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Bethe Ansatz and the Spectral Theory of Affine Lie Algebra-Valued Connections I. The simply-laced Case

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Abstract: We study the ODE/IM correspondence for ODE associated to $\widehat{\mathfrak{g}}$-valued connections, for a simply-laced Lie algebra $\mathfrak{g}$. We prove that subdominant solutions to the ODE defined in different fundamental representations satisfy a set of quadratic equations called Ψ-system. This allows us to show that the generalized spectral determinants satisfy the Bethe Ansatz equations.

Contents

1. Introduction

The ODE/IM correspondence, developed since the seminal papers [3,13], concerns the relations between the generalized spectral problems of linear ordinary differential equations (ODE) and two dimensional integrable quantum field theories (QFT), or continuous limit of quantum integrable systems. Accordingly, the acronym IM usually refers to integrable models (or integrals of motion). More recently, it was observed that the correspondence can also be interpreted as a relation between classical and quantum integrable systems [27], or as an instance of the Langlands duality [17].

The original ODE/IM correspondence states that the spectral determinant of certain Schrödinger operators coincides with the vacuum eigenvalue $Q(E)$ of the Q-operators

of the quantum KdV equation [4], when E is interpreted as a parameter of the theory. This correspondence has been proved [3,13] by showing that the spectral determinant and the eigenvalues of the Q operator solve the same functional equation, namely the Bethe Ansatz, and they have the same analytic properties.

The correspondence has soon been generalized to higher eigenvalues of the Quantum KdV equation—obtained modifying the potential of the Schrödinger operator [5]—as well as to Conformal Field Theory with extended $\mathcal{W}_n$-symmetries, by considering ODE naturally associated with the simple Lie algebra A_n (scalar linear ODEs of order $n + 1$) [10,34]. Within this approach, the original construction corresponds to the Lie algebra A_1, and the associated ODE is the Schrödinger equation.

After these results, it became clear that a more general picture including all simple Lie algebras—as well as massive (i.e., non conformal) Integrable QFT—was missing. While the latter issue was addressed in [20,27] (and more recently in [1,11,30]), the ODE/IM correspondence for simple Lie algebras different from A_n has not yet been fully understood.

The first work proposing an extension of the ODE/IM correspondence to classical Lie algebras other than A_n is [9], where the authors introduce a $\mathfrak{g}$-related scalar ODE as well as the important concept of Ψ-system. The latter is a set of quadratic relations that solutions of the $\mathfrak{g}$-related ODE are expected to satisfy in different representations. It was then recognized in [17] that the $\mathfrak{g}$-related scalar differential operators studied in [5,9] can be regarded as *affine opers* associated to the Langlands dual of the affine Lie algebra $\widehat{\mathfrak{g}}$. This discovery gave the theory a solid algebraic footing—based on Kac-Moody algebras and Drinfeld-Sokolov Lax operators—which was further investigated in [32] and that we hereby follow.

In the present paper we establish the ODE/IM correspondence for a simply laced Lie algebra $\mathfrak{g}$, in which case the affine Lie algebra $\widehat{\mathfrak{g}}$ coincides with its Langlands dual [17]. More concretely, we prove that for any simple Lie algebra $\mathfrak{g}$ of ADE type there exists a family of spectral problems, which are encoded by entire functions $Q^{(i)}(E)$, $i = 1, \ldots, n = \text{rank } \mathfrak{g}$, satisfying the ADE-Bethe Ansatz equations [2,31,35]:

$$\prod_{j=1}^{n} \Omega^{\beta_j C_{ij}} \frac{Q^{(j)}\left(\Omega^{\frac{C_{ij}}{2}} E^*\right)}{Q^{(j)}\left(\Omega^{-\frac{C_{ij}}{2}} E^*\right)} = -1, \tag{1.1}$$

for every $E^* \in \mathbb{C}$ such that $Q^{(i)}(E^*) = 0$. In equation (1.1), $C = (C_{ij})_{i,j=1}^{n}$ is the Cartan matrix of the algebra $\mathfrak{g}$, while the phase Ω and the numbers β_j are the free-parameters of the equation. More precisely, following [17] we introduce the $\widehat{\mathfrak{g}}$-valued connection

$$\mathcal{L}(x, E) = \partial_x + \frac{\ell}{x} + e + p(x, E)e_0, \tag{1.2}$$

where $\ell \in \mathfrak{h}$ is a generic element of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $e_0, e_1, \ldots, e_n$ are the positive Chevalley generators of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ and $e = \sum_{i=1}^{n} e_i$. In addition, the potential[1] is $p(x, E) = x^{Mh^\vee} - E$, where $M > 0$ and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

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[1] We stick to a potential of this form for simplicity. However, all proofs work with minor modifications for a more general potential discussed in [5,17].

After [32], for every fundamental representation[2] $V^{(i)}$ of $\widehat{\mathfrak{g}}$ we consider the differential equation

$$\mathcal{L}(x, E)\Psi(x, E) = 0. \tag{1.3}$$

The above equation has two singular points, namely a regular singularity in $x = 0$ and an irregular singularity in $x = \infty$, and a natural connection problem arises when one tries to understand the behavior of the solutions globally. We show that for every representation $V^{(i)}$ there exists a unique solution $\Psi^{(i)}(x, E)$ to Eq. (1.3) that is subdominant for $x \to +\infty$ (by subdominant we mean the solution that goes to zero the most rapidly). Then, we define the generalized spectral determinant $Q^{(i)}(E; \ell)$ as the coefficient of the most singular (yet algebraic) term of the expansion of $\Psi^{(i)}(x, E)$ around $x = 0$. Finally, we prove that for generic $\ell \in \mathfrak{h}$ the generalized spectral determinants $Q^{(i)}(E; \ell)$ are entire functions and satisfy the Bethe Ansatz equation (1.1), also known as Q-system.

The paper is organized as follows. In Sect. 2 we review some basic facts about the theory of finite dimensional Lie algebras, affine Kac-Moody algebras and their finite dimensional representations. Furthermore, we introduce the relevant representations that we will consider throughout the paper.

Section 3 is devoted to the asymptotic analysis of Eq. (1.3). The main result is provided by Theorem 3.4, from which the existence of the fundamental (subdominant) solution $\Psi^{(i)}(x, E)$ follows. The asymptotic properties of the solution turn out to depend on the spectrum of $\Lambda = e_0 + e \in \widehat{\mathfrak{g}}$, as observed in [32].

The main result of Sect. 4 is Theorem 4.6, which establishes the following quadratic relation among the functions $\Psi^{(i)}(x, E)$, known as Ψ-system, that was conjectured in [9]:

$$m_i\left(\Psi^{(i)}_{-\frac{1}{2}}(x, E) \wedge \Psi^{(i)}_{\frac{1}{2}}(x, E)\right) = \otimes_{j \in I} \Psi^{(j)}(x, E)^{\otimes B_{ij}}. \tag{1.4}$$

Here, $\Psi^{(i)}_{\pm\frac{1}{2}}(x, E)$ is the twisting of the solution $\Psi^{(i)}(x, E)$ defined in (3.8), while m_i is the morphism of $\widehat{\mathfrak{g}}$-modules defined by (2.11), and $B = (B_{ij})^n_{i,j=1}$ denotes the incidence matrix of $\mathfrak{g}$. In order to prove Theorem 4.6, we first establish in Proposition 4.4 some important properties of the eigenvalues of Λ on the relevant representations previously introduced. More precisely, we show that in each representation $V^{(i)}$ there exists a maximal positive eigenvalue $\lambda^{(i)}$ (see Definition 3.3), and that the following remarkable identity holds:

$$\left(e^{-\frac{\pi i}{h^\vee}} + e^{\frac{\pi i}{h^\vee}}\right)\lambda^{(i)} = \sum_{j=1}^n B_{ij}\lambda^{(j)}, \qquad i = 1, \ldots, n = \operatorname{rank} \mathfrak{g}. \tag{1.5}$$

Equation (1.5) shows that the vector $(\lambda^{(1)}, \ldots, \lambda^{(n)})$ is the Perron-Frobenius eigenvector of the incidence matrix B, and this implies that the $\lambda^{(i)}$'s are (proportional to) the masses of the Affine Toda Field Theory with the same underlying Lie algebra $\mathfrak{g}$ [18].

In Sect. 5 we derive the Bethe Ansatz equations (1.1). To this aim, we study the local behavior of Eq. (1.3) close to the Fuchsian singularity $x = 0$, and we define the generalized spectral determinants $Q^{(i)}(E; \ell)$ and $\widetilde{Q}^{(i)}(E; \ell)$ using some properties of the Weyl group of $\mathfrak{g}$. In addition, using the Ψ-system (1.4) we prove Theorem 5.3, which gives a set of quadratic relations among the generalized spectral determinants [Eq. (5.6)]. Evaluating these relations at the zeros of the functions $Q^{(i)}(E; \ell)$, we obtain the Bethe Ansatz equations (1.1).

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[2] Actually $V^{(i)}$ is an evaluation representation of the ith fundamental representation of $\mathfrak{g}$.

Finally, in Sect. 6, we briefly study an integral representation of the subdominant solution of Eq. (1.3) with a linear potential $p(x, E) = x$, and we compute the spectral determinants $Q^{(i)}(E; \ell)$. Some explicit computations of the eigenvectors of Λ are provided in Appendix A.

In this work we do not study the asymptotic behavior of the $Q^{(i)}(E)$'s, for $E \ll 0$, which was conjectured in [9], neither do we address the problem of solving the Bethe Ansatz equations (1.1)—for a given asymptotic behavior of the functions $Q^{(i)}(E)$'s— via the Destri-de Vega integral equation [8]. These problems are of a rather different mathematical nature with respect to the ones treated in this work, and they will be considered in a separate paper.

Note added in press. We proved the ODE/IM correspondence for non simply-laced Lie algebras in [29].

While we were writing the present paper, we became aware of the work [12] (see [30] for a more detailed exposition) that was being written by Patrick Dorey, Stefano Negro and Roberto Tateo. Their paper focuses on ODE /IM correspondence for massive models related to affine Lie algebras and deals with similar problems in representation theory. We thank Patrick, Stefano and Roberto for sharing a preliminary version of their paper.

2. Affine Kac-Moody Algebras and Their Finite Dimensional Representations

In this section we review some basic facts about simple Lie algebras, affine Kac-Moody algebras and their finite dimensional representations, which will be used throughout the paper. The discussion is restricted to simple Lie algebras of ADE type, but most of the results hold also in the non-simply laced case; we refer to [19,24] for further details. At the end of the section we introduce a class of $\widehat{\mathfrak{g}}$-valued connections and the associated differential equations.

2.1. Simple lie algebras and fundamental representations. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra of ADE type, and let n be its rank. The Dynkin diagrams associated to these algebras are given in Table 1. Let $C = (C_{ij})_{i,j \in I}$ be the Cartan matrix of $\mathfrak{g}$, and let us denote by $B = 2\mathbb{1}_n - C$ the corresponding incidence matrix. Since $\mathfrak{g}$ is simply-laced, it follows that $B_{ij} = 1$ if the nodes i, j of the Dynkin diagram of $\mathfrak{g}$ are linked by an edge, and $B_{ij} = 0$ otherwise.

Let $\{f_i, h_i, e_i \mid i \in I = \{1, \dots, n\}\} \subset \mathfrak{g}$ be the set of Chevalley generators of $\mathfrak{g}$. They satisfy the following relations ($i, j \in I$):

$$[h_i, h_j] = 0, \quad [h_i, e_j] = C_{ij} e_j, \quad [h_i, f_j] = -C_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_i, \quad (2.1)$$
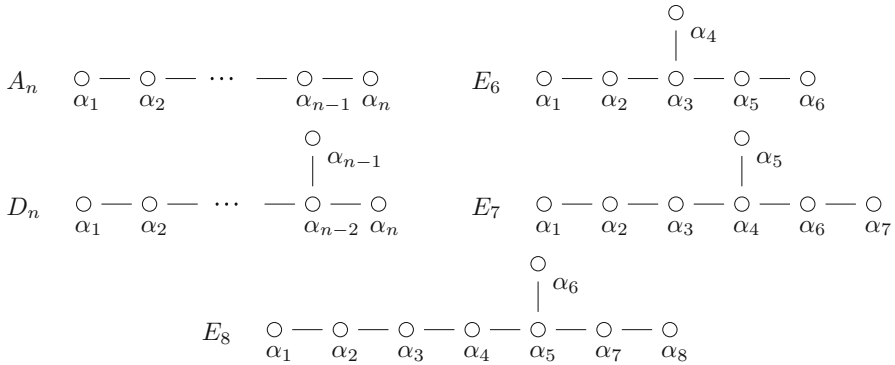
together with the Serre's identities. Recall that

$$\mathfrak{h} = \bigoplus_{i \in I} \mathbb{C} h_i \subset \mathfrak{g}$$

is a Cartan subalgebra of $\mathfrak{g}$ with corresponding Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right). \quad (2.2)$$

Table 1. Dynkin diagrams of simple Lie algebras of ADE type

As usual, the $\mathfrak{g}_\alpha$ are the eigenspaces of the adjoint representation and $R \subset \mathfrak{h}^*$ is the set of roots. For every $i \in I$, let $\alpha_i \in \mathfrak{h}^*$ be defined by $\alpha_i(h_j) = C_{ij}$, for every $j \in I$. Then $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_{\alpha_i}$. Hence, $\Delta = \{\alpha_i \mid i \in I\} \subset R$. The elements $\alpha_1, \ldots, \alpha_n$ are called the simple roots of $\mathfrak{g}$. They can be associated to each vertex of the Dynkin diagram as in Table 1. We denote by

$$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}, \ \forall i \in I\} \subset \mathfrak{h}^*$$

the set of weights of $\mathfrak{g}$ and by

$$P^+ = \{\lambda \in P \mid \lambda(h_i) \geq 0, \ \forall i \in I\} \subset P$$

the subset of dominant weights of $\mathfrak{g}$. We recall that we can define a partial ordering on P as follows: for $\lambda, \mu \in P$, we say that $\lambda < \mu$ if $\lambda - \mu$ is a sum of positive roots. For every $\lambda \in \mathfrak{h}^*$ there exists a unique irreducible representation $L(\lambda)$ of $\mathfrak{g}$ such that there is $v \in L(\lambda)\backslash\{0\}$ satisfying

$$h_i v = \lambda(h_i)v, \quad e_i v = 0, \quad \text{for every } i \in I.$$

$L(\lambda)$ is called the highest weight representation of weight λ, and v is the highest weight vector of the representation. We have that $L(\lambda)$ is finite dimensional if and only if $\lambda \in P^+$, and conversely, any irreducible finite dimensional representation of $\mathfrak{g}$ is of the form $L(\lambda)$ for some $\lambda \in P^+$. For $\lambda \in P^+$, the representation $L(\lambda)$ can be decomposed in the direct sum of its (finite dimensional) weight spaces $L(\lambda)_\mu$, with $\mu \in P$, and we have that $L(\lambda)_\mu \neq 0$ if and only if $\mu < \lambda$. We denote by P_λ the set of weights appearing in the weight space decomposition of $L(\lambda)$; the multiplicity of μ in the representation $L(\lambda)$ is defined as the dimension of the weight space $L(\lambda)_\mu$.

