Commun. Math. Phys. 341, 263–287 (2016)
Digital Object Identifier (DOI) 10.1007/s00220-015-2538-y

Communications in
Mathematical Physics

Quantization and Dynamisation of Trace-Poisson Brackets

Jean Avan[1], **Eric Ragoucy**[2], **Vladimir Rubtsov**[3,4]

[1] Laboratoire de Physique Théorique et Modélisation (CNRS UMR 8089), Université de Cergy-Pontoise, 95302 Cergy-Pontoise, France. E-mail: avan@u-cergy.fr
[2] LAPTh, CNRS and Université de Savoie, 9 Chemin de Bellevue, BP 110, 74941 Annecy le Vieux Cedex, France. E-mail: ragoucy@lapth.cnrs.fr
[3] Laboratoire Angevin de REcherche en MAthématiques (CNRS UMR 6093), UNAM et Université d'Angers, Faculté des Sciences, 2, Boulevard Lavoisier, 49045 Angers Cedex, France. E-mail: volodya@univ-angers.fr
[4] Theory Division, ITEP, 25, Bol. Tcheremushkinskaya, 117259, Moscow, Russia

Received: 4 February 2014 / Accepted: 19 October 2015
Published online: 22 December 2015 –

Abstract: The quantization problem for the trace-bracket algebra, derived from double Poisson brackets, is discussed. We obtain a generalization of the boundary YBE (or so-called ABCD-algebra) for the quantization of quadratic trace-brackets. A dynamical deformation is proposed on the lines of Gervais–Neveu–Felder dynamical quantum algebras.

1. Introduction

There has been a long-standing interest of both communities—mathematicians and physicists—in a class of objects known as *character varieties* or *spaces of representations*, and their equivalence classes—or "moduli spaces". A typical example of such objects is the moduli space $\mathcal{M}_{\Sigma,G}$ of flat connections in a principal bundle with the structure group G over a Riemann surface Σ. This space is defined as a quotient $\mathcal{M}_{\Sigma,G} = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{G})/\mathrm{G}$ of the representation space of the fundamental group $\pi_1(\Sigma)$ of the surface Σ in G with respect to conjugation by the group G.

There are many applications of the character varieties from a geometric viewpoint. The moduli spaces of representations can be used also for describing the moduli of stable vector bundles over a compact Riemann surface of genus > 2. This is the set of equivalence classes of irreducible unitary representations of the fundamental group. Another classical example is the Teichmüller space of Riemann surface complex structures—the set of equivalence classes of irreducible representations of the fundamental group in $PSL_2(\mathbb{R})$ or $SL_2(\mathbb{R})$.

From a physical point of view, such moduli spaces (or their reductions, subspaces etc.) act as an arena of various scenarios of modern classical and quantum field theories—(super)string theory, (super)gravity and tentatives to unify them. Many of them can be considered as phase spaces of interesting integrable systems (Beauville-Mukai and Hitchin models).

An algebraic avatar of the character varieties is provided by the representation space $\mathrm{Rep}_N(A)$ or more precisely the affine scheme $\mathbb{C}[\mathrm{Rep}_N(A)]^{GL_N(\mathbb{C})}$. Indeed in an algebraic context the representation spaces of a (non)commutative associative algebra A become algebraic varieties which "approximate", by Kontsevich philosophy, the underlying variety—the spectrum of the commutative counterpart of A. In particular, if A is a free associative algebra over $\mathbb{C}$, that is, $A = \mathbb{C} < x_1, ..., x_n >$ is given by a finite number of generators $x_1, \ldots, x_n$, the space of $N-$dimensional representations is

$$\mathrm{Rep}_N(A) = \{M = (M_1, \ldots, M_n) \in \mathbb{M}_N(\mathbb{C}) \oplus ... \oplus \mathbb{M}_N(\mathbb{C})\},$$

where $\mathbb{M}_N(\mathbb{C}) = \mathrm{End}(\mathbb{C}^N)$.

