c)}}{2(1+\sqrt{3m_c})} & 0 & \frac{\sqrt{3}(3+m_c)}{2(1+\sqrt{3m_c})} & 0 \\ 0 & \frac{3\sqrt{3m_c(3+m_c)}}{2(1+\sqrt{3m_c})} & 0 & \frac{\sqrt{3}(3+m_c)}{2(1+\sqrt{3m_c})} \end{pmatrix}, \tag{5.9}$$

was computed in [31]. Please note that there is a typo in (39) of [31]. Let

$$\mathcal{D}_{\mp} = \begin{pmatrix} \frac{1}{2} & \mp \frac{3\sqrt{3m_c(3+m_c)}}{2(1+\sqrt{3m_c})} \\ \mp \frac{3\sqrt{3m_c(3+m_c)}}{2(1+\sqrt{3m_c})} & \frac{\sqrt{3}(3+m_c)}{2(1+\sqrt{3m_c})} \end{pmatrix},$$

then $\mathcal{D} = \mathcal{D}_- \diamond \mathcal{D}_+$. Obviously the eigenvalues λ_{\pm} of \mathcal{D}_{\mp} is same, direct computation shows that

$$\lambda_{\pm}(m_c) = \frac{1}{2}(1 + \sqrt{3m_c})^{-1} \left[\sqrt{3m_c} + \frac{3\sqrt{3} + 1}{2} \pm \left(27(m_c^2 + 3m_c) + \left(\frac{3\sqrt{3} - 1}{2} \right)^2 \right)^{\frac{1}{2}} \right]. \tag{5.10}$$

Obviously $\lambda_+(m_c) > 0$ for $m_c \in [0, +\infty)$. Let $m_c(0) = \frac{\sqrt{3}}{24}, m_c(-1) = \frac{81+64\sqrt{3}}{249}$, then $\lambda_-(m_c(0)) = 0$ and $\lambda_-(m_c(-1)) = -1$, moreover

$$\begin{cases} \lambda_-(m_c) > 0 & \text{if } m_c \in [0, m_c(0)), \\ -1 < \lambda_-(m_c) < 0 & \text{if } m_c \in (m_c(0), m_c(-1)), \\ -\frac{9}{8} < \lambda_-(m_c) < -1, & \text{if } m_c \in (m_c(-1), +\infty). \end{cases} \tag{5.11}$$

Since $\lambda_1(R) = 1 + \lambda_-(m_c)$, it is obvious that Theorems 1.3 and 1.4 imply Corollary 1.5.

Remark 5.6. Inspired by the recent results obtained in [17], we conjecture that the non-degenerate minimal central configuration is collision nondegenerate and satisfies

$$i(V_d; l_+) = 0. \tag{5.12}$$

In a private communication with the first named author, Prof. Y. Long posed the following conjecture.

A smooth T -periodic non-collision solution of the planar N -body problem, with $N > 3$, is a smooth global minimizer of the action functional on the space of all T -periodic orbits having non-trivial winding number if and only if it is an elliptic motion corresponding to the global minimal central configuration of the potential restricted to the inertia ellipsoid.

We observe that, Long’s conjecture implies that, for any $e \in [0, 1)$ the Morse index for the ERE of any non-degenerate minimal central configuration is 0. In the case of brake symmetric central configurations, Long’s conjecture, imply our conjecture in the collision nondegenerate case. This conjecture is still open, and in the case $e = 0$, an interesting result from Chenciner and Desolneux shows that the minima of the action on the zero mean loop space, is the relative equilibrium corresponding to a minimal central configuration. [11].

5.2. *Stability analysis of Euler orbits.* The Euler orbits have been studied in [27,28], in this case, $R = \text{diag}(-\delta, 2\delta + 3)$, where $\delta \in [0, 7]$ only depends on mass m_1, m_2, m_3 . Please refer to Appendix A of [28] for the details. Although there is no physical meaning for $\delta > 7$, we will assume $\delta \geq 0$ to make the mathematical theory complete.

