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Global Existence to the Vlasov–Poisson System and Propagation of Moments Without Assumption of Finite Kinetic Energy

Zili Chen[1], **Xianwen Zhang**[2]

[1] Department of Mathematics, School of Science, Nanchang University, Nanchang 330031, Jiangxi, People's Republic of China. E-mail: chenzili_xiu@163.com
[2] School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, Hubei, People's Republic of China. E-mail: xwzhang@hust.edu.cn

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Abstract: We consider classical as well as weak solutions to the three dimensional Vlasov–Poisson system. Without assuming finiteness of kinetic energy, we prove global existence of classical solutions by assuming the initial datum is smooth enough and has a compact *velocity-spatial support*, which will be specified in Theorem 1.1. We also establish some propagation results for low moments of weak solutions.

1. Introduction

In this paper, we consider the three-dimensional Vlasov–Poisson system:

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \qquad f(0, x, v) = f_0(x, v), \tag{1.1}$$

$$U(t, x) = -\gamma \int_{\mathbb{R}^3} \frac{\rho(t, y)}{|x - y|} dy, \qquad \rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v) dv, \tag{1.2}$$

$$E(t, x) = -\nabla_x U(t, x). \tag{1.3}$$

In this system, the unknown $f(t, x, v) \geq 0$ denotes microscopic density of particles at time $t \geq 0$ and position $x \in \mathbb{R}^3$, moving with velocity $v \in \mathbb{R}^3$, evolving in a self-consistent potential $U(t, x)$. This potential is Coulomb potential or Newton potential created by macroscopic density $\rho(t, x)$, which is described by $\gamma = -1$ and $\gamma = 1$ respectively. E is the force field corresponding to the potential U. Since we will always consider $\rho(t, x) \in L^1 \cap L^{5/3}(\mathbb{R}^3_x)$, Eq. (1.2) can be written equivalently

$$\triangle_x U(t, x) = 4\pi\gamma\rho(t, x), \qquad \lim_{|x|\to\infty} U(t, x) = 0.$$

Global existence for the Vlasov–Poisson system in two space dimensions was established in [20,36]. The three dimensional case is more delicate, the existence of weak solutions is well known according to Arsenev [1], Illner and Neunzert [13] and Horst and Hunze [12]. Classical solutions were also studied extensively, Batt [3] gave a global existence result for spherically symmetric data and established an important continuation criterion: a local solution can be extended as long as its velocity support is under control. Then Bardos and Degond [2] proved global existence for small initial data. Almost simultaneously, Pfaffelmoser [28] and Lions and Perthame [17] independently established the global existence of classical solutions with general initial data. In [28], careful analysis of characteristic flows and an appropriate decomposition of phase space were used to control velocity support, which can obtain the global existence by the above continuation criterion. In [17], they were devoted to propagating the velocity moments of order higher than three, since by using it one can get the boundedness of velocity support in any finite time. We refer to [4,6,7,9,21–23,29,32] for further improvements and developments in both methods.

For the global existence of classical solutions, the above two celebrated methods focus on velocity support or velocity moments, and both need the boundedness of kinetic energy. However, we try to give an existence result without an assumption of finite kinetic energy. To this end, we first fix some notations used in the present paper. We shall denote by $C_0^1(\mathbb{R}^n)$ the function class of continuous differentiable functions $f(y)$ defined on $\mathbb{R}^n$ such that $\lim_{y\to\infty} f(y) = 0$. The symbol $C_c^1(\mathbb{R}^n)$ is the usual test function space of all continuous differentiable functions with compact support. Then, we can describe one of our main results in this paper as follows.

Theorem 1.1. *Let the initial datum* $0 \leq f_0 \in C_0^1 \cap L^1(\mathbb{R}^3 \times \mathbb{R}^3)$. *If there exists a positive constant* α *such that*

$$\sup\{|x - \alpha v|\colon (x, v) \in \operatorname{supp} f_0\} < \infty,$$

then there exists a unique global classical solution f *to* (1.1)–(1.3), *and for any* $T > 0$

$$\sup_{t\in[0,T]} \{|x - (t + \alpha)v|\colon (x, v) \in \operatorname{supp} f(t)\} < \infty.$$

Some weak existence results concerning infinite kinetic energy can be found in [15, 37]. Moreover, we refer to [5,8,19] for the Vlasov–Poisson system with point charges and [24,33,34] for the Vlasov–Poisson system with steady spatial asymptotic. A complete discussion of the literature concerning the Vlasov–Poisson system can be found in [10, 30] and the references therein.

This paper is arranged as follows: in Sect. 2, we give a new conservation law related to velocity-spatial moments of order two, and deduce a-priori estimates on the whole time for solutions to the Vlasov–Poisson system. Section 3 is devoted to showing a local existence of classical solutions and a continuation criterion: a local solution can be extended as long as its velocity-spatial support is under control. In Sect. 4, we prove the main existence result. In Sect. 5, the propagation of low moments is considered.

We will denote by C a generic constant that changes from line to line and independent of T. If a constant depends on T, we will give an interpretation. $C_0, C_1, \ldots$ denote fixed positive constants. For $1 \leq p \leq \infty$, $\|\cdot\|_p$ denotes either the norm of $L^p(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ or the norm of $L^p(\mathbb{R}_x^3)$, if the context makes it clear.

2. Conservation Laws, A-Priori Estimates and Moments Lemma

Definition 2.1. Let $f(t,x,v) \geq 0$ be a solution to (1.1)–(1.3) and $\alpha > 0$. We define for any $k > 0$:

$$H_k(t) := \iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-(t+\alpha)v|^k f(t,x,v)dvdx,$$
$$M_k(t) := \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v-\frac{x}{t+\alpha}|^k f(t,x,v)dvdx.$$

The kinetic energy and potential energy of this system at time t are respectively defined by

$$\mathcal{E}_k(t) := \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(t,x,v)dvdx$$

and

$$\mathcal{E}_p(t) := -\frac{\gamma}{8\pi}\int_{\mathbb{R}^3} |E(t,x)|^2 dx = -\frac{\gamma}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho(t,x)\rho(t,y)}{|x-y|}dydx.$$

The characteristic flow corresponding to the Vlasov equation (1.1) with a suitable field $E(t,x)$ [such as $E \in C(\mathbb{R}_+; C_b^1(\mathbb{R}^3))$] is defined by

$$\begin{cases} \frac{dX(s,t,x,v)}{ds} = V(s,t,x,v), & X(t,t,x,v) = x, \\ \frac{dV(s,t,x,v)}{ds} = E(s,X(s,t,x,v)), & V(t,t,x,v) = v. \end{cases}$$

For the sake of simplicity, we often use the shorthand

$$(X(s),V(s)) = (X(s,t,x,v),V(s,t,x,v)).$$

Following from

$$\frac{d}{ds}[(s+\alpha)V(s) - X(s)] = (s+\alpha)E(s,X(s)),$$

we obtain that for any $s,t \geq 0$

$$V(s) - \frac{X(s)}{s+\alpha} = \frac{t+\alpha}{s+\alpha}\left(v - \frac{x}{t+\alpha}\right) + \frac{1}{s+\alpha}\int_t^s (\tau+\alpha)E(\tau,X(\tau))d\tau,$$

and then

$$\left|V(s) - \frac{X(s)}{s+\alpha}\right| \leq \left|v - \frac{x}{t+\alpha}\right| + \int_t^s |E(s,X(s))|ds, \quad \forall 0 \leq t < s < \infty. \tag{2.1}$$

Firstly, we recall some interpolation inequalities.

Lemma 2.1. *Let $g := g(x,v) \in L^\infty_+(\mathbb{R}^3\times\mathbb{R}^3)$, and $k \geq l > -3$, $\alpha > 0$. Then for any $t \geq 0$ there exists a constant C depending only on k, l and $\|g\|_\infty$ such that*

$$\int_{\mathbb{R}^3} |x-(t+\alpha)v|^l g dv \leq C(t+\alpha)^{\frac{3(l-k)}{3+k}} \left(\int_{\mathbb{R}^3} |x-(t+\alpha)v|^k g dv\right)^{\frac{3+l}{3+k}} \tag{2.2}$$

and

$$\left\|\int_{\mathbb{R}^3} |x-(t+\alpha)v|^l g dv\right\|_{\frac{3+k}{3+l}} \leq C(t+\alpha)^{\frac{3(l-k)}{3+k}} \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-(t+\alpha)v|^k g dv dx\right)^{\frac{3+l}{3+k}}. \tag{2.3}$$

In particular,

$$\left\|\int_{\mathbb{R}^3} g dv\right\|_{\frac{k+3}{3}} \leq C(t+\alpha)^{-\frac{3k}{k+3}} \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-(t+\alpha)v|^k g dv dx\right)^{\frac{3}{k+3}}. \tag{2.4}$$

Proof. For any $R > 0$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |x-(t+\alpha)v|^l g dv \\
&= \int_{|x-(t+\alpha)v|\leq R} |x-(t+\alpha)v|^l g dv + \int_{|x-(t+\alpha)v|>R} |x-(t+\alpha)v|^l g dv \\
&\leq \|g\|_\infty \int_{|x-(t+\alpha)v|\leq R} |x-(t+\alpha)v|^l dv + R^{l-k}\int_{\mathbb{R}^3} |x-(t+\alpha)v|^k g dv \\
&\leq C(t+\alpha)^{-3} R^{l+3}\|g\|_\infty + R^{l-k}\int_{\mathbb{R}^3} |x-(t+\alpha)v|^k g dv.
\end{aligned}$$

Letting $R = (t+\alpha)^{\frac{3}{3+k}} \|g\|_\infty^{-\frac{1}{3+k}} \left(\int_{\mathbb{R}^3} |x-(t+\alpha)v|^k g dv\right)^{\frac{1}{k+3}}$, we get (2.2). □

Remark 2.2. (1) If $\alpha = 0$, the above estimates also hold for $t > 0$.
(2) Divide both sides of (2.2) by α^l, and then let $\alpha \to +\infty$, we can get that

$$\int_{\mathbb{R}^3} |v|^l g dv \leq C\left(\int_{\mathbb{R}^3} |v|^k g dv\right)^{\frac{3+l}{3+k}}, \tag{2.5}$$

which yields that

$$\left\|\int_{\mathbb{R}^3} |v|^l g dv\right\|_{\frac{3+k}{3+l}} \leq C\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k g dv dx\right)^{\frac{3+l}{3+k}}. \tag{2.6}$$

It is well known that when $f_0 \in C^1_c(\mathbb{R}^3\times\mathbb{R}^3)$, there exists a unique global classical solution to the system (1.1)–(1.3). For such solution, we will give a new conservation law, which is slightly different from the one obtained in [14,25]. However, using this conservation law we can obtain some better a-priori estimates.

