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On the Stability of Self-Similar Solutions to Nonlinear Wave Equations

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Abstract: We consider an explicit self-similar solution to an energy-supercritical Yang-Mills equation and prove its mode stability. Based on earlier work by one of the authors, we obtain a fully rigorous proof of the *nonlinear* stability of the self-similar blowup profile. This is a large-data result for a supercritical wave equation. Our method is broadly applicable and provides a general approach to stability problems related to self-similar solutions of nonlinear wave equations.

1. Introduction

The development of singularities in finite time is one of the most stunning features of nonlinear evolution equations. Singularity formation (or "blowup") of the solution signifies a dramatic change in the behavior of the underlying model or even the complete breakdown of the mathematical description. On the level of a fundamental physical theory, blowup occurs in Einstein's equation of general relativity to indicate the dynamical formation of a black hole. However, a rigorous treatment of Einstein's equation in this context is hopeless at the present stage of research. Consequently, it is a reasonable strategy to resort to simpler toy models that capture some of the features of the more complicated system. Natural candidates in this respect are energy-supercritical nonlinear wave equations with a geometric origin such as wave maps or Yang-Mills models.

The easiest way to demonstrate finite-time blowup in a given evolution equation is to construct self-similar solutions. In exceptional cases it is even possible to obtain closed-form expressions. The relevance of such solutions depends on their stability. After all, one would like to obtain information on the *generic* behavior of the system. However, already at the linear level the stability analysis of self-similar solutions to nonlinear wave equations is very challenging since one is confronted with highly nonself-adjoint

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spectral problems. Consequently, standard methods do not apply. This fact poses a serious obstacle to any rigorous analysis of the blowup dynamics.

In the present paper we develop a general approach that is capable of handling the difficult nonself-adjoint spectral problems related to self-similar blowup. For the sake of simplicity, however, we focus on the concrete example of an energy-supercritical Yang-Mills equation that displays blowup via an explicitly known self-similar solution.

1.1. An energy-supercritical Yang-Mills model. For $\mu \in \{0, 1, 2, \dots, 5\}$ let $A_\mu : \mathbb{R}^{1,5} \to \mathfrak{so}(5)$ be a collection of five fields on (1+5)-dimensional Minkowski space with values in the matrix Lie algebra of SO(5). In other words, for fixed μ and $(t, x) \in \mathbb{R}^{1,5}$, $A_\mu(t, x)$ is a skew-symmetric real (5×5)-matrix. One sets

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

and considers the action functional[1]

$$\int_{\mathbb{R}^{1,5}} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu}). \tag{1.1}$$

Formally, this is reminiscent of Maxwell's theory. However, the commutator in the definition of $F_{\mu\nu}$ introduces a very natural nonlinearity. In this sense, Yang-Mills theory can be viewed as a nonlinear generalization of electrodynamics. The Euler-Lagrange equations associated to the action (1.1) are

$$\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0$$

and the ansatz [5,17]

$$A_\mu^{jk}(t, x) = (\delta_\mu^k x^j - \delta_\mu^j x^k) \frac{\psi(t, |x|)}{|x|^2}$$

yields the scalar nonlinear wave equation

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{3\psi(\psi + 1)(\psi + 2)}{r^2} = 0, \tag{1.2}$$

$\psi = \psi(t, r)$, for the auxiliary function $\psi : \mathbb{R} \times [0, \infty) \to \mathbb{R}$. Eq. (1.2) has been proposed as a model for singularity formation in Einstein's equation [2,4,5,21]. In general, (classical) Yang-Mills fields attracted a lot of interest by both the physics and mathematics communities, see e.g. [1,3,11,18,19,22–25,27–30].

Equation (1.2) is energy-supercritical [4] and large-data solutions can develop singularities in finite time as is evidenced by the existence of self-similar solutions of the form $\psi(t, r) = f(\frac{r}{1-t})$, see [7]. Bizoń [5] found an explicit example of this kind given by

$$\psi_0(t, r) = f_0(\tfrac{r}{1-t}), \qquad f_0(\rho) = -\frac{8\rho^2}{5 + 3\rho^2}.$$

Numerical investigations [2,4,5] yield strong evidence that the solution ψ_0 gives rise to a stable self-similar blowup mechanism. Motivated by this, the second author [13] developed a complete nonlinear stability theory for the solution ψ_0, see also [12,14–16] for other types of nonlinear wave equations. However, the results in [13] are conditional in the sense that they depend on a spectral assumption which could not be verified rigorously so far. It is the aim of the present paper to close this gap.