The group $GL_N(\mathbb{C})$ acts on $\mathrm{Rep}_N(A)$ by conjugations : $g.M = \left(gM_1g^{-1}, ..., gM_ng^{-1}\right)$ and the quotient space $\mathrm{Rep}_N(A)^{GL_N(\mathbb{C})}$ describes isomorphism classes of semisimple representations of A. The classical Procesi theorem on GL_N-invariants [36] says that the coordinate ring $\mathbb{C}[\mathrm{Rep}_N(A)]^{Gl_N(\mathbb{C})}$ is generated by traces of "words" in generic matrices $M_1, \ldots, M_n$ (see Example 2.4 below).

A typical geometric character variety carries a Poisson or symplectic structure studied in many classical works by, e.g., Atiyah–Bott, Goldman, Hitchin, Beauville, Mukai et al. Similarly in the above described algebraic context the affine scheme $\mathbb{C}[\mathrm{Rep}_N(A)]^{GL_N(\mathbb{C})}$ can also be supplied with a natural Poisson algebra structure. Recently the wonderful algebraic mechanism governing the Poisson algebra structure on the representation spaces was discovered by Van den Bergh [42] and in a slightly different but closely related way by Crawley-Boevey [13].

The Van den Bergh construction is based on the notion of *double Lie bracket* on an associative algebra A, which is a $\mathbb{C}$-bilinear operation $\{\{-, -\}\} : A \otimes A \to A \otimes A$ satisfying certain *twisted* Lie algebra axioms. One may add a Leibniz-like property, different for left-hand and right-hand arguments in the correspondence with outer and inner A-bimodule structures on $A \otimes A$. Such an operation is then called a *double Poisson structure* on A. Precise definitions are given in Subsection 2.2.

A natural composition of a double Poisson bracket with the associative algebra multiplication μ defines an interesting operation on A and on the space of "traces" on A (the vector space $A/[A, A]$). This trace space $A/[A, A]$ is identified with the Hochschild homology of degree 0, $HH_0(A, A)$, and Crawley-Boevey had showed that the projection of $\mu \circ \{\{-, -\}\}$ defines a Lie algebra structure $< -, - >$ on $HH_0(A, A)$. This structure then defines a unique Poisson bracket on the invariant representation space $\mathbb{C}[\mathrm{Rep}(A)]^{GL_N(\mathbb{C})}$. The latter is identified with the trace map $\mathrm{Tr} : A/[A, A] \to \mathbb{C}[\mathrm{Rep}(A)]$ image. These Poisson brackets realize a "trace-covariant" object in the following sense:

$$\{\mathrm{Tr}(a), \mathrm{Tr}(b)\} = \mathrm{Tr} < \pi(a), \pi(b) >,$$

where $\pi : A \to A/[A, A]$ is the natural projection.

Such "trace-invariant" Poisson structures coming from $A/[A, A]$ appeared in [25] and [32] in the context of some integrable ordinary differential equations on associative algebras. Such brackets were constructed, their Hamiltonian nature in the proper framework of the Hamiltonian formalism was studied and examples of interesting noncommutative models on associative algebras were given. These works initiated the recent developments, including the formulation of analogues of Yang–Baxter conditions and their relevance to Hamiltonian formalism, together with related Poisson structures in theories of integrable systems with *matrix* variables [30]. The special role of the Associative Yang–Baxter Equation and (anti-) Frobenius algebra structures were uncovered.

In the case of free algebra the related notion of “trace brackets” was developed and studied extensively in [29] especially in the case of “quadratic” brackets. In the case of quiver path algebras the trace Poisson brackets on the moduli space of finite-dimensional representations are related to the pre-hierarchy of Hamiltonian structures of Ruijsenaar Schneider models [10]. On the same line, Massuyeau and Turaev [24] constructed a graded Poisson algebra structure on representation algebras associated with the loop algebra of any smooth oriented manifold M with non-empty boundary. When M is a Riemann surface Σ, the corresponding bracket coincides with the quasi-Poisson bracket on the representation space $\mathrm{Hom}(\pi_1(\Sigma), GL_N)$ defined in their previous work [23] via the described (slightly more general) algebraic construction of Van den Bergh.

