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# Classification of Quantum Groups and Belavin–Drinfeld Cohomologies

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**Abstract:** In the present article we discuss the classification of quantum groups whose quasi-classical limit is a given simple complex Lie algebra $\mathfrak{g}$. This problem is reduced to the classification of all Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$, where $\mathbb{K} = \mathbb{C}((\hbar))$. The associated classical double is of the form $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$, where $A$ is one of the following: $\mathbb{K}[\varepsilon]$, where $\varepsilon^2 = 0$, $\mathbb{K} \oplus \mathbb{K}$ or $\mathbb{K}[j]$, where $j^2 = \hbar$. The first case is related to quasi-Frobenius Lie algebras. In the second and third cases we introduce a theory of Belavin–Drinfeld cohomology associated to any non-skewsymmetric $r$-matrix on the Belavin–Drinfeld list (Belavin and Drinfeld in Soviet Sci Rev Sect C: Math Phys Rev 4:93–165, 1984). We prove a one-to-one correspondence between gauge equivalence classes of Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ and cohomology classes (in case II) and twisted cohomology classes (in case III) associated to any non-skewsymmetric $r$-matrix.

## 1. Introduction

Let $k$ be a field of characteristic 0. According to [4], a quantized universal enveloping algebra (or a quantum group) is a topologically free topological Hopf algebra $H$ over the formal power series ring $k[[\hbar]]$ such that $H/\hbar H$ is isomorphic to the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ over $k$.

The quasi-classical limit of a quantum group is a Lie bialgebra. By definition, a Lie bialgebra is a Lie algebra $\mathfrak{g}$ together with a cobracket $\delta$ which is compatible with the Lie bracket. Given a quantum group $H$, with comultiplication $\Delta$, the quasi-classical limit of $H$ is the Lie bialgebra $\mathfrak{g}$ of primitive elements of $H/\hbar H$ and the cobracket is the restriction of the map $(\Delta - \Delta^{21})/\hbar \pmod{\hbar}$ to $\mathfrak{g}$.

The operation of taking the semiclassical limit is a functor $SC : QUE \to LBA$ between categories of quantum groups and Lie bialgebras over $k$. The quantization problem raised by Drinfeld aims at finding a quantization functor, i.e., a functor $Q : LBA \to QUE$ such that $SC \circ Q$ is isomorphic to the identity. Moreover, a quantization functor is required to be universal, in the sense of props.

The existence of universal quantization functors was proved by Etingof and Kazhdan [5,6]. They used Drinfeld's theory of associators to construct quantization functors for any field $k$ of characteristic zero. Drinfeld introduced the notion of associators in relation to the theory of quasi-triangular quasi-Hopf algebras and showed that associators exist over any field $k$ of characteristic zero. Etingof and Kazhdan proved that for any fixed associator over $k$ one can construct a universal quantization functor. More precisely, let $(\mathfrak{g}, \delta)$ be a Lie bialgebra over $k$. Then it is possible to define a Lie bialgebra $\mathfrak{g}_\hbar$ over $k[[\hbar]]$ as $(\mathfrak{g} \otimes_k k[[\hbar]], \hbar\delta)$. According to Theorem 2.1 of [6] there exists an equivalence $\widehat{Q}$ between the category $LBA_0(k[[\hbar]])$ of topologically free Lie bialgebras over $k[[\hbar]]$ with $\delta = 0 \pmod{\hbar}$ and the category $HA_0(k[[\hbar]])$ of topologically free Hopf algebras cocommutative modulo $\hbar$. Moreover, for any $(\mathfrak{g}, \delta)$ over $k$, we have $\widehat{Q}(\mathfrak{g}_\hbar) = U_\hbar(\mathfrak{g})$.

The aim of the present article is the classification of quantum groups whose quasi-classical limit is a given simple complex Lie algebra $\mathfrak{g}$. Due to the equivalence between $HA_0(\mathbb{C}[[\hbar]])$ and $LBA_0(\mathbb{C}[[\hbar]])$, this problem is equivalent to the classification of Lie bialgebra structures on $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$. For simplicity, denote $\mathbb{O} := \mathbb{C}[[\hbar]]$, $\mathbb{K} := \mathbb{C}((\hbar))$, $\mathfrak{g}(\mathbb{O}) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{O}$ and $\mathfrak{g}(\mathbb{K}) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{K}$.

On the other hand, in order to classify cobrackets on $\mathfrak{g}(\mathbb{O})$ it is sufficient to classify cobrackets on $\mathfrak{g}(\mathbb{K})$. Indeed, if $\delta_0$ is a Lie bialgebra structure on $\mathfrak{g}(\mathbb{O})$, then it can be naturally extended to $\mathfrak{g}(\mathbb{K})$. Conversely, given a Lie bialgebra structure $\delta$ on $\mathfrak{g}(\mathbb{K})$, we can restrict $\hbar^n\delta$ to $\mathfrak{g}(\mathbb{O})$ for a sufficiently large $n$ since $\mathfrak{g}$ is finite dimensional.

From now on let $G$ be a connected split algebraic group with a reductive Lie algebra whose semisimple part is $\mathfrak{g}$. We will consider the adjoint action Ad of $G$ on $\mathfrak{g}$. We consider the equivalence classes of Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ with respect to the following equivalence: two bialgebra structures $\delta_1, \delta_2$ are equivalent if there exists an element $a \in \mathbb{K}^*$ and $X \in G(\mathbb{K})$ such that $\delta_1 = a(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)\delta_2$; here $((\mathrm{Ad}_X \otimes \mathrm{Ad}_X)\delta)(l) = (\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(\delta(\mathrm{Ad}_X^{-1} l))$. We will also use the term "gauge equivalence" or "$G$-equivalence" if there exists $X \in G(\mathbb{K})$ such that $\delta_1 = (\mathrm{Ad}_X \otimes \mathrm{Ad}_X)\delta_2$.

From the general theory of Lie bialgebras it is known that for each Lie bialgebra structure $\delta$ on a fixed Lie algebra $L$ one can construct the corresponding classical double $D(L, \delta)$, which is the vector space $L \oplus L^*$ together with a bracket which is induced by the bracket and the cobracket of $L$, and a non-degenerate invariant bilinear form, see [3]. We consider $L = \mathfrak{g}(\mathbb{K})$ and prove Proposition 1, which states that there exists an associative, unital, commutative algebra $A$, of dimension 2 over $\mathbb{K}$, such that $D(\mathfrak{g}(\mathbb{K}), \delta) \cong \mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$. In Proposition 2 we show that there are three possibilities for $A$: $A = \mathbb{K}[\varepsilon]$, where $\varepsilon^2 = 0$, $A = \mathbb{K} \oplus \mathbb{K}$ or $A = \mathbb{K}[j]$, where $j^2 = \hbar$.

Due to the correspondence between Lie bialgebras and Manin triples, to any Lie bialgebra structure $\delta$ on $L$ one can associate a certain Lagrangian subalgebra $W$ of $D(L, \delta)$ which is complementary to $L$. Conversely, any such $W$ produces a Lie cobracket on $L$. The main problem is to obtain a classification of all such subalgebras $W$ for the three choices of $A$ as above. We investigate separately each choice of $A$.

For $A = \mathbb{K}[\varepsilon]$, where $\varepsilon^2 = 0$, it turns out that the classification problem is related to that of quasi-Frobenius Lie subalgebras over $\mathbb{K}$.

In the case of $A = \mathbb{K} \oplus \mathbb{K}$, we introduce Belavin–Drinfeld cohomologies. Namely, for any non-skewsymmetric constant $r$-matrix $r_{BD}$ on the Belavin–Drinfeld list [1], we define a cohomology set $H^1_{BD}(r_{BD})$. This cohomology set will depend on a gauge group $G$ acting "naturally" on $\mathfrak{g}$. We will see that the choice of $G$ is important. Therefore, we will use the notation $H^1_{BD}(G, r_{BD})$. One should notice that in all the cases with exception for $GL(n)$, the Lie algebra of $G$ will be $\mathfrak{g}$.

We prove that there exists a one-to-one correspondence between any Belavin–Drinfeld cohomologies and gauge equivalence classes of Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$. Then we restrict our discussion to $\mathfrak{g} = sl(n)$ and show that all cohomologies $H^1_{BD}(GL(n), r_{BD})$ are trivial.

We also discuss the case of the orthogonal algebras $\mathfrak{g} = o(n)$, where it turns out that the cohomologies associated to the Drinfeld–Jimbo $r$-matrix are also trivial. We also give an example where the cohomology corresponding to a certain non-skewsymmetric constant $r$-matrix for $o(2n)$ is non-trivial.

We finally proceed with the classification of Lie bialgebras whose classical double is isomorphic to $\mathfrak{g}(\mathbb{K}[j])$ with $j^2 = \hbar$. We restrict ourselves to $\mathfrak{g} = sl(n)$ and show that in this case a cohomology theory can be introduced too. Our result states that there exists a one-to-one correspondence between Belavin–Drinfeld twisted cohomologies and gauge equivalence classes of Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$. We prove that the twisted cohomology corresponding to the Drinfeld–Jimbo $r$-matrix and a certain class of $r$-matrices (called generalized Cremmer–Gervais) is trivial.

In the last section of the article we compute Belavin–Drinfeld cohomology in certain cases for $\mathfrak{g} = sl(n)$ and $G = SL(n)$. In particular, we show that $H^1_{BD}(SL(n), r_{BD})$ is non-trivial for certain $r_{BD}$. Finally, we formulate a conjecture stating that the Belavin–Drinfeld cohomology associated to the Drinfeld–Jimbo $r$-matrix is trivial for any simple complex Lie algebra $\mathfrak{g}$. We also define the quantum Belavin–Drinfeld cohomology and formulate a second conjecture about the existence of a natural correspondence between classical and quantum cohomologies.

## 2. Lie Bialgebra Structures on $\mathfrak{g}(\mathbb{K})$

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra. Consider the Lie algebras $\mathfrak{g}(\mathbb{O}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{O}$ and $\mathfrak{g}(\mathbb{K}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{K}$.

We have seen that the classification of quantum groups with quasi-classical limit $\mathfrak{g}$ is equivalent to the classification of all Lie bialgebra structures on $\mathfrak{g}(\mathbb{O})$. Moreover, as explained in the introduction, in order to classify Lie bialgebra structures on $\mathfrak{g}(\mathbb{O})$, it is enough to classify them on $\mathfrak{g}(\mathbb{K})$.

Let us assume that $\delta$ is a Lie bialgebra structure on $\mathfrak{g}(\mathbb{K})$. This cobracket endows the dual of $\mathfrak{g}(\mathbb{K})$ with a Lie bracket. Then one can construct the corresponding classical double $D(\mathfrak{g}(\mathbb{K}), \delta)$. As a vector space, $D(\mathfrak{g}(\mathbb{K}), \delta) = \mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})^*$. As a Lie algebra, it is endowed with a bracket which is induced by the bracket and cobracket of $\mathfrak{g}(\mathbb{K})$. Moreover, the canonical symmetric non-degenerate bilinear form on this space is invariant.

Similarly to Lemma 2.1 in [8], one can prove that $D(\mathfrak{g}(\mathbb{K}), \delta)$ is a direct sum of regular adjoint $\mathfrak{g}$-modules. Combining this result with Proposition 2.2 in [2], we obtain

**Proposition 1.** *There exists an associative, unital, commutative algebra $A$ of dimension 2 over $\mathbb{K}$, such that $D(\mathfrak{g}(\mathbb{K}), \delta) \cong \mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$.*

*Remark 1.* The symmetric invariant non-degenerate bilinear form $Q$ on $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$ is given in the following way. For arbitrary elements $f_1, f_2 \in \mathfrak{g}(\mathbb{K})$ and $a, b \in A$ we have $Q(f_1 \otimes a, f_2 \otimes b) = K(f_1, f_2) \cdot t(ab)$, where $K$ denotes the Killing form on $\mathfrak{g}(\mathbb{K})$ and $t : A \longrightarrow \mathbb{K}$ is a trace function.

Let us investigate the algebra $A$. Since $A$ is unital and of dimension 2 over $\mathbb{K}$, one can choose a basis $\{e, 1\}$, where 1 denotes the unit. Moreover, there exist $p$ and $q$ in $\mathbb{K}$ such that $e^2 + pe + q = 0$. Let $\Delta = p^2 - 4q \in \mathbb{K}$. We distinguish the following cases:

(i) Assume $\Delta = 0$. Let $\varepsilon := e + \frac{p}{2}$. Then $\varepsilon^2 = 0$ and $A = \mathbb{K}\varepsilon \oplus \mathbb{K} = \mathbb{K}[\varepsilon]$.

(ii) Assume $\Delta \neq 0$ and has even order as an element of $\mathbb{K}$. This implies that $\Delta = \hbar^{2m}(a_0 + a_1\hbar + a_2\hbar^2 + \cdots)$, where $m$ is an integer, $a_i$ are complex coefficients and $a_0 \neq 0$.
One can easily check that the equation $x^2 = a_0 + a_1\hbar + a_2\hbar^2 + \cdots$ has two solutions $\pm x = x_0 + x_1\hbar + x_2\hbar^2 + \cdots$ in $\mathbb{O}$.
Then $e = -\frac{p}{2} \pm \frac{\hbar^m x}{2}$, which implies that $e \in \mathbb{K}$ and $A = \mathbb{K} \oplus \mathbb{K}$.

(iii) Assume $\Delta \neq 0$ and has odd order as an element of $\mathbb{K}$. We have $\Delta = \hbar^{2m+1}(a_0 + a_1\hbar + a_2\hbar^2 + \cdots)$, where $m$ is an integer, $a_i$ are complex coefficients and $a_0 \neq 0$. Again the equation $x^2 = a_0 + a_1\hbar + a_2\hbar^2 + \cdots$ has two solutions $\pm x = x_0 + x_1\hbar + x_2\hbar^2 + \ldots$ in $\mathbb{O}$. Since $a_0 \neq 0$, we have $x_0 \neq 0$ and thus $x$ is invertible in $\mathbb{O}$. Let $j = \hbar^{-m}(2e + p)x^{-1}$. Then $e^2 + pe + q = 0$ is equivalent to $j^2 = \hbar$. On the other hand, $A = \mathbb{K}e \oplus \mathbb{K}$ and $2e = \hbar^m xj - p$ imply that $A = \mathbb{K}j \oplus \mathbb{K}$. Therefore, we obtain that $A = \mathbb{K}[j]$ where $j^2 = \hbar$.