We will use the index theory to study the stability problem. Let $\gamma_{\delta,e}$ be the fundamental solutions of $\mathcal{B}(t)$ which is given by (2.2), that is $\dot{\gamma}_{\delta,e} = J_2 \mathcal{B}(t) \gamma_{\delta,e}, t \in [0, 2\pi], \gamma_{\delta,e}(0) = I_4$. For $\delta = 0$, the system degenerates to the Kepler problem, and it has been studied in [19], which proved that $\gamma_{0,e}(2\pi)$ with normal form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \diamond I_2$ and

$$i_\omega(\gamma_{0,e}) = \begin{cases} 0, & \text{if } \omega = 1, \\ 2, & \text{if } \omega \in \mathbb{U} \setminus \{1\}. \end{cases} \tag{5.13}$$

The Maslov-type index in the case $\delta > 0$ has been studied by Long and Zhou [27], then we review firstly their results. For any $j \in \mathbb{N}$, there exists 1-degenerate curves $\Gamma_j = Gr(\varphi_j(e))$, and we also let $\Gamma_0 = Gr(\varphi_0(e))$ with $\varphi_0(e) = 0$. Then $\gamma_{\delta,e}$ only degenerates at $\cup_{j=1}^\infty \Gamma_j$ and $\dim \ker(\gamma_{\delta,e}(2\pi) - I_4) = 2$ if $(\delta, e) \in \cup_{j=1}^\infty \Gamma_j$. The Maslov-type index satisfies

$$i_1(\gamma_{\delta,e}) = 2j + 3, \quad \text{if } \varphi_j(e) < \delta \leq \varphi_{j+1}(e), \quad j \in \mathbb{N} \cup \{0\}. \tag{5.14}$$

Similarly, for $\forall j \in \mathbb{N}$, there exists pair -1 -degenerate curves $\Upsilon_j^\pm = Gr(\psi_j^\pm(e))$.

Let $\psi_j^s(e) = \min\{\psi_j^+(e), \psi_j^-(e)\}$ and $\psi_j^l(e) = \max\{\psi_j^+(e), \psi_j^-(e)\}$. Moreover, we set $\psi_0^l = \psi_0^s = 0$, then for $k \in \mathbb{N}$ we have

$$i_{-1}(\gamma_{\delta,e}) = \begin{cases} 2j, & \text{if } \delta \in (\psi_{j-1}^l, \psi_j^s], \\ 2j + 1, & \text{if } \delta \in (\psi_j^s, \psi_j^l]. \end{cases} \tag{5.15}$$

Direct computation shows that $\psi_j^+(0) = \psi_j^-(0)$, but it is not clear if, for $e > 0$, there exist other intersection points. There is a monotonicity property for Maslov-type index, that is for $\omega \in \mathbb{U}$

$$i_\omega(\gamma_{\delta_1,e}) \leq i_\omega(\gamma_{\delta_2,e}), \quad \text{if } \delta_1 \leq \delta_2. \tag{5.16}$$

The ± 1 degenerate curves satisfies

$$0 < \psi_1^s \leq \psi_1^l < \varphi_1 < \psi_2^s \leq \psi_2^l < \dots < \psi_j^s \leq \psi_j^l < \varphi_j < \psi_j^s \leq \psi_j^l < \dots. \tag{5.17}$$

Moreover for the region between the ± 1 -degenerate curves, $\gamma_{\delta,e}(2\pi)$ is elliptic-hyperbolic and for the region between the pairs of -1 -degenerate curves $\gamma_{\delta,e}(2\pi)$ is hyperbolic.

As a continuous work of Long and Zhou [27], we use the near-collision index to study the limit case. From Theorem 1.1, we have

Theorem 5.7. *If $\delta \in (1/8, 7]$, let $\varepsilon = \frac{1}{2} \min\{\frac{\delta}{2\delta+5}, 1/8\}$, $\hat{e} = \frac{1-e^2}{2}$. For $\hat{e} < \varepsilon^3$, we have*

$$i_1(\gamma_e) \geq \frac{2}{\pi} \left(\delta - \frac{1}{8}\right)^{\frac{1}{2}} \ln\left(\frac{\varepsilon^2}{\sqrt{\hat{e}}}\right) - 6. \tag{5.18}$$

Hence

$$\sup\{\overline{\lim}_{e \rightarrow 1} \varphi_j(e), \overline{\lim}_{e \rightarrow 1} \psi_j^\pm(e), j \in \mathbb{N}\} \leq 1/8. \tag{5.19}$$