[1] Einstein's summation convention is in force. Greek indices take the values 0 to 5 whereas latin indices run from 1 to 5. Our convention for the Minkowski metric is $\eta = \operatorname{diag}(-1, 1, 1, 1, 1, 1)$.

1.2. The mode stability problem. The first important step in a stability analysis of the solution ψ_0 is to rule out unstable modes. To this end, one introduces *similarity coordinates* [2]

$$\tau = -\log(1-t), \qquad \rho = \frac{r}{1-t}.$$

Eq. (1.2) transforms into

$$\phi_{\tau\tau} + \phi_\tau + 2\rho\phi_{\tau\rho} - (1-\rho^2)(\phi_{\rho\rho} + \tfrac{2}{\rho}\phi_\rho) + \frac{3\phi(\phi+1)(\phi+2)}{\rho^2} = 0 \tag{1.3}$$

where $\phi(\tau,\rho) = \psi(1-e^{-\tau}, e^{-\tau}\rho)$. Due to finite speed of propagation one is mainly interested in the behavior inside the backward lightcone of the singularity, which corresponds to the coordinate domain $\tau \geq 0$, $\rho \in [0,1]$. Note that the self-similar solution is independent of τ and simply given by $f_0(\rho)$. Next, one inserts the *mode ansatz*

$$\phi(\tau,\rho) = f_0(\rho) + e^{\lambda\tau}u_\lambda(\rho), \qquad \lambda \in \mathbb{C}$$

and linearizes in u_λ. This yields the ODE spectral problem

$$-(1-\rho^2)(u_\lambda'' + \tfrac{2}{\rho}u_\lambda') + 2\lambda\rho u_\lambda' + \lambda(\lambda+1)u_\lambda + \frac{V(\rho)}{\rho^2}u_\lambda = 0 \tag{1.4}$$

for the function u_λ, where the potential V is given by

$$V(\rho) = 6 + 18f_0(\rho) + 9f_0(\rho)^2 = 6\frac{25 - 90\rho^2 + 33\rho^4}{(5+3\rho^2)^2}.$$

Observe that Eq. (1.4) has a singular point at the lightcone $\rho = 1$ which is a consequence of the fact that lightcones are the characteristic surfaces of Eq. (1.2).

Admissible solutions of Eq. (1.4) with $\operatorname{Re}\lambda \geq 0$ lead to instabilities of f_0 at the linear level. However, it is not entirely trivial to determine what "admissible" in this context means. This question can in fact only be answered once one has a suitable well-posedness theory for Eq. (1.3). The necessary framework is developed in [13] and it turns out that if $\operatorname{Re}\lambda \geq 0$, only smooth solutions are admissible. Consequently, a nonzero solution $u_\lambda \in C^\infty[0,1]$ of Eq. (1.4) with $\operatorname{Re}\lambda \geq 0$ is called an *unstable mode*. The corresponding λ is called an *(unstable) eigenvalue*. As a matter of fact, there exists an unstable mode. The function

$$u_1(\rho) := -\rho f_0'(\rho) = \frac{80\rho^2}{(5+3\rho^2)^2}$$

turns out to be a smooth solution of Eq. (1.4) with $\lambda = 1$, as one easily checks. However, this mode is not a "real" instability of the solution f_0 but rather a consequence of the time translation symmetry of Eq. (1.2). Indeed, the profile f_0 defines in fact a one-parameter family of blowup solutions given by

$$\psi^T(t,r) = f_0(\tfrac{r}{T-t})$$

where $T > 0$ is a free parameter. By the chain rule it follows that

$$\partial_T\psi^T(t,r)|_{T=1} = -\tfrac{r}{(1-t)^2}f_0'(\tfrac{r}{1-t}) = -e^\tau\rho f_0'(\rho)$$

solves the linearized equation. These observations lead to the following definition.

