



Robust Criterion for the Existence of Nonhyperbolic Ergodic Measures

Jairo Bochi¹, Christian Bonatti², Lorenzo J. Díaz³

¹ Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Santiago, Chile.
E-mail: jairo.bochi@mat.puc.cl
URL: <http://www.mat.uc.cl/~jairo.bochi>

² Institut de Mathématiques de Bourgogne, Dijon, France

³ Departamento de Matemática, Pontificia Universidade Católica do Rio de Janeiro, Rio de Janeiro, Brazil

Received: 23 February 2015 / Accepted: 23 February 2016
Published online: 12 May 2016 – © Springer-Verlag Berlin Heidelberg 2016

Abstract: We give explicit C^1 -open conditions that ensure that a diffeomorphism possesses a nonhyperbolic ergodic measure with positive entropy. Actually, our criterion provides the existence of a partially hyperbolic compact set with one-dimensional center and positive topological entropy on which the center Lyapunov exponent vanishes uniformly.

The conditions of the criterion are met on a C^1 -dense and open subset of the set of diffeomorphisms having a robust cycle. As a corollary, there exists a C^1 -open and dense subset of the set of non-Anosov robustly transitive diffeomorphisms consisting of systems with nonhyperbolic ergodic measures with positive entropy.

The criterion is based on a notion of a blender defined dynamically in terms of strict invariance of a family of discs.

1. Introduction

1.1. General context. Since the end of the sixties it is well known that there exist dynamical systems (diffeomorphisms and flows) that are C^r -robustly nonhyperbolic: every perturbation of the system fails to be hyperbolic (see for instance [N]). Nevertheless, hyperbolic-like techniques and concepts which are essentially weakened forms of hyperbolicity are still among the main tools for studying dynamical systems, even beyond uniform hyperbolicity.

Nonuniform hyperbolicity is an ergodic-theoretical version of hyperbolicity and has its origin in Oseledets' theorem about the existence of Lyapunov exponents. An invariant

The authors received support from CNPq, FAPERJ, PRONEX, and Ciência Sem Fronteiras CAPES (Brazil); Balzan–Palis Project, ANR (France); Fondecyt and PIA-Conicyt (Anillo ACT 1103 DySyRF) (Chile), and EU Marie-Curie IRSES Brazilian-European partnership in Dynamical Systems (FP7-PEOPLE-2012-IRSES 318999 BREUDS). The authors acknowledge the kind hospitality of Institut de Mathématiques de Bourgogne and Departamento de Matemática PUC-Rio while preparing this paper.

ergodic probability measure is called *hyperbolic* if all its Lyapunov exponents are different from zero, and *nonhyperbolic* otherwise. For $C^{1+\alpha}$ -systems, Pesin's theory recovers essential parts of hyperbolicity for generic points of hyperbolic measures, see [Pe]. One speaks of nonuniform hyperbolicity when studying systems endowed with some reference measure. For instance, in the conservative setting, a natural question is to know in what generality the volume is a hyperbolic measure. (For instance, see [SW, BB] for perturbative methods for removing zero Lyapunov exponents in the conservative setting.)

Another point of view, which is the one adopted in this paper, consists in looking at all ergodic invariant measures carried by the system. A diffeomorphism is called *completely nonuniformly hyperbolic* if all its ergodic invariant measures are hyperbolic. Let us observe that there are examples of nonhyperbolic systems that are completely nonuniformly hyperbolic and even have Lyapunov exponents far away from zero: see [BBS, CLR, LOR]. However, all of these examples are very fragile. Therefore one can naively ask whether every diffeomorphism can be approximated by completely nonuniformly hyperbolic ones, thus reclaiming in a weaker sense Smale's dream of denseness of hyperbolicity. Unfortunately, the answer to this question is negative, as was shown by Kleptsyn and Nalsky [KN].

Another question, in the opposite direction, is the following: Can every diffeomorphism be approximated either by uniformly hyperbolic ones or by diffeomorphisms with nonhyperbolic ergodic measures? The answer is positive: it was shown by Mañé [M2] that every C^1 -robustly nonhyperbolic diffeomorphism can be C^1 -approximated by diffeomorphisms that have nonhyperbolic periodic points¹ and in particular nonhyperbolic ergodic invariant measures. In particular, the existence of zero Lyapunov exponents is C^1 -dense in the complement of the closure of the set of hyperbolic diffeomorphisms. However, this dense subset is too "thin" (actually meager), since periodic orbits of C^r -generic diffeomorphisms (for any $r \geq 1$) are hyperbolic, by Kupka–Smale's theorem, [Ku, Sm]. Now if we consider not only measures supported on periodic orbits but arbitrary ergodic probability measures, we expect to have a much "thicker" dense set.

In order to be more precise, let us introduce some notation. Given a compact manifold M without boundary, define the following two disjoint C^r -open sets:

- $\mathcal{H}^r(M)$ is the set of hyperbolic C^r -diffeomorphisms, that is, those satisfying Axiom A with no cycles or, equivalently, with a hyperbolic chain recurrent set.
- $\mathcal{Z}^r(M)$ is the C^r -interior of the set of C^r -diffeomorphisms with a nonhyperbolic ergodic measure.

We pose the following question:

Question 1 (Hyperbolicity vs. robust zero exponents). *Given $r \geq 1$, is the (open) set $\mathcal{H}^r(M) \cup \mathcal{Z}^r(M)$ dense in $\text{Diff}^r(M)$?*

A positive answer to this question would mean that the existence of nonhyperbolic ergodic measures basically characterizes nonhyperbolic C^r -dynamics.

Let us remark that the first set $\mathcal{H}^r(M)$ is always nonempty, and the second set $\mathcal{Z}^r(M)$ is also nonempty provided $\dim M \geq 3$, as it is proved in the aforementioned paper [KN] (for $r = 1$ and therefore for any $r \geq 1$) as a continuation of the results from [GIKN].

In this paper, still assuming $\dim M \geq 3$, we prove that the open set $\mathcal{Z}^1(M)$ is not only nonempty but actually very large, thus providing evidence that Question 1 should

¹ Note that such a property of diffeomorphisms does not extend to vector fields, since there exist robustly nonhyperbolic Lorenz-like attractors whose periodic orbits are all robustly hyperbolic, see [GW].

have a positive answer. An answer in complete generality, even for $r = 1$, seems to be a hard problem. Before stating our results let us observe that Question 1 is motivated and closely related to the following conjecture of Palis [P] about topological characterizations of nonhyperbolicity:

Conjecture 1 (Hyperbolicity vs. cycles). *Every diffeomorphism $f \in \text{Diff}^r(M)$, $r \geq 1$, can be C^r -approximated by diffeomorphisms that are either hyperbolic or display homoclinic bifurcations (homoclinic tangencies or heterodimensional cycles).*

This conjecture is true for C^1 -diffeomorphisms of surfaces, see [PS]. See also [CP] for important progress in higher dimensions.

Note that the homoclinic bifurcations in this conjecture are associated to periodic points. By Kupka–Smale’s genericity theorem, most systems (i.e., generic ones) do not display homoclinic bifurcations. Bearing this fact in mind, a stronger version of the conjecture above was proposed in [BD2] and [B, Conjecture 7]. In order to state it, let us recall that two transitive hyperbolic basic sets Λ and Θ of a diffeomorphism f have a robust (heterodimensional) cycle if:

- Λ and Θ have different u-indices (i.e., their unstable bundles have different dimensions) and
- There is a C^1 -neighborhood \mathcal{U} of f so that for every $g \in \mathcal{U}$ one has

$$W^s(\Lambda_g) \cap W^u(\Theta_g) \neq \emptyset \quad \text{and} \quad W^u(\Lambda_g) \cap W^s(\Theta_g) \neq \emptyset,$$

where Λ_g and Θ_g are the hyperbolic continuations of Λ and Θ for g .

Observe that robust cycles can only occur in dimension 3 or larger. Let us denote by $\mathcal{RC}^1(M)$ the C^1 -open subset of $\text{Diff}^1(M)$ of diffeomorphisms with robust cycles. We can now state the following:

Conjecture 2 (Hyperbolicity vs. robust cycles). *The union of the disjoint open sets $\mathcal{H}^1(M)$ and $\mathcal{RC}^1(M)$ is dense in $\text{Diff}^1(M)$. In other words, every diffeomorphism in $\text{Diff}^1(M)$ can be C^1 -approximated by diffeomorphisms that are either hyperbolic or have robust cycles.²*

A consequence of the results in [DG] is that C^1 -generic diffeomorphisms with robust cycles have ergodic nonhyperbolic measures. Note that by Kupka–Smale’s theorem the nonhyperbolic measures in [DG] cannot be supported on periodic orbits. In the partially hyperbolic setting with one-dimensional central direction, [BDG] improves the results in [DG] by showing that these nonhyperbolic measures can be chosen with full support in the appropriate homoclinic class.

In this paper we strengthen the C^1 -generic conclusion of the result of [DG] and prove that C^1 -robust cycles yield C^1 -robust existence of nonhyperbolic ergodic measures. We also will see that these measures can be chosen with positive entropy.

Theorem 1. *Let M be a compact manifold without boundary and of dimension $d \geq 3$. Then the subset of $\text{Diff}^1(M)$ consisting of diffeomorphisms with a nonhyperbolic ergodic measure with positive entropy contains an open and dense subset of the C^1 -open set $\mathcal{RC}^1(M)$ of diffeomorphisms with robust cycles.*

² This conjecture involves robust cycles but does not involve homoclinic tangencies. The rationale behind this is the fact that most heterodimensional cycles can be made robust by small perturbations: see [BDK]. In contrast, the only known examples of C^1 -robust homoclinic tangencies occur in dimension at least 3 and are associated to C^1 -robust cycles: see [BD3]. Furthermore, C^1 -surface diffeomorphisms do not have robust homoclinic tangencies: see [Mo].

In the notations above, Theorem 1 implies that $\mathcal{Z}^1(M) \cap \mathcal{RC}^1(M)$ is dense in $\mathcal{RC}^1(M)$. Therefore, if Conjecture 2 is true, then Question 1 has a positive answer when $r = 1$.

Theorem 2 below is a more general (and technical) version of Theorem 1. The proof of that theorem is based on Theorem 5, which gives an explicit robust criterion for zero center Lyapunov exponents, which in turn relies on Theorem 4, which is an abstract ergodic-theoretical criterion for the existence of zero Birkhoff averages. The strategy used in this paper for the construction of nonhyperbolic measures is very different from the previous methods, which are reviewed in Sect. 1.5.

Let us present some consequences of Theorem 1. A diffeomorphism is called C^1 -robustly transitive if every C^1 -diffeomorphism nearby is transitive, that is, has a dense orbit. In dimension two, C^1 -robustly transitive diffeomorphisms are Anosov diffeomorphisms of the torus \mathbb{T}^2 (see [M2]), but there are non-Anosov robustly transitive diffeomorphisms on manifolds of dimension three or more (see [S,M1,BD1]). Combining the results in [H,BC,BD2], one obtains that there is a C^1 -open and dense subset of the set of nonhyperbolic robustly transitive diffeomorphisms consisting of diffeomorphisms with robust cycles. In other words, $\mathcal{H}^1(M) \cup \mathcal{RC}^1(M)$ contains an open and dense subset of the set of C^1 -robustly transitive diffeomorphisms. Thus the following is a corollary of Theorem 1.

Corollary 1. *The union of the set of Anosov diffeomorphisms and the set $\mathcal{Z}^1(M)$ is open and dense in the set of C^1 -robustly transitive diffeomorphisms.*

There are several *mutatis mutandis* versions of this corollary for robustly transitive sets and homoclinic or chain recurrence classes robustly containing saddles of different indices. One of these is the following. A diffeomorphism f is called C^1 -tame if each of its chain recurrence classes is C^1 -robustly isolated. In this case, the number of chain recurrence classes is finite and constant in a C^1 -neighborhood of f . The following corollary asserts that Question 1 with $r = 1$ has a positive answer if restricted to tame diffeomorphisms.

Corollary 2. *There is a C^1 -open and dense subset of the set of C^1 -tame diffeomorphisms consisting of diffeomorphisms whose chain recurrence classes are either hyperbolic or support a nonhyperbolic ergodic measure with positive entropy.*

1.2. Sharper results. A finite sequence of points $(x_i)_{i=0}^n$ is an ε -pseudo-orbit of a diffeomorphism $f : M \rightarrow M$ if $\text{dist}(f(x_i), x_{i+1}) < \varepsilon$ for all $i = 0, \dots, n - 1$. A point x is chain recurrent for f if for every $\varepsilon > 0$ there is an ε -pseudo-orbit $(x_i)_{i=0}^n$ starting and ending at x (i.e., with $x = x_0 = x_n$). The chain recurrent set is composed by all chain recurrent points of f and is denoted by $\mathcal{R}(f)$. This set splits into pairwise disjoint chain recurrence classes: the class $C(x, f)$ of $x \in \mathcal{R}(f)$ is the set of points y such that for every $\varepsilon > 0$ there are ε -pseudo-orbits joining x to y and y to x .

Given an f -invariant set Λ and a continuous Df -invariant line field $E = (E_x)_{x \in \Lambda}$ over Λ , the Lyapunov exponent of a point x along the direction E is defined as

$$\chi_E(x) := \lim_{|n| \rightarrow \infty} \frac{\log \|Df_x^n(v)\|}{n}, \quad v \in E_x \setminus \{0\},$$

where $\|\cdot\|$ stands for the Riemannian norm, whenever this limit exists.

Theorem 2. *Let M be a compact manifold without boundary and of dimension $d \geq 3$. Let $\mathcal{U} \subset \text{Diff}^1(M)$ be an open set of diffeomorphisms such that for every $f \in \mathcal{U}$ there are hyperbolic periodic points p_f and q_f , depending continuously on f , in the same chain recurrence class $C(p_f, f)$ and having respective u -indices $i_p > i_q$. Then there exists a C^1 -open and dense subset \mathcal{V} of \mathcal{U} with the following properties. For any $f \in \mathcal{V}$ and any integer i with $i_q < i \leq i_p$ there exists a compact f -invariant set $K_{f,i} \subset C(p_f, f)$ with a partially hyperbolic splitting*

$$T_{K_{f,i}}M = E^{uu} \oplus E^c \oplus E^{ss}$$

such that:

- E^{uu} is uniformly expanding and has dimension $i - 1 > 0$, E^c has dimension 1, and E^{ss} is uniformly contracting and has dimension $d - i > 0$;
- The Lyapunov exponent along the E^c direction of any point in $K_{f,i}$ is zero;
- The topological entropy of the restriction of f to $K_{f,i}$ is positive.

Additionally, the Lyapunov exponent along the central direction E^c is *uniformly* zero, in the sense that the limits that by definition are the Lyapunov exponent exist everywhere on $K_{f,i}$ and are uniform. Although this uniformity follows abstractly from the facts that $K_{f,i}$ is compact and E^c is one-dimensional, we will obtain it directly from the construction.

Let us now see how Theorem 2 implies Theorem 1. Given a pair of hyperbolic sets forming a robust cycle, their union is contained in the same chain recurrence class and contains hyperbolic periodic points of different indices, permitting us to apply Theorem 2. This allows us to conclude that diffeomorphisms f in an open and dense subset of $\mathcal{RC}^1(M)$ possess compact invariant partially hyperbolic sets K_f with positive topological entropy and with uniformly zero Lyapunov exponents along the center direction. By the variational principle for entropy (see e.g. [W]), each such K_f supports an ergodic measure of positive metric entropy, and so we obtain the statement of Theorem 1.

In the converse direction, if \mathcal{U} is an open set of diffeomorphisms satisfying the hypotheses of Theorem 2, that is, such that every $f \in \mathcal{U}$ has a chain recurrence class with periodic points p_f, q_f of respective u -indices $i_p > i_q$ then by results of [BC, ABCDW, BD2, BDK] there exists an open and dense subset \mathcal{W} of \mathcal{U} such that if $f \in \mathcal{W}$ then the chain recurrence class $C(p_f, f)$ contains periodic points of every intermediate u -index. Moreover, any two of these periodic points having consecutive u -indices belong to a pair of hyperbolic basic sets forming a robust cycle.³

So if in Theorem 2 we replace the hypothesis “ p_f and q_f are in the same chain recurrence class for every $f \in \mathcal{U}$ ” by the stronger hypothesis “ p_f and q_f have consecutive indices and are related by a robust cycle for every $f \in \mathcal{U}$ ”, we obtain a result that is not much weaker than its ancestor: the two theorems are equivalent modulo a nowhere dense closed subset of $\text{Diff}^1(M)$. To prove Theorem 2, we will actually work with these robust cycles associated to periodic points of consecutive indices.

Given an ergodic measure μ , by Oseledets’ Theorem we can define its *Lyapunov exponents* $\chi_1(\mu) \geq \chi_2(\mu) \geq \dots \geq \chi_d(\mu)$, where $d = \dim M$. While we will not recall

³ According to [BC], for C^1 -generic diffeomorphisms f in \mathcal{U} the homoclinic class and the chain recurrent class of p_f coincide, and the same occurs for q_f . Therefore the homoclinic classes $H(p_f)$ and $H(q_f)$ of p_f and q_f coincide. Now [BCDG, Corollary 2.4] claims that for every generic f for which $H(p_f)$ and $H(q_f)$ coincide, the class $H(p_f)$ contains hyperbolic sets K_i of u -index i for every $i_q \leq i \leq i_p$ so that K_i and K_{i+1} , $i_q \leq i < i_p$, have a robust cycle.

here the full statement of that theorem, let us remark that for μ -a.e. point x it is possible to choose linearly independent vectors v_1, \dots, v_d in $T_x M$ such that

$$\lim_{n \rightarrow \pm\infty} \frac{\log \|Df_x^n(v_i)\|}{n} = \chi_i(\mu) \quad \text{for each } i.$$

So Theorem 2 also has the following corollary:

Corollary 3. *Under the hypotheses of Theorem 2, there exists a C^1 -open and dense subset $\mathcal{V} \subset \mathcal{U}$ such that every diffeomorphism $f \in \mathcal{V}$ has ergodic measures $\mu_{i_p}, \dots, \mu_{i_q-1}$ such that each μ_i has positive entropy and its i -th Lyapunov exponent $\chi_i(\mu_i)$ is zero.*

As a matter of fact, our methods permit us to obtain not only a central Lyapunov exponent equal to zero, but a whole interval of central Lyapunov exponents. If p is a periodic point then we denote by $\chi_i(p)$ the i -th Lyapunov exponent of the ergodic measure supported on the orbit of p .

Theorem 3. *Let M be a compact manifold without boundary and of dimension $d \geq 3$. Let $\mathcal{U} \subset \text{Diff}^1(M)$ be an open set of diffeomorphisms such that for every $f \in \mathcal{U}$ there are hyperbolic periodic points p_f and q_f , depending continuously on f , in the same chain recurrence class $C(p_f, f)$ and having respective u -indices i and $i - 1$.*

Then there exists a C^1 -open and dense subset \mathcal{V} of \mathcal{U} such that for every $f \in \mathcal{V}$ and every $\chi \in (\chi_i(q_f), \chi_i(p_f))$, there exists a compact f -invariant set $K_{f,\chi} \subset C(p_f, f)$ with a partially hyperbolic splitting

$$T_{K_{f,\chi}} M = E^{uu} \oplus E^c \oplus E^{ss}$$

such that:

- E^{uu} is uniformly expanding and has dimension $i - 1 > 0$, E^c has dimension 1, and E^{ss} is uniformly contracting and has dimension $d - i > 0$;
- The Lyapunov exponent along the E^c direction of any point in $K_{f,\chi}$ equals χ ;
- the topological entropy of the restriction of f to $K_{f,\chi}$ is positive.

In particular, the i -th Lyapunov exponent of any measure supported on $K_{f,\chi}$ equals χ and there is an ergodic measure supported on $K_{f,\chi}$ with positive entropy.

The proof of this theorem has two parts. The construction of the sets $K_{f,\chi}$ for χ bounded away from zero follows from the arguments in [ABCDW] using Markov partitions and has a hyperbolic flavor. The new and more difficult part here is the construction of the sets $K_{f,\chi}$ for χ close to zero.

1.3. An abstract criterion for the existence of zero averages. Along the proof of Theorem 2, we develop some non-perturbative criteria for the existence of zero center Lyapunov exponents, or more generally, zero limit Birkhoff averages. This first criterion holds on a purely topological setting, and relies in the following concept (see Fig. 1):

Definition 1.1 (*Flip-flop family*). Let (X, d) be a metric space, $f : X \rightarrow X$ be a continuous map, K be a compact subset of X , and $\varphi : K \rightarrow \mathbb{R}$ be a continuous function.

A *flip-flop family* is a family \mathfrak{F} of compact subsets of K with uniformly bounded diameters that splits as $\mathfrak{F} = \mathfrak{F}^+ \cup \mathfrak{F}^-$ into two disjoint families satisfying the following properties:

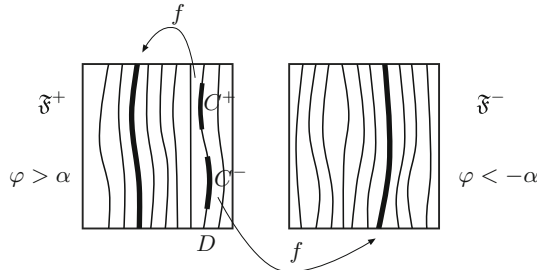


Fig. 1. Flip-flop family

(FF1) There is a constant $\alpha > 0$ such that for all members $D^+ \in \mathfrak{F}^+$, $D^- \in \mathfrak{F}^-$ and all points $x^+ \in D^+$, $x^- \in D^-$ we have

$$\varphi(x^-) < -\alpha < 0 < \alpha < \varphi(x^+).$$

(FF2) For every $D \in \mathfrak{F}$ there are compact subsets C^+ , C^- of D such that $f(C^+) \in \mathfrak{F}^+$ and $f(C^-) \in \mathfrak{F}^-$.