Theorem 2.2. *Let* $0 \le f_0 \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ *and* $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ *be the unique classical solution to the three dimensional Vlasov–Poisson system* (1.1)–(1.3)*, then for any* $t \ge 0$

$$\|f(t)\|_p = \|f_0\|_p, \quad \forall 1 \le p \le \infty. \tag{2.7}$$

Moreover, for any $\alpha > 0$ *and* $t \ge 0$ *we have the following conservation law:*

$$\frac{d}{dt}\left(\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-(t+\alpha)v|^2 f(t,x,v)dvdx + (t+\alpha)^2\mathcal{E}_p(t)\right) = (t+\alpha)\mathcal{E}_p(t) \tag{2.8}$$

and

$$\frac{d}{dt}\left(\frac{t+\alpha}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\left|v-\frac{x}{t+\alpha}\right|^2 f(t,x,v)dvdx + (t+\alpha)\mathcal{E}_p(t)\right) = -\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\left|v-\frac{x}{t+\alpha}\right|^2 f(t,x,v)dvdx. \tag{2.9}$$

Proof. Note that the characteristic flow is measure preserving and $f(s, X(s), V(s)) = f(t, x, v)$, so (2.7) is deduced immediately. Now, we devote to proving (2.8). Using (1.1)–(1.3) and Green's formula we can obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-(t+\alpha)v|^2 f(t,x,v)dvdx\right)\\
&= \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-(t+\alpha)v|^2\partial_t f dvdx - \iint_{\mathbb{R}^3\times\mathbb{R}^3}(x-(t+\alpha)v)\cdot vf dvdx\\
&= \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-(t+\alpha)v|^2\big(-\mathrm{div}_x(vf) - \mathrm{div}_v(Ef)\big)dvdx\\
&\quad - \iint_{\mathbb{R}^3\times\mathbb{R}^3}(x-(t+\alpha)v)\cdot vf dvdx\\
&= \iint_{\mathbb{R}^3\times\mathbb{R}^3}(x-(t+\alpha)v)\cdot vf dvdx - (t+\alpha)\iint_{\mathbb{R}^3\times\mathbb{R}^3}(x-(t+\alpha)v)\cdot Ef dvdx\\
&\quad - \iint_{\mathbb{R}^3\times\mathbb{R}^3}(x-(t+\alpha)v)\cdot vf dvdx\\
&= -(t+\alpha)\int_{\mathbb{R}^3} x\cdot E\rho dx + (t+\alpha)^2\int_{\mathbb{R}^3} E\cdot j dx,
\end{aligned}$$

where $j = \int_{\mathbb{R}^3} vf dv$. Note that $\partial_t\rho + \mathrm{div}_x j = 0$, so using Green's formula again we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-(t+\alpha)v|^2 f(t,x,v)dvdx\right)\\
&= \gamma(t+\alpha)\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{x-y}{|x-y|^3}\rho(t,y)dy\right)\cdot x\rho(t,x)dx + (t+\alpha)^2\int_{\mathbb{R}^3} U\mathrm{div}_x j dx\\
&= \gamma\frac{t+\alpha}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho(t,x)\rho(t,y)}{|x-y|}dxdy - (t+\alpha)^2\iint_{\mathbb{R}^3\times\mathbb{R}^3}(-\gamma)\frac{\rho(t,y)}{|x-y|}\partial_t\rho(t,x)dydx\\
&= -(t+\alpha)\mathcal{E}_p(t) - (t+\alpha)^2\frac{d}{dt}\mathcal{E}_p(t).
\end{aligned}$$

On the other hand,

$$\frac{d}{dt}\left((t+\alpha)^2\mathcal{E}_p(t)\right) = 2(t+\alpha)\mathcal{E}_p(t) + (t+\alpha)^2\frac{d}{dt}\mathcal{E}_p(t).$$

So, by denoting $H(t) = \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|x-(t+\alpha)v|^2 f(t,x,v)dvdx + (t+\alpha)^2\mathcal{E}_p(t)$ we have

$$\frac{dH(t)}{dt} = (t+\alpha)\mathcal{E}_p(t).$$

Then

$$\left(\frac{H(t)}{t+\alpha}\right)' = -\frac{H(t)}{(t+\alpha)^2} + \frac{H'(t)}{t+\alpha} = -\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\left|v - \frac{x}{t+\alpha}\right|^2 f(t,x,v)dvdx.$$

□

Following from the above conservation law and asymptotic method, we can establish some a-priori estimates even for much weaker initial datum. For such initial datum, we can only get a distributional solution with rough regularity, this kind of solution will be called weak solution in this paper.

Corollary 2.3. *Let* $0 \le f_0 \in L^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ *and* $\iint_{\mathbb{R}^3\times\mathbb{R}^3}(1+|x-\alpha v|^2)f_0(x,v)dvdx < \infty$ *for some* $\alpha > 0$*, then there exists a nonnegative weak solution* f *to the Vlasov–Poisson system* (1.1)–(1.3) *such that*

$$\|f(t)\|_\infty \le \|f_0\|_\infty, \quad \|f(t)\|_1 \le \|f_0\|_1, \quad \forall t \ge 0. \tag{2.10}$$

Furthermore, for all $t \ge 0$ *there exists a constant* C *depending only on* $\|f_0\|_1$, $\|f_0\|_\infty$, $M_2(0)$ *and* α *such that*

$$M_2(t), \quad |\mathcal{E}_p(t)|, \quad \|\rho(t)\|_{\frac{5}{3}} \le C. \tag{2.11}$$

Furthermore if $\gamma = -1$*, we have for all* $t \ge 0$

$$M_2(t), \quad \mathcal{E}_p(t) \le C(t+\alpha)^{-1}, \quad \|\rho(t)\|_{\frac{5}{3}} \le C(t+\alpha)^{-\frac{3}{5}}. \tag{2.12}$$

Proof. From the proof of Theorem 3.2 in [25] and (2.7), we can easily get a weak solution f with (2.10) by taking the limit of approximate solutions corresponding to regularized initial data. Now we devote ourselves to showing (2.11) and (2.12). Actually, we only need to establish these results for the corresponding approximate sequences, and then take the limits. For simplicity, we skip the last step.

For $\gamma = 1$, the Hardy-Littlewood-Sobolev inequality (Theorem 4.3 in [16] with $\lambda = 1, n = 3, p = r = 6/5$) and the Hölder inequality give that

$$|\mathcal{E}_p(t)| \le C\|\rho(t)\|_{6/5}^2 \le C\|\rho(t)\|_{5/3}^{5/6},$$

since $\|\rho(t)\|_1$ is uniformly bounded. Then using (2.4) with $k = 2$ we have

$$|\mathcal{E}_p(t)| \le C\|\rho(t)\|_{5/3}^{5/6} \le C\left[(t+\alpha)^{-\frac{6}{5}}H_2(t)^{\frac{3}{5}}\right]^{5/6} \le C(t+\alpha)^{-1}H_2(t)^{1/2}. \tag{2.13}$$

Combining (2.8) with (2.13), we can obtain

$$\frac{1}{2}H_2(0) \ge \frac{1}{2}H_2(t) + (t+\alpha)^2\mathcal{E}_p(t) \ge \frac{1}{2}H_2(t) - C(t+\alpha)H_2(t)^{1/2},$$

since $\mathcal{E}_p(s) \le 0$ for $s \ge 0$. So

$$H_2(t) \le C(t+\alpha)^2 + 2H_2(0),$$

and then by (2.13) we obtain for all $t \ge 0$

$$|\mathcal{E}_p(t)| \le C, \quad \|\rho(t)\|_{\frac{5}{3}} \le C.$$

For $\gamma = -1$, it follows from (2.9) that

$$\frac{H(t)}{t+\alpha} \le \frac{H(0)}{\alpha},$$

which implies that

$$H(t) \le (\frac{t}{\alpha} + 1)H(0).$$

Remember that (2.13), we have the boundedness of $\mathcal{E}_p(0)$ and then $H(0)$. As a consequence,

$$M_2(t) \le C(t+\alpha)^{-1}, \quad \mathcal{E}_p(t) \le C(t+\alpha)^{-1}.$$

Combining the above inequality with (2.4), we obtain

$$\|\rho(t)\|_{\frac{5}{3}} \le C(t+\alpha)^{-\frac{3}{5}}.$$

□

Remark 2.3. (1) The identity (2.8) holds also for $\alpha = 0$, which was discovered independently by Perthame [25] and Illner and Rein [14]. In their papers, by using (2.8) and (2.9) with $\alpha = 0$ they proved some time decay estimates for the repulsive Vlasov–Poisson system. However, Corollary 2.3 gives some a-priori estimates for both repulsive and attractive cases. It is worth noting that all these estimates are not singular when $t \to 0^+$.

(2) Dividing both sides of (2.8) by α^2 and integrating on t, and then letting $\alpha \to +\infty$, we can obtain that the total energy is conserved, if proper assumptions of the initial datum are given. So roughly speaking, (2.8) also contains

$$\frac{d}{dt}\left(\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(t,x,v)dvdx + \mathcal{E}_p(t)\right) = 0.$$

(3) For the free transport equation

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = 0, \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{2.14}$$

we have

$$\frac{d}{dt}\left(\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-(t+\alpha)v|^2 f(t,x,v)dvdx\right) = 0.$$

Similar to [25], we have the following a-priori estimate:

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v-u(t,x)|^2 f(t,x,v)dvdx \\ \leq \frac{1}{(t+\alpha)^2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-\alpha v|^2 f_0(x,v)dvdx, \quad \forall t\geq 0,$$

where the bulk velocity $u(t,x)$ is defined by $u(t,x)=\frac{\int_{\mathbb{R}^3} vf(t,x,v)dv}{\rho(t,x)}$.

As we know, a moments lemma is an important tool in kinetic equations, which can improve the obvious integrability derived from conservation laws. The classical one comes from Perthame [26], and is named "velocity moments lemma", that is, the solution of transport equation has three order velocity moments locally in space under the assumption of finite kinetic energy. Inspired by the methods in [26,27] and some new conservation laws like (2.8), we are able to give a new moments lemma, that is,

Theorem 2.4. *Let* $\alpha\geq 0$, $\beta>0$, $k\geq 1$ *and* $T>0$. *If* $0\leq f\in C^1([0,T];C_c^1(\mathbb{R}^3\times\mathbb{R}^3))$ *is a classical solution to the following Vlasov equation*

$$\partial_t f(t,x,v)+v\cdot\nabla_x f(t,x,v)+F(t,x)\cdot\nabla_v f(t,x,v)=0.$$

Assume that $(t+\alpha)^{\frac{3+2k}{3+k}}F(t,x)\in L^1([0,T];L^{k+3}(\mathbb{R}^3))$ *and* $\sup_{t\in[0,T]} H_k(t)<\infty$, *then*

$$\int_0^T\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|x-(t+\alpha)v|^{k+1}}{(1+|x|^\beta)^{1+\frac{1}{\beta}}} fdxdvdt \\ \leq (3T+2\alpha)\sup_{t\in[0,T]}H_k(t)+C(k,\|f\|_{L^\infty})\int_0^T (t+\alpha)^{\frac{3+2k}{3+k}}\|F(t)\|_{k+3}dt\cdot\sup_{t\in[0,T]}H_k(t)^{\frac{2+k}{3+k}}.$$

Proof. Denote the C^1 function ϕ as

$$\phi(t,x,v)=\frac{x\cdot z}{(1+|x|^\beta)^{\frac{1}{\beta}}}|z|^\delta,$$

where $z=x-(t+\alpha)v$ and $0\leq\delta\leq k-1$. Multiplying the Vlasov equation by ϕ and integrating

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi\partial_t fdvdx+\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi\,\mathrm{div}_x(vf)dvdx+\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi\,\mathrm{div}_v(Ff)dvdx=0.$$

Then integrating by parts over $\mathbb{R}^3\times\mathbb{R}^3$ we obtain

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi\partial_t fdvdx=\frac{d}{dt}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi fdvdx-\iint_{\mathbb{R}^3\times\mathbb{R}^3}\partial_t\phi fdvdx \\ =\frac{d}{dt}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi fdvdx+\frac{1}{t+\alpha}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|x|^2-x\cdot z}{(1+|x|^\beta)^{\frac{1}{\beta}}}|z|^\delta fdvdx \\ +\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{x\cdot z}{(1+|x|^\beta)^{\frac{1}{\beta}}}\frac{\delta v\cdot z}{|z|^{2-\delta}}fdvdx$$

and

$$\begin{aligned}
&\iint_{\mathbb{R}^3\times\mathbb{R}^3} \phi \mathrm{div}_x(vf)dvdx = -\iint_{\mathbb{R}^3\times\mathbb{R}^3} \nabla_x\phi\cdot vf dvdx \\
&= \frac{1}{t+\alpha}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|z|^2-|x|^2}{(1+|x|^\beta)^{\frac{1}{\beta}}}|z|^\delta f dvdx - \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{x\cdot z}{(1+|x|^\beta)^{\frac{1}{\beta}}}\frac{\delta v\cdot z}{|z|^{2-\delta}} f dvdx \\
&\quad + \frac{1}{t+\alpha}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|x|^2(x\cdot z)-(x\cdot z)^2}{(1+|x|^\beta)^{1+\frac{1}{\beta}}}|x|^{\beta-2}|z|^\delta f dvdx.
\end{aligned}$$