We can summarize the above facts:

**Proposition 2.** *Let $\delta$ be an arbitrary Lie bialgebra structure on $\mathfrak{g}(\mathbb{K})$. Then $D(\mathfrak{g}(\mathbb{K}), \delta)$ is isomorphic to $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$, where $A = \mathbb{K}[\varepsilon]$ and $\varepsilon^2 = 0$, $A = \mathbb{K} \oplus \mathbb{K}$ or $A = \mathbb{K}[j]$ and $j^2 = \hbar$.*

On the other hand, it is well-known, see for instance [4], that there is a one-to-one correspondence between Lie bialgebra structures on a Lie algebra $L$ and Manin triples $(D(L), L, W)$, where $D(L) = L \oplus W$ is equipped with a bilinear symmetric invariant non-degenerate form $Q$ such that both $L$ and $W$ are Lagrangian subalgebras of $D(L)$ with respect to $Q$. For $L = \mathfrak{g}(\mathbb{K})$, this fact implies the following

**Proposition 3.** *There exists a one-to-one correspondence between Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ for which the classical double is $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$ and Lagrangian subalgebras $W$ of $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$ transversal to $\mathfrak{g}(\mathbb{K})$.*

**Corollary 1.** (i) *There exists a one-to-one correspondence between Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ for which the classical double is $\mathfrak{g}(\mathbb{K}[\varepsilon])$, $\varepsilon^2 = 0$, and Lagrangian subalgebras $W$ of $\mathfrak{g}(\mathbb{K}[\varepsilon])$ that are transversal to $\mathfrak{g}(\mathbb{K})$.*
(ii) *There exists a one-to-one correspondence between Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ for which the classical double is $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$ and Lagrangian subalgebras $W$ of $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$ that are transversal to $\mathfrak{g}(\mathbb{K})$, embedded diagonally into $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$.*
(iii) *There exists a one-to-one correspondence between Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ for which the classical double is $\mathfrak{g}(\mathbb{K}[j])$, where $j^2 = \hbar$, and Lagrangian subalgebras $W$ of $\mathfrak{g}(\mathbb{K}[j])$ that are transversal to $\mathfrak{g}(\mathbb{K})$.*

## 3. Lie Bialgebra Structures in Case I

Here we study the Lie bialgebra structures $\delta$ on $\mathfrak{g}(\mathbb{K})$ for which the corresponding Drinfeld double is isomorphic to $\mathfrak{g}(\mathbb{K}[\varepsilon])$, $\varepsilon^2 = 0$. Our problem is to find all subalgebras $W$ of $\mathfrak{g}(\mathbb{K}[\varepsilon])$ satisfying the following conditions:

(i) $W \oplus \mathfrak{g}(\mathbb{K}) = \mathfrak{g}(\mathbb{K}[\varepsilon])$.

(ii) $W = W^{\perp}$ with respect to the non-degenerate symmetric bilinear form $Q$ on $\mathfrak{g}(\mathbb{K}[\varepsilon])$ given by

$$Q(f_1 + \varepsilon f_2, g_1 + \varepsilon g_2) = K(f_1, g_2) + K(f_2, g_1).$$

**Proposition 4.** *Any subalgebra $W$ of $\mathfrak{g}(\mathbb{K}[\varepsilon])$ satisfying conditions (i) and (ii) from above is uniquely defined by a subalgebra $L$ of $\mathfrak{g}(\mathbb{K})$ together with a non-degenerate 2-cocycle $B$ on $L$.*

*Proof.* The proof is similar to that of Theorem 3.2 and Corollary 3.3 in [10].

*Remark 2.* We recall that a Lie algebra is called quasi-Frobenius if there exists a non-degenerate 2-cocycle on it. It is called Frobenius if the corresponding 2-cocycle is a coboundary. Thus we see that the classification problem for the Lagrangian subalgebras we are interested in includes the classification of Frobenius subalgebras of $\mathfrak{g}(\mathbb{K})$. This question is quite complicated, as it is known from studying Frobenius subalgebras of $\mathfrak{g}$. However, for $\mathfrak{g} = sl(2)$ there is only one Frobenius subalgebra up to conjugation, the standard parabolic one.

## 4. Lie Bialgebra Structures in Case II and Belavin-Drinfeld Cohomologies

Our task now is to classify Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ for which the associated classical double is isomorphic to $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$.

**Lemma 1.** *Any Lie bialgebra structure $\delta$ on $\mathfrak{g}(\mathbb{K})$ for which the associated classical double is isomorphic to $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$ is a coboundary $\delta = dr$ given by an $r$-matrix satisfying $r + r^{21} = f\Omega$, where $f \in \mathbb{K}$ and* $\mathrm{CYB}(r) = 0$.

Without loss of generality we may suppose that $f = 1$. The corresponding $r$-matrices in the case of an algebraically closed field have been classified up to the $\mathrm{Ad}(G)$-equivalence in [1]; the classification is given in terms of admissible triples. (Recall that $G$ stands for a connected split algebraic group with a reductive Lie algebra whose semisimple part is $\mathfrak{g}$.)

Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and the associated root system. Fix a set of simple roots $\Gamma$. We choose a system of generators $e_{\alpha}, e_{-\alpha}, h_{\alpha}$ such that $K(e_{\alpha}, e_{-\alpha}) = 1$, for any positive root $\alpha$. Denote by $\Omega_0$ the Cartan part of $\Omega$. Suppose also that $H \subset G$ is a maximal torus with Lie algebra $\mathfrak{h}$.

Let us recall from [1,4] that any non-skewsymmetric $r$-matrix depends on certain discrete and continuous parameters. The discrete one is an admissible triple $(\Gamma_1, \Gamma_2, \tau)$, i.e., an isometry $\tau : \Gamma_1 \longrightarrow \Gamma_2$ where $\Gamma_1, \Gamma_2 \subset \Gamma$ are such that for any $\alpha \in \Gamma_1$ there exists $k \in \mathbb{N}$ satisfying $\tau^k(\alpha) \notin \Gamma_1$. The continuous parameter is a tensor $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ satisfying $r_0 + r_0^{21} = \Omega_0$ and $(\tau(\alpha) \otimes 1 + 1 \otimes \alpha)(r_0) = 0$ for any $\alpha \in \Gamma_1$. Then the associated $r$-matrix is given by the formula

$$r_{BD} = r_0 + \sum_{\alpha>0} e_{\alpha} \otimes e_{-\alpha} - \sum_{\alpha \in (\mathrm{Span}\,\Gamma_1)^+} \sum_{k \in \mathbb{N}} e_{-\alpha} \wedge e_{\tau^k(\alpha)}.$$

Now, let us consider an $r$-matrix corresponding to a Lie bialgebra structure on $\mathfrak{g}(\mathbb{K})$. Up to $\mathrm{Ad}(G(\overline{\mathbb{K}}))$-equivalence, we have the Belavin–Drinfeld classification. We may assume that our $r$-matrix is of the form $r_X = (\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$, where $X \in G(\overline{\mathbb{K}})$

and $r_{BD}$ satisfies the equations $r + r^{21} = \Omega$ and $\mathrm{CYB}(r) = 0$. The corresponding bialgebra structure is $\delta(a) = [r_X, a \otimes 1 + 1 \otimes a]$ for any $a \in \mathfrak{g}(\mathbb{K})$.

Let us take an arbitrary $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$. Then we have $(\sigma \otimes \sigma)(\delta(a)) = [\sigma(r_X), a \otimes 1 + 1 \otimes a]$ and $(\sigma \otimes \sigma)(\delta(a)) = \delta(a)$, which implies that $\sigma(r_X) = r_X + \lambda\Omega$, for some $\lambda \in \overline{\mathbb{K}}$. Let us show that $\lambda = 0$. Indeed, $\Omega = \sigma(\Omega) = \sigma(r_X) + \sigma(r_X^{21}) = r_X + r_X^{21} + 2\lambda\Omega$. Thus $\lambda = 0$ and $\sigma(r_X) = r_X$. Consequently, we get $(\mathrm{Ad}_{X^{-1}\sigma(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma(X)})(\sigma(r_{BD})) = r_{BD}$.

**Definition 1.** Let $r$ be an $r$-matrix. The *centralizer* $C(G, r)$ of $r$ is the set of all $X \in G(\overline{\mathbb{K}})$ satisfying $(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r) = r$.

**Theorem 1.** *For any simple Lie algebra $\mathfrak{g}$ and for any Belavin-Drinfeld matrix $r_{BD}$ we have*

$$C(G, r_{BD}) \subset H,$$

*where $H$ is a maximal torus of $G$.*

*Proof.* (1) Let us consider the map $\Phi : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}^* = \mathrm{End}(\mathfrak{g})$ induced by the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ given by the Killing form, i.e.

$$\Phi(a \otimes b)(u) = K(a, u)b.$$

Let $X \in C(G, r_{BD})$. We have

$$(\mathrm{Ad}_X a \otimes \mathrm{Ad}_X b)(u) = K(\mathrm{Ad}_X a, u)\mathrm{Ad}_X b = \mathrm{Ad}_X(K(a, \mathrm{Ad}_X u)b).$$

Thus, $X \in C(G, r_{BD})$ iff $\mathrm{Ad}_X \Phi(r) = \Phi(r)\mathrm{Ad}_X$.

(2) The fact that $\mathrm{Ad}_X$ commutes with $\Phi(r)$ implies that it commutes with semisimple and nilpotent parts of $\Phi(r)$. Our next aim is to compute them. The operator $\Phi(e_\alpha \otimes e_\beta)$ maps $e_{-\alpha}$ to $e_\beta$ and the rest of the Chevalley basis to zero. Hence, when $\alpha + \beta \neq 0$ the operator $\Phi(e_\alpha \otimes e_\beta)$ is nilpotent. Thus the operator $A = \Phi(\sum e_{\tau^k(\alpha)} \wedge e_{-\alpha})$ is nilpotent.

For any positive root $\alpha$, we have $\Phi(r_{DJ})e_\alpha = 0$, $\Phi(r_{DJ})e_{-\alpha} = e_{-\alpha}$ and $\Phi(r_{DJ})h_{\pm\alpha} = \frac{1}{2}h_{\pm\alpha}$. So when $\alpha$ and $\beta$ have opposite signs, $\Phi(r_{DJ})$ commutes with $\Phi(e_\alpha \otimes e_\beta)$. Therefore, $\Phi(r_{DJ})$ commutes with $A$. Clearly, $A(\mathfrak{h}) = 0$. Hence, both $A$ and $\Phi(r_{DJ})$ commute with $\Phi(s)$, where $s = r - r_{DJ} - \sum e_{\tau^k(\alpha)} \wedge e_{-\alpha} \in \mathfrak{h}^{\otimes 2}$.

So we have the decomposition of $\Phi(r_{BD})$ into the sum of three commuting operators: $\Phi(r_{BD}) = \Phi(r_{DJ}) + \Phi(s) + A$. If $\Phi(s) = \Phi(s)_d + \Phi(s)_n$ is the Jordan decomposition of $\Phi(s)$ then $D = \Phi(r_{DJ}) + \Phi(s)_d$ is semisimple, $N = A + \Phi(s)_n$ is nilpotent, and $D$ and $N$ commute. Thus, we have obtained the Jordan decomposition $\Phi(r_{BD}) = D + N$. Note that we have $De_\alpha = 0$, $De_{-\alpha} = e_{-\alpha}$ and $Dh_\alpha \in \mathfrak{h}$. It remains to show that the centralizer of $D$ lies in $H$.

(3) The zero eigenspace $V_0$ of the operator $D$ contains all positive root vectors and no negative root vectors. $\mathrm{Ad}_X$ commutes with $D$ and hence must preserve $V_0$. But it also must preserve its normalizer, which is the Borel subalgebra $\mathfrak{b}^+$. Similarly, considering $V_1$ instead of $V_0$, we obtain that $\mathrm{Ad}_X$ preserves $\mathfrak{b}^-$. Therefore, $\mathrm{Ad}_X$ preserves $\mathfrak{h}$. So, $X \in N_G(\mathfrak{h})$, the normalizer of the Cartan subalgebra. Consequently, $\mathrm{Ad}_X$ induces an element of the Weyl group $W$. It is well-known that $W$ acts transitively and without fixed points on the set of the Borel subalgebras containing $\mathfrak{h}$. But $\mathrm{Ad}_X$ preserves $\mathfrak{b}^+$. Therefore, $\mathrm{Ad}_X$ induces the unit of $W$ and thus, $X \in H$.

For any root $\alpha$ we denote by $e^\alpha$ the corresponding character of the torus $H$.

**Theorem 2.** *If* $(\Gamma_1, \Gamma_2, \tau)$ *is an admissible triple corresponding to a Belavin-Drinfeld r-matrix* $r_{BD}$ *then* $X \in C(G, r_{BD})$ *iff for any root* $\alpha \in \Gamma_1 \setminus \Gamma_2$ *and for any* $k \in \mathbb{N}$ *we have* $e^{\alpha}(X) = e^{\tau^k(\alpha)}(X)$*, i.e.,* $e^{\alpha}(X)$ *is constant on the strings of* $\tau$.

*Proof.* Vectors $e_\alpha \otimes e_{-\alpha}, h_\alpha \otimes h_\beta$ and $e_\gamma \wedge e_\delta$ for $\gamma + \delta \neq 0$ form a set of linearly independent eigenvectors of $\mathrm{Ad}_X$. Hence, $X \in C(G, r_{BD})$ if and only if $\mathrm{Ad}_X$ preserves $e_{-\gamma} \wedge e_{\tau^k(\gamma)}$ for $\gamma \in \Gamma_1$. But this is equivalent to $e^{\alpha}(X) = e^{\tau^k(\alpha)}(X)$ for any root $\alpha \in \Gamma_1 \setminus \Gamma_2$ and for any $k \in \mathbb{N}$.

**Theorem 3.** *Let* $r_{BD}$ *be an r-matrix on the Belavin–Drinfeld list for* $\mathfrak{g}(\overline{\mathbb{K}})$*. Suppose that*

$$(\mathrm{Ad}_{X^{-1}\sigma(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma(X)})(\sigma(r_{BD})) = r_{BD}.$$

*Then* $\sigma(r_{BD}) = r_{BD}$ *and* $X^{-1}\sigma(X) \in C(G, r_{BD})$.

*Proof.* Consider $r = r_{BD}$ which corresponds to an admissible triple $(\Gamma_1, \Gamma_2, \tau)$ and $r_0 \in \mathfrak{h}\otimes\mathfrak{h}$. Denote $Y := X^{-1}\sigma(X)$ and $s := r - r_0$. Then $(\mathrm{Ad}_Y \otimes \mathrm{Ad}_Y)(s + \sigma(r_0)) = s + r_0$.

Following [7] p. 43–47, let $\Phi(r) : \mathfrak{g} \longrightarrow \mathfrak{g}$ be defined as in Theorem 1. Let

$$\mathfrak{g}_r^{\lambda} = \bigcup_{n>0} \mathrm{Ker}(\Phi(r) - \lambda)^n.$$

Then

$$\mathfrak{g} = \mathfrak{g}_r^0 \oplus \mathfrak{g}_r' \oplus \mathfrak{g}_r^1, \quad \mathfrak{g}_r' = \bigoplus_{\lambda \neq 0,1} \mathfrak{g}_r^{\lambda}.$$

In our case, $\mathfrak{n}_- \subseteq \mathfrak{g}_{s+r_0}^0 \subseteq \mathfrak{b}_-$, $\mathfrak{n}_+ \subseteq \mathfrak{g}_{s+r_0}^1 \subseteq \mathfrak{b}_+$, $\mathfrak{g}_{s+r_0}' \subseteq \mathfrak{h}$, $\mathfrak{g}_{s+r_0}^0 + \mathfrak{g}_{s+r_0}' = \mathfrak{b}_-$ and $\mathfrak{g}_{s+r_0}^1 + \mathfrak{g}_{s+r_0}' = \mathfrak{b}_+$. Similarly for $s + \sigma(r_0)$.