Definition 1.1. The solution ψ_0 (or f_0) is said to be *mode stable* if u_1 is the only unstable mode.

1.3. The main result. With these preparations at hand we can formulate our main result.

Theorem 1.2. *The self-similar solution ψ_0 is mode stable.*

The first result of this kind was proved very recently for a similar problem related to the wave maps equation [10]. However, the method we develop here is different and much more effective. As a consequence, the main argument fits on a few pages and the method easily generalizes to other types of nonlinear wave equations. In view of the fact that rigorous research on self-similar blowup in supercritical wave equations was blocked for a long time by the difficulties related to these spectral problems, we hope that our method will trigger new developments in the field. In this respect we also remark that Theorem 1.2 in conjunction with the theory developed in [13] yields a fully rigorous proof of stable self-similar blowup dynamics for the Yang-Mills equation (1.2). The precise statement is given in [13], Theorem 1.3. We emphasize that this is a large-data result for an energy-supercritical wave equation.

2. Removal of the Symmetry Mode

Although the eigenvalue $\lambda = 1$ is not connected to a real instability of the solution ψ_0, it is still inconvenient for the further analysis. Consequently, it is desirable to "remove" it. This can be done by a suitable adaptation of a well-known procedure from supersymmetric quantum mechanics which we recall here briefly.

2.1. Interlude on SUSY quantum mechanics. Consider the Schrödinger operator $H = -\partial_x^2 + V$ on $L^2(\mathbb{R})$ with some nice potential V and suppose there exists a ground state $f_0 \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$, i.e., $f_0'' = V f_0$. Assume further that f_0 has no zeros. Then one has the factorization

$$-\partial_x^2 + V = \left(-\partial_x - \frac{f_0'}{f_0}\right)\left(\partial_x - \frac{f_0'}{f_0}\right) =: Q^* Q.$$

By interchanging the order of this factorization, one defines the SUSY partner $\tilde{H}$ of H, i.e., $\tilde{H} := Q Q^*$. Explicitly, the SUSY partner is given by

$$\tilde{H} = \left(\partial_x - \frac{f_0'}{f_0}\right)\left(-\partial_x - \frac{f_0'}{f_0}\right) = -\partial_x^2 - V + 2\frac{f_0'^2}{f_0^2} =: -\partial_x^2 + \tilde{V}$$

where $\tilde{V} = -V + 2\frac{f_0'^2}{f_0^2}$ is called the SUSY potential. The point of all this is the following. Suppose λ is an eigenvalue of H, i.e., $Hf = Q^* Q f = \lambda f$ for some (nontrivial) f. Applying Q to this equation yields $Q Q^* Q f = \lambda Q f$, i.e., $\tilde{H} Q f = \lambda Q f$. Thus, if $Qf \neq 0$, i.e., if $f \notin \ker Q$, λ is an eigenvalue of $\tilde{H}$ as well. Obviously, we have $\ker Q = \langle f_0 \rangle$ and thus, if $\lambda \neq 0$ is an eigenvalue of H, then it is also an eigenvalue of $\tilde{H}$. Moreover, 0 is not an eigenvalue of $\tilde{H}$ for if this were the case, we would have $Q Q^* f = 0$ for a nontrivial f, i.e., $f \in \ker Q^*$ or $Q^* f \in \ker Q$. The former is impossible since $\ker Q^* = \langle \frac{1}{f_0} \rangle$ but $\frac{1}{f_0} \notin L^2(\mathbb{R})$. The latter is impossible since $\operatorname{rg} Q^* \perp \ker Q$. In summary, $\tilde{H}$ has the same set of eigenvalues as H except for 0.