On the other hand, we have from (2.3) and $\|f(t)\|_\infty \le \|f_0\|_\infty$ that

$$\begin{aligned}
&\iint_{\mathbb{R}^3\times\mathbb{R}^3} \phi \mathrm{div}_v(Ff)dvdx = -\iint_{\mathbb{R}^3\times\mathbb{R}^3} \nabla_v\phi\cdot Ff dvdx \\
&= (t+\alpha)\left[\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{x\cdot F}{(1+|x|^\beta)^{\frac{1}{\beta}}}|z|^\delta f dvdx + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\delta x\cdot z}{(1+|x|^\beta)^{\frac{1}{\beta}}}\frac{z\cdot F}{|z|}|z|^{\delta-1} f dvdx\right] \\
&\ge -(1+\delta)(t+\alpha)\int_{\mathbb{R}^3}|F|\left(\int_{\mathbb{R}^3}|z|^\delta f dv\right)dx \\
&\ge -(1+\delta)(t+\alpha)\|F(t)\|_{\frac{k+3}{k-\delta}}\left\|\int_{\mathbb{R}^3}|z|^\delta f(t)dv\right\|_{\frac{k+3}{\delta+3}} \\
&\ge -C(1+\delta)(t+\alpha)^{\frac{3+3\delta-2k}{3+k}}\|F(t)\|_{\frac{k+3}{k-\delta}}\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3}|z|^k f(t)dvdx\right)^{\frac{\delta+3}{k+3}}.
\end{aligned}$$

Adding the above equations we have

$$\begin{aligned}
0 &= \iint_{\mathbb{R}^3\times\mathbb{R}^3} \phi\partial_t f dvdx + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \phi \mathrm{div}_x(vf)dvdx + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \phi \mathrm{div}_v(Ff)dvdx \\
&\ge \frac{d}{dt}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi f dvdx + \frac{1}{t+\alpha}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|z|^2-(x\cdot z)}{(1+|x|^\beta)^{\frac{1}{\beta}}}|z|^\delta f dvdx \\
&\quad + \frac{1}{t+\alpha}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|x|^2(x\cdot z)-(x\cdot z)^2}{(1+|x|^\beta)^{1+\frac{1}{\beta}}}|x|^{\beta-2}|z|^\delta f dvdx \\
&\quad - C(1+\delta)(t+\alpha)^{\frac{3+3\delta-2k}{3+k}}\|F(t)\|_{\frac{k+3}{k-\delta}}\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3}|z|^k f dvdx\right)^{\frac{\delta+3}{k+3}} \\
&\ge \frac{d}{dt}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi f dvdx + \frac{1}{t+\alpha}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|z|^2-(x\cdot z)}{(1+|x|^\beta)^{1+\frac{1}{\beta}}}|z|^\delta f dvdx \\
&\quad - C(1+\delta)(t+\alpha)^{\frac{3+3\delta-2k}{3+k}}\|F(t)\|_{\frac{k+3}{k-\delta}}\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3}|z|^k f dvdx\right)^{\frac{\delta+3}{k+3}}.
\end{aligned}$$

Multiplying $t+\alpha$ on both sides of the above inequality, we obtain

$$
\begin{aligned}
&\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|z|^{2+\delta}}{(1+|x|^\beta)^{1+\frac{1}{\beta}}} f\,dv\,dx \\
&\le \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{x\cdot z}{(1+|x|^\beta)^{1+\frac{1}{\beta}}}|z|^\delta f\,dv\,dx-(t+\alpha)\frac{d}{dt}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi f\,dv\,dx \\
&\quad +C(1+\delta)(t+\alpha)^{\frac{6+3\delta-k}{3+k}}\|F(t)\|_{\frac{k+3}{k-\delta}}\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3}|z|^k f\,dv\,dx\right)^{\frac{\delta+3}{k+3}} \\
&\le 2\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi f\,dv\,dx-\frac{d}{dt}\left((t+\alpha)\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi f\,dv\,dx\right) \\
&\quad +C(1+\delta)(t+\alpha)^{\frac{6+3\delta-k}{3+k}}\|F(t)\|_{\frac{k+3}{k-\delta}}\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3}|z|^k f\,dv\,dx\right)^{\frac{\delta+3}{k+3}}. \qquad (2.15)
\end{aligned}
$$

As a consequence, by choosing $\delta=k-1$ we have

$$
\begin{aligned}
&\int_0^T\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|z|^{k+1}}{(1+|x|^\beta)^{1+\frac{1}{\beta}}} f\,dv\,dx\,dt \\
&\le -(T+\alpha)\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi(T,x,v)f(T,x,v)dv\,dx+\alpha\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi(0,x,v)f(0,x,v)dv\,dx \\
&\quad +2\int_0^T\iint_{\mathbb{R}^3\times\mathbb{R}^3}\phi f\,dv\,dx\,dt+Ck\int_0^T(t+\alpha)^{\frac{3+2k}{3+k}}\|F(t)\|_{k+3}\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3}|z|^k f\,dv\,dx\right)^{\frac{k+2}{k+3}}dt \\
&\le (3T+2\alpha)\sup_{t\in[0,T]}H_k(t)+Ck\int_0^T(t+\alpha)^{\frac{3+2k}{3+k}}\|F(t)\|_{k+3}dt\cdot\sup_{t\in[0,T]}H_k(t)^{\frac{2+k}{3+k}},
\end{aligned}
$$

which yields our conclusion. □

Remark 2.4. (1) For the Vlasov–Poisson system (1.1)–(1.3), if $0\le f_0\in L^1\cap L^\infty$ and $H_2(0)<\infty$ with $\alpha>0$, the solution f stated in Corollary 2.3 satisfies

$$
\int_0^T\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|x-(t+\alpha)v|^{\frac{8}{3}}}{(1+|x|^\beta)^{1+\frac{1}{\beta}}} f\,dv\,dx\,dt\le C(T,\alpha,\|f_0\|_{L^1\cap L^\infty},H_2(0)), \qquad (2.16)
$$

where $\beta,T>0$. Actually, from Corollary 2.3 we obtain that for any $t\in[0,T]$, $\|f(t)\|_\infty\le\|f_0\|_\infty$, $\|f(t)\|_1\le\|f_0\|_1$, $H_2(t)\le C$ and $\|E(t)\|_{\frac{15}{4}}\le C\|\rho(t)\|_{\frac{5}{3}}\le C$ by the Hardy-Littlewood-Sobolev inequality (Theorem 4.53 in [11] with $a=3/2$, $q=15/4$, $p=5/3$). Then choosing $k=2$, $\delta=\frac{2}{3}$ in (2.15) and integrating on $[0,T]$, we obtain that (2.16) holds for the approximate solutions and then for f by taking the limits.

(2) Except establishing the a-priori estimate (2.16) for the Vlasov–Poisson system (1.1)–(1.3), Theorem 2.4 will not be used in the rest part of this paper. However, like other velocity moments lemmas, it may have potential applications in some nonlinear kinetic models.

3. Local Existence and Continuation Criterion

Before showing the local existence of classical solutions to the system (1.1)–(1.3), we give some known estimates of self-consistent field $E(t,x)$, which are slightly changed versions of those in [3,10,30] (see e.g. (4.22) (4.23) in [10]). So, we omit their proofs.

Lemma 3.1. *Let $\rho(x) \in L^1 \cap L^\infty(\mathbb{R}^3)$ and let $E(x) = \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3}\rho(y)dy$, then for any $x \in \mathbb{R}^3$ and $R > 0$,*

$$|E(x)| \leq 2\pi R \sup_{y\in B_R(x)} \rho(y) + R^{-2}\|\rho\|_1. \tag{3.1}$$

In particular, set $\pi R\|\rho\|_\infty = R^{-2}\|\rho\|_1$, we have

$$\|E\|_\infty \leq c_0\|\rho\|_1^{1/3}\|\rho\|_\infty^{2/3}. \tag{3.2}$$

If assume further that ρ is Lipschitz, then for any $x \in \mathbb{R}^3$ and $0 < d_1 \leq d_2 < \infty$

$$|\partial_{x_i} E(x)| \leq C[d_2^{-3/p}\|\rho\|_p + d_1 \mathrm{Lip}(\rho) + (1 + \ln(d_2/d_1)) \sup_{y\in B_{d_2}(x)} \rho(y)], \tag{3.3}$$

where $1 \leq p < \infty$. In particular, by choosing $d_2 = 1$ and $d_1 = \frac{1}{1+\mathrm{Lip}(\rho)}$ we have

$$\|\partial_{x_i} E\|_\infty \leq C[(1 + \|\rho\|_\infty)(1 + \ln(1 + \mathrm{Lip}(\rho))) + \|\rho\|_p]. \tag{3.4}$$

Next, we will need the remarkable uniqueness result established by Leoper [18].

Lemma 3.2 [18]. *Let $f_0(x, v)$ be a bounded positive measure. For any $T > 0$, there exists at most one weak solution to the Vlasov–Poisson system (1.1)–(1.3) such that*

$$\rho(t, x) \in L^\infty([0, T] \times \mathbb{R}^3).$$

Now we can establish the local existence to the Vlasov–Poisson system without the assumption of finite kinetic energy, as well as a new continuation criterion: a local solution can be extended as long as its velocity-spatial support is under control.

Theorem 3.3. *Let $0 \leq f_0 \in C_0^1 \cap L^1(\mathbb{R}^3\times\mathbb{R}^3)$ and $\sup\{|x-\alpha v| : (x, v) \in \mathrm{supp} f_0\} < \infty$ for a fixed $\alpha > 0$. Then on some time interval $[0, T)$ there exists a unique classical solution f to the system (1.1)–(1.3). If $T > 0$ is chosen maximal and if*

$$\sup\{|x - (t + \alpha)v| : (x, v) \in \mathrm{supp} f(t),\ t \in [0, T)\} < \infty,$$

then the solution is global, i.e., $T = \infty$.