On the other hand, it can be easily checked that

$$\Phi(\mathrm{Ad}_Y \otimes \mathrm{Ad}_Y)(r) = \mathrm{Ad}_Y \circ \Phi(r) \circ \mathrm{Ad}_Y^{-1}.$$

Hence, $\mathrm{Ad}_Y(\mathfrak{g}_{s+\sigma(r_0)}^i) = \mathfrak{g}_{s+r_0}^i$, $i = 0, 1$ and $\mathrm{Ad}_Y(\mathfrak{g}_{s+\sigma(r_0)}') = \mathfrak{g}_{s+r_0}'$. Therefore, $\mathrm{Ad}_Y(\mathfrak{b}_\pm) = \mathfrak{b}_\pm$ and $\mathrm{Ad}_Y \in H(\overline{\mathbb{K}})$ since $G$ is connected.

Let us analyse the equality $(\mathrm{Ad}_Y \otimes \mathrm{Ad}_Y)(s + \sigma(r_0)) = s + r_0$. It follows that $(\mathrm{Ad}_Y \otimes \mathrm{Ad}_Y)(s) + \sigma(r_0) = s + r_0$. Taking into account that $r_0, \sigma(r_0) \in \mathfrak{h}^{\otimes 2}$ and

$$(\mathrm{Ad}_Y \otimes \mathrm{Ad}_Y)(s) = \sum_{\alpha>0} e_\alpha \otimes e_{-\alpha} + \sum_{\beta \in (\mathbb{Z}\Gamma_1)^+} \sum_{n>0} k_{\beta,n} e_\beta \wedge e_{-\tau^n(\beta)},$$

for some integers $k_{\beta,n}$, we deduce that $\sigma(r_0) = r_0$. Thus, $\sigma(r) = r$ and $\mathrm{Ad}_Y \in C(G, r)$.

Henceforth we will assume that $r_{BD}$ is defined over $\mathbb{K}$, i.e., $r_0 \in \mathfrak{g}(\mathbb{K}) \otimes \mathfrak{g}(\mathbb{K})$.

In conclusion, $r_X = (\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$ induces a Lie bialgebra structure on $\mathfrak{g}(\mathbb{K})$ if and only if $X \in G(\overline{\mathbb{K}})$ satisfies the condition $X^{-1}\sigma(X) \in C(G, r_{BD})$, for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$.

**Definition 2.** Let $r_{BD}$ be a non-skewsymmetric $r$-matrix on the Belavin–Drinfeld list and $C(G, r_{BD})$ its centralizer. We say that $X \in G(\overline{\mathbb{K}})$ is a *Belavin–Drinfeld cocycle* associated to $r_{BD}$ if $X^{-1}\sigma(X) \in C(G, r_{BD})$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$.

We denote the set of Belavin–Drinfeld cocycles associated to $r_{BD}$ by $Z(G, r_{BD})$. This set is non-empty, since it always contains the identity.

**Definition 3.** Two cocycles $X_1$ and $X_2$ in $Z(G, r_{BD})$ are called *equivalent* ($X_1 \sim X_2$) if there exists $Q \in G(\mathbb{K})$ and $C \in C(G, r_{BD})$ such that $X_1 = QX_2C$.

**Definition 4.** Let $H^1_{BD}(G, r_{BD})$ denote the set of equivalence classes of cocycles in $Z(G, r_{BD})$. We call this set the *Belavin–Drinfeld cohomology* associated to the $r$-matrix $r_{BD}$. The Belavin–Drinfeld cohomology is said to be *trivial* if all cocycles are equivalent to the identity, and *non-trivial* otherwise.

We make the following remarks:

*Remark 3.* Assume that $X \in Z(G, r_{BD})$. Then for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$, $\sigma(X) = XC$, for some $C \in C(G, r_{BD})$. We get $(\mathrm{Ad}_{\sigma(X)} \otimes \mathrm{Ad}_{\sigma(X)})(r_{BD}) = (\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$.

Consequently, $(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$ induces a Lie bialgebra structure on $\mathfrak{g}(\mathbb{K})$.

*Remark 4.* Assume that $X_1$ and $X_2$ in $Z(G, r_{BD})$ are equivalent. Then $X_1 = QX_2C$, for some $Q \in G(\mathbb{K})$ and $C \in C(G, r_{BD})$. This implies that $(\mathrm{Ad}_{X_1} \otimes \mathrm{Ad}_{X_1})(r_{BD}) = (\mathrm{Ad}_{QX_2} \otimes \mathrm{Ad}_{QX_2})(r_{BD})$. In other words the $r$-matrices $(\mathrm{Ad}_{X_1} \otimes \mathrm{Ad}_{X_1})(r_{BD})$ and $(\mathrm{Ad}_{X_2} \otimes \mathrm{Ad}_{X_2})(r_{BD})$ are gauge equivalent over $\mathbb{K}$ via an element $Q \in G(\mathbb{K})$.

The above remarks imply the following result.

**Proposition 5.** *Let $r_{BD}$ be a non-skewsymmetric $r$-matrix over $\overline{\mathbb{K}}$. There exists a one-to-one correspondence between $H^1_{BD}(G, r_{BD})$ and gauge equivalence classes of Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ with classical double $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$ and $\overline{\mathbb{K}}$-isomorphic to $dr_{BD}$.*

## 5. Belavin-Drinfeld Cohomologies for $sl(n)$

Our next goal is to compute $H^1_{BD}(GL(n), r_{BD})$. Let us first restrict ourselves to the case of $\mathfrak{g} = sl(n)$ and the cohomology associated to the Drinfeld–Jimbo $r$-matrix $r_{DJ}$. In this section we assume that $G = GL(n)$.

**Lemma 2.** *Let $X \in GL(n, \overline{\mathbb{K}})$. Assume that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$, $X^{-1}\sigma(X) \in \mathrm{diag}(n, \overline{\mathbb{K}})$. Then there exist $Q \in GL(n, \mathbb{K})$ and $D \in \mathrm{diag}(n, \overline{\mathbb{K}})$ such that $X = QD$.*

*Proof.* Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ and $\sigma(X) = XD_\sigma$, where $D_\sigma = \mathrm{diag}(d_1, \ldots, d_n)$. Here the elements $d_i$ depend on $\sigma$. Then $\sigma(x_{ij}) = x_{ij}d_j$, for any $i, j$.

On the other hand, in each column of $X$ there exists a nonzero element. Let us denote these elements by $x_{i_1 1}, \ldots, x_{i_n n}$. For $j = 1$, $\sigma(x_{i1}) = x_{i1}d_1$ and $\sigma(x_{i_1 1}) = x_{i_1 1}d_1$. These relations imply that $\sigma(x_{i1}/x_{i_1 1}) = x_{i1}/x_{i_1 1}$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ and thus $x_{i1}/x_{i_1 1} \in \mathbb{K}$, for any $i$.

Similarly, $x_{i2}/x_{i_2 2} \in \mathbb{K}, \ldots, x_{in}/x_{i_n n} \in \mathbb{K}$, for any $i$. Let $Q = (k_{ij})$ be the matrix whose elements are $k_{ij} = x_{ij}/x_{i_j j}$, for any $i$ and $j$.

Thus $X = QD$, where $Q \in GL(n, \mathbb{K})$ and $D = \mathrm{diag}(x_{i_1 1}, \ldots, x_{i_n n})$.

**Proposition 6.** *For $\mathfrak{g} = sl(n)$, the Belavin–Drinfeld cohomology $H^1_{BD}(GL(n), r_{DJ})$ associated to $r_{DJ}$ and to the group $GL(n)$ is trivial.*

*Proof.* It easily follows from the proof of Theorem 1 that the centralizer of $r_{DJ}$ is $C(GL(n), r_{DJ}) = \mathrm{diag}(n, \overline{\mathbb{K}})$. Let us show that any cocycle is equivalent to the identity. Indeed, let $X = (x_{ij})$ be a cocycle in $Z(GL(n), r_{DJ})$, i.e., $X^{-1}\sigma(X) \in C(GL(n), r_{DJ})$, for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$.

It follows that $X^{-1}\sigma(X) \in \mathrm{diag}(n, \overline{\mathbb{K}})$. According to Lemma 2, there exists $Q \in GL(n, \mathbb{K})$ and $D \in \mathrm{diag}(n, \overline{\mathbb{K}})$ such that $X = QD$. This proves that $X$ is equivalent to the identity.

It turns out that the above result is true not only for $r_{DJ}$. Given an arbitrary $r$-matrix $r_{BD}$ on the Belavin–Drinfeld list, the corresponding cohomology is also trivial. First we will take a closer look at the centralizer $C(GL(n), r_{BD})$ of an $r$-matrix $r_{BD}$. Due to Theorem 1, the following result holds.

**Lemma 3.** *Let $r_{BD}$ be an arbitrary $r$-matrix on the Belavin–Drinfeld list. Then*

$$C(GL(n), r_{BD}) \subseteq \mathrm{diag}(n, \overline{\mathbb{K}}).$$

For $sl(n)$ we are now able to give the exact description of $C(GL(n), r_{BD})$.

**Lemma 4.** *$C(GL(n), r_{BD})$ consists of all diagonal matrices $T = \mathrm{diag}(t_1, \dots, t_n)$ such that $t_i = s_i s_{i+1} \dots s_n$, where $s_i \in \overline{\mathbb{K}}$ satisfy the condition: $s_i = s_j$ if $\alpha_i \in \Gamma_1$ and $\tau(\alpha_i) = \alpha_j$.*

*Proof.* Let us assume that $r_{BD}$ is associated to an admissible triple $(\Gamma_1, \Gamma_2, \tau)$, where $\Gamma_1, \Gamma_2 \subset \{\alpha_1, \dots, \alpha_{n-1}\}$. Let $T \in C(GL(n), r_{BD})$. According to Lemma 3, $T \in \mathrm{diag}(n, \overline{\mathbb{K}})$, therefore we put $T = \mathrm{diag}(t_1, \dots, t_n)$. Now we note that $T \in C(GL(n), r_{BD})$ if and only if $(\mathrm{Ad}_T \otimes \mathrm{Ad}_T)(e_{\tau^k(\alpha)} \wedge e_{-\alpha}) = e_{\tau^k(\alpha)} \wedge e_{-\alpha}$ for any $\alpha \in \Gamma_1$ and any positive integer $k$.

For simplicity, let us take an arbitrary $\alpha_i \in \Gamma_1$ and suppose that $\tau(\alpha_i) = \alpha_j$. Then we get $t_i t_{i+1}^{-1} = t_j t_{j+1}^{-1}$. Denote $s_j := t_j t_{j+1}^{-1}$ for each $j \le n-1$ and $s_n = t_n$. Then $t_j = s_j s_{j+1} \dots s_n$ and $s_i = s_j$.

**Theorem 4.** *For $\mathfrak{g} = sl(n)$, the Belavin–Drinfeld cohomology $H^1_{BD}(GL(n), r_{BD})$ associated to any $r_{BD}$ is trivial. Any Lie bialgebra structure on $\mathfrak{g}(\mathbb{K})$ is of the form $\delta(a) = [r, a \otimes 1 + 1 \otimes a]$, where $r$ is an $r$-matrix which is $GL(n, \mathbb{K})$–equivalent to a non-skewsymmetric $r$-matrix on the Belavin–Drinfeld list.*

*Proof.* Let $X$ be a cocycle associated to $r_{BD}$ which is a fixed $r$-matrix on the Belavin–Drinfeld list. Thus $X^{-1}\sigma(X)$ belongs to the centralizer of the $r_{BD}$. On the other hand, according to Lemma 3, $C(GL(n), r_{BD}) \subseteq \mathrm{diag}(n, \overline{\mathbb{K}})$.

Then we obtain that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$, $X^{-1}\sigma(X)$ is diagonal. By Lemma 2, we have a decomposition $X = QD$, where $Q \in GL(n, \mathbb{K})$ and $D \in \mathrm{diag}(n, \overline{\mathbb{K}})$. Since $\sigma(Q) = Q$, we have $X^{-1}\sigma(X) = (QD)^{-1}\sigma(QD) = D^{-1}Q^{-1}Q\sigma(D) = D^{-1}\sigma(D)$. Recall that $X^{-1}\sigma(X) \in C(GL(n), r_{BD})$. It follows that $D^{-1}\sigma(D) \in C(GL(n), r_{BD})$.

Let $D = \mathrm{diag}(d_1, \dots, d_n)$. Then $\mathrm{diag}(d_1^{-1}\sigma(d_1), \dots, d_n^{-1}\sigma(d_n)) \in C(GL(n), r_{BD})$. Denote $t_i = d_i^{-1}\sigma(d_i)$ and $T = \mathrm{diag}(t_1, \dots, t_n)$. According to Lemma 4, $T \in C(GL(n), r_{BD})$ if and only if $t_i t_{i+1}^{-1} = t_j t_{j+1}^{-1}$. Equivalently, $\sigma(d_i^{-1} d_{i+1} d_j d_{j+1}^{-1}) = d_i^{-1} d_{i+1} d_j d_{j+1}^{-1}$. It follows that $d_i^{-1} d_{i+1} d_j d_{j+1}^{-1} \in \mathbb{K}$. Let $s_i := d_i d_{i+1}^{-1}$ for any $i$ and $s_n = d_n$. Then we get $s_j s_i^{-1} \in \mathbb{K}$.

Let us fix a root $\alpha_{i_0} \in \Gamma_1 \backslash \Gamma_2$ and let $\tau^j(\alpha_{i_0}) = \alpha_j$. Then $s_j s_{i_0}^{-1} \in \mathbb{K}$, for any $j$. Denote $k_j := s_j s_{i_0}^{-1}$.

On the other hand, $d_j = s_j s_{j+1} \dots s_{n-1} s_n = k_j k_{j+1} \dots k_n s_{i_0}^{n-j+1}$. Let

$$K := \mathrm{diag}(k_1 k_2 \dots k_n, k_2 \dots k_n, \dots, k_n),$$
$$C := \mathrm{diag}(s_{i_0}^n, s_{i_0}^{n-1}, \dots, s_{i_0}).$$

Note that $D = KC$ and $K \in GL(n, \mathbb{K})$. Moreover, according to Lemma 4, $C \in C(GL(n), r_{BD})$.

Summing up, we have obtained that if $X$ is any cocycle associated to $r_{BD}$, then $X = QD = QKC$, with $QK \in GL(n, \mathbb{K})$, $C \in C(GL(n), r_{BD})$. This ends the proof.