2.2. The supersymmetric problem. Now we implement a version of this SUSY factorization trick for our problem. Note that the Frobenius indices of Eq. (1.4) at $\rho = 0$ are $\{-3, 2\}$ and at $\rho = 1$ we have $\{0, 1 - \lambda\}$. Suppose u_λ is an unstable mode of Eq. (1.4) and $\lambda \neq 0$. By definition, $u_\lambda \in C^\infty[0, 1]$ and from Frobenius theory it follows that $|u_\lambda(\rho)| \simeq \rho^2$ as $\rho \to 0+$ as well as $|u_\lambda(\rho)| \simeq 1$ as $\rho \to 1-$. We define a new function v_λ by[2]

$$u_\lambda(\rho) = \rho^{-1}(1 - \rho^2)^{-\lambda/2} v_\lambda(\rho).$$

From Eq. (1.4) it follows that v_λ satisfies

$$-v_\lambda'' + \frac{V(\rho)}{\rho^2(1 - \rho^2)} v_\lambda = \frac{\lambda(2 - \lambda)}{(1 - \rho^2)^2} v_\lambda. \tag{2.1}$$

For $\lambda = 1$ we have

$$v_1(\rho) = \rho(1 - \rho^2)^{\frac{1}{2}} u_1(\rho) = \frac{80\rho^3(1 - \rho^2)^{\frac{1}{2}}}{(5 + 3\rho^2)^2}.$$

We rewrite Eq. (2.1) as

$$-v_\lambda'' + V_1 v_\lambda = \frac{\lambda(2 - \lambda) - 1}{(1 - \rho^2)^2} v_\lambda$$

with

$$V_1(\rho) = \frac{V(\rho)}{\rho^2(1 - \rho^2)} - \frac{1}{(1 - \rho^2)^2}.$$

Then we have $v_1'' = V_1 v_1$ and thus, Eq. (2.1) may be factorized as

$$(-\partial_\rho - \tfrac{v_1'}{v_1})(\partial_\rho - \tfrac{v_1'}{v_1}) v_\lambda = \frac{\lambda(2 - \lambda) - 1}{(1 - \rho^2)^2} v_\lambda$$

or

$$-(1 - \rho^2)^2(\partial_\rho + \tfrac{v_1'}{v_1})(\partial_\rho - \tfrac{v_1'}{v_1}) v_\lambda = [\lambda(2 - \lambda) - 1] v_\lambda.$$

We set $\tilde{v}_\lambda = (\partial_\rho - \frac{v_1'}{v_1}) v_\lambda$ and apply the operator $\partial_\rho - \frac{v_1'}{v_1}$ to the equation which yields the supersymmetric problem

$$-(\partial_\rho - \tfrac{v_1'}{v_1})[(1 - \rho^2)^2(\partial_\rho + \tfrac{v_1'}{v_1})]\tilde{v}_\lambda = [\lambda(2 - \lambda) - 1]\tilde{v}_\lambda. \tag{2.2}$$

[2] Observe that this transformation depends on λ. This is the reason why Eq. (1.4) is not equivalent to a standard self-adjoint Sturm-Liouville problem. What happens is the following. Since $|u_\lambda(\rho)| \simeq 1$ as $\rho \to 1-$, the corresponding v_λ behaves like $|v_\lambda(\rho)| \simeq (1 - \rho)^{\mathrm{Re}\,\lambda/2}$. The Hilbert space in which the spectral problem for v_λ is symmetric is $L^2_w(0, 1)$ with the weight $w(\rho) = \frac{1}{(1-\rho^2)^2}$. Thus, if $\mathrm{Re}\,\lambda \leq 1$, the admissible solution v_λ does not belong to $L^2_w(0, 1)$! Consequently, for $\mathrm{Re}\,\lambda \leq 1$ the self-adjoint formulation does not yield any information. This shows that the spectral problem (1.4) is truly nonself-adjoint in nature. In particular, there can be nonreal eigenvalues. For $\mathrm{Re}\,\lambda > 1$, on the other hand, one can indeed use Sturm oscillation theory to exclude eigenvalues.

