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Variational Formulas and Cocycle solutions for Directed Polymer and Percolation Models

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Abstract: We discuss variational formulas for the law of large numbers limits of certain models of motion in a random medium: namely, the limiting time constant for last-passage percolation and the limiting free energy for directed polymers. The results are valid for models in arbitrary dimension, steps of the admissible paths can be general, the environment process is ergodic under spatial translations, and the potential accumulated along a path can depend on the environment and the next step of the path. The variational formulas come in two types: one minimizes over gradient-like cocycles, and another one maximizes over invariant measures on the space of environments and paths. Minimizing cocycles can be obtained from Busemann functions when these can be proved to exist. The results are illustrated through 1+1 dimensional exactly solvable examples, periodic examples, and polymers in weak disorder.

Contents

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1. Introduction

Existence of limit shapes has been foundational for the study of growth models and percolation type processes. These limits are complicated, often coming from subadditive sequences. Beyond a handful of exactly solvable models, very little information is available about the limit shapes. This article develops and studies variational formulas for the limiting free energies of directed random paths in a random medium, both for positive temperature directed polymer models and for zero-temperature last-passage percolation models. Earlier papers [55] and [57] proved variational formulas for positive temperature directed polymers, without addressing solutions of these formulas. Article [58] gives simpler proofs of some of the results of [57].

The present paper continues the project in two directions:

(i) We extend the variational formulas from positive to zero temperature; that is, we derive variational formulas for the limiting time constants of directed last-passage percolation models.
(ii) We develop an approach for finding minimizers for one type of variational formula in terms of cocycles, for both positive temperature and zero temperature models.

Our paper, and the concurrent and independent work of Krishnan [41,42], are the first to provide general formulas for the limits of first- and last-passage percolation models.

The variational formulas we present come in two types.

(a) One formula minimizes over gradient-like cocycle functions. In the positive temperature case this formula mimics the commonly known min–max formula of the Perron–Frobenius eigenvalue of a nonnegative matrix. In the case of a periodic environment this cocycle variational formula reduces to the min–max formula from linear algebra. The origins of this formula go back to the PhD thesis of Rosenbluth [60]. He adapted homogenization work [39] to deduce a formula of this type for the quenched large deviation rate function for random walk in a random environment.
(b) The second formula maximizes over invariant measures on the space of environments and paths. The positive temperature version of this formula is of the familiar type that gives the dual of entropy as a function of the potential. In zero temperature the entropy disappears and only the expected potential is left, maximized over invariant measures that are absolutely continuous with respect to the background measure. In a periodic environment this zero-temperature formula reduces to the maximal average circuit weight formula of a max-plus eigenvalue.

The next example illustrates the two types of variational formulas for the two-dimensional corner growth model. The notation and the details are made precise in the sequel.

Example 1.1. Let $\Omega = \mathbb{R}^{\mathbb{Z}^2}$ be the space of weight configurations $\omega = (\omega_x)_{x\in\mathbb{Z}^2}$ on the planar integer lattice $\mathbb{Z}^2$, and let $\mathbb{P}$ be an i.i.d. product probability measure on Ω. Assume $\mathbb{E}(|\omega_x|^p) < \infty$ for some $p > 2$. Let $h \in \mathbb{R}^2$ be an external field parameter. The point-to-line last-passage time is defined by

$$G^{\infty}_{0,(n)}(h) = \max_{x_{0,n}:\, x_0=0} \Big\{ \sum_{k=0}^{n-1} \omega_{x_k} + h \cdot x_n \Big\} \tag{1.1}$$

where the maximum is over paths $x_{0,n} = (x_0, \ldots, x_n)$ that begin at the origin $x_0 = 0$ and take directed nearest-neighbor steps $x_k - x_{k-1} \in \{e_1, e_2\}$. There is a law of large numbers

$$g_{\mathrm{pl}}^{\infty}(h) = \lim_{n\to\infty} n^{-1} G_{0,(n)}^{\infty}(h) \quad \mathbb{P}\text{-almost surely, simultaneously} \quad \forall\, h \in \mathbb{R}^2. \tag{1.2}$$

This defines a deterministic convex Lipschitz function $g_{\mathrm{pl}}^{\infty} : \mathbb{R}^2 \to \mathbb{R}$. (The subscript pl is for point-to-line and the superscript ∞ is for zero temperature.) The results to be described give the following two characterizations of the limit.

Theorem 3.2 gives the cocycle variational formula

$$g_{\mathrm{pl}}^{\infty}(h) = \inf_F \,\mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \max_{i=1,2} \big\{\omega_0 + h\cdot e_i + F(\omega, 0, e_i)\big\}. \tag{1.3}$$

The infimum is over centered stationary cocycles F. These are mean-zero functions $F : \Omega \times (\mathbb{Z}^2)^2 \to \mathbb{R}$ that satisfy additivity $F(\omega, x, y) + F(\omega, y, z) = F(\omega, x, z)$ and stationarity $F(T_z\omega, x, y) = F(\omega, z + x, z + y)$ (Definition 3.1).

The second formula is over measures and comes as a special case of Theorem 7.2:

$$g_{\mathrm{pl}}^{\infty}(h) = \sup\big\{E^{\mu}[\omega_0 + h\cdot z] : \mu \in \mathcal{M}_s(\Omega \times \{e_1, e_2\}),\ \mu|_{\Omega} \ll \mathbb{P},\ E^{\mu}[\omega_0^-] < \infty\big\}. \tag{1.4}$$

The supremum is over probability measures μ on pairs $(\omega, z) \in \Omega \times \{e_1, e_2\}$ that are invariant in a natural way (described in Proposition 7.1) and whose Ω-marginal is absolutely continuous with respect to the environment distribution $\mathbb{P}$. E^{μ} denotes expectation under μ.

As we will see, these formulas are valid quite generally in all dimensions, for general walks, ergodic environments, and more complicated potentials, provided certain moment assumptions are satisfied.

In addition to deriving the formulas, we develop a solution approach for the cocycle formula in terms of stationary cocycles suitably adapted to the potential. Such cocycles can be obtained from limits of gradients of free energies and last-passage times. These limits are called *Busemann functions*. Their existence is in general a nontrivial problem. Along the way we show that, once Busemann functions exist as almost sure limits, their integrability follows from the L^1 shape theorem, which a priori is a much cruder result.

Over the last two decades Busemann functions have become an important tool in the study of the geometry of percolation and invariant distributions of related particle systems. Study of Busemann functions is also motivated by fluctuation questions. One approach to quantifying fluctuations of free energy and the paths goes through control of fluctuations of Busemann functions. In 1+1 dimension these models are expected to lie in the Kardar–Parisi–Zhang (KPZ) universality class and there are well-supported conjectures for universal fluctuation exponents and limit distributions. Some of these conjectures have been verified for a handful of exactly solvable models. (See surveys [16,52,68,70].) In dimensions 3+1 and higher, high temperature behavior of directed polymers has been proved to be diffusive [15], but otherwise conjectures beyond 1+1 dimension are murky.

To summarize, the purpose of this paper is to develop the variational formulas, illustrate them with examples, and set an agenda for future study with the Busemann solution. We show how the formulas work in weak disorder, in exactly solvable 1+1 dimensional models, and in periodic environments. Applications that go beyond these cases cannot be covered within the scope of this paper and will follow in future work.

Minimizing cocycles for (1.3) have been constructed for the two-dimensional corner growth model with general i.i.d. weights in [27]. In the sequel [26] these cocycles are used to construct geodesics and to prove existence, uniqueness and coalescence properties of directional geodesics and to study the competition interface. In another direction of work on these formulas, article [58] proves the cocycle variational formula for the annealed free energy of a directed polymer and uses it to characterize the so-called weak disorder phase of the model.

Overview of related literature. Independently of the present work and with a different methodology, Krishnan [41,42] proves a variational formula for undirected first passage bond percolation with bounded ergodic weights. Taking an optimal control approach, he embeds the lattice problem into $\mathbb{R}^d$ and applies the recent stochastic homogenization results of Lions and Souganidis [45] to derive a variational formula. The resulting formula is a first passage percolation version of our formula (3.8). The homogenization parallel of our work is [39,40] rather than [2,45]. The quantity homogenized corresponds in our world to the finite-volume free energy.

We run through a selection of highlights from past study of limiting shapes and free energies. For directed polymers Vargas [72] proved the a.s. existence of the limiting free energy under moment assumptions similar to the ones we use. Earlier proofs with stronger assumptions appeared in [8,13]. In weak disorder the limiting polymer free energy is the same as the annealed one. In strong disorder no general formulas appeared in the literature before [55,57]. Carmona and Hu [8] gave some bounds in the Gaussian case. Lacoin [43] gave small-β asymptotics in dimensions $d = 1, 2$. The earliest explicit free energy for an exactly solvable directed polymer model is the calculation in [49] for the semi-discrete polymer in a Brownian environment. Explicit limits for the exactly solvable log-gamma polymer appear in [28,66].

The study of Lyapunov exponents and large deviations for random walks in random environments is a related direction of literature. [71,73] are two early papers in the multidimensional setting.

A seminal paper in the study of directed last-passage percolation is Rost 1981 [61]. He deduced the limit shape of the corner growth model with exponential weights in conjunction with a hydrodynamic limit for TASEP (totally asymmetric simple exclusion process) with the step initial condition. However, the last passage representation of this model was discovered only later. The study of directed last-passage percolation bloomed in the 1990s, with the first shape results for exactly solvable cases in [1,12,35,64,65]. Early motivation for [1] came from Hammersley [30]. The breakthroughs of [5,36] transformed the study of exactly solvable last-passage models and led to the first rigorous KPZ fluctuation results. The only universal shape result is the asymptotic result on the boundary of $\mathbb{R}^2_+$ for the corner growth model by Martin [47].

In undirected first passage percolation the fundamental shape theorem is due to Cox and Durrett [17]. A classic in the field is the flat edge result of Durrett and Liggett [22]. Marchand [46] sharpened this result and Auffinger and Damron [3] built on it to prove differentiability of the shape at the edge of the percolation cone.

Busemann functions came on the percolation scene in the work of Newman et al. [34,44,50]. Busemann functions were shown to exist as almost sure limits of passage time gradients as a consequence of uniqueness and coalescence of infinite directional geodesics, under uniform curvature assumptions on the limit shape. These assumptions were relaxed through a weak convergence approach of Damron and Hanson [18]. Busemann functions have been used to study competition in percolation models and properties of particle systems and randomly driven equations. For a selection of the literature, see [6,9–11,23,24,32,33,51].

Organization of the paper. Section 2 defines the models and states the existence theorems for the limiting free energies whose description is the purpose of the paper.

Section 3 derives the cocycle variational formula for the point-to-level case and develops an approach for solving these formulas.

Section 4 extends this to point-to-point free energy via a duality between tilt and velocity.

Section 5 demonstrates how minimizing cocycles arise from Busemann functions.

Section 6 explains how the theory of the paper works in explicitly solvable 1+1 dimensional models, namely the log-gamma polymer and the corner growth model with exponential weights.

Section 7 develops variational formulas in terms of measures. In the positive temperature case these formulas involve relative entropy.

Section 8 illustrates the results of the paper for periodic environments where our variational formulas become elements of Perron–Frobenius theory.

Notation and conventions. We collect here some items for later reference. $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, $\mathbb{R}_+ = [0, \infty)$. $|x| = (\sum_i |x_i|^2)^{1/2}$ denotes Euclidean norm. The standard basis vectors of $\mathbb{R}^d$ are $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, $\dots$, $e_d = (0, \dots, 0, 1)$. $\mathcal{M}_1(\mathcal{X})$ denotes the space of Borel probability measures on a space $\mathcal{X}$ and $b\mathcal{X}$ the space of bounded Borel functions $f : \mathcal{X} \to \mathbb{R}$. $\mathbb{P}$ is a probability measure on environments ω, with expectation operation $\mathbb{E}$. Expectation with respect to ω of a multivariate function $F(\omega, x, y)$ can be expressed as $\mathbb{E}F(\omega, x, y) = \mathbb{E}F(x, y) = \int F(\omega, x, y)\, \mathbb{P}(d\omega)$.

2. Free Energy in Positive and Zero Temperature

In this section we describe the setting and state the limit theorems for free energy and last-passage percolation. The positive temperature limits are quoted from past work and then extended to last-passage percolation via a zero-temperature limit.

Fix the dimension $d \in \mathbb{N}$. Let $p : \mathbb{Z}^d \to [0, 1]$ be a random walk probability kernel: $\sum_{z \in \mathbb{Z}^d} p(z) = 1$. Assume p has finite support $\mathcal{R} = \{z \in \mathbb{Z}^d : p(z) > 0\}$. $\mathcal{R}$ must contain at least one nonzero point, and $\mathcal{R}$ may contain 0. A path $x_{0,n} = (x_k)_{k=0}^n$ in $\mathbb{Z}^d$ is *admissible* if its steps satisfy $z_k \equiv x_k - x_{k-1} \in \mathcal{R}$. The probability of an admissible path from a fixed initial point x_0 is $p(x_{0,n}) = p(z_{1,n}) = \prod_{i=1}^n p(z_i)$. Let $\delta = \min_{z \in \mathcal{R}} p(z) > 0$.

$\mathcal{R}$ generates the additive subgroup $\mathcal{G} = \{\sum_{z \in \mathcal{R}} a_z z : a_z \in \mathbb{Z}\}$ of $\mathbb{Z}^d$. $\mathcal{G}$ is isomorphic to some $\mathbb{Z}^k$ (Prop. P1 on p. 65 in [67]). $\mathcal{U}$ is the convex hull of $\mathcal{R}$ in $\mathbb{R}^d$, and $\mathrm{ri}\,\mathcal{U}$ the relative interior of $\mathcal{U}$. The common affine hull of $\mathcal{R}$ and $\mathcal{U}$ is denoted by $\mathrm{aff}\,\mathcal{R} = \mathrm{aff}\,\mathcal{U}$.

An *environment* ω is a sample point from a Polish probability space $(\Omega, \mathfrak{S}, \mathbb{P})$ where $\mathfrak{S}$ is the Borel σ-algebra of Ω. Ω comes equipped with a group $\{T_x : x \in \mathcal{G}\}$ of measurable commuting bijections that satisfy $T_{x+y} = T_x T_y$ and T_0 is the identity. $\mathbb{P}$ is a $\{T_x\}_{x \in \mathcal{G}}$-invariant probability measure on $(\Omega, \mathfrak{S})$. This is summarized by the statement that $(\Omega, \mathfrak{S}, \mathbb{P}, \{T_x\}_{x \in \mathcal{G}})$ is a measurable dynamical system. We assume $\mathbb{P}$ *ergodic*. As usual this means that $\mathbb{P}(A) = 0$ or 1 for all events $A \in \mathfrak{S}$ that satisfy $T_z^{-1} A = A$ for all $z \in \mathcal{R}$. Occasionally we make stronger assumptions on $\mathbb{P}$. $\mathbb{E}$ denotes expectation under $\mathbb{P}$.

A *potential* is a measurable function $V : \Omega \times \mathcal{R}^\ell \to \mathbb{R}$ for some $\ell \in \mathbb{Z}_+$, denoted by $V(\omega, z_{1,\ell})$ for an environment ω and a vector of admissible steps $z_{1,\ell} = (z_1, \dots, z_\ell) \in \mathcal{R}^\ell$. The case $\ell = 0$ corresponds to a potential $V : \Omega \to \mathbb{R}$ that is a function of ω alone.

The variational formulas from [55] and [57] that this article relies upon were proved under the following assumption on V.

Definition 2.1 (*Class $\mathcal{L}$*). A function $V : \Omega \times \mathcal{R}^\ell \to \mathbb{R}$ is in class $\mathcal{L}$ if for every $\tilde{z}_{1,\ell} = (\tilde{z}_1, \dots, \tilde{z}_\ell) \in \mathcal{R}^\ell$ and for every nonzero $z \in \mathcal{R}$, $V(\,\cdot\,, \tilde{z}_{1,\ell}) \in L^1(\mathbb{P})$ and

$$\varlimsup_{\varepsilon \searrow 0} \varlimsup_{n\to\infty} \max_{x \in \mathcal{G}:|x|\le n} \frac{1}{n} \sum_{0\le k\le \varepsilon n} |V(T_{x+kz}\omega, \tilde{z}_{1,\ell})| = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega. \tag{2.1}$$

Membership $V \in \mathcal{L}$ depends on a combination of mixing of $\mathbb{P}$ and moments of V. See Lemma A.4 of [57] for a precise statement. Boundedness of V is of course sufficient.

Remark 2.2 (*Canonical settings*). Often the natural choice for Ω is a product space $\Omega = \mathcal{S}^{\mathbb{Z}^d}$ with a Polish space $\mathcal{S}$, product topology, and Borel σ-algebra $\mathfrak{S}$. A generic point of Ω is then denoted by $\omega = (\omega_x)_{x\in\mathbb{Z}^d}$. The mappings are shifts $(T_x\omega)_y = \omega_{x+y}$. For example, random weights assigned to the vertices of $\mathbb{Z}^d$ would be modeled by $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ and $V(\omega) = \omega_0$. In fact, it would be sufficient to take $\Omega = \mathbb{R}^{\mathcal{G}}$ since the coordinates outside $\mathcal{G}$ are not needed as long as paths begin at points in $\mathcal{G}$.

To represent directed edge weights we can take $\Omega = \mathcal{S}^{\mathcal{G}}$ with $\mathcal{S} = \mathbb{R}^{\mathcal{R}}$ where an element $s \in \mathcal{S}$ represents the weights of the admissible edges out of the origin: $s = (\omega_{(0,z)} : z \in \mathcal{R})$. Then $\omega_x = (\omega_{(x,x+z)} : z \in \mathcal{R})$ is the vector of edge weights out of vertex x. Shifts act by $(T_u\omega)_{(x,y)} = \omega_{(x+u,y+u)}$ for $u \in \mathcal{G}$. The potential is $V(\omega, z) = \omega_{(0,z)} =$ the weight of the edge $(0, z)$.

To have weights on undirected nearest-neighbor edges take $\Omega = \mathbb{R}^{\mathcal{E}}$ where $\mathcal{E} = \{\{x, y\} \subset \mathbb{Z}^d : |y - x| = 1\}$ is the set of undirected nearest-neighbor edges on $\mathbb{Z}^d$. Now $\mathcal{R} = \{\pm e_i : i = 1, \dots, d\}$, $V(\omega, z) = \omega_{\{0,z\}}$ and $(T_u\omega)_{\{x,y\}} = \omega_{\{x+u,y+u\}}$ for $u \in \mathbb{Z}^d$.