Further studies of *parameter-dependent* AYBE and Poisson structures were done in [31]. It is amusing to observe that the parameter-dependent AYBE were objects of interest and studies even earlier than the “constant” AYBE. They appeared in different contexts: as associativity conditions in the quadratic parameter dependent algebras related to Sklyanin elliptic algebras of Odesskii and Feigin [28] and as a triple Massey product expression for the associativity constraint in A_∞-category, which is the derived category of coherent sheaves on an elliptic curve [35]. The solutions of the AYBE relate to triple Massey products for simple vector bundles on elliptic curves and their degenerations. Further study of this connection has been continued in [11].

All these studies have been yet conducted purely on *classical* associative and trace algebras. A natural question is thus to construct a quantum version of these algebras. A second related question is to address the issue of consistent deformations of these classical and/or quantum algebra structures. The quantum case is much easier to handle since the underlying algebraic notions of deforming bialgebras are much simpler than the classical ones related to Poisson algebras. We shall tackle these two issues here, however restricting ourselves for the time being to *non-parametric* associative and trace algebras.

Let us briefly summarize the content of our work. We shall first describe the relevant structures, detailing in particular double-brackets (classical) and their subsequent trace-Poisson algebras. We shall then give a full r-matrix type description of a ‘parameter-independent” trace-Poisson algebra, taking the form of a generalized a, s Poisson structure à la Maillet [21,22]. We shall define its extension to the most general quadratic form, which lead us to a consistent quantum algebraic structure mimicking the Freidel–Maillet [18] quadratic exchange algebras, albeit with a new, third vector index. The bivector formulation introduced by Freidel and Maillet plays a crucial role here. By newly deriving the bivector formulation for the known three dynamical reflection algebras (without the extra vector index or “flavor index”) we identify several key features of a consistent dynamical deformation of quadratic exchange algebras, which we then extend in the most natural way to define a dynamical deformation of the quantum trace reflection algebra.

2. Double Brackets on Algebras and Their Representations

We introduce here the general, purely algebraic notions of double Lie and double Poisson brackets together with the associated general notion of trace Poisson brackets to be used extensively in the following. Let us first establish the algebraic framework.

2.1. Algebraic generalities. We suppose that A is an associative finite dimensional $\mathbb{C}$-algebra (with unity). We will consider $A \otimes A$ as an $A - A$ bimodule with respect to the outer and the inner structures, defined respectively by

$$a \cdot (\alpha \otimes \beta) \cdot b = (a\alpha) \otimes (\beta b), \qquad \alpha \otimes \beta \in A \otimes A \quad \text{and} \quad a, b \in A \tag{2.1}$$

$$a(\alpha \otimes \beta)b = (\alpha b) \otimes (a\beta). \tag{2.2}$$

The $A \otimes A$-valued derivations $D(A, A \otimes A)$ of A (w.r.t. the outer bimodule structure) will play the role of the usual derivations ("vector fields") on A.

Example 2.1. Let A be the free $\mathbb{C}$-algebra with n generators $A = \mathbb{C} < x_1, \ldots, x_n >$. The partial double derivations are defined for each generator x_α as $\partial_\alpha \in D(A, A \otimes A)$ such that $\partial_\alpha(x_\alpha) = 1 \otimes 1$ and $\partial_\alpha(x_\beta) = 0,\ \alpha \neq \beta, \quad 1 \leq \alpha \leq m$.