Proof. Let δ_0 be a positive constant and $L(t)$, $R(t)$ be positive and continuous functions, which will be fixed later. Define

$$\begin{aligned}\mathcal{X} = \{g \in C_b([0, \delta_0] \times \mathbb{R}^3 \times \mathbb{R}^3)\colon & \|g(t)\|_1 = \|f_0\|_1, \|g(t)\|_\infty \leq \|f_0\|_\infty, t \in [0, \delta_0],\\ & g \geq 0,\ \|\mathrm{Lip}_x g(t)\|_{L^\infty(\mathbb{R}^3_v)} \leq L(t)\ \text{ and }\ g(t, x, v) = 0\ \text{ if }\ |v - \frac{x}{t+\alpha}| \geq R(t)\},\end{aligned}$$

which is a closed and bounded convex subset of $C_b([0, \delta_0] \times \mathbb{R}^3 \times \mathbb{R}^3)$. Similarly, denote that

$$E_g(t, x) = \gamma \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3}\rho_g(t, y)dy, \quad \rho_g(t, x) = \int_{\mathbb{R}^3} g(t, x, v)dv.$$

For any $g \in \mathcal{X}$, we know that $\rho_g(t,x) \in C_b([0,\delta_0] \times \mathbb{R}^3)$ is Lipschitz on x, so from the theory of Poisson equation $E_g(t,x)$ is $C([0,\delta_0]; C^1(\mathbb{R}^3))$. Furthermore, using (3.2) and (3.4) with $p = 1$ we can get $E_g(t,x) \in C([0,\delta_0]; C_b^1(\mathbb{R}^3))$. Now the characteristic flow is well defined, that is

$$\begin{cases} \dot{X}(s,t,x,v) = V(s,t,x,v), & X(t,t,x,v) = x, \\ \dot{V}(s,t,x,v) = E_g(s, X(s,t,x,v)), & V(t,t,x,v) = v. \end{cases} \tag{3.5}$$

We consider an operator $\mathcal{T} : g \mapsto f$ with domain $\mathcal{X}$ as follows:

$$f(t,x,v) = f_0(X(0,t,x,v), V(0,t,x,v)).$$

From the definition of characteristic flow, we have

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + E_g(t,x) \cdot \nabla_v f(t,x,v) = 0. \tag{3.6}$$

Step 1. We show that the operator $\mathcal{T}$ maps $\mathcal{X}$ into itself. Remembering the assumption of f_0, we define $R_0 = \sup\{|v - \frac{x}{\alpha}| : (x,v) \in \operatorname{supp} f_0\}$, and then

$$f_0 = 0, \quad |v - \frac{x}{\alpha}| \geq R_0.$$

By the definition of $\mathcal{X}$, we have

$$\|\rho_g(t)\|_{L^\infty} \leq \frac{4\pi}{3} \|f_0\|_\infty R(t)^3, \tag{3.7}$$

which is deduced from

$$\left| v \in \mathbb{R}^3 \text{ such that } |v - \frac{x}{t+\alpha}| \leq R(t) \right| = \frac{4\pi}{3} R(t)^3.$$

We point out that the main new ingredient when adapting the methods in [2,3] is the fact that (3.7) holds not only for the velocity-compactly supported densities but also for the densities stated in this theorem. Combining (3.7) with (3.2), we can get

$$\|E_g(t)\|_\infty \leq c_0 \|\rho_g(t)\|_1^{1/3} \|\rho_g(t)\|_\infty^{2/3} \leq C_0 R^2(t), \tag{3.8}$$

where $C_0 = c_0 \left(\frac{4\pi}{3}\right)^{2/3} \|f_0\|_1^{1/3} \|f_0\|_\infty^{2/3}$. Now we choose $\delta = \frac{1}{C_0 R_0}$ and

$$R(t) = R_0 + C_0 \int_0^t R^2(s) ds,$$

which means that

$$R(t) = \frac{R_0}{1 - C_0 R_0 t}, \quad 0 \leq t < \delta.$$

For $0 \leq s \leq t < \delta$ and $(x,v) \in \operatorname{supp} f_0$, by (2.1) and (3.8) we know that

$$\begin{aligned} |V(s,0,x,v) - \frac{X(s,0,x,v)}{s+\alpha}| &\leq |v - \frac{x}{\alpha}| + \int_0^s \|E_g(\tau)\|_\infty d\tau \\ &\leq R_0 + C_0 \int_0^s R^2(\tau) d\tau \\ &\leq R_0 + C_0 \int_0^t R^2(\tau) d\tau = R(t). \end{aligned}$$

Note that $f(t, X(t,0,x,v), V(t,0,x,v)) = f_0(x,v)$, we obtain that

$$f(t,x,v) = 0 \quad \text{if } |v - \frac{x}{t+\alpha}| \geq R(t).$$

By choosing $0 < \delta_0 < \delta$, we immediately know that for any $t \in [0, \delta_0]$, $R(t) \leq C$ and then $\|E_g(t)\| \leq C$ for the constants C only dependent on $\|f_0\|_1$, $\|f_0\|_\infty$, R_0 and δ_0.

Then we choose a positive and continuous function $L(t)$ on $[0, \delta_0]$ such that

$$\|\text{Lip}_x f(t)\|_{L^\infty(\mathbb{R}^3_v)} \leq L(t) \quad \text{if } \|\text{Lip}_x g(t)\|_{L^\infty(\mathbb{R}^3_v)} \leq L(t).$$

Following from (3.5) we have

$$\begin{cases} \partial_x \dot{X}(s,t,x,v) = \partial_x V(s,t,x,v), & \partial_x X(t,t,x,v) = id, \\ \partial_x \dot{V}(s,t,x,v) = \partial_X E_g(s, X(s)) \partial_x X(s,t,x,v), & \partial_x V(t,t,x,v) = 0 \end{cases}$$

and

$$\begin{cases} \partial_v \dot{X}(s,t,x,v) = \partial_v V(s,t,x,v), & \partial_v X(t,t,x,v) = 0, \\ \partial_v \dot{V}(s,t,x,v) = \partial_X E_g(s, X(s)) \partial_v X(s,t,x,v), & \partial_v V(t,t,x,v) = id. \end{cases}$$

Notice that $(X(s), V(s)) = (X(s,t,x,v), V(s,t,x,v))$, so we have

$$|\partial_{x,v} X(s)| + |\partial_{x,v} V(s)| \leq 1 + \int_s^t (1 + \|\partial_x E_g(\tau)\|_\infty)(|\partial_{x,v} X(\tau)| + |\partial_{x,v} V(\tau)|) d\tau.$$

Gronwall's inequality gives that

$$|\partial_{x,v} X(s)| + |\partial_{x,v} V(s)| \leq \exp\left\{ \int_0^t (1 + \|\partial_x E_g(\tau)\|_\infty) d\tau \right\}. \tag{3.9}$$

Note that

$$\begin{aligned} \text{Lip}_x f(t,\cdot,v) &\leq \sup_{x,v} |\partial_x (f_0(X(0,t,x,v), V(0,t,x,v)))| \\ &\leq C(\|\partial_x X(0,t)\|_\infty + \|\partial_x V(0,t)\|_\infty) \\ &\leq C \exp\left\{ \int_0^t (1 + \|\partial_x E_g(\tau)\|_\infty) d\tau \right\} \end{aligned}$$

and then

$$\ln(1 + \|\text{Lip}_x f(t)\|_{L^\infty(\mathbb{R}^3_v)}) \leq C \int_0^t (1 + \|\partial_x E_g(\tau)\|_\infty) d\tau + C.$$

By (3.4) in Lemma 3.3 and the uniformly boundedness of $\|\rho_g(t)\|_\infty$ on $[0, \delta_0]$, we have

$$\|\partial_x E_g(t)\|_\infty \leq C[1 + \ln(1 + \text{Lip}_x \rho_g(t))] \leq C[1 + \ln(1 + \|\text{Lip}_x g(t)\|_{L^\infty(\mathbb{R}^3_v)})], \tag{3.10}$$

then

$$\ln(1 + \|\text{Lip}_x f(t)\|_{L^\infty(\mathbb{R}^3_v)}) \leq C\left(1 + \int_0^t \ln(1 + \|\text{Lip}_x g(s)\|_{L^\infty(\mathbb{R}^3_v)}) ds\right).$$

So there exists a positive and continuous function $L(t)$ on $[0, \delta_0]$ such that $\|\text{Lip}_x f(t)\|_{L^\infty(\mathbb{R}^3_v)} \leq L(t)$ if $\|\text{Lip}_x g(t)\|_{L^\infty(\mathbb{R}^3_v)} \leq L(t)$. Thus, the operator $\mathcal{T}$ maps $\mathcal{X}$ into itself.

Step 2. Next, we will show that

$$\mathcal{T}\colon \mathcal{X} \mapsto \{f \in C([0,\delta_0]; C_0^1(\mathbb{R}^3\times\mathbb{R}^3)) \cap C^1([0,\delta_0]; C_b^1(\mathbb{R}^3\times\mathbb{R}^3))\colon \forall t \in [0,\delta_0], \\ \|f(t)\|_1 = \|f_0\|_1,\quad \|f(t)\|_\infty \le \|f_0\|_\infty,\quad \|\partial_{x,v} f(t)\|_\infty \le C_1,\quad \text{and} \\ f(t,x,v) = 0 \quad \text{if } |v - \frac{x}{t+\alpha}| \ge R(t)\} \tag{3.11}$$

for the constant C_1 depending only on $\|f_0\|_1$, $\|f_0\|_\infty$, $\|\partial_{x,v} f_0\|_\infty$, R_0 and δ_0.

For any $f \in \mathcal{T}\mathcal{X}$, there exists $g \in \mathcal{X}$ such that $f = \mathcal{T}g$. Firstly, we point out that f belongs to $C^1([0,\delta_0]; C_b^1(\mathbb{R}^3\times\mathbb{R}^3))$ and all the estimates in (3.11) hold. Note that $E_g(t,x) \in C([0,\delta_0]; C_b^1(\mathbb{R}^3))$, we have $X, V \in C^1([0,\delta_0]\times[0,\delta_0]\times\mathbb{R}^3\times\mathbb{R}^3)$, which means that $f \in C^1([0,\delta_0]\times\mathbb{R}^3\times\mathbb{R}^3)$. Remember that $\|\mathrm{Lip}_x f(t)\|_{L^\infty(\mathbb{R}_v^3)} \le L(t)$ for any $t \in [0,\delta_0]$ and $\mathcal{T}\mathcal{X} \subset \mathcal{X}$, we only need to show that $\partial_v f(t,x,v)$ has a uniform bound. From (3.9) and (3.10) we get that $\partial_{x,v}X$, $\partial_{x,v}V$ are uniformly bounded on $[0,\delta_0]\times[0,\delta_0]\times\mathbb{R}^3\times\mathbb{R}^3$, and the bound is only dependent on $\|f_0\|_1$, $\|f_0\|_\infty$, R_0, δ_0. So there exists a positive constant C_1 only dependent on $\|f_0\|_1$, $\|f_0\|_\infty$, $\|\partial_v f_0\|_\infty$, R_0, δ_0 such that

$$\begin{aligned}|\partial_v f(t,x,v)| &= |\partial_v(f_0(X(0,t,x,v), V(0,t,x,v)))| \\ &\le |\partial_x f_0||\partial_v X(0,t,x,v)| + |\partial_v f_0||\partial_v X(0,t,x,v)| \le C_1.\end{aligned}$$

Then, we prove that $f \in C([0,\delta_0]; C_0(\mathbb{R}^3\times\mathbb{R}^3))$. Note that $\sup\{|x-\alpha v|\colon (x,v) \in \operatorname{supp} f_0\} < \infty$, we have

$$\lim_{\sqrt{|x|^2+|v|^2}\to\infty} f_0(x,v) = 0 \Longleftrightarrow \lim_{|v|\to\infty} f_0(x,v) = 0.$$

For any $f \in \mathcal{T}\mathcal{X}$, we know $f(t,x,v) = f_0(X(0,t,x,v), V(0,t,x,v))$ and $|V(0)| \ge |v| - |V(0)-v| \ge |v| - C$ for any $t \in [0,\delta_0]$. So we can deduce that $\lim\limits_{|v|\to\infty} f(t,x,v) = 0$. Using $\sup\{|x-(t+\alpha)v|\colon (x,v) \in \operatorname{supp} f(t),\ t \in [0,\delta_0]\} < \infty$, we can easily obtain that $f(t,x,v) \in C([0,\delta_0]; C_0(\mathbb{R}^3\times\mathbb{R}^3))$. Moreover, for any $\varepsilon > 0$ there exists $R > 0$ independent of f such that

$$|f(t,x,v)| < \varepsilon, \quad \text{if } (t,x,v) \in [0,\delta_0]\times(B_R\times B_R)^c. \tag{3.12}$$

Step 3. We prove that $\mathcal{T}\mathcal{X}$ is relatively compact and the operator $\mathcal{T}$ is continuous. Firstly, we show the relative compactness of $\mathcal{T}\mathcal{X}$. For any $f \in \mathcal{T}\mathcal{X}$ and any $\varepsilon > 0$ we can choose $R > 0$ such that

$$|f(t,x,v)| \le \frac{1}{6}\varepsilon, \quad \text{if } (t,x,v) \in [0,\delta_0]\times(B_R\times B_R)^c$$

by (3.12). For $(t,x,v) \in [0,\delta_0]\times B_R\times B_R$, by (3.6) and the boundedness of $\|\partial_{x,v} f(t)\|_\infty$, we have

$$|\partial_t f(t,x,v)| \le |v\cdot\nabla_x f(t,x,v)| + |E_g(t,x)\cdot\nabla_v f(t,x,v)| \le C(R+1).$$

Thus, $\mathcal{T}\mathcal{X}$ is equicontinuous in $[0,\delta_0]\times B_R\times B_R$ and then $\mathcal{T}\mathcal{X}$ is relatively compact.