Recall that the fundamental weights of $\mathfrak{g}$ are those elements $\omega_i \in P^+$, $i \in I$, satisfying

$$\omega_i(h_j) = \delta_{ij}, \quad \text{for every } j \in I. \tag{2.3}$$

The corresponding highest weight representations $L(\omega_i)$, $i \in I$, are known as fundamental representations of $\mathfrak{g}$, and for every $i \in I$ we denote by $v_i \in L(\omega_i)$ the highest weight vector of the representation $L(\omega_i)$. Since the simple roots and the fundamental weights of $\mathfrak{g}$ are related via the Cartan matrix (which is non-degenerate) by the relation

$$\omega_i = \sum_{j \in I} (C^{-1})_{ji}\alpha_j, \quad i \in I,$$

then we can associate to the ith vertex of the Dynkin diagram of $\mathfrak{g}$ the corresponding fundamental representation $L(\omega_i)$ of $\mathfrak{g}$.

2.2. The representations $L(\eta_i)$. Let us consider the dominant weights

$$\eta_i = \sum_{j \in I} B_{ij} \omega_j, \quad i \in I,$$

where $B = (B_{ij})_{i,j \in I}$ is the incidence matrix defined in Sect. 2.1, and let $L(\eta_i)$ be the corresponding irreducible finite dimensional representations. In addition, we consider the tensor product representations

$$\bigotimes_{j \in I} L(\omega_j)^{\otimes B_{ij}}, \quad i \in I.$$

We now show that we can find a copy of the irreducible representation $L(\eta_i)$ inside the representations $\bigwedge^2 L(\omega_i)$ and $\bigotimes_{j \in I} L(\omega_j)^{\otimes B_{ij}}$. This will be used in Section 4 to construct the so-called ψ-system.

Lemma 2.1. *The following facts hold in the representation $\bigwedge^2 L(\omega_i)$.*

(a) *For every $j \in I$, we have that*

$$e_j(f_i v_i \wedge v_i) = 0.$$

(b) *For every $h \in \mathfrak{h}$, we have that*

$$h(f_i v_i \wedge v_i) = \eta_i(h)(f_i v_i \wedge v_i).$$

(c) *Any other weight of the representation $\bigwedge^2 L(\omega_i)$ is smaller than η_i.*

Proof. Since v_i is a highest weight vector for $L(\omega_i)$ we have that $e_j v_i = 0$ for every $j \in I$. Moreover, using the relations $[e_j, f_i] = \delta_{ij} h_i$ together with Eq. (2.3), we have

$$e_j f_i v_i = f_i e_j v_i + \delta_{ij} h_i v_i = \delta_{ij} v_i,$$

and therefore $e_j(f_i v_i \wedge v_i) = \delta_{ij} v_i \wedge v_i = 0$, proving part (a). By linearity, it suffices to show part (b) for $h = h_i$, $i \in I$. Recall that, by Eq. (2.1), we have $[h_i, f_j] = -C_{ij} f_j$, for every $i, j \in I$. Then,

$$h_j f_i v_i = \left(f_i h_j + [h_j, f_i]\right) v_i = (\delta_{ij} - C_{ji}) f_i v_i.$$

Hence, $h_j(f_i v_i \wedge v_i) = (2\delta_{ij} - C_{ji})(f_i v_i \wedge v_i) = \eta_i(h_j)(f_i v_i \wedge v_i)$. Finally, let $w \neq \omega_i, \eta_i - \omega_i$ be a weight appearing in $L(\omega_i)$. It follows from representation theory of simple Lie algebras that

$$\omega < \eta_i - \omega_i = \omega_i - \sum_{j \in I} C_{ji} \omega_j < \omega_i.$$

Therefore, by part (b), the maximal weight of $\bigwedge^2 L(\omega_i)$ is $w_i + \eta_i - \omega_i = \eta_i$, thus proving part (c). $\square$

As a consequence of the above Lemma it follows that $f_i v_i \wedge v_i \in \bigwedge^2 L(\omega_i)$ is a highest weight vector of weight η_i. Since $\mathfrak{g}$ is simple, the subrepresentation of $\bigwedge^2 L(\omega_i)$ generated by the highest weight vector $f_i v_i \wedge v_i$ is irreducible. Therefore, it is isomorphic to $L(\eta_i)$. By the complete reducibility of $\bigwedge^2 L(\omega_i)$ we can decompose it as follows:

$$\bigwedge^2 L(\omega_i) = L(\eta_i) \oplus U ,$$

where U is the direct sum of all the irreducible representations different from $L(\eta_i)$. By an abuse of notation we denote with the same symbol the representation $L(\eta_i)$ and its copy in $\bigwedge^2 L(\omega_i)$. For every $i \in I$, let us denote by $w_i = \otimes_{j \in I} v_j^{\otimes B_{ij}}$. The following analogue of Lemma 2.1 in the case of the representation $\bigotimes_{j \in I} L(\omega_j)^{\otimes B_{ij}}$ holds true.

Lemma 2.2. *The following facts hold in the representation* $\bigotimes_{j \in I} L(\omega_j)^{\otimes B_{ij}}$.

(a) *For every $j \in I$, we have that*

$$e_j w_i = 0.$$

(b) *For every $h \in \mathfrak{h}$, we have that*

$$h w_i = \eta_i(h) w_i.$$

(c) *Any other weight of the representation $\bigotimes_{j \in I} L(\omega_j)^{\otimes B_{ij}}$ is smaller than η_i.*

Proof. Same as the proof of Lemma 2.1. $\square$

As for the previous discussion, by Lemma 2.2 it follows that $w_i \in \bigotimes_{j \in I} L(\omega_j)^{\otimes B_{ij}}$ is a highest weight vector which generates an irreducible subrepresentation isomorphic to $L(\eta_i)$. Lemmas 2.1 and 2.2, together with the Schur Lemma, imply that there exists a unique morphism of representations

$$m_i : \bigwedge^2 L(\omega_i) \to \bigotimes_{j \in I} L(\omega_j)^{\otimes B_{ij}} , \qquad (2.4)$$

such that $\operatorname{Ker} m_i = U$ and $m_i(f_i v_i \wedge v_i) = w_i$.

2.3. Affine Kac-Moody algebras and finite dimensional representations. Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$. Let us denote by κ the Killing form of $\mathfrak{g}$ and let us fix the following non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$ ($a, b \in \mathfrak{g}$):

$$(a|b) = \frac{1}{h^\vee} \kappa(a|b).$$

Let $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ be the space of Laurent polynomials with coefficients in $\mathfrak{g}$. For $a(t) = \sum_{i=-M}^{N} a_i t^i \in \mathbb{C}[t, t^{-1}]$, we let

$$\operatorname{Res}_{t=0} a(t) dt = a_{-1}.$$

The affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ is the vector space $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ endowed with the following Lie algebra structure ($a, b \in \mathfrak{g}$, $f(t), g(t) \in \mathbb{C}[t, t^{-1}]$):

$$[a \otimes f(t), b \otimes g(t)] = [a, b] \otimes f(t)g(t) + (a|b)\operatorname{Res}_{t=0}\left(f'(t)g(t)dt\right)c,$$
$$[c, \widehat{\mathfrak{g}}] = 0. \tag{2.5}$$

The set of Chevalley generators of $\widehat{\mathfrak{g}}$ is obtained by adding to the Chevalley generators of $\mathfrak{g}$ some new generators f_0, h_0, e_0 (for the construction, see for example [24]). The generator e_0, which plays an important role in the paper, can be constructed as follows. There exists a root $-\theta \in R$, known as the lowest root of $\mathfrak{g}$, such that $-\theta - \alpha_i \notin R$, for every $i \in I$. Then

$$e_0 = a \otimes t, \quad \text{for some } a \in \mathfrak{g}_{-\theta}. \tag{2.6}$$

We now consider an important class of representations of $\widehat{\mathfrak{g}}$. Let V be a finite dimensional representation of $\mathfrak{g}$. For $\zeta \in \mathbb{C}^*$ we define a representation of $\widehat{\mathfrak{g}}$, which we denote by $V(\zeta)$, as follows: as a vector space we take $V(\zeta) = V$, and the action of $\widehat{\mathfrak{g}}$ is defined by

$$(a \otimes f(t))v = f(\zeta)(av), \quad c\,v = 0, \quad \text{for } a \in \mathfrak{g}, \quad f(t) \in \mathbb{C}[t, t^{-1}], \quad v \in V.$$

The representation $V(\zeta)$ is called an evaluation representation of $\widehat{\mathfrak{g}}$. If V and W are representations of $\mathfrak{g}$ then any morphism of representations $f : V \to W$ can obviously be extended to a morphism of representations $f : V(\zeta) \to W(\zeta)$, which we denote by the same letter by an abuse of notation. Similarly, when referring to weights or weight vectors of the evaluation representation $V(\zeta)$, we mean the weights and weight vectors of the representation V with respect to the action of $\mathfrak{g}$.

For a representation V of $\mathfrak{g}$ and $k \in \mathbb{C}$, we denote $V_k = V(e^{2\pi i k})$ the evaluation representation of $\widehat{\mathfrak{g}}$ corresponding to $\zeta = e^{2\pi i k}$. Clearly, if $k \in \mathbb{Z}$ then $V_k = V_0 = V(1)$, and if $k \in \frac{1}{2} + \mathbb{Z}$ then $V_k = V_{\frac{1}{2}} = V(-1)$. Then, for each vertex $i \in I$ of the Dynkin diagram of $\mathfrak{g}$, we associate the following finite dimensional representation of $\widehat{\mathfrak{g}}$:

$$V^{(i)} = L(\omega_i)_{\frac{p(i)}{2}}, \tag{2.7}$$

where $L(\omega_i)$ is the fundamental representation of $\mathfrak{g}$ with highest weight ω_i, and the function $p : I \to \mathbb{Z}/2\mathbb{Z}$ is defined inductively as follows:

$$p(1) = 0, \quad p(i) = p(j) + 1, \text{ for } j < i \text{ such that } B_{ij} > 0. \tag{2.8}$$

By an abuse of language we call the representations (2.7), for $i \in I$, the finite dimensional fundamental representations of $\widehat{\mathfrak{g}}$. These representations will play a fundamental role in the present paper.

Example 2.3. In type A_n, $V^{(1)} = L(\omega_1)_0$ is the evaluation representation at $\zeta = 1$ of the standard representation $L(\omega_1) = \mathbb{C}^{n+1}$.

Moreover,

$$V^{(i)} = \bigwedge^i V^{(1)}_{\frac{i-1}{2}}, \quad \text{for } k = 1, \ldots, n.$$

Example 2.4. In type D_n, $V^{(1)} = L(\omega_1)_0$ is the evaluation representation at $\zeta = 1$ of the standard representation $L(\omega_1) = \mathbb{C}^{2n}$. Moreover,

$$V^{(i)} = \bigwedge^i V^{(1)}_{\frac{i-1}{2}}, \quad \text{for } k = 1, \ldots, n-2.$$

Finally, $V^{(n-1)} = L(\omega_{n-1})_{\frac{n}{2}}$ and $V^{(n)} = L(\omega_n)_{\frac{n}{2}}$ are the evaluation representations at $\zeta = (-1)^n$ of the so-called half-spin representations.

Example 2.5. In type E_6, $V^{(1)} = L(\omega_1)_0$ and $V^{(5)} = L(\omega_5)_0$ are the evaluation at $\zeta = 1$ of the two 27-dimensional representations (they are dual to each other). Moreover,

$$V^{(2)} = \bigwedge^2 V^{(1)}_{\frac{1}{2}}, \quad V^{(3)} = \bigwedge^3 V^{(1)}_0 \cong \bigwedge^3 V^{(5)}_0, \quad V^{(4)} = \bigwedge^2 V^{(5)}_{\frac{1}{2}}.$$

Finally, $V^{(6)} = L(\omega_6)_{\frac{1}{2}}$ is the evaluation representation at $\zeta = -1$ of the adjoint representation.

As a final remark, note that by Eq. (2.7) we have

$$\left(\bigwedge^2 L(\omega_i) \right)_{\frac{p(i)+1}{2}} = \bigwedge^2 V^{(i)}_{\frac{1}{2}}, \tag{2.9}$$

and if we introduce the evaluation representation

$$M^{(i)} = \bigotimes_{j \in I} \left(V^{(j)} \right)^{\otimes B_{ij}}, \quad \forall i \in I, \tag{2.10}$$

then by Eq. (2.7), and from the fact that $B_{ij} \neq 0$ implies $p(i) = p(j) + 1$, it follows that the morphism m_i defined by Eq. (2.4) extends to a morphism of evaluation representations

$$m_i : \bigwedge^2 V^{(i)}_{\frac{1}{2}} \to M^{(i)}. \tag{2.11}$$

The construction of the ψ-system will be provided by means of the above extended morphism. Due to Lemmas 2.1 and 2.2, the evaluation representation

$$W^{(i)} = L(\eta_i)_{\frac{p(i)+1}{2}}, \quad i \in I, \tag{2.12}$$

is a subrepresentation of both $\bigwedge^2 V^{(i)}_{\frac{1}{2}}$ and $M^{(i)}$, and the copies of $W^{(i)}$ in these representations are identified by means of the morphism m_i.