(FF3) There is a constant $\lambda > 1$ such that if E is contained in a member of \mathfrak{F} and $f(E)$ is a member of \mathfrak{F} then

$$d(f(x), f(y)) \geq \lambda d(x, y) \quad \text{for every } x, y \in E.$$

The motivation for the definition above is the following result:

Theorem 4 (Abstract criterion for zero limit Birkhoff averages). *Consider a continuous map $f : X \rightarrow X$ on a metric space X having a flip-flop family \mathfrak{F} associated to a continuous function $\varphi : K \rightarrow \mathbb{R}$ defined on a compact subset K of X .*

Then there exists a compact forward invariant subset $\Omega \subset K$ such that the Birkhoff averages of φ converge to zero uniformly on Ω . Moreover, the restriction of f to Ω has positive topological entropy.

In particular, there exist ergodic f -invariant measures μ of positive entropy such that $\int \varphi d\mu = 0$.

Let us emphasize that the function φ is only continuous. If φ were assumed to be more regular (say, Hölder) then the corresponding theorem would be considerably simpler to prove (see Remark 2.5 in this regard), but it would not be sufficient to obtain C^1 -openness in Theorems 1 and 2 above or Theorem 5 below.

1.4. A criterion for the existence of nonhyperbolic measures. Coming back to diffeomorphisms, let us explain our next result, which gives explicit C^1 -open sufficient conditions for the existence of partially hyperbolic sets with zero center exponents.

We assume the existence of a *dynamical blender*. This basically means a partially hyperbolic set together with a family of embedded discs tangent to a strong unstable cone such that the image of any perturbation of a disc in the family contains a disc in the family. See Sect. 3 for the precise definition. Dynamical blenders are a more flexible version of previous notions of blenders, see Sect. 1.6.

In addition, we assume that there exists a saddle that forms together with the dynamical blender a certain *split flip-flop configuration*. Roughly this means that the dynamics

associated to the blender and the saddle is partially hyperbolic with one-dimensional center; see Definition 4.7 for the precise definition.

We prove the following:

Theorem 5 (Criterion for zero center Lyapunov exponents). *Let M be a compact manifold without boundary and of dimension $d \geq 3$. Assume that $f \in \text{Diff}^1(M)$ has a periodic saddle q and a dynamical blender Γ in a split flip-flop configuration. Let $i - 1$ be the u -index of q .*

Then there exists a compact f -invariant set K contained in the chain recurrence class $C(q, f)$ and admitting a partially hyperbolic splitting

$$T_K M = E^{uu} \oplus E^c \oplus E^{ss}$$

such that:

- E^{uu} is uniformly expanding and has dimension $i - 1 > 0$, E^c has dimension 1, and E^{ss} is uniformly contracting and has dimension $d - i > 0$;
- The Lyapunov exponent along the E^c direction of any point in K is zero;
- The topological entropy of the restriction of f to K is positive.

We will deduce Theorem 5 from Theorem 4: starting from split flip-flop configurations, we will construct flip-flop families \mathfrak{F} composed of discs contained in the strong unstable manifolds of a partially hyperbolic set with one dimensional center bundle; the function φ will be essentially the logarithm of the center Jacobian.

To prove Theorem 2, we basically show that in the presence of a robust cycle the hypothesis of Theorem 5 are satisfied after a suitable C^1 -perturbation.

1.5. Discussion on the methods. The literature contains a number of results on the existence of nonhyperbolic measures. Let us briefly compare these results with those of the present paper.

The paper [GIKN] deals with certain partially hyperbolic skew-products. It constructs nonhyperbolic ergodic measures with nondiscrete support as limits of sequences of measures supported on periodic orbits. Each periodic orbit in the sequence shadows the previous one for a large proportion of time — this is the key property to obtain ergodicity. Each periodic orbit has a proportionally small tail far from the previous orbits, which is chosen in order to make the center Lyapunov exponent smaller. That *method of periodic approximations* was also used in subsequent papers [KN,DG,BDG,BBD] to find nonhyperbolic measures for partially hyperbolic dynamics. In [BBD], the method was extended to higher center dimension (in the skew-product setting) so to yield multiple zero exponents. On the other hand, the nonhyperbolic ergodic measures constructed by the method of periodic approximations are highly “repetitive” and are likely to have zero entropy.

The strategy developed in this paper is completely different from the periodic approximations one. Using a recursive construction we find a point x whose central expansion is controlled at all time scales. Then its omega-limit set $\omega(x)$ is an invariant compact set which is *completely nonhyperbolic*, meaning that every invariant measure supported on this set is nonhyperbolic. Moreover, the orbit of x can be chosen “noisy” enough so that the restriction of f to $\omega(x)$ has positive topological entropy.

The strategy of recursive control at all time scales does not need much regularity: the diffeomorphism f in Theorem 5 is only C^1 , and the function φ in Theorem 4 is only continuous. Under stronger regularity assumptions it is possible to use a simpler strategy

and obtain sharper conclusions: assuming that φ is Hölder, for example, it is possible to obtain a set Ω in Theorem 4 over which the Birkhoff *sums* are uniformly bounded (see Remark 2.5). Actually that simpler strategy already appears in the proof of another result from the paper [BBD], which constructs completely nonhyperbolic compact sets with positive topological entropy in the context of skew-products.

With the method of periodic approximations, it is relatively easy to obtain nonhyperbolic measures with “large” or sometimes full support. On the other hand, the supports of the nonhyperbolic measures constructed in this paper are completely nonhyperbolic and therefore should be relatively “small”. Nevertheless, it seems reasonable to conjecture that our methods could be sharpened (by dropping uniformity) so to yield nonhyperbolic measures with bigger support.

Another natural question concerns the abundance of ergodic measures that have a zero Lyapunov exponent with multiplicity as high as possible.

1.6. Dynamical blenders. The paper [BD1] introduced a dynamical mechanism called *blender*, which provides the existence of a hyperbolic set whose stable set behaves as if its dimension were greater than the dimension of its stable bundle: the stable set intersects all elements of an open family of embedded discs of low dimension. Important applications of blenders are the construction of C^1 -robust cycles, see [BD2], and C^1 -robust transitive sets, see [BD1]. Afterwards, the paper [BD3] introduced a variation of this concept called *blender-horseshoe* to obtain robust tangencies.

These definitions of blender require quite specific mechanisms and are not flexible enough for the purposes of this paper. So we introduce here a new concept, what we call *dynamical blender*, which only requires an invariance property of a family of disks, see Definition 3.11. This property can be checked more easily and also has the clear advantage of being intrinsically robust. Also, it easily implies the aforementioned property of the stable set (though we do not use this property directly).

1.7. Organization of the paper. In Sect. 2 we prove the abstract zero averages criterion (Theorem 4). This is done by finding special points whose orbits have controlled (i.e., small) Birkhoff averages at every time scale, and are noisy enough so to produce positive entropy.

In Sect. 3, we introduce *dynamical blenders* and establish some properties for later use. Then in Sect. 4 we introduce *split flip-flop configurations*, show how they permit us to find flip-flop families associated to a certain function φ related to the central Jacobian, and complete the proof of Theorem 5.

In Sect. 5 we prove Theorem 2: Starting from a robust cycle associated to saddles of consecutive indices, we show that with an additional perturbation a flip-flop configuration appears, and therefore the theorem follows from the previous results.

Finally, in Sect. 6 we explain how to obtain Theorem 3.

2. Flip-flop Families and Birkhoff Averages: Proof of Theorem 4

The aim of this section is to prove Theorem 4, the zero averages result for flip-flop families. In fact, we will prove a slightly finer version of this result (Theorem 2.1).

In all of this section, let (X, d) be a metric space, $f: X \rightarrow X$ be a continuous map, K be a compact subset of X , $\varphi: K \rightarrow \mathbb{R}$ be a continuous function, and $\mathfrak{F} = \mathfrak{F}^+ \cup \mathfrak{F}^-$ be a flip-flop family as in Definition 1.1.

Let us fix some notation: Denote $\bigcup \mathfrak{F} := \bigcup_{D \in \mathfrak{F}} D$ and analogously for $\bigcup \mathfrak{F}^+, \bigcup \mathfrak{F}^-$. The Birkhoff sums of φ will be denoted as

$$\varphi_n(x) := \sum_{i=0}^{n-1} \varphi(f^i(x)) \quad \text{if } n \in \mathbb{N}, x \in \bigcap_{i=0}^{n-1} f^{-i}(K).$$

2.1. Control of Birkhoff averages at all scales. We begin with some definitions that will be central to our constructions.

Definition 2.1 (Control). Given $\beta > 0, t \in \mathbb{N}^*$, and $T \in \mathbb{N}^* \cup \{\infty\}$, we say that a point $x \in K$ is (β, t, T) -controlled if $f^i(x) \in K$ for $0 \leq i < T$ and there exists a subset $\mathcal{P} \subset \mathbb{N}$ of control times such that

- $0 \in \mathcal{P}$,
- $T \in \mathcal{P}$ if $T < \infty$, and \mathcal{P} is infinite if $T = \infty$, and
- If $k < \ell$ are two consecutive control times in \mathcal{P} then

$$\ell - k \leq t \quad \text{and} \quad \frac{1}{\ell - k} \left| \varphi_{\ell-k}(f^k(x)) \right| \leq \beta.$$

The point x is *controlled at all scales* (with respect to φ) if there exist monotone sequences $(t_i)_i$ of natural numbers and $(\beta_i)_i$ of positive numbers, $t_i \rightarrow \infty$ and $\beta_i \rightarrow 0^+$, such that x is (β_i, t_i, T) -controlled for every i .

One can easily prove the following:

Lemma 2.2. *If $x \in K$ is controlled at all scales then every point $y \in \omega(x)$ satisfies*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \varphi_n(y) = 0.$$

Moreover, the limit is uniform over the ω -limit set $\omega(x)$.

Although we will not need it, let us remark that the converse holds: if the Birkhoff averages converge uniformly to zero over an f -invariant set Ω , then every point in Ω is controlled at all scales.

In this section, we will prove the following result:

Theorem 2.1. *If \mathfrak{F} is a flip-flop family associated to the map f and the function φ then every member $D \in \mathfrak{F}$ contains a point x that is controlled at all scales with respect to φ and such that the restriction of f to the ω -limit set $\omega(x)$ has positive topological entropy.*

Note that Theorem 4 is a direct consequence of Theorem 2.1 together with Lemma 2.2 and the variational principle for entropy.

The idea of the proof of Theorem 2.1 is roughly as follows: For any member D of \mathfrak{F} , it is possible to find points $x \in D$ that have a large number of iterates n_1 in the positive region $\bigcup \mathfrak{F}^+$, and then a large number of iterates n_2 in the negative region $\bigcup \mathfrak{F}^-$ in such a way that the Birkhoff average $\varphi_{n_1+n_2}(x)/(n_1+n_2)$ is positive and small, but not exceedingly small. So we obtain some control at the first time scale: see Fig. 2. Analogously we can obtain small (but not exceedingly small) negative Birkhoff averages at this same scale. We then pass to a second time scale where much smaller (say, positive)

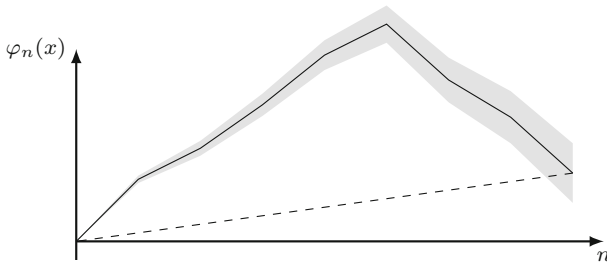


Fig. 2. Control of Birkhoff averages at the first time scale

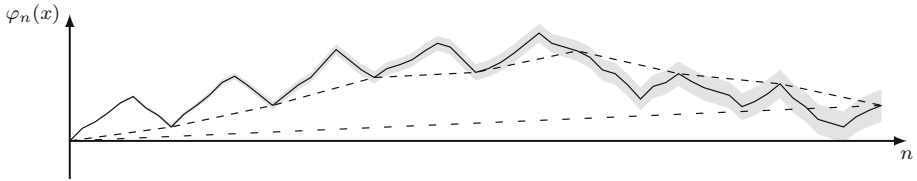


Fig. 3. Control of Birkhoff averages at the second time scale

Birkhoff averages are obtained by concatenating several controlled segments of the first scale, the initial ones being of positive type, and the later ones being of negative type: see Fig. 3. The construction then proceeds recursively in order to control longer and longer time scales. Moreover, we can incorporate some periodic noise in the construction and obtain positive entropy.

The sketch above is imprecise in many ways. We need to make sense of what concatenation and noise mean. Since the desired point x is not known a priori, at each time scale we need to control simultaneously not only a single point but many of them. Thus some “uncertainty clouds” appear; they are shown in gray in Figs. 2 and 3. Uncertainty increases with time due to the expanding character of the dynamics imposed by condition (FF3). Nevertheless it does not have a big effect on Birkhoff averages over long time scales.

We now proceed with precise proofs. We need a few preparatory lemmas and definitions before proving Theorem 2.1.

2.2. *Segments and concatenations.* Recall that \mathfrak{F} is a fixed flip-flop family with respect to a dynamics $f : X \rightarrow X$ and a function $\varphi : K \rightarrow \mathbb{R}$.

Definition 2.3 (*Segments*). Let $T \in \mathbb{N}^*$. A \mathfrak{F} -segment of length T is a sequence $\mathbf{D} = \{D_i\}_{0 \leq i \leq T}$ such that $f(D_i) = D_{i+1}$, each D_i is contained in a member of \mathfrak{F} , and D_T is a member of \mathfrak{F} . The sets D_0 and D_T are respectively called the *entrance* and the *exit* of \mathfrak{F} .

Lemma 2.4 (*Birkhoff averages over long segments have small distortion*). Let \mathfrak{F} be a flip-flop family. For every $\eta > 0$ there exists $N = N(\eta) \in \mathbb{N}$ such that if $\mathbf{D} = \{D_i\}_{0 \leq i \leq T}$ is a segment of length $T \geq N$ then for every pair of points x, y in the entrance D_0 we have

$$\frac{1}{T} |\varphi_T(x) - \varphi_T(y)| < \eta.$$

Proof. Let $\beta_1 := \sup_K |\varphi|$. Given $\eta > 0$, let $\varrho > 0$ be such that

$$|\varphi(z) - \varphi(w)| < \eta/2, \quad \text{for every } z, w \in K \text{ with } d(z, w) < \varrho.$$

Let d_0 be an upper bound for the diameters of the members of \mathfrak{F} , and let $\lambda > 1$ be the expansivity constant given by condition (FF3). Take $n_0 \in \mathbb{N}$ such that $\lambda^{-n_0} d_0 < \varrho$. Fix an integer $N > n_0$ such that

$$\frac{\beta_1 n_0}{N} < \frac{\eta}{4}.$$

Now suppose that $\mathbf{D} = \{D_i\}_{0 \leq i \leq T}$ is a segment of length $T \geq N$, and that x and y are points in the entrance D_0 . By condition (FF3) in definition of flip-flop family, for each $i = 0, \dots, T - 1$ we have

$$d_0 \geq d(f^T x, f^T y) \geq \lambda^{T-i} d(f^i x, f^i y).$$

In particular, if $i < T - n_0$ then $d(f^i x, f^i y) < \varrho$. Therefore we estimate:

$$|\varphi_T(x) - \varphi_T(y)| \leq \sum_{i=0}^{T-n_0-1} \underbrace{|\varphi(f^i x) - \varphi(f^i y)|}_{\leq \eta/2} + \sum_{i=T-n_0}^{T-1} \underbrace{|\varphi(f^i x) - \varphi(f^i y)|}_{\leq 2\beta_1} < T\eta,$$

which proves the lemma. \square

Remark 2.5. Notice that in Lemma 2.4 we are comparing Birkhoff averages, and not Birkhoff sums. If the function φ were not only continuous but, say, Hölder, then by the classical (and easy) bounded distortion argument, a much stronger result would hold: there would exist a uniform (not depending on the length T) upper bound for $|\varphi_T(x) - \varphi_T(y)|$. With such estimates the proof of Theorem 2.1 would be considerably easier. Actually, as mentioned in Sect. 1.5, one could strengthen the conclusion and obtain a set Ω where the Birkhoff sums are uniformly bounded.

Definition 2.6 ((β, t) -controlled segments). Given $\beta > 0$ and $t \leq T \in \mathbb{N}^*$, a \mathfrak{F} -segment $\mathbf{D} = \{D_i\}_{0 \leq i \leq T}$ is said to be (β, t) -controlled if there exists a set of control times $\mathcal{P} \subset \{0, \dots, T\}$ such that

- $0, T \in \mathcal{P}$ and
- If $k < \ell$ are two consecutive control times in \mathcal{P} then

$$\ell - k \leq t \quad \text{and} \quad \frac{1}{\ell - k} \left| \varphi_{\ell-k}(f^k(x)) \right| \leq \beta \quad \text{for all } x \in D_0.$$

That is, every point in the entrance D_0 is (β, t, T) -controlled and moreover we can take an uniform set of control times.

Definition 2.7 (Concatenations). Consider \mathfrak{F} -segments $\mathbf{D} = \{D_i\}_{0 \leq i \leq T}$ and $\mathbf{E} = \{E_j\}_{0 \leq j \leq L}$ of lengths T and L such that the exit of the first contains the entrance of the second, that is, $D_T \supset E_0$. Then the concatenation of \mathbf{D} and \mathbf{E} is \mathfrak{F} -segment $\mathbf{D} * \mathbf{E} = \{F_k\}_{0 \leq k \leq T+L}$ of length $T + L$ defined by

$$F_k := \begin{cases} (f^{T-k}|_{D_k})^{-1}(E_0) & \text{if } 0 \leq k < T, \\ E_{k-T} & \text{if } T \leq k \leq T + L. \end{cases}$$

The concatenation of \mathfrak{F} -segments is an associative operation on \mathfrak{F} -segments. This allows us to define multiple concatenations.

The proof of the next lemma is straightforward and thus omitted.

Lemma 2.8 (Concatenation preserves control). *Consider $\beta > 0$ and integers $t > 0$, $T > t$, and $L > t$. Assume that $\mathbf{D} = \{D_i\}_{0 \leq i \leq T}$ and $\mathbf{E} = \{E_j\}_{0 \leq j \leq L}$ are (β, t) -controlled \mathfrak{F} -segments with \mathcal{P} and \mathcal{Q} as respective sets of control times. Suppose that $D_T \supset E_0$. Then the concatenation $\mathbf{D} * \mathbf{E}$ is (β, t) -controlled with $\mathcal{P} \cup (T + \mathcal{Q})$ as a set of control times.*

2.3. Patterns.

Definition 2.9 (Pattern of an orbit). Given a point x in $\bigcup \mathfrak{F}$, a sequence of signs $\mathbf{s} = (s_n) \in \{+, -\}^{\mathbb{N}}$, and $T \in \mathbb{N}^* \cup \{\infty\}$, we say that x follows the τ -pattern \mathbf{s} up to time T if

$$\left. \begin{array}{l} 0 \leq n < T \\ n \equiv 0 \pmod{\tau} \end{array} \right\} \Rightarrow f^{n+1}(x) \in \bigcup \mathfrak{F}^{s_n}.$$

Consider a \mathfrak{F} -segment $\mathbf{D} = \{D_i\}_{0 \leq i \leq T}$ of length T . Note that if a point D_0 follows the τ -pattern $\mathbf{s} \in \{+, -\}^{\mathbb{N}}$ up to time T then every point in D_0 follows the same τ -pattern. If in addition τ divides T then we say that the \mathfrak{F} -segment \mathbf{D} of length T follows the τ -pattern \mathbf{s} .

Let $\sigma : \{+, -\}^{\mathbb{N}} \rightarrow \{+, -\}^{\mathbb{N}}$ denote the one-sided shift to the left. The proof of the next lemma is straightforward.

Lemma 2.10 (Concatenation and patterns). *Let $\tau \in \mathbb{N}^*$ and $\mathbf{s} \in \{+, -\}^{\mathbb{N}}$. Suppose \mathbf{D} and \mathbf{E} are \mathfrak{F} -segments of respective lengths T and L such that:*

- \mathbf{D} follows the τ -pattern \mathbf{s} ,
- \mathbf{E} follows the τ -pattern $\sigma^{T/\tau}(\mathbf{s})$, and
- $E_0 \subset D_T$.