Proof. Since $R = \text{diag}(-\delta, 2\delta + 3)$, then for $\delta > 1/8$, $\lambda_1(R) < -1/8$. Now Eq. (5.18) follows from (1.5) of Theorem 1.1. To prove (5.19), let $\hat{\delta}_j = \overline{\lim}_{e \rightarrow 1} \varphi_j(e)$, then there exists $e_l \rightarrow 1$ such that $\lim_{l \rightarrow \infty} \varphi_j(e_l) = \hat{\delta}_j$. If $\hat{\delta}_j > 1/8$, then we choose $\delta_\varepsilon \in (1/8, \hat{\delta}_j)$, for l large enough, $\delta_\varepsilon < \varphi_j(e_l)$, by the monotone property (5.16), so we have $i_1(\gamma_{\delta_\varepsilon, e_l}) \leq 2j + 1$, which contradict to (5.18). The proof for $\overline{\lim}_{e \rightarrow 1} \psi_j^\pm(e) \leq 1/8$ is similar. \square

Let $N = \text{diag}(1, -1)$, then $NR = RN$, we will compute the collision index for $\delta \in (0, 1/8)$ by the decomposition property. By the brake symmetry, from (4.3) we have

$$\begin{aligned} \dim \ker(\gamma_{\delta,e}(2\pi) + 1) &= \dim(V^-(\hat{N}) \cap \gamma_{\delta,e}(2\pi)V^+(\hat{N})) \\ &\quad + \dim(V^+(\hat{N}) \cap \gamma_{\delta,e}(2\pi)V^-(\hat{N})). \end{aligned}$$

We always set ψ_k^+ to be the degenerate curve in the sense that $V^+(\hat{N}) \cap \gamma_{\delta,e}(2\pi)V^-(\hat{N})$ nontrivial and similarly ψ_k^- to be the degenerate curve in the sense that $V^-(\hat{N}) \cap \gamma_{\delta,e}(2\pi)V^+(\hat{N})$ nontrivial.

We get the collision index on l_+^- numerically from (4.21) and the step in Remark 3.17. With the help of matlab, we have

Numerical result A: For the Euler orbits is collision nondegenerate for $\delta \in (0, \frac{1}{8})$, and on l_+^-

$$i_-(V^+(\hat{N}); l_+^-) = i_-(V^-(\hat{N}); l_+^-) = 1. \tag{5.20}$$

It is obvious that $\phi(R) = 1$ for $\delta \in (0, \frac{1}{8})$, then from (4.18), we have

Corollary 5.8. *For $\delta \in (0, \frac{1}{8})$, under the condition of numerical result A, we have*

$$\lim_{e \rightarrow 1} i_1(\gamma) = \lim_{e \rightarrow 1} i_{-1}(\gamma) = 3. \tag{5.21}$$

From (3.39–3.40), easy computation shows that for $\delta \in (0, \frac{1}{8})$,

$$i(V^+(\hat{N}), V^-(\hat{N}); l_0^-) = 1, \quad i(V^-(\hat{N}), V^+(\hat{N}); l_0^+) = 0.$$

So for $1 - e$ small enough,

$$\mu(V^+(\hat{N}), \gamma_{\delta,e}V^-(\hat{N})) = 2, \tag{5.22}$$

$$\mu(V^-(\hat{N}), \gamma_{\delta,e}V^+(\hat{N})) = 1. \tag{5.23}$$

Theorem 5.9. *Under the assumption of numerical fact A, for the ± 1 -degenerate curve, we have*

$$\lim_{e \rightarrow 1} \varphi_j(e) = \lim_{e \rightarrow 1} \psi_{j+1}^\pm(e) = 1/8, \text{ for } j \in \mathbb{N}, \tag{5.24}$$

$$\lim_{e \rightarrow 1} \psi_1^+(e) = 0, \quad \lim_{e \rightarrow 1} \psi_1^-(e) = 1/8. \tag{5.25}$$

Proof. To prove (5.24), from Theorem 5.7, we only need to show

$$\inf\{\lim_{e \rightarrow 1} \varphi_j(e), \lim_{e \rightarrow 1} \psi_{j+1}^\pm(e), j \in \mathbb{N}\} \geq 1/8. \tag{5.26}$$