2.4. $\widehat{\mathfrak{g}}$-Valued connections and differential equations. Let $\{f_i, h_i, e_i \mid i = 0, \ldots, n\} \subset \widehat{\mathfrak{g}}$ be the set of Chevalley generators of $\widehat{\mathfrak{g}}$, and let us denote $e = \sum_{i=1}^{n} e_i$. Fix an element $\ell \in \mathfrak{h}$. Following [17] (see also [32]), we consider the $\widehat{\mathfrak{g}}$-valued connection

$$\mathcal{L}(x, E) = \partial_x + \frac{\ell}{x} + e + p(x, E)e_0, \tag{2.13}$$

where $p(x, E) = x^{Mh^\vee} - E$, with $M > 0$ and $E \in \mathbb{C}$. Since e is a principal nilpotent element, it follows from the Jacobson-Morozov Theorem together with Eq. (2.6) that there exists an element $h \in \mathfrak{h}$ such that

$$[h, e] = e, \quad [h, e_0] = -(h^\vee - 1)e_0, \tag{2.14}$$

see for instance [7]. Under the isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$ provided by the Killing form, the element h corresponds to the Weyl vector (half sum of positive roots). Let $k \in \mathbb{C}$ and introduce the quantities

$$\omega = e^{\frac{2\pi i}{h^\vee(M+1)}}, \quad \Omega = e^{\frac{2\pi i M}{M+1}} = \omega^{h^\vee M}.$$

Moreover, let $\mathcal{M}_k$ be the automorphism of $\widehat{\mathfrak{g}}$-valued connections which fixes ∂_x, $\mathfrak{g}$ and c and sends $t \to e^{2\pi i k} t$. Then from Eqs. (2.5), (2.13) and (2.14) we get

$$\mathcal{M}_k \left(\omega^{k\,\mathrm{ad}\,h} \mathcal{L}(x, E) \right) = \omega^k \mathcal{L}(\omega^k x, \Omega^k E). \tag{2.15}$$

We denote $\mathcal{L}_k(x, E) = \mathcal{M}_k (\mathcal{L}(x, E))$, for every $k \in \mathbb{C}$. Note in particular that $\mathcal{L}_0(x, E) = \mathcal{L}(x, E)$. Equation (2.15) implies that for a given finite dimensional representation V of $\mathfrak{g}$, the action of $\mathcal{L}_k(x, E)$ on the evaluation representation V_0 of $\widehat{\mathfrak{g}}$ is the same as the action of $\mathcal{L}(x, E)$ on V_k, for every $k \in \mathbb{C}$. In addition, let $\widehat{\mathbb{C}}$ be the universal cover of $\mathbb{C}^*$, suppose $\varphi(x, E) : \widehat{\mathbb{C}} \to V_0$ is a family—depending on the parameter E—of solutions of the (system of) ODE

$$\mathcal{L}(x, E)\varphi(x, E) = 0, \tag{2.16}$$

and for $k \in \mathbb{C}$ introduce the function

$$\varphi_k(x, E) = \omega^{-kh} \varphi(\omega^k x, \Omega^k E). \tag{2.17}$$

Since on any evaluation representation of $\widehat{\mathfrak{g}}$ we have

$$e^{\mathrm{ad}\,a} b = e^a b e^{-a}, \quad a, b \in \widehat{\mathfrak{g}}, \tag{2.18}$$

then by Eqs. (2.15) and (2.18), we have that

$$\mathcal{L}_k(x, E)\varphi_k(x, E) = 0, \tag{2.19}$$

namely $\varphi_k(x, E) : \widehat{\mathbb{C}} \to V_k$ is a solution of (2.16) for the representation V_k.

3. Asymptotic Analysis

Let V be an evaluation representation of $\widehat{\mathfrak{g}}$. Let us denote by $\widetilde{\mathbb{C}} = \mathbb{C} \backslash \mathbb{R}_{\leq 0}$, the complex plane minus the semi-axis of real non-positive numbers, and let us consider a solution $\Psi : \widetilde{\mathbb{C}} \to V$ of

$$\mathcal{L}(x, E)\Psi(x) = \Psi'(x) + \left(\frac{\ell}{x} + e + p(x, E)e_0 \right) \Psi(x) = 0. \tag{3.1}$$

Equation (3.1) is a (system of) linear ODEs, with a Fuchsian singularity at $x = 0$ and an irregular singularity at $x = \infty$. Our goal is to prove the existence of solutions of Eq. (3.1) with a prescribed asymptotic behavior at infinity in a Stokes sector of the complex plane. In the present form, however, Eq. (3.1) falls outside the reach of classical results such as the Levinson theorem [26] or the complex Wentzel-Kramers-Brillouin (WKB) method [16], since the most singular term $p(x, E)e_0$ is nilpotent. In order to study the asymptotic behavior of solutions to (3.1) we use a slight modification of a gauge transform originally introduced in [32], and we then prove the existence of the desired solution for the new ODE by means of a Volterra integral equation.

We now consider the required transformation. First, we note that the function $p(x, E)^{\frac{1}{h^\vee}}$ has the asymptotic expansion

$$p(x, E)^{\frac{1}{h^\vee}} = q(x, E) + O(x^{-1-\delta}),$$

where

$$\delta = M(h^\vee(1+s) - 1) - 1 > 0, \quad s = \left\lfloor \frac{M+1}{h^\vee M} \right\rfloor, \tag{3.2}$$

and

$$q(x, E) = x^M + \sum_{j=1}^{s} c_j(E) x^{M(1-h^\vee j)}. \tag{3.3}$$

For every $j = 1, \ldots, s$, the function $c_j(E)$ is a monomial of degree j in E.

Lemma 3.1. *Let $\mathcal{L}(x, E)$ be the $\widehat{\mathfrak{g}}$-valued connection defined in* (2.13) *and let $h \in \mathfrak{h}$ satisfy relations* (2.14). *Then, we have the following gauge transformation*

$$q(x, E)^{\mathrm{ad}\, h} \mathcal{L}(x, E) = \partial_x + q(x, E)\Lambda + \frac{\ell - Mh}{x} + O(x^{-1-\delta}), \tag{3.4}$$

where $\Lambda = e_0 + e$, the function $q(x, E)$ is given by (3.3) *and δ by* (3.2).

Proof. It follows by a straightforward computation using relations (2.14). □

Remark 3.2. The transformation (3.4) differs from the one used in [32] and given by

$$x^{M\, \mathrm{ad}\, h} \mathcal{L}(x, E) = \partial_x + x^M \Lambda + \frac{\ell - Mh}{x} - \frac{e_0 E}{x^{M(h^\vee - 1)}}.$$

The latter transformation fails to be the correct if $M(h^\vee - 1) \le 1$. In fact, in this case the term $e_0 E / x^{M(h^\vee - 1)}$ goes to zero slowly and alters the asymptotic behavior.

We are then left to analyze the equation

$$\Psi'(x) + \left(q(x, E)\, \Lambda + \frac{\ell - Mh}{x} + A(x) \right) \Psi(x) = 0, \tag{3.5}$$

where $A(x) = O(x^{-1-Mh^\vee})$ is a certain matrix-valued function. From the general theory we expect the asymptotic behavior to depend on the primitive of $q(x, E)$, which is known as the action. In the generic case $\frac{M+1}{h^\vee M} \notin \mathbb{Z}_+$, the action is defined as

$$S(x, E) = \int_0^x q(y, E)dy, \quad x \in \widetilde{\mathbb{C}}, \tag{3.6}$$

where we chose the branch of $q(x, E)$ satisfying $q \sim |x|^M$ for x real. In the case $\frac{M+1}{h^\vee M} \in \mathbb{Z}_+$ then formula (3.6) becomes

$$S(x, E) = \sum_{j=0}^{s-1} \int_0^x c_j(E) y^{M(1-h^\vee j)} dy + c_s(E) \log x, \quad s = \frac{M+1}{h^\vee M}.$$

Existence and uniqueness of solutions to Eq. (3.1) [or equivalently, of (3.5)] depend on the properties of the element $\Lambda = e + e_0$ in the representation considered. We therefore introduce the following

Definition 3.3. Let A be an endomorphism of a vector space V. We say that a eigenvalue λ of A is maximal if it is real, its algebraic multiplicity is one, and $\lambda > \mathrm{Re}\, \mu$ for every eigenvalue μ of A.

We are now in the position to state the main result of this section.

Theorem 3.4. *Let V be an evaluation representation of $\widehat{\mathfrak{g}}$, and let the matrix represent-ing $\Lambda \in \widehat{\mathfrak{g}}$ in V have a maximal eigenvalue λ. Let $\psi \in V$ be the corresponding unique (up to a constant) eigenvector. Then, there exists a unique solution $\Psi(x, E) : \widetilde{\mathbb{C}} \to V$ to Eq. (3.1) with the following asymptotic behavior:*

$$\Psi(x, E) = e^{-\lambda S(x,E)} q(x, E)^{-h} \big(\psi + o(1) \big) \quad as \quad x \to +\infty.$$

Moreover, the same asymptotic behavior holds in the sector $|\arg x| < \frac{\pi}{2(M+1)}$, that is, for any $\delta > 0$ it satisfies

$$\Psi(x, E) = e^{-\lambda S(x,E)} q(x, E)^{-h} \big(\psi + o(1) \big), \quad in \ the \ sector \ |\arg x| < \frac{\pi}{2(M+1)} - \delta. \tag{3.7}$$

The function $\Psi(x, E)$ is an entire function of E.

Remark 3.5. We will prove in Sect. 4 that the hypothesis of the theorem are satisfied in the ADE case for the representations $V^{(i)}$, $\bigwedge^2_{\frac{1}{2}} V_1^{(i)}$, $M^{(i)}$ and $W^{(i)}$, which were introduced in the previous section.

Remark 3.6. In the case $M(h^\vee - 1) > 1$, the asymptotic expansion of the solution Ψ in the above theorem reduces to

$$\Psi(x, E) = e^{-\lambda \frac{x^{M+1}}{M+1}} q(x, E)^{-h} \big(\psi + o(1) \big), \quad in \ the \ sector \ |\arg x| < \frac{\pi}{2(M+1)}.$$

Before proving Theorem 3.4, we apply it to prove the existence of linearly independent solutions to Eq. (3.1). First, note that for any $k \in \mathbb{R}$ such that $|k| < \frac{h^\vee(M+1)}{2}$, the function

$$\Psi_k(x, E) = \omega^{-kh} \Psi(\omega^k x, \Omega^k E), \quad x \in \mathbb{R}_+ \tag{3.8}$$

defines, by analytic continuation, a solution $\Psi_k : \widetilde{\mathbb{C}} \to V_k$ of Eq. (3.1) in the represen-tation V_k. Theorem 3.4 implies the following result.

Corollary 3.7. *For any $k \in \mathbb{R}$ such that $|k| < \frac{h^\vee(M+1)}{2}$, on the positive real semi-axis the function Ψ_k has the asymptotic behaviour*

$$\Psi_k(x, E) = e^{-\lambda \gamma^k S(x,E)} q(x, E)^{-h} \gamma^{-kh} (\psi + o(1)), \quad x \gg 0, \tag{3.9}$$

where ψ is defined as in Theorem 3.4 and $\gamma = e^{\frac{2\pi i}{h^\vee}}$. Moreover, let $l \in \mathbb{Z}_+$ be such that $1 \leq l \leq \frac{h^\vee}{2}$. Then, the functions $\Psi_{\frac{1-l}{2}}, \ldots, \Psi_{\frac{l-1}{2}}$ are linearly independent solutions to Eq. (3.1) in the representation $V_{\frac{l+1}{2}}$.

Proof. A simple computation shows that $S(\omega^k x, \Omega^k E) = \gamma^k S(x, E)$, and similarly $q(\omega^k x, \Omega^k E) = \omega^M q(x, E)$. Therefore by Theorem 3.4,

$$\Psi(\omega^k x, \Omega^k E) = \omega^{-Mh} e^{-\gamma^k \lambda S(x,E)} q(x, E)^{-h} (\psi + o(1)), \quad x \gg 0.$$

Hence, on the positive real semi axis we have

$$\Psi_k(x, E) = e^{-\gamma^k \lambda S(x,E)} q(x, E)^{-h} \gamma^{-kh} (\psi + o(1)), \quad x \gg 0,$$

where we have used the identity $\omega^{M+1} = \gamma$. Note that γ^k, $k \in \{\frac{1-l}{2}, \ldots, \frac{l-1}{2}\}$, are pairwise distinct. Therefore, the vectors ψ_k are linearly independent. This concludes the proof. $\square$

Remark 3.8. A slight variation of the proof of Theorem 3.4 presented below allows one to prove that the asymptotic behavior of Ψ holds on a larger sector of the complex plane, consisting of three adjacent Stokes sectors. This refined result is necessary to consider the lateral connection problem, where one considers the linear relations among solutions that are subdominant in different Stokes sectors. Such a problem leads naturally to another instance of the ODE/IM correspondence, namely to spectral determinants related to the transfer matrix of integrable systems, see e.g. [28].

3.1. Proof of Theorem 3.4. To complete our analysis, we need to further simplify Eq. (3.5) with a new gauge transformation. Let us denote by $\widetilde{\mathcal{L}}(x, E) = q(x, E)^{\mathrm{ad}\, h} \mathcal{L}(x, E)$, where $\mathcal{L}(x, E)$ and h are as in Lemma 3.1.

Lemma 3.9. *Let* $\ell = \sum_{i \in I} \ell_i h_i$ *and* $h = \sum_{i \in I} a_i h_i$, *with* $\ell_i, a_i \in \mathbb{C}$, *and consider the element* $N = \sum_{i \in I} (\ell_i - M a_i) f_i \in \mathfrak{g}$. *Then we have the following gauge transformation:*

$$e^{\alpha(x)\,\mathrm{ad}\,N} \widetilde{\mathcal{L}}(x, E) = \partial + q(x, E)\, \Lambda + O(x^{-1-M}),$$

where $\alpha(x) = (x\, q(x, E))^{-1}$.