*Then the concatenation $\mathbf{D} * \mathbf{E}$ follows the τ -pattern \mathbf{s} .*

The purpose of the next lemma is to allow us to introduce periodic “noise” in the patterns followed by our orbits, while still allowing us to control Birkhoff averages:

Lemma 2.11. *Fix α_1 with $0 < \alpha_1 < \alpha$, where α is as in condition (FF1). Then there exists an integer $\tau = \tau(\alpha_1) > 1$ with the following properties. Given any member D of \mathfrak{F} , there exist four \mathfrak{F} -segments $\mathbf{D}^{+,+}$, $\mathbf{D}^{-,+}$, $\mathbf{D}^{+,-}$, $\mathbf{D}^{-,-}$ such that:*

- (a) *The entrances $D^{+,+}$, $D^{-,+}$, $D^{+,-}$, $D^{-,-}$ of the segments are contained in D ;*
- (b) *The segments have length τ ;*
- (c) *The segments follow the respective 1-patterns:*

$$(+, +, \dots, +), \quad (-, +, +, \dots, +), \quad (+, -, -, \dots, -), \quad (-, -, \dots, -);$$

- (d) *For all $x \in D^{+,+} \cup D^{-,+}$ and $y \in D^{+,-} \cup D^{-,-}$, we have*

$$\frac{1}{\tau} \varphi_\tau(x) \geq \alpha_1 > 0 \quad \text{and} \quad \frac{1}{\tau} \varphi_\tau(y) \leq -\alpha_1 < 0.$$

Proof. Note that $\alpha \leq \beta_1 := \sup_K |\varphi|$. Choose an integer $\tau > 1$ such that

$$\alpha_1 < \frac{-\beta_1 + \alpha(\tau - 1)}{\tau}.$$

Take $D \in \mathfrak{F}$. Fix $s_1, s_2 \in \{+, -\}$. By condition (FF2) in the definition of flip-flop family, the image $f(D)$ contains a member D_1 of \mathfrak{F}^{s_1} . By induction, assume that a member D_i of \mathfrak{F} (where $0 < i < \tau$) has already been defined. Again by definition of flip-flop family, the image $f(D_i)$ contains an element D_{i+1} of \mathfrak{F}^{s_2} . Continuing in this way we eventually define a member D_τ of \mathfrak{F}^{s_2} . Let \mathbf{D}^{s_1, s_2} be the segment of length τ that has D_τ as exit. Then this segment has the required properties. \square

2.4. Constructing controlled sets. Fix sequences (β_k) and (α_k) of positive numbers converging to zero of the form

$$\beta_1 > \alpha_1 > \beta_2 > \alpha_2 > \dots$$

and such that $\beta_1 := \sup_K |\varphi|$ and α_1 is less than the constant α given by condition (FF1). Let $\tau = \tau(\alpha_1)$ be given by Lemma 2.11.

The core of the proof of Theorem 2.1 is the following lemma.

Lemma 2.12. *There exists a sequence of integers $t_0 < t_1 < t_2 < \dots$, where $t_0 = 1$, $t_1 = \tau$ and each element of the sequence is a multiple of its predecessor, such that the following properties hold:*

For every integer $k \geq 1$, every member D of \mathfrak{F} , and every pattern $\mathbf{s} \in \{+, -\}^{\mathbb{N}}$, there exist numbers $T_+, T_- \in \mathbb{N}^$ and \mathfrak{F} -segments \mathbf{D}^+ and \mathbf{D}^- of respective lengths T_+ and T_- such that:*

- (a) *The entrances of \mathbf{D}^+ and \mathbf{D}^- are contained in D ;*
- (b) *The lengths T_\pm are multiples of τ and satisfy $t_{k-1} < T_\pm \leq t_k$;*
- (c) *The segments \mathbf{D}^+ and \mathbf{D}^- are (β_i, t_i) -controlled for $i = 1, 2, \dots, k - 1$;*
- (d) *For all x in the entrance of \mathbf{D}^+ and all y in the entrance of \mathbf{D}^- , we have*

$$\alpha_k \leq \frac{1}{T_+} \varphi_{T_+}(x) \leq \beta_k, \tag{2.1}$$

$$-\beta_k \leq \frac{1}{T_-} \varphi_{T_-}(y) \leq -\alpha_k; \tag{2.2}$$

in particular, the segment \mathbf{D}^\pm is also (β_k, t_k) -controlled with $\mathcal{P}^\pm = \{0, T_\pm\}$ as a set of control times;

- (e) *The segments \mathbf{D}^+ and \mathbf{D}^- follow the τ -pattern \mathbf{s} .*

Proof. The sequence (t_k) is constructed by induction. Since $t_1 = \tau$, the conclusion of the lemma for $k = 1$ follows from Lemma 2.11, taking $T_\pm = \tau$.

Let $k \geq 2$ and assume that $t_1 < t_2 < \dots < t_{k-1}$ are already defined and that the conclusions of the lemma are met up to this point. Fix an element D of \mathfrak{F} . We will explain how to construct the announced segment \mathbf{D}^+ . The construction of \mathbf{D}^- is analogous and hence omitted.

The segment will be obtained as a concatenation of $m + \ell$ \mathfrak{F} -segments \mathbf{D}_i^+ , $i = 1, \dots, m$, and \mathbf{D}_{m+j}^- , $j = 1, \dots, \ell$, which are taken with the following properties:

- (I1) For each $i \in \{1, \dots, m\}$, \mathbf{D}_i^+ is a \mathfrak{F} -segment given by the induction hypothesis associated to $k - 1$, so its length $T_i^+ \leq t_{k-1}$ is a multiple of τ and all points in the entrance of the segment satisfies the inequality (2.1) with $k - 1$ in the place of k .
- (I2) For each $j \in \{1, \dots, \ell\}$, \mathbf{D}_{m+j}^- is a \mathfrak{F} -segment given by the induction hypothesis associated to $k - 1$, so its length $T_{m+j}^- \leq t_{k-1}$ is a multiple of τ and all points in the entrance of the segment satisfies the inequality (2.2) with $k - 1$ in the place of k .
- (I3) The following concatenation conditions hold:
 1. D contains the entrance of \mathbf{D}_1^+ ;
 2. For each $i \in \{1, \dots, m - 1\}$, the exit of \mathbf{D}_i^+ contains the entrance of \mathbf{D}_{i+1}^+ ;
 3. The exit of \mathbf{D}_m^+ contains the entrance of \mathbf{D}_{m+1}^- ;
 4. For each $j \in \{1, \dots, \ell - 1\}$, the exit of \mathbf{D}_{m+j}^- contains the entrance of \mathbf{D}_{m+j+1}^- ;
- (I4) Each segment \mathbf{D}_i^\pm follows the τ -pattern $\sigma^{S_{i-1}/\tau}(s)$, where $S_0 := 0$ and

$$S_i := \sum_{j=1}^i T_j^\pm, \text{ the } j\text{-th sign } \pm \text{ being } + \text{ if } j \leq m \text{ and } - \text{ otherwise.}$$

It follows from the induction hypothesis on k that for every $m > 0$ and $\ell > 0$ there are families of \mathfrak{F} -segments satisfying properties (I1)–(I4) above; just reason by induction on m and ℓ .

Given m and ℓ and families \mathbf{D}_i^+ , $i = 1, \dots, m$, and \mathbf{D}_{m+j}^- , $j = 1, \dots, \ell$ as above, consider their concatenation

$$\mathbf{D} = \mathbf{D}_1^+ * \dots * \mathbf{D}_m^+ * \mathbf{D}_{m+1}^- * \dots * \mathbf{D}_{m+\ell}^-.$$

Note that for every choice of m and ℓ and of the families \mathbf{D}_i^+ and \mathbf{D}_{m+j}^- , the concatenated \mathfrak{F} -segment \mathbf{D} satisfies conditions (a), (b) (taking $t_k = (m + \ell)t_{k-1}$, say), (c), and (e) in Lemma 2.12. The difficult part is to show that we can also obtain item (d) with an uniform bound on $m + \ell$. This involves careful choices of these numbers and of the families.

Before going into the details of this construction let us give a heuristic explanation. The average of φ along the first m segments is between α_{k-1} and β_{k-1} , in particular larger than the desired average (between α_k and β_k). The average along the next ℓ segments is negative, between $-\beta_{k-1}$ and $-\alpha_{k-1}$, and so the total average decreases. The idea is to stop when this average is between α_k and β_k . We implement this idea in precise terms in the next paragraphs.

2.4.1. Control of the distortion of the Birkhoff averages The first issue to be dealt with is that the Birkhoff averages depend on the point in the entrance of the concatenated segment \mathbf{D} . Lemma 2.4 provides the solution.

Choose a number η with

$$0 < \eta < \frac{\beta_k - \alpha_k}{3}$$

and let $N = N(\eta)$ be given by Lemma 2.4. Then, for all integers $m > N$ and $j > 0$, the variation of the Birkhoff averages of φ over the points in the entrance of a segment obtained by concatenation of $m + j$ segments is less than η .

Suppose we find ℓ such that the concatenation of the $m + \ell$ segments has a point in its entrance whose Birkhoff average $\varphi_{S_{m+\ell}/S_{m+\ell}}$ hits the interval $[\alpha_k + \eta, \beta_k - \eta]$,

which we call the *target*. In that case, it follows from the definition of η that the Birkhoff averages of φ at *all* points in the entrance of the segment belong to the interval (α_k, β_k) , as desired.

2.4.2. Hitting the target Note that as the Birkhoff sums corresponding to the segments \mathbf{D}_{m+j}^- are less than $-T_{m+j}^- \alpha_{k-1} < -\alpha_{k-1}$, if ℓ is large enough then the Birkhoff average $\varphi_{S_{m+\ell}}/S_{m+\ell}$ will be less than α_k . We do not want that these averages go from above β_k to below α_k without hitting the target $[\alpha_k + \eta, \beta_k - \eta]$.

To deal with this issue, consider a segment \mathbf{D}_{m+j}^- and recall that by property (I2) the length of the segment is $T_{m+j}^- \leq t_{k-1}$ and the Birkhoff sums $\varphi_{T_{m+i}^-}$ of points in the entrance of this segment belong to the interval $[-T_{m+j}^- \beta_{k-1}, -T_{m+j}^- \alpha_{k-1}]$.

An average $\varphi_{S_{m+j}}/S_{m+j}$ belongs to the target interval $[\alpha_k + \eta, \beta_k - \eta]$ if, and only if, the Birkhoff sum $\varphi_{S_{m+j}}$ belongs to interval $[S_{m+j}(\alpha_k + \eta), S_{m+j}(\beta_k - \eta)]$, whose length is $S_{m+j}(\beta_k - \alpha_k - 2\eta)$. This is the size of gap that we want to be large enough so that it cannot be jumped over in just one step.

By the definition of η we have

$$\beta_k - \alpha_k - 2\eta \geq \frac{\beta_k - \alpha_k}{3},$$

while a crude lower bound for S_{m+j} is m , so the gap is larger than

$$\frac{m(\beta_k - \alpha_k)}{3}.$$

On the other hand, any step $|\varphi_{S_{m+j}} - \varphi_{S_{m+j+1}}|$ is at most $T_{m+j}^- \beta_{k-1} \leq t_{k-1} \beta_{k-1}$. Thus choosing

$$m > \frac{3 t_{k-1} \beta_{k-1}}{\beta_k - \alpha_k},$$

the averages $\varphi_{S_{m+j}}/S_{m+j}$ cannot go from a value bigger than β_k to a value less than α_k without hitting the target. We now choose and fix such a number m that additionally satisfies $m > \max(N(\eta), t_{k-1})$, so that the previous reasoning applies. It follows that there exists an integer $\ell > 0$ such that the target is hit and so the desired (2.1) estimate holds for all points in the entrance of the segment $\mathbf{D} = \mathbf{D}_1^+ * \dots * \mathbf{D}_m^+ * \mathbf{D}_{m+1}^- * \dots * \mathbf{D}_{m+\ell}^-$. We take the least such number ℓ .

2.4.3. Upper bound for the hitting time The final step is to bound ℓ (independently of D etc) and thus be able to define t_k . Fix an integer

$$\ell_0 > \frac{m t_{k-1} \beta_{k-1}}{\alpha_{k-1}}.$$

We claim that for any point in the entrance of the concatenated segment $\mathbf{D}_1^+ * \dots * \mathbf{D}_m^+ * \mathbf{D}_{m+1}^- * \dots * \mathbf{D}_{m+\ell_0}^-$, its Birkhoff sum $\varphi_{S_{m+\ell_0}}$ is negative. Indeed, this follows by breaking the sum into two parts, the first being at most $S_m \beta_k \leq m t_{k-1} \beta_{k-1}$, and the second being at most $(S_{m+\ell_0} - S_m)(-\alpha_{k-1}) \leq -\ell_0 \alpha_{k-1}$.

It follows that $\ell < \ell_0$ and therefore $t_k := (m + \ell_0)t_{k-1}$ is an upper bound for the length $S_{m+\ell}$ of the segment \mathbf{D} . Notice that t_k does not depend on the member D nor the pattern s .

This completes the inductive construction, and so Lemma 2.12 is proved. \square

2.5. *End of the proof.* In this subsection we use the previous lemmas to prove Theorem 2.1. For simplicity we break the proof into two parts: in the first part we show that every $D \in \mathfrak{F}$ contains a point x that is controlled at all scales, and in the second part we explain how to find such a point with the additional property that $f|_{\omega(x)}$ has positive entropy.

2.5.1. *Finding a point that is controlled at all scales* Fix any member D of the flip-flop family \mathfrak{F} . By Lemma 2.12, we can find a monotone sequence (β_i) of positive numbers converging to zero, a monotone sequence (t_i) of positive integers converging to infinity, and a sequence of \mathfrak{F} -segments (\mathbf{D}_k^+) such that:

- The entrance of \mathbf{D}_k^+ is contained in D ;
- The length T_k^+ of the segment \mathbf{D}_k^+ goes to infinity with k ;
- Every point in the entrance of \mathbf{D}_k^+ is (β_i, t_i, T_k^+) -controlled, for every i with $1 \leq i \leq k$.

Choose a point x_k in the entrance of \mathbf{D}_k^+ . Let $x \in D$ be any accumulation point of the sequence (x_k) . Since $x_k \in \bigcap_{i=0}^{T_k^+} f^{-i}(K)$ and $T_k^+ \rightarrow \infty$, we conclude that the orbit of x does not leave K . We will show that x is (β_i, t_i, ∞) -controlled for every i .

Suppose x is the limit of a subsequence (x_{k_j}) . Let i be fixed. By construction, for every sufficiently large j , the point x_{k_j} is $(\beta_i, t_i, T_{k_j}^+)$ -controlled. Let $\mathcal{P}_j \subset \{0, \dots, T_{k_j}^+\}$ be the corresponding set of control times. The n -th element of \mathcal{P}_j is bounded by nt_i , so that there are finitely many possibilities for this value. By a standard diagonal argument, there exists a (strictly increasing) subsequence $\{j_\ell\}_{\ell \in \mathbb{N}}$ and an infinite set $\mathcal{P} \subset \mathbb{N}$ such that the first ℓ elements of \mathcal{P}_{j_ℓ} and \mathcal{P} coincide.

By continuity, the average of φ along the orbit of x between two successive times in \mathcal{P} is the limit as $\ell \rightarrow \infty$ of the averages of φ along the orbit of x_{k_ℓ} between the same times. This implies that all these averages belong to $[-\beta_i, \beta_i]$, proving that x is (β_i, t_i, ∞) -controlled with set of control times \mathcal{P} .

We have proved the existence of a point $x \in D$ that is controlled at all scales, thus proving Theorem 2.1 up to the part of positive entropy,

2.5.2. *Positive entropy* Let $\mathbf{s} = (s_0, s_1, \dots)$ be a point in the symbolic space $\{+, -\}^{\mathbb{N}}$ whose orbit under the shift σ is dense.

By construction, the lengths T_k^+ of the \mathfrak{F} -segments \mathbf{D}_k^+ considered above are all multiples of a fixed integer τ . Moreover, by property (e) in Lemma 2.12, we can choose each segment \mathbf{D}_k^+ following the τ -pattern $(s_0, s_1, \dots, s_{T_k^+})$.

Let $\alpha > 0$ be the constant given by condition (FF1), and define two disjoint compact subsets of K :

$$K^+ := \{x \in K; \varphi(x) \geq \alpha\} \quad \text{and} \quad K^- := \{x \in K; \varphi(x) \leq -\alpha\}.$$

By definition, every member of \mathfrak{F} is contained either in K^+ or in K^- . As a consequence, every point in the orbit of x belongs either to K^+ or to K^- . Thus

$$\omega(x) \subset \bigcap_{n \in \mathbb{N}} f^{-n}(K^+ \cup K^-).$$

Define a map $\Pi_\tau : \omega(x) \rightarrow \{+, -\}^{\mathbb{N}}$ by

$$\Pi_\tau(y) := (\pi_i(y))_{i \in \mathbb{N}} \quad \text{where} \quad \pi_i(y) = \begin{cases} + & \text{if } f^{\tau i}(y) \in K^+, \\ - & \text{if } f^{\tau i}(y) \in K^-. \end{cases}$$

In other words, $\Pi_\tau(y)$ is the itinerary of y under iterations of f^τ with respect to the partition $\{\omega(x) \cap K^+, \omega(x) \cap K^-\}$ of the set $\omega(x)$. The map Π_τ is continuous and satisfies $\Pi_\tau \circ f^\tau = \sigma^\tau \circ \Pi_\tau$.

Let us show that Π_τ is also onto. It is sufficient to show that its image intersects every cylinder in $\{+, -\}^{\mathbb{N}}$. Consider a finite word $\varepsilon = (\varepsilon_0, \dots, \varepsilon_n)$ over the alphabet $\{+, -\}$. As the pattern $\mathbf{s} = (s_i)_{i \in \mathbb{N}}$ has a dense orbit under the shift, there is a sequence $k_i \rightarrow +\infty$ so that $(s_{k_i}, \dots, s_{k_i+n}) = \varepsilon$. Let y be any accumulation point of the sequence $(f^{\tau k_i}(x))_i$, so that in particular $y \in \omega(x)$. Then $(\pi_0(y), \dots, \pi_n(y)) = \varepsilon$. This shows that the image of Π_τ intersects the cylinder corresponding to ε , and we conclude that Π_τ is onto.

Hence we have proved that the restriction of f^τ to $\omega(x)$ is topologically semiconjugate to the one-sided full-shift on 2 symbols, and in particular has topological entropy $h_{\text{top}}(f^\tau|_{\omega(x)}) \geq \log 2$. In particular, $h_{\text{top}}(f|_{\omega(x)}) \geq \tau^{-1} \log 2$ is positive, as we wanted to show.

The proof of Theorem 2.1 is complete. As explained before, Theorem 4 follows.

3. Blenders

In this section we introduce the notion of a *dynamical blender*. This definition involves three main ingredients: a space of discs (Sect. 3.1), invariant families of discs (Sect. 3.2), and invariant cone fields (Sect. 3.3). Thereafter with these ingredients on hand we will define dynamical blenders in Sect. 3.4.

3.1. The space of discs. Let M be a compact Riemannian manifold of dimension d . For each $i \in \{1, \dots, d - 1\}$ we denote by $\mathcal{D}^i(M)$ the set of i -dimensional (closed) discs C^1 -embedded in M . We endow the space $\mathcal{D}^i(M)$ with the following C^1 -topology: given a disc $D \in \mathcal{D}^i(M)$ that is the image of an embedding $\varphi : \mathbb{D}^i \rightarrow M$ (here \mathbb{D}^i is the closed unit disc in \mathbb{R}^i) a basis of neighborhoods of it consists of the sets $\{\psi(\mathbb{D}^i) : \psi \in \mathcal{V}\}$, where \mathcal{V} runs over the neighborhoods of φ in the space of embeddings. Alternatively, we can use the sets of the form $\{f(D) : f \in \mathcal{W}\}$, where \mathcal{W} runs over the neighborhoods of the identity in $\text{Diff}^1(M)$.

We now show that the topological space $\mathcal{D}^i(M)$ can be metrized with a distance that behaves nicely with respect to the composition of diffeomorphisms.

Proposition 3.1. *There is a distance $\delta(\cdot, \cdot)$ inducing the C^1 -topology in $\mathcal{D}^i(M)$ that satisfies the following property. For every $\varepsilon > 0$ and $K > 0$ there exists $\eta > 0$ such that for every pair of diffeomorphisms $f, g \in \text{Diff}^1(M)$ whose derivatives and their inverses are bounded by K it holds*

$$d_{C^1}(f, g) < \eta \implies \delta(f(D), g(D)) < \varepsilon$$

for every disc $D \in \mathcal{D}^i(M)$.

The rest of this subsection is dedicated to the proof of this proposition.

3.1.1. *The distance δ* For each $j \in \{1, \dots, d - 1\}$ let $\mathcal{G}_j(M) \rightarrow M$ be the fiber bundle over M whose fiber over a point $x \in M$ is the j^{th} -Grassmannian manifold of the tangent space $T_x M$. In other words, a point $P \in \mathcal{G}_j(M)$ whose projection in M is the point x is a subspace of dimension j of the vector space $T_x M$. Recall that $\mathcal{G}_j(M)$ is a compact manifold naturally endowed with a metric associated to the metric on M .

Given a disc $D \in \mathcal{D}^i(M)$, we use the following notations:

- $TD \subset \mathcal{G}_i(M)$ denotes the compact subset $TD := \{(x, T_x D)\}_{x \in D}$;
- $T\partial D \subset \mathcal{G}_{i-1}(M)$ denotes the compact subset $T\partial D := \{(x, T_x \partial D)\}_{x \in \partial D}$.

The distance δ is defined as follows, given a pair of discs $D_1, D_2 \in \mathcal{D}^i(M)$ we let

$$\delta(D_1, D_2) := d_{\text{Haus}}(TD_1, TD_2) + d_{\text{Haus}}(T\partial D_1, T\partial D_2),$$

where d_{Haus} denotes the Hausdorff distance between compact subsets of a metric space.

Remark 3.2.

- (a) δ defines a distance in $\mathcal{D}^i(M)$.
- (b) The distance δ is continuous with respect to the C^1 -topology, that is, the map $(D_1, D_2) \mapsto \delta(D_1, D_2)$ is continuous in $\mathcal{D}^i(M)^2$. This follows noting that the maps $D \mapsto TD$ and $D \mapsto T\partial D$ are both continuous from $\mathcal{D}^i(M)$ (endowed with the C^1 -topology) to the spaces $\mathcal{G}_i(M)$ and $\mathcal{G}_{i-1}(M)$, respectively, (endowed with the Hausdorff topologies).