## 6. Belavin-Drinfeld Cohomologies for Orthogonal Algebras

The next step in our investigation of Belavin–Drinfeld cohomologies is for orthogonal algebras $o(m)$. We begin with the case of the Drinfeld–Jimbo $r$-matrix. In what follows, we will use the following split form of the orthogonal algebra $o(n, \mathbb{C})$ and $o(n, \mathbb{K})$:

$$o(n) = \{A \in gl(n) : A^T S + SA = 0\},$$

where $S$ is the matrix with 1 on the second diagonal and zero elsewhere. The group

$$SO(n) = \{X \in SL(n) : X^T SX = S\}$$

acts naturally on $o(n)$. It follows from Theorem 1 that $C(SO(n), r_{DJ})$ coincides with the maximal torus of $SO(n)$. Our main result about Belavin-Drinfeld cohomologies for orthogonal algebras is the following:

**Theorem 5.** *Let $\mathfrak{g} = o(m)$ and $r_{DJ}$ be the Drinfeld–Jimbo $r$-matrix. Then $H^1_{BD}(SO(m), r_{DJ})$ is trivial.*

*Proof.* (i) Assume $m = 2n$ and fix the bilinear form

$$B(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} x_i y_{m+1-i}$$

on $\overline{\mathbb{K}}^m$.

Let $X \in SO(m, \overline{\mathbb{K}})$ be a cocycle associated to $r_{DJ}$. Thus $X^{-1}\sigma(X) \in C(SO(m), r_{DJ})$. Recall that $C(SO(m), r_{DJ}) = \mathrm{diag}(m, \overline{\mathbb{K}}) \cap SO(m, \overline{\mathbb{K}})$. Therefore $X^{-1}\sigma(X) \in \mathrm{diag}(m, \overline{\mathbb{K}})$. By Lemma 2, one has the decomposition $X = QD$, where $Q \in GL(m, \mathbb{K})$ and $D \in \mathrm{diag}(m, \overline{\mathbb{K}})$. Let us write $D = \mathrm{diag}(d_1, \dots, d_{2n})$ and denote by $q_i$ the columns of $Q$. Then $X = QD$ is equivalent to $Q^T SQ = D^{-1} SD^{-1}$, which in turn implies that $B(q_i, q_{i'}) d_i d_{i'} = \delta_i^{2n+1-i'}$. We get $B(q_i, q_{i'}) = 0$ if $i + i' \neq 2n + 1$ and $B(q_i, q_{2n+1-i}) d_i d_{2n+1-i} = 1$. Let $k_i := B(q_i, q_{2n+1-i})$. Since $Q \in GL(2n, \mathbb{K})$, we have $k_i \in \mathbb{K}$. Because $k_i^{-1} = d_i d_{2n+1-i}$, it follows that $D = Q_1 D_1$, where

$$Q_1 = \mathrm{diag}(k_1^{-1}, \dots, k_n^{-1}, 1 \dots, 1),$$
$$D_1 = \mathrm{diag}(d_1 k_1, \dots, d_n k_n, d_{n+1}, \dots, d_{2n}).$$

We note that $X = (QQ_1)D_1$, $D_1 \in SO(2n)$ and hence, $D_1 \in C(SO(2n), r_{DJ})$. Then, clearly, we have $QQ_1 \in SO(2n, \mathbb{K})$, which proves that $X$ is equivalent to the identity.

(ii) Now consider $m = 2n + 1$. By Lemma 2, we may write again $X = QD$, where $Q \in GL(m, \mathbb{K})$ and $D \in \text{diag}(m, \overline{\mathbb{K}})$.

Let $k_i := B(q_i, q_{2n+2-i}) \in \mathbb{K}$. Repeating the computations as in (i), we obtain $k_i^{-1} = d_i d_{2n+2-i}$. If $i = n+1$, $d_{n+1}^2 = k_{n+1}^{-1} \in \mathbb{K}$. This implies that either $d_{n+1} \in \mathbb{K}$ or $d_{n+1} \in j\mathbb{K}$, where $j^2 = \hbar$.

Actually we can prove that the second case is impossible.

Let us denote $R = Q^{-1}$ and its rows by $r_1, \ldots, r_{2n+1}$. Then the relation $X^T S X = S$ is equivalent to $RSR^T = DSD$, which in turn gives the following: $B(r_i, r_{i'}) = 0$, for all $i \neq i'$, $B(r_i, r_i) = d_i d_{2n+2-i}$ for all $i$.

Let us take an arbitrary orthogonal basis $v_1, \ldots, v_{2n+1}$ in $\mathbb{K}^{2n+1}$ and denote $B(v_i, v_i) = A_i$.

The matrix $V$ with rows $v_i$ satisfies $VSV^T = \text{diag}(A_1, \ldots, A_{2n+1})$. This relation implies that $A_1 \ldots A_{2n+1} = (-1)^n \det(V)^2 = ((\sqrt{-1})^n \det(V))^2$. Therefore $A_1 \ldots A_{2n+1} = l^2$ is a square of some $l \in \mathbb{K}$.

On the other hand, if $M$ is the change of basis matrix from $r_i$ to $v_i$, then

$$M^T \text{diag}(A_1, \ldots, A_{2n+1}) M = \text{diag}(d_1 d_{2n+1}, \ldots, d_{n+1}^2, \ldots, d_{2n+1} d_1).$$

By taking the determinant on both sides, we obtain

$$\det(M)^2 A_1 \ldots A_{2n+1} = (d_1 d_{2n+1})^2 \ldots (d_n d_{n+2})^2 d_{n+1}^2$$

which implies that $d_{n+1}^2$ is a square in $\mathbb{K}$, and consequently, $d_{n+1} \in \mathbb{K}$.

Let us show that $X$ is equivalent to the trivial cocycle. Consider

$$Q_1 = \text{diag}(k_1^{-1}, \ldots, k_n^{-1}, d_{n+1}, 1, \ldots, 1),$$
$$D_1 = \text{diag}(d_1 k_1, \ldots, d_n k_n, 1, d_{n+2}, \ldots, d_{2n+1}).$$

We have $D = Q_1 D_1$ and $D_1 \in SO(2n+1, \overline{\mathbb{K}})$. Thus $X = (QQ_1)D_1$, $QQ_1 \in SO(2n+1, \mathbb{K})$, $D_1 \in C(SO(2n+1), r_{DJ})$, i.e., $X$ is equivalent to the trivial cocycle, which completes the proof of triviality of $H^1_{BD}(SO(m), r_{DJ})$.

Regarding Belavin–Drinfeld cohomology $H^1_{BD}(SO(2n), r_{BD})$ for an arbitrary $r_{BD}$, we can give an example where this set is non-trivial. Let us denote the simple roots of $o(2n)$ by $\alpha_i = \epsilon_i - \epsilon_{i+1}$, for $i < n$, $\alpha_n = \epsilon_{n-1} + \epsilon_n$, where $\{\epsilon_i\}$ is an orthonormal basis of $\mathfrak{h}^*$. Let $\Gamma_1 = \{\alpha_{n-1}\}$, $\Gamma_2 = \{\alpha_n\}$ and $\tau(\alpha_{n-1}) = \alpha_n$. Denote by $r_{BD}$ the $r$-matrix corresponding to the triple $(\Gamma_1, \Gamma_2, \tau)$ and $s$, where $s \in \mathfrak{h} \wedge \mathfrak{h}$ satisfies $((\alpha_{n-1} - \alpha_n)) \otimes 1)(2s) = ((\alpha_{n-1} + \alpha_n)) \otimes 1)\Omega_0$.

**Lemma 5.** *The centralizer $C(SO(2n), r_{BD})$ consists of all diagonal matrices of the form*

$$T = \text{diag}(t_1, \ldots, t_{n-1}, \pm 1, \pm 1, t_{n-1}^{-1}, \ldots, t_1^{-1}),$$

*for arbitrary nonzero $t_1, \ldots, t_{n-1} \in \overline{\mathbb{K}}$.*

*Proof.* We already have the inclusion $C(SO(2n), r_{BD}) \subseteq \text{diag}(2n, \overline{\mathbb{K}}) \cap O(2n, \overline{\mathbb{K}})$. Let $T \in C(SO(2n), r_{BD})$, where $T = \text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1})$. Since $T$ commutes with $r_0$ and $r_{DJ}$, $T \in C(SO(2n), r_{BD})$ if and only if $(\text{Ad}_T \otimes \text{Ad}_T)(e_{\alpha_n} \wedge e_{\alpha_{n-1}}) = e_{\alpha_n} \wedge e_{\alpha_{n-1}}$. One can check that $(\text{Ad}_T \otimes \text{Ad}_T)(e_{\alpha_n} \wedge e_{\alpha_{n-1}}) = t_n^{-2} e_{\alpha_n} \wedge e_{\alpha_{n-1}}$. Therefore we get $t_n^{-2} = 1$ and the conclusion follows.

**Proposition 7.** *Let $\mathfrak{g} = o(2n)$, and $r_{BD}$ be the $r$-matrix corresponding to the triple $(\Gamma_1, \Gamma_2, \tau)$ and some $s \in \mathfrak{h} \wedge \mathfrak{h}$, where $\Gamma_1 = \{\alpha_{n-1}\}$, $\Gamma_2 = \{\alpha_n\}$ and $\tau(\alpha_{n-1}) = \alpha_n$, and $((\alpha_{n-1} - \alpha_n)) \otimes 1)(2s) = ((\alpha_{n-1} + \alpha_n)) \otimes 1)\Omega_0$. Then $H^1_{BD}(SO(2n), r_{BD})$ is non-trivial.*

*Proof.* Assume that $X^{-1}\sigma(X) \in C(SO(2n), r_{BD})$ for all $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$. By the above lemma, $X^{-1}\sigma(X) = \text{diag}(t_1, \ldots, t_{n-1}, \pm 1, \pm 1, t_{n-1}^{-1}, \ldots, t_1^{-1})$.

On the other hand, since $X^{-1}\sigma(X)$ is diagonal, it follows from Theorem 5 that there exist $Q \in SO(2n, \mathbb{K})$ and a diagonal matrix $D \in SO(2n, \overline{\mathbb{K}})$ such that $X = QD$. Let us write $D = \text{diag}(s_1, \ldots, s_n, s_n^{-1}, \ldots, s_1^{-1})$. Since $Q \in O(2n, \mathbb{K})$, for any $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$, $\sigma(Q) = Q$. We obtain $X^{-1}\sigma(X) = D^{-1}Q^{-1}Q\sigma(D) = D^{-1}\sigma(D)$, which is equivalent to the following: $s_i^{-1}\sigma(s_i) = t_i$ for all $i \leq n-1$, and $s_n^{-1}\sigma(s_n) = \pm 1$.

Assume first that there exists $\sigma$ such that $\sigma(s_n) = -s_n$. Then $s_n \in j\mathbb{K}$. One can check that $X$ is equivalent to $X_0 = \text{diag}(1, \ldots, 1, j, j^{-1}, 1, \ldots, 1)$, which is a non-trivial cocycle.

If $\sigma(s_n) = s_n$ for all $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$, then $s_n \in \mathbb{K}$. In this case,

$$D = \text{diag}(s_1, \ldots, s_{n-1}, 1, 1, s_{n-1}^{-1}, \ldots, s_1^{-1}) \cdot \text{diag}(1, \ldots, 1, s_n, s_n^{-1}, 1, \ldots, 1),$$

where the first matrix is in $C(SO(2n), r_{BD})$ and the second in $SO(2n, \mathbb{K})$. This proves that $X$ is equivalent to the identity cocycle.

## 7. Lie Bialgebra Structures in Case III and Twisted Belavin-Drinfeld Cohomologies

Throughout this section we restrict our discussion to $\mathfrak{g} = sl(n)$ and consider $GL(n)$ as the gauge group. Here we analyse Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ for which the corresponding Drinfeld double is isomorphic to $\mathfrak{g}(\mathbb{K}[j])$, where $j^2 = \hbar$. Our aim is to find all subalgebras $W$ of $\mathfrak{g}(\mathbb{K}[j])$ satisfying the following conditions:

(i) $W \oplus \mathfrak{g}(\mathbb{K}) = \mathfrak{g}(\mathbb{K}[j])$.
(ii) $W = W^{\perp}$ with respect to the non-degenerate symmetric bilinear form $Q$ given by

$$Q(f_1 + jf_2, g_1 + jg_2) = K(f_1, g_2) + K(f_2, g_1).$$

We begin with the following remark. The field $\mathbb{K}[j]$ is endowed with a conjugation. For any element $a = f_1 + jf_2$, its conjugate is $\overline{a} := f_1 - jf_2$. By the norm of an element $a \in \mathbb{K}[j]$ we will understand the element $a\overline{a} \in \mathbb{K}$.

If $A = A_1 + jB_1$ and $B = A_2 + jB_2$ are two matrices in $sl(n, \mathbb{K}[j])$, then $Q(A, B) = \text{Tr}(A_1B_2 + B_1A_2)$, i. e., the coefficient of $j$ in $\text{Tr}(AB)$.

**Lemma 6.** *Let $L$ be the subalgebra of $sl(n, \mathbb{K}[j])$ which consists of all matrices $Z = (z_{ij})$ satisfying $z_{ij} = \overline{z}_{n+1-i,n+1-j}$. Then $L$ and $sl(n, \mathbb{K})$ are isomorphic via conjugation in $sl(n, \mathbb{K}[j])$.*

*Proof.* Assume that $Z = (z_{ij})$ satisfies $z_{ij} = \overline{z}_{n+1-i,n+1-j}$. Then $Z = S\overline{Z}S$, where $S$ is the matrix with 1 on the second diagonal and zero elsewhere.

Choose a matrix $X \in GL(n, \mathbb{K}[j])$ such that $\overline{X} = XS$. Then $\overline{XZX^{-1}} = XS\overline{Z}SX^{-1} = XZX^{-1}$, which implies $XZX^{-1} \in sl(n, \mathbb{K})$. Conversely, if $A \in sl(n, \mathbb{K})$, then $Z = X^{-1}AX$ satisfies the condition $Z = S\overline{Z}S$.

From now on we will identify $sl(n, \mathbb{K})$ with $L$. Let us find a complementary subalgebra to $L$ in $sl(n, \mathbb{K}[j])$. Let us denote by $H$ the Cartan subalgebra of $L$. If we identify the Cartan subalgebra of $sl(n, \mathbb{K}[j])$ with $\mathbb{K}^{2(n-1)}$, then $H$ is a Lagrangian subspace of $\mathbb{K}^{2(n-1)}$. Choose a Lagrangian subspace $H_0$ of $\mathbb{K}^{2(n-1)}$ such that $H_0$ has trivial intersection with $H$. Let $N^+$ be the algebra of upper triangular matrices of $sl(n, \mathbb{K}[j])$ with zero diagonal. Consider $W_0 = H_0 \oplus N^+$. We immediately obtain the following

**Lemma 7.** *The subalgebra $W_0$ as above satisfies conditions (i) and (ii), where $sl(n, \mathbb{K})$ is identified with $L$ as in Lemma 6.*

**Proposition 8.** *Any Lie bialgebra structure on $sl(n, \mathbb{K})$ for which the classical double is isomorphic to $sl(n, \mathbb{K}[j])$ is given by an $r$-matrix which satisfies $CYB(r) = 0$ and $r + r^{21} = j\Omega$.*

*Proof.* Let $W_0$ be as in the above lemma. By choosing two dual bases in $W_0$ and $sl(n, \mathbb{K})$ respectively, one can construct the corresponding $r$-matrix $r_0$ over $\overline{\mathbb{K}}$. It is easily seen that $r_0$ satisfies the system $CYB(r_0) = 0$ and $r_0 + r_0^{21} = j\Omega$.