The A-bimodule Ω_A of 1-differentials is generated by da, $a \in A$, with relations $d(ab) = a(db) + (da)b$ for $a, b \in A$ and we assume $d^2a = d(da) = 0$ and $d(a) = da$. We have

$$D(A, A \otimes A) = \mathrm{Hom}_{A \otimes A^o}(\Omega_A, A \otimes A).$$

Here A^o denotes the "opposite" algebra: if $\mu : A \otimes A \to A$ is the multiplication law in $A : \mu(a \otimes b) = ab$, then $\mu^o : A^o \otimes A^o \to A^o$ is $\mu^o(a \otimes b) = ba$. We shall denote by $A^e := A \otimes A^o$ the enveloping algebra of A.

2.2. Double Lie and Poisson brackets.

Definition 2.2. Let V be any $\mathbb{C}$-vector space. A *double Lie bracket* is a $\mathbb{C}$-linear map $\{\!\{,\}\!\} : V \otimes V \mapsto V \otimes V$ satisfying the following two conditions:

$$skew\text{-}symetry: \quad \{\!\{u, v\}\!\} = -\{\!\{v, u\}\!\}^o, \tag{2.3}$$

and

$$\sigma\text{-}associativity: \quad \{\!\{u, \{\!\{v, w\}\!\}\}\!\}_l + \sigma\big(\{\!\{u, \{\!\{v, w\}\!\}\}\!\}_l\big) + \sigma^2\big(\{\!\{u, \{\!\{v, w\}\!\}\}\!\}_l\big) = 0, \tag{2.4}$$

where $\sigma \in S_3$ is the cyclic permutation $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, whose action is defined as follows:

for $v = v_1 \otimes v_2 \otimes v_3 \in V_1 \otimes V_2 \otimes V_3$ one has $\sigma(v) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes v_{\sigma^{-1}(3)}$.

The bracket in (2.4), $\{\!\{u, \{\!\{v, w\}\!\}\}\!\}_l$ has to be understood as an extension of the operation $\{\!\{-, -\}\!\} \in \mathrm{End}(V \otimes V)$ by

$$\{\!\{u, v \otimes w\}\!\}_l := \{\!\{u, v\}\!\} \otimes w$$

and defines an element in $V \otimes V \otimes V$.

If we replace the vector space V by an associative $\mathbb{C}$-algebra A we must examine the compatibility of the double Lie bracket (2.2) with the associative multiplication. We suppose, following Van den Bergh [42], that the double Lie bracket satisfies the "usual" Leibniz rule on the right argument:

$$\{\!\{a, bc\}\!\} = b\,\{\!\{a, c\}\!\} + \{\!\{a, b\}\!\}\,c, \tag{2.5}$$

for any $a, b, c \in A$. In other words the operation $\{\!\{a, -\}\!\} : A \to A \otimes A$ is a double derivation with values in the bimodule $A \otimes A$ with the outer A-bimodule structure. The derivation property with respect to the left argument follows from the skew-symmetry of the double Lie bracket and from (2.5):

$$\{\!\{ab, c\}\!\} = a\,\{\!\{b, c\}\!\} + \{\!\{a, c\}\!\}\,b. \tag{2.6}$$

This means that the operation $\{\!\{-, b\}\!\} : A \to A \otimes A$ is a double derivation with values in the bimodule $A \otimes A$ with the inner A-bimodule structure.

Definition 2.3. A double Lie bracket $\{\{-,-\}\} : A \times A \to A \otimes A$ on an associative $\mathbb{C}$-algebra A satisfying (2.5) is called a *double Poisson bracket on A*.