Note the asymptotics

$$\frac{v_1'}{v_1}(\rho) = 3\rho^{-1} + O(\rho) \qquad (\rho \to 0+)$$
$$\frac{v_1'}{v_1}(\rho) \sim -\tfrac{1}{2}(1-\rho)^{-1} \qquad (\rho \to 1-).$$

Consequently, from the representation $v_\lambda(\rho) = \rho^3 h_\lambda(\rho^2)$, where h_λ is analytic near 0, we get $\tilde{v}_\lambda(\rho) = O(\rho^4)$ near $\rho = 0$ and from $v_\lambda(\rho) \sim c(1-\rho)^{\lambda/2}$ we infer $\tilde{v}_\lambda(\rho) \sim c(1-\rho)^{\lambda/2-1}$ near $\rho = 1$ (unless $\lambda = 1$). Writing out Eq. (2.2) explicitly yields

$$-(1-\rho^2)^2\tilde{v}_\lambda'' + 4\rho(1-\rho^2)\tilde{v}_\lambda' + \frac{(1-\rho^2)\tilde{V}(\rho)}{\rho^2}\tilde{v}_\lambda = \lambda(2-\lambda)\tilde{v}_\lambda \tag{2.3}$$

with the supersymmetric potential

$$\tilde{V}(\rho) = 20\frac{15-2\rho^2+3\rho^4}{(5+3\rho^2)^2}.$$

Setting $\tilde{u}_\lambda(\rho) = \rho^{-1}(1-\rho^2)^{1-\lambda/2}\tilde{v}_\lambda(\rho)$ we find the equation

$$-(1-\rho^2)(\tilde{u}_\lambda'' + \tfrac{2}{\rho}\tilde{u}_\lambda') + 2\lambda\rho\tilde{u}_\lambda' + (\lambda^2+\lambda-2)\tilde{u}_\lambda + \frac{\tilde{V}(\rho)}{\rho^2}\tilde{u}_\lambda = 0. \tag{2.4}$$

Note that the Frobenius indices of Eq. (2.4) are $\{-4, 3\}$ at 0 and $\{0, 1-\lambda\}$ at $\rho = 1$. With minor modifications the same procedure can be performed in the case $\lambda = 0$. As before, we say that $\lambda \in \mathbb{C}$ is an unstable eigenvalue of Eq. (2.4) if $\operatorname{Re}\lambda \geq 0$ and there exists a nontrivial solution $\tilde{u}_\lambda \in C^\infty[0, 1]$ of Eq. (2.4). In summary, we have proved the following result.

Proposition 2.1. *Let $\lambda \neq 1$ be an unstable eigenvalue of Eq.* (1.4)*. Then λ is an unstable eigenvalue of Eq.* (2.4)*.*

3. Absence of Unstable Eigenvalues for the Supersymmetric Problem

In this section we exclude unstable eigenvalues of Eq. (2.4). Via Proposition 2.1 this implies the main result Theorem 1.2.

Theorem 3.1. *The supersymmetric problem Eq.* (2.4) *does not have unstable eigenvalues.*

The Frobenius indices of (2.4) at 0 are -4 and 3, hence the solution analytic at 0 has the power series representation

$$\sum_{n=0}^{\infty} a_n(\lambda)\rho^{2n+3}, \quad a_0 \neq 0. \tag{3.1}$$

Note that λ is an eigenvalue of (2.4) if and only if the radius of convergence of (3.1) is greater than 1. Therefore, our aim is to prove that for any λ in the closed right half-plane (which from now on we denote by $\overline{\mathbb{H}}$), (3.1) cannot be analytically extended through $\rho = 1$.

By substituting (3.1) into (2.4) we obtain a four term recurrence relation (with the initial condition $a_0 = 1$ and $a_n = 0$ for $n < 0$)

$$p_3(n)a_{n+3} + p_2(n)a_{n+2} + p_1(n)a_{n+1} + p_0(n)a_n = 0, \tag{3.2}$$

where

$$\begin{aligned}
p_3(n) &= -100n^2 - 950n - 1950,\\
p_2(n) &= -20n^2 + (100\lambda - 150)n + 25\lambda^2 + 375\lambda - 370,\\
p_1(n) &= 84n^2 + (120\lambda + 462)n + 30\lambda^2 + 330\lambda + 630,\\
p_0(n) &= 36n^2 + (36\lambda + 126)n + 9\lambda^2 + 63\lambda + 90.
\end{aligned}$$