Let us suppose that $W$ is another subalgebra of $sl(n, \mathbb{K}[j])$ satisfying conditions (i) and (ii). Then the corresponding $r$-matrix over $\overline{\mathbb{K}}$ is obtained by choosing dual bases in $W$ and $sl(n, \mathbb{K})$ respectively. We have $r + r^{21} = a\Omega$ for some $a \in \mathbb{K}[j]$. On the other hand, the classical double of the Lie bialgebras corresponding to $r$ and $r_0$ is the same. This implies that $r$ and $r_0$ are classical twists of each other and therefore $a = j$.

On the other hand, over $\overline{\mathbb{K}}$, all $r$-matrices are gauge equivalent to the ones on the Belavin–Drinfeld list. It follows that there exists a non-skewsymmetric $r$-matrix $r_{BD}$ and $X \in GL(n, \overline{\mathbb{K}})$ such that $r = j(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$.

Denote by $\sigma_0$ an arbitrary lift of the conjugation on $\mathbb{K}[j]$ to $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$. We recall, see [9], that $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ is generated by $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}[j])$ and $\sigma_0$.

Consider an arbitrary $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$. Since $\delta$ is a cobracket on $sl(n, \mathbb{K})$, $(\sigma \otimes \sigma)(\delta(a)) = \delta(a)$ and $(\sigma \otimes \sigma)(\delta(a)) = [\sigma(r), a \otimes 1 + 1 \otimes a]$.

Let us assume that $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}[j])$. Exactly as in Sect. 4, it follows that $\sigma(r) = r$ and if $r = (\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(jr_{BD})$ with $X \in GL(n, \overline{\mathbb{K}})$, then $\sigma(X) = XD(\sigma)$.

By the same arguments as in the proof of Lemma 2, the following result is established.

**Lemma 8.** *Let $X \in GL(n, \overline{\mathbb{K}})$. Assume that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}[j])$, $X^{-1}\sigma(X) \in \mathrm{diag}(n, \overline{\mathbb{K}})$. Then there exists $P \in GL(n, \mathbb{K}[j])$ and $D \in \mathrm{diag}(n, \overline{\mathbb{K}})$ such that $X = PD$.*

Now let us consider the action of $\sigma_0 \in \mathrm{Gal}(\mathbb{K}[j]/\mathbb{K})$. Our identities imply that $\sigma_0(r) = r + \alpha\Omega$, for some $\alpha \in \overline{\mathbb{K}}$. Let us show that $\alpha = -j$. Indeed, since $r + r^{21} = j\Omega$, we also have $\sigma_0(r) + \sigma_0(r^{21}) = -j\Omega$. Combining these relations with $\sigma_0(r) = r + \alpha\Omega$, we get $\alpha = -j$ and therefore $\sigma_0(r) = r - j\Omega = -r^{21}$.

Recall now that $r = j(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$. It follows that $X \in GL(n, \overline{\mathbb{K}})$ must satisfy the identity $(\mathrm{Ad}_{X^{-1}\sigma_0(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma_0(X)})(\sigma_0(r_{BD})) = r_{BD}^{21}$. Using the same arguments as in the proof of Theorem 3 in Sect. 4, we obtain

**Proposition 9.** *Any Lie bialgebra structure on $sl(n,\mathbb{K})$ for which the classical double is $sl(n,\mathbb{K}[j])$ is given by an $r$-matrix $r = j(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$, where $r_{BD}$ is a non-skewsymmetric $r$-matrix on the Belavin–Drinfeld list and $X \in GL(n,\overline{\mathbb{K}})$ satisfies*

$$(\mathrm{Ad}_{X^{-1}\sigma_0(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma_0(X)})(r_{BD}) = r_{BD}^{21}$$

*and, for $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}[j])$,*

$$(\mathrm{Ad}_{X^{-1}\sigma(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma(X)})(r_{BD}) = r_{BD}.$$

From now on we assume that $r_{BD}$ is defined over $\mathbb{K}$ (i.e. its Cartan part $r_0$ is defined over $\mathbb{K}$).

**Definition 5.** Let $r_{BD}$ be a non-skewsymmetric $r$-matrix on the Belavin–Drinfeld list. We call $X \in G(\overline{\mathbb{K}})$ a *Belavin–Drinfeld twisted cocycle* associated to $r_{BD}$ if $(\mathrm{Ad}_{X^{-1}\sigma_0(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma_0(X)})(r_{BD}) = r_{BD}^{21}$ and for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}[j])$, $(\mathrm{Ad}_{X^{-1}\sigma(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma(X)})(r_{BD}) = r_{BD}$.

The set of Belavin–Drinfeld twisted cocycles associated to $r_{BD}$ will be denoted by $\overline{Z}(G, r_{BD})$.

Now let us restrict ourselves to the case $r_{BD} = r_{DJ}$. In order to continue our investigation, let us prove the following

**Lemma 9.** *Let $S$ be the matrix with 1 on the second diagonal and zero elsewhere. Then*

$$r_{DJ}^{21} = (\mathrm{Ad}_S \otimes \mathrm{Ad}_S) r_{DJ}.$$

*Proof.* We recall that $r_{DJ}$ is given by the following formula:

$$r_{DJ} = \sum_{\alpha>0} e_\alpha \otimes e_{-\alpha} + \frac{1}{2}\Omega_0$$

where $\Omega_0$ is the Cartan part of $\Omega$.

First note that $(\mathrm{Ad}_S \otimes \mathrm{Ad}_S)(e_{ij} \otimes e_{ji}) = e_{n+1-i,n+1-j} \otimes e_{n+1-j,n+1-i}$, which is a term in $r_{DJ}^{21}$, if $i > j$ (here $e_{ij}$ is a matrix with 1 on the $(i, j)$ position and zero elsewhere). On the other hand, since $\Omega_0$ is the Cartan part of the invariant element $\Omega$, we get $(\mathrm{Ad}_S \otimes \mathrm{Ad}_S)\Omega_0 = \Omega_0$. This could also be proved by using the following: $\Omega_0 = n\sum_{i=1}^n e_{ii} \otimes e_{ii} - I \otimes I$, where $I$ denotes the identity matrix of $GL(n,\mathbb{K})$. Then the identity $r_{DJ}^{21} = (\mathrm{Ad}_S \otimes \mathrm{Ad}_S) r_{DJ}$ holds.

**Definition 6.** Denote $m = n/2$ if $n$ is even, and $m = (n+1)/2$ if $n$ is odd. By $J$ we denote the matrix with elements $a_{kk} = 1$ for $k \le m$, $a_{kk} = -j$ for $k \ge m+1$, $a_{k,n-k+1} = 1$ for $k \le m$, $a_{k,n-k+1} = j$ for $k \ge m+1$, and other elements vanish.

**Lemma 10.** *$\overline{Z}(GL(n), r_{DJ})$ is non-empty.*

*Proof.* Indeed, $\sigma_0(J) = JS$, $J \in GL(n,\mathbb{K}[j])$.

**Corollary 2.** *Let $X$ be a Belavin–Drinfeld twisted cocycle associated to $r_{DJ}$. Then $X = PD$, where $P \in GL(n,\mathbb{K}[j])$ and $D \in \mathrm{diag}(n,\overline{\mathbb{K}})$. Moreover, $\sigma_0(P) = PSD_1$, where $D_1 \in \mathrm{diag}(n,\mathbb{K}[j])$.*

*Proof.* Since $X$ is a twisted cocycle, for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}[j])$, $X^{-1}\sigma(X) \in C(GL(n), r_{DJ})$. Recall that $C(GL(n), r_{DJ}) = \mathrm{diag}(n, \overline{\mathbb{K}})$. By Lemma 8, we have $X = PD$, where $P \in GL(n, \mathbb{K}[j])$ and $D \in \mathrm{diag}(n, \overline{\mathbb{K}})$. Lemma 9 implies that $S^{-1}X^{-1}\sigma_0(X) =: D_2 \in \mathrm{diag}(n, \overline{\mathbb{K}})$. Since $X = PD$, $S^{-1}D^{-1}P^{-1}\sigma_0(P)\sigma_0(D) = D_2$. Hence $P^{-1}\sigma_0(P) = DSD_0\sigma_0(D^{-1})$.

Let $D_1 := S^{-1}DSD_2\sigma_0(D^{-1}) \in \mathrm{diag}(n, \overline{\mathbb{K}})$. Then $\sigma_0(P) = PSD_1$ and $D_1 \in \mathrm{diag}(n, \mathbb{K}[j])$.

**Definition 7.** Let $X_1$ and $X_2$ be two Belavin–Drinfeld twisted cocycles associated to $r_{BD}$. We say that they are *equivalent* if there exist $Q \in GL(n, \mathbb{K})$ and $D \in C(GL(n), r_{BD})$ such that $X_1 = QX_2D$.

*Remark 5.* Assume that $X$ is a twisted cocycle associated to $r_{DJ}$. By Corollary 2, $X = PD$ and is equivalent to the twisted cocycle $P \in GL(n, \mathbb{K}[j])$.

**Definition 8.** Let $\overline{H}^1_{BD}(GL(n), r_{BD})$ denote the set of equivalence classes of twisted cocycles associated to $r_{BD}$. We call this set the *Belavin–Drinfeld twisted cohomology* associated to the $r$-matrix $r_{BD}$.

*Remark 6.* If $X_1$ and $X_2$ are equivalent, then the corresponding $r$-matrices $r_1 = j(\mathrm{Ad}_{X_1} \otimes \mathrm{Ad}_{X_1})(r_{DJ})$ and $r_2 = j(\mathrm{Ad}_{X_2} \otimes \mathrm{Ad}_{X_2})(r_{DJ})$ are gauge equivalent via $Q \in GL(n, \mathbb{K})$.

**Proposition 10.** *There is a one-to-one correspondence between $\overline{H}^1_{BD}(GL(n), r_{BD})$ and gauge equivalence classes of Lie bialgebra structures on $sl(n, \mathbb{K})$ with classical double $sl(n, \mathbb{K}[j])$ and $\overline{\mathbb{K}}$-isomorphic to $dr_{BD}$.*

**Proposition 11.** *For $\mathfrak{g} = sl(n)$, the Belavin–Drinfeld twisted cohomology $\overline{H}^1_{BD}(GL(n), r_{DJ})$ is non-empty and consists of one element.*

*Proof.* Let $X$ be a twisted cocycle associated to $r_{DJ}$. By Remark 5, $X$ is equivalent to a twisted cocycle $P \in GL(n, \mathbb{K}[j])$, associated to $r_{DJ}$. We may therefore assume from the beginning that $X \in GL(n, \mathbb{K}[j])$ and it remains to prove that all such cocycles are equivalent.

We will prove that $X$ and $J$ are equivalent, i.e., $X = QJD'$, for some $Q \in GL(n, \mathbb{K})$ and $D' \in \mathrm{diag}(n, \mathbb{K}[j])$. The proof will be done by induction.

For $n = 2$, we have $J = \begin{pmatrix} 1 & 1 \\ j & -j \end{pmatrix}$ and let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{K}[j])$ satisfy $\overline{X} = XSD$ with $D = \mathrm{diag}(d_1, d_2) \in GL(2, \mathbb{K}[j])$. This equation is equivalent to the system $\overline{a} = bd_1$, $\overline{b} = ad_2$, $\overline{c} = dd_1$, $\overline{d} = cd_2$. Assume that $cd \neq 0$. Let $a/c = a' + b'j$. Then $b/d = a' - b'j$. One can immediately check that $X = QJD'$, where $Q = \begin{pmatrix} a' & b' \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{K})$, $D' = \mathrm{diag}(c, d) \in \mathrm{diag}(2, \mathbb{K}[j])$.

For $n = 3$, consider $J = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ j & 0 & -j \end{pmatrix}$ and let $X = (a_{ij}) \in GL(3, \mathbb{K}[j])$ satisfy $\overline{X} = XSD$, with $D = \mathrm{diag}(d_1, d_2, d_3) \in GL(3, \mathbb{K}[j])$. This equation is equivalent to the system $\overline{a_{11}} = d_1a_{13}$, $\overline{a_{21}} = d_1a_{23}$, $\overline{a_{31}} = d_1a_{33}$, $\overline{a_{12}} = d_2a_{12}$, $\overline{a_{22}} = d_2a_{22}$, $\overline{a_{32}} = d_2a_{32}$, $\overline{a_{13}} = d_3a_{11}$, $\overline{a_{23}} = d_3a_{21}$, $\overline{a_{33}} = d_3a_{31}$. Assume that $a_{21}a_{22}a_{23} \neq 0$.

Let $a_{11}/a_{21} = b_{11} + b_{13}j$ and $a_{31}/a_{21} = b_{31} + b_{33}j$. Then $a_{13}/a_{23} = b_{11} - b_{13}j$ and $a_{33}/a_{23} = b_{31} - b_{33}j$. On the other hand, let $b_{12} := a_{12}/a_{22}$ and $b_{32} := a_{32}/a_{22}$.

Note that $b_{12} \in \mathbb{K}$, $b_{32} \in \mathbb{K}$. One can immediately check that $X = QJD'$, where $Q = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 1 & 1 & 0 \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \in GL(3, \mathbb{K})$, $D' = \mathrm{diag}(a_{21}, a_{22}, a_{23}) \in \mathrm{diag}(3, \mathbb{K}[j])$.

For $n > 3$, we proceed by induction. Let us denote $J \in GL(n, \mathbb{K}[j])$, which was defined above, by $J_n$. We are going to prove that if $X \in GL(n, \mathbb{K}[j])$ satisfies $\overline{X} = XSD$, then using elementary row operations with entries in $\mathbb{K}$ and multiplying columns by proper elements in $\mathbb{K}[j]$ we can transform $X$ to $J_n$.

We will need the following operations on a matrix

$$M = (m_{pq}) \in \mathrm{Mat}(n):$$

1. $u_n(M) = (m_{pq}) \in \mathrm{Mat}(n-2)$, $p, q = 2, 3, \dots, n-1$;
2. $g_n(M) = (m_{pq}) \in \mathrm{Mat}(n+2)$, where $m_{pq}$ are already defined for $p, q = 1, 2, \dots n$, $m_{00} = m_{n+1,n+1} = 1$ and the rest $m_{0,a} = m_{a,0} = m_{n+1,a} = m_{a,n+1} = 0$.