The proof of (5.26) is similar to (5.19). Let $\bar{\delta}_j = \lim_{e \rightarrow 1} \varphi_j(e)$. If $\bar{\delta}_j < 1/8$, then we choose $e_l \rightarrow 1$, such that $\varphi_j(e_l) \rightarrow \bar{\delta}_j$. Choose $\varepsilon < 1/8 - \bar{\delta}_j$, for l large enough, $\varphi_j(e_l) < \bar{\delta}_j + \varepsilon$, so we have

$$i_1(\gamma_{\bar{\delta}_j + \varepsilon, e_l}) > i_1(\gamma_{\varphi_j(e_l), e_l}) = 2j + 3, \tag{5.27}$$

which is contradict to (5.21). It is totally similar that $\lim_{e \rightarrow 1} \psi_{j+1}^\pm(e) \geq 1/8$ for $j \in \mathbb{N}$.

Direct computation shows that for $\delta \in (0, 1/8)$, $\mu(V^-(\hat{N}), \gamma_{\delta, 0} V^+(\hat{N})) = \mu(V^+(\hat{N}), \gamma_{\delta, 0} V^-(\hat{N})) = 1$. By monotone property,

$$\mu(V^+(\hat{N}), \gamma_{\delta, e} V^-(\hat{N})) = \begin{cases} 1, & \text{if } \delta \in (0, \psi_1^+], \\ 2, & \text{if } \delta \in (\psi_1^+, \psi_2^+]. \end{cases} \tag{5.28}$$

From (5.22), we get $\lim_{e \rightarrow 1} \psi_1^+(e) = 0$. The proof for $\lim_{e \rightarrow 1} \psi_1^-(e) = 1/8$ is from (5.23) and the step is similar. \square

This theorem shows that the system is hyperbolic for $\delta \in (0, \frac{1}{8})$, and $1 - e$ small enough. To explain the results, we use the pictures which are taken from [29] (Fig. 7, Fig. 8).

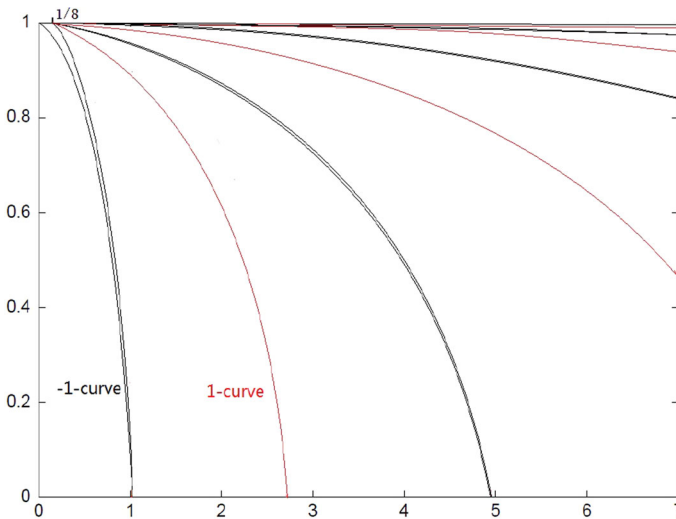


Fig. 7. Stability bifurcation diagram

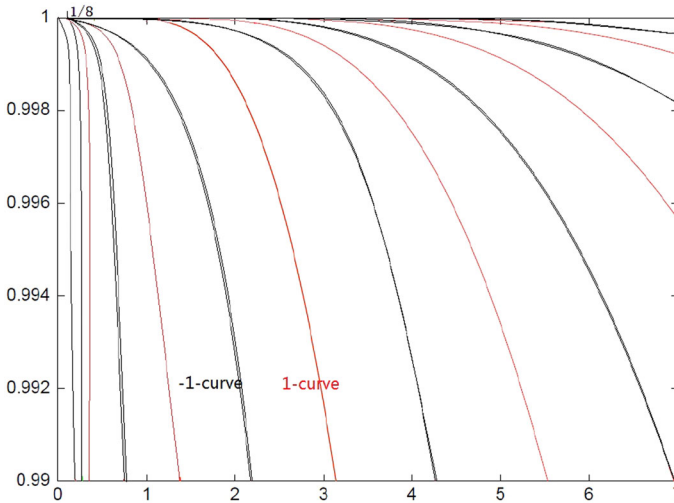


Fig. 8. A magnification of Fig. 7 for $1 - \epsilon$ small

6. Numerical Results for Collision Index

As shown in Remark 3.17, in order compute the collision index, we only need to count the zeros of a determinant function. We use the exterior algebra representation from [7–9] to do the computation. For reader’s convenience, we give a brief review in the four-dimensional case, here.