Now we show that $\mathcal{T}$ is continuous. Let $\mathcal{T}\tilde{g} = \tilde{f}$, the characteristic flow corresponding to the field $E_{\tilde{g}}$ is defined by

$$\begin{cases} \dot{\tilde{X}}(s,t,x,v) = \tilde{V}(s,t,x,v), & \tilde{X}(t,t,x,v) = x, \\ \dot{\tilde{V}}(s,t,x,v) = E_{\tilde{g}}(s,\tilde{X}(s,t,x,v)), & \tilde{V}(t,t,x,v) = v. \end{cases}$$

Then, for any $0 \le s < t \le \delta_0$

$$
\begin{aligned}
&|\tilde{X}(s) - X(s)| + |\tilde{V}(s) - V(s)| \\
&\le \int_s^t |\tilde{V}(\tau) - V(\tau)| d\tau + \int_s^t |E_{\tilde{g}}(\tau, \tilde{X}(\tau)) - E_g(\tau, X(\tau))| d\tau \\
&\le \int_s^t |\tilde{V}(\tau) - V(\tau)| d\tau + C \int_s^t |\tilde{X}(\tau) - X(\tau)| d\tau + \int_s^t \|E_{(\tilde{g}-g)}(\tau)\|_\infty d\tau \\
&\le C \int_s^t (|\tilde{X}(\tau) - X(\tau)| + |\tilde{V}(\tau) - V(\tau)|) d\tau + C \int_s^t \|(\tilde{g} - g)(\tau)\|_\infty^{2/3} d\tau,
\end{aligned}
$$

which implies that for any $0 \le s < t \le \delta_0$

$$
|\tilde{X}(s) - X(s)| + |\tilde{V}(s) - V(s)| \le C e^{Ct} \int_0^t \|(\tilde{g} - g)(\tau)\|_\infty^{2/3} d\tau.
$$

So, for any $t \in [0, \delta_0]$

$$
\begin{aligned}
&|\tilde{f}(t, x, v) - f(t, x, v)| \\
&\quad = |f_0(\tilde{X}(0, t, x, v), \tilde{V}(0, t, x, v)) - f_0(X(0, t, x, v), V(0, t, x, v))| \\
&\quad \le C(|\tilde{X}(0, t, x, v) - X(0, t, x, v)| + |\tilde{V}(0, t, x, v) - V(0, t, x, v)|) \\
&\quad \le C \int_0^t \|(\tilde{g} - g)(\tau)\|_\infty^{2/3} d\tau \le C \sup_{t \in [0, \delta_0]} \|\tilde{g}(t) - g(t)\|_\infty^{2/3},
\end{aligned}
$$

which gives the continuity of $\mathcal{T}$. Thus, Schauder's theorem ensures that $\mathcal{T}$ has a fixed point f in $\mathcal{X}$. From (3.6) and (3.11) we can deduce that f is a classical solution on $[0, \delta_0]$ to (1.1)–(1.3). The uniqueness follows from Lemma 3.2. Note that all the above arguments hold on any compact subinterval of $[0, \delta)$, so the solution f exists on $[0, \delta)$.
Step 4. Lastly, we prove the continuation criterion. Assume that $[0, T)$ is the maximal existence interval of the unique classical solution of (1.1)–(1.3). Let

$$
R^* = \sup\{|v - \frac{x}{t + \alpha}| : (x, v) \in \operatorname{supp} f(t),\ t \in [0, T)\} < \infty.
$$

Any $t_0 \in (0, T)$ can be viewed as the initial time. From the proof of local existence result especially the computation in step 1, there exists $\delta_1 = \frac{1}{2C_0 R^*}$ such that the solution can be extended to $[0, t_0 + \delta_1]$. Choosing t_0 sufficiently close to T, we can get the contradiction. □

Remark 3.1. If the nonnegative initial datum $f_0 \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$, the local existence and uniqueness result was given in [3,30].

4. Global Existence of Classical Solutions

To prove Theorem 1.1, we can follow the approach in [17]. Specifically, we will propagate velocity-spatial moments of order higher than three.

Theorem 4.1. *Let the initial datum* $0 \leq f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *and assume that for some* $\alpha > 0$ *and some* $m_0 > 3$

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x - \alpha v|^{m_0} f_0(x, v)dvdx < \infty. \tag{4.1}$$

Then there exists a weak solution $f(t, x, v)$ *to* (1.1)–(1.3)*, and for any* $3 < m < m_0$ *there exists a positive and continuous function* $C(t)$ *such that*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x - (t+\alpha)v|^m f(t, x, v)dvdx \leq C(t).$$

Proof. Since the method in [17] is well known, our proof at the first two steps is given roughly.

Step 1. We can deduce from (1.1) and (2.3) that for any $n > 0$

$$\begin{aligned}\frac{d}{dt}H_n(t) &\leq n(t+\alpha)\iint_{\mathbb{R}^3\times\mathbb{R}^3} |E(t, x)||x - (t+\alpha)v|^{n-1} f(t, x, v)dvdx\\ &\leq n(t+\alpha)^{\frac{n}{3+n}}\|E(t)\|_{n+3}H_n(t)^{\frac{2+n}{3+n}},\end{aligned} \tag{4.2}$$

which implies that

$$H_n(t) \leq C\left[H_n(0) + \left(\int_0^t (s+\alpha)^{\frac{n}{3+n}}\|E(s)\|_{n+3}ds\right)^{n+3}\right]. \tag{4.3}$$

Step 2. By (4.3), to propagate the moments $H_m(t)$ we need to estimate $\|E(t)\|_{m+3}$, where $m > 3$. Following from the inequality (28) and its proof in [17], we can get

$$\|E(t)\|_{m+3} \leq C\|\rho_0(t)\|_{\frac{3(m+3)}{m+6}} + C\|\sigma(t)\|_{m+3}, \tag{4.4}$$

where $\rho_0(t, x) = \int_{\mathbb{R}^3} f_0(x - tv, v)dv$ and $\sigma(t, x) = \int_0^t (s-t)\int (Ef)(s, x + (s - t)v, v)dvds$.

Using Hölder's inequality, we obtain

$$\|\rho_0(t)\|_{\frac{3(m+3)}{m+6}} \leq \|\rho_0(t)\|_{\frac{m+3}{3}}^{\frac{2m+3}{3m}}\|\rho_0(t)\|_1^{\frac{m-3}{3m}} \leq C\|\rho_0(t)\|_{\frac{m+3}{3}}^{\frac{2m+3}{3m}}$$

since $m > 3$ and $\|\rho_0(t)\|_1 \leq \|f_0\|_1$. Then, it follows from (2.4) that

$$\begin{aligned}&\left\|\int_{\mathbb{R}^3} f_0(x - tv, v)dv\right\|_{\frac{3+m}{3}}\\ &\leq C(t+\alpha)^{-\frac{3m}{3+m}}\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x - (t+\alpha)v|^m f_0(x - tv, v)dvdx\right)^{\frac{3}{3+m}}\\ &\leq C(t+\alpha)^{-\frac{3m}{3+m}}\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x - \alpha v|^m f_0(x, v)dvdx\right)^{\frac{3}{3+m}}\\ &\leq C(t+\alpha)^{-\frac{3m}{3+m}}H_m(0)^{\frac{3}{3+m}}.\end{aligned}$$

So, we have

$$\|\rho_0(t)\|_{\frac{3(m+3)}{m+6}} \leq C(t+\alpha)^{-\frac{2m+3}{m+3}}. \tag{4.5}$$

For the estimate of $\|\sigma(t)\|_{m+3}$, we divide the integral into two parts,

$$\begin{aligned}\|\sigma(t)\|_{m+3} &= \left\| \int_0^t s \int_{\mathbb{R}^3} (Ef)(t-s, x-sv, v) dv ds \right\|_{m+3} \\ &\leq \left\| \int_0^{t_0} s \int_{\mathbb{R}^3} (Ef)(t-s, x-sv, v) dv ds \right\|_{m+3} \\ &\quad + \left\| \int_{t_0}^t s \int_{\mathbb{R}^3} (Ef)(t-s, x-sv, v) dv ds \right\|_{m+3} \\ &=: I_1 + I_2, \end{aligned} \tag{4.6}$$

where $t_0 \in (0, t)$ will be given later on.

Let $r' < 3$ be close enough to 3 such that $r' \geq \max\{\frac{15}{11}, \frac{3(m+3)}{m_0+3}, 6-m\}$. So $r = \frac{r'}{1-r'} > \frac{3}{2}$, and then Hölder's inequality and the boundedness of f give that

$$\begin{aligned} I_1 &\leq C \left\| \int_0^{t_0} s \left(\int_{\mathbb{R}^3} |E(t-s, x-sv)|^r dv \right)^{1/r} \left(\int_{\mathbb{R}^3} f(t-s, x-sv, v) dv \right)^{1/r'} ds \right\|_{m+3} \\ &\leq C \int_0^{t_0} s^{1-3/r} \|E(t-s)\|_r \left\| \int_{\mathbb{R}^3} f(t-s, x-sv, v) dv \right\|_{\frac{m+3}{r'}}^{\frac{1}{r'}} ds. \end{aligned} \tag{4.7}$$

It follows from the Hardy-Littlewood-Sobolev inequality (Theorem 4.5.3 in [11] with $q = r, a = \frac{3}{2}, p = \frac{3r}{3+r}$) and (2.11) that

$$\|E(t-s)\|_r \leq C \|\rho(t-s)\|_{\frac{3r}{3+r}} \leq C \tag{4.8}$$

since $1 < \frac{3r}{3+r} \leq \frac{5}{3}$. On the other hand, there exists $k \in (m, m_0]$ such that $\frac{k+3}{3} = \frac{m+3}{r'}$, since $\frac{3(m+3)}{m_0+3} \leq r' < 3$. It follows from (2.4), (4.1) and (4.3) that

$$\begin{aligned} &\left\| \int_{\mathbb{R}^3} f(t-s, x-sv, v) dv \right\|_{\frac{m+3}{r'}}^{\frac{1}{r'}} = \left\| \int_{\mathbb{R}^3} f(t-s, x-sv, v) dv \right\|_{\frac{k+3}{3}}^{\frac{1}{r'}} \\ &\leq C(t+\alpha)^{-\frac{3k}{(3+k)r'}} \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |x-(t+\alpha)v|^k f(t-s, x-sv, v) dv dx \right)^{\frac{3}{(3+k)r'}} \\ &= C(t+\alpha)^{-\frac{k}{m+3}} H_k(t-s)^{\frac{1}{m+3}} \\ &\leq C(t+\alpha)^{-\frac{k}{m+3}} \left(1 + \int_0^{t-s} (\tau+\alpha)^{\frac{k}{3+k}} \|E(\tau)\|_{k+3} d\tau \right)^{\frac{3+k}{3+m}}. \end{aligned}$$