It is clear that $u_n(X)$ satisfies the twisted cocycle condition. However, its determinant might vanish. To avoid this complication, we note that columns $2, 3, \dots, n-1$ of $X$ are linearly independent. Applying elementary row operations (in fact, they are permutations) we obtain a new cocycle $X_1$, which is equivalent to $X$ and such that $u_n(X_1)$ is a cocycle in $GL(n-2, \mathbb{K}[j])$. Then, by induction, there exist $Q_{n-2} \in GL(n-2, \mathbb{K})$ and a diagonal matrix $D_{n-2}$ such that

$$Q_{n-2} \cdot u_n(X_1) \cdot D_{n-2} = J_{n-2}$$

Let us consider $X_n = g_{n-2}(Q_{n-2}) \cdot X_1 \cdot g_{n-2}(D_{n-2})$. Clearly, $X_n$ is a twisted cocycle equivalent to $X$ and $u_n(X_n) = J_{n-2}$.

Applying elementary row operations with entries in $\mathbb{K}$ and multiplying by a proper diagonal matrix, we can obtain a new cocycle $Y_n = (y_{pg})$ equivalent to $X$ with the following properties:

1. $u_n(Y_n) = J_{n-2}$;
2. $y_{12} = y_{13} = \cdots = y_{1,n-1} = 0$ and $y_{n2} = y_{n3} = \cdots = y_{n,n-1} = 0$;
3. $y_{11} = y_{1n} = 1$, here we use the fact that if $y_{pq} = 0$, then $y_{p,n+1-q} = 0$.

It follows from the cocycle condition $\overline{Y_n} = Y_n \cdot S \cdot \mathrm{diag}(h_1, \dots, h_n)$ that $h_1 = h_n = 1$ and hence, $y_{n1} = \overline{y_{nn}}$.

Now, we can use the first row to achieve $y_{n1} = -y_{nn} = j$ and after that, we use the first and the last rows to get $y_{k1} = 0$, $k = 2, \dots, n-1$. Then the elements $y_{kn}$, $k = 2, \dots, n-1$ will vanish automatically. Thus, $X$ is equivalent to $J_n$.

*Example 1.* For $\mathfrak{g} = sl(2)$, the Belavin–Drinfeld list of non-skewsymmetric constant $r$-matrices consists of only one class, $r_{DJ} = e \otimes f + \frac{1}{4} h \otimes h$, where $e = e_{12}$, $f = e_{21}$ and $h = e_{11} - e_{22}$. We can easily determine the corresponding class of gauge equivalent Lie bialgebra structures on $sl(2, \mathbb{K})$ with classical double $sl(2, \mathbb{K}[j])$ and $\overline{\mathbb{K}}$-isomorphic to $dr_{DJ}$. Indeed, we have seen that the corresponding Lie bialgebra structure equals $\delta = dr$, where the $r$-matrix is $r = j(\mathrm{Ad}_X \otimes \mathrm{Ad}_X) r_{DJ}$ and $X$ is a twisted cocycle. On the other hand, according to the above result, any such $X$ is equivalent to

$$J = \begin{pmatrix} 1 & 1 \\ j & -j \end{pmatrix}.$$

Therefore a class representative is $\delta_0 = dr_0$, where $r_0 = j(\mathrm{Ad}_J \otimes \mathrm{Ad}_J)r_{DJ}$. A straightforward computation gives

$$r_0 = \frac{j\Omega}{2} + \frac{1}{4}h \wedge e + \frac{\hbar}{4}f \wedge h.$$

We conclude that any Lie bialgebra structure on $sl(2, \mathbb{K})$ with classical double $sl(2, \mathbb{K}[j])$ is gauge equivalent to the one given by $a \cdot dr_0,\ a \in \mathbb{K}$.

*Remark 7.* In the case $\mathfrak{g} = sl(2)$, it follows that the Drinfeld–Jimbo $r$-matrix multiplied by $a \in \mathbb{K}$ along with $ar_0$, $r_0 = \frac{j\Omega}{2} + \frac{1}{4}h \wedge e + \frac{\hbar}{4}f \wedge h$, provides all $GL(n)$ non-equivalent Lie bialgebra structures on $sl(2, \mathbb{K})$ of types II and III and, consequently, two families of non-isomorphic Hopf algebra structures on $U(sl(2, \mathbb{C}))[[\hbar]]$. Moreover, in some sense these two structures exhaust all Hopf algebra structures on $U(sl(2, \mathbb{C}))[[\hbar]]$ with a non-trivial Drinfeld associator (see also conjectures below).

*Remark 8.* The next step would be to compute the Belavin–Drinfeld twisted cohomology corresponding to an arbitrary $r$-matrix $r_{BD}$. Unlike untwisted cohomology, it might happen that even $\overline{Z}(G, r_{BD})$ is empty as we will see in the next section.

## 8. Twisted Cohomologies for $sl(n)$ of Cremmer-Gervais Type

In this section the gauge group $G$ is always $GL(n)$. We have seen that $\overline{H}^1_{BD}(GL(n), r_{DJ})$, where $r_{DJ}$ is the Drinfeld–Jimbo $r$-matrix, consists of one element. We will now turn our attention to other non-skewsymmetric $r$-matrices and analyse the corresponding twisted cohomology set. Let us consider an arbitrary admissible triple $(\Gamma_1, \Gamma_2, \tau)$, and a tensor $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ satisfying $r_0 + r_0^{21} = \Omega_0$ and $(\tau(\alpha) \otimes 1 + 1 \otimes \alpha)(r_0) = 0$ for any $\alpha \in \Gamma_1$. We recall that the associated $r$-matrix is given by the following formula

$$r = r_0 + \sum_{\alpha>0} e_\alpha \otimes e_{-\alpha} + \sum_{\alpha \in (\mathrm{Span}\Gamma_1)^+} \sum_{k \in \mathbb{N}} e_\alpha \wedge e_{-\tau^k(\alpha)}.$$

Assume now that there exists $X \in \overline{Z}(GL(n), r)$. Then $r$ and $r^{21}$ are gauge equivalent since $(\mathrm{Ad}_{X^{-1}\sigma_0(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma_0(X)})(r) = r^{21}$.

Let $S \in GL(n, \mathbb{K})$ be the matrix with 1 on the second diagonal and 0 elsewhere. Let us denote by $s$ the automorphism of the Dynkin diagram given by $s(\alpha_i) = \alpha_{n-i}$ for all $i = 1, \dots, n-1$. Clearly, $\mathrm{Ad}_S(e_\alpha) = e_{-s(\alpha)}$ and $\mathrm{Ad}_S(e_{-\tau^k(\alpha)}) = e_{s\tau^k(\alpha)}$. Thus

$$(\mathrm{Ad}_S \otimes \mathrm{Ad}_S)(r) = (\mathrm{Ad}_S \otimes \mathrm{Ad}_S)(r_0) + \sum_{\alpha>0} e_{-s(\alpha)} \otimes e_{s(\alpha)}$$
$$+ \sum_{\alpha \in (\mathrm{Span}\Gamma_1)^+} \sum_{k \in \mathbb{N}} e_{-s(\alpha)} \wedge e_{s\tau^k(\alpha)}.$$

On the other hand, since $r$ and $r^{21}$ are gauge equivalent, $(\mathrm{Ad}_S \otimes \mathrm{Ad}_S)(r)$ and $r^{21}$ must be gauge equivalent as well. The following condition has to be fulfilled for all $k$: $s(\alpha) = \tau^k(\beta)$ if $\beta = s\tau^k(\alpha)$. We get $s\tau = \tau^{-1}s$, $s(\Gamma_1) = \Gamma_2$ (and $s(\Gamma_2) = \Gamma_1$). In conclusion we have obtained

**Proposition 12.** *Let $r$ be a non-skewsymmetric $r$-matrix associated to an admissible triple $(\Gamma_1, \Gamma_2, \tau)$. If $\overline{Z}(GL(n), r)$ is non-empty, then $s(\Gamma_1) = \Gamma_2$ and $s\tau = \tau^{-1}s$.*

The following two results will prove to be quite useful for the investigation of the twisted cohomologies for arbitrary non-skewsymmetric $r$-matrices.

**Lemma 11.** *Assume* $X \in \overline{Z}(GL(n), r)$. *Then there exists a twisted cocycle* $Y \in GL(n, \mathbb{K}[j])$, *associated to* $r$, *and equivalent to* $X$.

*Proof.* We have $X \in GL(n, \overline{\mathbb{K}})$ and for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}[j])$, $X^{-1}\sigma(X) \in C(GL(n), r)$. On the other hand, the Belavin–Drinfeld cohomology for $sl(n)$ associated to $r$ is trivial. This implies that $X$ is equivalent to the identity, where in the equivalence relation we consider $\mathbb{K}[j]$ instead of $\mathbb{K}$. So there exists $Y \in GL(n, \mathbb{K}[j])$ and $C \in C(GL(n), r)$ such that $X = YC$. Since $(\mathrm{Ad}_{X^{-1}\sigma_0(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma_0(X)})(r) = r^{21}$, $(\mathrm{Ad}_{Y^{-1}\sigma_0(Y)} \otimes \mathrm{Ad}_{Y^{-1}\sigma_0(Y)})(r) = r^{21}$. Thus $Y$ is also a twisted cocycle associated to $r$.

Recall that $J \in GL(n, \mathbb{K}[j])$ denotes the matrix with entries $a_{kk} = 1$ for $k \leq m$, $a_{kk} = -j$ for $k \geq m+1$, $a_{k,n+1-k} = 1$ for $k \leq m$, $a_{k,n+1-k} = j$ for $k \geq m+1$, where $m = [\frac{n+1}{2}]$; other entries vanish.

**Lemma 12.** *Let* $r$ *be a non-skewsymmetric* $r$*-matrix associated to an admissible triple* $(\Gamma_1, \Gamma_2, \tau)$ *satisfying* $s(\Gamma_1) = \Gamma_2$ *and* $s\tau = \tau^{-1}s$. *If* $X \in \overline{Z}(GL(n), r)$, *then there exist* $R \in GL(n, \mathbb{K})$ *and* $D \in \mathrm{diag}(n, \overline{\mathbb{K}})$ *such that* $X = RJD$.

*Proof.* According to Lemma 11, $X = YC$, where $Y \in GL(n, \mathbb{K}[j])$ and $C \in C(GL(n), r)$. Since $(\mathrm{Ad}_{Y^{-1}\sigma_0(Y)} \otimes \mathrm{Ad}_{Y^{-1}\sigma_0(Y)})(r) = r^{21}$ and $(\mathrm{Ad}_S \otimes \mathrm{Ad}_S)(r) = r^{21}$, it follows that $S^{-1}Y^{-1}\sigma_0(Y) \in C(GL(n), r)$. On the other hand, by Lemma 3, $C(GL(n), r) \subset \mathrm{diag}(n, \overline{\mathbb{K}})$. We get $S^{-1}Y^{-1}\sigma_0(Y) \in \mathrm{diag}(n, \overline{\mathbb{K}})$. Now Proposition 11 implies that $Y = RJD_0$, where $R \in GL(n, \mathbb{K})$ and $D_0 \in \mathrm{diag}(n, \overline{\mathbb{K}})$. Consequently, $X = RJD_0C = RJD$ with $D = D_0C \in \mathrm{diag}(n, \overline{\mathbb{K}})$.

We will now look for admissible triples which satisfy condition $s\tau = \tau^{-1}s$. Let us consider the Cremmer–Gervais triple: $\Gamma_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-2}\}$, $\Gamma_2 = \{\alpha_2, \alpha_3, \ldots, \alpha_{n-1}\}$ and $\tau(\alpha_i) = \alpha_{i+1}$. Clearly, $s\tau = \tau^{-1}s$. Denote by $r_{CG}$ the Cremmer–Gervais $r$-matrix corresponding to the above triple and whose Cartan part is given by the following expression:

$$r_0 = \frac{1}{2}\sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{1 \leq i < k \leq n} \frac{n + 2(i-k)}{2n} e_{ii} \otimes e_{kk}.$$

We intend to describe $\overline{H}^1_{BD}(GL(n), r_{CG})$. Let us first analyse the case $\mathfrak{g} = sl(3)$. The centralizer $C(GL(n), r_{CG})$ consists of diagonal matrices $\mathrm{diag}(a, b, c)$ such that $b^2 = ac$. Consider

$$J = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ j & 0 & -j \end{pmatrix}.$$

**Lemma 13.** *Let* $X \in GL(3, \mathbb{K}[j])$. *Then* $\overline{X} = XSC$, *where* $C \in C(GL(n), r_{CG})$ *if and only if* $X = RJ\mathrm{diag}(p, q, r)$, *with* $R \in GL(3, \mathbb{K})$ *and* $prq^{-2} = k \in \mathbb{K}$.

*Proof.* According to Lemma 12, there exist $R \in GL(3, \mathbb{K})$ and $D = \mathrm{diag}(p, q, r)$, $p, q, r \in \mathbb{K}[j]$ such that $X = RJD$. We get $\overline{X} = RJS\overline{D} = RJDD^{-1}S\overline{D} = XS\mathrm{diag}(\overline{p}r^{-1}, \overline{q}q^{-1}, \overline{r}p^{-1})$. Let $C = \mathrm{diag}(\overline{p}r^{-1}, \overline{q}q^{-1}, \overline{r}p^{-1})$. Then $C \in C(GL(n), r_{CG})$ if and only if $\overline{pr}(pr)^{-1} = (\overline{q}q^{-1})^2$, which is equivalent to $\overline{prq^{-2}} = prq^{-2}$, i.e., $prq^{-2} \in \mathbb{K}$.

**Proposition 13.** *$\overline{H}^1_{BD}(GL(3), r_{CG})$ consists of one element, namely $J$ can be chosen as a representative.*

*Proof.* Let $X \in \overline{Z}(GL(3), r_{CG})$. According to the preceding lemma, $X = RJ\mathrm{diag}(p, q, r)$, with $R \in GL(3, \mathbb{K})$ and $prq^{-2} = k \in \mathbb{K}$. We distinguish the following cases:

*Case 1* Let $k = l^{-2}$, where $l \in \mathbb{K}$. Then we have a particular solution to the equation $prq^{-2} = l^{-2}$, namely $p_0 = r_0 = 1, q_0 = l$. By setting $p = p_0p_1, q = q_0q_1, r = r_0r_1$, we see that $\mathrm{diag}(p_1, q_1, r_1) \in C(GL(n), r_{CG})$ and $\mathrm{diag}(p_0, q_0, r_0) = \mathrm{diag}(1, l, 1)$, which commutes with $J$. It follows that $X = RJ\mathrm{diag}(1, l, 1) \cdot \mathrm{diag}(p_1, q_1, r_1)$, or, equivalently, $X = R_1J\mathrm{diag}(p_1, q_1, r_1)$, where $R_1 := R \cdot \mathrm{diag}(1, l, 1)$. Consequently, $X$ is equivalent to $J$.