Proof. It follows by a straightforward computation using the explicit form of $\widetilde{\mathcal{L}}(x, E)$ given in Eq. (3.4), together with the definitions of N and $\alpha(x)$ and the commutation relations [see (2.1)]

$$[f_i, e_0] = 0 \quad \text{and} \quad [f_i, e_j] = \delta_{ij} h_i, \quad \text{for every } i, j \in I.$$

□

By Lemma 3.9 and Eq. (2.18), it follows that the ODE (3.5) is transformed into

$$\widetilde{\Psi}'(x) + \big(q(x, E)\, \Lambda + \widetilde{A}(x)\big)\, \widetilde{\Psi}(x) = 0, \qquad (3.10)$$

where $\widetilde{A}(x) = O(x^{-1-M})$, and $\widetilde{\Psi}(x) = G(x)\Psi(x)$, with $G(x) = e^{\alpha(x)N}$. Since $G(x) = \mathbb{1} + o(x^{-M})$, then we look for a solution to Eq. (3.10) with the same asymptotic behavior (3.7) as the solution $\Psi(x)$ to Eq. (3.5). Finally, we define $\Phi(x) = e^{\lambda S(x, E)} \widetilde{\Psi}(x)$. Then, it is easy to show that Eq. (3.10) takes the form

$$\Phi'(x) + \big(q(x, E)\, \tilde{\Lambda} + \widetilde{A}(x)\big)\, \Phi(x) = 0, \qquad (3.11)$$

where $\tilde{\Lambda} = \Lambda - \lambda \mathbb{1}$. Note that $\tilde{\Lambda}\, \psi = 0$. The problem is now reduced to prove the existence of a solution to Eq. (3.11) satisfying, for every $\varepsilon > 0$, the asymptotic condition

$$\Phi(x) = \psi + o(1), \quad \text{in the sector } |\arg x| < \frac{\pi}{2(M + 1)} - \varepsilon. \qquad (3.12)$$

To proceed with the proof we need to introduce some elements of WKB analysis, for which we refer to [16, Chap. 3]. We fix $\kappa > 0$ (possibly depending on E) big enough such that the Stokes sector

$$\Sigma = \{x \in \widetilde{\mathbb{C}} \mid \mathrm{Re}\, S(x, E) \geq \mathrm{Re}\, S(\kappa, E)\}$$

has the following two properties: S is a bi-holomorphic map from Σ to the right half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}\, z \geq 0\}$, and Σ coincides asymptotically with the sector $|\arg x| < \frac{\pi}{2(M+1)}$. In

other words, any ray of argument $|\arg x| < \frac{\pi}{2(M+1)}$ eventually lies inside Σ, while any ray of argument $|\arg x| > \frac{\pi}{2(M+1)}$ eventually lies in the complement of Σ. For any $\varepsilon > 0$, we define $D_\varepsilon = S^{-1}\{|\arg z| \leq \frac{\pi}{2} - \varepsilon\}$, and we introduce the space $\mathcal{B}$ of holomorphic bounded functions $U : D_\varepsilon \to V$ for which the limit $\lim_{x\to\infty} U(x) = U(\infty)$ is well-defined. The space $\mathcal{B}$ is complete with respect to the norm $\|U\|_\infty = \sup_{x\in D_\varepsilon} |U(x)|$. We construct the solution of Eq. (3.11) with the desired asymptotic behavior (3.12) as the solution of the following integral equation in the Banach space $\mathcal{B}$:

$$\Phi(x) = K[\Phi](x) + \psi^{(i)}, \tag{3.13}$$

where the Volterra integral operator K is defined as

$$K[\Phi](x) = \int_x^\infty e^{-\tilde{\Lambda}\left(S(x,E)-S(y,E)\right)} \widetilde{A}(y)\Phi(y)dy.$$

The integral above is computed along any admissible path c, and we say that a (piecewise differentiable) parametric curve c is admissible if $\int_0^t |\dot{c}(s)|ds = O(|c(t)|)$ and $\mathrm{Re}\, S(c(t), E)$ is a non decreasing function. For example, $S^{-1}(t + x)$ is an admissible path [16]. Note that a solution $\Phi(x)$ of Eq. (3.11), with the asymptotic behavior (3.12) belongs to $\mathcal{B}$. Moreover, it satisfies Eq. (3.10) if and only if it satisfies the Volterra integral equation (3.13). Indeed, provided that K is a continuous operator, we have that $\lim_{x\to\infty} K[\Phi](x) = 0$ and

$$\begin{aligned} K[\Phi]'(x) &= -\widetilde{A}(x)\Phi(x) - q(x, E)\,\tilde{\Lambda}K[\Phi](x) \\ &= -\widetilde{A}(x)\Phi(x) - q(x, E)\,\tilde{\Lambda}\left(\Phi(x) - \psi^{(i)}\right) \\ &= -\widetilde{A}(x)\Phi(x) - q(x, E)\,\tilde{\Lambda}\Phi(x), \end{aligned}$$

where the last equality follows from the fact that $\tilde{\Lambda}\psi^{(i)} = 0$.

In order to prove that the solution of the integral Eq. (3.13) exists and it is unique we start showing that K is a continuous operator on $\mathcal{B}$ and its spectral radius is zero, i.e. $\lim_{n\to\infty} \|K^n\|^{\frac{1}{n}} = 0$. By hypothesis, all eigenvalues of Λ have non-positive real part. Since Λ is semi-simple, i.e. diagonalizable, we deduce that there exists a $\beta > 0$ such that for any $\kappa \geq 0$ and any $\varphi \in V$,

$$|e^{\kappa\tilde{\Lambda}}\varphi| \leq \beta|\varphi|. \tag{3.14}$$

Since $|\widetilde{A}(x)| = O(x^{-1-M})$, we have that $|\widetilde{A}| \in L^1(c, |\dot{c}(t)|dt)$ for every admissible path c, and the function

$$\rho_c(x) = \beta \int_x^\infty |\widetilde{A}(c(s))||\dot{c}(s)|ds \tag{3.15}$$

is well-defined and satisfies $\lim_{x\to\infty} \rho_c(x) = 0$. Moreover, the inequality (3.14) implies that

$$\left| \int_x^\infty e^{-\tilde{\Lambda}\left(S(x,E)-S(y,E)\right)} \widetilde{A}(y)U(y)dy \right| \leq \rho_c(x)\|U\|_\infty,$$

for every $U \in \mathcal{B}$. By (3.14) and (3.15) it follows [16] that $K[U]$ is path-independent and thus well defined. Furthermore, we have

$$|K[U](x)| \leq \|U\|_\infty \rho(x) \quad \text{and} \quad \|K\| = \sup_{|U|_\infty=1} \|K[U]\|_\infty \leq \bar{\rho},$$

where

$$\rho(x) = \inf_{c \text{ admissible}} \rho_c(x), \quad \bar{\rho} = \sup_{x \in D_\varepsilon} \rho(x).$$

Note that $\rho(x)$ is a bounded function such that $\lim_{x \to \infty} \rho(x) = 0$. Hence, we conclude that K is a well-defined bounded operator on $\mathcal{B}$. By definition, $K^n[U]$ can be written in terms of $K(x, y)$ as

$$K^n[U](x) = \int_x^\infty \cdots \int_{y_{n-1}}^\infty K(x, y_1) \ldots K(y_{n-1}, y_n)U(y_n)dy_1 \ldots dy_n,$$

where $K(x, y)$ is the matrix-valued function

$$K(x, y) = e^{-\tilde{\Lambda}\left(S(x,E) - S(y,E)\right)} \tilde{A}(y),$$

and from this it follows that $K^n[U](x)$ can be estimated as

$$|K^n[U](x)| = \|U\|_\infty \int_x^\infty \cdots \int_{y_{n-1}}^\infty |\tilde{A}(y_1)| \ldots |\tilde{A}(y_n)|dy_1 \ldots dy_n \le \|U\|_\infty \frac{\rho^n(x)}{n!}.$$

Thus

$$\|K^n\| \le \frac{\bar{\rho}^n}{n!}, \tag{3.16}$$

and the series

$$\Phi = \sum_{n \in \mathbb{Z}_+} K^n[\psi]$$

converges in $\mathcal{B}$. Clearly Φ satisfies the integral Eq. (3.13), since

$$K[\Phi] = \sum_{n=1}^\infty K^n[\psi] = \Phi - \psi.$$

Moreover, Φ is the unique solution of Eq. (3.13), for if $\tilde{\Phi}$ is another solution then

$$K^n[\Phi - \tilde{\Phi}] = \Phi - \tilde{\Phi}, \quad \text{for every } n \in \mathbb{Z}_+.$$

By Eq. (3.16), it follows that $\Phi = \tilde{\Phi}$.

It remains to prove that Φ is an entire function of E. This follows, by a standard perturbation theory argument, from the fact that $\tilde{A}$ and S are holomorphic functions of E in D_ε. Theorem 3.4 is proved.

4. The Ψ-system

In this section, for a simple Lie algebra $\mathfrak{g}$ of ADE type, we prove an algebraic identity known as Ψ-system, which was conjectured first in [9]. Recall from Sect. 2.3 that we can write

$$\bigwedge^2 V_{\frac{1}{2}}^{(i)} \cong W^{(i)} \oplus U, \quad M^{(i)} = \bigotimes_{j \in I} \left(V^{(j)}\right)^{\otimes B_{ij}} \cong W^{(i)} \oplus \widetilde{U}, \qquad (4.1)$$

where the representation $W^{(i)} = L(\eta_i)_{\frac{p(i)+1}{2}}$, $i \in I$, was defined in Eq. (2.12), and U and $\widetilde{U}$ are direct sums of all the irreducible representations different from $W^{(i)}$. Due to (2.11), for every $i \in I$ we have a morphism of representations

$$m_i : \bigwedge V_{\frac{1}{2}}^{(i)} \to M^{(i)}, \quad \mathrm{Ker}\, m_i = U.$$

The Ψ-system is a set of quadratic relations, realized by means of the morphisms m_i, among the subdominant solutions to the linear ODE (3.1) defined on the representations $\bigwedge V_{\frac{1}{2}}^{(i)}$ and $M^{(i)}$.

In order to prove the main result of this section, we need to study the maximal eigenvalue (and the corresponding eigenvector) of Λ in the representations $\bigwedge^2 V_{\frac{1}{2}}^{(i)}$, $M^{(i)}$ and $W^{(i)}$. To this aim, we recall some further facts from the theory of simple Lie algebras and simple Lie groups. First, we need the following

Lemma 4.1. *Let $h \in \mathfrak{h}$ satisfy relations (2.14), and let us set $\gamma = e^{\frac{2\pi i}{h^\vee}}$. Then, for every $k \in \mathbb{C}$, the following formula holds:*

$$\gamma^{k\,\mathrm{ad}\,h} \Lambda = \gamma^k \mathcal{M}_{-k}(\Lambda). \qquad (4.2)$$

In particular, if ψ is an eigenvector of Λ in an evaluation representation V, then $\psi_k = \gamma^{-kh}\psi$ is an eigenvector of Λ in the representation V_k, and we have

$$\Lambda\psi = \lambda\psi \quad \text{if and only if} \quad \Lambda\psi_k = \gamma^k\lambda\psi_k. \qquad (4.3)$$

Proof. It follows directly from the commutation relations (2.14). $\quad\square$

Now consider the element $\bar{\Lambda} = \Lambda|_{t=1} \in \mathfrak{g}$, which is well known [25] to be a regular semisimple element of $\mathfrak{g}$. Therefore, its centralizer $\mathfrak{g}^{\bar{\Lambda}} = \{a \in \mathfrak{g} \mid [\bar{\Lambda}, a] = 0\}$ is a Cartan subalgebra of $\mathfrak{g}$, which we denote $\mathfrak{h}'$. Due to Lemma 4.1, it follows that

$$\gamma^{\mathrm{ad}\,h} \bar{\Lambda} = \gamma \bar{\Lambda}, \qquad (4.4)$$

where $h \in \mathfrak{h}$ is the element introduced in Sect. 2 satisfying relations (2.14). Equation (4.4) says that $\bar{\Lambda}$ is an eigenvector of $\gamma^{\mathrm{ad}\,h}$, and this in turn implies that $\mathfrak{h}'$ is stable under the action of $\gamma^{\mathrm{ad}\,h}$. Since $\gamma^{\mathrm{ad}\,h}$ is a Coxeter-Killing automorphism of $\mathfrak{g}$ [25], we can regard $\gamma^{\mathrm{ad}\,h}$ as a Coxeter-Killing automorphism of $\mathfrak{h}'$. From these considerations it follows—as proved in [6]—that we can choose a set of Chevalley generators of $\mathfrak{g}$, say $\{f_i', h_i', e_i' \mid i \in I\} \subset \mathfrak{g}$, such that $\mathfrak{h}' = \bigoplus_{i \in I} \mathbb{C}h_i'$ and that the identity

$$2\mathbb{1}_n + \gamma^{\mathrm{ad}\,h} + \gamma^{-\mathrm{ad}\,h} = (B^t)^2, \qquad (4.5)$$

holds as operators on $\mathfrak{h}'$. Here B denotes, as usual, the incidence matrix of $\mathfrak{g}$. Following [18], relation (4.5) can be used to express—in the basis $\{h_i', \, i \in I\}$—the eigenvectors of $\gamma^{\mathrm{ad}\,h}$ in terms of the eigenvectors of B. Now it is well known that the incidence matrix B is a non-negative irreducible matrix [24]. Therefore, it has a maximal (in the sense of the Perron-Frobenius theory) eigenvalue, which is $\gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}}$, and a corresponding eigenvector, say $(x_1, x_2, \ldots, x_n) \in \mathbb{R}_{>0}^n$. If we fix $x_1 = 1$, we then have the following relation for every $i \in I$:

$$\sum_{j \in I} B_{ij} x_j = (\gamma^{-\frac{1}{2}} + \gamma^{\frac{1}{2}}) x_i, \quad x_i > 0. \tag{4.6}$$

Following [18], and essentially using (4.5), we finally obtain

$$\bar{\Lambda} = \sum_{i \in I} \gamma^{\frac{p(i)}{2}} x_i h_i', \tag{4.7}$$

where x_i, $i \in I$, are defined in (4.6) and the function $p(i)$, $i \in I$, is given in (2.8).