Consider the linear system

$$\dot{x} = A(\tau)x, \quad x \in \mathbb{R}^4, \quad \tau \in [0, +\infty), \tag{6.1}$$

where $A(+\infty)$ is hyperbolic. Let $\wedge^2(\mathbb{R}^4)$ be the vector space of 2-vector space in \mathbb{R}^4 . Supposing $e_j, j = 1, \dots, 4$ is basis of \mathbb{R}^4 , then $\hat{e}_1 = e_1 \wedge e_2, \hat{e}_2 = e_1 \wedge e_3, \hat{e}_3 = e_1 \wedge e_4, \hat{e}_4 = e_2 \wedge e_3, \hat{e}_5 = e_2 \wedge e_4, \hat{e}_6 = e_3 \wedge e_4$ is basis of $\wedge^2(\mathbb{R}^4)$. There is a induced system from (6.1)

$$\dot{y} = A^{(2)}(\tau)y, \quad y \in \wedge^2(\mathbb{R}^4). \tag{6.2}$$

Suppose $A = (a_{i,j})$, then $A^{(2)}$ could be expressed by $(a_{i,j})$ (cfr. [9, Equation (2.8)] for the expression). Let σ be the sum of the eigenvalues of $A(\infty)$ with positive real part. Let $\hat{y}(\tau) = e^{-\sigma\tau}y(\tau)$, then

$$\frac{d\hat{y}}{d\tau} = (A^{(2)}(\tau) - \sigma I_4)\hat{y}. \tag{6.3}$$

To compute the Maslov index $\mu(V_d, \gamma(\tau)V_0)$, we choose a basis $\xi_1(0), \xi_2(0)$ of V_0 , and let $\hat{y}(0) = y(0) = \xi_1(0) \wedge \xi_2(0) = \sum_{j=1}^6 y_j(0)\hat{e}_j$. Then $\hat{y}(\tau)$ could be computed by matlab from Equation (6.3). Let γ be the fundamental solution of (6.1), then $\gamma(\tau)V_0$ could be expressed by $\hat{y}(\tau)$. We choose e_1, e_2 to be the basis of V_d , then it is obvious that $V_d \cap \gamma(\tau)V_0$ is nontrivial if and only if $e_1 \wedge e_2 \wedge \hat{y}(\tau) = 0$, which is equivalent to $\hat{y}_6(\tau) = 0$. So we can draw the picture of $\hat{y}_6(\tau)$ and count the number of zero points to get the Maslov index.