*Case 2* Suppose $k$ is not a square of an element of $\mathbb{K}$. In this case, without loss of generality, we can set $l = j$ and $k = \hbar$. We want to prove that $J \cdot \mathrm{diag}(1, j, 1) = R'JC'$, for some $R' \in GL(3, \mathbb{K})$ and some $C' = \mathrm{diag}(x, y, z)$ with $xy^{-2}z = 1$. Equivalently, $J \cdot \mathrm{diag}(x^{-1}, jy^{-1}, z^{-1})J^{-1} = R'$. Since $\overline{R'} = R'$, we get $\overline{J}\mathrm{diag}(\overline{x}^{-1}, -j\overline{y}^{-1}, \overline{z}^{-1})\overline{J}^{-1} = J\mathrm{diag}(x^{-1}, jy^{-1}, z^{-1})J^{-1}$. Thus $\mathrm{diag}(\overline{x}^{-1}, -j\overline{y}^{-1}, \overline{z}^{-1}) = \mathrm{diag}(x^{-1}, jy^{-1}, z^{-1})$. We obtained that $x = \overline{z}$ and $y = kj$, with $k \in \mathbb{K}$. Hence, we have to find $x$ and $k$ so that $x\overline{x} = k^2\hbar$. Clearly, it is sufficient to find $\alpha \in \mathbb{K}[j]$ with norm $\hbar$ (recall that the norm of an element $a \in \mathbb{K}[j]$ is the element $a\overline{a} \in \mathbb{K}$). The latter is trivial because we can for instance choose $\alpha = \sqrt{-1}j$. Thus the existence of $R' \in GL(3, \mathbb{K})$ and $C' = \mathrm{diag}(x, y, z)$ is proved and therefore we conclude that $X$ is equivalent to $J$.

The above result can be generalized to $sl(n), n > 3$. Let us first note that the centralizer $C(GL(n), r_{CG})$ consists of diagonal matrices $\mathrm{diag}(p_1, p_2, \ldots, p_n)$ such that $p_{i+1} = p_2^i p_1^{1-i}$ for all $i$. Let $m = [\frac{n+1}{2}]$.

**Lemma 14.** *Let $X \in GL(n, \mathbb{K}[j])$. Then $\overline{X} = XSC$, where $C \in C(GL(n), r_{CG})$ if and only if $X = RJ\mathrm{diag}(d_1, \ldots, d_n)$, with $R \in GL(n, \mathbb{K})$, $d_1, \ldots, d_n \in \mathbb{K}[j]$ and $d_{n-i+1} = \overline{d_i}r^{i-2}q^{-1}$ for $i \leq m$, where $r, q$ are such that $r^{n-3} = q\overline{q}$.*

*Proof.* According to Lemma 12, there exist $R \in GL(n, \mathbb{K})$, $D = \mathrm{diag}(d_1, \ldots, d_n)$, $d_i \in \mathbb{K}[j]$ such that $X = RJD$. We get $\overline{X} = RJS\overline{D} = RJDD^{-1}S\overline{D} = XS(SD^{-1}S\overline{D})$. On the other hand, $SD^{-1}S\overline{D} = \mathrm{diag}(\overline{d_1}d_n^{-1}, \overline{d_2}d_{n-1}^{-1}, \ldots, \overline{d_n}d_1^{-1})$. Denote $p_i = \overline{d_i}d_{n+1-i}^{-1}$. Obviously, $p_{n+1-i} = (\overline{p_i})^{-1}$. But $\mathrm{diag}(p_1, p_2, \ldots, p_n)$ belongs to $C(GL(n), r_{CG})$ if and only if $p_{i+1} = p_2^i p_1^{1-i}$ for all $i$. It follows that $p_2^{n-i} p_1^{1+i-n} = (\overline{p_2})^{-i+1}(\overline{p_1})^{i-2}$ must be fulfilled for all $i$. For $i = 1$ we get $p_2^{n-1} = p_1^{n-1}\overline{p_1}^{-1}$ (note that if this identity holds then the other identities also hold for all $i$). This identity is also equivalent to $p_1^{n-3} = p_2^{n-2}\overline{p_2}$. Set $p_1 = qr$, $p_2 = q$. Then $r^{n-3} = q\overline{q}$. We obtain $d_{n-i+1} = \overline{d_i}r^{i-2}q^{-1}$, for all $i \leq m$. Let us note that if $n = 2m - 1$, we have $d_m(\overline{d_m})^{-1} = r^{m-2}q^{-1}$. Since the norm of $r^{m-2}q^{-1}$ is 1, this condition is self-consistent.

*Remark 9.* It follows from the above lemma that $X = RJ$, where $R \in GL(n, \mathbb{K})$, is a twisted cocycle associated to $r_{CG}$. All such cocycles are equivalent to $J$.

**Proposition 14.** *$\overline{H}^1_{BD}(GL(n), r_{CG})$ consists of one element, namely $J$ can be chosen as a representative.*

*Proof.* Let $X \in \overline{Z}(GL(n), r_{CG})$. According to the previous lemma, $X = RJ\mathrm{diag}(d_1, \ldots, d_n)$, where $d_{n-i+1} = \overline{d_i} r^{i-2} q^{-1}$ for $i \le m$, and $r^{n-3} = q\overline{q}$. We are looking for $Q \in GL(n, \mathbb{K})$ and $C \in C(GL(n), r_{CG})$ such that $X = QJC$. We get $RJD = QJC$. By taking the conjugate, we obtain $RJS\overline{D} = QJS\overline{C}$, which implies $SD^{-1}S\overline{D} = SC^{-1}S\overline{C}$. Let $C = \mathrm{diag}(c_1, \ldots, c_n)$ with $c_{i+1} = c_2^i c_1^{1-i}$ for all $i$. Therefore $c_i$ must fulfill the system $\overline{d_i} d_{n+1-i}^{-1} = \overline{c_i} c_{n+1-i}^{-1}$. Equivalently, $\frac{\overline{c_2}^{i-1} c_1^{n-i-1}}{\overline{c_1}^{i-2} c_2^{n-i}} = \frac{q}{r^{i-2}}$ must hold for all $i$. Substituting $c_1 = xy, c_2 = y$, we immediately obtain $x\overline{x} = r$ and $x^{n-3}\overline{y}y^{-1} = q$. The first equation clearly has a solution in $\mathbb{K}[j]$. Since $q/x^{n-3}$ has norm 1, Hilbert's Theorem 90 implies that there exists a solution $y \in \mathbb{K}[j]$ to the equation $\overline{y}/y = q/x^{n-3}$. Thus we find a solution to the system which in turn provides us with a matrix $C \in C(GL(n), r_{CG})$ that satisfies $SD^{-1}S\overline{D} = SC^{-1}S\overline{C}$. Finally we note that if we let $Q = XC^{-1}J^{-1}$, then $Q \in GL(n, \mathbb{K})$ because of the way $C$ was chosen.

The Cremmer–Gervais case can be further generalized. We call a triple $(\Gamma_1, \Gamma_2, \tau)$ *generalized Cremmer–Gervais* if $\Gamma_1 = \{\alpha_1, \ldots, \alpha_k\}$. Without loss of generality, such a triple has one of the forms:

Type 1: $\Gamma_1 = \{\alpha_1, \ldots, \alpha_k\}$, $\Gamma_2 = \{\alpha_{n-k}, \ldots, \alpha_{n-1}\}$ and $\tau(\alpha_i) = \alpha_{n-k+i-1}$.

Type 2: $\Gamma_1 = \{\alpha_1, \ldots, \alpha_k\}$, $\Gamma_2 = \{\alpha_{n-k}, \ldots, \alpha_{n-1}\}$ and $\tau(\alpha_i) = \alpha_{n-i}$.

Let us recall that a necessary condition for $\overline{Z}(SL(n), r)$ to be non-empty is that the corresponding admissible triple satisfies $s(\Gamma_1) = \Gamma_2$ and $s\tau = \tau^{-1}s$, where $s$ is given by $s(\alpha_i) = \alpha_{n-i}$ for all $i = 1, \ldots, n-1$. If the triple is generalized Cremmer–Gervais then this condition is satisfied.

**Theorem 6.** *Let $r$ be a non-skewsymmetric $r$-matrix corresponding to a generalized Cremmer–Gervais triple $(\Gamma_1, \Gamma_2, \tau)$. Then $\overline{H}^1_{BD}(GL(n), r)$ consists of one element, the class of $J$.*

*Proof.* First let us describe the centralizer $C(GL(n), r)$.

For type 1, i.e. $\Gamma_1 = \{\alpha_1, \ldots, \alpha_k\}$, $\Gamma_2 = \{\alpha_{n-k}, \ldots, \alpha_{n-1}\}$ and $\tau(\alpha_i) = \alpha_{n-k+i-1}$, the centralizer $C(GL(n), r)$ consists of matrices $\mathrm{diag}(p_1, \ldots, p_n)$ such that $p_{i-1}p_i^{-1} = p_{n-k+i-1}p_{n-k+i}^{-1}$ for all $i \le k$.

For type 2, i.e. $\Gamma_1 = \{\alpha_1, \ldots, \alpha_k\}$, $\Gamma_2 = \{\alpha_{n-k}, \ldots, \alpha_{n-1}\}$ and $\tau(\alpha_i) = \alpha_{n-i}$, the corresponding $C(GL(n), r)$ consists of matrices $\mathrm{diag}(p_1, \ldots, p_n)$ such that $p_i p_{i+1}^{-1} = p_{n-i}p_{n-i+1}^{-1}$ for all $i \le k$. We note that $k \le [\frac{n-1}{2}]$, since otherwise $\tau$ has fixed points.

Let us assume that $X \in \overline{Z}(GL(n), r)$ for a triple $(\Gamma_1, \Gamma_2, \tau)$ of the first type. Then $X = RJD$, where $R \in GL(n, \mathbb{K})$ and $D = \mathrm{diag}(d_1, \ldots, d_n)$ is such that $SD^{-1}S\overline{D} \in C(GL(n), r)$. Let $p_i = \overline{d_i} d_{n+1-i}^{-1}$. Then $p_{n+1-i} = \overline{p_i}^{-1}$. On the other hand, since $\mathrm{diag}(p_1, \ldots, p_n) \in C(L(n), r)$, we have $p_{i-1}p_i^{-1} = p_{n-k+i-1}p_{n-k+i}^{-1}$ for all $i \le k$. This further implies $p_i p_{n-k+i}^{-1} = p_{k-i+1}p_{n+1-i}^{-1}$ for all $i \le k$. Thus we get $p_i\overline{p}_{k-i+1} = p_{k-i+1}\overline{p_i}$, which is equivalent to $p_i/p_{k-i+1} \in \mathbb{K}$. Equivalently, $\frac{d_i d_{n+1-i}}{d_{k-i+1}d_{n-k+i}} \in \mathbb{K}$ for $i \le k$.

Let us prove that $X$ is equivalent to $J$. For this, it is enough to determine $C \in C(GL(n), r)$ which satisfies $SD^{-1}S\overline{D} = SC^{-1}S\overline{C}$. Let $C = \mathrm{diag}(c_1, \ldots, c_n)$. The preceding condition is equivalent to the system $\overline{c_i} c_{n+1-i}^{-1} = \overline{d_i} d_{n+1-i}^{-1}$, where $i \le n$.

On the other hand, since $C \in C(GL(n), r)$, we have $c_{i-1}c_i^{-1} = c_{n-k+i-1}c_{n-k+i}^{-1}$ for $i \le k$. It follows that $c_i c_{n-k+1} = c_1 c_{n-k+i}$ and $c_{k-i+1}c_{n-k+1} = c_1 c_{n-i+1}$. Consequently, $c_i c_{n-i+1} = c_{k-i+1}c_{n-k+i}$. Furthermore, $\frac{\overline{c}_{k-i+1}c_{k-i+1}}{\overline{c_i}c_i} = \frac{\overline{d}_{k-i+1}d_{n+1-i}}{d_{n-k+i}\overline{d_i}} =: \lambda_i$. We note that $\lambda_i \in \mathbb{K}$ since $\frac{d_i d_{n+1-i}}{d_{k-i+1}d_{n-k+i}} \in \mathbb{K}$, for $i \le k$. Thus we have obtained that the norm $c_{k-i+1}/c_i$ should be $\lambda_i$. Now, if $c_1, \dots, c_{[\frac{k}{2}]}$ are fixed, then we can determine $c_{[\frac{k}{2}]+1}, \dots, c_k$ since we can solve equations of the type $x\overline{x} = \lambda_i$. The remaining unknowns $c_{n-i+1}$ are determined by the relation $c_{k-i+1}c_{n-k+1} = c_1 c_{n-i+1}$. Thus we have proved the existence of $C \in C(GL(n), r)$ and in conclusion $X$ and $J$ are equivalent.

Now let us consider $X \in \overline{Z}(SL(n), r)$, where the triple $(\Gamma_1, \Gamma_2, \tau)$ is of the second type. Again we have a decomposition $X = RJD$, where $R \in GL(n, \mathbb{K})$ and $D = \mathrm{diag}(d_1, \dots, d_n)$ is such that $SD^{-1}S\overline{D} \in C(GL(n), r)$. Let $p_i = \overline{d_i}d_{n+1-i}^{-1}$. Since $\mathrm{diag}(p_1, \dots, p_n) \in C(GL(n), r)$, we have $p_i p_{i+1}^{-1} = p_{n-i}p_{n-i+1}^{-1}$ for all $i \le k$. Since $p_{n+1-i} = \overline{p_i}^{-1}$, we easily get $p_i/p_{i+1} \in \mathbb{K}$, or equivalently, $\frac{d_i d_{n-i}}{d_{i+1}d_{n-i+1}} \in \mathbb{K}$ for $i \le k$.

Let us show that $X$ is equivalent to $J$. As in the preceding case, the problem is reduced to solving the following system: $\overline{c_i}c_{n+1-i}^{-1} = \overline{d_i}d_{n+1-i}^{-1}$, for $i \le n$. On the other hand, since $C \in C(GL(n), r)$, $c_i c_{i+1}^{-1} = c_{n-i}c_{n-i+1}^{-1}$ for all $i \le k$. We immediately get that the norm of $c_i/c_{n-i}$ is $\lambda_i := \frac{\overline{d_i}d_{i+1}}{d_{n-i}d_{n+1-i}}$, which belongs to $\mathbb{K}$ since $\frac{d_i d_{n-i}}{d_{i+1}d_{n-i+1}} \in \mathbb{K}$ for $i \le k$. If we fix $c_i$ and solve equations $x\overline{x} = \lambda_i$, we can determine $c_{n-i}$. The remaining unknowns $c_{k+1}, \dots, c_{n-k}$ can be arbitrarily chosen satisfying the condition $\overline{c_i}c_{n+1-i}^{-1} = \overline{d_i}d_{n+1-i}^{-1}$. Thus $C$ exists and therefore the twisted cohomology set consists of the class of $J$.

## 9. Other Gauge Groups and Conjectures

*9.1. Computation of $H^1_{BD}(SL(n), r_{BD})$.* The group $SL(n)$ is a subgroup of $GL(n)$ consisting of matrices with determinant one. Let $H$ be the subgroup of diagonal matrices in $SL(n)$. Simple roots are given by the formula $e^{\alpha_i} = d_i d_{i+1}^{-1}$, where $\mathrm{diag}(d_1, \dots, d_n) \in H$. We will first prove the cohomology triviality for the Drinfeld-Jimbo $r$-matrix.