Remark 4.2. Note that Eq. (4.7) corresponds to a precise normalization of $\bar{\Lambda}$. In fact, it is always possible to find an automorphism of $\mathfrak{g}$, under which $\mathfrak{h}$ is stable, in such a way that $\bar{\Lambda} = c \sum_{i \in I} \gamma^{\pm \frac{p(i)}{2}} x_i h_i'$ for any $c \in \mathbb{C}^*$ and any choice of the relative phase $\gamma^{\pm \frac{p(i)}{2}}$. The choice of the relative phase is however immaterial. Indeed, under the transformation $\gamma^{\frac{p(i)}{2}} \to \gamma^{-\frac{p(i)}{2}}$ the right hand side of (4.7) represents a different element in the Cartan subalgebra $\mathfrak{h}'$ which is however conjugated to $\bar{\Lambda}$. Hence, their spectra coincide.

The decomposition (4.7) allows one to compute the spectrum of Λ using the Perron-Frobenius eigenvector of the incidence matrix B. We will also need the following:

Lemma 4.3. *Let α_j, $j \in I$, be the positive roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}'$. Then $\bar{\Lambda}$ satisfies*

$$\alpha_j(\gamma^{-\frac{p(i)}{2}} \bar{\Lambda}) = x_j \gamma^{\frac{p(j)-p(i)}{2}} (1 - \gamma^{-2p(j)+1}), \quad \text{for all } i, j \in I, \tag{4.8}$$

where the x_i are defined in (4.6). This implies that

$$\mathrm{Re}\,\alpha_j(\gamma^{-\frac{p(i)}{2}} \bar{\Lambda}) > 0, \quad \text{if} \quad p(i) = p(j),$$
$$\mathrm{Re}\,\alpha_j(\gamma^{-\frac{p(i)}{2}} \bar{\Lambda}) = 0, \quad \text{if} \quad p(i) \neq p(j).$$

Proof. Since $B_{ij} \neq 0$ if and only if $p(i) \neq p(j)$, Eq. (4.6) implies that

$$\sum_{j \in I} \gamma^{-\frac{p(j)}{2}} B_{ij} x_j = (\gamma^{-\frac{1}{2}} + \gamma^{\frac{1}{2}}) \gamma^{\frac{p(i)-1}{2}} x_i. \tag{4.9}$$

Equation (4.9) and the decomposition (4.7) imply the thesis. $\quad\square$

Using the above lemma, together with Eq. (4.7), we compute the maximal eigenvalue of Λ in the representations $V^{(i)}$, $\bigwedge V^{(i)}$, $W^{(i)}$ and $M^{(i)}$.

Proposition 4.4. *Let $i \in I$. If $\mathfrak{g}$ is a simple Lie algebra of ADE type, then the following facts hold.*

(a) *For the fundamental representation $V^{(i)}$ there exists a unique maximal eigenvalue $\lambda^{(i)}$ of Λ. The eigenvalues $\lambda^{(i)}, i \in I$, satisfy the following identity:*

$$\sum_{j \in I} B_{ij} \lambda^{(j)} = (\gamma^{-\frac{1}{2}} + \gamma^{\frac{1}{2}}) \lambda^{(i)}. \tag{4.10}$$

We denote by $\psi^{(i)} \in V^{(i)}$ the unique (up to a constant factor) eigenvector of Λ corresponding to the eigenvalue $\lambda^{(i)}$.

(b) *For the representation $M^{(i)}$, we have that $\sum_{j \in I} B_{ij} \lambda^{(i)}$ is a maximal eigenvalue of Λ. Moreover,*

$$\psi_\otimes^{(i)} = \bigotimes_{j \in I} \psi^{(j) \otimes B_{ij}} \in M^{(i)}$$

is the corresponding eigenvector.

(c) *For the representation $\bigwedge^2 V_{\frac{1}{2}}^{(i)}$, we have that $(\gamma^{-\frac{1}{2}} + \gamma^{\frac{1}{2}}) \lambda^{(i)}$ is a maximal eigenvalue of Λ. Moreover,*

$$\gamma^{-\frac{1}{2}h} \psi^{(i)} \wedge \gamma^{\frac{1}{2}h} \psi^{(i)} \in \bigwedge^2 V_{\frac{1}{2}}^{(i)}$$

is the corresponding eigenvector.

(d) *The elements $\gamma^{-\frac{1}{2}h} \psi^{(i)} \wedge \gamma^{\frac{1}{2}h} \psi^{(i)} \in \bigwedge^2 V_{\frac{1}{2}}^{(i)}$ and $\psi_\otimes^{(i)} \in M^{(i)}$ belong to the subrepresentation $W^{(i)}$. Hence, for the representation $W^{(i)}$ there exists a maximal eigenvalue $\mu^{(i)}$ of Λ, given by*

$$\mu^{(i)} = \sum_{j \in I} B_{ij} \lambda^{(i)} = (\gamma^{-\frac{1}{2}} + \gamma^{\frac{1}{2}}) \lambda^{(i)},$$

and the relation

$$m_i \left(\gamma^{-\frac{1}{2}h} \psi^{(i)} \wedge \gamma^{\frac{1}{2}h} \psi^{(i)} \right) = c \psi_\otimes^{(i)}$$

holds for some $c \in \mathbb{C}^*$.

Proof. Recall that the weights appearing in the representation $L(\omega_i)$ are of the following type:

$$\omega_i, \qquad \omega_i - \alpha_i, \qquad \omega_i - \alpha_i - \alpha_j - \omega,$$

where in the latter case the index j is such that $B_{ij} \neq 0$ and ω is an integral non-negative linear combination of the positive roots. Clearly for any $\tilde{h} \in \mathfrak{h}'$ such that $\operatorname{Re} \alpha_j(\tilde{h}) \geq 0$, $\forall j \in I$ we have

$$\operatorname{Re} \omega_i(\tilde{h}) \geq (\omega_i - \alpha_i)(\tilde{h}) \geq (\omega_i - \alpha_i - \alpha_j - \omega)(\tilde{h}).$$

Therefore for any such $\tilde{h}$ the eigenvalues with maximal real part is $\omega_i(\tilde{h})$, followed by $(\omega_i - \alpha_i)(\tilde{h})$, $(\omega_i - \alpha_i - \alpha_j)(\tilde{h})$, and so on.

We use this simple argument to study the maximal eigenvalues of Λ in the representation $V^{(i)}$. By definition of Λ and $\bar{\Lambda}$, and using Eq. (4.2), the action of Λ on $V^{(i)}$ coincides

with the action of $\gamma^{\frac{p(i)}{2}(-1+\text{ad}\, h)}\bar{\Lambda}$ on $L(\omega_i)$. By Lemma 4.3, $\alpha_j(\gamma^{-\frac{p(i)}{2}}\bar{\Lambda}) \geq 0$, for every $j \in I$, and $\alpha_i(\gamma^{-\frac{p(i)}{2}}\bar{\Lambda}) = x_i(1 - \gamma^{1-2p(i)})$. It follows that in the representation $V^{(i)}$, Λ has the maximal eigenvalue

$$\omega_i(\gamma^{-\frac{p(i)}{2}}\bar{\Lambda}) = x_i.$$

Therefore, by definition of the x_i's, the $\lambda^{(i)}$ satisfy Eq. (4.6).

Moreover we have

$$\psi^{(i)} = \gamma^{\frac{p(i)}{2}h}v_i' \tag{4.11}$$

and

$$f_i'\psi^{(i)} = c\gamma^{-h}\psi^{(i)}, \tag{4.12}$$

for some $c \in \mathbb{C}^*$. This completes the proof of part (a). Part (b) follows directly from part (a). To prove part (c), we follow the same lines we used to prove part (a). In fact, the action of Λ on $V^{(i)}_{\frac{1}{2}}$ coincides with the action of $\gamma^{\frac{p^*(i)}{2}(-1+\text{ad}\, h)}\bar{\Lambda}$ on $L(\omega_i)$, where $p^*(i) = 1 - p(i)$. By Eq. (4.8) we have $\text{Re}\,\alpha_i(\gamma^{\frac{p^*(i)}{2}(-1+\text{ad}\, h)}\bar{\Lambda}) = 0$, while $\text{Re}\,\alpha_j(\gamma^{\frac{p^*(i)}{2}(-1+\text{ad}\, h)}\bar{\Lambda}) > 0$ if $p(i) \neq p(j)$. It follows that the two eigenvalues with maximal real part correspond to the weights ω_i and $\omega_i - \alpha_i$. By Eq. (4.8), these eigenvalues are $x_i\gamma^{\frac{1}{2}}$ and $x_i\gamma^{-\frac{1}{2}}$, and the corresponding eigenvectors are $\gamma^{\mp\frac{1}{2}h}\psi^{(i)}$. Therefore $\bigwedge V^{(i)}$ has a unique maximal eigenvalue $(\gamma^{\frac{1}{2}} + \gamma^{-\frac{1}{2}})x_i$, with eigenvector

$$\gamma^{-\frac{1}{2}h}\psi^{(i)} \wedge \gamma^{\frac{1}{2}h}\psi^{(i)} = c\gamma^{\frac{(1+p(i))}{2}h}(v_i' \wedge f_i'v_i'), \tag{4.13}$$

for some $c \neq 0$. Here we have used Eqs. (4.11) and (4.12).

Let us prove (d). By Eq. (4.11) and Lemma 2.2, it follows that $\psi_{\otimes}^{(i)} \in W^{(i)}$, where $W^{(i)}$ is seen as a subrepresentation of $M^{(i)}$. Therefore the maximal eigenvalue of $W^{(i)}$ and $M^{(i)}$ coincide. Finally by Eqs. (4.11), (4.13) and Lemma 2.1, we have that $\gamma^{-\frac{1}{2}h}\psi^{(i)} \wedge \gamma^{\frac{1}{2}h}\psi^{(i)} \in W^{(i)}$ and $m_i(\gamma^{-\frac{1}{2}h}\psi^{(i)} \wedge \gamma^{\frac{1}{2}h}\psi^{(i)}) = c\psi_{\otimes}^{(i)}$, $c \in \mathbb{C}^*$, thus proving part (d) and concluding the proof. $\square$

Remark 4.5. We note that we can always choose a normalization of the eigenvectors $\psi^{(i)} \in V^{(i)}$, $i \in I$, such that

$$m_i(\gamma^{-\frac{1}{2}h}\psi^{(i)} \wedge \gamma^{\frac{1}{2}h}\psi^{(i)}) = \psi_{\otimes}^{(i)}, \tag{4.14}$$

for every $i \in I$. From now on, we will assume that Proposition 4.4(d) holds with the normalization constant $c = 1$.

By Proposition 4.4(a), for any fundamental representation $V^{(i)}$, $i \in I$ of $\widehat{\mathfrak{g}}$, there exists a maximal eigenvalue $\lambda^{(i)}$ with eigenvector $\psi^{(i)}$. Therefore, by Theorem 3.4 there exists a unique subdominant solution $\Psi^{(i)}(x, E) : \mathbb{C} \to V^{(i)}$ of the differential equation (3.1) in the representation $V^{(i)}$ with asymptotic behavior

$$\Psi^{(i)}(x, E) = e^{-\lambda^{(i)}S(x,E)}q(x, E)^{-h}(\psi^{(i)} + o(1)), \quad \text{in the sector } |\arg x| < \frac{\pi}{2(M+1)}. \tag{4.15}$$

Using Proposition 4.4 we can prove the following theorem establishing the Ψ-system associated to the Lie algebra $\mathfrak{g}$.

Theorem 4.6. *Let $\mathfrak{g}$ be a simple Lie algebra of ADE type, and let the solutions $\Psi^{(i)}(x, E) : \widetilde{\mathbb{C}} \to V^{(i)}$, $i \in I$, have the asymptotic behavior (4.15). Then, the following identity holds:*

$$m_i\left(\Psi^{(i)}_{-\frac{1}{2}}(x, E) \wedge \Psi^{(i)}_{\frac{1}{2}}(x, E)\right) = \otimes_{j \in I} \Psi^{(j)}(x, E)^{\otimes B_{ij}}, \quad \text{for every } i \in I. \quad (4.16)$$

Proof. Due to Proposition 4.4(b) and Theorem 3.4, the unique subdominant solution to Eq. (3.1) in $M^{(i)}$ is

$$\otimes_{j \in I} \Psi^{(j)}(x, E)^{\otimes B_{ij}} = e^{-\mu^{(i)} S(x,E)} q(x, E)^{-h} \left(\psi^{(i)}_{\otimes} + o(1)\right), \quad \text{for } x \gg 0.$$

Moreover, by Eq. (3.8), Corollary 3.7 and Proposition 4.4(c) we have that

$$\Psi^{(i)}_{-\frac{1}{2}}(x, E) \wedge \Psi^{(i)}_{\frac{1}{2}}(x, E) = e^{-\mu^{(i)} S(x,E)} q(x, E)^{-h} \left(\psi^{(i)}_{-\frac{1}{2}} \wedge \psi^{(i)}_{\frac{1}{2}} + o(1)\right), \quad \text{for } x \gg 0.$$

The proof follows by Eq. (4.14) and the uniqueness of the subdominant solution. $\square$

Example 4.7. In type A_n, Eq. (4.16) becomes

$$m_i\left(\Psi^{(i)}_{-\frac{1}{2}} \wedge \Psi^{(i)}_{\frac{1}{2}}\right) = \Psi^{(i-1)} \otimes \Psi^{(i+1)}, \quad \text{for } i = 1, \dots, n,$$

where we set $\Psi^{(0)} = \Psi^{(n+1)} = 1$.

Example 4.8. In type D_n, Eq. (4.16) becomes

$$m_i\left(\Psi^{(i)}_{-\frac{1}{2}} \wedge \Psi^{(i)}_{\frac{1}{2}}\right) = \Psi^{(i-1)} \otimes \Psi^{(i+1)}, \quad \text{for } i = 1, \dots, n-3, \quad (4.17)$$

$$m_{n-2}\left(\Psi^{(n-2)}_{-\frac{1}{2}} \wedge \Psi^{(n-2)}_{\frac{1}{2}}\right) = \Psi^{(n-3)} \otimes \Psi^{(n-1)} \otimes \Psi^{(n)}, \quad (4.18)$$

$$m_{n-1}\left(\Psi^{(n-1)}_{-\frac{1}{2}} \wedge \Psi^{(n-1)}_{\frac{1}{2}}\right) = \Psi^{(n-2)} = m_n\left(\Psi^{(n)}_{-\frac{1}{2}} \wedge \Psi^{(n)}_{\frac{1}{2}}\right), \quad (4.19)$$

where we set $\Psi^{(0)} = 1$.