$\mathbb{P}$ is an *i.i.d.* or *product measure* if the coordinates $\{\omega_x\}_{x\in\mathbb{Z}^d}$ (or $\{\omega_x\}_{x\in\mathcal{G}}$ or $\{\omega_e\}_{e\in\mathcal{E}}$) are independent and identically distributed (i.i.d.) random variables under $\mathbb{P}$. With an i.i.d. $\mathbb{P}$ and *local* V (that is, V depends on only finitely many coordinates of ω), for $V \in \mathcal{L}$ it suffices to assume $V(\cdot\,, z_{1,\ell}) \in L^p(\mathbb{P})$ for some $p > d$ and all $z_{1,\ell} \in \mathcal{R}^\ell$.

For inverse temperature parameter $0 < \beta < \infty$ define the n-step *quenched partition function*

$$Z^\beta_{0,(n)} = \sum_{x_{0,n+\ell-1}:\, x_0=0} p(x_{0,n+\ell-1})\, e^{\beta \sum_{k=0}^{n-1} V(T_{x_k}\omega,\, z_{k+1,k+\ell})}. \tag{2.2}$$

The sum is over admissible $(n + \ell - 1)$-step paths $x_{0,n+\ell-1}$ that start at $x_0 = 0$. The second argument of V is the ℓ-vector $z_{k+1,k+\ell} = (z_{k+1}, z_{k+2}, \dots, z_{k+\ell})$ of steps, and it is not present if $\ell = 0$. The corresponding free energy is defined by

$$G^\beta_{0,(n)} = \beta^{-1} \log Z^\beta_{0,(n)}. \tag{2.3}$$

In the $\beta \to \infty$ limit this turns into the n-step *last-passage time*

$$G^\infty_{0,(n)} = \max_{x_{0,n+\ell-1}:\, x_0=0} \sum_{k=0}^{n-1} V(T_{x_k}\omega,\ z_{k+1,k+\ell}). \tag{2.4}$$

As in the definitions above we shall consistently use the subscript (n) with parentheses to indicate number of steps.

In the most basic situation where $d = 2$ and $\mathcal{R} = \{e_1, e_2\}$ the quantity $G^{\infty}_{0,(n)}$ is a *point-to-line* last-passage value because admissible paths $x_{0,n}$ go from 0 to the line $\{(i, j) : i + j = n\}$. We shall call the general case (2.3)–(2.4) *point-to-level*.

The n-step *quenched point-to-point partition function* is for $x \in \mathbb{Z}^d$

$$Z^{\beta}_{0,(n),x} = \sum_{x_{0,n+\ell-1}:\, x_0=0,\, x_n=x} p(x_{0,n+\ell-1})\, e^{\beta \sum_{k=0}^{n-1} V(T_{x_k}\omega,\, z_{k+1,k+\ell})} \tag{2.5}$$

with free energy

$$G^{\beta}_{0,(n),x} = \beta^{-1} \log Z^{\beta}_{0,(n),x}.$$

Its zero-temperature limit is the n-step *point-to-point last-passage time*

$$G^{\infty}_{0,(n),x} = \max_{x_{0,n+\ell-1}:\, x_0=0,\, x_n=x} \sum_{k=0}^{n-1} V(T_{x_k}\omega,\ z_{k+1,k+\ell}). \tag{2.6}$$

Remark 2.3. The formulas for limits presented in this paper are for the case where the length of the path is restricted, as in (2.5) and (2.6), so that only those paths that reach x from 0 in exactly n steps are considered. This is indicated by the subscript (n). Extension to paths of unrestricted length from 0 to x or from 0 to a hyperplane is left for future work. In the most-studied directed models this restriction can be dropped because each path between two given points has the same number of steps. Examples where this is the case are $\mathcal{R} = \{e_1, \dots, e_d\}$ and $\mathcal{R} = \{(z', 1) : z' \in \mathcal{R}'\}$ for a finite subset $\mathcal{R}' \subset \mathbb{Z}^{d-1}$.

To take limits of point-to-point quantities we specify lattice points $\hat{x}_n(\xi)$ that approximate $n\xi$ for $\xi \in \mathcal{U}$. For each point $\xi \in \mathcal{U}$ fix weights $\alpha_z(\xi) \in [0, 1]$ such that $\sum_{z\in\mathcal{R}} \alpha_z(\xi) = 1$ and $\xi = \sum_{z\in\mathcal{R}} \alpha_z(\xi) z$. Then define a path

$$\hat{x}_n(\xi) = \sum_{z\in\mathcal{R}} \big(\lfloor n\alpha_z(\xi)\rfloor + b^{(n)}_z(\xi)\big) z, \quad n \in \mathbb{Z}_+, \tag{2.7}$$

where $b^{(n)}_z(\xi) \in \{0, 1\}$ are arbitrary but subject to these constraints: if $\alpha_z(\xi) = 0$ then $b^{(n)}_z(\xi) = 0$, and $\sum_{z\in\mathcal{R}} b^{(n)}_z(\xi) = n - \sum_{z\in\mathcal{R}} \lfloor n\alpha_z(\xi)\rfloor$. In other words, $\hat{x}_n(\xi)$ is a lattice point that approximates $n\xi$ to within a constant independent of n, can be reached in n $\mathcal{R}$-steps from the origin, and uses only those steps that appear in the pre-specified convex representation $\xi = \sum_z \alpha_z z$. When $\xi \in \mathcal{U} \cap \mathbb{Q}^d$ we require that $\alpha_z(\xi)$ be rational. This is possible by Lemma A.1 of [57].

The next theorem defines the limits whose study is the purpose of the paper. We state it so that it covers simultaneously both the positive temperature ($0 < \beta < \infty$) and the zero-temperature case (last-passage percolation, or $\beta = \infty$). The subscripts are pl for point-to-level and pp for point-to-point.

Theorem 2.4. *Let $V \in \mathcal{L}$ and assume $\mathbb{P}$ ergodic. Let $\beta \in (0, \infty]$.*

(a) *The nonrandom limit*

$$g^{\beta}_{\mathrm{pl}} = \lim_{n\to\infty} n^{-1} G^{\beta}_{0,(n)} \tag{2.8}$$

exists $\mathbb{P}$-a.s. in $(-\infty, \infty]$.

(b) *There exists an event* Ω_0 *with* $\mathbb{P}(\Omega_0) = 1$ *such that the following holds for all* $\omega \in \Omega_0$. *For all* $\xi \in \mathcal{U}$ *and any choices made in the definition of* $\hat{x}_n(\xi)$ *in (2.7), the limit*

$$g^{\beta}_{\mathrm{pp}}(\xi) = \lim_{n\to\infty} n^{-1} G^{\beta}_{0,(n),\hat{x}_n(\xi)} \tag{2.9}$$

exists in $(-\infty, \infty]$. *For a particular* ξ *the limit is independent of the choice of convex representation* $\xi = \sum_z \alpha_z(\xi) z$ *and the numbers* $b^{(n)}_z(\xi)$ *that define* $\hat{x}_n(\xi)$ *in (2.7). We have the almost sure identity*

$$g^{\beta}_{\mathrm{pl}} = \sup_{\xi\in\mathbb{Q}^d\cap\mathcal{U}} g^{\beta}_{\mathrm{pp}}(\xi) = \sup_{\xi\in\mathcal{U}} g^{\beta}_{\mathrm{pp}}(\xi). \tag{2.10}$$

Proof. The case $0 < \beta < \infty$ is covered by Theorem 2.2 of [55]. (The kernel there is the uniform one $p(z) = |\mathcal{R}|^{-1}$ but this makes no difference to the arguments. Alternatively, the kernel can be moved into the potential.)

For any $0 < \beta < \infty$,

$$\begin{aligned} G^{\infty}_{0,(n)} + \beta^{-1}(n+\ell-1)\log\delta \ &\le\ \beta^{-1}\log Z^{\beta}_{0,(n)} \ \le\ G^{\infty}_{0,(n)} \\ \text{and}\quad G^{\infty}_{0,(n),x} + \beta^{-1}(n+\ell-1)\log\delta \ &\le\ \beta^{-1}\log Z^{\beta}_{0,(n),x} \ \le\ G^{\infty}_{0,(n),x}. \end{aligned}$$

Divide by n, let first $n \to \infty$ and then $\beta \to \infty$. This gives the existence of the limits for the case $\beta = \infty$. We also get these bounds, uniformly in ω and $\xi \in \mathcal{U}$:

$$\begin{aligned} g^{\infty}_{\mathrm{pl}} + \beta^{-1}\log\delta \le g^{\beta}_{\mathrm{pl}} \ &\le\ g^{\infty}_{\mathrm{pl}} \\ \text{and}\quad g^{\infty}_{\mathrm{pp}}(\xi) + \beta^{-1}\log\delta \le g^{\beta}_{\mathrm{pp}}(\xi) \ &\le\ g^{\infty}_{\mathrm{pp}}(\xi). \end{aligned} \tag{2.11}$$

These bounds extend (2.10) from $0 < \beta < \infty$ to $\beta = \infty$. □

Since our hypotheses are fairly general, we need to address the randomness, finiteness, and regularity of the limits. For $0 < \beta < \infty$ the remarks below repeat claims proved in [55]. The properties extend to $\beta = \infty$ by way of bounds (2.11) as $\beta \to \infty$.

Remark 2.5 ($\mathbb{P}$ *ergodic*). If we only assume $\mathbb{P}$ ergodic and place no further restrictions on admissible paths then we need to begin by assuming that $g^{\beta}_{\mathrm{pl}} \in \mathbb{R}$. An obvious way to guarantee this would be to assume that V is bounded above (in addition to what is assumed to have $V \in \mathcal{L}$). Under the assumption $g^{\beta}_{\mathrm{pl}} \in \mathbb{R}$ the point-to-point limit $g^{\beta}_{\mathrm{pp}}(\xi)$ is a nonrandom, real-valued, concave and continuous function on the relative interior $\mathrm{ri}\,\mathcal{U}$. Boundary values $g^{\beta}_{\mathrm{pp}}(\xi)$ for $\xi \in \mathcal{U}\smallsetminus\mathrm{ri}\,\mathcal{U}$ can be random, but on the whole of $\mathcal{U}$, for $\mathbb{P}$-a.e. ω, the (possibly random) function $\xi \mapsto g^{\beta}_{\mathrm{pp}}(\xi;\omega)$ is lower semicontinuous and bounded. The upper semicontinuous regularization of g^{β}_{pp} and its unique continuous extension from $\mathrm{ri}\,\mathcal{U}$ to $\mathcal{U}$ are equal and nonrandom.

Remark 2.6 (*Directed i.i.d.* $L^{d+\varepsilon}$ *case*). Assume the canonical setting from Remark 2.2: Ω is a product space, $\mathbb{P}$ is i.i.d., V is local, and $\mathbb{E}[|V(\omega, z_{1,\ell})|^p] < \infty$ for some $p > d$ and $\forall z_{1,\ell} \in \mathcal{R}^{\ell}$. Assume additionally that $0 \notin \mathcal{U}$. We call this the *directed i.i.d.* $L^{d+\varepsilon}$ *case*. Then $V \in \mathcal{L}$, $g^{\beta}_{\mathrm{pl}} \in \mathbb{R}$, and the point-to-point limit $g^{\beta}_{\mathrm{pp}}(\xi)$ is a nonrandom, real-valued, concave and continuous function on all of $\mathcal{U}$ (Theorem 3.2(a) of [55]).

3. Cocycle Variational Formula for the Point-to-Level Case

In Sects. 3–5 we study potentials of the form

$$V(\omega, z) = V_0(\omega, z) + h \cdot z, \quad (\omega, z) \in \Omega \times \mathcal{R} \tag{3.1}$$

for a measurable function $V_0 : \Omega \times \mathcal{R} \to \mathbb{R}$ and a vector $h \in \mathbb{R}^d$. We think of V_0 as fixed and h as a variable and hence amend our notation as follows. As before the steps of admissible paths are $z_k = x_k - x_{k-1} \in \mathcal{R}$.

$$G^{\beta}_{0,(n)}(h) = \beta^{-1} \log \sum_{x_{0,n}:\, x_0=0} p(x_{0,n})\, e^{\beta \sum_{k=0}^{n-1} V_0(T_{x_k}\omega,\, z_{k+1}) + \beta h\cdot x_n} \tag{3.2}$$

for $0 < \beta < \infty$,

$$G^{\infty}_{0,(n)}(h) = \max_{x_{0,n}:\, x_0=0} \Big\{ \sum_{k=0}^{n-1} V_0(T_{x_k}\omega, z_{k+1}) + h \cdot x_n \Big\}, \tag{3.3}$$

and

$$g^{\beta}_{\mathrm{pl}}(h) = \lim_{n\to\infty} n^{-1} G^{\beta}_{0,(n)}(h) \quad \text{a.s. for all } \ 0 < \beta \le \infty. \tag{3.4}$$

Limit (3.4) is a special case of (2.8).

By (2.11), if $g^{\beta}_{\mathrm{pl}}(0)$ is finite for one $\beta \in (0, \infty]$, it is finite for all $\beta \in (0, \infty]$. This can be guaranteed by assuming V_0 bounded above, or by the directed i.i.d. $L^{d+\varepsilon}$ assumption of Remark 2.6, or by some other case-specific assumption. If $g^{\beta}_{\mathrm{pl}}(0)$ is finite, it is clear from the expressions above that $g^{\beta}_{\mathrm{pl}}(h)$ is a real-valued convex Lipschitz function of $h \in \mathbb{R}^d$.

We develop a variational formula for $g^{\beta}_{\mathrm{pl}}(h)$ for $\beta \in (0, \infty]$ in terms of gradient-like cocycles, and identify a condition that singles out extremal cocycles. For $0 < \beta < \infty$ this variational formula appeared in [57] and here we extend it to $\beta = \infty$. The solution proposal is new for all β.

Definition 3.1 (*Cocycles*). A measurable function $F : \Omega \times \mathcal{G}^2 \to \mathbb{R}$ is a *stationary cocycle* if it satisfies these two conditions for $\mathbb{P}$-a.e. ω and all $x, y, z \in \mathcal{G}$:

$$\begin{aligned} F(\omega, z + x, z + y) &= F(T_z\omega, x, y) && \text{(stationarity)}\\ F(\omega, x, y) + F(\omega, y, z) &= F(\omega, x, z) && \text{(additivity)}. \end{aligned}$$

If $\mathbb{E}|F(x, y)| < \infty$ $\forall x, y \in \mathcal{G}$ then F is an $L^1(\mathbb{P})$ *cocycle*, and if also $\mathbb{E}[F(x, y)] = 0$ $\forall x, y \in \mathcal{G}$ then F is *centered*. $\mathcal{K}$ denotes the space of stationary $L^1(\mathbb{P})$ cocycles, and $\mathcal{K}_0$ denotes the subspace of centered stationary $L^1(\mathbb{P})$ cocycles.

As illustrated above, ω can be dropped from the notation $F(\omega, x, y)$. The term cocyle is borrowed from differential forms terminology, see e.g. [37]. One could also use the term *conservative flow* or *curl-free flow* following vector fields terminology.

The space $\mathcal{K}_0$ is the $L^1(\mathbb{P})$ closure of gradients $F(\omega, x, y) = \varphi(T_y\omega) - \varphi(T_x\omega)$ [57, Lemma C.3]. For $B \in \mathcal{K}$ there exists a vector $h(B) \in \mathbb{R}^d$ such that

$$\mathbb{E}[B(0, z)] = -h(B) \cdot z \quad \text{for all } \ z \in \mathcal{R}. \tag{3.5}$$

Existence of $h(B)$ follows because $c(x) = \mathbb{E}[B(0, x)]$ is an additive function on the group $\mathcal{G} \cong \mathbb{Z}^k$. $h(B)$ is not unique unless $\mathcal{R}$ spans $\mathbb{R}^d$, but the inner products $h(B) \cdot x$ for $x \in \mathcal{G}$ are uniquely defined. Then

$$F(\omega, x, y) = h(B) \cdot (x - y) - B(\omega, x, y), \quad x, y \in \mathcal{G} \tag{3.6}$$

is a centered stationary $L^1(\mathbb{P})$ cocycle.

Theorem 3.2. *Let $V_0 \in \mathcal{L}$ and assume $\mathbb{P}$ ergodic. Then the limits in (3.4) have these variational representations: for* $0 < \beta < \infty$

$$g_{\text{pl}}^{\beta}(h) = \inf_{F \in \mathcal{K}_0} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \; \beta^{-1} \log \sum_{z \in \mathcal{R}} p(z) e^{\beta V_0(\omega,z) + \beta h \cdot z + \beta F(\omega,0,z)} \tag{3.7}$$

and

$$g_{\text{pl}}^{\infty}(h) = \inf_{F \in \mathcal{K}_0} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \; \max_{z \in \mathcal{R}} \{V_0(\omega, z) + h \cdot z + F(\omega, 0, z)\}. \tag{3.8}$$

A minimizing $F \in \mathcal{K}_0$ exists for each $0 < \beta \le \infty$ *and* $h \in \mathbb{R}^2$.

Proof. Theorem 2.1 of [58] gives formula (3.7) for $0 < \beta < \infty$. The kernel in that reference is the uniform one $p(z) = |\mathcal{R}|^{-1}$ but changing the kernel makes no difference to the proof. To get the formula for $\beta = \infty$, note that for $\beta > 0$ and $F \in \mathcal{K}_0$,

$$\begin{aligned} &\beta^{-1} \log \sum_z p(z) e^{\beta V(\omega,z) + \beta F(\omega,0,z)} \\ &\quad \le \max_z \{V(\omega, z) + F(\omega, 0, z)\} \\ &\quad \le \beta^{-1} \log \sum_z p(z) e^{\beta V(\omega,z) + \beta F(\omega,0,z)} + \beta^{-1} \log \delta^{-1}. \end{aligned}$$

Thus

$$g_{\text{pl}}^{\beta} \le \inf_{F \in \mathcal{K}_0} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \; \max_z \{V(\omega, z) + F(\omega, 0, z)\} \le g_{\text{pl}}^{\beta} + \beta^{-1} \log \delta^{-1}.$$

Formula (3.8) follows from this and (2.11), upon letting $\beta \to \infty$. Theorem 2.3 of [58] gives the existence of a minimizer for $0 < \beta < \infty$, and the same proof works also for $\beta = \infty$. □

Assuming $g_{\text{pl}}^{\beta}(0)$ finite is not necessary for Theorem 3.2. By the assumption $V_0 \in \mathcal{L}$, any $F \in \mathcal{K}_0$ that makes the right-hand side of (3.8) finite satisfies the ergodic theorem (Theorem A.1) in the appendix. Then potential $V(\omega, z)$ can be replaced by $V(\omega, z) + F(\omega, 0, z)$ without altering $g_{\text{pl}}^{\beta}(h)$, and consequently $g_{\text{pl}}^{\beta}(h)$ is finite.

Formulas (3.7) and (3.8) can be viewed as infinite-dimensional versions of the min–max variational formula for the Perron–Frobenius eigenvalue of a nonnegative matrix. This connection is discussed in Sect. 8.