2.3. From double Poisson brackets to trace Poisson brackets.

2.3.1. The affine varieties of representations and their coordinates. Following [30,42] and [23] we now define classes of Poisson structures on representation spaces of the given associative algebra A. For any natural number $N \geq 1$ we define an algebra A_N whose commutative generators are defined by a correspondence from elements $a \in A : a \to a_{ij}$, $1 \leq i, j \leq N$ which satisfy the standard $\mathbb{C}$-matrix element relations:

$$(a+b)_{ij} = a_{ij} + b_{ij}, \quad (ab)_{ij} = \sum_k a_{ik} b_{kj}, \quad 1_{ij} = \delta_{ij},$$

where $a, b \in A$ and $1 \leq i, j \leq N$. In other words, there is a canonical bijection between $A_N^* = \mathrm{Hom}_{\mathbb{C}}(A_N, \mathbb{C})$ and $\mathrm{Hom}(A, \mathbb{M}_N(\mathbb{C}))$ which assign to each linear functional $l \in A_N^*$ the algebra morphism $\hat{l} : A \to \mathbb{M}_N(\mathbb{C})$ defined by $[\hat{l}(a)]_{ij} = l(a_{ij})$. Reciprocally if we have a map $l : A \to \mathbb{M}_N(\mathbb{C})$ then it corresponds by the bijection to the linear map $A_N \to \mathbb{C}$ such that the generator $a_{ij} \in A_N$ is transformed in $i, j-$component of the matrix $l(a)$. Geometrically, one can say that there exists an affine variety $\mathrm{Rep}_N(A)$ whose coordinate algebra $\mathbb{C}[\mathrm{Rep}_N(A)]$ is the commutative algebra A_N.

2.3.2. Trace map. The usual matrix trace defines correctly the map $\mathrm{Tr} : A \to A_N$ by $\mathrm{Tr}(a) := \sum_i a_{ii}$ for any $a \in A$. This map annihilates the commutators : $\mathrm{Tr}(ab - ba) = 0$ for any $a, b \in A$ and extends to the map $\mathrm{Tr} : A_\natural \to A_N$ where $A_\natural := A/[A, A]$ is the quotient vector space (the "trace space of A"). We shall denote in what follows the projection of $a \in A$ in $A_\natural$ by $p(a)$.

Example 2.4. Consider the case of the free associative algebra $A = \mathbb{C} < x_1, \ldots, x_m >$.

The coordinate algebra $\mathbb{C}[\mathrm{Rep}_N(A)]$ in this case is the polynomial ring of mN^2 variables $x^j_{i,\alpha}$, where

$$x_\alpha \to M_\alpha = \begin{pmatrix} x^1_{1,\alpha} & \cdot & x^N_{1,\alpha} \\ \cdot & \cdot & \cdot \\ x^1_{N,\alpha} & \cdot & x^N_{N,\alpha} \end{pmatrix}, \qquad 1 \leq \alpha \leq m.$$

The map Tr gives the following interpretation of the variables $x^j_{i,\alpha}$: if e_{ij} denotes the (i, j)-matrix unit (i.e. the $N \times N$ matrix with 0 everywhere except the i-th row and j-th column) then $x^j_{i,\alpha} = \mathrm{Tr}(e_{ij} M_\alpha)$.

2.3.3. Trace Poisson brackets from double brackets. Van den Bergh defines a bracket operation on A_N starting from a double Poisson structure on it. We shall use Sweedler's notations: an element $\alpha \in A \otimes A$ shall be denoted by $\alpha = \alpha^{(1)} \otimes \alpha^{(2)}$ meaning that there is in fact a finite family $(\alpha_i^{(1)}, \alpha_i^{(2)})$ in $A \times A$ such that $\alpha = \sum_i \alpha_i^{(1)} \otimes \alpha_i^{(2)}$.

Theorem 2.5 [42].