**Lemma 15.** *The Belavin-Drinfeld cohomology $H^1_{BD}(SL(n), r_{DJ})$ is trivial.*

*Proof.* Let $X \in Z^1(SL(n), r_{DJ})$. We have $X = QD$, where $Q \in GL(n, \mathbb{K})$, $D \in H(\overline{\mathbb{K}})$. Then $D^{-1}\sigma(D) \in H(\mathbb{K})$ for any $\sigma$ in the absolute Galois group of $\mathbb{K}$. Thus $\det D = k \in \mathbb{K}$. Let $D' = \mathrm{diag}(1, 1, \dots, k)$. Then $X = (QD')I(D'^{-1}D)$ is the desired decomposition, which provides an equivalence between $X$ and $I$.

Given an $r$-matrix on the Belavin–Drinfeld list, let $\tau : \Gamma_1 \to \Gamma_2$ be the corresponding admissible triple for $sl(n)$. Let $\alpha_{i_1}, \dots, \alpha_{i_k}$ be a string for $\tau$, $\tau(\alpha_{i_p}) = \alpha_{i_{p+1}}$. If $\tau(\alpha_{i_p})$ is not defined, then anyway we define the corresponding string, which consists of one element $\{\alpha_{i_p}\}$ only. Moreover, for any Belavin-Drinfeld triple we will also consider a string $\{\alpha_n\}$ with weight $n$. For any string $S = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ of $\tau$, we define the weight of $S$ by $w_S = \sum_p i_p$. Let $t_1, \dots, t_n$ be the ends of the strings with weights $w_1, \dots, w_n$. We note that some indices in $w_1, \dots, w_n$ are missing unless $\Gamma_1$ is an empty set and $w_n = n$ is always present. Let $N = GCD(w_1, \dots, w_n)$.

**Theorem 7.** *The number of elements of $H^1_{BD}(SL(n), r)$ is $N$. Each cohomology class contains a diagonal matrix $D = A_1A_2$, where $A_2 \in C(GL(n), r)$ and $A_1 \in \mathrm{diag}(n, \mathbb{K})$. Two such diagonal matrices $D_1 = A_1A_2$ and $D_2 = B_1B_2$ are contained in the same class of $H^1_{BD}(SL(n), r)$ if and only if $\det(A_1) = \det(B_1)$ in $\mathbb{K}^*/(\mathbb{K}^*)^N$.*

*Proof.* Let $X \in SL(n, \overline{\mathbb{K}})$ be a representative of a cohomology class of $H^1_{BD}(SL(n), r)$. Then we can find $Q \in SL(n, \mathbb{K})$ and a diagonal matrix $D$ such that $X = QD$. Therefore, $\det(D) = 1$ and $X \sim D$. Using the fact that $H^1_{BD}(GL(n), r)$ is trivial we can find a decomposition $D = A_1 A_2$ such that $A_1$ is diagonal and has $\mathbb{K}$-entries while $A_2 \in C(GL(n), r)$.

Let two diagonal matrices $D_1 = A_1 A_2$ and $D_2 = B_1 B_2$ be equivalent. Then we have

$$A_1 B_1^{-1} C_1 = A_2^{-1} B_2 C_2, \ C_1 \in \operatorname{diag}(n, \mathbb{K}), \ C_2 \in C(SL(n), r).$$

We see that $A_1 B_1^{-1} C_1 = A_2^{-1} B_2 C_2 = K \in C(GL(n), r) \cap GL(n, \mathbb{K})$. Then $A_1 K^{-1} = B_1 C_1^{-1}$, $D_1 = (A_1 K^{-1})(A_2 K)$. Since $\det(C_1) = \det(C_2) = 1$, it follows that the class of $D_1$ uniquely defines $\det(A_1)$ in $\mathbb{K}^*$ modulo the subgroup generated by determinants of elements of $C(GL(n), r) \cap GL(n, \mathbb{K})$.

Let $K = \operatorname{diag}(k_1, \dots, k_n) \in C(GL(n), r) \cap GL(n, \mathbb{K})$. Then it is easy to check that $\det(K) = s_{t_1}^{w_1} s_{t_2}^{w_2} \dots s_{t_n}^{w_n}$ (where $s_p = k_p / k_{p+1}$, $s_n = k_n$) is the $N$th power of an element of $\mathbb{K}$.

Conversely, let $D = \operatorname{diag}(d_1, \dots, d_n) \in Z(SL(n), r)$ and $D = A_1 A_2$ as above. It is sufficient to show that if $\det(A_1) = u^N$ for some $u \in \mathbb{K}^*$, then $D \sim I$. There are integers $m_i$ such that $\sum m_i w_i = N$. Set again $s_p = d_p / d_{p+1}$, $s_n = d_n$ and choose a string. If $t_p$ is the end of the string, set $s_i = s_p = u^{m_p}$ along the string. Solving the corresponding system for $\{d_i\}$, we find $d_1, d_2, \dots, d_n \in \mathbb{K}$ (each $d_i$ will be a power of $u$), such that the corresponding diagonal matrix $C = \operatorname{diag}(d_1, \dots, d_n)$ has determinant $u^N$ and by construction $C \in C(r, GL(n)) \cap GL(n, \mathbb{K})$. Then $D = (A_1 C^{-1})(C A_2)$ and $D \sim I$.

*9.2. Computation of* $\overline{H}^1_{BD}(SL(n), r_{CG})$. In this section we will compute Belavin-Drinfeld twisted cohomology for the Cremmer-Gervais $r$-matrix when the gauge group is $SL(n)$. The definition of this cohomology is exactly the same as in the $GL(n)$ case.

**Lemma 16.** *Any element of* $\overline{Z}(SL(n), r_{CG})$ *is equivalent to an element of the form* $\alpha h_m J$, *where* $\alpha \in \overline{\mathbb{K}}$, $h_m = \operatorname{diag}(\hbar^m, 1, 1, \dots, 1)$.

*Proof.* By Proposition 14, an arbitrary cocycle can be written as $RJC$, where $R \in GL(n, \mathbb{K})$, $C \in C(GL(n), r_{CG})$. We can write $C = x C_1$, where $x \in \overline{\mathbb{K}}$, $C \in C(SL(n), r_{CG})$. Also we have $R = y h_m R_1$, where $y \in \mathbb{K}$, $R_1 \in C(SL(n), r_{CG})$. Therefore $RJC = R_1 \alpha h_m J C_1 \sim \alpha h_m J$.

**Lemma 17.** *If* $\alpha_1 h_{m_1} J$ *is equivalent to* $\alpha_2 h_{m_2} J$ *then* $m_2 \equiv m_1 \pmod{n/2}$ *if* $n$ *is even and* $m_2 \equiv m_1 \pmod{n}$ *if* $n$ *is odd.*

*Proof.* The condition $\alpha_1 h_{m_1} J \sim \alpha_2 h_{m_2} J$ is equivalent to $\alpha_2 h_{m_2} J = R \alpha_1 h_{m_1} J C$, where $R \in SL(n, \mathbb{K})$, $C \in C(SL(n), r_{CG})$. This in turn is equivalent to $h_{m_1}^{-1} R h_{m_2} = J C_1 J^{-1}$, where $C_1 = \alpha_1 \alpha_2^{-1} C \in C(GL(n), r_{CG})$. Since $h_m, R, J$ are defined over $\mathbb{K}[j]$, we see that $C_1$ is defined over $K[j]$. Let $C_1 = \operatorname{diag}(c_1, \dots, c_n)$ (recall that all elements of $C(SL(n), r_{CG})$ are diagonal). Applying conjugation we get $J C_1 J^{-1} = h_{m_1}^{-1} R h_{m_2} = \overline{h_{m_1}^{-1} R h_{m_2}} = J S \overline{C_1} S J^{-1}$. Thus $S \overline{C_1} S = C_1$, i.e., $c_i = \overline{c_{n+1-i}}$. From the structure of the centralizer we have $c_i / c_{i+1} = c_{n+1-(i+1)} / c_{n+1-i}$ so $c_i / c_{i+1} = \overline{c_{i+1} / c_i}$. It follows that the norms of all diagonal elements are equal to $\gamma \in \mathbb{K}$. If $n$ is odd then considering the central element we get that the norms of all diagonal elements are in fact equal to $\gamma^2$, for some $\gamma \in \mathbb{K}$. Finally we have $\hbar^{m_2-m_1} = \det(h_{m_1}^{-1} R h_{m_2}) = \det(JC_1J^{-1}) = \gamma^k$, where $k = n/2$ for even $n$ and $k = n$ for odd $n$. The result follows.

**Theorem 8.** *$\overline{H}_{BD}^1(SL(n), r_{CG})$ consists of $k$ elements where $k = n/2$ for even $n$ and $k = n$ for odd $n$.*

*Proof.* Note that if $X \in SL(n)$ commutes with all elements of the centralizer then the condition $A \sim B$ implies $AX \sim BX$. Indeed, from $A = RBC$ we get $AX = RBCX = RBXC$. Note that the matrices $h_m$ commute with the centralizer. Therefore, to prove the theorem we need to show that $\alpha h_k J \sim \beta J$, for some scalars $\alpha, \beta$ (the scalars are defined uniquely in such a way that the cocycles are elements of $SL(n)$). We will consider the cases of odd and even $n$ separately.

Let $n$ be even. We need to find $R \in SL(n, \mathbb{K})$ and $C \in C(SL(n), r_{CG})$ such that $\alpha h_k J = \beta RJC$. Let us denote $C_1 = \beta\alpha^{-1}C \in C(GL(n), r_{CG})$. Then the equation becomes $h_k J = RJC_1$. Take $C_1 = \text{diag}(j, -j, j, \ldots, -j)$. Then $R = h_k JC_1^{-1}J^{-1}$. $\det R = 1$, $\overline{R} = h_k JS(-C_1)SJ^{-1} = h_k JC_1J^{-1} = R$. Therefore $R \in SL(n, \mathbb{K})$ and we are done.

Now assume $n$ is odd. Again we need to find $R \in SL(n, \mathbb{K})$ and $C \in C(SL(n), r_{CG})$ such that $\alpha h_k J = \beta RJC$. Let $C_1 = \beta\alpha^{-1}C \in C(GL(n), r_{CG})$. Then we get $h_k J = RJC_1$. Take $C_1 = \hbar$. Then $R = h_k JC_1^{-1}J^{-1}$, $\det R = 1$. Finally $\overline{R} = h_k JSC_1^{-1}SJ^{-1} = R$.

*9.3. Belavin–Drinfeld cohomology conjecture.*

**Conjecture 1.** *Let $\mathfrak{g}$ be a simple Lie algebra and $r_{DJ}$ the Drinfeld–Jimbo r-matrix. For any connected split algebraic group $G$ which has $\mathfrak{g}$ as its Lie algebra, $H_{BD}^1(G, r_{DJ})$ is trivial.*

*9.4. Quantization conjecture.* Let $L$ be a finite dimensional Lie algebra over $\mathbb{C}$ and $\delta$ a Lie bialgebra structure on $L(\mathbb{K})$ such that $\delta = 0 \pmod \hbar$.

Let $(U_\hbar(L), \Delta_\hbar)$ be the corresponding quantum group, in other words the dequantization functor $\widehat{Q}$ sends $(U_\hbar(L), \Delta_\hbar)$ to $(L(\mathbb{K}), \delta)$. Let $G$ be a connected algebraic group with the Lie algebra $L$. We assume that $G$ acts on $L$ by the adjoint action. Consider $G(\overline{\mathbb{K}})$. Let us define the centralizer $C(\overline{\mathbb{K}}, \delta)$.

**Definition 9.** The centralizer $C(\overline{\mathbb{K}}, \delta)$ consists of all $X \in G(\overline{\mathbb{K}})$ such that for any $l \in L$

$$(\text{Ad}_X \otimes \text{Ad}_X)\delta(\text{Ad}_X^{-1}(l)) = \delta(l).$$

**Definition 10.** We say that $X \in G(\overline{\mathbb{K}})$ is a *Belavin–Drinfeld cocycle* associated to $\delta$ if for any $\sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ there exists $C \in C(\overline{\mathbb{K}}, \delta)$ such that $\sigma(X) = XC$.

Two cocycles $X_1$ and $X_2$, associated to $\delta$, are *equivalent* if $X_1 = QX_2C$, where $Q \in G(\mathbb{K})$ and $C \in C(\overline{\mathbb{K}}, \delta)$.

The set of equivalence classes will be denoted by $H_{BD}^1(G, \delta)$.

Now let us define quantum Belavin–Drinfeld cohomology. The quantum group $(U_\hbar(L), \Delta_\hbar)$ is defined over $\mathbb{O} = \mathbb{C}[[\hbar]]$. We extend the Hopf structures of $U_\hbar(L)$ to $U_\hbar(L, \mathbb{K}) = U_\hbar(L) \otimes_{\mathbb{O}} \mathbb{K}$ and $U_\hbar(L, \overline{\mathbb{K}}) = U_\hbar(L) \otimes_{\mathbb{K}} \overline{\mathbb{K}}$. By abuse of notation, $\Delta_\hbar$ denotes all three comultiplications.

**Definition 11.** Let $P$ be an invertible element of $U_\hbar(L, \overline{\mathbb{K}})$. We say that it belongs to $C(U_\hbar(L), \Delta_\hbar)$ if

$$(P \otimes P)\Delta_\hbar(P^{-1}aP)(P^{-1} \otimes P^{-1}) = \Delta_\hbar(a)$$

for all $a \in U_\hbar(L)$.

Denote

$$F_P := (P \otimes P)\Delta_\hbar(P^{-1}) \in U_\hbar(L, \overline{\mathbb{K}})^{\otimes 2}.$$

**Definition 12.** $P$ is called a *quantum Belavin–Drinfeld cocycle* if for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ there exists $C \in C(U_\hbar(L), \Delta_\hbar)$ such that $\sigma(P) = PC$.

Two quantum cocycles $P_1$ and $P_2$ are *equivalent* if $P_2 = QP_1C$, where $Q$ is an invertible element of $U_\hbar(L, \mathbb{K})$ and $C \in C(U_\hbar(L), \Delta_\hbar)$.

*Remark 10.* On $U_\hbar(L)$ consider the comultiplications $\Delta_{\hbar,P_1}(a) = F_{P_1}\Delta_\hbar(a)F_{P_1}^{-1}$ and $\Delta_{\hbar,P_2}(a) = F_{P_2}\Delta_\hbar(a)F_{P_2}^{-1}$. Clearly, $\Delta_{\hbar,P_2}(a) = (Q \otimes Q)\Delta_{\hbar,P_1}(Q^{-1}aQ) \cdot (Q^{-1} \otimes Q^{-1})$. Since $Q \in U_\hbar(L(\mathbb{K}))$, it is natural to call $\Delta_{\hbar,P_1}$ and $\Delta_{\hbar,P_2}$ $\mathbb{K}$–equivalent comultiplications on $U_\hbar(L(\mathbb{K}))$.

The set of equivalence classes of quantum Belavin–Drinfeld cocycles associated to $\Delta_\hbar$ will be denoted by $H^1_{q-BD}(\Delta_\hbar)$.

**Conjecture 2.** *There is a natural correspondence between* $H^1_{BD}(G, \delta)$ *and* $H^1_{q-BD}(\Delta_\hbar)$.

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