Remark 4.9. The Ψ-system in Example 4.8 is a complete set of relations among the subdominant solutions. The analogue system of equations obtained in [32, Eqs. (3.19,3.20,3.21)] lacks of the Eq. (4.18) for the node $n - 2$ of the Dynkin diagram. Note that equations [32, Eqs. (3.19,3.20)] are equivalent to (4.17) and (4.19) of the present paper. Indeed, it follows from the direct sum decomposition 4.1 that $m_i \circ \iota = \mathbb{1}_{V^{(n-2)}}$, for $i = n - 1, n$, where ι is the embedding $V^{(n-2)} \overset{\iota}{\hookrightarrow} \bigwedge^2 V^{(i)}_{\frac{1}{2}}$ (in fact, $V^{(n-2)} \cong W^{(n-1)} \cong W^{(n)}$ in this particular case). The further identity [32, Eq. (3.21)], which can be obtained from (A.4), requires the introduction of another unknown and therefore equations [32, Eqs. (3.19,3.20,3.21)] are an incomplete system of n equations in $n + 1$ unknowns.

Example 4.10. In type E_n, $n = 6, 7, 8$, Eq. (4.16) becomes

$$m_i\left(\Psi^{(i)}_{-\frac{1}{2}} \wedge \Psi^{(i)}_{\frac{1}{2}}\right) = \Psi^{(i-1)} \otimes \Psi^{(i+1)}, \quad \text{for } i = 1, \ldots, n - 4,$$

$$m_{n-3}\left(\Psi^{(n-3)}_{-\frac{1}{2}} \wedge \Psi^{(n-3)}_{\frac{1}{2}}\right) = \Psi^{(n-4)} \otimes \Psi^{(n-2)} \otimes \Psi^{(n-1)},$$

$$m_{n-2}\left(\Psi^{(n-2)}_{-\frac{1}{2}} \wedge \Psi^{(n-2)}_{\frac{1}{2}}\right) = \Psi^{(n-3)},$$

$$m_{n-1}\left(\Psi^{(n-1)}_{-\frac{1}{2}} \wedge \Psi^{(n-1)}_{\frac{1}{2}}\right) = \Psi^{(n-3)} \otimes \Psi^{(n)},$$

$$m_n\left(\Psi^{(n)}_{-\frac{1}{2}} \wedge \Psi^{(n)}_{\frac{1}{2}}\right) = \Psi^{(n-1)},$$

where we set $\Psi^{(0)} = 1$.

5. The Q-System

In this section, for any simple Lie algebra of ADE type, we define the generalized spectral determinants $Q^{(i)}(E; \ell)$, $i \in I$, and we prove that they satisfy the Bethe Ansatz equations, also known as Q-system. The spectral determinants are entire functions of the parameter E, and they are defined by the behavior of the functions $\Psi^{(i)}(x, E)$ close to the Fuchsian singularity. More precisely, $Q^{(i)}(E; \ell)$ is the coefficient of the most singular term of the expansion of $\Psi^{(i)}(x, E)$ in the neighborhood of $x = 0$.

In the case of the Lie algebra $\mathfrak{sl}_2$, the spectral determinant $Q(E; \ell)$ was originally introduced in [3,14], while the generalization to $\mathfrak{sl}_n$ has been given in [10,34]. An incomplete construction for Lie algebras of BCD type can be found in [9,32]. The terminology generalized spectral determinants is motivated by the $\mathfrak{sl}_2$ case. Indeed, for the Lie algebra $\mathfrak{sl}_2$, the linear ODE (3.1) is equivalent to a Schrödinger equation with a polynomial potential and a centrifugal term, and the spectral determinant $Q(E; \ell)$ vanishes at the eigenvalues of the Schrödinger operator.

Before proving the main result of this section, we briefly review some well-known fundamental results from the theory of Fuchsian singularities. Here we follow [22, Chap. III].

Remark 5.1. In this Section we assume that the potential $p(x, E) = x^{Mh^\vee}$ is analytic in $x = 0$, namely $Mh^\vee \in \mathbb{Z}_+$. With this assumption, the point $x = 0$ is simply a Fuchsian singularity of the equation. The case of a potential with a branch point at $x = 0$ can formally be treated without any modification—see [3] for the case A_1—but the mathematical theory is less developed.

5.1. Monodromy about the Fuchsian singularity. For any linear ODE with Fuchsian singularity at $x = 0$, namely of the type

$$\Psi'(x) = \left(\frac{A}{x} + B(x)\right)\Psi(x), \tag{5.1}$$

where A is a constant matrix and $B(x)$ is a regular function, the monodromy operator is the endomorphism on the space of solutions of (5.1) which associates to any solution its analytic continuation around a small Jordan curve encircling $x = 0$. More concretely, if

we fix a matrix solution $Y(x)$ of the linear ODE (5.1) with non-vanishing determinant (i.e. a basis of solutions), then the monodromy matrix M is defined by the relation

$$Y(e^{2\pi i}x) = Y(x)M.$$

Here, $Y(e^{2\pi i}x)$ is the customary notation for the analytic continuation of $Y(x)$. We are interested in the invariant subspaces of the monodromy matrix M. By the general theory, the algebraic eigenvalues of the monodromy matrix M are of the form $e^{2\pi i a}$, for any eigenvalue a of A.

We said that an eigenvalue a of A is non-resonant if there exists no other eigenvalue of a' such that $a - a' \in \mathbb{Z}$. In this case the eigenvalue $e^{2\pi i a}$ has multiplicity one and there exists a solution of equation (5.1) which is an eigenvector of the monodromy matrix M with eigenvalue $e^{2\pi i a}$. Such a solution has the form

$$\chi_a(x) = x^a\big(\chi_a + O(x)\big) \tag{5.2}$$

for any χ_a eigenvector of A with eigenvalue a. If the eigenvalues $a_1, \ldots, a_k$ are resonant, meaning that $a_i - a_j \in \mathbb{Z}$ for $i, j = 1 \ldots, k$, then the eigenvalue $e^{2\pi i a_1} = \cdots = e^{2\pi i a_k}$ has multiplicity k. The matrix M, in general, is not diagonalizable when restricted to the k-dimensional invariant subspace corresponding to this eigenvalue. The corresponding solutions do not have the form (5.2) due to the appearance of logarithmic terms.

We are interested in the linear ODE (3.1), which has a Fuchsian singularity with $A = -\ell, \ell \in \mathfrak{h}$. For any representation $V^{(i)}$, $i \in I$, the eigenvalues for the action of ℓ on $V^{(i)}$ have the form $\lambda(\ell)$, where $\lambda \in P$ is a weight of the representation $V^{(i)}$. Note that for a generic choice of $\ell \in \mathfrak{h}$ and for distinct weights λ_1 and λ_2, the corresponding eigenvalues $\lambda_1(\ell), \lambda_2(\ell)$ are non-resonant. Indeed, generically, two eigenvalues are resonant if and only if they coincide, and therefore they correspond to a weight of multiplicity bigger than one.

5.2. The dominant term at $x = 0$. Let $\omega \in P^+$, and let us denote by

$$l^* = \max_{\lambda \in P_\omega} \text{Re}\, \lambda(\ell).$$

Let also $\lambda^* \in P$ be such that $l^* = \text{Re}\, \lambda^*(\ell)$. If λ^* is a weight of multiplicity one in $L(\omega)$, then—by the discussion in Sect. 5.1—the most singular behavior at $x = 0$ of a solution to Eq. (3.1) is x^{-l^*}. In this section, for any irreducible representation $L(\omega)$ of $\mathfrak{g}$ and a generic $\ell \in \mathfrak{h}$, we characterize the weight $\lambda^* \in P$ maximizing $\lambda(\ell) \in \mathbb{Z}$. Moreover, we show that it has multiplicity one. These facts will be used to define the generalized spectral determinant $Q^{(i)}(E; \ell)$.

It is well-known from the general theory of simple Lie algebras (see [19, Appendix D]), that given a generic element $\ell \in \mathfrak{h}$, we can decompose the set of the roots R of $\mathfrak{g}$ in two distinct and complementary sets (of positive and negative roots)

$$R_\ell^+ = \{\alpha \in R \mid \text{Re}\, \alpha(\ell) > 0\}, \quad R_\ell^- = \{\alpha \in R \mid \text{Re}\, \alpha(l) < 0\}.$$

Consequently, we can associate to this $\ell \in \mathfrak{h}$ a Weyl Chamber $\mathfrak{W}_\ell$, as well as an element w_ℓ of the Weyl group (it is the element which maps the original Weyl chamber into $\mathfrak{W}_\ell$) and a (possibly new) set of simple roots $\Delta_\ell = \{w_\ell(\alpha_i) \mid i \in I\}$. The weight $w_\ell(\omega)$ is the highest weight of $L(\omega)$ with respect to the new set of simple roots Δ_ℓ, and due to the definition of R_ℓ^-, the action of any negative simple root $w_\ell(-\alpha_i)$, $i \in I$, decreases

the value of Re $w_\ell(\omega)(\ell)$. Therefore, $\lambda^* = w_\ell(\omega)$ is the unique weight maximizing the function Re $\lambda(\ell)$, $\lambda \in P_\omega$. This weight has multiplicity one since it lies in the Weyl orbit of ω.

In the case of a fundamental representation $L(\omega_i)$, $i \in I$, the weight $w_\ell(\omega_i - \alpha_i)$ belongs to P_{ω_i}, and it has multiplicity one. Moreover, it is possible to show that all the weights in P_{ω_i} different from $w_\ell(\omega_i)$ are obtained from $w_\ell(\omega_i - \alpha_i)$ by a repeated action of the negative simple roots $w_\ell(-\alpha_i)$. We conclude that $w_\ell(\omega_i - \alpha_i)$ is the unique weight maximizing Re $\lambda(\ell)$ on $P_{\omega_i} \setminus \{w_\ell(\omega_i)\}$. We have thus proved the following result:

Proposition 5.2. *Let $\ell \in \mathfrak{h}$ be a generic element, and let w_ℓ the associated element of the Weyl group. Then the weight $w_\ell(\omega) \in P_\omega$, $\omega \in P^+$, has multiplicity one and*

$$\text{Re } w_\ell(\lambda)(\ell) > \text{Re } \lambda(\ell),$$

for any weight $\lambda \in P_\omega$, $\lambda \neq w_\ell(\omega)$. In the case of a fundamental weight $\omega = \omega_i$, $i \in I$, the weight $w_\ell(\omega_i - \alpha_i) \in P_{\omega_i}$ has multiplicity one and

$$\text{Re } w_\ell(\omega_i)(\ell) > \text{Re } w_\ell(\omega_i - \alpha_i)(\ell) > \text{Re } \lambda(\ell),$$

for any $\lambda \in P_{\omega_i}$, $\lambda \neq w_\ell(\omega_i)$, $w_\ell(\omega_i - \alpha_i)$.

Let $\chi^{(i)}, \varphi^{(i)} \in L(\omega_i)$ be weight vectors corresponding to the weights $w_\ell(\omega_i)$ and $w_\ell(\omega_i - \alpha_i)$ respectively. As done in Lemmas 2.1 and 2.2 it is possible to show that $L(\eta_i)$ is a subrepresentation of $\bigwedge^2 L(\omega_i)$ (respectively $\bigotimes L(\omega_i)^{\otimes B_{ij}}$) and that $\chi^{(i)} \wedge \varphi^{(i)} \in L(\eta_i)$ (respectively $\otimes_{j \in I} \chi^{(j) \otimes B_{ij}} \in L(\eta_i)$). Moreover, $\chi^{(i)} \wedge \varphi^{(i)}$ and $\otimes_{j \in I} \chi^{(j) \otimes B_{ij}}$ are highest weight vectors (with respect to Δ_ℓ) of the subrepresentation $L(\eta_i)$. Hence, we can identify these vectors (up to a constant) using the morphism m_i defined in Eq. (2.4). From now on we fix the normalization of $\chi^{(i)}, \varphi^{(i)}$, $i \in I$, in such a way that

$$m_i(\chi^{(i)} \wedge \varphi^{(i)}) = \otimes_{j \in I} \chi^{(j) \otimes B_{ij}}. \tag{5.3}$$

5.3. Decomposition of $\Psi^{(i)}$. From the above discussion, and in particular form Proposition 5.2 and Eq. (5.2), we have that for any evaluation representation $V_k^{(i)}$, $i \in I$ and $k \in \mathbb{C}$, there exist distinguished (and normalized) solutions $\chi_k^{(i)}(x, E)$ and $\varphi_k^{(i)}(x, E)$ of the linear ODE (3.1) such that they have the most singular behavior at $x = 0$ and they are eigenvectors of the associated monodromy matrix.

Explicitly, these solutions have the asymptotic expansion

$$\chi_k^{(i)}(x, E) = x^{-w_\ell(\omega_i)(\ell)}\big(\chi^{(i)} + O(x)\big),$$
$$\varphi_k^{(i)}(x, E) = x^{-w_\ell(\omega_i - \alpha_i)(\ell)}\big(\varphi^{(i)} + O(x)\big).$$

Since the parameter E appears linearly in the ODE (3.1) and the asymptotic behavior at $x = 0$ does not depend on E, by the general theory [22] it follows that $\chi_k^{(i)}(x, E; \ell)$ and $\varphi_k^{(i)}(x, E; \ell)$ are entire functions of E.