- *Given a double Poisson bracket on A one defines a bracket* $[-,-] : A \times A \to A$

$$[a, b] := \{\{a, b\}\}^{(1)} \{\{a, b\}\}^{(2)} = \sum_i \{\{a, b\}\}_i^{(1)} \{\{a, b\}\}_i^{(2)} \tag{2.7}$$

such that

- (2.7) *satisfies the following derivation property:*

$$[a,[b,c]] = [a,b],c]] + [b,[a,c]];$$

- *The restriction* $[-,-] : A_\natural \times A_\natural \to A_\natural$ *defines on* $A_\natural$ *a Lie algebra structure:*

$$[p(a), p(b)] := p([a,b]); \tag{2.8}$$

- *The map* $[p(a),-] \in \mathrm{End}(A_\natural)$ *is induced by a derivation of A.*

- *Given a double Poisson bracket on A one defines a Poisson structure on the representation variety* $\mathrm{Rep}_N(A)$ *i.e. a Poisson bracket* $\{-,-\} : A_N \times A_N \to A_N$ *such that on generators of* A_N *(defined by elements a and b of A) we have*

$$\{a_{ij}, b_{kl}\} := \{\!\{a,b\}\!\}^{(1)}_{kj}\{\!\{a,b\}\!\}^{(2)}_{il} \tag{2.9}$$

- *The map* $\mathrm{Tr} : A_\natural \to A_N$ *is a morphism of Lie algebras* $A_\natural$ *and* $A_N = \mathbb{C}[\mathrm{Rep}_N(A)]$. *Namely:*

$$\{\mathrm{Tr}\,p(a), \mathrm{Tr}\,p(b)\} = \mathrm{Tr}([p(a), p(b)]). \tag{2.10}$$

Definition 2.6. We shall refer to the Poisson brackets (2.9) (following the suggestion in [29]) as *trace Poisson brackets*.

It is an easy exercise to check that the trace Poisson brackets are defined in fact on the invariant part of A_N or, more precisely, on conjugation classes $\mathbb{C}[\mathrm{Rep}_N(A)]^{GL_N(\mathbb{C})}$, where $GL_N(\mathbb{C})$ acts on A_N by conjugations. This is the unique Poisson structure on $\mathbb{C}[\mathrm{Rep}_N(A)]^{GL_N(\mathbb{C})}$ such that (2.10) holds.

We shall be mostly interested by the trace Poisson brackets induced by double brackets on a free associative algebra such as considered in Example 2.4.

3. Associative Yang–Baxter Equation

We are now interested in double bracket structures induced by endomorphisms $r \in \mathrm{End}(V \otimes V)$ (and not simply maps) later identified with "classical r-matrices". Associativity conditions on the double Poisson algebra induces conditions of Yang–Baxter type on their structure constants encapsulated in r [2]. Note that the general situation of r maps would give rise to structures analog to the "set-theoretical YB equations" studied in e.g. [1,12,33].

3.1. AYBE and double Lie brackets. Schedler [39] proposed the following existence criterion for double Lie brackets (2.2):

Proposition 3.1. *Let* $r \in \mathrm{End}(V \otimes V)$ *defines the operation*

$$\{\!\{u,v\}\!\} := r(u \otimes v). \tag{3.1}$$

This operation induces a double Lie bracket on V iff r is skew-symmetric *and satisfies the Associative Yang–Baxter Equation (AYBE) in* $V \otimes V \otimes V$:

$$AYBE(r) := r^{12}r^{13} - r^{23}r^{12} + r^{13}r^{23} = 0, \tag{3.2}$$

where, as usual, r^{ij} *acts in* $V^{\otimes 3}$, *non trivially on* (i,j) *spaces and as identity elsewhere.*

Here the skew-symmetry of r means that it satisfies the condition

$$r(v \otimes u) = -r(u \otimes v)^o, \tag{3.3}$$

which implies (2.3).

Conjugating (3.2) by the permutation operator P_{13} and using the skew-symmetry property of r implies:

$$AYBE^*(r) = \quad r^{23}r^{12} + r^{31}r^{23} + r^{12}r^{31} = 0. \tag{3.4}$$

Now $r \in \mathrm{End}(V)$ satisfies both (3.2) and (3.4). Such r then satisfies the full *Skew-Symmetric Classical Yang–Baxter Equation*:

$$\begin{aligned}[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] &= AYBE(r) - AYBE^*(r) \\ &= r^{12}r^{13} - r^{23}r^{12} + r^{13}r^{23} - (r^{23}r^{12} + r^{31}r^{23} + r^{12}r^{31}) = 0 - 0 = 0. \end{aligned} \tag{3.5}$$

The full classical Yang–Baxter equation for a non-skew symmetric r matrix exhibits a different display of indices 32–13 in the third term.