Remember the definition of k and $r' \geq 6-m$, there exists $m_1 \in (3, m]$ such that $\frac{3(k+3)}{k+6} = \frac{m_1+3}{3}$. Using the Hardy-Littlewood-Sobolev inequality (Theorem 4.5.3 in [11] with $q = k+3, a = \frac{3}{2}, p = \frac{3(k+3)}{6+k}$) and (2.4) again we have

$$\begin{aligned} \|E(\tau)\|_{k+3} &\leq C \|\rho(\tau)\|_{\frac{3(k+3)}{k+6}} = C \|\rho(\tau)\|_{\frac{m_1+3}{3}} \\ &\leq C(\tau+\alpha)^{-\frac{3m_1}{3+m_1}} H_{m_1}(\tau)^{\frac{3}{3+m_1}} \leq C(\tau+\alpha)^{-\frac{3m_1}{3+m_1}} H_m(\tau)^{\frac{3m_1}{m(3+m_1)}}, \end{aligned}$$

where the last inequality comes from the Hölder inequality and the boundedness of $\|f(\tau)\|_1$. As a consequence, by noting that $\frac{k}{3+k} - \frac{3m_1}{3+m_1} = -1$ we have

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^3} f(t-s, x-sv, v)dv \right\|_{\frac{m+3}{r'}}^{\frac{1}{r'}} \\
&\leq C(t+\alpha)^{-\frac{k}{m+3}} \left(1 + \sup_{\tau\in[0,t-s]} H_m(\tau)^{\frac{3m_1}{m(3+m_1)}} \int_0^{t-s} (\tau+\alpha)^{-1} d\tau \right)^{\frac{3+k}{3+m}} \\
&\leq C(t+\alpha)^{-\frac{k}{m+3}} \left[1 + \left(\ln(1+\frac{t-s}{\alpha}) \right)^{\frac{3+k}{3+m}} \sup_{\tau\in[0,t-s]} H_m(\tau)^{\frac{3+2k}{(3+m)m}} \right]. \qquad (4.9)
\end{aligned}$$

Following from (4.7)–(4.9), we have that

$$\begin{aligned}
I_1 &\leq C t_0^{2-3/r} (t+\alpha)^{-\frac{k}{m+3}} \left(1 + (\ln(1+t/\alpha))^{\frac{3+k}{3+m}} \sup_{s\in[0,t]} H_m(\tau)^{\frac{3+2k}{(3+m)m}} \right) \\
&\leq C t_0^{2-3/r} (t+\alpha)^{-\frac{k}{m+3}} [1+\ln(1+t/\alpha)]^{\frac{3+k}{3+m}} \left(1 + \sup_{s\in[0,t]} H_m(\tau)^{\frac{3+2k}{(3+m)m}} \right). \qquad (4.10)
\end{aligned}$$

For the estimation of I_2, we firstly compute $\int (Ef)(t-s, x-sv, v)dv$. Using Lemma 1.13 in [30] we can obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} |Ef|(t-s, x-sv, v)dv \\
&\leq C\|E(t-s, x-sv)\|_{L_w^{3/2}(\mathbb{R}_v^3)} \left(\int_{\mathbb{R}^3} f(t-s, x-sv, v)dv \right)^{1/3} \|f(t-s)\|_\infty^{2/3} \\
&\leq Cs^{-2}\|E(t-s, x)\|_{L_w^{3/2}(\mathbb{R}_x^3)} \left(\int_{\mathbb{R}^3} f(t-s, x-sv, v)dv \right)^{1/3} \\
&\leq Cs^{-2} \left(\int_{\mathbb{R}^3} f(t-s, x-sv, v)dv \right)^{1/3}, \qquad (4.11)
\end{aligned}$$

since $\|E(t)\|_{L_w^{3/2}(\mathbb{R}^3)} \leq C\|\rho(t)\|_1$ (see Theorem 1 of Section V.1.2 in [35] with $\alpha = 1, n = 3, p = 1$ and $q = \frac{3}{2}$). Here, $L_w^{3/2}(\mathbb{R}^3)$ is the weak $L^{3/2}$-space which is defined as the space of all measurable functions h on $\mathbb{R}^3$ such that $\|h\|_{L_w^{3/2}(\mathbb{R}^3)} := \sup_{\tau>0} \tau|\{x \in \mathbb{R}^3 : |h(x)| > \tau\}|^{2/3} < \infty$. Consequently, by (4.6), (4.11) and (2.4) we have

$$\begin{aligned}
I_2 &\leq C \left\| \int_{t_0}^t s^{-1} \left(\int_{\mathbb{R}^3} f(t-s, x-sv, v)dv \right)^{1/3} ds \right\|_{m+3} \\
&\leq C \int_{t_0}^t s^{-1} \left\| \int_{\mathbb{R}^3} f(t-s, x-sv, v)dv \right\|_{(m+3)/3}^{1/3} ds \\
&\leq C(t+\alpha)^{-\frac{m}{3+m}} \int_{t_0}^t s^{-1} H_m(t-s)^{\frac{1}{3+m}} ds \\
&\leq C(t+\alpha)^{-\frac{m}{3+m}} \ln(t/t_0) \sup_{s\in[0,t]} H_m(s)^{\frac{1}{3+m}}. \qquad (4.12)
\end{aligned}$$

By (4.6) (4.10) and (4.12), we obtain

$$\|\sigma(t)\|_{m+3} \le Ct_0^{2-3/r}(t+\alpha)^{-\frac{k}{m+3}}[1+\ln(1+t/\alpha)]^{\frac{3+k}{3+m}}\left(1+\sup_{s\in[0,t]} H_m(s)^{\frac{3+2k}{(3+m)m}}\right)$$
$$+C(t+\alpha)^{-\frac{m}{3+m}}\ln(t/t_0)\sup_{s\in[0,t]} H_m(s)^{\frac{1}{3+m}}. \tag{4.13}$$

So, it follows from (4.4), (4.5) and (4.13) that

$$\|E(t)\|_{m+3} \le Ct_0^{2-3/r}(t+\alpha)^{-\frac{k}{m+3}}[1+\ln(1+t/\alpha)]^{\frac{3+k}{3+m}}\left(1+\sup_{s\in[0,t]} H_m(s)^{\frac{3+2k}{(3+m)m}}\right)$$
$$+C(t+\alpha)^{-\frac{m}{3+m}}\ln(t/t_0)\sup_{s\in[0,t]} H_m(s)^{\frac{1}{3+m}}+C(t+\alpha)^{-\frac{2m+3}{m+3}}. \tag{4.14}$$

Step 3. Now we estimate $H_m(t)$. Combining the above estimate with (4.2), we have

$$\frac{d}{dt}H_m(t) \le Ct_0^{2-3/r}(t+\alpha)^{\frac{m-k}{m+3}}[1+\ln(1+t/\alpha)]^{\frac{3+k}{3+m}}\left(1+\sup_{s\in[0,t]} H_m(s)^{\frac{3+2k}{(3+m)m}}\right)H_m(t)^{\frac{2+m}{3+m}}$$
$$+C\ln(t/t_0)\sup_{s\in[0,t]} H_m(s)^{\frac{1}{3+m}}H_m(t)^{\frac{2+m}{3+m}}+C(t+\alpha)^{-1}H_m(t)^{\frac{2+m}{3+m}}.$$

For simplicity, $H_m(t)$ will be still used to denote $\sup_{s\in[0,t]} H_m(s)$. By the definition of k, we have $2-\frac{3}{r}=\frac{k-m}{m+3}$, and then

$$\frac{d}{dt}(H_m(t)+1) \le C\left(\frac{t_0}{t+\alpha}\right)^{2-3/r}[1+\ln(1+t/\alpha)]^{\frac{3+k}{3+m}}(H_m(t)+1)^{1+\mu}$$
$$+C\ln(t/t_0)(H_m(t)+1)+C(H_m(t)+1), \tag{4.15}$$

where $\mu=\frac{3+2k}{(3+m)m}-\frac{1}{3+m}>0$. If for any $t\ge 0$

$$3(H_m(t)+1)^{\mu} \le \left(\frac{t}{t+\alpha}\right)^{3/r-2},$$

then let $t\to\infty$, we have $3\le \lim_{t\to\infty} 3(H_m(t)+1)<1$, which is a contradiction. Thus, there exists $t^*>0$ such that

$$3(H_m(t)+1)^{\mu} > \left(\frac{t}{t+\alpha}\right)^{3/r-2}, \quad \forall t>t^*,$$

and

$$3(H_m(t)+1)^{\mu} < \left(\frac{t}{t+\alpha}\right)^{3/r-2}, \quad \forall t<t^*, \tag{4.16}$$

since the left side is increasing on t and the right one is decreasing. So we have

$$3(t+\alpha)^{3/r-2}(H_m(t)+1)^{\mu} > t^{3/r-2}, \quad \forall t>t^*.$$

For any fixed $t>t^*$ we choose $t_0\in(0,t)$ such that

$$t_0^{3/r-2}=3(t+\alpha)^{3/r-2}(H_m(t)+1)^{\mu},$$

which means that

$$t_0 = 3^{-\frac{r}{2r-3}}(H_m(t)+1)^{-\frac{r\mu}{2r-3}}(t+\alpha) < 2^{-1}(t+\alpha),$$

since $3/2 < r \le 15/4$. Then putting this t_0 into (4.15), we have that for any $t \ge t^*$

$$\frac{d}{dt}(H_m(t)+1) \le C\left[(1+\ln(1+t/\alpha))^{\frac{3+k}{3+m}} + \ln(H_m(t)+1)\right](H_m(t)+1). \qquad (4.17)$$

Note that $t^* \le (3^{\frac{r}{2r-3}} - 1)^{-1}\alpha < \alpha$, since

$$3(H_m(t^*)+1)^{\mu} = \left(\frac{t^*}{t^*+\alpha}\right)^{3/r-2} \ge 3.$$

So, it follows from (4.17) that any $t \ge \alpha$,

$$H_m(t) \le (H_m(\alpha)+1)^{e^{C(t-\alpha)}} e^{(e^{C(t-\alpha)}-1)(\ln(1+t/\alpha))^{\frac{3+k}{3+m}}} - 1. \qquad (4.18)$$

Now, we need to show $H_m(\alpha) \le C$. For any $t \in [0,\alpha]$, we have from (4.15) that

$$\frac{d}{dt}(H_m(t)+1) \le Ct_0^{2-3/r}(H_m(\alpha)+1)^{\mu}(H_m(t)+1) + C(|\ln t_0|+1)(H_m(t)+1).$$

By choosing $t_0 \in (0,\alpha)$ such that

$$t_0^{2-\frac{3}{r}} = \alpha^{2-\frac{3}{r}}[1+H_m(\alpha)]^{-\mu},$$

we can obtain that for all $t \in [0,\alpha]$.

$$\frac{d}{dt}(H_m(t)+1) \le C(H_m(t)+1)[1+\ln(H_m(t)+1)].$$

So Gronwall's inequality gives that $H_m(\alpha) \le C$. Combining this estimate with (4.18), we find a positive and continuous function $C(t)$ such that $H_m(t) \le C(t)$ for all $t \ge 0$. □

Remark 4.1. The case of $\alpha = 0$ was considered in [6], where the propagation of spatial moments of order higher than three was obtained with an additional assumption that $\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^{\varepsilon} f_0 dv dx < \infty$ for some arbitrarily small $\varepsilon > 0$. Since the singularity of $t \to 0^+$ is removed in all a-priori estimates for the case of $\alpha > 0$, we can give a more direct result, which is very similar to the one in [17]. However, the upper bound $C(t)$ for the velocity-spatial moments is better than the one for the velocity moments in [17].