The next definition and theorem offer a way to identify a minimizing F for (3.7) and (3.8). Later we explain how Busemann functions provide minimizers that match this recipe. That this approach is feasible will be demonstrated by examples: weak disorder (Example 3.7), the exactly solvable log-gamma polymer (Sect. 6.1 below) and the corner growth model with exponential weights (Sect. 6.2). This strategy is carried out for the two-dimensional corner growth model with general weights (a non-solvable case) in article [27].

Definition 3.3. Fix $\beta \in (0, \infty]$. A stationary L^1 cocycle B *is adapted to* potential V_0 if the following condition holds. If $0 < \beta < \infty$ the requirement is

$$\sum_{z\in\mathcal{R}} p(z)\, e^{\beta V_0(\omega,z)-\beta B(\omega,0,z)} = 1 \quad \text{for } \mathbb{P}\text{-a.e. } \omega, \tag{3.9}$$

while if $\beta = \infty$ then the condition is

$$\max_{z\in\mathcal{R}}\{V_0(\omega, z) - B(\omega, 0, z)\} = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega. \tag{3.10}$$

Theorem 3.4. *Fix $\beta \in (0, \infty]$, assume $\mathbb{P}$ ergodic and $V_0 \in \mathcal{L}$. Suppose we have a stationary L^1 cocycle B that is adapted to V_0 in the sense of Definition 3.3. Define $h(B)$ and F as in* (3.5)–(3.6). *Then we have conclusions* (i)–(ii) *below.*

(i) $g^{\beta}_{\mathrm{pl}}(h(B)) = 0$. *$g^{\beta}_{\mathrm{pl}}(h)$ is finite for all $h \in \mathbb{R}^d$.*

(ii) *F solves the variational formula. Precisely, assume $h \in \mathbb{R}^d$ satisfies*

$$(h - h(B)) \cdot (z - z') = 0 \quad \textit{for all } z, z' \in \mathcal{R}. \tag{3.11}$$

Under this assumption we have the two cases below.

(ii-a) *Case $0 < \beta < \infty$. F is a minimizer in* (3.7) *for potential $V(\omega, z) = V_0(\omega, z) + h \cdot z$. The essential supremum in* (3.7) *disappears and we have, for $\mathbb{P}$-a.e. ω and any $z' \in \mathcal{R}$,*

$$g^{\beta}_{\mathrm{pl}}(h) = \beta^{-1} \log \sum_{z\in\mathcal{R}} p(z)\, e^{\beta V_0(\omega,z)+\beta h\cdot z+\beta F(\omega,0,z)} = (h - h(B)) \cdot z'. \tag{3.12}$$

(ii-b) *Case $\beta = \infty$. Then F is a minimizer in* (3.8) *for potential $V(\omega, z) = V_0(\omega, z) + h \cdot z$. The essential supremum in* (3.8) *disappears and we have, for $\mathbb{P}$-a.e. ω and any $z' \in \mathcal{R}$,*

$$g^{\infty}_{\mathrm{pl}}(h) = \max_{z\in\mathcal{R}} \{V_0(\omega, z) + h \cdot z + F(\omega, 0, z)\} = (h - h(B)) \cdot z'. \tag{3.13}$$

Condition (3.11) says that $h - h(B)$ is orthogonal to the affine hull of $\mathcal{R}$ in $\mathbb{R}^d$. If $0 \in \mathcal{U}$ this affine hull is the linear span of $\mathcal{R}$ in $\mathbb{R}^d$.

Remark 3.5 (*Correctors*). A mean-zero cocycle that minimizes in (3.7) or (3.8) without the essential supremum (that is, satisfies the first equality of (3.12) or (3.13)) could be called a *corrector* by analogy with the homogenization literature (see for example Section 7 in [38] and top of page 468 in [2]). These correctors have been useful in the study of infinite geodesics in the corner growth model [26] and infinite directed polymers [28].

Proof of Theorem 3.4. *Case* $0 < \beta < \infty$. From assumption (3.9) and definition (3.6) of F

$$\log \sum_{z\in\mathcal{R}} p(z)\, e^{\beta V_0(\omega,\, z)+\beta h(B)\cdot z+\beta F(\omega,0,z)} = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega. \tag{3.14}$$

Iterating this gives (with $x_k = z_1 + \cdots + z_k$)

$$\log \sum_{z_{1,n}\in\mathcal{R}^n} p(x_{0,n})\, e^{\beta \sum_{k=0}^{n-1} V_0(T_{x_k}\omega,\, z_{k+1})+\beta h(B)\cdot x_n+\beta F(\omega,0,x_n)} = 0. \tag{3.15}$$

Assumption (3.9) gives the bound $F(\omega, 0, z) \le V_0^*(\omega) + C$ for $z \in \mathcal{R}$, with

$$V_0^*(\omega) = \max_{z\in\mathcal{R}} |V_0(\omega, z)| \tag{3.16}$$

that satisfies $V_0^* \in \mathcal{L}$ and a constant C. By Theorem A.1 in the appendix, $F(\omega, 0, x_n) = o(n)$ uniformly in $z_{1,n}$, $\mathbb{P}$-almost surely. It follows from (3.15) that $g_{\mathrm{pl}}^{\beta}(h(B)) = 0$. Since the steps of the walks are bounded, finiteness of $g_{\mathrm{pl}}^{\beta}(h)$ for all h follows from the definition (3.2).

Assume (3.11) for h. Then (3.14) gives

$$\beta^{-1}\log\sum_{z\in\mathcal{R}} p(z)\, e^{\beta V_0(\omega,\, z)+\beta h\cdot z+\beta F(\omega,0,z)} = (h - h(B))\cdot z'$$

while from (3.2) and (3.4)

$$g_{\mathrm{pl}}^{\beta}(h) = g_{\mathrm{pl}}^{\beta}(h(B)) + (h - h(B))\cdot z' = (h - h(B))\cdot z'.$$

We have verified (3.12).

Case $\beta = \infty$. From assumption (3.10) and definition (3.6) of F

$$\max_{z\in\mathcal{R}}\{V_0(\omega, z) + h(B)\cdot z + F(\omega, 0, z)\} = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega. \tag{3.17}$$

Iterating this gives (with $x_k = z_1 + \cdots + z_k$)

$$\max_{z_{1,n}\in\mathcal{R}^n}\Big\{\sum_{k=0}^{n-1} V_0(T_{x_k}\omega, z_{k+1}) + h(B)\cdot x_n + F(\omega, 0, x_n)\Big\} = 0. \tag{3.18}$$

By Theorem A.1, $F(\omega, 0, x_n) = o(n)$ uniformly in $z_{1,n}$ $\mathbb{P}$-a.s. It follows that $g_{\mathrm{pl}}(h(B)) = 0$.

Assume (3.11) for h. Then (3.17) gives

$$\max_{z\in\mathcal{R}}\{V_0(\omega, z) + h\cdot z + F(\omega, 0, z)\} = (h - h(B))\cdot z'$$

while from (3.3)–(3.4)

$$g_{\mathrm{pl}}^{\infty}(h) = g_{\mathrm{pl}}^{\infty}(h(B)) + (h - h(B))\cdot z' = (h - h(B))\cdot z'. \qquad \square$$

Remark 3.6. The results of this section extend to the more general potentials of the form $V(T_{x_k}\omega, z_{k+1,k+\ell})$ discussed in Sect. 2. For the definition of the cocycle see Definition 2.2 of [57]. We do not pursue these generalizations to avoid becoming overly technical and because presently we do not have an interesting example of this more general potential.

The remainder of this section discusses an example that illustrates Theorem 3.4.

Example 3.7 (*Directed polymer in weak disorder*). We consider the standard $k+1$ dimensional directed polymer in an i.i.d. random environment, or "bulk disorder". (For references see [14,15,20].) We show that the condition of weak disorder itself gives the corrector that solves the variational formula for the point-to-level free energy. The background walk is a simple random walk in $\mathbb{Z}^k$, and we use an additional $(k+1)$st coordinate to represent time. So $d = k + 1$, $\Omega = \mathbb{R}^{\mathbb{Z}^d}$, $\mathbb{P}$ is i.i.d., $\mathcal{R} = \{(\pm e_i, 1) : 1 \le i \le k\}$ and $p(z) = |\mathcal{R}|^{-1}$ for $z \in \mathcal{R}$. The potential is simply the environment at the site: $V_0(\omega) = \omega_0$.

Define the logarithmic moment generating functions

$$\lambda(\beta) = \log \mathbb{E}(e^{\beta\omega_0}) \quad \text{for} \quad \beta \in \mathbb{R} \tag{3.19}$$

and

$$\kappa(h) = \log \sum_{z\in\mathcal{R}} p(z)\, e^{h\cdot z} \quad \text{for} \quad h \in \mathbb{R}^d. \tag{3.20}$$

Consider only β-values such that $\lambda(\beta) < \infty$. The normalized partition function

$$W_n = e^{-n(\lambda(\beta)+\kappa(\beta h))} \sum_{x_{0,n}} p(x_{0,n})\, e^{\beta \sum_{k=0}^{n-1} \omega_{x_k} + \beta h\cdot x_n}$$

is a positive mean 1 martingale. The weak disorder assumption is this:

$$\text{the martingale } W_n \text{ is uniformly integrable.} \tag{3.21}$$

Given $h \in \mathbb{R}^d$, this can be guaranteed by taking $k \ge 3$ and small enough $\beta > 0$ (see Lemma 5.3 in [54]). Then $W_n \to W_\infty$ a.s. and in $L^1(\mathbb{P})$, $W_\infty \ge 0$ and $\mathbb{E}W_\infty = 1$. The event $\{W_\infty > 0\}$ is a tail event in the product space of environments, and hence by Kolmogorov's $0-1$ law we must have $\mathbb{P}(W_\infty > 0) = 1$. This gives us the limiting point-to-level free energy:

$$\begin{aligned} g_{\mathrm{pl}}^{\beta}(h) &= \lim_{n\to\infty} n^{-1}\beta^{-1} \log \sum_{x_{0,n}} p(x_{0,n})\, e^{\beta \sum_{k=0}^{n-1} \omega_{x_k} + \beta h\cdot x_n} \\ &= \lim_{n\to\infty} n^{-1}\beta^{-1} \log W_n + \beta^{-1}(\lambda(\beta) + \kappa(\beta h)) \\ &= \beta^{-1}(\lambda(\beta) + \kappa(\beta h)). \end{aligned} \tag{3.22}$$

Decomposition according to the first step (Markov property) gives

$$W_n(\omega) = e^{-\lambda(\beta)-\kappa(\beta h)} \sum_{z\in\mathcal{R}} p(z) e^{\beta\omega_0 + \beta h\cdot z} W_{n-1}(T_z\omega)$$

and a passage to the limit

$$W_\infty(\omega) = e^{-\lambda(\beta)-\kappa(\beta h)} \sum_{z\in\mathcal{R}} p(z) e^{\beta\omega_0 + \beta h\cdot z} W_\infty(T_z\omega) \quad \mathbb{P}\text{-a.s.} \tag{3.23}$$

Combining (3.22) and (3.23) gives

$$g_{\mathrm{pl}}^{\beta}(h) = \beta^{-1} \log \sum_{z\in\mathcal{R}} p(z) e^{\beta\omega_0 + \beta h\cdot z + \beta F(\omega,0,z)} \quad \mathbb{P}\text{-a.s.} \tag{3.24}$$

with the gradient

$$F(\omega, x, y) = \beta^{-1} \log W_\infty(T_y\omega) - \beta^{-1} \log W_\infty(T_x\omega). \tag{3.25}$$

In order to check that F is a centered cocycle it remains to verify that $F(\omega, 0, z)$ is integrable and mean-zero. Equation (3.24) gives an upper bound that shows $\mathbb{E}[F(\omega, 0, z)^+] < \infty$. We argue indirectly that also $\mathbb{E}[F(\omega, 0, z)^-] < \infty$. The first limit in probability below comes from stationarity.

$$\begin{aligned} 0 &\overset{\text{prob}}{=} \lim_{n\to\infty} \left[n^{-1}\beta^{-1} \log W_\infty(T_{nz}\omega) - n^{-1}\beta^{-1} \log W_\infty(\omega)\right] \\ &= \lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} F(T_{kz}\omega, 0, z). \end{aligned}$$

Since $\mathbb{E}[F(\omega, 0, z)^+] < \infty$, the assumption $\mathbb{E}[F(\omega, 0, z)^-] = \infty$ and the ergodic theorem would force the limit above to $-\infty$. Hence it must be that $F(\omega, 0, z) \in L^1(\mathbb{P})$. The limit above then gives $\mathbb{E}[F(\omega, 0, z)] = 0$.

To summarize, (3.24) shows that the centered cocycle F satisfies (3.12) for $V(\omega, z) = \omega_0 + h \cdot z$ for this particular value (β, h). F is the corrector given in Theorem 3.4, from the cocycle B that is adapted to V_0 given by

$$B(\omega, x, y) = g^{\beta}_{\text{pl}}(h)e_d \cdot (y - x) - h \cdot (y - x) - F(\omega, x, y)$$

with $h(B) = h - g^{\beta}_{\text{pl}}(h)e_d$. A vector $\tilde{h}$ satisfies (3.11) if and only if $\tilde{h} = h + \alpha e_d$ for some $\alpha \in \mathbb{R}$. The conclusion of the theorem, that F is a corrector for potential $V_0(\omega) = \omega_0$ and all such tilts $\tilde{h}$, is obvious because $e_d \cdot x_n = n$ for admissible paths.

4. Tilt-Velocity Duality

Section 3 gave a variational description of the point-to-level limit in terms of stationary cocycles. Theorem 4.4 below extends this description to point-to-point limits via tilt-velocity duality. Tilt-velocity duality is the familiar idea from large deviation theory that pinning the path is dual to tilting the energy by an external field. In the positive temperature setting this is exactly the convex duality of the quenched large deviation principle for the endpoint of the path (see Remark 4.2 in [55]).

We continue to consider potentials of the form $V(\omega, z) = V_0(\omega, z) + h \cdot z$ in general dimension $d \in \mathbb{N}$, with $\beta \in (0, \infty]$ and $\mathbb{P}$ ergodic. As above, the point-to-level limits $g^{\beta}_{\text{pl}}(h)$ are defined by (3.4). For the point-to-point limits $g^{\beta}_{\text{pp}}(\xi)$ we use only the V_0-part of the potential. So for $\xi \in \mathcal{U}$ define

$$g^{\beta}_{\text{pp}}(\xi) = \lim_{n\to\infty} n^{-1}\beta^{-1} \log \sum_{\substack{x_{0,n}:\, x_0=0,\\ x_n=\hat{x}_n(\xi)}} p(x_{0,n})\, e^{\beta \sum_{k=0}^{n-1} V_0(T_{x_k}\omega,\, z_{k+1})} \tag{4.1}$$

for $0 < \beta < \infty$ and

$$g^{\infty}_{\text{pp}}(\xi) = \lim_{n\to\infty} \max_{\substack{x_{0,n}:\, x_0=0,\\ x_n=\hat{x}_n(\xi)}} n^{-1} \sum_{k=0}^{n-1} V_0(T_{x_k}\omega, z_{k+1}). \tag{4.2}$$

In this context we call the vector $h \in \mathbb{R}^d$ a *tilt* and elements $\xi \in \mathcal{U}$ *directions* or *velocities*. Let us assume $g_{\mathrm{pl}}^{\beta}(0)$ finite. Then for $\xi \in \mathrm{ri}\,\mathcal{U}$, the a.s. point-to-point limits (4.1)–(4.2) define nonrandom, bounded, concave, continuous functions $g_{\mathrm{pp}}^{\beta} : \mathrm{ri}\,\mathcal{U} \to \mathbb{R}$ for $\beta \in (0,\infty]$ (see Theorem 2.4 and 2.6 and Remark 2.5 of [55]). The results of this section do not touch the relative boundary of $\mathcal{U}$. Consequently we do not need additional assumptions that guarantee regularity of g_{pp}^{β} up to the boundary. One sufficient assumption would be the directed i.i.d. $L^{d+\varepsilon}$ of Remark 2.6 (Theorem 3.2 of [55]).

Remark 4.1. To illustrate what can go wrong on the boundary of $\mathcal{U}$, suppose $z \in \mathcal{R}$ is an extreme point of $\mathcal{U}$. Then the only path from 0 to nz is $x_k = kz$, and we get $g_{\mathrm{pp}}^{\beta}(z) = \beta^{-1}\log p(z) + \mathbb{E}[V_0(\omega, z)\,|\,\mathcal{I}_z]$ where $\mathcal{I}_z$ is the σ-algebra of events invariant under the mapping T_z. This can be random even if $\mathbb{P}$ is assumed ergodic under the full group $\{T_x\}$. In general g_{pp}^{β} is lower semicontinuous on all of $\mathcal{U}$, for a.e. fixed ω (Theorem 2.6 of [55]).

With definitions (3.2)–(3.4) and (4.1)–(4.2), Eq. (2.10) becomes

$$g_{\mathrm{pl}}^{\beta}(h) = \sup_{\xi\in\mathcal{U}}\bigl\{g_{\mathrm{pp}}^{\beta}(\xi) + h\cdot\xi\bigr\}, \quad h \in \mathbb{R}^d. \tag{4.3}$$

In order to invert this relationship between g_{pl}^{β} and g_{pp}^{β} we turn it into a convex (or rather, concave) duality. First extend g_{pp}^{β} outside $\mathcal{U}$ via $g_{\mathrm{pp}}^{\beta}(\xi) = -\infty$ for $\xi \in \mathcal{U}^c$, and then replace g_{pp}^{β} with its upper semicontinuous regularization $\bar{g}_{\mathrm{pp}}^{\beta}(\xi) = g_{\mathrm{pp}}^{\beta}(\xi) \vee \overline{\lim}_{\zeta\to\xi}\, g_{\mathrm{pp}}^{\beta}(\zeta)$. Now (4.3) extends to

$$g_{\mathrm{pl}}^{\beta}(h) = \sup_{\xi\in\mathbb{R}^d}\{\bar{g}_{\mathrm{pp}}^{\beta}(\xi) + h\cdot\xi\}, \quad h \in \mathbb{R}^d,$$

which standard convex duality [59] inverts to

$$\bar{g}_{\mathrm{pp}}^{\beta}(\xi) = \inf_{h\in\mathbb{R}^d}\{g_{\mathrm{pl}}^{\beta}(h) - h\cdot\xi\}, \quad \xi \in \mathbb{R}^d.$$

By the continuity of g_{pp}^{β} on $\mathrm{ri}\,\mathcal{U}$, the last display gives

$$g_{\mathrm{pp}}^{\beta}(\xi) = \inf_{h\in\mathbb{R}^d}\bigl\{g_{\mathrm{pl}}^{\beta}(h) - h\cdot\xi\bigr\} \quad \text{for } \xi \in \mathrm{ri}\,\mathcal{U}. \tag{4.4}$$

Equations (4.3) and (4.4) suggest the next definition, and then Lemma 4.3 answers part of the natural next question.