We now see that the distance δ satisfies the ‘‘continuity’’ property in Proposition 3.1.

Lemma 3.3. *For every $\varepsilon > 0$ and $K > 0$ there exists $\eta > 0$ such that for every pair of diffeomorphisms $f, g \in \text{Diff}^1(M)$ whose derivatives and their inverses are bounded by K it holds*

$$d_{C^1}(f, g) < \eta \implies \delta(f(D), g(D)) < \varepsilon$$

for every disc $D \in \mathcal{D}^i(M)$.

Proof. It is enough to note that given any K and $\varepsilon > 0$ there is $\eta > 0$ such that given any $f, g \in \text{Diff}^1(M)$ whose derivatives and their inverses of are bounded by K and are η -close one has that for every $j \in \{1, \dots, d - 1\}$ and every point $(x, P) \in \mathcal{G}_j(M)$ it holds

$$d(Df(x)(P), Dg(x)(P)) < \varepsilon.$$

□

3.1.2. *The distance δ defines the C^1 -topology.* To conclude the proof of Proposition 3.1 we are left to prove the following:

Lemma 3.4. *Given any disk $D_0 \in \mathcal{D}^i(M)$ and any C^1 -neighborhood \mathcal{U} of D_0 there is $\varepsilon > 0$ such that if $\delta(D_1, D_0) < \varepsilon$ then $D_1 \in \mathcal{U}$.*

Proof. Choose a smooth embedding $\mathbb{D}^i \times \mathbb{D}^{d-i} \rightarrow M$ such that D_0 is contained in the (interior of) image of the graph Γ of a map from \mathbb{D}^i to \mathbb{D}^{d-i} . For $\varepsilon > 0$ small enough every disc D_1 with $\delta(D_1, D_0) < \varepsilon$ is contained in $\mathbb{D}^i \times \mathbb{D}^{d-i}$ and is transverse to the fibers $\{x\} \times \mathbb{D}^{d-i}$. We will prove the following:

Lemma 3.5. *With the notation above, for every $\varepsilon > 0$ small enough, the projection of D_1 in the graph Γ is a diffeomorphism of D_1 into a disc contained in Γ whose boundary is C^1 -close to ∂D_0 .*

Let us observe that this lemma implies Lemma 3.4. For that just note that the disc D_1 is the image by a diffeomorphisms C^1 -close to the identity of a disc in Γ which is C^1 -close to D_0 , this implies the proposition. We are left to prove Lemma 3.5. \square

Proof of Lemma 3.5. Consider a small tubular neighborhood $\Delta \subset \mathbb{D}^i \times \mathbb{D}^{d-i}$ of ∂D_0 whose projection $\Delta \rightarrow \partial D_0$ commutes with the projection on Γ .

As, by hypothesis, $T\partial D_1$ is δ -close to $T\partial D_0$ the boundary ∂D_1 is contained in the tubular neighborhood Δ and is transverse to the fibers of Δ . As a consequence, the projection π from ∂D_1 to ∂D_0 along the fibers of Δ is a C^1 -covering map.

Claim 3.6. *The projection $\pi : \partial D_1 \rightarrow \partial D_0$ is a C^1 -diffeomorphism.*

Proof. If $i > 2$ then ∂D_0 is simply connected and then the projection is a diffeomorphisms. Hence the unique case where the claim is not trivial is when $i = 2$. Thus assume $i = 2$ and that the projection is a covering with k sheets. We need to see that $k = 1$.

Consider the projection of ∂D_1 in \mathbb{D}^2 along the vertical fibers $\{x\} \times \mathbb{D}^{d-2}$. On the one hand, this projection is a closed immersed curved of \mathbb{D}^2 whose index is precisely k , where the index is the number of turns made by the tangent direction of the curve when one goes around the curve. On the other hand, the index of the boundary of an immersed disc of \mathbb{D}^2 is 1. Thus $k = 1$, ending the proof of the claim. \square

Using the claim we get that the projection of ∂D_1 in Γ is an embedded $i - 1$ sphere that bounds an i -disc \tilde{D} in Γ . The disc D_1 is a graph of a map defined on Γ close to the identity. This implies that D_1 is in a small neighborhood of D_0 , ending the proof the lemma. \square

The proof of Proposition 3.1 is now complete.

3.2. Strictly invariant families of discs and robustness. In this section we introduce strictly invariant families of discs and see that this property persists after small perturbations of the diffeomorphisms.

Given a family of discs $\mathcal{D} \subset \mathcal{D}^i(M)$ and $\eta > 0$ we denote by $\mathcal{V}_\eta^\delta(\mathcal{D})$ the open η -neighborhood of \mathcal{D} with respect to the distance δ , that is,

$$\mathcal{V}_\eta^\delta(\mathcal{D}) := \{D \in \mathcal{D}^i(M) : \delta(D, \mathcal{D}) < \eta\}.$$

Definition 3.7 (*Strictly f -invariant families of discs*). Let f be a diffeomorphism. A family of discs $\mathcal{D} \subset \mathcal{D}^i(M)$ is *strictly f -invariant* if there is $\varepsilon > 0$ such that for every disc $D_0 \in \mathcal{V}_\varepsilon^\delta(\mathcal{D})$ there is a disc $D_1 \in \mathcal{D}$ with $D_1 \subset f(D_0)$. The number ε is *the strength* of the strict invariance.

To be strictly invariant is a C^1 -robust property of a family of discs:

Lemma 3.8. *Let f be a diffeomorphism and $\mathcal{D} \subset \mathcal{D}^i(M)$ a strictly f -invariant family of discs of strength $\varepsilon > 0$. Then for every $0 < \mu < \varepsilon$ there exists $\eta > 0$ such that the family $\mathcal{D}_\mu = \mathcal{V}_\mu^\delta(\mathcal{D})$ is strictly g -invariant with strength $\varepsilon - \mu$ for every $g \in \text{Diff}^1(M)$ which is η - C^1 -close to f .*

Proof. Let η (associated to μ) be given by Lemma 3.3, then for every diffeomorphism g which is η - C^1 -close to f and every disc in $D \in \mathcal{D}^i(M)$ it holds $\delta(f(D), g(D)) < \mu$.

Take any disc $D \in \mathcal{V}_{\varepsilon-\mu}^\delta(\mathcal{D}_\mu)$ and note that $D \in \mathcal{V}_\varepsilon^\delta(\mathcal{D})$. As the family \mathcal{D} is strictly f -invariant with strength ε , the set $f(D)$ contains a disc in $D_1 \in \mathcal{D}$. Consider the disc $D_0 = f^{-1}(D_1) \subset D$. Now, by the choice of η , $\delta(g(D_0), f(D_0)) = \delta(g(D_0), D_1) < \mu$. Thus $g(D_0) \in \mathcal{D}_\mu$ and hence $g(D)$ contains a disc of \mathcal{D}_μ , concluding the proof of the lemma. \square

3.3. Strictly invariant cone fields. A subset C of a vector space E is a *cone of index i* if there are a splitting $E = E_1 \oplus E_2$ with $\dim E_1 = i$ and a norm $\|\cdot\|$ on E such that

$$C = \{v_1 + v_2 : v_i \in E_i, \|v_2\| \leq \|v_1\|\}.$$

A cone C' is *strictly contained* in the cone C above if there exists $\alpha > 1$ such that

$$C' \subset C_\alpha = \{v_1 + v_2 : v_i \in E_i, \|v_2\| \leq \alpha^{-1}\|v_1\|\} \subset C.$$

A *cone field of index i* defined on a subset V of a compact manifold M is a continuous assignment $x \mapsto \mathcal{C}(x) \subset T_x M$ of a cone of index i for each $x \in V$. Given $f \in \text{Diff}^1(M)$, we say that this cone field is *strictly Df -invariant* if $Df(x)(\mathcal{C}(x))$ is strictly contained in $\mathcal{C}(f(x))$ for every $x \in V \cap f^{-1}(V)$.

Lemma 3.9. *Let f be a diffeomorphism and \mathcal{C} a strictly Df -invariant cone field defined on a compact set $V \subset M$. Then there is a C^1 -neighborhood \mathcal{U} of f so that \mathcal{C} is strictly Dg -invariant for every $g \in \mathcal{U}$.*

Proof. It suffices to note that, since V is compact, the set $g^{-1}(V) \cap V$ depends upper semi-continuously on g , that is, for g C^0 -close to f , that set is contained in a small neighborhood of $V \cap f^{-1}(V)$. \square

3.4. Geometric and dynamically defined blenders. We next introduce the notion of a *dynamical blender* and state some of its properties which are pertinent in our context. As a motivation let us first recall the definition of a *geometric blender* as defined in [BDV, Definitions 6.9 and 6.11].

Definition 3.10 (Geometric blender). Let f be a diffeomorphism and Λ a compact f -invariant set. We say that Λ is a *geometric cu-blender of uu-index i* if it is (uniformly) hyperbolic with u-index strictly larger than i and there exist

- An open family $\mathcal{D} \subset \mathcal{D}^i(M)$ of discs and
- A C^1 -neighborhood \mathcal{U} of f

such that $D \cap W^s(\Lambda_g) \neq \emptyset$ for every $g \in \mathcal{U}$ and every $D \in \mathcal{D}$, here Λ_g is the continuation of Λ for g .

The family \mathcal{D} is called the *superposition region* and the C^1 -neighborhood \mathcal{U} is the *validity domain* of the blender.

The definition of a geometric blender is well suited for the study of the generation of robust cycles [BD2]. A geometric blender is also an important mechanism to obtain robust transitivity [BD1]. Indeed, these constructions were the main motivations for its definition. However, the fact that robustness forms part of its definition makes difficult

in general to check if a given hyperbolic set is a geometric blender, restraining further applications. On the other hand, all known mechanisms producing geometric blenders involve the dynamical property of the existence of a strictly invariant family of discs.⁴ This motivates the following definition:

Definition 3.11 (*Dynamical blender*). Let f be a diffeomorphism. A compact f -invariant set Λ is a *dynamically defined cu-blender* (or simply *dynamical blender*) of *uu-index* i if the following holds:

- (a) There is an open neighborhood U of Λ such that Λ is the maximal invariant set of f in the closure of U ,

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(\overline{U});$$

- (b) The set Λ is (uniformly) hyperbolic with u-index strictly larger than i ;
- (c) The set Λ is transitive (thus it is a hyperbolic basic set);
- (d) There is a strictly Df -invariant cone field C^{uu} of index i defined on \overline{U} ; and
- (e) There is a strictly f -invariant family of discs $\mathcal{D} \subset \mathcal{D}^i(M)$ with strength $\varepsilon > 0$ such that every disc in $\mathcal{V}_\varepsilon^\delta(\mathcal{D})$ is contained in U and tangent to C^{uu} .

We say that U is the *domain of the blender*, C^{uu} is its *strong unstable cone field*, and \mathcal{D} is its *strictly invariant family of discs* with strength ε . To emphasize the role of these objects in the definition of a geometrical blender we write $(\Lambda, U, C^{uu}, \mathcal{D})$. To emphasize the strength ε of the family of discs we write $(\Lambda, U, C^{uu}, \mathcal{D}, \varepsilon)$.

Remark 3.12. In the definition of dynamical blender above, one does not require that the discs in the strictly invariant family \mathcal{D} intersect the stable manifold $W^s[\Lambda]$ (which is the main property of the usual blenders): this property is indeed a consequence of the invariance, as proved in Lemma 3.14 below.

Remark 3.13. Note that the hyperbolic splitting of a dynamical blender Λ as above can be refined in order to get a partially hyperbolic splitting of the form $T_\Lambda M = E^{uu} \oplus E^c \oplus E^{ss}$, where E^{uu} has dimension i and E^c has positive dimension and is expanding. Actually, all dynamical blenders we consider in this paper have one-dimensional central (unstable) bundle E^c .

We have that dynamical blenders are geometric ones and are robust:

Lemma 3.14 (*Blenders are robust*). Assume that $(\Lambda, U, C^{uu}, \mathcal{D}, \varepsilon)$ is a dynamical blender of a diffeomorphism f . Let $\mathcal{D}_{\varepsilon/2} = \mathcal{V}_{\varepsilon/2}^\delta(\mathcal{D})$ and for g C^1 -close to f let Λ_g be the hyperbolic continuation of Λ for g . Then there is a C^1 -neighborhood \mathcal{U} of f such that for every diffeomorphism $g \in \mathcal{U}$ the following holds:

- The 4-tuple $(\Lambda_g, U, C^{uu}, \mathcal{D}_{\varepsilon/2})$ is a dynamical blender;
- The hyperbolic set Λ_g is a geometric blender with superposition region $\mathcal{D}_{\varepsilon/2} = \mathcal{V}_{\varepsilon/2}^\delta(\mathcal{D})$ and validity domain \mathcal{U} .

By analogy with the terminology for geometric blenders, the neighborhood \mathcal{U} is called a *validity domain* of the dynamical blender.

⁴ Besides the references above, see for instance the *criterion of the recurrent compact set* given by [ACW, Proposition 7.3], which is based on the previous works [MS,MY].

Proof of Lemma 3.14. We choose a small neighborhood \mathcal{U} of f such that for every g in \mathcal{U} the hyperbolic continuation Λ_g of Λ is well defined and equal to the maximal g -invariant set in \overline{U} . By shrinking \mathcal{U} if necessary, we can also assume that the cone field \mathcal{C}^{uu} is still strictly Dg -invariant (Lemma 3.9) and that the family $\mathcal{D}_{\varepsilon/2} = \mathcal{V}_{\varepsilon/2}^{\delta}(\mathcal{D})$ is still strictly g -invariant with strength $\varepsilon/2$ (Lemma 3.8). Summarizing, the 4-tuple $(\Lambda_g, U, \mathcal{C}^{\text{uu}}, \mathcal{D}_{\varepsilon/2}, \varepsilon/2)$ is a dynamical blender.

To prove the second part of the lemma, first note that by definition $\mathcal{D}_{\varepsilon/2}$ is an open family of discs. Take any $g \in \mathcal{U}$ and consider any disc $D_0 \in \mathcal{D}_{\varepsilon/2}$. By the strict g -invariance of this family (first part of the lemma) we have that $g(D_0)$ contains a disc $D_1 \in \mathcal{D}_{\varepsilon/2}$. Arguing inductively we construct a sequence of discs $(D_n)_{n \in \mathbb{N}}$ in $\mathcal{D}_{\varepsilon/2}$ so that $D_{n+1} \subset g(D_n)$.

By construction, the set $\bigcap g^{-n}(D_n)$ is nonempty as it is the intersection of a decreasing sequence of nonempty compact sets. Moreover, the forward orbit of any point x in such an intersection is contained in U . Hence, as Λ_g is the maximal invariant set in U , $x \in W^s(\Lambda_g)$. This shows that every disc in the family $\mathcal{D}_{\varepsilon/2}$ intersects $W^s(\Lambda_g)$. This proves that Λ_g is a geometrical blender with validity domain \mathcal{U} and superposition region $\mathcal{D}_{\varepsilon/2}$. \square

Scholium 3.15. If $(\Lambda, U, \mathcal{C}^{\text{uu}}, \mathcal{D})$ is a dynamical blender then there exists a disc $D_{\infty} \in \mathcal{D}^i(M)$ contained in the local strong unstable manifold of a point of Λ that is C^1 -accumulated by discs in \mathcal{D} .

Caveat 3.16. In the definition of a dynamical blender, there are two properties that we require only for convenience in this paper and that may be removed if necessary for further uses:

- (a) *Transitivity of the hyperbolic set Λ* We will use this condition in Proposition 4.5 to show that a flip-flop configuration contains a robust cycle.
- (b) *The set Λ is maximal invariant in \overline{U} and contained in U* If this hypothesis is removed then Lemma 3.14 must be slightly modified as follows: For every g C^1 -close to f the blender for g is the maximal invariant set of g in \overline{U} instead of the continuation of Λ for g .

3.5. Dynamical blenders and towers. In many situations, as in this paper, it is more natural and convenient to construct dynamical blenders for some induced maps instead of blenders for the given diffeomorphism. In this subsection, we show that a dynamical blender for an induced map of a diffeomorphism f leads to a dynamical blender of the initial f and see how these blenders are naturally and appropriately related.

3.5.1. Induced maps Let f be a diffeomorphism of a compact manifold M , $U, U_1, \dots, U_k \subset M$ be nonempty open sets, and n_1, \dots, n_k be positive integers such that:

- The sets \overline{U}_i are pairwise disjoint and are contained in U ;
- $f^j(\overline{U}_i) \cap \overline{U} = \emptyset$ for all $0 < j < n_i$; and
- $f^{n_i}(\overline{U}_i) \subset U$.

The map $F: \bigcup_1^k \overline{U}_i \rightarrow U$ defined by $F(x) := f^{n_i}(x)$ if $x \in \overline{U}_i$ is called an *induced map* of f . We say that U_1, \dots, U_k are the *domains* of F and that n_1, \dots, n_k are the associated *return times*. For such an induced map we use the notation $F = [f, U, (U_i)_{i=1}^k, (n_i)_{i=1}^k]$.

A *safety domain* of the induced map $F = [f, U, (U_i)_{i=1}^k, (n_i)_{i=1}^k]$ is a set of the form

$$V = \bigcup_{i=1}^k \left(\bigcup_{j=0}^{n_i-1} V_{i,j} \right),$$

where the sets $V_{i,j}$ are open and satisfy the following conditions:

- The sets $\bar{V}_{i,j}, i \in \{1, \dots, k\}$ and $j \in \{0, \dots, n_i - 1\}$, are pairwise disjoint;
- $\bar{U}_i \subset V_{i,0} \subset U$ for every $i \in \{1, \dots, k\}$;
- $f(\bar{V}_{i,j}) \subset V_{i,j+1}$ for every $i \in \{1, \dots, k\}$ and $j \in \{0, \dots, n_i - 2\}$; and
- $f(\bar{V}_{i,n_i-1}) \subset U$ for every $i \in \{1, \dots, k\}$.

Lemma 3.17 (Existence of safety domains). *Let $F = [f, U, (U_i)_{i=1}^k, (n_i)_{i=1}^k]$ be an induced map. Then for every family of neighborhoods \tilde{U}_i of $\bar{U}_i, i = 1, \dots, k$, there is a safety domain*

$$V = \bigcup_{i=1}^k \left(\bigcup_{j=0}^{n_i-1} V_{i,j} \right)$$

such that $\bar{V}_{i,j} \subset f^j(\tilde{U}_i)$ for every $i \in \{1, \dots, k\}$ and $j \in \{0, \dots, n_i - 1\}$.

Proof. By definition, the compact set \bar{U}_i is contained in the open set U and the compact sets $f^j(\bar{U}_i), j \in \{1, \dots, n_i - 1\}$, are pairwise disjoint and disjoint from \bar{U} . Hence there is a decreasing basis of open neighborhoods V_i^n of \bar{U}_i having the same properties and such that \bar{V}_i^{n+1} is contained in V_i^n . Now, for any k the union of the family of opens $V_{i,j} = f^j(V_i^{k+j})$ is a safety domain. Moreover, if k is large enough then $\bar{V}_{i,j} \subset f^j(\tilde{U}_i)$ as announced. \square

3.5.2. Induced map and strictly invariant cone fields Consider an induced map $F = [f, U, (U_i)_{i=1}^k, (n_i)_{i=1}^k]$. We say that a cone field \mathcal{C} defined on U is strictly invariant for F if for every $i \in \{1, \dots, k\}$ and every $x \in \bar{U}_i$ the cone $Df^{n_i}(\mathcal{C}(x)) = DF(\mathcal{C}(x))$ is strictly contained in the cone $\mathcal{C}(f^{n_i}(x)) = \mathcal{C}(F(x))$.

Lemma 3.18 (Induced extended cone field). *Let $F = [f, U, (U_i)_{i=1}^k, (n_i)_{i=1}^k]$ be an induced map and \mathcal{C} a cone field defined on U that is strictly invariant under F . Then there is a safety domain $V = \bigcup_{i=1}^k \left(\bigcup_{j=0}^{n_i-1} V_{i,j} \right)$ of F and a cone field $\tilde{\mathcal{C}}$ defined on \bar{V} such that*

- $\tilde{\mathcal{C}}$ coincides with \mathcal{C} in \bar{U} and
- The cone field $\tilde{\mathcal{C}}$ is strictly Df -invariant.

Proof. Fix $N > \max_{i \in \{1, \dots, k\}} n_i$. By compactness and uniform continuity, one may find a sequence of cone fields $\mathcal{C}_j, j \in \{0, \dots, N\}$, defined on \bar{U} such that the cones of \mathcal{C} are strictly contained in cones of \mathcal{C}_0 , the cones of \mathcal{C}_j are strictly contained in cones of \mathcal{C}_{j+1} for every $j \in \{0, \dots, N - 1\}$, and the cones of $Df^{n_i}(\mathcal{C}_N(x))$ are strictly contained in cones of $\mathcal{C}(F(x))$ for every $i \in \{1, \dots, k\}$ and every $x \in \bar{U}_i$. Again by compactness, the cone $Df^{n_i}(\mathcal{C}_N(x))$ is strictly contained in $\mathcal{C}(F(x))$ for x in a small neighborhood on \bar{U}_i .