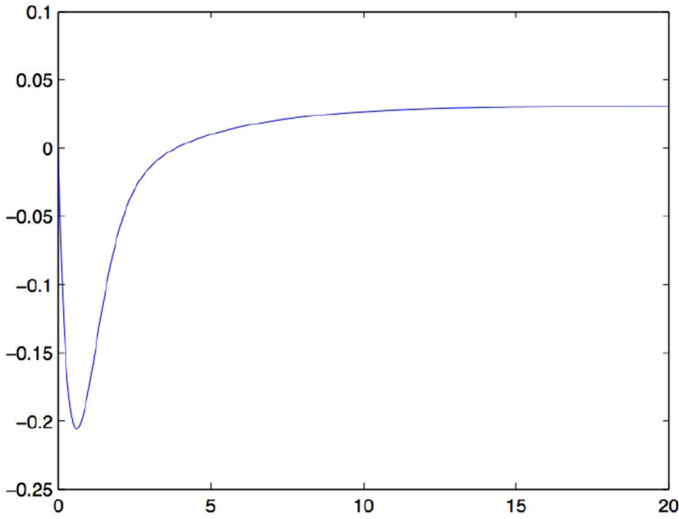


Fig. 9. $\hat{y}_6^+(\tau)$ for $\delta = 0.1$

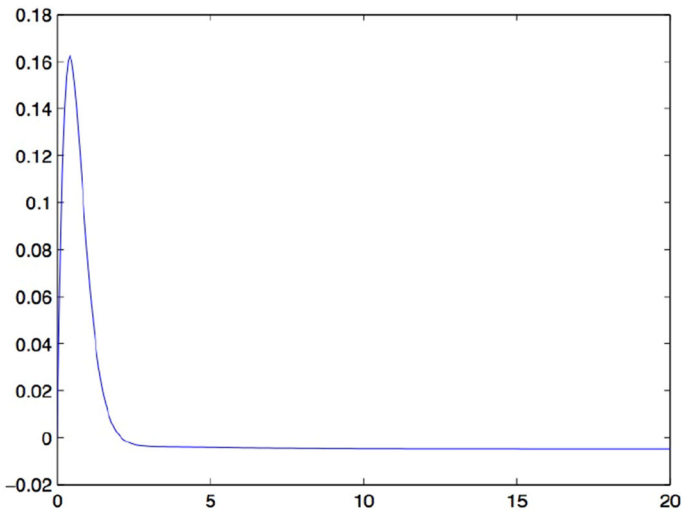


Fig. 10. $\hat{y}_6^-(\tau)$ for $\delta = 0.1$

We will compute $i_-(V^\pm(\hat{N}); l_+^-)$ for Euler and Lagrangian orbits. From Lemma 4.6, we only need to count the points of $V_d \cap \gamma(-\tau_j^\pm)V^\pm(\hat{N})$. From (4.20), the linear system with form with $\dot{x}(\tau) = -J\hat{B}(-\tau)x(\tau)$, let $A(\tau) = -J\hat{B}(-\tau)$, then we can get $A^{(2)}(\tau)$. We choose e_1, e_4 to be the basis of $V^+(\hat{N})$ and e_2, e_3 to be the basis of $V^-(\hat{N})$. Let $\hat{y}^+(\tau)$ be the solution of (6.3) with initial condition $\hat{y}(0) = \hat{e}_3$ and $\hat{y}^-(\tau)$ be the solution with initial condition $\hat{y}(0) = \hat{e}_4$, then $i_-(V^\pm(\hat{N}); l_+^-)$ just is the zero points of $\hat{y}_6^\pm(\tau)$.

We firstly give some numerical pictures for Euler orbits:

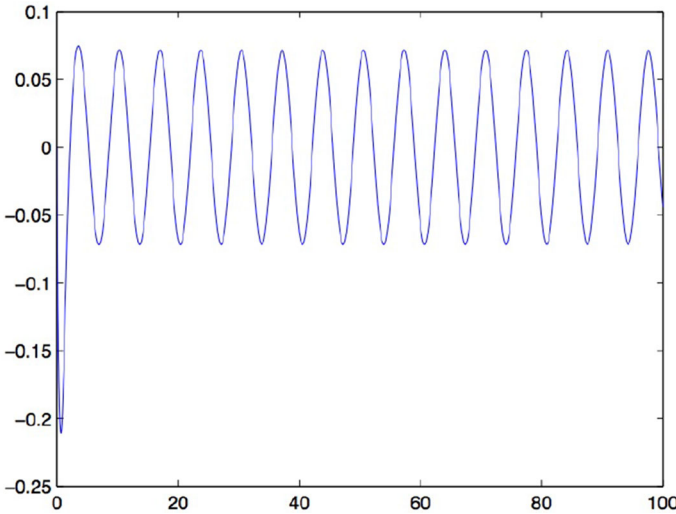


Fig. 11. $\hat{y}_6^+(\tau)$ for $\delta = 1$

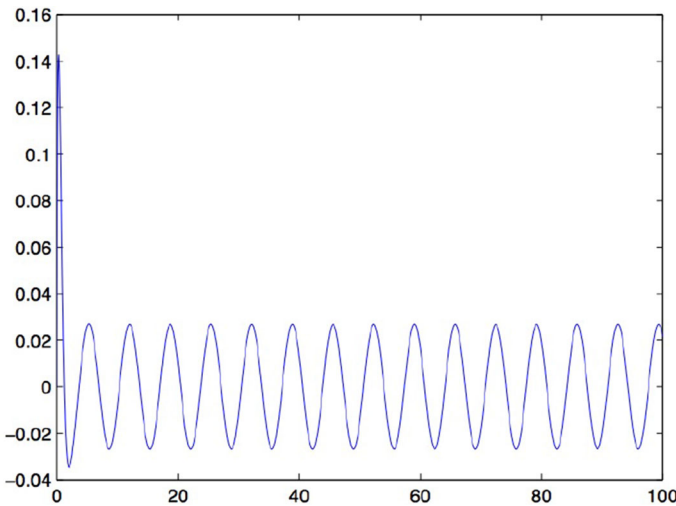


Fig. 12. $\hat{y}_6^-(\tau)$ for $\delta = 1$

It is obvious that there is only one zero point in Figs. 9 and 10, and we have computed it for many value of $\delta \in (0, 1/8)$ and for time large as $\tau = 1000$. All the pictures shows that there is only one zero point. This is why we gave Numerical result A.

If $\delta > 1/8$, Theorem 1.1 shows that the collision index is infinity, which is corresponding to the picture of Figs. 11 and 12 that the number of zero points growth in direct proportion to the time.

For the Lagrangian orbits, Proposition 5.4 shows that the collision index is zero, which corresponds to the following pictures showing no zero point (Figs. 13, 14).

It is quite difficult in a concrete situation to establish if an orbit is collision non-degenerate. However, if the collision index depends on one parameter having a jump,

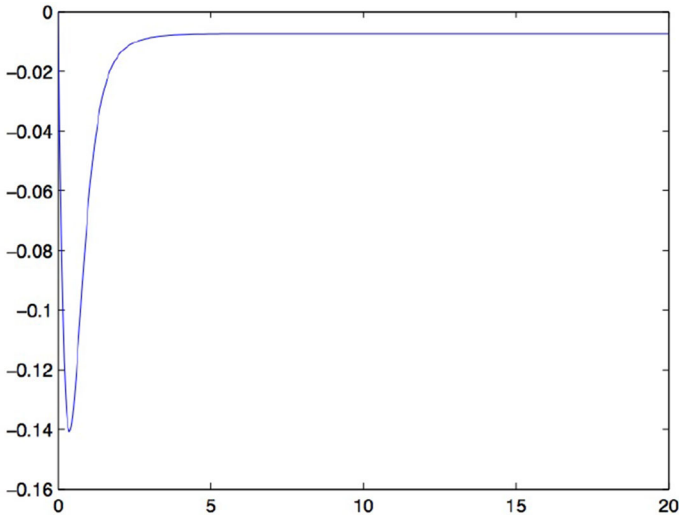


Fig. 13. $\hat{y}_6^+(\tau)$ for $\beta = 6$

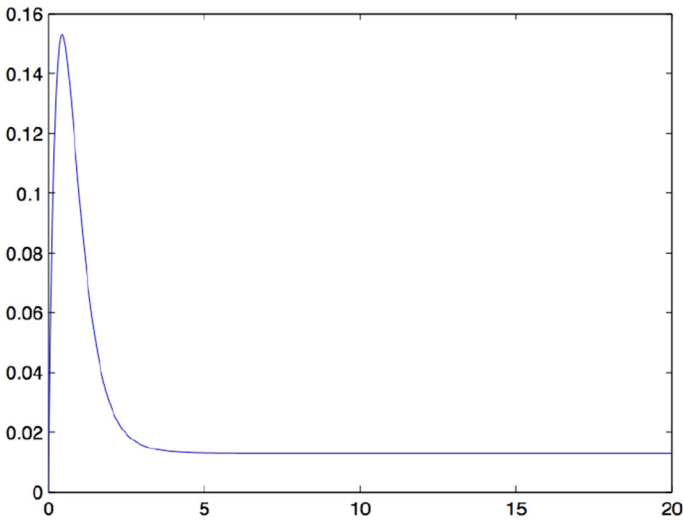


Fig. 14. $\hat{y}_6^-(\tau)$ for $\beta = 6$

then there is a collision degenerate point. During our computations, we found that the Kepler case ($\delta = 0$) is collision degenerate. We guess that every nondegenerate central configuration satisfying the condition $\lambda_1(R) > -1/8$ is collision nondegenerate.

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