Definition 4.2. At a fixed $\beta \in (0,\infty]$, we say that tilt $h \in \mathbb{R}^d$ and velocity $\xi \in \mathrm{ri}\,\mathcal{U}$ are *dual* to each other if

$$g_{\mathrm{pl}}^{\beta}(h) = g_{\mathrm{pp}}^{\beta}(\xi) + h\cdot\xi. \tag{4.5}$$

Lemma 4.3. *Fix $\beta \in (0,\infty]$. Assume $\mathbb{P}$ ergodic, $V_0 \in \mathcal{L}$ and $g_{\mathrm{pl}}^{\beta}(0) < \infty$. Then every $\xi \in \mathrm{ri}\,\mathcal{U}$ has a dual $h \in \mathbb{R}^d$. Furthermore, if h is dual to $\xi \in \mathcal{U}$ and h' is such that*

$$(h - h')\cdot(z - z') = 0 \quad \textit{for all } z, z' \in \mathcal{R} \tag{4.6}$$

then h' is also dual to ξ.

Proof. We start with the proof of the second claim. If (4.6) holds then directly from (3.2)–(3.4), $g_{\mathrm{pl}}^{\beta}(h') = g_{\mathrm{pl}}^{\beta}(h) + (h'-h)\cdot z$ for all $z \in \mathcal{R}$. Hence, $g_{\mathrm{pl}}^{\beta}(h') - h'\cdot\xi = g_{\mathrm{pl}}^{\beta}(h) - h\cdot\xi$ and h is dual to ξ if and only if h' is.

The equality above also implies that any h in (4.4) can be replaced by any h' satisfying (4.6). Fix $z_0 \in \mathcal{R}$. One way to satisfy (4.6) is to let h' be the orthogonal projection of h onto the linear span $\mathcal{V}$ of $\mathcal{R} - z_0$. Consequently we can restrict the infimum in (4.4) to $h \in \mathcal{V}$. (This can be all of $\mathbb{R}^d$.)

For any $z \in \mathcal{R}$, $h \in \mathbb{R}^d$, and $\beta \in (0,\infty]$,

$$g_{\mathrm{pl}}^{\beta}(h) \geq \mathbb{E}[V_0(\omega, z)] + h\cdot z + \beta^{-1}\log p(z).$$

To see this, for $z \neq 0$ consider the path $x_k = kz$ and use the ergodic theorem. For $z = 0$ consider a path that finds $V_0(T_x\omega, 0)$ within ε of $\operatorname{ess\,sup} V_0(\cdot\,, 0)$ and stays there.

Furthermore, (2.10) gives $g_{\mathrm{pp}}^{\beta}(\xi) \leq g_{\mathrm{pl}}^{\beta}(0)$. Consequently we can restrict the infimum in (4.4) to $h \in \mathcal{V}$ that satisfy

$$h\cdot(z-\xi) \leq g_{\mathrm{pl}}^{\beta}(0) + 1 - \mathbb{E}[V_0(\omega,z)] - \beta^{-1}\log p(z) \;\leq\; c$$

for all $z \in \mathcal{R}$ and a constant c. Convex combinations over z lead to $h\cdot(\eta-\xi) \leq c$ for all $\eta \in \mathcal{U}$. By the definition of relative interior, $\xi \in \operatorname{ri}\mathcal{U}$ implies that for some $\varepsilon > 0$, $\zeta \in \mathcal{U}$ for all $\zeta \in \operatorname{aff}\mathcal{U}$ such that $|\xi - \zeta| \leq \varepsilon$. Since $h \in \mathcal{V}$, $\eta = \xi + \varepsilon|h|^{-1}h$ lies in $\operatorname{aff}\mathcal{U}$ and then by choice of ε also in $\mathcal{U}$. We conclude that $\varepsilon|h| \leq c$ and thereby that the infimum in (4.4) can be restricted to a compact set. Continuity of g_{pl}^{β} implies that the infimum is achieved and existence of an h dual to ξ has been established. $\square$

With these preliminaries we extend Theorem 3.2 to the point-to-point case. Recall Definition 3.1 of the space $\mathcal{K}$ of stationary L^1 cocycles.

Theorem 4.4. *Assume $V_0 \in \mathcal{L}$, $\mathbb{P}$ ergodic and $g_{\mathrm{pl}}^{\beta}(0)$ finite. Then we have these variational formulas for $\xi \in \operatorname{ri}\mathcal{U}$.*

$$g_{\mathrm{pp}}^{\beta}(\xi) \;=\; \inf_{B\in\mathcal{K}} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\; \beta^{-1}\log\sum_{z\in\mathcal{R}} p(z)e^{\beta V_0(\omega,z)-\beta B(\omega,0,z)-\beta h(B)\cdot\xi} \tag{4.7}$$

for $0 < \beta < \infty$ and

$$g_{\mathrm{pp}}^{\infty}(\xi) \;=\; \inf_{B\in\mathcal{K}} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\; \max_{z\in\mathcal{R}}\{V_0(\omega,z) - B(\omega,0,z) - h(B)\cdot\xi\}. \tag{4.8}$$

The infimum in (4.7)–(4.8) can be restricted to $B \in \mathcal{K}$ such that $h(B)$ is dual to ξ. For each $\xi \in \operatorname{ri}\mathcal{U}$ and $0 < \beta \leq \infty$, there exists a minimizing $B \in \mathcal{K}$ such that $h(B)$ is dual to ξ.

Proof. We write the proof for $0 < \beta < \infty$, the case $\beta = \infty$ being similar enough. The right-hand side of (4.7) equals

$$\begin{aligned}
&\inf_h\Big\{\inf_{B:h(B)=h} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\; \beta^{-1}\log\sum_{z\in\mathcal{R}} p(z)e^{\beta V_0(\omega,z)-\beta B(\omega,0,z)} - h\cdot\xi\Big\}\\
&\quad= \inf_h\Big\{\inf_{F\in\mathcal{K}_0} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\; \beta^{-1}\log\sum_{z\in\mathcal{R}} p(z)e^{\beta V_0(\omega,z)+\beta h\cdot z+\beta F(\omega,0,z)} - h\cdot\xi\Big\}\\
&\quad= \inf_h\{\,g_{\mathrm{pl}}^{\beta}(h) - h\cdot\xi\} = g_{\mathrm{pp}}^{\beta}(\xi).
\end{aligned}$$

The middle equality is true because B is a cocycle with $h(B) = h$ if and only if $F(\omega, 0, z) = -B(\omega, 0, z) - h \cdot z$ is a centered cocycle.

For the existence, use Lemma 4.3 to pick h dual to ξ, and then Theorem 3.2 to find a minimizing $F \in \mathcal{K}_0$ for $g^{\beta}_{\mathrm{pl}}(h)$. Then $B(\omega, 0, z) = -h \cdot z - F(\omega, 0, z)$ is a minimizer for $g^{\beta}_{\mathrm{pp}}(\xi)$ and $h(B) = h$. $\square$

Combining Theorems 3.4 and 4.4 with (4.5) gives:

Corollary 4.5. *Assume $V_0 \in \mathcal{L}$, $\mathbb{P}$ ergodic and $g^{\beta}_{\mathrm{pl}}(0)$ finite. Let $\beta \in (0, \infty]$ and $\xi \in \mathrm{ri}\,\mathcal{U}$. Suppose there exists $B \in \mathcal{K}$ adapted to V_0 (Definition 3.3) and such that $h(B)$ is dual to ξ. Then B minimizes in (4.7) or (4.8) without the essential supremum over ω and*

$$g^{\beta}_{\mathrm{pp}}(\xi) = -h(B) \cdot \xi. \tag{4.9}$$

If $\nabla g^{\beta}_{\mathrm{pp}}$ exists at ξ, the duality of $h(B)$ and ξ implies that

$$\nabla g^{\beta}_{\mathrm{pp}}(\xi) = -h(B). \tag{4.10}$$

In some situations $\mathcal{U}$ has empty interior but g^{β}_{pp} extends as a homogeneous function to an open neighborhood of $\mathcal{U}$, and (4.10) makes sense for the extended function. Such is the case for example when $\mathcal{R} = \{e_1, \dots, e_d\}$. In the 1+1 dimensional exactly solvable models discussed in Sect. 6 below, for each $\xi \in \mathrm{ri}\,\mathcal{U}$ there exists a cocycle $B = B^{\xi}$ that satisfies (4.9) and (4.10). Modulo some regularity issues, this is the case also for the 1+1 dimensional corner growth model with general weights [27].

5. Cocycles from Busemann Functions

The solution approach advanced in this paper for the cocycle variational formulas relies on cocycles that are adapted to V_0 (Definition 3.3). This section describes how to obtain such cocycles from limits of gradients of free energy, called *Busemann functions*, provided such limits exist. Busemann functions come in two variants, point-to-point and point-to-level. These are treated in the next two theorems. Proofs of the theorems are at the end of the section.

We assume now that every admissible path between two given points x and y has the same number of steps. This prevents loops. The natural examples are $\mathcal{R} = \{e_1, e_2, \dots, e_d\}$ and $\mathcal{R} = \{(z', 1) : z' \in \mathcal{R}'\}$ for some finite $\mathcal{R}' \subset \mathbb{Z}^{d-1}$. For $x, y \in \mathbb{Z}^d$ such that y can be reached from x define the free energy

$$G^{\beta}_{x,y} = \beta^{-1} \log \sum_{\substack{n \ge 1 \\ x_{0,n} : x_0 = x,\, x_n = y}} p(x_{0,n})\, e^{\beta \sum_{k=0}^{n-1} V_0(T_{x_k}\omega, z_{k+1})} \quad \text{for } 0 < \beta < \infty \tag{5.1}$$

and the last-passage time

$$G^{\infty}_{x,y} = \max_{\substack{n \ge 1 \\ x_{0,n} : x_0 = x,\, x_n = y}} \sum_{k=0}^{n-1} V_0(T_{x_k}\omega, z_{k+1}). \tag{5.2}$$

The sum and the maximum are taken over all admissible paths from x to y, and then there is a unique n, namely the number of steps from x to y.

Recall definition (2.7) of the path $\hat{x}_n(\xi)$. A point-to-point Busemann function in direction $\xi \in \operatorname{ri}\mathcal{U}$ is defined by

$$B^{\xi}_{\mathrm{pp}}(x, y) = \lim_{n\to\infty}\big[G^{\beta}_{x,\,\hat{x}_n(\xi)+z} - G^{\beta}_{y,\,\hat{x}_n(\xi)+z}\big], \quad x, y \in \mathcal{G},\ z \in \mathcal{R}\cup\{0\}, \tag{5.3}$$

provided that the limit exists $\mathbb{P}$-almost surely and does not depend on z. The extra perturbation by z on the right-hand side will be used to establish stationarity of the limit. β is now fixed and we omit the dependence of B^{ξ}_{pp} on β from the notation. To ensure that paths to $\hat{x}_n(\xi)$ from both x and y exist in (5.3), in the definition (2.7) of $\hat{x}_n(\xi)$ pick $\alpha_z(\xi) > 0$ for all $z \in \mathcal{R}$. (For $\xi \in \operatorname{ri}\mathcal{U}$ this is possible by Theorem 6.4 in [59].) Then, any point $x \in \mathcal{G}$ can reach $\hat{x}_n(\xi)$ with steps in $\mathcal{R}$ for large enough n.

Theorem 5.1. *Let $\beta \in (0, \infty]$, $V_0 \in \mathcal{L}$, $\mathbb{P}$ ergodic and $g^{\beta}_{\mathrm{pl}}(0)$ finite. Assume that every admissible path between two given points x and y has the same number of steps.*

Fix $\xi \in \operatorname{ri}\mathcal{U}$ and choose $\alpha_z(\xi) > 0$ for each $z \in \mathcal{R}$ in (2.7). Assume that for all $x, y \in \mathcal{G}$ and $\mathbb{P}$-a.e. ω, the limits (5.3) exist for $z \in \mathcal{R}\cup\{0\}$ and are independent of z. Then $B^{\xi}_{\mathrm{pp}}(x, y)$ is a stationary cocycle that is adapted to V_0 in the sense of Definition 3.3.

Assume additionally

$$\varlimsup_{n\to\infty} n^{-1}\mathbb{E}[G^{\beta}_{0,\,\hat{x}_n(\xi)}] \le g^{\beta}_{\mathrm{pp}}(\xi). \tag{5.4}$$

Then $B^{\xi}_{\mathrm{pp}}(x, y) \in L^1(\mathbb{P})$ $\forall x, y \in \mathcal{G}$, $h(B^{\xi}_{\mathrm{pp}})$ is dual to ξ (Definition 4.2), and $g^{\beta}_{\mathrm{pp}}(\xi) = -h(B^{\xi}_{\mathrm{pp}})\cdot\xi$.

The point of the theorem is that the Busemann function furnishes correctors for the variational formulas. Once the assumptions of Theorem 5.1 are satisfied, (i) Theorem 3.4 implies that $F(x, y) = h(B^{\xi}_{\mathrm{pp}})\cdot(x - y) - B^{\xi}_{\mathrm{pp}}(x, y)$ is a corrector for $g^{\beta}_{\mathrm{pl}}(h)$ for any h such that $h - h(B^{\xi}_{\mathrm{pp}}) \perp \operatorname{aff}\mathcal{R}$, and (ii) depending on β, B^{ξ}_{pp} minimizes either (4.7) or (4.8) without the $\mathbb{P}$-essential supremum.

In the point-to-level case the free energy and last-passage time for paths of length n started at x are defined by a shift $G^{\beta}_{x,(n)}(h)(\omega) = G^{\beta}_{0,(n)}(h)(T_x\omega)$. Point-to-level Busemann functions are defined by

$$B^{h}_{\mathrm{pl}}(0, z) = \lim_{n\to\infty}\big[G^{\beta}_{0,(n)}(h) - G^{\beta}_{z,(n-1)}(h)\big], \quad z \in \mathcal{R}, \tag{5.5}$$

omitting again the β-dependence from the notation.

Theorem 5.2. *Let $\beta \in (0, \infty]$, $V_0 \in \mathcal{L}$, $\mathbb{P}$ ergodic and $g^{\beta}_{\mathrm{pl}}(0)$ finite. Assume that every admissible path between any two given points x and y has the same number of steps.*

Fix $h \in \mathbb{R}^d$. Assume the $\mathbb{P}$-a.s. limits (5.5) exist for all $z \in \mathcal{R}$. Then we can extend $\{B^{h}_{\mathrm{pl}}(0, z)\}_{z\in\mathcal{R}}$ to a stationary cocycle $\{B^{h}_{\mathrm{pl}}(x, y)\}_{x,y\in\mathcal{G}}$, and cocycle $B^{h}_{\mathrm{pl}}(x, y) - h\cdot(y - x)$ is adapted to V_0 in the sense of Definition 3.3.

Assume additionally

$$\varliminf_{n\to\infty} n^{-1}\mathbb{E}[G^{\beta}_{0,(n)}(h)] \le g^{\beta}_{\mathrm{pl}}(h). \tag{5.6}$$

Then $B^{h}_{\mathrm{pl}}(x, y) \in L^1(\mathbb{P})$ for $x, y \in \mathcal{G}$. $F(\omega, x, y) = h(B^{h}_{\mathrm{pl}})\cdot(x - y) - B^{h}_{\mathrm{pl}}(\omega, x, y)$ is a minimizer in (3.7) for $g^{\beta}_{\mathrm{pl}}(h)$ if $0 < \beta < \infty$ and in (3.8) if $\beta = \infty$.

Remark 5.7 below indicates how the theorem could be upgraded to state that the minimizer F is also a corrector, in other words satisfies (3.12) or (3.13).

Remark 5.3. Assumptions (5.4) and (5.6) need to be verified separately for the case at hand. In the directed i.i.d. $L^{d+\varepsilon}$ case of Remark 2.6, we can use lattice animal bounds: Lemma 3 from page 85 of [25] gives $\sup_n \mathbb{E}\big[\big(n^{-1}G^{\beta}_{0,\hat{x}_n(\xi)}\big)^2\big] < \infty$ and $\sup_n \mathbb{E}\big[\big(n^{-1}G^{\beta}_{0,(n)}(h)\big)^2\big] < \infty$, which imply L^1 convergence in (4.1)–(4.2) and (3.4), respectively. A completely general sufficient condition is to have V_0 bounded above.

Remark 5.4. All of the assumptions and conclusions of Theorems 5.1–5.2 can be verified in the exactly solvable cases. In the explicitly solvable 1+1 dimensional cases the Busemann limits B^{ξ}_{pp} and B^{h}_{pl} are connected by the duality of ξ and h, and lead to the same set of cocycles, as described in the next section. This also holds for the general 1+1 dimensional corner growth model under local regularity assumptions on the shape that ensure the existence of Busemann functions [27]. We would expect this feature to be true very generally.

Remark 5.5. According to (5.3), B^{ξ}_{pp} is a microscopic gradient of free energy and passage times in direction ξ, and by (4.10) its average gives the macroscopic gradient. This form of (4.10) was anticipated in [34] in the context of Euclidean first passage percolation (FPP), where $g_{\text{pp}}(x, y) = c\sqrt{x^2 + y^2}$ for some $c > 0$. (See the paragraph after the proof of Theorem 1.13 in [34].) A version of the formula also appears in Theorem 3.5 of [18] in the context of nearest-neighbor FPP.

Example 5.6 (*Directed polymer in weak disorder*). The directed polymer in weak disorder illustrates Theorem 5.2. We continue with the notation from Example 3.7 and take $\beta > 0$ small enough. Then $\mathbb{P}$-almost surely for $z \in \mathcal{R}$,

$$\begin{aligned}
&G^{\beta}_{0,(n)}(h) - G^{\beta}_{z,(n-1)}(h)\\
&\quad = \beta^{-1}\log W_n - \beta^{-1}\log W_{n-1}\circ T_z + \beta^{-1}(\lambda(\beta) + \kappa(\beta h))\\
&\quad \underset{n\to\infty}{\longrightarrow}\ \beta^{-1}\log W_\infty - \beta^{-1}\log W_\infty \circ T_z + \beta^{-1}(\lambda(\beta) + \kappa(\beta h))\\
&\quad = -F(0, z) + g^{\beta}_{\text{pl}}(h),
\end{aligned}$$

with F defined by (3.25). Thus the Busemann function is $B^{h}_{\text{pl}}(0, z) = -F(0, z) + g^{\beta}_{\text{pl}}(h)$. By Theorem 5.2, cocycle $B^{h}_{\text{pl}}(0, z) - h \cdot z$ is adapted to V_0, as already observed in Example 3.7. The Busemann function recovers the corrector F identified in Example 3.7.

In the remainder of the section we prove Theorems 5.1 and 5.2 and then comment on getting a corrector in Theorem 5.2.