Using Lemma 3.17 we can choose a sufficiently small safety neighborhood V of F with elements $V_{i,j}$ and define the cone field $\tilde{\mathcal{C}}$ as $Df^j(\mathcal{C}_j)$ on the set $\bar{V}_{i,j}$. By construction the cone field $\tilde{\mathcal{C}}$ is strictly Df -invariant. \square

3.5.3. Induced maps and dynamical blenders We now state a proposition that relates the dynamical blender of an induced map of a diffeomorphism (see the definition below) and the dynamical blenders of the initial dynamics. We begin with a remark about families of invariant discs. For the following statements, recall that $\mathcal{V}_\varepsilon^\delta(\cdot)$ denotes the open ε -neighborhood for the distance δ in the space of discs in Sect. 3.1.1.

Remark 3.19. Let $\mathfrak{D}, \mathfrak{D}_1, \dots, \mathfrak{D}_k \subset \mathcal{D}^i(M)$ be families of i -dimensional discs and n_1, \dots, n_k positive integers such that

$$\mathfrak{D} = \mathfrak{D}_1 \cup \dots \cup \mathfrak{D}_k$$

and there is $\varepsilon > 0$ such that for every $i \in \{1, \dots, k\}$ the image $f^{n_i}(D)$ of any disc $D \in \mathcal{V}_\varepsilon^\delta(\mathfrak{D}_i)$ contains a disc in \mathfrak{D} . Then the family of discs

$$\bigcup_{i=1}^k \mathfrak{D}_i \cup \bigcup_{i=1}^k \left(\bigcup_{j=1}^{n_i-1} f^j(\mathcal{V}_{\frac{j\varepsilon}{n_i}}^\delta(\mathfrak{D}_i)) \right)$$

is strictly f -invariant.

Proposition 3.20 (Induced blenders induce blenders). *Consider an induced map $F = [f, U, (U_i)_{i=1}^k, (n_i)_{i=1}^k]$ such that:*

- *The maximal invariant set*

$$\Lambda_F = \bigcap_{n \in \mathbb{Z}} F^n \left(\bigcup_{i=1}^k \bar{U}_i \right) \subset \bigcup_{i=1}^k U_i$$

is hyperbolic (for F) with u-index strictly larger than i .

- *There is a cone field \mathcal{C}^{uu} of index i defined on \bar{U} that is strictly invariant under F .*
- *There are families $\mathfrak{D}, \mathfrak{D}_1, \dots, \mathfrak{D}_k \subset \mathcal{D}^i(M)$ of i -dimensional discs and $\varepsilon > 0$ such that:*
 - The families $\mathcal{V}_\varepsilon^\delta(\mathfrak{D}), \mathcal{V}_\varepsilon^\delta(\mathfrak{D}_1), \dots, \mathcal{V}_\varepsilon^\delta(\mathfrak{D}_k)$ are contained in U, U_1, \dots, U_k , respectively, and are tangent to the cone field \mathcal{C}^{uu} ;*
 - Given any disc $D \in \mathfrak{D}$ there are $i \in \{1, \dots, k\}$ and a disc $D_0 \in \mathfrak{D}_i$ such that $D_0 \subset D$;*
 - For every $i \in \{1, \dots, k\}$ and every $D \in \mathcal{V}_\varepsilon^\delta(\mathfrak{D}_i)$, the set $f^{n_i}(D)$ contains a disc $D_0 \in \mathfrak{D}$.*

Then there is a safety domain V of F such that the maximal invariant set Λ_f of f in \bar{V} is hyperbolic and the 4-tuple $(\Lambda_f, V, \tilde{\mathcal{C}}^{\text{uu}}, \tilde{\mathfrak{D}})$ is a dynamical blender, where

- *$\tilde{\mathcal{C}}^{\text{uu}}$ is a strictly Df -invariant cone field defined over \bar{V} that extends \mathcal{C}^{uu} and*
- *$\tilde{\mathfrak{D}}$ is a strictly f -invariant family of discs contained in V such that $\{D \in \tilde{\mathfrak{D}}; D \subset \bar{U}\} = \bigcup_{i=1}^k \mathfrak{D}_i$.*

With the notation above, we say that $(\Lambda_F, U_1 \cup \dots \cup U_k, \mathcal{C}^{\text{uu}}, \mathfrak{D}_1 \cup \dots \cup \mathfrak{D}_k)$ is dynamical blender of the induced map F .

Proof. Using Lemma 3.18 we find a safety domain $V = \bigcup_{i=1}^k \left(\bigcup_{j=0}^{n_i-1} V_{i,j} \right)$ of F and extend the cone field \mathcal{C}^{uu} to a strictly Df -invariant cone field $\tilde{\mathcal{C}}^{\text{uu}}$ defined on \bar{V} . By assumption, for every disc $D \in \mathcal{V}_\varepsilon^\delta(\mathfrak{D}_i)$ the set $f^{n_i}(D)$ contains a disc $D_0 \in \mathfrak{D}$. As $V_{i,0}$

is a neighborhood of \overline{U}_i , the discs in $\mathcal{V}_\varepsilon^\delta(\mathcal{D}_i)$ are contained $V_{i,0}$. Thus, by definition of the safety neighborhood, the discs in $f^j(\mathcal{D}_i)$ are contained in $V_{i,j}$ and are tangent to the cone field $\tilde{\mathcal{C}}^{\text{uu}}$ (note that the cone field is strictly invariant). According to Remark 3.19,

$$\bigcup_{i=1}^k \mathcal{D}_i \cup \bigcup_{i=1}^k \left(\bigcup_{j=1}^{n_i-1} f^j(\mathcal{V}_{j\varepsilon}^\delta(\mathcal{D}_i)) \right)$$

is a strictly f -invariant family of discs contained in V and tangent to \mathcal{C}^{uu} that coincides with $\bigcup_{i=1}^k \mathcal{D}_i$ in U . This concludes the proof of the proposition. \square

4. From Flip-flop Configurations to Flip-flop Families: Proof of Theorem 5

We next introduce *flip-flop configurations* and prove that they are robust and generate robust cycles.

4.1. Flip-flop configurations.

Definition 4.1 (*Flip-flop configuration*). Let $(\Gamma, V, \mathcal{C}^{\text{uu}}, \mathcal{D})$ be a dynamical blender of uu-index i of a diffeomorphism f . Suppose q is a periodic point of u-index i . We say that $(\Gamma, V, \mathcal{C}^{\text{uu}}, \mathcal{D})$ and q form a *flip-flop configuration* if there exist:

- A disc Δ^u contained in the unstable manifold $W^u(q)$;
- A compact submanifold with boundary $\Delta^s \subset V \cap W^s(q)$

such that:

- (FFC0) The disc Δ^u belongs to the interior of the family \mathcal{D} ;
- (FFC1) $f^{-n}(\Delta^u) \cap \overline{V} = \emptyset$ for all $n > 0$;
- (FFC2) There is $N > 0$ such that $f^n(\Delta^s) \cap \overline{V} = \emptyset$ for all $n > N$. Moreover, if $x \in \Delta^s$ and $j > 0$ are such that $f^j(x) \notin V$ then $f^i(x) \notin \overline{V}$ for every $i \geq j$;
- (FFC3) $T_y W^s(q) \cap \mathcal{C}^{\text{uu}}(y) = \{0\}$ for every $y \in \Delta^s$;
- (FFC4) There exists a compact set K contained in the relative interior of Δ^s and $\varepsilon > 0$ such that every element D of \mathcal{D} intersects the set K at a point x whose distance to ∂D is larger than ε .

The sets Δ^u and Δ^s are the *unstable and stable connecting sets* of the flip-flop configuration, respectively. The compact set K is an ε -safe *stable connecting set*.

4.1.1. Flip-flop configurations are robust

The aim of this section is the following proposition.

Proposition 4.2. *Consider a dynamical blender $(\Gamma, V, \mathcal{C}^{\text{uu}}, \mathcal{D})$ and a saddle q of a diffeomorphism f in a flip-flop configuration. Consider a validity domain \mathcal{U} of the blender such that the hyperbolic continuations Γ_g of Γ and q_g of q are defined for every $g \in \mathcal{U}$. Then there exists a neighborhood \mathcal{V} of f contained in \mathcal{U} such that the dynamical blender $(\Gamma_g, V, \mathcal{C}^{\text{uu}}, \mathcal{D})$ and the saddle q_g are in a flip-flop configuration for every $g \in \mathcal{V}$.*

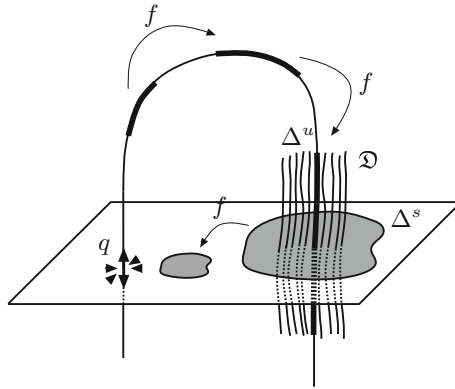


Fig. 4. Flip-flop configuration

Proof. Let Δ^s and Δ^u be the stable and unstable connecting set of the configuration and consider an ε -safe stable connecting set K ($\varepsilon > 0$ will be fixed below).

Note that compact parts of the unstable manifold $W^u(q_g, g)$ depend continuously on the diffeomorphism g . Thus condition (4.1) for f implies that a local unstable manifold $W_{loc}^u(q_g, g)$ of any g C^1 -close enough to f contains a disc Δ_g^u arbitrarily C^1 -close to $\Delta_f^u = \Delta^u$ (these discs depend continuously on f). Therefore the discs Δ_g^u are in the interior of \mathfrak{D} , obtaining the robustness of condition (4.1). Furthermore, the negative iterates of Δ_g^u by g remain close to the ones of Δ^u by f , hence they are disjoint from \bar{V} , proving (4.1).

Let $N > 0$ as in (4.1) for f . In the same way as above, for every g close enough to f , we can choose a continuation Δ_g^s of Δ^s that is simultaneously contained in a local stable manifold of q_g and in V and whose positive iterates $g^j(\Delta_g^s)$ are disjoint from ∂V and from \bar{V} for every $j > N$. Since the set $\bigcup_0^N g^j(\Delta_g^s)$ is compact and disjoint of the compact set ∂V we have that for every g close enough to f the points of $g^j(\Delta_g^s)$ cannot return to V after a first exit. In this way we get (4.1).

Note that condition (4.1) is robust and therefore Δ_g^s transverse to the cone field \mathcal{C}^{uu} .

For g close to f consider a continuation K_g of the safety set K contained in Δ_g^s . We prove property (4.1) with ε replaced by $\varepsilon/2$: the set K_g intersects every disc $D \in \mathfrak{D}$ at a point whose distance to ∂D is larger than $\varepsilon/2$. That is precisely the content of the next lemma.

Lemma 4.3. *Let \mathcal{C} be a cone field of index i and K_0 a sub-manifold of dimension $n - i$ with boundary that is transverse to \mathcal{C} . Consider $\varepsilon > 0$ and K a compact subset contained the interior of K_0 . Then there is a C^1 -neighborhood \mathcal{W} of K_0 such that any disc D of radius ε tangent to \mathcal{C} centered at some point $x \in K$ intersects every sub-manifold in \mathcal{W} .*

Sketch of proof. It is enough to prove the lemma for some small $\varepsilon > 0$. Note that after shrinking ε , if necessary, we can assume that the distance from K to ∂K_0 is larger than $\varepsilon > 0$. We can also assume that the distance between K_0 and the boundary of any disc of radius ε centered at some point of K and tangent to \mathcal{C} is larger than some $\mu > 0$.

Fix any disc D of radius ε as in the statement of the lemma. Note that every sub-manifold that is sufficiently C^1 -close to K_0 and is transverse to \mathcal{C} admits an isotopy to K_0 by sub-manifolds $K_t \in \mathcal{W}$. Thus, for small t , the intersection $K_t \cap D$ is just a point

x_t that depends continuously on t . The unique way of this point “vanishing” is to cross either the boundary of D or the boundary of K_t . We now see that these two possibilities are forbidden.

To discard the first possibility note that the choices of ε and μ above imply that if the neighborhood \mathcal{W} of K_0 is small enough then $x_t \notin \partial D$.

To eliminate the second possibility note that the choice of $\varepsilon > 0$ implies that if \mathcal{W} is small enough then the distance between K to ∂K_t is larger than $\mu/2$ for every K_t and thus $x_t \notin \partial K_t$. This concludes the proof of the lemma. \square

The proof of the robustness of flip-flop configurations is now complete. \square

The following presentation of flip-flop configurations will be very useful for us.

Proposition 4.4. *Consider a dynamical blender $(\Gamma, V, \mathcal{C}^{uu}, \mathcal{D})$ of uu-index i of a diffeomorphism f . Suppose that there are a periodic point q of f of u-index i , a disc Δ^u contained in the unstable manifold $W^u(q)$, and a compact sub-manifold with boundary $\Delta^s \subset V \cap W^s(q)$ satisfying conditions FFC1, FFC2, FFC3, FFC4, and*

(FFC0') Δ^u belongs to the family \mathcal{D} .

Then there is $\eta > 0$ such that $(V, \Gamma, \mathcal{C}^{uu}, \mathcal{V}_\eta^\delta(\mathcal{D}))$ is a dynamical blender in a flip-flop configuration with q such that Δ^u and Δ^s are its unstable and stable connecting sets.

Proof. First, by Lemma 3.14, $(\Gamma, V, \mathcal{C}^{uu}, \mathcal{V}_\eta^\delta(\mathcal{D}))$ is a dynamical blender of f for every $\eta > 0$ sufficiently small. We now see that this blender and q are in a flip-flop configuration.

To check that property FFC0 holds just note that condition FFC0p implies that Δ^u is a disc in the interior of $\mathcal{V}_\eta^\delta(\mathcal{D})$.

Note that conditions FFC1, FFC2 and FFC3 remain unchanged and thus there is nothing to prove.

Finally, the fact that property FFC4 still holds for the new blender is a direct consequence of Lemma 4.3. \square

4.1.2. Flip-flop configurations generate robust cycles

Proposition 4.5. *Consider a dynamical blender $(\Gamma, V, \mathcal{C}^{uu}, \mathcal{D})$ and a saddle q of a diffeomorphism f in a flip-flop configuration. Then Γ and q form a robust cycle and thus Γ is contained in the chain recurrence class of q .*

Proof. We first see that f has a cycle associated to Γ and q . We will see that the invariant manifolds of q and Λ meet cyclically. As Γ is transitive then q and Γ form a cycle.

Consider the connecting sets Δ^s and Δ^u and the ε -safety stable set of this configuration. Note that $\Delta^u \subset W^u(q)$ belongs to the interior of \mathcal{D} and thus it intersects $W^s(\Gamma)$. Hence $W^u(q) \cap W^s(\Gamma) \neq \emptyset$.

To see that $W^s(q) \cap W^u(\Gamma) \neq \emptyset$ note that by Scholium 3.15 there is a sequence of discs $D_n \in \mathcal{D}$ converging to a disc D contained in the unstable manifold of some point in Γ . Every disc D_n contains a point in the compact set $K \subset \Delta^s$. Thus $\Delta^s \cap D \neq \emptyset$ and hence $W^s(q) \cap W^u(\Gamma) \neq \emptyset$.

The robustness of the cycle now follows from the robustness of the flip-flop configuration (see Proposition 4.2). \square

4.2. *Partially hyperbolic neighborhoods for flip-flop configurations.* The next lemma is a standard consequence of transversality properties. Recall first Remark 3.13 that claims that a dynamical blender $(\Gamma, V, \mathcal{C}^{\text{uu}}, \mathcal{D})$ has a partially hyperbolic splitting $T_\Gamma M = E^{\text{uu}} \oplus E^c \oplus E^{\text{ss}}$, where E^{uu} and E^c are expanding bundles and E^{uu} is contained in \mathcal{C}^{uu} .

Lemma 4.6 (Flip-flop configurations and partial hyperbolicity). *Consider a diffeomorphism f having a periodic point q and a dynamical blender $(\Gamma, V, \mathcal{C}^{\text{uu}}, \mathcal{D})$ in a flip-flop configuration with connecting sets $\Delta^u \subset W^u(q)$ and $\Delta^s \subset W^s(q)$.*

Consider the closed set

$$\mathcal{O}(q) \cup \bar{V} \cup \bigcup_{k \geq 0} f^k(\Delta^s) \cup \bigcup_{k \leq 0} f^k(\Delta^u)$$

and for every small compact neighborhood U of it the maximal invariant set $\Lambda(U)$ of f in U . If the neighborhood U is sufficiently small then the set $\Lambda(U)$ contains the blender Γ and has a partially hyperbolic splitting of the form

$$T_{\Lambda(U)}M = \tilde{E}^{\text{uu}} \oplus \tilde{E}^{\text{cs}},$$

where \tilde{E}^{uu} is uniformly expanding and extends the sub-bundle E^{uu} defined over Γ and \tilde{E}^{cs} is a dominated bundle that extends the bundle $E^c \oplus E^{\text{ss}}$ defined over Γ .

Moreover, there is a strictly Df -invariant cone field over U that extends the cone field \mathcal{C}^{uu} of the bundle defined \bar{V} and whose vectors are uniformly expanded by Df .

For notational simplicity, we also denote the extension of the cone field in the lemma by \mathcal{C}^{uu} . The set U in the lemma is called a *partially hyperbolic neighborhood of the flip-flop configuration*.

Proof of Lemma 4.6. Since the blender Γ is, by definition, the maximal invariant set of f in the set \bar{V} , it is contained in $\Lambda(U)$. The existence of the partially hyperbolic splitting over $\Lambda(U)$ is a standard consequence of the transversality between Δ^s and the cone field \mathcal{C}^{uu} defined on \bar{V} , and the fact that the disc Δ^u is tangent to the strong unstable cone field \mathcal{C}^{uu} of Γ . \square

If in Lemma 4.6 we can write $\tilde{E}^{\text{cs}} = \tilde{E}^c \oplus \tilde{E}^{\text{ss}}$, where \tilde{E}^c is one-dimensional and expanding and \tilde{E}^{ss} is contracting, then the flip-flop configuration is called *split*. More precisely:

Definition 4.7 (*Split flip-flop configuration*). Consider a flip-flop configuration of a dynamical blender $(\Gamma, V, \mathcal{C}^{\text{uu}}, \mathcal{D})$ and a saddle q . This configuration is *split* if it has a partially hyperbolic neighborhood U such that $T_{\Lambda(U)}M = \tilde{E}^{\text{uu}} \oplus \tilde{E}^c \oplus \tilde{E}^{\text{ss}}$ is a dominated splitting where \tilde{E}^c is one-dimensional and uniformly expanding, \tilde{E}^{uu} is uniformly expanding, \tilde{E}^{ss} is uniformly contracting, and $\tilde{E}^{\text{uu}} \oplus \tilde{E}^c$ and \tilde{E}^{ss} extend the unstable and stable hyperbolic bundles of Γ , respectively.

The neighborhood U is called a *strict partially hyperbolic neighborhood of the split flip-flop configuration*.

Remark 4.8 (Robustness of split flip-flop configurations). It follows from Proposition 4.2 and the persistence of partially hyperbolic splittings that to have a split flip-flop configuration is a robust property.

4.3. Proof of Theorem 5. The aim of this section is to prove the criterion for zero center Lyapunov exponents in Theorem 5. Basically, we show that split flip-flop configurations “embody” flip-flop families associated to the logarithm of the central Jacobian. Thus Theorem 5 follows from the abstract criterion in Theorem 4. Actually, the split condition is only used to define the central Jacobian and the bulk of the work consists in proving the following:

Proposition 4.9 (Flip-flop families associated to flip-flop configurations). *Consider a diffeomorphism f having a periodic point q and a dynamical blender Γ in a flip-flop configuration. Let U be a partially hyperbolic neighborhood of this flip-flop configuration, $\Lambda(U)$ the maximal invariant set of f in U , and C^{uu} the associated strong unstable cone field in U .*

Suppose $\varphi : U \rightarrow \mathbb{R}$ is a continuous function that is positive on the blender Γ and negative on the periodic orbit $\mathcal{O}(q)$ of q .

Then there exist an integer $N \geq 1$ and a flip-flop family \mathfrak{F} with respect to f^N and the function

$$\varphi_N := \sum_{j=0}^{N-1} \varphi \circ f^j \quad \text{defined on the set} \quad \bigcap_{j=0}^{N-1} f^{-j}(U).$$

Moreover, given any $\delta > 0$, we can choose the flip-flop family $\mathfrak{F} = \mathfrak{F}^+ \cup \mathfrak{F}^-$ such that $\bigcup \mathfrak{F}^+$ (resp. $\bigcup \mathfrak{F}^-$) is contained in the δ -neighborhood of Γ (resp. $\mathcal{O}(q)$).

We postpone the proof of Proposition 4.9 to Sect. 4.4. We now derive Theorem 5 from this proposition.

Proof of Theorem 5. Suppose q is a saddle of u-index $i - 1$ and Γ is a dynamical blender of u-index i in a split flip-flop configuration. Let U be a strict partially hyperbolic neighborhood of this split configuration and $\tilde{E}^{uu} \oplus \tilde{E}^c \oplus \tilde{E}^{ss}$ the corresponding partially hyperbolic splitting defined over the maximal invariant subset $\Lambda(U)$ of f in U . This implies that $\dim \tilde{E}^{uu} = i - 1$ and $\dim \tilde{E}^c = 1$.

Since $\Gamma \subset \Lambda(U)$ is a hyperbolic set whose unstable bundle E^u is the restriction of $\tilde{E}^{uu} \oplus \tilde{E}^c$, we can take an adapted metric such that the vectors in E^u are uniformly expanded by Df (with respect to such a metric). In particular, the center Jacobian map $J^c := \|Df|_{E^c}\|$, which is well defined on $\Lambda(U)$, is uniformly bigger than 1 on Γ . Since the saddle q has u-index $i - 1$, after a new change of metric that does not affect the previous one, we can assume that $J^c < 1$ on the orbit $\mathcal{O}(q)$. Applying Tietze extension theorem, we continuously extend the function $\log J^c$ on $\Lambda(U)$ to a function φ defined on the set U .