Using Eq. (2.17) and comparing the asymptotic behaviors we have that

$$\omega^{-kh}\chi_{k'}^{(i)}(\omega^k x, \Omega^k E) = \omega^{-kw_\ell(\omega_i)(\ell+h)}\chi^{(i)}(x, E)_{k+k'}$$
$$\omega^{-kh}\varphi_{k'}^{(i)}(\omega^k x, \Omega^k E) = \omega^{-kw_\ell(\omega_i - \alpha_i)(\ell+h)}\varphi^{(i)}(x, E)_{k+k'}, \tag{5.4}$$

for every $k, k' \in \mathbb{C}$. For generic $\ell \in \mathfrak{h}$, the eigenvalues $-w_\ell(\omega_i)(\ell)$, and $-w_\ell(\omega_i - \alpha_i)(\ell)$ are non-resonant and therefore we can unambiguously define two functions $Q^{(i)}(E; \ell)$ and $\widetilde{Q}^{(i)}(E; \ell)$ as follows. We write

$$\Psi^{(i)}(x, E, \ell) = Q^{(i)}(E; \ell)\chi^{(i)}(x, E) + \widetilde{Q}^{(i)}(E; \ell)\varphi^{(i)}(x, E) + v^{(i)}(x, E), \quad (5.5)$$

where $v^{(i)}(x, E)$ belongs to an invariant subspace of the monodromy matrix corresponding to lower weights. We call $Q^{(i)}(E; \ell)$ and $\widetilde{Q}^{(i)}(E; \ell)$ the generalized spectral determinants of the Eq. (2.13). Since $\Psi^{(i)}(x, E, \ell)$, $\chi^{(i)}(x, E)$ and $\varphi^{(i)}(x, E)$ are entire functions of the parameter E, then also $Q^{(i)}(E; \ell)$ and $\widetilde{Q}^{(i)}(E; \ell)$ are entire functions of the parameter E.

Finally, the Ψ-system (4.16) implies some remarkable functional relations among the generalized spectral determinants. Indeed, we have the following result.

Theorem 5.3. *Let $\ell \in \mathfrak{h}$ be a generic element. Then, the spectral determinants $Q^{(i)}(E; \ell)$ and $\widetilde{Q}^{(i)}(E; \ell)$ are entire functions of E and they satisfy the following $Q\widetilde{Q}$-system:*

$$\prod_{j \in I} Q^{(j)}(E; \ell)^{B_{ij}} = \omega^{-\frac{1}{2}\theta_i} Q^{(i)}(\Omega^{-\frac{1}{2}}E; \ell)\widetilde{Q}^{(i)}(\Omega^{\frac{1}{2}}E; \ell)$$

$$- \omega^{\frac{1}{2}\theta_i} Q^{(i)}(\Omega^{\frac{1}{2}}E; \ell)\widetilde{Q}^{(i)}(\Omega^{-\frac{1}{2}}E; \ell), \qquad (5.6)$$

where $\theta_i = w_\ell(\alpha_i)(\ell + h)$.

Proof. We already showed that $Q^{(i)}(E; \ell)$ and $\widetilde{Q}^{(i)}(E; \ell)$ are entire functions of the parameter E.

By the definition of $\Psi^{(i)}_{\pm\frac{1}{2}}(x, E; \ell)$ given in (3.8) and from Eqs. (5.5) and (5.4) we have that

$$\Psi^{(i)}_{\pm\frac{1}{2}}(x, E; \ell) = \omega^{\pm\frac{1}{2}w_\ell(\omega_i)(\ell+h)} Q^{(i)}(\Omega^{\pm\frac{1}{2}}E; \ell)\chi^{(i)}_{\frac{1}{2}}(x, E)$$

$$+ \omega^{\pm\frac{1}{2}w_\ell(\omega_i - \alpha_i)(\ell+h)} \widetilde{Q}^{(i)}(\Omega^{\pm\frac{1}{2}}E; \ell)\varphi^{(i)}_{\frac{1}{2}}(x, E) + \tilde{v}^{(i)}(x, E). \qquad (5.7)$$

Here we used the fact that, since $V^{(i)}_{-\frac{1}{2}} = V^{(i)}_{\frac{1}{2}}$, then $\chi^{(i)}_{\frac{1}{2}}(x, E) = \chi^{(i)}_{-\frac{1}{2}}(x, E)$ and $\varphi^{(i)}_{\frac{1}{2}}(x, E) = \varphi^{(i)}_{-\frac{1}{2}}(x, E)$. Note that $\tilde{v}^{(i)}(x, E)$ belongs to an invariant subspace of the monodromy matrix corresponding to lower weights. Equation (5.6) is then obtained by equating the component in $W^{(i)}$ in the ψ-system (4.16) by means of Eqs. (5.5), (5.7) and the normalization (5.3). □

We now derive the Q-system (Bethe Ansatz) associated to the Lie algebras of ADE type. Let us denote by $E_i \in \mathbb{C}$ any zero of $Q^{(i)}(E; \ell)$. Evaluating Eq. (5.6) at $E = \Omega^{\pm\frac{1}{2}}E_i$ we get the following relations

$$\prod_{j \in I} Q^{(j)}(\Omega^{\frac{1}{2}}E; \ell)^{B_{ij}} = -\omega^{\frac{1}{2}\theta_i} Q^{(i)}(\Omega E_i; \ell)\widetilde{Q}^{(i)}(E_i; \ell),$$

$$\prod_{j \in I} Q^{(j)}(\Omega^{-\frac{1}{2}}E; \ell)^{B_{ij}} = \omega^{-\frac{1}{2}\theta_i} Q^{(i)}(\Omega^{-1}E_i; \ell)\widetilde{Q}^{(i)}(E_i; \ell). \qquad (5.8)$$

Assuming that for generic $\ell \in \mathfrak{h}$ the functions $Q^{(i)}(E; \ell)$ and $\widetilde{Q}^{(i)}(E; \ell)$ do not have common zeros, and taking the ratio of the two identities in (5.8), we get, for any zero E_i of $Q^{(i)}(E; \ell)$, the Q-system

$$\prod_{j=1}^{n} \Omega^{\beta_j C_{ij}} \frac{Q^{(j)}\left(\Omega^{\frac{C_{ij}}{2}} E^*\right)}{Q^{(j)}\left(\Omega^{-\frac{C_{ij}}{2}} E^*\right)} = -1, \tag{5.9}$$

where $C = \left(C_{ij}\right)_{i,j \in I}$ is the Cartan matrix of the Lie algebra $\mathfrak{g}$, and

$$\beta_j = \frac{1}{2Mh^\vee} \sum_{i \in I} (C^{-1})_{ij} \theta_i = \frac{1}{2Mh^\vee} w_\ell(\omega_j)(\ell + h), \quad j \in I.$$

Remark 5.4. The construction of the $Q\widetilde{Q}$ system (5.6) works for any affine Kac-Moody algebra. However, formula (5.9) holds only in the ADE case. In order to define the analogue of Eq. (5.9) for the Cartan matrix of a non-simply laced algebra $\mathfrak{g}$, one has to consider, as suggested in [17], a connection with values in the Langlands dual of the affine Lie algebra $\widehat{\mathfrak{g}}$.

5.4. The case $\ell = 0$. The case of $\ell = 0$ is not generic and thus the Q functions cannot be defined as in Sect. 5. It is however straightforward to generalize the generic case to $\ell = 0$, either directly or as the limit $\ell \to 0$. Indeed, we can take the limit $\ell \to 0$ along sequences such that the element of the Weyl group $w_\ell = w$ is fixed. In this way, the limit solutions $\chi^{(i)}(x, E, \ell = 0)$, $\varphi^{(i)}(x, E, \ell = 0)$ of (3.1) satisfy the Cauchy problem

$$\chi_k^{(i)}(x, E, \ell = 0) = v_i^w, \quad \varphi_k^{(i)}(x, E, \ell = 0) = f_i^w v_i^w. \tag{5.10}$$

Here v_i^w and f_i^w are respectively the highest weight vector of $V^{(i)}$ and the negative Chevalley generator corresponding to the element w of the Weyl group. Equations (5.10) define uniquely the two solutions. Then the spectral determinants $Q^{(i)}$, $\widetilde{Q}^{(i)}$ are exactly the coefficients of $\Psi^{(i)}(x = 0, E)$ with respect to the basis element v_i^w, $f_i^w v_i$. Note that this implies a certain freedom in the choice of the spectral determinants $Q^{(i)}$, $\widetilde{Q}^{(i)}$. In fact, the freedom in the choice of $Q^{(i)}$ corresponds to the elements of the orbit of ω_i under the Weyl group action, while the freedom in the choice of $\widetilde{Q}^{(i)}$ corresponds to the elements of the orbit of the ordered pair (ω_i, α_i) under the same group action.

For any choice of $w_\ell = w$, the functions $Q^{(i)}$, $\widetilde{Q}^{(i)}$ satisfy (5.8) with $\ell = 0$, $w_\ell = w$ and the corresponding Bethe Ansatz equation (5.9), provided they do not have common zeros.

6. $\mathfrak{g}$-Airy function

We consider in more detail the special case of (3.1) with a linear potential $p(x, E) = x$ and with $\ell = 0$, which is particularly interesting because the subdominant solutions and the spectral determinants have an integral representation. Since in the $\mathfrak{sl}_2$ case the subdominant solution coincides with the Airy function, we call the solution obtained in the general case the $\mathfrak{g}$-Airy function. The classical Airy function models the local behavior of the subdominant solution of the Schrödinger equation with a generic potential close to a turning point [16]; we expect the $\mathfrak{g}$-Airy function to play a similar role for Eq. (3.1).

6.1. Q functions. In the special case of (3.1) with a linear potential $p(x, E) = x$ and with $\ell = 0$, it is straightforward to compute the Q functions. By the discussion in Section 5.4, in order to define the Q functions we need to chose an element w of the Weyl group and thus the distinguished element v_i^w of the basis of $V^{(i)}$. The spectral $Q^{(i)}(E)$ is then defined as the coefficient with respect to v_i^w of $\Psi^{(i)}(x = 0, E)$. Due to the special form of the potential we have

$$\Psi^{(i)}(x = 0, E) = \Psi^{(i)}(-E, 0).$$

and thus, using formula (4.15), we get

$$Q^{(i)}(E) = e^{-\lambda^{(i)}\frac{h^\vee}{h^\vee+1}} |E|^{\frac{h^\vee+1}{h^\vee}} C_i\big(1 + o(1)\big) \text{ as } E \to -\infty, \tag{6.1}$$

where C_i is the coefficient of the vector $|E|^{-h}\psi^{(i)}$ with respect to the basis vector v_i^w. It is important to note that the asymptotic behavior of the functions $Q^{(i)}$ is expressed via the eigenvalues $\Lambda^{(i)}$ which coincide with the masses of the (classical) affine Toda field theory, as we proved in Proposition 4.4. This is precisely the same behavior predicted for the vacuum eigenvalue Q of the corresponding conformal field theory [31].

We conclude that the ODE/IM correspondence depends on the relation among the element $\Lambda \in \widehat{\mathfrak{g}}$, the Perron-Frobenius eigenvalue of the incidence matrix B and the masses of the affine Toda field theory.

Remark 6.1. Note that formula (6.1) does not coincide with the asymptotic behavior conjectured in [9, Eq. 2.6] for a general potential. In fact, Suzuki communicated us [33] that the latter formula is conjectured to hold only for potentials such that $\frac{M+1}{Mh^\vee} < 1$, in which case the Hadamard's product [9, Eq. 2.7] converges. Clearly in case of a linear potential $\frac{M+1}{Mh^\vee} = \frac{h^\vee+1}{h^\vee} > 1$.

Remark 6.2. A connection between generalized Airy equations and integrable systems appears, among other places, in [23], where the orbit of the KdV topological tau–function in the Sato Grassmannian is described. We note that an operator of the form (2.13)— specialized to the case of algebras of type $\mathfrak{sl}_n$ and for $\ell = -h/h^\vee$, $M = 1/h^\vee$—also appears in that paper. It would be interesting to obtain a more precise relation between these results and those of the present paper.

6.2. Integral representation. Given an evaluation representation of $\widehat{\mathfrak{g}}$, we look for solutions of the equation

$$\Psi'(x) + \big(e + xe_0\big)\Psi(x) = 0, \tag{6.2}$$

in the form

$$\Psi(x) = \int_c e^{-xs}\Phi(s)ds, \tag{6.3}$$

where $\Phi(s)$ is an analytic function and c is some path in the complex plane. Differentiating and integrating by parts we get

$$\int_c e^{-xs}\big(-s + e + e_0\frac{d}{ds}\big)\Phi(s)ds + e^{-xs}e_0\Phi(s)\big|_c = 0, \tag{6.4}$$

where the last term is the evaluation of the integrand at the end points of the path. We are therefore led to the study of the simpler linear equation

$$\left(-s + e + e_0 \frac{d}{ds}\right)\Phi(s) = 0,\tag{6.5}$$

which we solve below for two important examples.

6.3. A_{n-1} in the representation $V^{(1)}$. In this case Eq. (6.2) is already known in the literature as the n-Airy equation [16]. Using the explicit form of Chevalley generators e_i, $i = 0, \ldots, n-1$, which is provided in Appendix A.1, and denoting by $\Phi_i(s)$ the component of $\Phi(s)$ in the standard basis of $\mathbb{C}^n$, we find the general solution of (6.5) as

$$\Phi_1(s) = \kappa\, e^{\frac{s^{n+1}}{n+1}}, \quad \Phi_i(s) = s^{i-1}\Phi_1(s), \quad i = 2, \ldots, n,$$

where κ is an arbitrary complex number. If we choose the path c as the one connecting $e^{-\frac{\pi i}{n+1}}\infty$ with $e^{\frac{\pi i}{n+1}}\infty$, then the integral formula (6.3) is well-behaved, and the boundary terms in (6.4) vanish. Thus, the $\mathfrak{sl}_n$-Airy function has the integral representation

$$\Psi_j(x) = \frac{1}{2\pi i}\int_{e^{-\frac{\pi i}{n+1}}\infty}^{e^{+\frac{\pi i}{n+1}}\infty} s^{j-1}e^{-xs + \frac{s^{n+1}}{n+1}}\,ds, \quad j = 1, \ldots, n.\tag{6.6}$$

In case $n = 2$, the latter definition coincides with the definition of the standard Airy function. By the method of steepest descent we get the following asymptotic expansion of Ψ for $x \gg 0$:

$$\Psi_j(x) \sim \sqrt{\frac{1}{2\pi n}}x^{\frac{2j-1-n}{2n}} e^{-\frac{n}{n+1}x^{\frac{n+1}{n}}}, \quad j = 1, \ldots, n.\tag{6.7}$$

Due to Theorem 3.4, the $\mathfrak{sl}_n$-Airy function coincides with the fundamental solution $\Psi^{(1)}$ of the linear ODE (6.2). In fact the asymptotic behavior of the Airy function coincides with the asymptotic behavior of the function $\Psi^{(1)}$ [for this computation one needs to use the explicit formula for h which is given in equation (A.1)]. The solutions $\Psi_k^{(1)}$, $k \in \mathbb{Z}$, are obtained by integrating (6.3) along the contour obtained rotating c by $e^{\frac{2k\pi i}{n+1}}$.