Proof of Theorem 5.1. To check stationarity, for $z \in \mathcal{R}$

$$\begin{aligned}
B^{\xi}_{\text{pp}}(z + x, z + y) &= \lim_{n\to\infty}[G^{\beta}_{z+x,\, z+\hat{x}_n(\xi)} - G^{\beta}_{z+y,\, z+\hat{x}_n(\xi)}]\\
&= \lim_{n\to\infty}[G^{\beta}_{x,\, \hat{x}_n(\xi)} - G^{\beta}_{y,\, \hat{x}_n(\xi)}] \circ T_z = B^{\xi}_{\text{pp}}(x, y) \circ T_z.
\end{aligned}$$

Additivity is satisfied by telescoping sums. The condition of Definition 3.3 is readily checked. For example, in the $\beta = \infty$ case, if x is reachable from 0 and from every $z \in \mathcal{R}$, $\max_{z\in\mathcal{R}}\{V_0(\omega, z) + G^{\infty}_{z,x} - G^{\infty}_{0,x}\} = 0$ because some $z \in \mathcal{R}$ is the first step of a maximizing path from 0 to x.

Assume (5.4). Recall (3.16). Fix $\ell \in \mathbb{N}$ large enough so that, for each $k \ge m \ge 1$, there exists an admissible path $\{y_i^{m,k}\}_{i=0}^{\ell}$ from $\hat{x}_{k-m}(\xi)$ to $\hat{x}_{k+\ell}(\xi) - \hat{x}_m(\xi)$. Then

$$G^{\beta}_{0,\,\hat{x}_{k+\ell}(\xi)-\hat{x}_m(\xi)}(\omega) \;\ge\; G^{\beta}_{0,\,\hat{x}_{k-m}(\xi)}(\omega) + \beta^{-1}\log p(y^{m,k}_{0,\ell}) - \sum_{i=0}^{\ell-1} V_0^*(T_{y_i^{m,k}}\omega). \tag{5.7}$$

By (5.7), for $0 < m < n$,

$$\begin{aligned}
&\frac{1}{(m+\ell)n}\sum_{k=m}^{n}\mathbb{E}\big[G^{\beta}_{0,\,\hat{x}_{k+\ell}(\xi)} - G^{\beta}_{\hat{x}_m(\xi),\,\hat{x}_{k+\ell}(\xi)}\big]\\
&= \frac{1}{(m+\ell)n}\sum_{k=m}^{n}\mathbb{E}\big[G^{\beta}_{0,\,\hat{x}_{k+\ell}(\xi)} - G^{\beta}_{0,\,\hat{x}_{k+\ell}(\xi)-\hat{x}_m(\xi)}\big]\\
&\le \frac{1}{(m+\ell)n}\sum_{k=m}^{n}\mathbb{E}\big[G^{\beta}_{0,\,\hat{x}_{k+\ell}(\xi)} - G^{\beta}_{0,\,\hat{x}_{k-m}(\xi)}\big] - \frac{\log p(y^{m,k}_{0,\ell})}{\beta(m+\ell)} + \frac{\ell\mathbb{E}(V_0^*)}{m+\ell}\\
&\le \frac{1}{(m+\ell)n}\sum_{k=n-m+1}^{n+\ell}\mathbb{E}[G^{\beta}_{0,\,\hat{x}_k(\xi)}] - \frac{1}{(m+\ell)n}\sum_{k=0}^{m+\ell-1}\mathbb{E}[G^{\beta}_{0,\,\hat{x}_k(\xi)}] + \frac{C}{m}
\end{aligned}$$

where the last C depends on the fixed ℓ. By (5.4) we get the upper bound

$$\varlimsup_{n\to\infty}\frac{1}{(m+\ell)n}\sum_{k=m}^{n}\mathbb{E}\big[G^{\beta}_{0,\,\hat{x}_{k+\ell}(\xi)} - G^{\beta}_{\hat{x}_m(\xi),\,\hat{x}_{k+\ell}(\xi)}\big] \le g^{\beta}_{\mathrm{pp}}(\xi) + \frac{C}{m}. \tag{5.8}$$

On the other hand, by superadditivity,

$$G^{\beta}_{0,\,\hat{x}_{k+\ell}(\xi)} - G^{\beta}_{\hat{x}_m(\xi),\,\hat{x}_{k+\ell}(\xi)} \ge G^{\beta}_{0,\,\hat{x}_m(\xi)}$$

and hence $\frac{1}{(m+\ell)n}\big[\sum_{k=m}^{n}\big(G^{\beta}_{0,\,\hat{x}_{k+\ell}(\xi)} - G^{\beta}_{\hat{x}_m(\xi),\,\hat{x}_{k+\ell}(\xi)}\big)\big]^-$ is uniformly integrable as $n \to \infty$. Since by assumption (5.3)

$$\frac{1}{m+\ell}\,B^{\xi}_{\mathrm{pp}}(0, \hat{x}_m(\xi)) = \lim_{n\to\infty}\frac{1}{(m+\ell)n}\sum_{k=m}^{n}\big[G^{\beta}_{0,\,\hat{x}_{k+\ell}(\xi)} - G^{\beta}_{\hat{x}_m(\xi),\,\hat{x}_{k+\ell}(\xi)}\big] \quad \mathbb{P}\text{-a.s.}$$

we can apply Lemma A.2 from the appendix to conclude that $B^{\xi}_{\mathrm{pp}}(0, \hat{x}_m(\xi))$ is integrable and satisfies

$$\frac{1}{m+\ell}\,\mathbb{E}[B^{\xi}_{\mathrm{pp}}(0, \hat{x}_m(\xi))] \le g^{\beta}_{\mathrm{pp}}(\xi) + \frac{C}{m}. \tag{5.9}$$

Now we can show $B^{\xi}_{\mathrm{pp}}(0, z) \in L^1(\mathbb{P})$ $\forall z \in \mathcal{R}$. We have assumed that each step z appears along the path $\hat{x}_m(\xi)$, so it suffices to observe that

$$B^{\xi}_{\mathrm{pp}}(0, \hat{x}_m(\xi) - \hat{x}_{m-1}(\xi)) \circ T_{\hat{x}_{m-1}(\xi)} = B^{\xi}_{\mathrm{pp}}(0, \hat{x}_m(\xi)) - B^{\xi}_{\mathrm{pp}}(0, \hat{x}_{m-1}(\xi)) \;\in\; L^1(\mathbb{P}).$$

We have established that B^{ξ}_{pp} is a stationary $L^1(\mathbb{P})$ cocycle that is adapted to V_0 in the sense of Definition 3.3. By definition (3.5), the left-hand side of (5.9) equals

$$-(m+\ell)^{-1}h(B^{\xi}_{\text{pp}})\cdot\hat{x}_m(\xi) \;\to\; -h(B^{\xi}_{\text{pp}})\cdot\xi \quad \text{as } m\to\infty.$$

We have $-h(B^{\xi}_{\text{pp}})\cdot\xi \le g^{\beta}_{\text{pp}}(\xi)$. Since $g^{\beta}_{\text{pl}}(h(B^{\xi}_{\text{pp}})) = 0$ by Theorem 3.4, variational formula (4.4) gives the opposite inequality $-h(B^{\xi}_{\text{pp}})\cdot\xi \ge g^{\beta}_{\text{pp}}(\xi)$. Duality of $h(B^{\xi}_{\text{pp}})$ and ξ has been established.

Proof of Theorem 5.2. We check that limits (5.5) define a stationary cocycle $B^{h}_{\text{pl}}(\omega, x, y)$. Fix $x, y \in \mathcal{G}$ such that there is a path $x_{0,\ell}$ with increments $z_i = x_i - x_{i-1} \in \mathcal{R}$ that goes from $x = x_0$ to $y = x_\ell$. By shifting the n-index,

$$\begin{aligned}
\sum_{i=0}^{\ell-1} B^{h}_{\text{pl}}(T_{x_i}\omega, 0, z_{i+1}) &= \lim_{n\to\infty}\sum_{i=0}^{\ell-1}[G^{\beta}_{x_i,(n)}(h) - G^{\beta}_{x_{i+1},(n-1)}(h)] \\
&= \lim_{n\to\infty}\sum_{i=0}^{\ell-1}[G^{\beta}_{x_i,(n-i)}(h) - G^{\beta}_{x_{i+1},(n-i-1)}(h)] \\
&= \lim_{n\to\infty}[G^{\beta}_{x_0,(n)}(h) - G^{\beta}_{x_\ell,(n-\ell)}(h)] \\
&= \lim_{n\to\infty}[G^{\beta}_{0,(n)}(h) - G^{\beta}_{y-x,(n-\ell)}(h)]\circ T_x.
\end{aligned} \tag{5.10}$$

By assumption each path from x to y has the same number ℓ of steps. Hence we can define $B^{h}_{\text{pl}}(\omega, x, y) = \sum_{i=0}^{\ell-1} B^{h}_{\text{pl}}(T_{x_i}\omega, 0, z_{i+1})$ independently of the particular steps z_i taken, and with the property $B^{h}_{\text{pl}}(\omega, x, y) = B^{h}_{\text{pl}}(T_x\omega, 0, y-x)$.

If y is not accessible from x, pick a point $\bar{x}$ from which both x and y are accessible and set $B^{h}_{\text{pl}}(\omega, x, y) = B^{h}_{\text{pl}}(\omega, \bar{x}, y) - B^{h}_{\text{pl}}(\omega, \bar{x}, x)$. This definition is independent of the point $\bar{x}$. Now we have a stationary cocyle B^{h}_{pl}.

A first step decomposition of $G^{\beta}_{0,(n)}(h)$ shows that cocycle

$$\widetilde{B}(0, z) = B^{h}_{\text{pl}}(0, z) - h\cdot z \tag{5.11}$$

satisfies Definition 3.3.

Under assumption (5.6) the integrability of $B^{h}_{\text{pl}}(0, z)$ is proved exactly as in the proof of Theorem 5.1. First an upper bound:

$$\begin{aligned}
&\varlimsup_{n\to\infty} n^{-1}\sum_{k=1}^{n}\mathbb{E}[G^{\beta}_{0,(k)}(h) - G^{\beta}_{z,(k-1)}(h)] \\
&= \varlimsup_{n\to\infty} n^{-1}\sum_{k=1}^{n}\mathbb{E}[G^{\beta}_{0,(k)}(h) - G^{\beta}_{0,(k-1)}(h)] \\
&= \varlimsup_{n\to\infty} n^{-1}\mathbb{E}[G^{\beta}_{0,(n)}(h)] \le g^{\beta}_{\text{pl}}(h).
\end{aligned} \tag{5.12}$$

Then uniform integrability of $\big[n^{-1}\sum_{k=1}^{n}(G^{\beta}_{0,(k)}(h) - G^{\beta}_{z,(k-1)}(h))\big]^{-}$ from the lower bound

$$G^{\beta}_{0,(n)}(h) - G^{\beta}_{z,(n-1)}(h) \ge V_0(\omega, z) + h\cdot z + \beta^{-1}\log p(z). \tag{5.13}$$

By Lemma A.2, $B^h_{\text{pl}}(0,z) \in L^1(\mathbb{P})$ and

$$-h(B^h_{\text{pl}}) \cdot z = \mathbb{E}[B^h_{\text{pl}}(0,z)] \le g^{\beta}_{\text{pl}}(h) \quad \text{for} \quad z \in \mathcal{R}. \tag{5.14}$$

Define the centered stationary L^1 cocycle

$$F(\omega,x,y) = h(\widetilde{B}) \cdot (x-y) - \widetilde{B}(\omega,x,y) = h(B^h_{\text{pl}}) \cdot (x-y) - B^h_{\text{pl}}(\omega,x,y). \tag{5.15}$$

By variational formula (3.7), (5.11), (5.14), and (3.9) applied to $\widetilde{B}$,

$$\begin{aligned} g^{\beta}_{\text{pl}}(h) &\le \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\ \beta^{-1} \log \sum_{z\in\mathcal{R}} p(z) e^{\beta V_0(\omega,z)+\beta h\cdot z+\beta F(\omega,0,z)} \\ &= \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\ \beta^{-1} \log \sum_{z\in\mathcal{R}} p(z) e^{\beta V_0(\omega,z)-\beta h(B^h_{\text{pl}})\cdot z-\beta \widetilde{B}(\omega,0,z)} \\ &\le g^{\beta}_{\text{pl}}(h) + \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega}\ \beta^{-1} \log \sum_{z\in\mathcal{R}} p(z) e^{\beta V_0(\omega,z)-\beta \widetilde{B}(\omega,0,z)} = g^{\beta}_{\text{pl}}(h). \end{aligned} \tag{5.16}$$

This shows that F is a minimizer in (3.7). A similar proof works for the case $\beta = \infty$. ☐

Remark 5.7 (*Corrector in Theorem* 5.2). Continue with the assumptions of Theorem 5.2. We point out two sufficient conditions for concluding that F of (5.15) is not merely a minimizing cocycle for $g^{\beta}_{\text{pl}}(h)$ as stated in Theorem 5.2, but also a corrector for $g^{\beta}_{\text{pl}}(h)$. By Theorem 3.4, F is a corrector for $g^{\beta}_{\text{pl}}(h')$ for any h' such that $h' - h(\widetilde{B}) \perp \operatorname{aff}\mathcal{R}$. Since $h' - h(\widetilde{B}) = h' - h(B^h_{\text{pl}}) - h$, for $h' = h$ the condition is $h(B^h_{\text{pl}}) \perp \operatorname{aff}\mathcal{R}$, or equivalently that $h(B^h_{\text{pl}}) \cdot z$ is constant over $z \in \mathcal{R}$. (5.14) and (5.16) (and its analogue for $\beta = \infty$) imply that $-h(B^h_{\text{pl}}) \cdot z = g^{\beta}_{\text{pl}}(h)$ for at least one $z \in \mathcal{R}$. Hence the condition is

$$-h(B^h_{\text{pl}}) \cdot z = g^{\beta}_{\text{pl}}(h) \quad \text{for all} \quad z \in \mathcal{R}. \tag{5.17}$$

Here are two ways to satisfy (5.17).

(a) By the first two equalities in (5.12), (5.17) would follow from convergence of expectations in (3.4) and Cesàro convergence of expectations in (5.5):

$$\begin{aligned} \mathbb{E}[B^h_{\text{pl}}(0,z)] &= \lim_{n\to\infty} \mathbb{E}\Big[\frac{1}{n}\sum_{k=1}^{n}\big(G^{\beta}_{0,(k)}(h) - G^{\beta}_{z,(k-1)}(h)\big)\Big] \\ &= \lim_{n\to\infty} \mathbb{E}\big[n^{-1} G^{\beta}_{0,(n)}(h)\big] = g^{\beta}_{\text{pl}}(h). \end{aligned}$$

(b) Suppose h is dual to some $\bar{\xi} \in \operatorname{ri}\mathcal{U}$. Then (5.17) follows by this argument. First $g^{\beta}_{\text{pl}}(h(B^h_{\text{pl}}) + h) = g^{\beta}_{\text{pl}}(h(\widetilde{B})) = 0$ by Theorem 3.4. Then combining (4.3) and (5.14) gives

$$g_{\text{pp}}(\xi) + h \cdot \xi \le -h(B^h_{\text{pl}}) \cdot \xi \le g^{\beta}_{\text{pl}}(h) \quad \forall \xi \in \mathcal{U}. \tag{5.18}$$

From this $-h(B^h_{\text{pl}}) \cdot \bar{\xi} = g^{\beta}_{\text{pl}}(h)$. Since $\bar{\xi} \in \operatorname{ri}\mathcal{U}$ we can write $\bar{\xi} = \sum_{z\in\mathcal{R}} \alpha_z z$ where each $\alpha_z > 0$, and now (5.14) forces (5.17). ☐

6. Exactly Solvable Models in 1+1 Dimensions

We describe how the theory developed manifests itself in two well-known 1+1 dimensional exactly solvable models. The setting is the canonical one with $\Omega = \mathbb{R}^{\mathbb{Z}^2}$, $\mathcal{R} = \{e_1, e_2\}$, $\mathcal{U} = \{(s, 1-s) : 0 \le s \le 1\}$, and i.i.d. weights $\{\omega_x\}_{x\in\mathbb{Z}^2}$ under $\mathbb{P}$. The distributions of the weights are specified in the examples below.

6.1. Log-gamma polymer. The log-gamma polymer [66] is an explicitly solvable 1+1 dimensional directed polymer model for which the approach of this paper can be carried out explicitly. Some details are in [28]. We describe the results briefly.

Fix $0 < \rho < \infty$ and let ω_x be Gamma(ρ)-distributed, i.e. $\mathbb{P}\{\omega_x \le r\} = \Gamma(\rho)^{-1}\int_0^r t^{\rho-1}e^{-t}\,dt$ for $0 \le r < \infty$. Inverse temperature is fixed at $\beta = 1$. (Parameter ρ can be viewed as temperature, see Remark 3.2 in [28].) The potential is $V_0(\omega) = -\log\omega_0 + \log 2$. Let $\Psi_0 = \Gamma'/\Gamma$ and $\Psi_1 = \Psi_0'$ be the digamma and trigamma function.

Utilizing the stationary version of the log-gamma polymer one can compute the point-to-point limit for $\xi = (s, 1-s)$ as

$$\begin{aligned} g^1_{\mathrm{pp}}(\xi) &= \inf_{\theta\in(0,\rho)}\{-s\Psi_0(\theta) - (1-s)\Psi_0(\rho-\theta)\} \\ &= -s\Psi_0(\theta(\xi)) - (1-s)\Psi_0(\rho-\theta(\xi)) \end{aligned} \tag{6.1}$$

where $\theta = \theta(\xi) \in (0,\rho)$ is the unique solution of the equation

$$s\Psi_1(\theta) - (1-s)\Psi_1(\rho-\theta) = 0.$$

(See Theorem 2.4 in [66] or Theorem 2.1 in [29].) From this we solve the tilt-velocity duality explicitly: tilt $h = (h_1, h_2) \in \mathbb{R}^2$ and velocity $\xi \in \mathrm{ri}\,\mathcal{U}$ are dual (Definition 4.2) if and only if

$$h_1 - h_2 = \Psi_0(\theta(\xi)) - \Psi_0(\rho-\theta(\xi)). \tag{6.2}$$

Then

$$g^1_{\mathrm{pl}}(h) = h_1 - \Psi_0(\theta(\xi)) = h_2 - \Psi_0(\rho-\theta(\xi)). \tag{6.3}$$

For all $\xi \in \mathrm{ri}\,\mathcal{U}$ and $h \in \mathbb{R}^2$, the point-to-point and point-to-line Busemann functions $B^\xi_{\mathrm{pp}}(\omega, 0, z)$ and $B^h_{\mathrm{pl}}(\omega, 0, z)$ exist as the a.s. limits defined by (5.3) and (5.5) (Theorems 4.1 and 6.1 in [28]). They satisfy

$$B^h_{\mathrm{pl}}(\omega, 0, z) = B^\xi_{\mathrm{pp}}(\omega, 0, z) + h\cdot z \quad \text{for } z \in \mathcal{R} \tag{6.4}$$

whenever ξ and h are dual ([28], Theorem 6.1). All the assumptions and conclusions of Theorems 5.1–5.2 and Remark 5.7 are valid.