Fix $\delta > 0$ small enough such that φ is defined and positive (resp., negative) on the δ -neighborhood of Γ (resp., of $\mathcal{O}(q)$). Applying Proposition 4.9 to the map φ , we obtain a flip-flop family \mathfrak{F} associated to a power f^N of f and the corresponding Birkhoff sum φ_N . Applying Theorem 4 we obtain an f^N -invariant compact set $\Upsilon \subset U \cap f^{-1}(U) \cap \dots \cap f^{-N+1}(U)$ such that $h_{\text{top}}(f^N|_{\Upsilon}) > 0$ and the Birkhoff averages of φ_N with respect to f^N converge to zero uniformly on Υ . Then the Birkhoff averages of φ with respect to f also converge to zero uniformly on the set $\Omega := \bigcup_{j=0}^{N-1} f^j(\Upsilon)$, which is contained in the maximal invariant set $\Lambda(U)$. This means that the Lyapunov exponent of f along the E^c direction vanishes on Ω . Moreover, $h_{\text{top}}(f|_{\Omega})$ is also positive.

To conclude the proof of Theorem 5 it remains to see that we can choose the set Ω contained in the chain recurrent class $C(q, f)$ of q . First note that we can assume that Ω

consists of chain recurrent points. To see that $\Omega \subset C(q, f)$ fix any point $x \in \Omega$. If the number $\delta > 0$ in Proposition 4.9 is small enough then the orbit of x has infinitely many iterates close to q (otherwise the φ -average would be positive). Hence, as δ is small, the strong unstable manifold of x intersects the stable manifold of q . Similarly, the orbit of x also has infinitely many iterates close to Γ (otherwise the φ -average would be negative). Hence, if $\delta > 0$ is small enough, the strong stable manifold of x intersects the unstable manifold of a point in Γ .

By Proposition 4.5, the saddle q and the blender Γ form a robust cycle, therefore they are in the same chain recurrence class. Properties $W^{uu}(x) \cap W^s(q) \neq \emptyset$ and $W^{ss}(x) \cap W^u(\Gamma) \neq \emptyset$ immediately imply that x belongs to such a class. We have shown $\Omega \subset C(q, f)$, ending the proof of Theorem 5. \square

4.4. Proof of Proposition 4.9. We divide the proof of Proposition 4.9 into several lemmas. Consider a diffeomorphism f with a dynamical blender $(\Gamma, V, \mathcal{C}^{uu}, \mathfrak{D}_\Gamma)$ and a saddle q in a flip-flop configuration with unstable and stable connecting sets Δ^u and Δ^s . Consider also a partially hyperbolic neighborhood U of this configuration endowed with the extended cone field, that for notational simplicity we denote by \mathcal{C}^{uu} . Let u be the u -index of q , by definition the number u is also the uu -index of Γ and the index of the cone field \mathcal{C}^{uu} .

Let us fix some ingredients of our construction. We take a Riemannian metric and a constant $\mu > 1$ such that for every $x \in U \cap f^{-1}(U)$ the vectors in $\mathcal{C}^{uu}(x) \setminus \{\bar{0}\}$ are expanded by a ratio at least μ by Df .

Remark 4.10 (Choice of δ). Let $\delta > 0$ be small enough such that the closed δ -neighborhood of Γ (resp. $\{\mathcal{O}(q)\}$) is contained in U and the function φ is bigger than some constant $\alpha_\Gamma > 0$ (resp. less than some constant $-\alpha_q < 0$) in such a neighborhood. Reducing δ , if necessary, we can assume that the local manifold $W_\delta^s(f^i(q))$ is contained in U and tangent to \mathcal{C}^{uu} .

In the next lemmas we will introduce auxiliary families of discs that will be used to define the sets of the flip-flop family. The first step is to define a family \mathfrak{D}_q of discs in the δ -neighborhood of the orbit $\mathcal{O}(q)$. In this way, we have two preliminary families of discs, the discs \mathfrak{D}_Γ in the blender and the discs \mathfrak{D}_q . The second step is to define “transitions” between these two families.

Lemma 4.11 (The family \mathfrak{D}_q). *There is a family $\mathfrak{D}_q \subset D \in \mathcal{D}^u(M)$ of C^1 -embedded discs of dimension u containing the discs $W_{\delta/4}^u(f^i(q))$ in its interior and consisting of discs $D \in \mathfrak{D}_q$ that satisfy the following properties:*

- (a) $D \subset U$;
- (b) D is tangent to \mathcal{C}^{uu} ;
- (c) $\text{diam } D < \delta$;
- (d) D transversely intersects $W_\delta^s(f^i(q))$ for some i ;
- (e) $\|Df(v)\| \geq \mu \|v\|$ for every vector v tangent to D ;
- (f) $f(D)$ contains a disc in \mathfrak{D}_q .

The existence of the family \mathfrak{D}_q follows easily from the strict Df -invariance of the cone field \mathcal{C}^{uu} , the uniform expansion of the vectors in \mathcal{C}^{uu} by Df , and the fact that the discs of \mathfrak{D}_q transversely intersect the local stable manifold of $\mathcal{O}(q)$.

We now study the transitions between the families \mathfrak{D}_q and \mathfrak{D}_Γ . The first step is the following preliminary result:

Lemma 4.12. *Let $\mathfrak{K} \subset \mathcal{D}^u(M)$ be a family of C^1 -embedded compact discs of dimension u that are contained in U and tangent to \mathcal{C}^{uu} . Assume that there is $\varepsilon > 0$ such that every disc $D \in \mathfrak{K}$ contains a point $x \in W_\delta^s(f^i(q))$, for some i , whose distance to the boundary of D is larger than ε .*

Then there exists $n_0 = n_0(\mathfrak{K}) \geq 0$ such that for every disc $D \in \mathfrak{K}$ and every $n \geq n_0$ the set $f^n(D)$ contains a disc $D_1 \in \mathfrak{D}_q$ such that $f^{-i}(D_1) \subset U$ for every $i \in \{0, \dots, n\}$. Moreover, if $\mathfrak{K} = \mathfrak{D}_q$ then we can take $n_0 = 0$.

Proof. In the case $\mathfrak{K} = \mathfrak{D}_q$, the assertion of the lemma with $n_0 = 0$ follows arguing recursively from property (4.11) in the definition of the family \mathfrak{D}_q .

Next consider an arbitrary family \mathfrak{K} satisfying the hypotheses of the lemma. Recall that the neighborhood U contains all the discs of the family \mathfrak{D}_q and also contains the discs $W_\delta^s(f^i(q))$.

Consider the family of discs of radius ε centered at some point in $W_\delta^s(f^i(q))$ and contained in some disc of \mathfrak{K} . As these discs are tangent to the cone field \mathcal{C}^{uu} , they are Lipschitz graphs over the local unstable manifold of $f^i(q)$, where the Lipschitz constant depends only on \mathcal{C}^{uu} . Using the Lambda Lemma we find a uniform $n_0 = n_0(\mathfrak{K}) \geq 0$ such that the image $f^{n_0}(D)$ of any disc $D \in \mathfrak{K}$ contains a disc D_0 close enough to $W_{\delta/4}^u(f^{i+n_0}(q))$ such that $D_0 \in \mathfrak{D}_q$ and, moreover, $f^{-i}(D_0) \subset U$ for every $i \in \{0, \dots, n_0\}$.

Applying the first case $\mathfrak{K} = \mathfrak{D}_q$ to the disc D_0 we conclude that, for every $n \geq n_0$, the image $f^{n-n_0}(D_0)$ contains a disc $D_1 \in \mathfrak{D}_q$ such that $f^{-i}(D_1) \subset U$ for every $i \in \{0, \dots, n - n_0\}$. Thus the disc D_1 satisfies the required properties. \square

We now study the transition from the family \mathfrak{D}_q to the family \mathfrak{D}_Γ .

Lemma 4.13 (Going from \mathfrak{D}_q to \mathfrak{D}_Γ). *There is $N_q > 0$ such that for every disc $D \in \mathfrak{D}_q$ and every $n \geq N_q$, the image $f^n(D)$ contains a disc $D_1 \in \mathfrak{D}_\Gamma$ such that $f^{-i}(D_1) \subset U$ for every $i \in \{0, \dots, n\}$.*

Proof. The first step is the following claim.

Claim. There is N_q such that for every $D_0 \in \mathfrak{D}_q$ there is $m \in \{1, \dots, N_q\}$ such that the set $f^m(D_0)$ contains a disc $D_1 \in \mathfrak{D}_\Gamma$ with $f^{-i}(D_1) \subset U$ for every $i \in \{0, \dots, m\}$.

Proof. Fix n_0 such that the connecting disc Δ^u is contained in $f^{n_0}(W_{\delta/4}^u(q))$. There is a neighborhood \mathcal{V} of $W_{\delta/4}^u(q)$ contained in \mathfrak{D}_q consisting of discs whose image $f^{n_0}(D)$ contains a disc D_1 ε -close to Δ^u in the C^1 -distance for some small ε . By definition of a flip-flop configuration, Δ^u belongs to the interior of the family \mathfrak{D}_Γ , thus if $\varepsilon > 0$ is small enough the same holds for the disc D_1 . Moreover, since the negative iterates of Δ^u are contained in U , by shrinking ε if necessary, we get that $f^{-i}(D_1) \subset U$ for all $i = 0, 1, \dots, n_0$.

Applying the Lambda Lemma to the discs in \mathfrak{D}_q and by the definition of \mathfrak{D}_q , we obtain $n_1 > 0$ such that for every disc $D_0 \in \mathfrak{D}_q$ there is a sequence $D_{0,i} \in \mathfrak{D}_q$, $i = 0, \dots, n_1$, such that $D_{0,0} = D_0$, $D_{0,i+1} \subset f(D_{0,i})$, and $D_{0,n_1} \in \mathcal{V}$.

By construction to prove the claim it is enough to take $N_q = n_0 + n_1$. \square

Take N_q as in the claim. Consider $D_0 \in \mathfrak{D}_q$ and $n \geq N_q$. Associated to D_0 consider $m \leq N_q$ and the disc $D_1 \subset f^m(D_0)$ given by the claim. By the strict f -invariance of the family \mathfrak{D}_Γ , there is a sequence $D_i \subset f(D_{i-1})$, $i = 2, \dots, n+1-m$, such that $D_i \in \mathfrak{D}_\Gamma$.

By construction, D_{n-m+1} is contained in \mathfrak{D}_Γ and the negative iterates $f^{-j}(D_{n-m+1})$ for $j \in \{0, \dots, n - m\}$ are contained in $D_{n-m-j+1} \in \mathfrak{D}_\Gamma$, hence contained in U . Further negative iterates $f^{-j}(D_{n-m+1})$, $j \in \{n - m, \dots, n\}$ are contained in the negative iterates $f^{-(j-(n-m))}(D_1)$. By the claim and the choice of m and D_1 , these backwards iterates of D_1 are contained in U . This ends the proof of the lemma. \square

In the next lemma we study the transition from the family \mathfrak{D}_Γ to the family \mathfrak{D}_q .

Lemma 4.14. (Going from \mathfrak{D}_Γ to \mathfrak{D}_q). *There is $N_\Gamma > 0$ such that for every disc $D \in \mathfrak{D}_\Gamma$ and every $n \geq N_\Gamma$, the set $f^n(D)$ contains a disc $D_1 \in \mathfrak{D}_q$ such that $f^{-i}(D_1) \subset U$ for every $i \in \{0, \dots, n\}$.*

Proof. Recall that the stable connecting set Δ^s of the flip-flop configuration given by (4.1) is contained in $W^s(q)$, there exists $n_1 > 0$ such that $f^{n_1}(\Delta^s) \subset W^s_{\delta/2}(q)$. Moreover, since Δ^s is compact and by definition of a partially hyperbolic neighborhood of the configuration one has $\bigcup_{j \geq 0} f^j(\Delta^s) \subset U$, there is a neighborhood B of Δ^s such that $\bigcup_{j=0}^{n_1} f^j(B) \subset U$.

Recall also that there are $\varepsilon > 0$ and an ε -safe stable connecting set of the configuration, that is a compact subset K contained in the interior of Δ^s such that every disc $D \in \mathfrak{D}_\Gamma$ contains a disc D_ε of radius ε centered at some point of K . By shrinking ε if necessary, we can assume that the discs D_ε are contained in B .

Consider the family

$$\mathfrak{K} := \{f^{n_1}(D_\varepsilon) : D \in \mathfrak{D}_\Gamma\}.$$

This family satisfies the hypotheses of Lemma 4.12. Thus let $n_0 = n_0(\mathfrak{K})$ as in this lemma and define $N_\Gamma := n_1 + n_0$. We claim that this number satisfies the conclusions of the lemma.

Let $n \geq N_\Gamma$ and take any disc $D \in \mathfrak{D}_\Gamma$ and consider $E := f^{n_1}(D_\varepsilon) \in \mathfrak{K}$. By Lemma 4.12, the image $f^{n-n_1}(E)$ contains a disc $E_1 \in \mathfrak{D}_q$ such that $f^{-i}(E_1) \subset U$ for all $i \in \{0, 1, \dots, n - n_1\}$. Since $f^{-j}(E) \subset U$ for all $i \in \{0, 1, \dots, n_1\}$, the disc $D_1 := E_1$ satisfies all the required properties. The proof of the lemma is now complete. \square

Let us summarize our constructions up to this point. We have defined two families of discs \mathfrak{D}_q and \mathfrak{D}_Γ such that one can go from each family to the other in times N_q or N_Γ (according to the case and in the sense of Lemmas 4.13 and 4.14), and from each family to itself in time 1 [Property (4.11) of \mathfrak{D}_q and strict f -invariance property of \mathfrak{D}_Γ]. Moreover, during these transitions the orbits of these discs remain in a partially hyperbolic neighborhood U of the flip-flop configuration.

Recall now that our goal is to construct a flip-flop family associated to a Birkhoff sum of the function φ . This function is bounded away from zero in the discs of \mathfrak{D}_q , but this is not necessarily true for the discs of \mathfrak{D}_Γ . Thus we need to shrink the discs of the family \mathfrak{D}_Γ while keeping the transition properties between the families above. This is the reason why we introduce the new family \mathfrak{D}_Γ^m below. Let us now go to the details of this construction.

Recall the choice of δ in Remark 4.10 and that $\varphi > \alpha_\Gamma > 0$ in the δ -neighborhood of the blender Γ . Since the blender Γ is the maximal invariant set of f in \bar{V} , there is an integer $m \geq 0$ such that the set $\bigcap_{i=-m}^m f^i(\bar{V})$ is contained in the δ -neighborhood of Γ , thus in a region where $\varphi > \alpha_\Gamma > 0$.

Definition 4.15 (The family \mathfrak{D}_Γ^m). Let \mathfrak{D}_Γ^m be the family of discs D^+ such that there exist $E_{-m}, E_{-m+1}, \dots, E_m \in \mathfrak{D}_\Gamma$ satisfying the following conditions:

- $E_i \subset f(E_{i-1})$ for each $i \in \{-m + 1, -m + 2, \dots, m\}$;
- $E_m = f^m(D^+)$.

In particular, $f^i(D^+) \subset E_i$ for each $i \in \{-m, -m + 1, \dots, m\}$.

Remark 4.16. Every disc $D \in \mathfrak{D}_\Gamma^m$ is contained in the δ -neighborhood of Γ , thus $\varphi(x) > \alpha_\Gamma$ for all $x \in D$.

Using recursively the strict f -invariance of the family \mathfrak{D}_Γ , we get that the image $f^m(D)$ of any disc $D \in \mathfrak{D}_\Gamma$ contains a disc of \mathfrak{D}_Γ^m . In particular, the family \mathfrak{D}_Γ^m is nonempty.

Let $C := \sup |\varphi|$ and recall that $\alpha_q, \alpha_\Gamma > 0$ (see Remark 4.10). Fix an integer

$$N \geq \max(N_q, N_\Gamma) + m$$

such that the number

$$\alpha := \min \{ (N - N_q - m)\alpha_q - (N_q + m)C, (N - N_\Gamma - m)\alpha_\Gamma - (N_\Gamma + m)C \} > 0.$$

In the next two lemmas we obtain transitions between the families \mathfrak{D}_q and \mathfrak{D}_Γ^m : from each family it is possible to go to the other family and to itself. In what follows, we denote by D^- the discs in \mathfrak{D}_q and by D^+ the ones in \mathfrak{D}_Γ^m .

Lemma 4.17 (Transitions of discs of \mathfrak{D}_q). *Every disc $D^- \in \mathfrak{D}_q$ contains subdiscs D_-^-, D_+^- such that:*

(Tq1) $f^N(D_-^-) \in \mathfrak{D}_q$ and $f^i(D_-^-) \subset U$ for every $i \in \{0, 1, \dots, N - 1\}$. Moreover;

$$\varphi_N(x) < -N\alpha_q \leq -\alpha, \text{ for every } x \in D_-^-.$$

(Tq2) $f^N(D_+^-) \in \mathfrak{D}_\Gamma^m$ and $f^i(D_+^-) \subset U$ for every $i \in \{0, 1, \dots, N - 1\}$. Moreover,

$$\varphi_N(x) < -(N - N_q - m)\alpha_q + (N_q + m)C \leq -\alpha, \text{ for every } x \in D_+^-.$$

Proof. Take any disc D^- of the family \mathfrak{D}_q . Applying recursively property (4.11) of the family \mathfrak{D}_q we find a sequence of discs $(D_i)_{i \geq 0}$ in \mathfrak{D}_q such that $D_0 = D^-$ and $D_{i+1} \subset f(D_i)$ for each i . Let

$$D_-^- := f^{-N}(D_N).$$

By construction and the definitions of \mathfrak{D}_q and α_q , the disc D_-^- satisfies the properties in (4.17).

To prove the second part of the lemma, recall that by Lemma 4.13, the set $f^{N_q}(D_{N-N_q-m})$ contains a disc $F \in \mathfrak{D}_\Gamma$ such that $f^{-j}(F) \subset U$ for every $j \in \{0, \dots, N_q\}$. Applying recursively the strict f -invariance of the family \mathfrak{D}_Γ , we get a sequence of discs $(F_i)_{i \geq 0}$ in \mathfrak{D}_Γ such that $F_0 = F$ and $F_{i+1} \subset f(F_i)$ for each i . Let

$$D_+^- := f^{-N-m}(F_{2m}).$$

Notice that $f^N(D_+^-) = f^{-m}(F_{2m})$ belongs to the family \mathfrak{D}_Γ^m . Indeed the associated sequence of discs E_{-m}, \dots, E_m in Definition 4.15 is given by $E_i = F_{m+i}$. The inclusion properties and the upper bound for the sum φ_N in (4.17) follow straightforwardly by construction. The proof of the lemma is now complete. \square

Lemma 4.18 (Transitions of discs of \mathfrak{D}_Γ). *Every disc $D^+ \in \mathfrak{D}_\Gamma^m$ contains subdiscs D_+^+, D_-^+ such that:*

(TG1) $f^N(D_+^+) \in \mathfrak{D}_\Gamma^m$ and $f^i(D_+^+) \subset U$ for every $i \in \{0, 1, \dots, N - 1\}$. Moreover,

$$\varphi_N(x) > N\alpha_\Gamma \geq \alpha, \text{ for every } x \in D_+^+.$$

(TG2) $f^N(D_-^+) \in \mathfrak{D}_q$ and $f^i(D_-^+) \subset U$ for every $i \in \{0, 1, \dots, N - 1\}$. Moreover,

$$\varphi_N(x) > (N - N_\Gamma - m)\alpha_\Gamma - (N_\Gamma + m)C \geq \alpha, \text{ for every } x \in D_-^+.$$

Proof. Given a disc D^+ in the family \mathfrak{D}_Γ^m , let $E_{-m}, E_{-m+1}, \dots, E_m$ be the discs associated to D^+ in \mathfrak{D}_Γ given by Definition 4.15 with $E_m = f^m(D^+)$. Applying recursively the f -invariance property of family \mathfrak{D}_Γ , we find new discs $E_{m+1}, E_{m+2}, \dots \in \mathfrak{D}_\Gamma$ such that $E_i \subset f(E_{i-1})$ for each $i > m$. Notice that by construction the disc $f^{-m}(E_j)$ is a member of \mathfrak{D}_Γ^m for every $j \geq m$.

Let

$$D_+^+ := f^{-N-m}(E_{N+m}).$$

Note that

$$D_+^+ \subset f^{-N-m}(f^N(E_m)) = f^{-m}(E_m) = D^+$$

and that

$$f^N(D_+^+) = f^{-m}(E_{N+m}) \in \mathfrak{D}_\Gamma^m$$

as required. To see that the sets $f^i(D_+^+)$ are contained in U for all $i \in \{0, 1, \dots, N\}$ note that $f^i(D_+^+)$ is contained in $f^{-m}(E_{m+i}) \in \mathfrak{D}_\Gamma^m$ thus contained in U . By Remark 4.16 this implies that $\varphi(f^i(x)) > \alpha_\Gamma$ for every $x \in D_+^+$ and therefore $\varphi_N(x) > N\alpha_\Gamma$ for every $x \in D_+^+$. This ends the proof of (4.18).