One can check that $a_n = (-3/5)^n$ is an exact solution to (3.2), hence the order of the recurrence (3.2) can be reduced by one through the substitution

$$b_n = a_{n+1} + \tfrac{3}{5}\, a_n. \tag{3.3}$$

This yields a three term recurrence relation for b_n

$$q_2(n)b_{n+2} + q_1(n)b_{n+1} + q_0(n)b_n = 0, \tag{3.4}$$

where

$$\begin{aligned}
q_2(n) &= p_3(n),\\
q_1(n) &= p_2(n) - \tfrac{3}{5}\, p_3(n),\\
q_0(n) &= p_1(n) - \tfrac{3}{5}\, p_2(n) + \tfrac{9}{25}\, p_3(n).
\end{aligned}$$

After substituting for $p_i(n)$ in the last three relations, dividing all of them by 5 and using the q_i notation for the new coefficients, we get

$$\begin{aligned}
q_2(n) &= -20n^2 - 190n - 390,\\
q_1(n) &= 8n^2 + (20\lambda + 84)n + 5\lambda^2 + 75\lambda + 160,\\
q_0(n) &= 12n^2 + (12\lambda + 42)n + 3\lambda^2 + 21\lambda + 30.
\end{aligned}$$

By letting $A_n = q_1(n)/q_2(n)$ and $B_n = q_0(n)/q_2(n)$, (3.4) becomes equivalent to

$$b_{n+2} + A_n b_{n+1} + B_n b_n = 0, \tag{3.5}$$

with the initial condition $b_{-2} = 0$ and $b_{-1} = 1$.

Lemma 3.2. *Given λ in the complex plane, either*

$$\lim_{n\to\infty} \frac{b_{n+1}(\lambda)}{b_n(\lambda)} = 1, \tag{3.6}$$

or

$$\lim_{n\to\infty} \frac{b_{n+1}(\lambda)}{b_n(\lambda)} = -\frac{3}{5}. \tag{3.7}$$

Proof. Since $\lim_{n\to\infty} A_n(\lambda) = -2/5$ and $\lim_{n\to\infty} B_n(\lambda) = -3/5$, the characteristic equation associated to (3.5) is

$$t^2 - \tfrac{2}{5}t - \tfrac{3}{5} = 0. \tag{3.8}$$

As the solutions to (3.8) (1 and $-3/5$) have distinct moduli, by a theorem of Poincaré (see, for example, [20], p. 343, or [6]), either b_n is zero eventually in n, or $\lim_{n\to\infty} b_{n+1}(\lambda)/b_n(\lambda)$ exists and it is equal to either 1 or $-3/5$. Now, for a fixed λ, b_n cannot be zero eventually in n, since by backward induction from (3.5) one would get $b_{-1} = 0$, hence the claim follows. □

Note that in order to prove Theorem 3.1, it suffices to show that (3.6) holds for all λ in $\overline{\mathbb{H}}$, for that implies non-analyticity of (3.1) at 1. Indeed, defining f_λ by (3.1) and g_λ by $g_\lambda(\rho) = a_0(\lambda)\rho + \sum_{n=0}^{\infty} b_n(\lambda)\rho^{2n+3}$, one easily checks that

$$f_\lambda(\rho) = \frac{5\rho^2}{3\rho^2+5} g_\lambda(\rho). \tag{3.9}$$

So if (3.6) holds and therefore g_λ is singular at 1, then, by (3.9), so is f_λ.

Let $r_n = b_{n+1}/b_n$. Then from (3.5) we obtain

$$r_{n+1} = -A_n - \frac{B_n}{r_n}, \tag{3.10}$$

where

$$r_{-1} = \frac{b_0}{b_{-1}} = -A_{-2}(\lambda) = \frac{1}{18}\lambda^2 + \frac{7}{18}\lambda + \frac{4}{15}. \tag{3.11}$$

The idea is to find a "simple", provably close approximation to r_n in $\overline{\mathbb{H}}$, that converges to 1 for any fixed λ, which would then imply (3.6).