The marginal distributions of the Busemann functions are given by

$$e^{-B^\xi_{\mathrm{pp}}(x,x+e_1)} \sim \text{Gamma}(\theta(\xi)) \quad \text{and} \quad e^{-B^\xi_{\mathrm{pp}}(x,x+e_2)} \sim \text{Gamma}(\rho-\theta(\xi)).$$

Vector

$$h(B^\xi_{\mathrm{pp}}) = -\big(\mathbb{E}[B^\xi_{\mathrm{pp}}(0,e_1)]\,,\mathbb{E}[B^\xi_{\mathrm{pp}}(0,e_2)]\big) = \big(\Psi_0(\theta(\xi)), \Psi_0(\rho-\theta(\xi))\big)$$

is dual to ξ and $g^1_{\mathrm{pp}}(\xi) = -h(B^\xi_{\mathrm{pp}})\cdot\xi$ gives the point-to-point free energy (6.1). From (6.4) we deduce $\mathbb{E}[B^h_{\mathrm{pl}}(0,z)] = g^1_{\mathrm{pl}}(h)$ for $z \in \{e_1, e_2\}$.

6.2. Corner growth model with exponential weights. This is last-passage percolation on $\mathbb{Z}^2$ with admissible steps $\{e_1, e_2\}$ and i.i.d. weights $\{\omega_x\}_{x\in\mathbb{Z}^2}$ with rate 1 exponential distribution. That is, $\mathbb{P}\{\omega_x > s\} = e^{-s}$ for $0 \le s < \infty$. The potential is $V_0(\omega) = \omega_0$ and then $G^{\infty}_{x,y}$ is as in (5.2). This model can be viewed as the zero-temperature limit of the log-gamma polymer (Remark 4.3 in [28]).

Since the limit shape of the exponential corner growth model is known explicitly and has curvature, Busemann functions can be derived with the approach of Newman et al. by first proving coalescence of geodesics. This approach was carried out by Ferrari and Pimentel [24] (see also Sect. 8 of [10]). An alternative approach that begins by constructing stationary cocycles from queueing fixed points is in [27].

Velocity $\xi = (s, 1-s)$ now selects a parameter $\alpha(\xi) = \frac{\sqrt{s}}{\sqrt{s}+\sqrt{1-s}} \in (0,1)$ that characterizes the marginal distributions of the Busemann functions:

$$B^{\xi}_{\text{pp}}(x, x+e_1) \sim \text{Exp}(\alpha(\xi)) \quad \text{and} \quad B^{\xi}_{\text{pp}}(x, x+e_2) \sim \text{Exp}(1-\alpha(\xi)).$$

A tilt dual to $\xi \in \mathcal{U}$ is given by

$$h(\xi) = -\big(\mathbb{E}[B^{\xi}_{\text{pp}}(0, e_1)], \mathbb{E}[B^{\xi}_{\text{pp}}(0, e_2)]\big) = -\Big(\frac{1}{\alpha(\xi)}, \frac{1}{1-\alpha(\xi)}\Big).$$

Substituting in (4.9) we obtain the well-known limit formula from Rost [61]:

$$g^{\infty}_{\text{pp}}(s, 1-s) = 1 + 2\sqrt{s(1-s)}.$$

7. Variational Formulas in Terms of Measures

In this section we derive variational formulas for last-passage percolation in terms of probability measures on the spaces $\Omega_\ell = \Omega \times \mathcal{R}^\ell$ for $\ell \in \mathbb{Z}_+$. This section contains no new results for positive temperature models, but positive temperature results are recalled and rewritten for taking a zero-temperature limit. The formulas we obtain are zero-temperature limits of polymer variational formulas that involve entropy. A maximizing measure can be identified for polymers in weak (enough) disorder (Example 7.7 below). In the final Sect. 8 we relate these measure variational formulas to Perron–Frobenius theory, the classical one for $0 < \beta < \infty$ and max-plus theory for $\beta = \infty$.

Return now to the setting of Sect. 2, with general $\ell \in \mathbb{Z}_+$ and measurable potential $V : \Omega_\ell \to \mathbb{R}$. For $\beta \in (0, \infty]$ define the point-to-level and point-to-point limits g^{β}_{pl} and $g^{\beta}_{\text{pp}}(\xi)$ by Theorem 2.4. A generic element of Ω_ℓ is denoted by $\eta = (\omega, z_{1,\ell})$ with $\omega \in \Omega$ and $z_{1,\ell} = (z_1, \dots, z_\ell) \in \mathcal{R}^\ell$. For $1 \le j \le \ell$ let $Z_j(\omega, z_{1,\ell}) = z_j$ denote the jth step variable on Ω_ℓ. On Ω_ℓ introduce the mappings

$$S_z(\omega, z_{1,\ell}) = (T_{z_1}\omega, (z_{2,\ell-1}, z)), \quad z \in \mathcal{R}. \tag{7.1}$$

When $\ell = 0$, always take $\Omega_0 = \Omega$, $\eta = \omega$ and $S_z = T_z$. In general, let $b\mathcal{X}$ denote the space of bounded measurable real-valued functions on the space $\mathcal{X}$.

The probability measures that appear in the variational formula possess a natural invariance. This is described in the next proposition, proved at the end of the section. One manifestation of the invariance will be the following property of a probability measure $\mu \in \mathcal{M}_1(\Omega_\ell)$ for any $\ell \in \mathbb{Z}_+$:

$$E^{\mu}\big[\max_{z\in\mathcal{R}} f \circ S_z\big] \ge E^{\mu}[f] \quad \forall f \in b\Omega_\ell. \tag{7.2}$$

If $\ell \geq 1$ and $\mu \in \mathcal{M}_1(\Omega_\ell)$, let $\mu_\ell(\cdot \,|\, \omega, z_{1,\ell-1})$ denote the conditional distribution of Z_ℓ under μ, given $(\omega, z_{1,\ell-1})$. We associate to μ the following Markov transition kernel on the space Ω_ℓ:

$$q_z(\omega, z_{1,\ell}) \equiv q\big((\omega, z_{1,\ell}), (T_{z_1}\omega, (z_{2,\ell}, z))\big) = \mu_\ell(z \,|\, T_{z_1}\omega, z_{2,\ell}). \tag{7.3}$$

The first notation is a convenient abbreviation. Under this kernel the state of the Markov chain on Ω_ℓ jumps from $(\omega, z_{1,\ell})$ to $(T_{z_1}\omega, (z_{2,\ell}, z))$ with probability $\mu_\ell(z \,|\, T_{z_1}\omega, z_{2,\ell})$ for $z \in \mathcal{R}$.

Let $z_{k,\infty} = (z_i)_{k \leq i < \infty}$ denote an infinite sequence of steps indexed by $\{k, k+1, k+2, \dots\}$. It is an element of $\mathcal{R}^{\{k,k+1,k+2,\dots\}}$ which we identify with $\mathcal{R}^{\mathbb{N}}$ in the obvious way. On the space $\Omega_{\mathbb{N}} = \Omega \times \mathcal{R}^{\mathbb{N}}$ define a shift mapping S by $S(\omega, z_{1,\infty}) = (T_{z_1}\omega, z_{2,\infty})$. Let $\mathcal{M}_s(\Omega_{\mathbb{N}})$ denote the set of S-invariant probability measures on $\Omega_{\mathbb{N}}$.

Proposition 7.1. Case (a). *Let $\ell \in \mathbb{N}$ and $\mu \in \mathcal{M}_1(\Omega_\ell)$. Then properties* (a.i)–(a.iv) *below are equivalent.*

(a.i) *μ is invariant under kernel* (7.3) *defined in terms of μ itself.*

(a.ii) *μ is the Ω_ℓ-marginal of an S-invariant probability measure $\nu \in \mathcal{M}_s(\Omega_{\mathbb{N}})$.*

(a.iii) *μ has property* (7.2)*.*

(a.iv) *μ satisfies this condition:*

$$E^\mu[f(\omega, Z_{1,\ell-1})] = E^\mu[f(T_{Z_1}\omega, Z_{2,\ell})] \quad \forall\, f \in b\Omega_{\ell-1}. \tag{7.4}$$

Case (b). *Let $\ell = 0$ and $\mu \in \mathcal{M}_1(\Omega)$. Then properties* (b.i)–(b.iii) *below are equivalent.*

(b.i) *There exists a Markov kernel of the form $\{q_z(\omega) \equiv q(\omega, T_z\omega) : z \in \mathcal{R}\}$ on Ω that fixes μ.*

(b.ii) *μ is the Ω-marginal of an S-invariant probability measure $\nu \in \mathcal{M}_s(\Omega_{\mathbb{N}})$.*

(b.iii) *μ has property* (7.2) *with $S_z = T_z$.*

For $\ell \in \mathbb{Z}_+$ let $\mathcal{M}_s(\Omega_\ell)$ denote the space of probability measures described in Proposition 7.1 above. To illustrate, if $\ell = 0$ then $\mathcal{M}_s(\Omega)$ contains all $\{T_x\}$-invariant measures, and if also $0 \in \mathcal{R}$ then $\mathcal{M}_s(\Omega)$ contains all probability measures on Ω.

We can now state the measure variational formulas for point-to-level and point-to-point last-passage percolation limits. For a probability measure μ on Ω_ℓ, μ_0 denotes the Ω-marginal: $\mu_0(A) = \mu(A \times \mathcal{R}^\ell)$. If $\ell = 0$ then $\mu_0 = \mu$. $V^- = -\min\{V, 0\}$ is the negative part of the function V.

Theorem 7.2. *Let $\mathbb{P}$ be ergodic, $\ell \in \mathbb{Z}_+$, and assume $V \in \mathcal{L}$. Then*

$$g_{\mathrm{pl}}^\infty = \sup\Big\{E^\mu[V] : \mu \in \mathcal{M}_s(\Omega_\ell),\ \mu_0 \ll \mathbb{P},\ E^\mu[V^-] < \infty\Big\}. \tag{7.5}$$

The set in braces in (7.5) is not empty because the measure $\mu(d\omega, z_{1,\ell}) = \mathbb{P}(d\omega)\alpha(z_1) \cdots \alpha(z_\ell)$ is a member of $\mathcal{M}_s(\Omega_\ell)$ for any probability α on $\mathcal{R}$ and $V(\cdot\,, z_{1,\ell}) \in L^1(\mathbb{P})$ by the assumption $V \in \mathcal{L}$.

We state the point-to-point version only for the directed i.i.d. $L^{d+\varepsilon}$ case defined in Remark 2.6.

Theorem 7.3. *Let $\Omega = \mathcal{S}^{\mathbb{Z}^d}$ be a product of Polish spaces with shifts $\{T_x\}_{x \in \mathbb{Z}^d}$ and an i.i.d. product measure $\mathbb{P}$. Let $\ell \in \mathbb{N}$ and assume $0 \notin \mathcal{U}$. Assume that $\forall z_{1,\ell} \in \mathcal{R}^\ell$, $V(\omega, z_{1,\ell})$ is a local function of ω and a member of $L^p(\mathbb{P})$ for some $p > d$. Then for all $\xi \in \mathcal{U}$,*

$$g_{\mathrm{pp}}^\infty(\xi) = \sup\Big\{E^\mu[V] : \mu \in \mathcal{M}_s(\Omega_\ell),\ \mu_0 \ll \mathbb{P}, E^\mu[V^-] < \infty, E^\mu[Z_1] = \xi\Big\}. \tag{7.6}$$

Note that even if V is a function on Ω only, variational formula (7.6) uses measures on Ω_ℓ with $\ell \geq 1$ in order for the mean step condition $E^\mu[Z_1] = \xi$ to make sense. Remark 7.8 below explains why Theorem 7.3 is stated only for the directed i.i.d. $L^{d+\varepsilon}$ case. In the general setting of Theorem 7.2 the point-to-point formula (7.6) is valid for compact Ω and $\xi \in \mathrm{ri}\,\mathcal{U}$. It can be derived by applying the argument given below to the results in [54].

To prepare for the proofs we discuss the positive temperature setting. In the end we take $\beta \to \infty$ to prove Theorems 7.2–7.3. Recall the random walk kernel p from the beginning of Sect. 2 with ellipticity constant $\delta = \min_{z\in\mathcal{R}} p(z) > 0$. It acts as a Markov transition kernel on Ω_ℓ through

$$p(\eta, S_z\eta) = p(z) \quad \text{for } z \in \mathcal{R} \quad \text{and} \quad \eta = (\omega, z_{1,\ell}) \in \Omega_\ell. \tag{7.7}$$

This kernel defines a joint Markovian evolution $(T_{X_n}\omega, Z_{n+1,n+\ell})$ of the environment seen by the p-walk X_n and the vector $Z_{n+1,n+\ell} = (Z_{n+1}, \ldots, Z_{n+\ell})$ of the next ℓ steps $Z_k = X_k - X_{k-1}$ of the walk. As before if $\ell = 0$ then $S_z = T_z$ and the Markov chain is $T_{X_n}\omega$.

We define an entropy $\bar{H}(\mu)$ for probability measures $\mu \in \mathcal{M}_1(\Omega_\ell)$, associated to this Markov chain and the background measure $\mathbb{P}$. If $q(\eta, \cdot)$ is a Markov kernel on Ω_ℓ such that $q(\eta, \cdot) \ll p(\eta, \cdot)$ μ-a.s., then $q(\eta, \cdot)$ is supported on $\{S_z\eta\}_{z\in\mathcal{R}}$ and the familiar relative entropy is

$$H(\mu \times q \,|\, \mu \times p) = \int_{\Omega_\ell} \sum_{z\in\mathcal{R}} q(\eta, S_z\eta) \log \frac{q(\eta, S_z\eta)}{p(\eta, S_z\eta)}\, \mu(d\eta).$$

Set

$$\bar{H}(\mu) = \begin{cases} \inf\limits_{q:\,\mu q = \mu} H(\mu \times q \,|\, \mu \times p) & \text{if } \mu_0 \ll \mathbb{P} \\ \infty & \text{otherwise,} \end{cases} \tag{7.8}$$

where the infimum is over Markov kernels q on Ω_ℓ that fix μ, i.e. $\mu q(\cdot) \equiv \int q(\eta, \cdot)\mu(d\eta) = \mu(\cdot)$. The function $\bar{H} : \mathcal{M}_1(\Omega_\ell) \to [0, \infty]$ is convex [53, Sect. 4].

Remark 7.4. When $\mu \in \mathcal{M}_s(\Omega_\ell)$ for some $\ell \geq 1$ and $\mu_0 \ll \mathbb{P}$, the minimizing kernel in (7.8) is the one defined in (7.3), and

$$\bar{H}(\mu) = H(\mu \,|\, \mu_{\ell-1} \otimes p) = \int_{\Omega_\ell} \mu(d\omega, dz_{1,\ell}) \log \frac{\mu_\ell(z_\ell \,|\, \omega, z_{1,\ell-1})}{p(z_\ell)} \tag{7.9}$$

where $\mu_{\ell-1}$ is the distribution of $(\omega, Z_{1,\ell-1})$ under μ and $\mu_{\ell-1} \otimes p$ is the product measure on Ω_ℓ.

Here is the argument. Let $q(\eta, S_z\eta) = q_z(\eta)$ be an arbitrary kernel that fixes μ and is supported on $\{S_z\eta\}_{z\in\mathcal{R}}$. The first equality below is the convex dual representation of relative entropy (see for example Theorem 5.4 in [56]). In the second last equality use both q-invariance and (7.4).

$$\begin{aligned} &H(\mu \times q \,|\, \mu \times p) \\ &= \sup_{h\in b\Omega_\ell^2} \Big\{ \sum_z \int_{\Omega_\ell} h(\eta, S_z\eta)\, q_z(\eta)\, \mu(d\eta) - \log \sum_z p(z) \int_{\Omega_\ell} e^{h(\eta, S_z\eta)}\, \mu(d\eta) \Big\} \end{aligned}$$

$$
\begin{aligned}
&\geq \sup_{f \in b\Omega_\ell} \Big\{ \sum_z \int_{\Omega_\ell} f(S_z\eta)\, q_z(\eta)\, \mu(d\eta) - \log \sum_z p(z) \int_{\Omega_\ell} e^{f(T_{z_1}\omega, (z_{2,\ell}, z))}\, \mu(d\omega, dz_{1,\ell}) \Big\} \\
&= \sup_{f \in b\Omega_\ell} \Big\{ \int_{\Omega_\ell} f\, d\mu - \log \sum_z p(z) \int_{\Omega_{\ell-1}} e^{f(\omega, (z_{1,\ell-1}, z))}\, \mu_{\ell-1}(d\omega, dz_{1,\ell-1}) \Big\} \\
&= H(\mu \,|\, \mu_{\ell-1} \otimes p).
\end{aligned}
$$

We state the measure variational formulas for point-to-level and point-to-point polymers in positive temperature. These are slightly altered versions of Theorem 2.3 of [57] and Theorem 5.3 of [55].

Theorem 7.5. *Let $\mathbb{P}$ be ergodic, $\ell \in \mathbb{Z}_+$, $0 < \beta < \infty$, and assume $V \in \mathcal{L}$. Then*

$$
g_{\mathrm{pl}}^{\beta} = \sup \Big\{ E^{\mu}[V] - \beta^{-1} \bar{H}(\mu) : \mu \in \mathcal{M}_s(\Omega_\ell),\ \mu_0 \ll \mathbb{P},\ E^{\mu}[V^-] < \infty \Big\}. \tag{7.10}
$$

The quantity inside the braces cannot be $\infty - \infty$ for the following reason. By Proposition 7.1 every $\mu \in \mathcal{M}_s(\Omega_\ell)$ is fixed by some kernel q supported on shifts. Thereby, if also $\mu_0 \ll \mathbb{P}$, the definition of entropy gives

$$
0 \leq \bar{H}(\mu) \leq \log \delta^{-1}. \tag{7.11}
$$

As above, we state the point-to-point version only for the directed i.i.d. $L^{d+\varepsilon}$ case defined in Remark 2.6. See Remark 7.8 below for an explanation.

Theorem 7.6. *Repeat the assumptions of Theorem 7.3. Then for $0 < \beta < \infty$ and $\xi \in \mathcal{U}$,*

$$
g_{\mathrm{pp}}^{\beta}(\xi) = \sup \Big\{ E^{\mu}[V] - \beta^{-1} \bar{H}(\mu) : \mu \in \mathcal{M}_s(\Omega_\ell),\ \mu_0 \ll \mathbb{P},\ E^{\mu}[V^-] < \infty,\ E^{\mu}[Z_1] = \xi \Big\}. \tag{7.12}
$$

We illustrate formulas (7.10) and (7.12) in the case of weak disorder.