As $E_{N-N_\Gamma} \in D_\Gamma$, Lemma 4.14 implies that the disc $f^{N_\Gamma}(E_{N-N_\Gamma})$ contains a disc $F \in \mathfrak{D}_q$ such that $f^{-j}(F) \subset U$ for every $j \in \{0, \dots, N_\Gamma\}$. Let

$$D_-^+ := f^{-N}(F).$$

Notice that $D_-^+ \subset f^{-N+N_\Gamma}(E_{N-N_\Gamma}) \subset D^+$ (since $E_m = f^m(D^+)$ and $N - N_\Gamma \geq m$). It is clear that $f^i(D_-^+) \subset U$ for every $i \in \{0, 1, \dots, N\}$. This completes the proof of the inclusion properties.

To get the estimate for the Birkhoff sum note that for each $i \in \{0, 1, \dots, N - N_\Gamma - m\}$, the set $f^i(D_-^+)$ is contained in $f^{-m}(E_{m+i}) \in \mathfrak{D}_\Gamma^m$ and in particular, by Remark 4.16, in the part of U where $\varphi > \alpha_\Gamma$. So the lower bound for the Birkhoff sum φ_N on D_-^+ follows. The proof of (4.18) is now complete. \square

4.4.1. *End of the proof of Proposition 4.9* Let N and α be as above. The flip-flop family $\mathfrak{F} = \mathfrak{F}^+ \cup \mathfrak{F}^-$ is defined as follows:

- \mathfrak{F}^+ is the family of unions $\{D_+^+ \cup D_-^+\}_{D^+ \in \mathfrak{D}_\Gamma^m}$, where D_+^+ and D_-^+ are associated to $D^+ \in \mathfrak{D}_\Gamma^m$ and given by Lemma 4.18;
- \mathfrak{F}^- is the family of unions $\{D_+^- \cup D_-^-\}_{D^- \in \mathfrak{D}_q}$, where D_+^- and D_-^- are associated to $D^- \in \mathfrak{D}_q$ and given by Lemma 4.17.

Lemmas 4.18 and 4.17 provide properties (FF1) and (FF2) of a flip-flop family with respect to f^N and the function φ_N . Property (FF3) follows from the fact that the discs in \mathfrak{F} are tangent to C^{uu} .

Finally, the construction implies that the discs of \mathfrak{F}^+ are contained in the δ -neighborhood of Γ and the discs of \mathfrak{F}^- are contained in the δ -neighborhood of $\mathcal{O}(q)$. The proof of the proposition is now complete. \square

5. From Robust Cycles to Split Flip-flop Configurations via Spawners

In this section we see how robust cycles generate flip-flop configurations. This generation is done throughout a special class of partially hyperbolic sets called *spawners* that we will introduce in the next subsection. The advantage of spawners for us is that they spawn split flip-flop configurations. The organization of this section is the following:

$$\text{Robust cycles} \xrightarrow{5.2} \text{Spawners} \xrightarrow{5.3} \text{Split flip-flop configurations.}$$

The corresponding steps are done in Sects. 5.1 and 5.2.

5.1. *From robust cycles to spawners.* Given a natural number i , an i -box is a product of i non-degenerate compact intervals.

Definition 5.1 (*Spawner*). Let $f \in \text{Diff}^1(M)$ and u and s be positive integers with $u + 1 + s = d = \dim M$. Suppose $C \subset M$ is an embedded d -dimensional cube. For notational simplicity, let us identify C with $[-1, 1]^d$. Suppose there are disjoint subsets $L_1, L_2, L_3 \subset C$ of the form

$$L_i = I_i^u \times [-1, 1] \times [-1, 1]^s,$$

where each $I_i^u \subset (-1, 1)^u$ is a u -box, and positive integers n_1, n_2, n_3 such that:

- $f^j(L_i) \cap C = \emptyset$ for $0 < j < n_i$ and $f^{n_i}(L_i) \subset C$;
- $f^{n_i}(L_i) = [-1, 1]^u \times [-1, 1] \times I_i^s$, where I_i^s is an s -box;
- The restriction of f^{n_i} to L_i is of the form

$$f^{n_i}(x^u, x, x^s) = (A_i^u(x^u), x, A_i^s(x^s)),$$

where A_i^u is an expanding affine map of \mathbb{R}^u and A_i^s is a contracting affine map of \mathbb{R}^s .

Let Σ be the maximal invariant set of f in the set

$$Q_{123} := \bigcup_{i=1,2,3} \left(\bigcup_{k=0}^{n_i-1} f^k(L_i) \right). \tag{5.1}$$

The set Σ is called a *spawner of u -index u* , the set C is its *reference cube*, and the sets L_1, L_2 and L_3 are its *legs*. The numbers n_1, n_2, n_3 are the *first return times of the legs*. (See the first part of Fig. 5 in Sect. 5.3.1).

Note that each set $f^{n_i}(L_i)$ intersects the legs L_1, L_2, L_3 in a Markovian way and that the set Σ is partially hyperbolic with a splitting of the form $T_\Sigma M = E^{uu} \oplus E^c \oplus E^{ss}$ with bundles of respective dimensions $u, 1,$ and s , where E^{uu} is uniformly expanding and E^{ss} is uniformly contracting. In the cube C these three bundles are of the form $\mathbb{R}^u \times \{0\}^{s+1}, \{0\}^u \times \mathbb{R} \times \{0\}^s,$ and $\{0\}^{u+1} \times \mathbb{R}^s,$ respectively. Note that the strong stable manifold $W^{ss}(\Sigma)$ and the strong unstable manifold $W^{uu}(\Sigma)$ of Σ are well defined.

The spawners that we consider are generated by means of the following result:

Proposition 5.2 (From robust cycles to spawners). *Consider a diffeomorphism f with a pair of hyperbolic basic sets Λ_f and Θ_f of respective u -indices i and $i - 1$ forming a robust cycle. Then there exists a diffeomorphism g arbitrarily C^1 -close to f having a spawner Σ of u -index $i - 1$ such that*

$$W^u(\Lambda_g) \pitchfork W^{ss}(\Sigma) \neq \emptyset \text{ and } W^s(\Theta_g) \pitchfork W^{uu}(\Sigma) \neq \emptyset, \tag{5.2}$$

i.e. there exist points of transverse intersection between these manifolds.

This proposition follows from a sequence of previous results and indeed is a reformulation of results in [BD2]. For completeness, let us briefly explain the steps involved in this construction.

Proof (Sketch of the proof of Proposition 5.2). First, the existence of a robust cycle implies that there is a C^1 -neighborhood \mathcal{V} of f such that for every $g \in \mathcal{V}$ there are saddles $p_g \in \Lambda_g$ and $q_g \in \Theta_g$ depending continuously on g such that their chain recurrence classes are equal and non trivial.

By [BC] C^1 -generically the chain recurrence class of a hyperbolic periodic point is its homoclinic class. This fact allows us to get an open and dense subset of the neighborhood \mathcal{V} of f consisting of diffeomorphisms g such that homoclinic classes of p_g and q_g are both non trivial. Next, after a new perturbation, if necessary, we can replace these saddles by a pair of saddles homoclinically related to them whose eigenvalues are all real and have multiplicity 1 and different moduli (i.e., the linear map $Df^\pi(a), \pi$ the period of $a,$ satisfies such a property). For this standard property see for instance [ABCDW]. Using the terminology in [BD2, BD3] we say that these new periodic points have real center eigenvalues. As these new saddles are homoclinically related to the initial ones, they are still in the same chain recurrence class and contained in a pair of hyperbolic sets with a robust cycle. For simplicity, we continue to denote these new saddles by p_g and q_g .

In the above setting, [BD2, Theorem 2.3] claims that by an arbitrarily small C^1 -perturbation one can get a saddle-node or a flip periodic point with a strong homoclinic intersection (called *strong connection*): the strong stable and strong unstable manifolds of the saddle-node/flip point meet quasi-transversely, meaning that the sum of the tangent spaces at the intersection point is $\dim M - 1$. Furthermore, [BD3, Proposition 5.9] asserts that the strong stable (resp. unstable) manifold of such nonhyperbolic periodic point intersects transversely the unstable manifold of p_g (resp. the stable manifold of q_g). With the terminology in [BD3], this nonhyperbolic periodic point is *strong intermediate*. As p_g and q_g are robustly in the same chain recurrence class, the strong intermediate point r also belongs to this class.

Finally, [BD2, Theorem 2.4] shows that these strong homoclinic intersections and intermediate points yield geometric blenders and robust cycles. To get such a property in [BD2, pp. 501, 502] it is shown that an arbitrarily small C^1 -perturbation of a strong connection associated to a saddle node generates a dynamical configuration that is exactly what we call here a spawner. A key point is that this spawner is by construction strong

intermediate with relation to p_g and q_g , thus it is contained in the chain recurrence class of p_g . The case of flip periodic points is solved using [BD2, Remark 4.5] that asserts that the strong connection associated to a flip periodic point generates a strong connection associated to a saddle node satisfying the strong intermediate properties. This concludes our sketch of proof. \square

5.2. *From spawners to split flip-flop configurations: proof of Theorem 2.* Recall the definitions of the sets L_i and the numbers n_i , and define the sets

$$Q_{12} := \bigcup_{i=1,2} \bigcup_{k=0}^{n_i-1} f^k(L_i) \quad \text{and} \quad Q_3 := \bigcup_{k=0}^{n_3-1} f^k(L_3). \tag{5.3}$$

Proposition 5.3 (From spawners to split flip-flop configurations). *Suppose f has a spawner Σ of u -index u . Then every neighborhood \mathcal{V} of f contains a nonempty open set $\mathcal{U} \subset \mathcal{V}$ such that every $g \in \mathcal{U}$ has*

- A dynamical blender Γ_g of uu -index u whose domain is contained in Q_{12} ,
- A unique hyperbolic periodic orbit $\mathcal{O}(r_g)$ of u -index u contained in Q_3 ,

that form a split flip-flop configuration.

Moreover, this split flip-flop configuration has a strict partially hyperbolic neighborhood U contained in the domain of the spawner.

We postpone the proof of this proposition to the next subsection and prove now Theorem 2.

5.2.1. *Proof of Theorem 2* Let $\mathcal{U} \subset \text{Diff}^1(M)$ be an open set of diffeomorphisms such that every $f \in \mathcal{U}$ has a pair of hyperbolic periodic points p_f and q_f that depend continuously on f , have respective u -indices $i_p > i_q$, and are in the same chain recurrence class $C(q_f, f)$.

In the introduction we recalled that there is a C^1 -dense open subset \mathcal{U}_0 of \mathcal{U} such that for every $f \in \mathcal{U}_0$ and every number $i_q \leq i \leq i_p$ there is a family of hyperbolic transitive sets $\Lambda_{i,f}$ of u -index i depending continuously on f , contained in $C(q_f, f)$, and such that for every $i < i_p$ the sets $\Lambda_{i,f}$ and $\Lambda_{i+1,f}$ form a robust cycle. Thus, without loss of generality, we can assume that $i_q = i, i_p = i + 1$, and that p_f and q_f belong to hyperbolic transitive sets $\Lambda_{i+1,f}$ and $\Lambda_{i,f}$, respectively, forming a robust cycle.

By Proposition 5.2 there is a C^1 -dense subset $\mathcal{D} \subset \mathcal{U}$ such that every $f \in \mathcal{D}$ has a spawner of u -index i such that the strong unstable (resp. stable) manifold of any point of the spawner intersects transversally the stable manifold of $\Lambda_{i,f}$ (resp. unstable manifold of $\Lambda_{i+1,f}$).

By Proposition 5.3, given any diffeomorphism $f \in \mathcal{D}$ (with a spawner) there is an arbitrarily small C^1 -perturbation g of it f with dynamical blender Γ_g of uu -index i and a saddle r_g of u -index i forming a split flip flop configuration. Moreover, this split flip-flop configuration has a strict partially hyperbolic neighborhood U contained in the domain of the spawner. Recall that to have a split flip-flop configuration is a robust property, see Remark 4.8, thus such diffeomorphisms g form a dense open subset \mathcal{U}_1 of \mathcal{U}_0 hence of \mathcal{U} .

This implies, in particular, that the maximal invariant set in U is contained in the chain recurrence class $C(q_g, g)$. Theorem 5 now implies that the maximal g -invariant

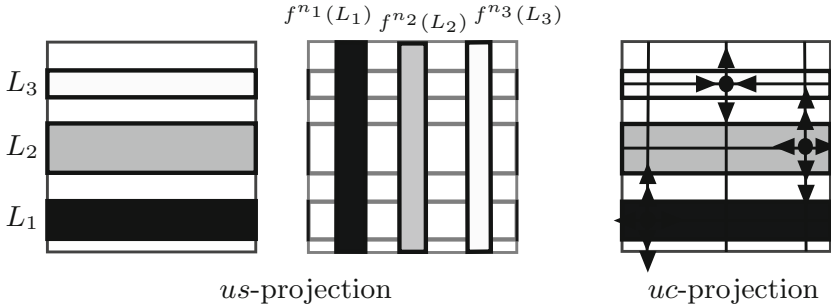


Fig. 5. The maps f and f_λ

set in U contains a partially hyperbolic set $K_g \subset C(q_g, f)$ with a partially hyperbolic splitting of the form

$$T_K M = E^{uu} \oplus E^c \oplus E^{ss},$$

where E^{uu} is uniformly expanding and has dimension $i - 1 > 0$, E^c has dimension 1, and E^{ss} is uniformly contracting, such that the Lyapunov exponent of any point of K_g along E^c is zero. Moreover, the topological entropy of the restriction of g to K_g is positive. This concludes the proof Theorem 2. \square

5.3. Proof of Proposition 5.3. The aim of the rest of the section is the proof of Proposition 5.3.

5.3.1. A family of perturbations of a spawner and their induced maps We now assume that f has a spawner with reference cube C , legs L_1, L_2, L_3 and first return times to C n_1, n_2, n_3 , respectively. By definition, the restriction of f^{n_i} to L_i is of the form

$$f^{n_i}(x^u, x, x^s) = (A_i^u(x^u), x, A_i^s(x^s)).$$

For every $i \in \{1, 2, 3\}$ and $k \in \{1, \dots, n_i - 1\}$, we fix coordinates in $f^k(L_i)$ such that the expression of the restriction $f^k|_{L_i}$ in these coordinates is the identity map, $f^k(x^u, x, x^s) = (x^u, x, x^s)$.

For $\lambda > 1$, define one-dimensional maps

$$g_{\lambda,1}(x) := \lambda x + (-1 + \lambda)/2; \quad g_{\lambda,2}(x) := \lambda x + (1 - \lambda)/2; \quad g_{\lambda,3}(x) := \lambda^{-1}x.$$

Note that the maps $g_{\lambda,1}, g_{\lambda,2}, g_{\lambda,3}$ have fixed points $-1/2, 1/2, 0$, respectively.

Consider a neighborhood \mathcal{V} of f . For $\lambda > 1$ sufficiently close to 1 define a diffeomorphism f_λ as follows,

$$f_\lambda(x^u, x, x^s) = \begin{cases} f(x^u, x, x^s), & \text{if } (x^u, x, x^s) \in \bigcup_{i=1}^3 \left(\bigcup_{k=1}^{n_i-2} f^k(L_i) \right), \\ (A_i^u(x^u), g_{\lambda,i}(x), A_i^s(x^s)), & \text{if } (x^u, x, x^s) \in f^{n_i-1}(L_i), \quad i \in \{1, 2, 3\}. \end{cases}$$

We take $\lambda > 1$ sufficiently close to 1 such that $f_\lambda \in \mathcal{V}$. See Fig. 5.

Remark 5.4. The map f_λ has a periodic point $q = (q^u, 0, q^s)$ (independent of λ) in L_3 , of period n_3 and u -index u . The local invariant manifolds of q are $W_{loc}^s(q) = \{q^u\} \times [-1, 1]^{s+1}$ and $W_{loc}^u(q) = [-1, 1]^u \times \{(0, q^s)\}$.

5.3.2. *Dynamical blenders for induced maps* We fix a small open neighborhood U of C such that, for $\lambda > 1$ sufficiently close to 1, $f_\lambda^j(L_i)$ is disjoint from \overline{U} for every $0 < j < n_i$, and $f_\lambda^{n_i}(L_i)$ is contained in U . Define the map F_λ by

$$F_\lambda: L_1 \cup L_2 \rightarrow U, \quad F_\lambda(x) := f_\lambda^{n_i}(x), \quad \text{if } x \in L_i, i = 1, 2.$$

In this way we get an induced map $F_\lambda = [f_\lambda, U, (L_i)_{i=1}^2, (n_i)_{i=1}^2]$ (or for short simply F_λ) of f_λ . The next step is to get a dynamical blender for this induced map (see Proposition 5.5). For that we need to introduce some ingredients as domains, cone fields, and families of discs.

Let Γ_λ be the maximal invariant set of F_λ in $L_1 \cup L_2$. Note that this set is hyperbolic and contained in the interior of $L_1 \cup L_2$. For every $0 < \alpha < 1$ consider the cone field

$$C_\alpha^{\text{uu}} := \{(v^u, v^c, v^s); \|(v^u, v^s)\| \leq \alpha \|v^u\|\}.$$

As the maps A_i^s are affine contractions, the maps A_i^u are affine expansions, and λ is close to 1, any cone field C_α^{uu} is strictly DF_λ -invariant. We fix constants $0 < \alpha_0 < \alpha_1 < \frac{1}{8\sqrt{u}}$ such that $DF(C_{\alpha_1}^{\text{uu}})$ is strictly contained in $C_{\alpha_0}^{\text{uu}}$.

Recall that $I_i^s = A_i^s([-1, 1]^s)$ and $I_i^u = (A_i^u)^{-1}([-1, 1]^u)$, for $i = 1, 2, 3$. We fix compact discs $J^u, J_0^u \subset (-1, 1)^u$ whose interiors contain $I_1^u \cup I_2^u \cup I_3^u$ and such that J^u is contained in the interior of J_0^u . Similarly, we fix compact discs $J^s, J_0^s \subset (-1, 1)^s$ whose interiors contain $I_1^s \cup I_2^s \cup I_3^s$ and such that J^s is contained in the interior of J_0^s .

We consider the following set of graphs of C^1 -maps with Lipschitz constant less than α_0 :

- \mathfrak{D} is the set of graphs of C^1 -maps $J^u \rightarrow [-\frac{1}{4}, \frac{1}{4}] \times J^s$;
- \mathfrak{D}_1 is the set of graphs of C^1 -maps $(A_1^u)^{-1}(J_0^u) \rightarrow [-\frac{1}{4}, \frac{1}{8}] \times J^s$;
- \mathfrak{D}_2 is the set of graphs of C^1 -maps $(A_2^u)^{-1}(J_0^u) \rightarrow [-\frac{1}{8}, \frac{1}{4}] \times J^s$.

Finally, choose open discs U^u and U^s such that

$$J_0^u \subset U^u \subset \overline{U^u} \subset (-1, 1)^u \quad \text{and} \quad J_0^s \subset U^s \subset \overline{U^s} \subset (-1, 1)^s$$

and consider the sets

$$U_1 := (A_1^u)^{-1}(U^u) \times g_{\lambda,1}^{-1} \left(\left(-\frac{3}{4}, \frac{3}{4} \right) \right) \times U^s \subset L_1,$$

$$U_2 := (A_2^u)^{-1}(U^u) \times g_{\lambda,2}^{-1} \left(\left(-\frac{3}{4}, \frac{3}{4} \right) \right) \times U^s \subset L_2.$$

As $U_i \subset L_i, i = 1, 2$, the iterates $f_\lambda^j(\overline{U}_i), j \in \{1, \dots, n_i - 1\}$, are disjoint from \overline{U} and $f_\lambda^{n_i}(\overline{U}_i) \subset U$. Thus the induced map F_λ of f_λ can be extended to $U_1 \cup U_2$.

Proposition 5.5 (Dynamical blenders for F_λ). *Let Γ_λ be the maximal invariant set of F_λ in $L_1 \cup L_2$. Then $(\Gamma_\lambda, U_1 \cup U_2, C_\alpha^{\text{uu}}, \mathfrak{D}_1 \cup \mathfrak{D}_2)$ is a dynamical blender of F_λ .*

Proof. The main step of the proof is the following lemma.

Lemma 5.6. *Consider $\lambda > 1$ close enough to 1 and the induced map $F_\lambda = [f_\lambda, U, (L_i)_{i=1}^2, (n_i)_{i=1}^2]$. Then the cone field $C_{\alpha_1}^{\text{uu}}$ and the families of discs $\mathfrak{D}, \mathfrak{D}_1, \mathfrak{D}_2$ satisfies the hypotheses of Proposition 3.20.*

Proof. The invariance of the cone field was obtained above. It remains to check that the families of discs satisfy items (a), (b), and (c) in Proposition 3.20.

For item (a), just note that there are neighborhoods of the families \mathfrak{D} , \mathfrak{D}_1 , \mathfrak{D}_2 formed by discs contained in U and tangent to $\mathcal{C}_{\alpha_1}^{uu}$.

To prove item (b), note that a disc $D \in \mathfrak{D}$ is a graph over $J^u \supset I_1^u \cup I_2^u$. We see that either the restriction of this graph to I_1^u belongs to \mathfrak{D}_1 or the restriction to I_2^u is in \mathfrak{D}_2 . Suppose, by contradiction, that the first case does not hold. This means that the center coordinate of some point is larger than $\frac{1}{8}$. As the graph is α_0 -Lipschitz with $\alpha_0 < \frac{1}{8\sqrt{u}}$ and the diameter of J^u is strictly less than $2\sqrt{u}$ one gets that the central coordinates of this graph are larger than $-\frac{1}{8}$. Thus the central part is contained in $[-\frac{1}{8}, \frac{1}{4}]$ and thus the restriction to I_2^u belongs to \mathfrak{D}_2 .