Proof of Theorem 1.1. Following from the a-priori estimate (2.7), $\|f(t)\|_1$ and $\|f(t)\|_{\infty}$ are uniformly bounded for any $t \ge 0$. And by (2.4), we have $\|\rho(t)\|_2 \le CH_3(t)^{1/2}$. Then by using Lemma 4.5.4 of [11] with $p = 2$, $a = 3/2$ and the estimate (3.7), we know that

$$\|E(t)\|_{\infty} \le C\|\rho(t)\|_2^{\frac{2}{3}}\|\rho(t)\|_{\infty}^{\frac{1}{3}} \le CH_3(t)^{1/3}R(t),$$

where $R(t)$ is defined by

$$R(t) = \sup\{|v - \frac{x}{t+\alpha}| : (x,v) \in \operatorname{supp} f(t)\}.$$

For $0 \le s \le T$ and $(x, v) \in \operatorname{supp} f_0$, using (2.1) we have

$$|V(t,0,x,v) - \frac{X(t,0,x,v)}{t+\alpha}| \le |v - \frac{x}{\alpha}| + \int_0^t \|E(\tau)\|_\infty d\tau$$
$$\le R(0) + C\int_0^t H_3(\tau)^{1/3} R(\tau) d\tau.$$

Since $f(t, X(t,0,x,v), V(t,0,x,v)) = f_0(x,v)$, we know for any $0 \le t \le T$

$$R(t) \le R(0) + C\int_0^t H_3(\tau)^{1/3} R(\tau) d\tau,$$

which means that

$$R(t) \le R(0) e^{C\int_0^t H_3(s)^{1/3} ds}. \tag{4.19}$$

From the above theorem we have $\sup_{t\in[0,T]} H_m(t) \le C_T$ for some $m > 3$, and by conservation of mass we also have

$$H_3(t) = \left(\iint_{|x-(t+\alpha)v|>1} + \iint_{|x-(t+\alpha)v|\le 1}\right) |x-(t+\alpha)v|^3 f dv dx$$
$$\le \iint_{\mathbb{R}^3\times\mathbb{R}^3} (|x-(t+\alpha)v|^m + 1) f dv dx \le H_m(t) + \|f_0\|_1.$$

So, $\sup_{t\in[0,T]} H_3(t) \le C_T$. Combining it with (4.19), we have $R(t) \le C_T$ for any $0 \le t \le T$. Thus, Theorem 3.3 gives our conclusion. □

5. Propagation of Low Moments

According to the continuation criterion in Theorem 3.3, the key to prove the global existence is controlling the velocity-spatial support. The above method utilizes the fact that such support can be bounded through propagation of high order velocity-spatial moments, which follows the Eulerian approach in [17]. However, a more direct way is the Lagrangian approach due to Pfaffelmoser [28]: fixing a $(x^*, v^*) \in \operatorname{supp} f(t)$ and estimating $\int_0^T |E(s, X(s,t,x^*,v^*))| ds$ by appropriately splitting $(t,x,v) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$ into suitable subsets. Using this method one can get a much better estimate than (4.19) for the velocity-spatial support.

Recently, some remarkable propagation results were given by Pallard [22,23]. He combined the above two approaches to propagate velocity moments of order higher than two. The critical issue is that the upper bound of $\int_0^T |E(s, X(s))| ds$ can be independent of the velocity support of $f(t)$ but depend on $\sup_{t\in[0,T]} \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^{2+0} f(t) dv dx$. And the upper bound was further proved to depend just on T, k, $\|f_0\|_1$, $\|f_0\|_\infty$, $\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f_0 dv dx$ for any $k > 2$. Then spatial moments of order higher than two were also propagated in [23], where a similar but weaker estimate was obtained for the self-generated field, we put both estimates (Proposition 3 in [22], Theorem 4 in [23]) together into the following lemma.

Lemma 5.1 [22,23]. *Suppose given a nonnegative* $f_0 \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$ *and let* $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ *be the unique classical solution to* (1.1)–(1.3)*. Then for any* $k > 2$ *and* $T > 0$

$$\sup_{x,v} \int_0^T |E(s, X(s,t,x,v))| ds \le C(T, k, \|f_0\|_1, \|f_0\|_\infty, \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f_0 dv dx) \tag{5.1}$$

and

$$\sup_{x,v} \int_t^{t+\delta} s|E(s, X(s,t,x,v))| ds \le C(T, \varepsilon, k, \|f_0\|_1, \|f_0\|_\infty, \iint_{\mathbb{R}^3\times\mathbb{R}^3} |x|^k f_0 dv dx)\delta^{\frac{1}{2+\varepsilon}}, \tag{5.2}$$

where $\varepsilon > 0$ *and* $0 \le t \le t + \delta \le T$.

Note that (5.1) is stronger than (5.2) for integrability of $|E(s, X(s,t,x,v))|$. Now, we point out that a similar estimate like (5.1) can be established without boundedness of kinetic energy. Before doing this, we give a new notation for simplicity.

Definition 5.1. we define for any $t \ge 0$ and $\delta > 0$

$$Q(t,\delta) = \sup_{x,v} \int_t^{t+\delta} |E(s, X(s,t,x,v))| ds.$$

It follows from (2.1) that

$$\left| V(s) - \frac{X(s)}{s+\alpha} \right| \le \left| v - \frac{x}{t+\alpha} \right| + Q(t,\delta), \quad \forall s \in (t, t+\delta]. \tag{5.3}$$

In particular,

$$\left| V(t+\delta) - \frac{X(t+\delta)}{t+\delta+\alpha} \right| \le \left| v - \frac{x}{t+\alpha} \right| + Q(t,\delta).$$

Notice that we use the forward characteristics but not the backward characteristics, and $Q(t,\delta) \le Q(t,\delta_0) + Q(t+\delta_0, \delta-\delta_0)$ for $0 < \delta_0 < \delta$.

Theorem 5.2. *Let the initial datum* $0 \le f_0 \in C_0^1 \cap L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ *and assume that*

$$\sup\{|x - \alpha v| : (x,v) \in \operatorname{supp} f_0\} < \infty$$

for some $\alpha > 0$*. Let* f *be the unique classical solution to* (1.1)–(1.3)*, Then for any* $k > 2$ *and* $0 < \varepsilon < 1$ *there exists a positive constant* C *depending on* $M_k(0)$, $\|f_0\|_1$, $\|f_0\|_\infty$, k, α *and* ε *such that for any* $t > 0$

$$Q(0,t) \le Ct^{\frac{1}{2}}(1+t)^{\frac{1}{2}+\varepsilon}. \tag{5.4}$$

Lemma 5.3. *For any* $t \ge 0$, $\delta > 0$ *and* $\epsilon > 0$*, there exists a positive constant* C *depending on* $M_2(0)$, $\|f_0\|_1$, $\|f_0\|_\infty$, α *and* ϵ *such that*

$$Q(t,\delta) \le C\big[\delta Q(t,\delta)^{4/3} + \delta^{1/2}(1 + M_{2+\epsilon}(t))^{1/2}\big]. \tag{5.5}$$

Proof. For any fixed $(t, x^*, v^*) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$, $\delta > 0$, we devote to estimating the following integral:

$$\begin{aligned}\int_t^{t+\delta} |E(s, X(s, t, x^*, v^*))| ds &\le \int_t^{t+\delta} \int_{\mathbb{R}^3} \frac{\rho(s, x)}{|x - X^*(s)|^2} dx ds \\ &= \int_t^{t+\delta} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{f(s, x, v)}{|x - X^*(s)|^2} dv dx ds \\ &=: I\end{aligned}$$

where $(X^*(s), V^*(s)) = (X(s, t, x^*, v^*), V(s, t, x^*, v^*))$. Similar to the decomposition in [14,21–23,32], the integral area is divided into three sets:

$$\begin{aligned}G &= \{(s, x, v) \in (t, t+\delta) \times \mathbb{R}^3 \times \mathbb{R}^3 : |v - V^*(s)| \le P \\ &\quad \text{or } \ |v - x/(s+\alpha)| \le P\}, \\ B &= \{(s, x, v) \in (t, t+\delta) \times \mathbb{R}^3 \times \mathbb{R}^3 : colon|x - X^*(s)| \\ &\quad \le r(s, x, v)\} \backslash G, \\ U &= (t, t+\delta) \times \mathbb{R}^3 \times \mathbb{R}^3 \backslash (G \cup B),\end{aligned}$$

with $P = 5Q(t, \delta)$ and $r(s, x, v) = L\left(1 + \left|v - \frac{x}{s+\alpha}\right|^{2+\epsilon}\right)^{-1} |v - V^*(s)|^{-1}$. When $\alpha = 0$, $M_{2+\epsilon}(t)$ is singular for small times. So $H_{2+\epsilon}(t)$ instead of $M_{2+\epsilon}(t)$ was used to control $Q(t, \delta)$ in [23] and r was given by $r(s, x, v) = L\left|v - \frac{x}{s}\right|^{-2-\epsilon} |v - V^*(s)|^{-1}$ correspondingly, however, the final estimate is also singular for small times (see Proposition 1 in [23]). Since $M_{2+\epsilon}(t)$ is good enough for $\alpha > 0$, we can establish the estimation of $Q(t, \delta)$ paralleled to Proposition 1 in [22]. Now we give precise computation.

Denote

$$\bar{\rho}(s, x) = \int_{|v - V^*(s)| \le P \ \text{or} \ |v - x/(s+\alpha)| \le P} f(s, x, v) dv.$$

Following from (2.11), we have $\|\bar{\rho}(s, \cdot)\|_{5/3} \le C$. And using Lemma 4.5.4 of [11] with $p = 5/3$, $a = 3/2$ we obtain that

$$\begin{aligned}I_G \triangleq \iiint_G \frac{f(s, x, v)}{|x - X^*(s)|^2} dv dx ds &\le \int_t^{t+\delta} \int_{\mathbb{R}^3} \frac{\bar{\rho}(s, x)}{|x - X^*(s)|^2} dx ds \\ &\le C \int_t^{t+\delta} \|\bar{\rho}(s, \cdot)\|_{5/3}^{5/9} \|\bar{\rho}(s, \cdot)\|_\infty^{4/9} ds \\ &\le C P^{4/3} \delta. \end{aligned} \tag{5.6}$$

For $I_B \triangleq \iiint_B \frac{f(s,x,v)}{|x - X^*(s)|^2} dv dx ds$, we decompose B into two parts as that in [23]:

$$B_1 = B \cap \{s, x, v : |v - \frac{x}{s+\alpha}| \ge |v - V^*(s)|\}, \quad B_2 = B_1^c \cap B.$$

Since $r(s, x, v) \le L\left(1 + |v - V^*(s)|^{2+\epsilon}\right)^{-1} |v - V^*(s)|^{-1}$ when $(s, x, v) \in B_1$, by integrating in the space variable first we obtain that

$$\iiint_{B_1} \frac{f(s, x, v)}{|x - X^*(s)|^2} dv dx ds \le C \int_t^{t+\delta} \int_{\mathbb{R}^3} \frac{L}{\left(1 + |v - V^*(s)|^{2+\epsilon}\right)|v - V^*(s)|} dv ds \le C\delta L.$$

On the other hand, using variable substitution $w = v - \frac{x}{s+\alpha}$ and then do a similar computation as above, we have that

$$\iiint_{B_2} \frac{f(s,x,v)}{|x - X^*(s)|^2} dvdxds \le C \int_t^{t+\delta} \int_{\mathbb{R}^3} \frac{L}{(1+|w|^{2+\epsilon})|w|} dwds \le C\delta L.$$

Thus, there exists a positive constant C depending on ϵ such that

$$I_B \le C\delta L. \tag{5.7}$$

Note that the characteristic flow is measure preserving and $f(s, X(s), V(s)) = f(t,x,v)$, we have

$$\iiint_U \frac{f(s,x,v)}{|x - X^*(s)|^2} dvdxds = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left(\int_t^{t+\delta} \frac{1_U(s, X(s), V(s))}{|X(s) - X^*(s)|^2} ds \right) f(t,x,v)dvdx.$$

Using the classical method to deal with the ugly set U (see e.g. the proof of (33) in [23]), we can deduce from (5.3) that

$$\int_t^{t+\delta} \frac{1_U(s, X(s), V(s))}{|X(s) - X^*(s)|^2} ds \le C \frac{1 + |v - \frac{x}{t+\alpha}|^{2+\epsilon}}{L}.$$

As a consequence,

$$I_U \triangleq \iiint_U \frac{f(s,x,v)}{|x - X^*(s)|^2} dvdxds \le CL^{-1}(1 + M_{2+\epsilon}(t)). \tag{5.8}$$

It follows from (5.6)–(5.8) that

$$\int_t^{t+\delta} |E(s, X(s,t,x^*,v^*))|ds \le I \le C\big[\delta Q(t,\delta)^{4/3} + \delta L + L^{-1}(1 + M_{2+\epsilon}(t))\big].$$

By the definition of $Q(t,\delta)$, we have

$$Q(t,\delta) \le C\big[\delta Q(t,\delta)^{4/3} + \delta L + L^{-1}(1 + M_{2+\epsilon}(t))\big].$$

Choose $L = \delta^{-1/2}(1 + M_{2+\epsilon}(t))^{1/2}$, we get the desired result. □

Proof of Theorem 5.2. Following from Lemma 5.3, we can obtain the conclusion by adapting the method used in [22,23], so we only give a very sketchy proof.