Example 7.7. (*Directed polymer in weak disorder*) We identify first the measure μ that maximizes variational formula (7.10) for the directed polymer in weak disorder, with potential $V(\omega, z) = V_0(\omega) + h \cdot z = \omega_0 + h \cdot z$ and small enough $0 < \beta < \infty$. This measure will be invariant for the Markov transition implicitly contained in Eq. (3.23). We continue with the notation and assumptions from Example 3.7.

To define the measure we need a backward path and a martingale in the reverse time direction. The backward path $(x_k)_{k \leq 0}$ satisfies $x_0 = 0$ and $z_k = x_k - x_{k-1} \in \mathcal{R}$, and the corresponding martingale is

$$
W_n^- = e^{-n(\lambda(\beta) + \kappa(\beta h))} \sum_{x_{-n,0}} |\mathcal{R}|^{-n}\, e^{\beta \sum_{k=-n}^{-1} \omega_{x_k} - \beta h \cdot x_{-n}}.
$$

W_n^- is the same as W_n composed with the reflection $\omega_x \mapsto \omega_{-x}$, and so (3.21) guarantees also $W_n^- \to W_\infty^-$ with the same properties. (Recall that in this example we took the uniform kernel $p(z) = |\mathcal{R}|^{-1}$.)

By (3.23)

$$
q_0^h(\omega, z) = p(z)\, e^{\beta \omega_0 - \lambda(\beta) + \beta h \cdot z - \kappa(\beta h)} \frac{W_\infty(T_z \omega)}{W_\infty(\omega)}
$$

defines a stochastic kernel from Ω to $\mathcal{R}$. Define a Markov transition kernel on $\Omega \times \mathcal{R}$ by

$$q^h((\omega, z_1), (T_{z_1}\omega, z)) = q_0^h(T_{z_1}\omega, z). \tag{7.13}$$

Define the probability measure μ^h on $\Omega \times \mathcal{R}$ as follows. For a bounded Borel function φ

$$\sum_{z\in\mathcal{R}} \int_\Omega \varphi(\omega, z)\, \mu^h(d\omega, z) = \sum_{z\in\mathcal{R}} \int_\Omega W_\infty^-(\omega)\, W_\infty(\omega)\, q_0^h(\omega, z)\, \varphi(\omega, z)\, \mathbb{P}(d\omega).$$

Using the 1-step decomposition of W_∞^- (analogue of (3.23)) one shows that q^h fixes μ^h.

Let us strengthen assumption (3.21) to also include $\mathbb{E}[W_\infty \log^+ W_\infty] < \infty$. This is true for small enough β. Then the entropy can be calculated:

$$\begin{aligned} &H(\mu^h \times q^h | \mu^h \times p) \\ &\quad= \beta E^{\mu^h}[V] - \lambda(\beta) - \kappa(\beta h) \\ &\qquad + \sum_z \int \mu_0^h(d\omega) q_0^h(\omega, z) \log \frac{W_\infty(T_z\omega)}{W_\infty(\omega)} \\ &\quad= \beta E^{\mu^h}[V] - \lambda(\beta) - \kappa(\beta h) \end{aligned}$$

because the last term of the middle member vanishes by the invariance. $E^{\mu^h}[V]$ is finite because, by independence and Fatou's lemma,

$$E^{\mu^h}(|\omega_0|) = \mathbb{E}(|\omega_0| W_\infty^-\, W_\infty) \le \varliminf_{n\to\infty} \mathbb{E}(|\omega_0| W_n)$$

while the last sequence is bounded, as can be seen by utilizing the 1-step decomposition (3.23) and by taking β in the interior of the region $\lambda(\beta) < \infty$. Consequently

$$E^{\mu^h}[V] - \beta^{-1} H(\mu^h \times q^h | \mu^h \times p) = \beta^{-1}(\lambda(\beta) + \kappa(\beta h)) = g_{\mathrm{pl}}^\beta(h). \tag{7.14}$$

The pair (μ^h, q^h) is the unique one that satisfies (7.14), by virtue of the strict convexity of entropy.

The maximizer for the point-to-point formula (7.12) can also be found. Let $g_{\mathrm{pp}}^\beta(\xi)$ be as in (4.1) with $V_0(\omega) = \omega_0$. Given $\xi \in \mathrm{ri}\,\mathcal{U}$, $h \in \mathbb{R}^d$ can be chosen so that $\nabla\kappa(\beta h) = \xi$. If β is small enough, uniform integrability of the martingales W_n can be ensured, and thereby μ^h and q^h are again well-defined. The choice of h implies that $E^{\mu^h}[Z_1] = \xi$, and we can turn (7.14) into

$$\begin{aligned} E^{\mu^h}[V_0] - \beta^{-1} H(\mu^h \times q^h | \mu^h \times p) &= -h \cdot E^{\mu^h}[Z_1] + \beta^{-1}(\lambda(\beta) + \kappa(\beta h)) \\ &= \beta^{-1}\lambda(\beta) - \beta^{-1}\kappa^*(\xi) = g_{\mathrm{pp}}^\beta(\xi). \end{aligned}$$

The last equality can be seen for example from duality (4.4).

Markov chain (7.13) appeared in [15]. Under some restrictions on the environment and with $h = 0$, [48] showed that μ_0^0 is the limit of the environment seen by the particle.

We prove the theorems of this section, beginning with the positive temperature statements.

Proof of Theorems 7.5 and 7.6. Let $V : \Omega_\ell \to \mathbb{R}$ be a member of $\mathcal{L}$ (Definition 2.1), $\mathbb{P}$ ergodic and $0 < \beta < \infty$. Theorem 2.3 of [57] gives the variational formula

$$g_{\text{pl}}^{\beta} = \sup\Big\{E^{\mu}[\min(V,c)] - \beta^{-1}\bar{H}(\mu) : \mu \in \mathcal{M}_1(\Omega_\ell),\ c > 0\Big\}. \tag{7.15}$$

Note that [57] used the uniform kernel $p(z) = |\mathcal{R}|^{-1}$ but this makes no difference to the proofs, and in any case the kernel can be included in the potential to extend the result to an arbitrary kernel supported on $\mathcal{R}$. We convert (7.15) into (7.10) in a few steps.

The measure $\mu = \mathbb{P}\otimes\alpha$ with $\alpha(z_{1,\ell}) = p(z_{1,\ell})$ satisfies $\mu \in \mathcal{M}_s(\Omega_\ell)$, $\mu p = \mu$, and $\bar{H}(\mu) = 0$. Since $V(\cdot\,, z_{1,\ell}) \in L^1(\mathbb{P})$, this gives the finite lower bound $g_{\text{pl}}^{\beta} \geq E^{\mathbb{P}\otimes\alpha}[V]$ for (7.15). (If $\ell = 0$ the α-factor is not there.) Hence we can restrict the supremum in (7.15) to μ such that $E^{\mu}[V^-] + \bar{H}(\mu) < \infty$. Since $E^{\mu}[V]$ is well-defined in $(-\infty, \infty]$ for all such μ, we can drop the truncation at c.

Entropy has the following representation: for $\mu \in \mathcal{M}_1(\Omega_\ell)$,

$$\inf_{q:\mu q=\mu} H(\mu \times q \,|\, \mu \times p) = -\inf_{f\in b\Omega_\ell} E^{\mu}\Big[\log\sum_z p(z)e^{f\circ S_z - f}\Big]. \tag{7.16}$$

The infimum on the left is over Markov kernels q on Ω_ℓ that fix μ. S_z is the shift mapping defined in (7.1). For a proof of (7.16) see Theorem 2.1 of [21], Lemma 2.19 of [63], or Theorem 14.2 of [56].

Recall the definition of $\bar{H}$ in (7.8). From the inequality

$$\log\sum_z p(z)e^{f\circ S_z - f} \leq \max_z\{f\circ S_z - f\} \leq \log\sum_z p(z)e^{f\circ S_z - f} + \log\delta^{-1}$$

follows, for $\mu_0 \ll \mathbb{P}$,

$$\bar{H}(\mu) - \log\delta^{-1} \leq -\inf_{f\in b\Omega_\ell} E^{\mu}\big[\max_z\{f\circ S_z - f\}\big] \leq \bar{H}(\mu). \tag{7.17}$$

If there exists $f \in b\Omega_\ell$ such that $E^{\mu}[\max_z\{f\circ S_z - f\}] < 0$ then replacing f by cf and taking $c \to \infty$ shows that the infimum over f is actually $-\infty$. This makes $\bar{H}(\mu) = \infty$. Thus, relevant measures μ in (7.15) are ones that satisfy (7.2) and so we can insert the restriction $\mu \in \mathcal{M}_s(\Omega_\ell)$ into (7.15). (7.15) has been converted into (7.10).

Assuming the directed i.i.d. $L^{d+\varepsilon}$ setting described in Theorem 7.3, Theorem 5.3 of [55] gives the point-to-point version: for $\xi \in \mathcal{U}$,

$$g_{\text{pp}}^{\beta}(\xi) = \sup\big\{E^{\mu}[\min(V,c)] - \beta^{-1}\bar{H}(\mu) : \mu \in \mathcal{M}_1(\Omega_\ell),\ E^{\mu}[Z_1] = \xi,\ c > 0\big\}. \tag{7.18}$$

This is converted into (7.12) by the same reasoning used above. $\square$

Remark 7.8. We can state (7.18) only for the directed i.i.d. $L^{d+\varepsilon}$ setting for the following reason. The point-to-level formula (7.15) is proved directly in [57]. By contrast, the point-to-point formula (7.18) is derived in [55] via a contraction applied to a quenched large deviation principle (LDP) for polymer measures. This LDP is proved in [57]. In the general setting the upper bound of this LDP has been proved only for compact sets (weak LDP). However, in the directed i.i.d. case the LDP is a full LDP, and the contraction works without additional assumptions. Consequently in the directed i.i.d. $L^{d+\varepsilon}$ setting (7.18) is valid for Polish spaces Ω, but in the general setting Ω would need to be compact.

Proof of Theorems 7.2 and 7.3. Take $\beta \to \infty$ in (7.10) and (7.12), utilizing bounds (7.11) and (2.11). $\square$

Proof of Proposition 7.1. Each f below is a $b\Omega_\ell$ test function on the appropriate space Ω_ℓ. First we work with the case $\ell \geq 1$. We argue the implications (a.i)⇒(a.ii)⇒(a.iii)⇒(a.iv)⇒(a.i).

(a.i)⇒(a.ii): An S-invariant probability measure ν on $\Omega_{\mathbb{N}} = \Omega \times \mathcal{R}^{\mathbb{N}}$ that extends μ can be defined by writing, for any $m \geq \ell$,

$$\int f(\omega, z_{1,m})\, d\nu = \sum_{z_{1,m}} \int_\Omega f(\omega, z_{1,m}) \prod_{i=\ell+1}^{m} q_{z_i}(T_{x_{i-\ell-1}}\omega, z_{i-\ell,i-1})\, \mu(d\omega, z_{1,\ell}). \tag{7.19}$$

(a.ii)⇒(a.iii): From the S-invariance of ν,

$$\begin{aligned} E^\mu\big[\max_z f(T_{Z_1}\omega, (Z_{2,\ell}, z))\big] &= E^\nu\big[\max_z f(T_{Z_1}\omega, (Z_{2,\ell}, z))\big] \\ &= E^\nu\big[\max_z f(\omega, (Z_{1,\ell-1}, z))\big] \\ &\geq E^\nu\big[f(\omega, Z_{1,\ell})\big] = E^\mu[f]. \end{aligned}$$

(a.iii)⇒(a.iv): If f is only a function of $(\omega, z_{1,\ell-1})$, then $f(S_z(\omega, z_{1,\ell})) = f(T_{z_1}\omega, z_{2,\ell})$ does not depend on z. (7.2) then implies $E^\mu\big[f(T_{Z_1}\omega, Z_{2,\ell})\big] \geq E^\mu[f]$. Replacing f by $-f$ makes this an equality and (7.4) follows.

(a.iv)⇒(a.i): Use property (a.iv) in the second equality below to show that $\mu q = \mu$.

$$\begin{aligned} &\int_{\Omega\times\mathcal{R}^\ell} \sum_z q_z(\omega, z_{1,\ell}) f(T_{z_1}\omega, (z_{2,\ell}, z))\, \mu(d\omega, dz_{1,\ell}) \\ &= \sum_z \int_{\Omega\times\mathcal{R}^\ell} f(T_{z_1}\omega, (z_{2,\ell}, z))\, \mu_\ell(z \,|\, T_{z_1}\omega, z_{2,\ell})\, \mu(d\omega, dz_{1,\ell}) \\ &= \sum_z \int_{\Omega\times\mathcal{R}^\ell} f(\omega, (z_{1,\ell-1}, z))\, \mu_\ell(z \,|\, \omega, z_{1,\ell-1})\, \mu(d\omega, dz_{1,\ell}) \\ &= \int_{\Omega\times\mathcal{R}^\ell} f(\omega, z_{1,\ell})\, \mu(d\omega, dz_{1,\ell}). \end{aligned}$$

We turn to the case $\ell = 0$ and show (b.i)⇒(b.ii)⇒(b.iii)⇒(b.i).

(b.i)⇒(b.ii): Now define ν on $\Omega \times \mathcal{R}^{\mathbb{N}}$ by

$$E^\nu[f(\omega, Z_{1,m})] = \sum_{z_{1,m}} \int f(\omega, z_{1,m}) \prod_{i=1}^{m} q_{z_i}(T_{x_{i-1}}\omega)\, \mu(d\omega).$$

(b.ii)⇒(b.iii): Analogously to (a.ii)⇒(a.iii) above,

$$E^\mu\big[\max_z f(T_z\omega)\big] = E^\nu\big[\max_z f(T_z\omega)\big] \geq E^\nu\big[f(T_{Z_1}\omega)\big] = E^\nu[f(\omega)] = E^\mu[f].$$

(b.iii)⇒(b.i): Observe that for $f \in b\Omega$ we have

$$E^\mu\big[\max_z\{f \circ T_z - f\}\big] \leq E^\mu\big[\log \sum_z p(z) e^{f\circ T_z - f}\big] + \log \delta^{-1}.$$

By assumption (7.2) the left-hand side is nonnegative. Then by (7.16)

$$\inf\{H(\mu \times q \,|\, \mu \times p) : \mu q = \mu\} = -\inf_{f \in b\Omega} E^{\mu}\Big[\log \sum_z p(z) e^{f\circ T_z - f}\Big] \le \log \delta^{-1}.$$

Since the infimum is not $+\infty$ there must exist a Markov kernel q that fixes μ and for which $H(\mu \times q \,|\, \mu \times p) < \infty$. This implies that for μ-a.e. ω the kernel is supported on $\{T_z\omega : z \in \mathcal{R}\}$. $\square$

8. Periodic Environments

The case of finite Ω provides explicit illustration of the theory developed in the paper. The point-to-level limits and solutions to the variational formulas come from Perron–Frobenius theory, the classical theory for $0 < \beta < \infty$ and the max-plus theory for $\beta = \infty$. (See [4,7,31,62] for expositions.) We consider a potential $V(\omega, z) = V_0(\omega) + h \cdot z$ for $(\omega, z) \in \Omega \times \mathcal{R}$, $h \in \mathbb{R}^d$.

Let Ω be a finite set of m elements. As all along, $\{T_x\}_{x\in\mathcal{G}}$ is a group of commuting bijections on Ω that act irreducibly. That is, for each pair $(\omega, \omega') \in \Omega \times \Omega$ there exist $z_1, \dots, z_k \in \mathcal{R}$ such that $T_{z_1+\cdots+z_k}\omega = \omega'$. The ergodic probability measure is $\mathbb{P}(\omega) = m^{-1}$.

A basic example is a periodic environment indexed by $\mathbb{Z}^d$. Take a vector $a > 0$ in $\mathbb{Z}^d$ (coordinatewise inequalities), define the rectangle $\Lambda = \{x \in \mathbb{Z}^d : 0 \le x < a\}$, fix a finite configuration $(\bar{\omega}_x)_{x\in\Lambda}$, and then extend $\bar{\omega}$ to all of $\mathbb{Z}^d$ periodically: $\bar{\omega}_{x+k\circ a} = \bar{\omega}_x$ for $k \in \mathbb{Z}^d$, where $k \circ a = (k_i a_i)_{1\le i\le d}$ is the coordinatewise product of two vectors. Irreducibility holds for example if $\mathcal{R}$ contains $\{e_1, \dots, e_d\}$.

8.1. Case $0 < \beta < \infty$. We take $\beta = 1$ and drop it from the notation. Define a nonnegative irreducible matrix indexed by Ω by

$$A_{\omega,\omega'} = \sum_{z\in\mathcal{R}} p(z)\, \mathbb{1}\{T_z\omega = \omega'\} e^{V_0(\omega)+h\cdot z} \quad \text{for } \omega, \omega' \in \Omega. \tag{8.1}$$

Let ρ be the Perron–Frobenius eigenvalue (spectral radius) of A. Then by standard asymptotics the limiting point-to-level free energy is

$$\begin{aligned} g_{\mathrm{pl}}(h) &= \lim_{n\to\infty} n^{-1} \log \sum_{x_{0,n}:\, x_0=0} p(x_{0,n}) e^{\sum_{k=0}^{n-1} V_0(T_{x_k}\omega)+h\cdot x_n} \\ &= \lim_{n\to\infty} n^{-1} \log \sum_{\omega'\in\Omega} A^n_{\omega,\omega'} = \log \rho. \end{aligned} \tag{8.2}$$

On a finite Ω every cocycle is a gradient (proof left to the reader). Hence we can replace the general cocycle F with a gradient $F(\omega, 0, z) = f(T_z\omega) - f(\omega)$ and write the cocycle variational formula (3.7) as

$$g_{\mathrm{pl}}(h) = \inf_{f\in\mathbb{R}^{\Omega}} \max_{\omega} \log \sum_{z\in\mathcal{R}} p(z) e^{V_0(\omega)+h\cdot z+f(T_z\omega)-f(\omega)}. \tag{8.3}$$

This is now exactly the same as the following textbook characterization of the Perron–Frobenius eigenvalue:

$$\rho = \inf_{\varphi \in \mathbb{R}^{\Omega}:\, \varphi > 0} \max_{\omega} \frac{1}{\varphi(\omega)} \sum_{\omega'} A_{\omega,\omega'} \varphi(\omega'). \tag{8.4}$$

Let σ and τ be the left and right (strictly positive) Perron–Frobenius eigenvectors of A normalized so that $\sum_{\omega \in \Omega} \sigma(\omega)\tau(\omega) = 1$. For each $\omega \in \Omega$ the left eigenvector equation is

$$\sum_{z \in \mathcal{R}} p(z)\, e^{V_0(T_{-z}\omega) + h\cdot z} \sigma(T_{-z}\omega) = \rho\sigma(\omega) \tag{8.5}$$

and the right eigenvector equation is

$$e^{V_0(\omega)} \sum_{z \in \mathcal{R}} p(z) e^{h\cdot z} \tau(T_z\omega) = \rho\tau(\omega). \tag{8.6}$$

The right eigenvector equation (8.6) says that the gradient

$$F(\omega, x, y) = \log \tau(T_y\omega) - \log \tau(T_x\omega) \tag{8.7}$$

minimizes in (8.3) without the maximum over ω (the right-hand side of (8.3) is constant in ω). In other words, F is a corrector for $g_{\mathrm{pl}}(h)$. Compare this to (3.12).