We use the quasi-solution approach, initially developed for ordinary differential equations in [8,9], which we here, in a sense, extend to difference equations of type (3.10). Namely, as a quasi-solution to (3.10) we define

$$\tilde{r}_n(\lambda) = \frac{\lambda^2}{4n^2+31n+43} + \frac{\lambda}{n+4} + \frac{n+2}{n+4}. \tag{3.12}$$

Of course, the choice is not arbitrary, and in Sect. 4.1 we describe in some detail how to obtain such an approximate solution. The quasi-solution $\tilde{r}_n$ turns out to be a good approximation to r_n in the whole of $\overline{\mathbb{H}}$.

Lemma 3.3. *r_1 and $(\tilde{r}_n)^{-1}$ for $n \geq 1$, are analytic in $\overline{\mathbb{H}}$.*

Proof. From (3.10) and (3.11) we compute

$$r_1(\lambda) = \frac{1}{78}\frac{25\lambda^6 + 825\lambda^5 + 10945\lambda^4 + 69735\lambda^3 + 207694\lambda^2 + 260856\lambda + 96192}{25\lambda^4 + 450\lambda^3 + 2735\lambda^2 + 5070\lambda + 2016}.$$

The denominator of r_1 and the polynomials $\tilde{r}_n(\lambda)$ for $n \geq 1$ are Hurwitz-stable i.e., all of their zeros are in the (open) left half-plane, which can be straightforwardly checked by, say, the Routh-Hurwitz criterion or its reformulation by Wall (see [32] or Sect. 4.2).[3] The conclusion follows. □

[3] There are, of course, elementary ways of proving this claim. However, the suggested approach is more general.

Now, let

$$\delta_n = \frac{r_n}{\tilde{r}_n} - 1. \tag{3.13}$$

Substitution of (3.13) into (3.10) leads to the following recurrence relation for δ_n,

$$\delta_{n+1} = \varepsilon_n + C_n \frac{\delta_n}{1+\delta_n}, \tag{3.14}$$

where

$$\varepsilon_n = \frac{-A_n \tilde{r}_n - B_n}{\tilde{r}_n \tilde{r}_{n+1}} - 1 \quad \text{and} \quad C_n = \frac{B_n}{\tilde{r}_n \tilde{r}_{n+1}}. \tag{3.15}$$

Lemma 3.4. *The following estimates hold in* $\overline{\mathbb{H}}$*,*

$$|\delta_1| \leq \frac{1}{4}, \tag{3.16}$$

$$|\varepsilon_n| \leq \frac{1}{20}, \quad n \geq 1, \tag{3.17}$$

$$|C_n| \leq \frac{3}{5}, \quad n \geq 1. \tag{3.18}$$

Proof. The method of proof is the same for all three quantities, so we illustrate it only on C_n.

Lemma 3.3 and (3.15) imply that C_n is analytic in $\overline{\mathbb{H}}$. Also, being a rational function, C_n is evidently polynomially bounded in $\overline{\mathbb{H}}$. Hence, according to the Phragmén-Lindelöf principle,[4] it suffices to prove that (3.18) holds on the imaginary line. To that end, we first bring $C_{n+1}(\lambda)$ to the form of the ratio of two polynomials $P_1(n, \lambda)$ and $P_2(n, \lambda)$.[5] Then, for t real, $|C_{n+1}(it)|^2$ is equal to the quotient of two polynomials, $Q_1(n, t^2) = |P_1(n, it)|^2$ and $Q_2(n, t^2) = |P_2(n, it)|^2$. In order to show that $|C_{n+1}(it)| \leq 3/5$, for all real t and $n \geq 0$, all we need is to show that $|C_{n+1}(it)|^2 = Q_1(n, t^2)/Q_2(n, t^2) \leq 9/25$, or equivalently $9/25 \cdot Q_2 - Q_1 \geq 0$. Using elementary calculations, we see that $9/25 \cdot Q_2 - Q_1$ has manifestly positive coefficients, and the variable t appears with even powers only. Thus, (3.18) holds on the whole imaginary line, and the result follows. □

Proof of the Theorem 3.1. From (3.14) and Lemma 3.4, a simple inductive argument implies that

$$|\delta_n| \leq \frac{1}{4}, \quad \text{for all} \quad n \geq 1, \text{and} \quad \lambda \in \overline{\mathbb{H}}. \tag{3.19}$$

Since for any fixed λ, $\lim_{n\to\infty} \tilde{r}_n(\lambda) = 1$, (3.13) and (3.19) exclude the possibility of (3.7). Hence, (3.6) holds in $\overline{\mathbb{H}}$, and the claim follows. □

[4] We use the sectorial formulation of this principle, see, for example, [31], p. 177.