It remains to check item (c), which is an immediate consequence of the following claim:

Claim 5.7. Fix $\lambda > 1$ close to 1. Then there is $\varepsilon = \varepsilon(\lambda) > 0$ such that for every disc $D \in \mathcal{V}_\varepsilon^\delta(\mathfrak{D}_i)$, $i = 1, 2$, the set $F_\lambda(D)$ contains a disc in \mathfrak{D} .

Proof. We prove the claim for the family \mathfrak{D}_1 ; the proof for the family \mathfrak{D}_2 is identical and hence omitted.

We first prove the claim for discs in \mathfrak{D}_1 (next we extend the proof for discs in a neighborhood). Recall that $F_\lambda = (A_1^u, g_{\lambda,1}, A_1^s)$ and that a disc D in \mathfrak{D}_1 is the graph of map $\varphi = (\varphi^c, \varphi^s): (A_1^u)^{-1}(J_0^u) \rightarrow [-\frac{1}{4}, \frac{1}{8}] \times J^s$. Recalling the definition of $g_{\lambda,1}$ we get that $F_\lambda(D)$ is the graph over J_0^u of a map φ_* given by

$$x^u \mapsto \left(g_{\lambda,1}(\varphi^c(A_1^u)^{-1}(x^u)), A_1^s(\varphi^s(A_1^u)^{-1}(x^u)) \right).$$

Thus

$$\varphi_*: J_0^u \rightarrow \left[-\frac{1}{4} + \frac{\lambda - 1}{2}, \frac{1}{4} - \frac{6 - 3\lambda}{8} \right] \times A_1^s(J^s).$$

As $\lambda > 1$ is close to 1 and $J^s \subset (-1, 1)^s$, we get that

$$\left[-\frac{1}{4} + \frac{\lambda - 1}{2}, \frac{1}{4} - \frac{6 - 3\lambda}{8} \right] \times A_1^s(J^s) \subset \text{int} \left(\left[-\frac{1}{4}, \frac{1}{4} \right] \times I_1^s \right) \subset \text{int} \left(\left[-\frac{1}{4}, \frac{1}{4} \right] \times J^s \right).$$

Furthermore, the disc D is tangent to the cone field $\mathcal{C}_{\alpha_0}^{uu}$ which is strictly DF_λ -invariant. Thus the disc $F_\lambda(D)$ is the graph over J_0^u of the map φ_* whose Lipschitz constant is strictly less than α_0 . Therefore the graph of the restriction of φ_* to J^u is a disc in \mathfrak{D} and contained in $F_\lambda(D)$. This completes the proof for discs in \mathfrak{D}_1 .

To extend the result to a small neighborhood of \mathfrak{D}_1 note that there is $\varepsilon > 0$ such that any disc $\tilde{D} \in \mathcal{V}_\varepsilon^\delta(\mathfrak{D}_1)$ is a graph of a map φ whose definition domain contains $(A_1^u)^{-1}(J^u)$, with Lipschitz constant α_1 , and image in $g_{\lambda,1}^{-1}([-\frac{1}{4}, \frac{1}{4}]) \times J_0^s$. The image by F_λ of \tilde{D} contains a graph over J^u of a C^1 -map with Lipschitz constant α_0 whose image is contained in $[-\frac{1}{4}, \frac{1}{4}] \times J^s$, thus $F_\lambda(\tilde{D})$ contains a disc in \mathfrak{D} . This ends the proof of the claim for the family \mathfrak{D}_1 . \square

The proof of Lemma 5.6 is now complete. \square

We are now ready to finish the proof of the proposition. By construction, the maximal invariant set Γ_{F_λ} of F_λ in $L_1 \cup L_2$ is transitive, hyperbolic, and contained in the interior $U_1 \cup U_2$. Note that the discs of the family \mathfrak{D}_i are contained in the interior of the sets U_i , $i = 1, 2$, and satisfies the F_λ -invariance properties. The cone field $\mathcal{C}_{\alpha_1}^{uu}$ is also invariant. The proof is now complete. \square

5.3.3. *Dynamical blenders for f_λ* By Proposition 3.20, every neighborhood of the closure of $\bigcup_{i=1,2} \bigcup_{j=0}^{n_i-1} f_\lambda^j(U_i)$ contains a safety neighborhood V which is the domain of a dynamical blender Λ_λ of f_λ whose strictly invariant family of discs \mathcal{D}_λ is such that the discs $D \in \mathcal{D}_\lambda$ contained in U are precisely $\mathcal{D}_1 \cup \mathcal{D}_2$. We can in particular assume that \bar{V} is contained in $Q_{1,2}$.

Corollary 5.8 (Dynamical blender for f_λ). *The diffeomorphism f_λ has a dynamical blender $(\Lambda_\lambda, V, C_{\alpha_1}^{uu}, \mathcal{D}_\lambda)$ such that the discs of \mathcal{D}_λ contained in U belong to the family $\mathcal{D}_1 \cup \mathcal{D}_2$ and V is contained in $Q_{1,2}$.*

5.3.4. *Generation of split flip-flop configurations* Recall the definition of the saddle q of u -index u of f_λ , see Remark 5.4.

Proposition 5.9. *For every $\lambda > 1$ close enough to 1, the saddle q and the dynamical blender $(\Gamma_\lambda, V, C_{\alpha_1}^{uu}, \mathcal{D}_\lambda)$ satisfy conditions FFC0', FFC1, FFC2, FFC3, and FFC4.*

In view of Proposition 4.4 we get the following corollary of Proposition 5.9 that ends the proof of Proposition 5.3.

Corollary 5.10. *For every sufficiently small $\eta > 0$, the saddle q and the dynamical blender $(\Gamma_\lambda, V, C_{\alpha_1}^{uu}, \mathcal{V}_\eta^\delta(\mathcal{D}_\lambda))$ are in a split flip-flop configuration.*

Let us observe that the property of the configuration being split follows from the partial hyperbolicity (with one dimensional center direction) of the initial diffeomorphism f .

Proof of Proposition 5.9. To define the connecting sets Δ^u and Δ^s of the flip-flop configuration recall the definitions of the local invariant manifolds $W_{loc}^s(q)$ and $W_{loc}^u(q)$ in Remark 5.4. Let $\Delta^u := (A_1^u)^{-1}(J_0^u) \times \{(0, q^s)\}$. This set is a disc contained in $W_{loc}^u(q)$ that belongs to \mathcal{D}_1 and whose negative iterates are contained in Q_3 . Hence it is disjoint from V (as V is contained in $Q_{1,2}$). This proves properties (4.4) and (4.1).

Consider $F_\lambda^{-1}(W_{loc}^s(q))$. Note that there are points $x_1^u \in (A_1^u)^{-1}(J_0^u)$ and $x_1^s \in (A_2^s)^{-1}(J_0^s)$ such that $F_\lambda^{-1}(W_{loc}^s(q)) = \Sigma_1 \cup \Sigma_2$ where

$$\Sigma_1 := \{x_1^u\} \times g_{\lambda,1}^{-1}([-1, 1]) \times [-1, 1]^s \quad \text{and} \quad \Sigma_2 := \{x_2^u\} \times g_{\lambda,2}^{-1}([-1, 1]) \times [-1, 1]^s.$$

Consider the intersections $\tilde{\Delta}_1 := \Sigma_1 \cap U_1$ and $\tilde{\Delta}_2 := \Sigma_2 \cap U_2$ (recall that $U_1 \cap U_2$ is the domain of the dynamical blender of F_λ , see Proposition 5.5). By definition of a dynamical blender (existence of a safe stable connecting set) there are a compact set K contained in the interior of $\tilde{\Delta}_1 \cup \tilde{\Delta}_2$ and $\varepsilon > 0$ so that K intersects any disc $D \in \mathcal{D}_1 \cup \mathcal{D}_2$ at point a distance larger than ε from the boundary ∂D . Recall the definitions of the return times n_1, n_2 and consider the set

$$\tilde{\Delta}^s := \bigcup_{i=1,2} \left(\bigcup_{j=0}^{n_i-1} f_\lambda^j(\tilde{\Delta}_i) \right).$$

The set $\tilde{\Delta}^s$ is a manifold with boundary and corners contained in the interior of the safety neighborhood V , transverse to C^{uu} , and intersects any disc of the strictly f_λ -invariant family \mathcal{D}_λ at distance uniformly bounded away from 0 from its boundary.

Furthermore, $f_\lambda^i(\tilde{\Delta}^s)$ is disjoint from \bar{V} for every $i > n_1 + n_2$ and any point of $\tilde{\Delta}^s$ that exits from V does not return to V . In other words, as a set $\tilde{\Delta}^s$ satisfies conditions

FFC2, FFC3, and FFC4. We need to remove the corners of $\tilde{\Delta}^s$. For that it is enough to consider a sufficiently large submanifold with boundary $\Delta^s \subset \tilde{\Delta}^s$, in this way we get a stable connecting set Δ^s satisfying conditions FFC2, FFC3, and FFC4. This ends the proof of the proposition. \square

6. An Interval of Lyapunov Exponents: Proof of Theorem 3

In this section we prove Theorem 3. Consider an open set $\mathcal{U} \subset \text{Diff}^1(M)$ of diffeomorphisms with periodic points p_f, q_f (depending continuously on f) with u-indices i and $i - 1$, respectively, which are in the same chain recurrence class. We will prove the existence of a C^1 -open and dense subset \mathcal{V} of \mathcal{U} consisting of diffeomorphisms f such that for every $\chi \in (\chi_i(q_f), \chi_i(p_f))$ there is a partially hyperbolic compact set $K_{f,\chi} \subset C(p_f, f)$, with 1-dimensional center direction and positive entropy such that the center Lyapunov exponent of any point in $K_{f,\chi}$ is χ .

The proof of this result is different for χ close to zero and for large $|\chi|$. The case of χ close to zero is the more interesting one, and requires split flip-flop families. The case when χ is away from zero follows from simpler and essentially hyperbolic arguments.

6.1. An interval of small Lyapunov exponents. The constructions in Sect. 5 give an open and dense subset \mathcal{U}_0 of \mathcal{U} such that every $f \in \mathcal{U}_0$ has a saddle r of u-index $i - 1$ and a dynamical blender Γ of u-index i which are in a split flip-flop configuration. The following conditions hold:

- The saddle r and the blender Γ are contained in a partially hyperbolic set with $(i - 1)$ -dimensional strong unstable bundle and one-dimensional center. This set is contained in the chain recurrence class of p_f .
- There is $\alpha_f > 0$ such that the logarithm of the center Jacobian is less than $-\alpha_f$ on the orbit of r and larger than α_f on the blender. Moreover, since the flip-flop configuration is robust (Proposition 4.2), the constant $\alpha_f > 0$ can be chosen locally constant with f .

Recall that in the proof of Theorem 5 (see Sect. 4) we consider a continuous extension φ of the logarithm of the central Jacobian and apply Proposition 4.9 to get a flip-flop family with respect to a power f^N and the corresponding N -th Birkhoff sum φ_N of φ . Thereafter using Theorem 4 we obtain a compact set K_f with positive entropy whose i -th exponent is zero.

Now for any given $\chi \in [-\alpha_f, \alpha_f]$, we apply Proposition 4.9 to the function $\varphi + \chi$ instead, thus obtaining a flip-flop family with respect to a Birkhoff sum of this function. So Theorem 4 provides a compact set $K_{f,\chi}$ with positive entropy whose i -th exponent is χ . This completes the proof of the part the theorem about Lyapunov exponents close to zero.

6.2. Intervals of large Lyapunov exponents. We now fix f as above and the constant $\alpha_f = \alpha$. The proof below follows exactly as the one of [ABCDW, Theorem 1], thus we just explain the main steps.

By hypotheses, the saddles p_f and q_f are involved in a robust cycle. Recalling the sketch of the proof of Proposition 5.2 in Sect. 5.1, after an arbitrarily small C^1 -perturbation we can assume that the homoclinic classes of these saddles are both non-trivial and, after replacing p_f and q_f by some point of the class, that the eigenvalues corresponding to p_f and q_f are all positive and different.

The occurrence of a robust cycle implies that after a new arbitrarily small C^1 -perturbation we can obtain a heterodimensional cycle associated to p_g and q_g . The unfolding of this cycle generates a saddle \bar{p}_g homoclinically related to p_g (thus of u-index i) and such that $\chi_i(\bar{p}_g)$ is close to 0, in particular, $\chi_i(\bar{p}_g) \in (0, \frac{\alpha}{2})$. This construction implies that $\chi_i(\bar{p}_g) > \chi_{i+1}(\bar{p}_g)$. Note that this configuration is robust.

Since the initial cycle associated to p_f and q_f is robust and g is close to f , the saddles p_g and q_g also belong to hyperbolic sets involved in a robust cycle. Arguing as above, a perturbation generates a new heterodimensional cycle that yields a saddle \bar{q}_h homoclinically related to q_h (thus of u-index $i - 1$) with $\chi_i(\bar{q}_h) \in (-\frac{\alpha}{2}, 0)$. Moreover, we have that $\chi_i(\bar{q}_g) > \chi_{i+1}(\bar{q}_g)$.

As p_h and \bar{p}_h are homoclinically related, there is a hyperbolic basic set Λ_h containing p_h and \bar{p}_h . Since $\chi_i(p_h) > \chi_{i+1}(p_h)$ and $\chi_i(\bar{p}_h) > \chi_{i+1}(\bar{p}_h)$. Taking care that the intersection of the parts of the invariant manifolds of p_h and \bar{p}_h involved in the construction of the set Λ_h to be in general position, we can assume that the unstable bundle E^u of Λ_h has a dominated splitting $E^u = E^{uu} \oplus E^{cu}$, where E^{cu} is one-dimensional. This implies that the logarithm of the center Jacobian is well defined and continuous along the strong unstable manifold of Λ_h .

A standard argument involving Markov partitions implies that, for every $\chi \in (\chi_i(\bar{p}_h), \chi_i(p_h))$ there is an invariant compact set $K_{h,\chi}$ contained in Λ_h having positive entropy and such that the i -th Lyapunov exponent of every point in $K_{h,\chi}$ is χ . Here the use of Markov partitions substitutes the flip-flop-like arguments.

Fixed now $\chi \in (\chi_i(q_h), \chi_i(\bar{q}_h))$, the same construction as above with q_h and \bar{q}_h provides invariant compact subset $K_{h,\chi}$ of the chain recurrence class of q_h with positive entropy that consists of points whose i -th Lyapunov exponent is χ .

Since by construction

$$(\chi_i(q_h), \chi_i(\bar{q}_h)) \cup [-\alpha, \alpha] \cup (\chi(\bar{p}_h), \chi_i(p_h)) = (\chi_i(q_h), \chi_i(p_h)),$$

we get the announced family of compact sets $K_{h,\chi}$, $\chi \in (\chi_i(q_h), \chi_i(p_h))$. This ends the proof of Theorem 3.

References

- [ABCDW] Abdenur, F., Bonatti, C., Crovisier, S., Díaz, L.J., Wen, L.: Periodic points and homoclinic classes. *Ergodic Theory Dynam. Syst.* **27**(1), 1–22 (2007)
- [ACW] Avila, A., Crovisier, S., Wilkinson, A.: Diffeomorphisms with positive metric entropy. [arXiv:1408.4252](https://arxiv.org/abs/1408.4252)
- [BBS] Baladi, V., Bonatti, C., Schmitt, B.: Abnormal escape rates from nonuniformly hyperbolic sets. *Ergodic Theory Dynam. Syst.* **19**(5), 1111–1125 (1999)
- [BB] Baraviera, A., Bonatti, C.: Removing zero Lyapunov exponents. *Ergodic Theory Dynam. Syst.* **23**(6), 1655–1670 (2003)
- [BBD] Bochi, J., Bonatti, C., Díaz, L.J.: Robust vanishing of all Lyapunov exponents for iterated function systems. *Math. Z.* **276**(1-2), 469–503 (2014)
- [B] Bonatti, C.: Towards a global view of dynamical systems, for the C^1 -topology. *Ergodic Theory Dynam. Syst.* **31**(4), 959–993 (2011)
- [BC] Bonatti, C., Crovisier, S.: Réurrence et genericité. *Invent. Math.* **158**(1), 33–104 (2004)
- [BCDG] Bonatti, C., Crovisier, S., Díaz, L.J., Gourmelon, N.: Internal perturbations of homoclinic classes: non-domination, cycles, and self-replication. *Ergodic Theory Dynam. Syst.* **33**(3), 739–776 (2013)
- [BD1] Bonatti, C., Díaz, L.J.: Persistent nonhyperbolic transitive diffeomorphisms. *Ann. Math.* **143**(2), 357–396 (1996)
- [BD2] Bonatti, C., Díaz, L.J.: Robust heterodimensional cycles and C^1 -generic dynamics. *J. Inst. Math. Jussieu* **7**(3), 469–525 (2008)

- [BD3] Bonatti, C., Díaz, L.J.: Abundance of C^1 -robust homoclinic tangencies. *Trans. Am. Math. Soc.* **364**(10), 5111–5148 (2012)
- [BDG] Bonatti, C., Díaz, L.J., Gorodetski, A.: Non-hyperbolic ergodic measures with large support. *Nonlinearity* **23**(3), 687–705 (2010)
- [BDK] Bonatti, C., Díaz, L.J., Kiriki, S.: Stabilization of heterodimensional cycles. *Nonlinearity* **25**(4), 931–960 (2012)
- [BDV] Bonatti, C., Díaz, L.J., Viana, M.: Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective. *Encyclopaedia of Mathematical Sciences. Mathematical Physics*, III, vol. 102. Springer, Berlin (2005)
- [CLR] Cao, Y., Luzzatto, S., Rios, I.: Some non-hyperbolic systems with strictly non-zero Lyapunov exponents for all invariant measures: horseshoes with internal tangencies. *Discrete Contin. Dyn. Syst.* **15**(1), 61–71 (2006)
- [CP] Crovisier, S., Pujals, E.R.: Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms. *Invent. Math.* **201**(2), 385–517 (2015)
- [DG] Díaz, L.J., Gorodetski, A.: Non-hyperbolic ergodic measures for non-hyperbolic homoclinic classes. *Ergodic Theory Dynam. Syst.* **29**(5), 1479–1513 (2009)
- [GIKN] Gorodetski, A., Ilyashenko, Yu.S., Kleptsyn, V.A., Nalsky, M.B.: Nonremovability of zero Lyapunov exponents. (Russian) *Funktsional. Anal. i Prilozhen.* **39**(1), 27–38 [translation in *Funct. Anal. Appl.* **39**(1), 21–30 (2005)]
- [GW] Guckenheimer, J., Williams, R.: Structural stability of Lorenz attractors. *Inst. Hautes Études Sci. Publ. Math.* **50**, 59–72 (1979)
- [H] Hayashi, S.: Connecting invariant manifolds and the solution of the C^1 stability and Ω -stability conjectures for flows. *Ann. Math.* **145**(1), 81–137 (1997)
- [KN] Kleptsyn, V.A., Nalsky, M. B.: Stability of the existence of nonhyperbolic measures for C^1 -diffeomorphisms. *Funktsional. Anal. i Prilozhen.* **41**(4), 30–45 (2007) [translation in *Funct. Anal. Appl.* **41**(4), 271–283 (2007)]
- [Ku] Kupka, I.: Contribution à la théorie des champs génériques. *Contrib. Differ. Equations* **2**, 457–484 (1963)
- [LOR] Leplaideur, R., Oliveira, K., Rios, I.: Equilibrium states for partially hyperbolic horseshoes. *Ergodic Theory Dynam. Syst.* **31**(1), 179–195 (2011)
- [M1] Mañé, R.: Contributions to stability conjecture. *Topology* **17**(4), 383–396 (1978)
- [M2] Mañé, R.: An ergodic closing lemma. *Ann. Math.* **116**(3), 503–540 (1982)
- [Mo] Moreira, C.G.: There are no C^1 -stable intersections of regular Cantor sets. *Acta Math.* **206**(2), 311–323 (2011)
- [MS] Moreira, C.G., Silva, W.: On the geometry of horseshoes in higher dimensions. [arXiv:1210.2623](https://arxiv.org/abs/1210.2623) (preprint).
- [MY] Moreira, C.G., Yoccoz, J.-C.: Stable intersections of regular Cantor sets with large Hausdorff dimensions. *Ann. Math.* **154**(1), 45–96 (2001)
- [N] Newhouse, S.: Diffeomorphisms with infinitely many sinks. *Topology* **13**, 9–18 (1974)
- [P] Palis, J.: A global view of dynamics and a conjecture on the denseness and finitude of attractors. *Astérisque* **261**, 335–347 (2000)
- [Pe] Pesin, Y.: Families of invariant manifolds for diffeomorphisms with non-zero characteristic Lyapunov exponents. *Izvestiya Akademii Nauk SSSR* **40**, 1332–1379 (1976)
- [PS] Pujals, E.R., Sambarino, M.: Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. *Ann. Math.* **151**(3), 961–1023 (2000)
- [S] Shub, M.: Topological transitive diffeomorphisms on T^4 . *Lecture Notes Math.* **206**, 39–40 (1971)
- [SW] Shub, M., Wilkinson, A.: Pathological foliations and removable zero exponents. *Invent. Math.* **139**(3), 495–508 (2000)
- [Sm] Smale, S.: Stable manifolds for differential equations and diffeomorphisms. *Ann. Scuola Norm. Sup. Pisa* **17**, 97–116 (1963)
- [W] Walters, P.: An Introduction to Ergodic Theory. *Graduate Texts in Mathematics*, vol. 79. Springer, New York (1981)