Similar to the proof of Proposition 2 in [22], using Lemma 5.3 we can deduce

$$Q(0,t) \le C[t(1 + M_{2+\epsilon}(t))]^{\frac{1}{2}} + Ct(1 + M_{2+\epsilon}(t))^{\frac{4}{7}} \tag{5.9}$$

by the forward characteristics. Note that the characteristic flow is measure preserving and $f(s, X(s), V(s)) = f(t,x,v)$, we get from (5.3) that

$$\begin{aligned} M_k(t) &= \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v - \frac{x}{t+\alpha}|^k f(t,x,v)dvdx \\ &= \iint_{\mathbb{R}^3\times\mathbb{R}^3} |V(t,0,x,v) - \frac{V(t,0,x,v)}{t+\alpha}|^k f(t, X(t,0,x,v), V(t,0,x,v))dvdx \\ &\le \iint_{\mathbb{R}^3\times\mathbb{R}^3} \big(|v - \frac{x}{\alpha}| + Q(0,t)\big)^k f_0(x,v)dvdx \\ &\le 2^k M_k(0) + 2^k \|f_0\|_1 Q(0,t)^k. \end{aligned} \tag{5.10}$$

Then following the proof of Proposition 3 in [22] we can get our conclusion. □

Now we give some corollaries of Theorem 5.2. Using (5.4) and (5.10) we can propagate the corresponding velocity-spatial moments of order higher than two.

Corollary 5.4. *Let the initial datum* $0 \le f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *and assume that for some* $\alpha > 0$ *and some* $k > 2$

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-\alpha v|^k f_0(x,v)dvdx < \infty.$$

Then there exists a weak solution $f(t,x,v)$ *to* (1.1)–(1.3) *such that for some positive and continuous function* $C(t)$

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-(t+\alpha)v|^k f(t,x,v)dvdx \le C(t), \quad t \ge 0.$$

Combining (5.4) with Lemma 3.2 we have a uniqueness result. We refer to [17,22, 23,31] for other uniqueness results.

Corollary 5.5. *Let the initial datum* $0 \le f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *and assume that for some* $\alpha > 0$ *and some* $k > 2$

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-\alpha v|^k f_0(x,v)dvdx < \infty.$$

Moreover, if for some $0 < \varepsilon < 1$

$$\begin{aligned}&\mathrm{supess}\{f_0(y-tv,w) : |y-x| \le Ct^{\frac{3}{2}}(1+t)^{\frac{1}{2}+\varepsilon}, |w-v| \le Ct^{\frac{1}{2}}(1+t)^{\frac{1}{2}+\varepsilon}\}\\ &\quad \in L^\infty_{\mathrm{loc}}([0,\infty); L^\infty(\mathbb{R}^3_x; L^1(\mathbb{R}^3_v))),\end{aligned}$$

where C *is the precise constant in* (5.4)*. Then the weak solution* $f(t,x,v)$ *to* (1.1)–(1.3) *is unique and satisfies for any* $T > 0$

$$\sup_{t\in[0,T]} \|\rho(t)\|_\infty < +\infty.$$

From (5.1) and (5.4) we can simultaneously propagate velocity and velocity-spatial moments.

Corollary 5.6. *Let the initial datum* $0 \le f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *and assume that for some* $\alpha > 0$, $\iint_{\mathbb{R}^3\times\mathbb{R}^3}(|v|^m+|x-\alpha v|^l)f_0 dvdx < \infty$ *with* $m > 2, l > 0$ *or* $m > 0, l > 2$, *then the solution* $f(t,x,v)$ *of* (1.1)–(1.3) *satisfies for any* $t \in [0,T]$

$$\begin{aligned}&\iint_{\mathbb{R}^3\times\mathbb{R}^3}(|v|^m + |x-(t+\alpha)v|^l)f(t,x,v)dvdx\\ &\quad \le C(T, \|f_0\|_1, \|f_0\|_\infty, \|(|v|^m + |x-\alpha v|^l)f_0\|_1, \alpha).\end{aligned}$$

The last corollary almost reaches the natural cases: $l = 2$ or $m = 2$ for $\alpha > 0$, so we focus on the remaining case, $\alpha = 0$. For $m > 3, l > 0$ or $l > 3, m > 0$, these propagation results were proved by Castella [6]. Then with an additional assumption of the initial datum, the propagation was established for the case of $m > 2, l > \frac{1}{3}$ in [9]. As a consequence of (5.1) in [22], the conclusion also holds for the case of $m > 2, l > 0$. So there is an open question stated in [23], that is, how to propagate the velocity ($0 < m < 2$) and spatial ($2 < l \le 3$) moments. At last, we try to give a partial answer of it.

Theorem 5.7. *Let the initial datum* $0 \le f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *satisfy*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} (|v|^m + |x|^l) f_0(x,v)dvdx < +\infty$$

with $2 < l \le 3$ *and* $0 < m < \frac{3l^2}{(l+3)^2} + \frac{l}{l+3}$, $m \le 1$. *Then there exists a weak solution* f *to the Cauchy problem* (1.1)–(1.3) *such that for any* $T > 0$ *and any* $t \in [0, T]$ *we have*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} (|v|^m + |x-vt|^l) f(t,x,v)dvdx \le C(T, \|f_0\|_1, \|f_0\|_\infty, \|(|v|^m + |x|^l) f_0(x,v)\|_1).$$

Proof. Firstly, we use Theorem 1 in [23] to get that

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-tv|^l f(t,x,v)dvdx \le C(T, \|f_0\|_1, \|f_0\|_\infty, \||x|^l f_0(x,v)\|_1).$$

Then we bound $\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^m f dxdv$ for $m \le 1$. From (2.5) we have

$$\begin{aligned}\left|\frac{d}{dt}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^m f(t,x,v)dvdx\right| &\le m\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}|v|^{m-1}fdv\right)|E|dx\\ &\le C\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}fdv\right)^{\frac{2+m}{3}}|E|dx.\end{aligned}$$

Using (2.2) we obtain that for some $0 \le k \le l$ (which will be chosen later)

$$\int_{\mathbb{R}^3} fdv \le Ct^{-\frac{3k}{k+3}}\left(\int_{\mathbb{R}^3}|x-tv|^k fdv\right)^{\frac{3}{k+3}}.$$

So, the above estimates give that

$$\begin{aligned}&\left|\frac{d}{dt}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^m f(t,x,v)dvdx\right|\\ &\le Ct^{-\frac{(2+m)k}{k+3}}\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}|x-tv|^k fdv\right)^{\frac{2+m}{k+3}}|E|dx\\ &\le Ct^{-\frac{(2+m)k}{k+3}}\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3}|x-tv|^k fdvdx\right)^{\frac{2+m}{k+3}}\|E(t)\|_{\frac{\frac{1+k-m}{k+3}}{\frac{k+3}{1+k-m}}}\\ &\le Ct^{-\frac{(2+m)k}{k+3}}\|E(t)\|_{\frac{\frac{1+k-m}{k+3}}{\frac{k+3}{1+k-m}}}.\end{aligned} \tag{5.11}$$

Using the Hardy–Littlewood–Sobolev inequality (Theorem 4.5.3 in [11] with $a = \frac{3}{2}$, $p = \frac{m+3}{3}$, $q = \frac{3(m+3)}{6-m}$) and (2.6) with $l = 0$, $k = m$, we get that

$$\|E(t)\|_{\frac{3(m+3)}{6-m}} \le C\|\rho(t)\|_{\frac{m+3}{3}} \le C\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3}|v|^m fdvdx\right)^{\frac{3}{m+3}}.$$

Similarly, for $2 \le n \le l$

$$\|E(t)\|_{\frac{3(n+3)}{6-n}} \leq C\|\rho(t)\|_{\frac{n+3}{3}} \leq Ct^{-\frac{3n}{n+3}} \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-vt|^n f dv dx \right)^{\frac{3}{n+3}} \leq Ct^{-\frac{3n}{n+3}}.$$

If we further choose k such that $\frac{3(m+3)}{6-m} \leq \frac{k+3}{1+k-m} \leq \frac{3(n+3)}{6-n}$, we have

$$\|E(t)\|_{\frac{k+3}{1+k-m}} \leq \|E(t)\|^{1-\theta}_{\frac{3(m+3)}{6-m}} \|E(t)\|^{\theta}_{\frac{3(n+3)}{6-n}} \leq Ct^{-\frac{3n\theta}{n+3}} \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^m f dv dx \right)^{\frac{3(1-\theta)}{m+3}},$$

where $\theta \in [0, 1]$. Put this estimate into (5.11), we can obtain

$$\left| \frac{d}{dt} \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^m f(t,x,v) dv dx \right| \leq Ct^{-\alpha(m,k)} \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^m f dv dx \right)^{\beta(m,k)}$$

for some $0 \leq \beta(m,k) < 1$ and

$$\alpha(m,k) = \frac{(2+m)k}{k+3} + \frac{3n(1+k-m)}{(n+3)(k+3)}\theta. \tag{5.12}$$

Now we need to choose k and n such that $\alpha(m,k) < 1$ for $0 < m < \frac{3l^2}{(l+3)^2} + \frac{l}{l+3}$.

For $0 < m < \frac{1}{5}$, by choosing $k = 0$ and $n = 2$ we can get $\alpha(m,k) = \frac{2(1-m)(m^2+m+3)}{9(2-m)} < 1$.

For $\frac{1}{5} \leq m \leq \frac{22}{25}$, let $r = \frac{k+3}{1+k-m}$, then from (5.12) we have

$$\alpha(m,k) = m - 1 + \frac{3}{r} + \frac{27n(n-m)}{(n+3)^2(6r - mr - 3m - 9)}.$$

So, we fix $r = \frac{k+3}{1+k-m} = \frac{3(n+3)}{6-n}$, then $\theta = 1$ and we can obtain

$$\alpha(m,k) = m - \frac{3n^2}{(n+3)^2} + \frac{3}{n+3} < 1,$$

where $2 < n < l$.

For $\frac{22}{25} < m < \frac{3l^2}{(l+3)^2} + \frac{l}{l+3}$, from the above arguments we fix $n = l$ and $\frac{k+3}{1+k-m} = \frac{3(l+3)}{6-l}$, then $\alpha(m,k) < 1$. □

Remark 5.2. From the above proof, m can be at least in $(0, \frac{22}{25}]$ for any $l > 2$. And if $l > \frac{1+\sqrt{13}}{2}$, we know that $\frac{3l^2}{(l+3)^2} + \frac{l}{l+3} > 1$, so m can be any point of $(0, 1]$. But if $l \leq 3$, this method can not be used to propagate the velocity moments of $1 < m < 2$.

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