[5] For all three quantities, straightforward calculations would lead to the form that we used. However, to prevent possible ambiguity, in Sect. 4.3 we give the explicit form (as a ratio of polynomials) for all three quantities.

4. Appendix

4.1. Description of how to obtain a quasi-solution. First, the minimax polynomial approximation[6] of degree two to r_n over an interval $[0, 10]$ is found, where n ranges from 0 to 20. Then, appropriate rational functions in n are fitted to the coefficients of the approximation polynomials.

We should point out that interval of polynomial approximation and the range of values of n can vary, and the ones from the description are just our choice. We choose quadratic polynomial approximations due to the fact that r_n is a ratio of two polynomials whose degrees differ by two.

4.2. Wall's criterion for Hurwitz-stability. Let $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be a polynomial with real coefficients, and let $Q(z) = a_1 z^{n-1} + a_3 z^{n-3} + \cdots$ be the polynomial that contains exactly those terms of $P(z)$ that have odd-indexed coefficient. Then all the zeros of $P(z)$ have negative real parts if and only if the quotient $Q(z)/P(z)$ can be represented in a finite continued fraction form

$$1/(a_1 + 1/(a_2 + 1/(a_3 + \cdots + 1/a_n) \ldots),$$

where $a_1 = c_1 z + 1$, $a_2 = c_2 z, \ldots, a_n = c_n z$, and the coefficients $c_1, c_2, \ldots, c_n$ are all positive.

In our case, for the denominator of r_1, the coefficients c_i are $c_1 = 1/18$, $c_2 = 135/736$, $c_3 = 33856/64863$ and $c_4 = 36035/15456$, and for $\tilde{r}_n$, $c_1 = (n+4)/(4n^2+31n+43)$, and $c_2 = 1/(n+2)$.

4.3. Detailed expressions for C_n, ε_n and δ_1. We give details of these quantities in order to fully clarify the notations. We have

$$C_{n+1} = P_1(n, \lambda)/P_2(n, \lambda),$$

where

$$P_1(n, \lambda) = -3(n+5)(n+6)(4n^2+39n+78)(4n^2+47n+121) \\ \times [\lambda^2 + (4n+11)\lambda + 4n^2 + 22n + 28]$$

and

$$P_2(n, \lambda) = 10(2n^2+23n+60)[(n+5)\lambda^2 + (4n^2+39n+78)(\lambda+n+3)] \\ \times [(n+6)\lambda^2 + (4n^2+47n+121)(\lambda+n+4)],$$

respectively. Furthermore, $\varepsilon_{n+1} = P_3(n, \lambda)/P_2(n, \lambda)$, where

$$\begin{aligned} P_3(n, \lambda) = {} & 5(n+1)(n+5)(n+6)\lambda^4 \\ & - 5(8n^4 + 158n^3 + 1095n^2 + 3171n + 3162)\lambda^3 \\ & - (112n^5 + 2364n^4 + 17243n^3 + 48805n^2 + 33244n - 36060)\lambda^2 \\ & - 4(4n^2+39n+78)(4n^2+47n+121) \\ & \times [(3n^2+5n-3)\lambda - 4n^2 - 3n + 36]. \end{aligned}$$

[6] The minimax polynomial approximation of degree n to a continuous function f on a given finite interval $[a, b]$ is defined to be the best approximation, among the polynomials of degree n, to f in the uniform sense on $[a, b]$. For the proof of existence and uniqueness of this approximation and an algorithm to obtain it, see [26], §2.4.

Finally,

$$\delta_1 = \frac{-5\lambda^2(15\lambda^3 - 20\lambda^2 - 939\lambda + 1412) - 36(1093\lambda - 256)}{(5\lambda^2 + 78\lambda + 234)(25\lambda^4 + 450\lambda^3 + 2735\lambda^2 + 5070\lambda + 2016)}.$$

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