6.4. D_n in the standard representation. The second example is the D_n-Airy function in the representation $V^{(1)}$. Using the Chevalley generators e_i, $i = 0, \ldots, n$, as described in Appendix A.2, and denoting Φ_j the jth-component of Φ in the standard basis of $\mathbb{C}^{2n}$, we find that the general solution of (6.5) reads

$$\Phi_1 = \kappa\, s^{-\frac{1}{2}}e^{\frac{s^{2n-1}}{2n-1}}$$

$$\Phi_j = s^{j-1}\Phi_1(s), \quad j \le n-1,$$

$$\Phi_n = \frac{s^{n-1}}{2}\Phi_1(s), \quad \Phi_{n+1} = s^{n-1}\Phi_1(s),$$

$$\Phi_{n+j} = s^{n+j-2}\Phi_1(s), \quad j \le n-1,$$

$$\Phi_{2n} = \left(\frac{s^{2n-2}}{2} + \frac{1}{4s}\right)\Phi_1(s).$$

where $\kappa \in \mathbb{C}$ is an arbitrary constant. Let c be the path connecting $e^{-\frac{\pi i}{2n-1}} \infty$ with $e^{\frac{\pi i}{2n-1}} \infty$. Then along c the integral formula (6.3) is well-behaved and it thus defines a solution of (6.2) that we call D_n-Airy function. Moreover, the boundary terms in (6.4) vanish. As in the case A_n, the standard method of steepest descent shows that such solution is exactly the fundamental solution $\Psi^{(1)}$ to Eq. (6.2). The solutions $\Psi_k^{(1)}$, $k \in \mathbb{Z}$, are obtained by integrating (6.3) along the contour obtained rotating c by $e^{\frac{k\pi i}{2n-1}}$.

Appendix A. Action of Λ in the fundamental representations

In this appendix we give an explicit description of the maximal eigenvalues of Λ in the fundamental representations and of the corresponding eigenvector.

A.1. The A_n case. The simple Lie algebra of type A_n, $n \geq 1$, can be realized as the algebra of $(n+1) \times (n+1)$ traceless matrices

$$\mathfrak{g} = \mathfrak{sl}_{n+1} = \{A \in \mathrm{Mat}_{n+1}(\mathbb{C}) \mid \mathrm{Tr}\, A = 0\},$$

where the Lie bracket is the usual commutator of matrices. The dual Coxeter number of $\mathfrak{g}$ is $h^\vee = n+1$. Let us consider the following Chevalley generators of $\mathfrak{g}$ ($i \in I = \{1, \ldots, n\}$):

$$f_i = E_{i+1,i}, \quad h_i = E_{ii} - E_{i+1,i+1}, \quad e_i = E_{i,i+1},$$

where E_{ij} denotes the elementary matrix with 1 in position (i, j) and 0 elsewhere. It is well-known that the representation $L(\omega_1)$ is given by the natural action of $\mathfrak{g}$ on $L(\omega_1) = \mathbb{C}^{n+1}$. Moreover, we have that

$$L(\omega_i) = \bigwedge^i L(\omega_1), \quad i \in I.$$

We denote by u_j, $j = 1, \ldots, n+1$, the standard basis of $\mathbb{C}^{n+1}$, and by v_i, $i \in I$, the highest weight vector of the representation $L(\omega_i)$. Then, we have:

$$v_i = u_1 \wedge u_2 \wedge \cdots \wedge u_i.$$

The set of Chevalley generators for $\widehat{\mathfrak{g}}$ is obtained by adding to the Chevalley generators of $\mathfrak{g}$ the following elements:

$$f_0 = E_{1,n+1}t^{-1}, \quad h_0 = 2c - E_{11} + E_{n+1,n+1}, \quad e_0 = E_{n+1,1}t.$$

The element $h \in \mathfrak{h}$ satisfying relations (2.14) is

$$h = \mathrm{diag}\left(-\frac{n}{2}, -\frac{n-1}{2}, \ldots, \frac{n-1}{2}, \frac{n}{2}\right). \tag{A.1}$$

Recall that $\Lambda = e_0 + e_1 + \cdots + e_n$ and, from Example 2.3, that $V^{(i)} = \bigwedge^i L(\omega_1)^{(1)}_{\frac{i-1}{2}}$, for $i \in I$. In particular, $V^{(1)} = \mathbb{C}^{n+1}$, and we set

$$\psi^{(1)} = \sum_{j=1}^{n+1} u_j \in V^{(1)}. \tag{A.2}$$

Then, it is easy to check that $\Lambda \psi^{(1)} = \psi^{(1)}$. By Proposition 4.4(a), $\psi^{(1)} \in V^{(1)}$ is the unique (up to a constant) eigenvector corresponding to the maximal eigenvalue $\lambda^{(1)} = 1$. Furthermore, using Eq. (4.10), we can check that for every $i \in I$,

$$\lambda^{(i)} = \frac{\sin\left(\frac{i\pi}{n+1}\right)}{\sin\left(\frac{\pi}{n+1}\right)}, \tag{A.3}$$

is a maximal eigenvalue of Λ in the representation $V^{(i)}$. Using Eqs. (4.2) and (4.3), the corresponding eigenvector is easily checked to be

$$\psi^{(i)} = \psi^{(1)}_{-\frac{i-1}{2}} \wedge \psi^{(1)}_{-\frac{i-3}{2}} \wedge \cdots \wedge \psi^{(i)}_{\frac{i-3}{2}} \wedge \psi^{(1)}_{\frac{i-1}{2}} \in V^{(i)}.$$

A.2. The D_n case. Let $n \in \mathbb{Z}_+$, and consider the involution on the set $\{1, \ldots, 2n\}$ defined by $i \to i' = 2n + 1 - i$. Given a matrix $A = \left(A_{ij}\right)^{2n}_{i,j=1} \in \mathrm{Mat}_{2n}(\mathbb{C})$ we define its anti-transpose (the transpose with respect to the antidiagonal) by

$$A^{\mathrm{at}} = \left(A^{\mathrm{at}}_{ij}\right)^{2n}_{i,j=1}, \quad \text{where} \quad A^{\mathrm{at}}_{ij} = A_{j'i'}.$$

Let us set

$$S = \sum_{k=1}^{n} (-1)^{k+1} \left(E_{kk} + E_{k'k'}\right).$$

Following [15], the simple Lie algebra of type D_n can be realized as the algebra

$$\mathfrak{g} = \mathfrak{o}_{2n} = \{A \in \mathrm{Mat}_{2n}(\mathbb{C}) \mid AS + SA^{\mathrm{at}} = 0\},$$

where the Lie bracket is the usual commutator of matrices.
The dual Coxeter number of $\mathfrak{g}$ is $h^{\vee} = 2n - 2$.
For $i, j \in I = \{1, 2, \ldots, n\}$, we define

$$F_{ij} = E_{ij} + (-1)^{i+j+1} E_{j'i'}, \quad \widetilde{F}_{ij} = E_{ij'} + (-1)^{i+j+1} E_{ji'},$$

and we consider the following Chevalley generators of $\mathfrak{g}$:

$$f_i = F_{i+1,i}, \qquad h_i = F_{ii} - F_{i+1,i+1}, \qquad e_i = F_{i,i+1}, \qquad i = 1, \ldots, n-1,$$

$$f_n = 2\widetilde{F}_{n',(n-1)'}, \quad h_n = F_{n-1,n-1} + F_{n,n}, \quad e_n = \frac{1}{2}\widetilde{F}_{n-1,n}.$$

It is well-known that the representation $L(\omega_1)$ is given by the natural action of $\mathfrak{g}$ on $L(\omega_1) = \mathbb{C}^{2n}$. Moreover we have that

$$L(\omega_i) = \bigwedge^i L(\omega_1), \quad i = 1, \ldots, n - 2,$$

while $L(\omega_{n-1})$ and $L(\omega_n)$ are the so-called half-spin representations of $\mathfrak{g}$.
We denote by u_j, $j = 1, \ldots, 2n$, the standard basis of $\mathbb{C}^{2n}$, and by v_i, $i \in I$, the highest weight vector of the representation $L(\omega_i)$. It is well-known that

$$v_i = u_1 \wedge u_2 \wedge \cdots \wedge u_i, \quad i = 1, \ldots, n - 2.$$

The set of Chevalley generators for $\widehat{\mathfrak{g}}$ is obtained by adding to the Chevalley generators of $\mathfrak{g}$ the following elements:

$$f_0 = 2\widetilde{F}_{(2n)',(2n-1)'}t^{-1}, \quad h_0 = 2c - F_{11} - F_{22}, \quad e_0 = \frac{1}{2}\widetilde{F}_{2n-1,2n}t.$$

Recall that $\Lambda = e_0 + e_1 + \cdots + e_n$ and, from Example 2.3, that $V^{(i)} = \bigwedge^i L(\omega_1)^{(1)}_{\frac{i-1}{2}}$, for $i = 1, \ldots, n - 2$, $V^{(n-1)} = L(\omega_{n-1})_{\frac{n}{2}}$ and $V^{(n)} = L(\omega_n)_{\frac{n}{2}}$ are the evaluation representations at $\zeta = (-1)^n$ of the half-spin representations.
In particular, $V^{(1)} = \mathbb{C}^{2n}$, and we set

$$\psi^{(1)} = \sum_{j=1}^{n-1} \left(u_j + u_{n+j}\right) + \frac{1}{2}\left(u_n + u_{2n}\right).$$

Then, it is easy to check that $\Lambda\psi^{(1)} = \psi^{(1)}$. By Proposition 4.4(a), $\psi^{(1)} \in V^{(1)}$ is the unique (up to a constant) eigenvector corresponding to the maximal eigenvalue $\lambda^{(1)} = 1$. Furthermore, using Eq. (4.10), we can check that for every $i = 1, \ldots, n - 2$,

$$\lambda^{(i)} = \frac{\sin\left(\frac{i\pi}{2n-2}\right)}{\sin\left(\frac{\pi}{2n-2}\right)},$$

is the maximal eigenvalue of Λ in the representation $V^{(i)}$, and the corresponding eigenvector is easily checked to be

$$\psi^{(i)} = \psi^{(1)}_{-\frac{i-1}{2}} \wedge \psi^{(1)}_{-\frac{i-3}{2}} \wedge \cdots \wedge \psi^{(i)}_{\frac{i-3}{2}} \wedge \psi^{(1)}_{\frac{i-1}{2}} \in V^{(i)},$$

as follows from Eqs. (4.3) and (4.2). For the fundamental representations $V^{(n-1)}$ and $V^{(n)}$ (see [19] for the definition), it is easy to check, using again Eq. (4.10), that

$$\lambda^{(n-1)} = \lambda^{(n)} = \frac{1}{2\sin\left(\frac{\pi}{2n-2}\right)}.$$

Finally, we note that there is another important irreducible representation for the Lie algebra $\mathfrak{g}$, which we denote by $U^{(n-1)} = \bigwedge^{n-1} V^{(1)}_{\frac{n}{2}}$.

Table 2. Maximal eigenvalues for simple Lie algebras of E type

	E_6	E_7	E_8
$\lambda^{(1)}$	1	1	1
$\lambda^{(2)}$	$\dfrac{\sin\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{12}\right)}$	$\dfrac{\sin\left(\frac{\pi}{9}\right)}{\sin\left(\frac{\pi}{18}\right)}$	$\dfrac{\sin\left(\frac{2\pi}{30}\right)}{\sin\left(\frac{\pi}{30}\right)}$
$\lambda^{(3)}$	$\dfrac{\sin\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{12}\right)}$	$\dfrac{\sin\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{18}\right)}$	$\dfrac{\sin\left(\frac{\pi}{10}\right)}{\sin\left(\frac{\pi}{30}\right)}$
$\lambda^{(4)}$	$\dfrac{\sin\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{12}\right)}$	$\dfrac{\sin\left(\frac{2\pi}{9}\right)}{\sin\left(\frac{\pi}{18}\right)}$	$\dfrac{\sin\left(\frac{2\pi}{15}\right)}{\sin\left(\frac{\pi}{30}\right)}$
$\lambda^{(5)}$	$\dfrac{\sin\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{12}\right)}$	$\dfrac{\sin\left(\frac{2\pi}{9}\right)}{\sin\left(\frac{\pi}{9}\right)}$	$\dfrac{\sin\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{30}\right)}$
$\lambda^{(6)}$	1	$\dfrac{\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)}{\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{18}\right)}$	$\dfrac{\sin\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{15}\right)}$
$\lambda^{(7)}$		$\dfrac{\sin\left(\frac{2\pi}{9}\right)}{\sin\left(\frac{\pi}{6}\right)}$	$\dfrac{\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{15}\right)}{\sin\left(\frac{\pi}{10}\right)\sin\left(\frac{\pi}{30}\right)}$
$\lambda^{(8)}$			$\dfrac{\sin\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{10}\right)}$

It is not hard to show that $\widetilde{\lambda}^{(n-1)} = 2\lambda^{(n-1)}$ is the maximal eigenvalue for the action of Λ on $U^{(n-1)}$. Its corresponding eigenvector is

$$\widetilde{\psi}^{(n-1)} = \psi^{(1)}_{-\frac{n-2}{2}} \wedge \psi^{(1)}_{-\frac{n-3}{2}} \wedge \cdots \wedge \psi^{(i)}_{\frac{n-3}{2}} \wedge \psi^{(1)}_{\frac{n-2}{2}}.$$

Moreover, the following isomorphism of representation (see [19])

$$V^{(n-1)} \otimes V^{(n)} \cong U^{(n-1)} \oplus V^{(n-3)} \oplus \cdots,$$

implies that there exists a morphism of representations $\widetilde{m}$ such that

$$\widetilde{m}\left(\widetilde{\psi}^{(n-1)}\right) = \psi^{(n-1)} \otimes \psi^{(n)}. \tag{A.4}$$

A.3. The E case. In Table 2 we give the explicit form of the eigenvalues $\lambda^{(i)}$'s, obtained from (4.10). The corresponding eigenvectors are computed in a GAP [21] file available upon request to the authors.

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