Define a probability measure on Ω by $\mu_0(\omega) = \sigma(\omega)\tau(\omega)$. The left eigenvector equation (8.5) says that μ_0 is invariant under the stochastic kernel

$$q_0(\omega, \omega') = \sum_{z \in \mathcal{R}} p(z)\, \mathbb{1}\{T_z\omega = \omega'\} e^{V_0(\omega) + h\cdot z + F(\omega,0,z) - g_{\mathrm{pl}}(h)}, \quad \omega, \omega' \in \Omega. \tag{8.8}$$

Using this one can check that the measure

$$\mu(\omega, z_1) = p(z)\mu_0(\omega) e^{V_0(\omega) + h\cdot z_1 + F(\omega,0,z_1) - g_{\mathrm{pl}}(h)}$$

is a member of $\mathcal{M}_s(\Omega \times \mathcal{R})$ and invariant under the kernel

$$q((\omega, z_1), (T_{z_1}\omega, z)) = p(z) e^{V_0(T_{z_1}\omega) + h\cdot z + F(T_{z_1}\omega,0,z) - g_{\mathrm{pl}}(h)}.$$

Another computation checks that

$$E^{\mu}[V_0(\omega) + h \cdot Z_1] - H(\mu \times q \,|\, \mu \times p) = g_{\mathrm{pl}}(h).$$

Hence μ is a maximizer in the entropy variational formula (7.15).

Assume additionally that matrix A is aperiodic on Ω. Then A is primitive, that is, A^n is strictly positive for large enough n. Perron–Frobenius asymptotics (for example, Theorem 1.2 in [62]) give the Busemann function B^h_{pl} of (5.5).

$$\begin{aligned} B^h_{\mathrm{pl}}(\omega, 0, z) = \lim_{n\to\infty} \Bigg\{ &\log \sum_{x_{0,n}:\, x_0 = 0} p(x_{0,n}) e^{\sum_{k=0}^{n-1} V_0(T_{x_k}\omega) + h\cdot x_n} \\ &- \log \sum_{x_{0,n-1}:\, x_0 = z} p(x_{0,n-1}) e^{\sum_{k=0}^{n-2} V_0(T_{x_k}\omega) + h\cdot(x_{n-1} - z)} \Bigg\} \\ = \lim_{n\to\infty} \Bigg\{ &\log \sum_{\omega' \in \Omega} A^n_{\omega,\omega'} - \log \sum_{\omega' \in \Omega} A^{n-1}_{T_z\omega,\omega'} \Bigg\} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\Big\{\log\rho + \log\Big(\sum_{\omega'\in\Omega}\tau(\omega)\sigma(\omega') + o(1)\Big) \\
&\quad - \log\Big(\sum_{\omega'\in\Omega}\tau(T_z\omega)\sigma(\omega') + o(1)\Big)\Big\} \\
&= \log\rho + \log\tau(\omega) - \log\tau(T_z\omega).
\end{aligned}
$$

If we assume that all admissible paths between two given points have the same number of steps, then $B^h_{\text{pl}}(\omega, 0, z)$ extends to a stationary L^1 cocycle, as showed in Theorem 5.2. Then this situation fits the development of Sects. 3–5. Equation (8.6) shows that cocycle

$$
\widetilde{B}(\omega, 0, z) = B^h_{\text{pl}}(\omega, 0, z) - h\cdot z \tag{8.9}
$$

is adapted to V_0, illustrating Theorem 5.2. Definition (3.5) applied to the explicit formulas above gives

$$
h(\widetilde{B})\cdot z = -\mathbb{E}[\widetilde{B}(\omega, 0, z)] = -\log\rho + h\cdot z \quad \text{for each } z\in\mathcal{R}.
$$

Consequently $h(B^h_{\text{pl}}) \perp \text{aff}\,\mathcal{R}$. By Theorem 3.4 the cocycle

$$
\begin{aligned}
\widetilde{F}(\omega, 0, z) &= -h(\widetilde{B})\cdot z - \widetilde{B}(\omega, 0, z) = \log\rho - B^h_{\text{pl}}(\omega, 0, z) \\
&= \log\tau(T_z\omega) - \log\tau(\omega),
\end{aligned} \tag{8.10}
$$

that appeared in (8.7), is the minimizer in (8.3) for any tilt h' such that $(h' - h(\widetilde{B}))\cdot z = (h'-h)\cdot z + \log\rho$ is constant over $z\in\mathcal{R}$.

Connection (8.2) between the limiting free energy and the Perron–Frobenius eigenvalue is standard fare in textbook treatments of the large deviation theory of finite Markov chains [19,56,69].

8.2. Point-to-level last-passage case. The *max-plus algebra* is the semiring $\mathbb{R}_{\max} = \mathbb{R}\cup\{-\infty\}$ under the operations $x\oplus y = x\vee y$ and $x\otimes y = x+y$. Define an irreducible $\mathbb{R}_{\max}$-valued matrix by

$$
A(\omega, \omega') = \begin{cases} V_0(\omega) + \max\limits_{z: T_z\omega=\omega'} h\cdot z, & \omega'\in\{T_z\omega : z\in\mathcal{R}\} \\ -\infty, & \omega'\notin\{T_z\omega : z\in\mathcal{R}\}. \end{cases} \tag{8.11}
$$

As an irreducible matrix A has a unique finite max-plus eigenvalue λ together with a (not necessarily unique even up to an additive constant) finite eigenvector σ that satisfy

$$
\max_{\omega'\in\Omega}\,[A(\omega,\omega') + \sigma(\omega')] = \lambda + \sigma(\omega), \quad \omega\in\Omega. \tag{8.12}
$$

Inductively

$$
\max_{\omega=\omega_0,\,\omega_1,\dots,\,\omega_n}\Big\{\sum_{k=0}^{n-1} A(\omega_k, \omega_{k+1}) + \sigma(\omega_n)\Big\} = n\lambda + \sigma(\omega), \quad \omega\in\Omega. \tag{8.13}
$$

The last-passage value from (3.3) can be expressed as

$$
G^\infty_{0,(n)}(h) = \max_{x_{0,n}}\sum_{k=0}^{n-1}\big(V_0(T_{x_k}\omega) + h\cdot(x_{k+1}-x_k)\big) = \max_{\omega=\omega_0,\,\omega_1,\dots,\,\omega_n}\sum_{k=0}^{n-1} A(\omega_k, \omega_{k+1}). \tag{8.14}
$$

Dividing through (8.13) by n gives the limit

$$g_{\text{pl}}^{\infty}(h) = \lim_{n\to\infty} n^{-1} G_{0,(n)}^{\infty}(h) = \lambda.$$

The eigenvalue equation (8.12) now rewrites as

$$g_{\text{pl}}^{\infty}(h) = \max_{z\in\mathcal{R}} \{V_0(\omega) + h \cdot z + \sigma(T_z\omega) - \sigma(\omega)\}. \tag{8.15}$$

This is the cocycle variational formula (3.8) (without the supremum over ω) and shows that a corrector is given by the gradient

$$F(\omega, 0, z) = \sigma(T_z\omega) - \sigma(\omega). \tag{8.16}$$

Compare (8.15) to (3.13).

The measure variational formula (7.10) links with an alternative characterization of the max-plus eigenvalue as the maximal average weight of an elementary circuit. To describe this, consider the directed graph $(\Omega, \mathcal{E})$ with vertex set Ω and edges $\mathcal{E} = \{(\omega, T_z\omega) : \omega \in \Omega, z \in \mathcal{R}\}$. This allows multiple edges from ω to ω' and loops from ω to itself. Loops happen in particular if $0 \in \mathcal{R}$. Identify edge $(\omega, T_z\omega)$ with the pair (ω, z). An *elementary circuit* of length N is a sequence of edges $(\omega_0, z_1), (\omega_1, z_2), \dots, (\omega_{N-1}, z_N)$ such that $\omega_i = T_{z_i}\omega_{i-1}$ with $\omega_N = \omega_0$, but $\omega_i \neq \omega_j$ for $0 \le i < j < N$.

Given any fixed ω, all elementary circuits can be represented as admissible paths $x_0, x_1, \dots, x_N$ in $\mathcal{G}$ by choosing x_0 so that $\omega_0 = T_{x_0}\omega$ and $x_i = x_{i-1} + z_i$ for $1 \le i \le N$. Conversely, an admissible path $x_0, x_1, \dots, x_N$ in $\mathcal{G}$ represents an elementary circuit if $T_{x_0}\omega, T_{x_1}\omega, \dots, T_{x_{N-1}}\omega$ are distinct, but $T_{x_0}\omega = T_{x_N}\omega$. Let $\mathcal{C}$ denote the set of elementary circuits. The average weight formula for the eigenvalue is (Thm. 2.9 in [31])

$$\lambda = \max_{N\in\mathbb{N},\, x_{0,N}\in\mathcal{C}} N^{-1} \sum_{k=0}^{N-1} \big(V_0(T_{x_k}\omega) + h \cdot z_{k+1}\big). \tag{8.17}$$

The right-hand side is independent of ω because switching ω amounts to translating the circuit, by the assumption of irreducible action by $\{T_z\}_{z\in\mathcal{R}}$.

It is elementary to verify from definitions that $g_{\text{pl}}^{\infty}(h)$ equals the right-hand side of (8.17). (The sum on the right-hand side of (8.14) decomposes into circuits and a bounded part, while an asymptotically optimal path finds a maximizing circuit and repeats it forever.) If we take (8.17) as the definition of λ, then the identity

$$\lambda = \max\Big\{ \sum_{(\omega,z)\in\Omega\times\mathcal{R}} \mu(\omega, z)(V_0(\omega) + h \cdot z) : \mu \in \mathcal{M}_s(\Omega \times \mathcal{R})\Big\} \tag{8.18}$$

follows from the fact that the extreme points of the convex set $\mathcal{M}_s(\Omega \times \mathcal{R})$ are exactly those uniform probability measures whose support is a single elementary circuit. We omit the proof. Equation (8.18) is the measure variational formula (7.10) which has now been (re)derived in the finite setting from max-plus theory.

As in the finite temperature case, existence of point-to-level Busemann functions follows from asymptotics of matrices. The *critical graph* of the max-plus matrix A is the subgraph of $(\Omega, \mathcal{E})$ consisting of those nodes and edges that belong to elementary circuits that maximize in (8.17). Matrix A is *primitive* if it is irreducible and if its critical graph has a unique strongly connected component with cyclicity 1 (that is, a unique irreducible and aperiodic component in Markov chain terminology). This implies that the eigenvector is unique up to an additive constant and these asymptotics hold as $n \to \infty$:

$$G^{\infty}_{0,(n)}(h) - G^{\infty}_{z,(n-1)}(h) = (A^{\otimes n} \otimes \mathbf{0})(\omega) - (A^{\otimes(n-1)} \otimes \mathbf{0})(T_z\omega)$$
$$\longrightarrow \lambda + \sigma(\omega) - \sigma(T_z\omega) \equiv B^h_{\mathrm{pl}}(\omega, 0, z). \qquad (8.19)$$

(From [31] apply Thm. 3.9 with cyclicity 1 and section 4.3.) Above $\mathbf{0} = (0, \dots, 0)^T$ and operations $\otimes$ are in the max-plus sense. Equation (8.15) shows that cocycle $\widetilde{B}(\omega, 0, z) = B^h_{\mathrm{pl}}(\omega, 0, z) - h \cdot z$ is adapted to V_0, as an example of Theorem 5.2 for $\beta = \infty$.

The next simple example illustrates the max-plus case. All the previous results of this paper identify correctors that solve the variational formulas of Theorem 3.2 so that the essential supremum over ω can be dropped. This example shows that there can be additional minimizing cocycles F for which the function of ω on the right in (3.8) is not constant in ω.

Example 8.1. Take $d = 2$ and a two-point environment space $\Omega = \{\omega^{(1)}, \omega^{(2)} = T_{e_1}\omega^{(1)}\}$ where $\omega^{(1)}_{i,j} = \frac{1}{2}(1 + (-1)^i)$ for $(i, j) \in \mathbb{Z}^2$ is a vertically striped configuration of zeroes and ones, with a one at the origin (Fig. 1). Admissible steps are $\mathcal{R} = \{e_1, e_2\}$ and T_{e_2} acts as an identity. The ergodic measure is $\mathbb{P} = \frac{1}{2}(\delta_{\omega^{(1)}} + \delta_{\omega^{(2)}})$ and the potential $V_0(\omega) = \omega_0$ with tilts $h = (h_1, h_2) \in \mathbb{R}^2$.

Matrix $A(\omega^{(i)}, \omega^{(j)})$ of (8.11) is

$$A = \begin{bmatrix} 1 + h_2 & 1 + h_1 \\ h_1 & h_2 \end{bmatrix}$$

and the directed graph $(\Omega, \mathcal{E})$ is in Fig. 2.

Since A is irreducible its unique max-plus eigenvalue is the maximum average value of elementary circuits and this gives the point-to-line last-passage limit:

$$g^{\infty}_{\mathrm{pl}}(h) = \lambda = \max\{\tfrac{1}{2} + h_1, 1 + h_2\}. \qquad (8.20)$$

There are two cases to consider, and in both cases there is a unique eigenvector (up to an additive constant) $\sigma = (\sigma(\omega^{(1)}), \sigma(\omega^{(2)}))$:

(i) $\frac{1}{2} + h_1 \le 1 + h_2 = \lambda$, $\sigma = (1, h_1 - h_2)$, the critical graph has cyclicity 1.
(ii) $1 + h_2 < \frac{1}{2} + h_1 = \lambda$, $\sigma = (1, \frac{1}{2})$, the critical graph has cyclicity 2.

1	0	1	0	1	0	1	0
1	0	1	0	1	0	1	0
1	0	1	0	$\boxed{1}$	0	1	0
1	0	1	0	1	0	1	0

Fig. 1. Environment configuration $\omega^{(1)}$ indexed by $\mathbb{Z}^2$ in Example 8.1. The origin is *shaded* in a *thick frame*

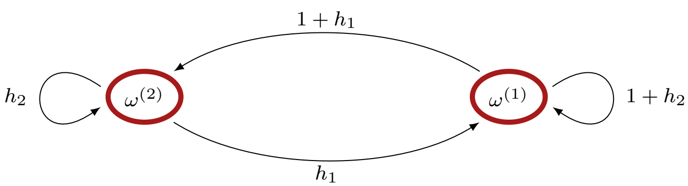

Fig. 2. Graph $(\Omega, \mathcal{E})$ for Example 8.1

Case (i). One can verify by hand that variational formula (3.8) is minimized by the cocycles

$$F(\omega^{(1)}, 0, e_1) = a = -F(\omega^{(2)}, 0, e_1), \quad F(\omega^{(1)}, 0, e_2) = F(\omega^{(2)}, 0, e_2) = 0 \tag{8.21}$$

for $a \in [h_1 - h_2 - 1, h_2 - h_1]$. Let $\widetilde{F}$ denote the cocycle for $a = h_1 - h_2 - 1$ which is the one consistent with (8.16) for the eigenvector σ. Among the minimizing cocycles only $\widetilde{F}$ satisfies (3.8) without $\max_\omega$, that is, in the form (3.13). And indeed this corrector comes from Theorem 3.4(ii-b). $\widetilde{F}$ is given by Eq. (3.6) with a cocycle $\widetilde{B}$ that is adapted to V_0 (as defined in (3.10)) if and only if $1+h_2 \geq \frac{1}{2}+h_1$. In case (i) matrix A is primitive and limit (8.19) gives an explicit Busemann function $B^h_{\mathrm{pl}}(\omega, 0, z)$. From this Busemann function (8.9) gives cocycle $\widetilde{B}$.

Case (ii). In this case there is a unique minimizing corrector $\check{F}$ which is (8.21) with $a = -1/2$, the one that satisfies (8.16) for the eigenvector σ. $\check{F}$ comes via Eq. (3.6) from a cocycle that is adapted to V_0 if and only if $\frac{1}{2} + h_1 \geq 1 + h_2$. So the variational formula (3.8) is again satisfied without $\max_\omega$. However, this time $\check{F}$ cannot come from Busemann functions because some Busemann functions do not exist. Maximizing n-step paths use only e_1-steps and consequently

$$G^\infty_{0,(n)}(h) - G^\infty_{e_2,(n-1)}(h) = h_1 + \mathbb{1}\{n \text{ is odd}\}$$

does not converge as $n \to \infty$.

Note that $\check{F}$ is a minimizing cocycle in both cases (i) and (ii), but only in case (ii) it satisfies (3.8) without $\max_\omega$. □

A. Auxiliary Lemmas

Centered cocycles satisfy a uniform ergodic theorem. The following is a special case of Theorem 9.3 of [28]. Note that a one-sided bound suffices for a hypothesis. Recall Definition 2.1 for class $\mathcal{L}$ and Definition 3.1 for the space $\mathcal{K}_0$ of centered cocycles.

Theorem A.1. *Assume $\mathbb{P}$ is ergodic under the transformations $\{T_z : z \in \mathcal{R}\}$. Let $F \in \mathcal{K}_0$. Assume there exists $V \in \mathcal{L}$ such that $\max_{z\in\mathcal{R}} F(\omega, 0, z) \leq V(\omega)$ for $\mathbb{P}$-a.e. ω. Then for $\mathbb{P}$-a.e. ω*

$$\lim_{n\to\infty} \max_{\substack{x=z_1+\cdots+z_n \\ z_{1,n}\in\mathcal{R}^n}} \frac{|F(\omega, 0, x)|}{n} = 0.$$

Lemma A.2. *Let $X_n \in L^1$, $X_n \to X$ a.s., $\varliminf_{n\to\infty} EX_n \leq c < \infty$, and X_n^- uniformly integrable. Then $X \in L^1$ and $EX \leq c$.*

Proof. Since $X_n^- \to X^-$ a.s. and X_n^- is uniformly integrable, $X_n^- \to X^-$ in L^1 and in particular $X^- \in L^1$. By Fatou's lemma and by the assumption,

$$E(X^+) = E(\lim_{n\to\infty} X_n^+) \leq \varliminf_{n\to\infty} E(X_n^+) = \varliminf_{n\to\infty} E(X_n + X_n^-) \leq c + E(X^-) < \infty$$

from which we conclude that $X \in L^1$ and then $EX \leq c$. □

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