3.1.1. AYBE and double Poisson brackets. The following result of Schedler [39] is a direct corollary from the definitions and (3.1)

Theorem 3.2. *Let A be any $\mathbb{C}$-algebra. An element $r \in \mathrm{End}_{\mathbb{C}}(A \otimes A)$ induces a double Poisson bracket iff r is a skew element satisfying the AYBE and $r \in \mathrm{Der}_{A^e \otimes A^e}((A \otimes A)_{l,r}, (A \otimes A)_{in,out})$.*

Remark 3.3. We consider $(A \otimes A)_{l,r}$ as an $A^e \otimes A^e$-module by having the first A^e act on the first component, and the second on the second component:

$$\big((u \otimes u^o) \otimes (v \otimes v^o)\big)(a \otimes b) = uau^o \otimes vbv^o,$$

and consider $(A \otimes A)_{in,out}$ as an $A^e \otimes A^e$-module by having the first A^e act by inner multiplication and the second A^e act by outer multiplication:

$$\big((x \otimes x^o) \otimes (y \otimes y^o)\big)(a \otimes b) = yax^o \otimes xby^o.$$

4. The Trace Poisson Brackets of the Free Associative Algebras

We shall from now on consider the situation of Example 2.4. In addition we restrict ourselves to particular choices of double brackets and their derived trace Poisson brackets, namely constant, linear and quadratic brackets.

4.1. Three particular brackets.

4.1.1. Constant, linear and quadratic brackets. Let $A = \mathbb{C} < x_1, \ldots, x_m >$ be the free associative algebra. If the double brackets $\{\{x_\alpha, x_\beta\}\}$ between all generators are fixed, then the bracket between two arbitrary elements of A is uniquely defined by identities (2.3) and (2.4).

The constant, linear, and quadratic double brackets are defined respectively by

$$\{\!\{x_\alpha, x_\beta\}\!\} = c_{\alpha\beta} 1 \otimes 1, \quad \text{with} \quad c_{\alpha,\beta} = -c_{\beta,\alpha}, \tag{4.1}$$

$$\{\!\{x_\alpha, x_\beta\}\!\} = b^{\gamma}_{\alpha\beta} x_\gamma \otimes 1 - b^{\gamma}_{\beta\alpha} 1 \otimes x_\gamma, \tag{4.2}$$

and

$$\{\!\{x_\alpha, x_\beta\}\!\} = r^{uv}_{\alpha\beta}\, x_u \otimes x_v + a^{vu}_{\alpha\beta}\, x_u x_v \otimes 1 - a^{uv}_{\beta\alpha}\, 1 \otimes x_v x_u, \tag{4.3}$$

where

$$r^{\sigma\epsilon}_{\alpha\beta} = -r^{\epsilon\sigma}_{\beta\alpha}. \tag{4.4}$$

The summation with respect to repeated indexes is assumed.

It is easy to verify that the bracket (4.1) satisfies (2.4) for any skew-symmetric tensor $c_{\alpha\beta}$.

The following observations of [34] gives us that the condition (2.4) is equivalent for the bracket (4.2) to the identity

$$b^{\mu}_{\alpha\beta} b^{\sigma}_{\mu\gamma} = b^{\sigma}_{\alpha\mu} b^{\mu}_{\beta\gamma}, \tag{4.5}$$

which means that $b^{\sigma}_{\alpha\beta}$ are structure constants of an associative algebra structure on a vector space $V = \oplus_{i=1}^{N} \mathbb{C} x_i$.

The corresponding statement for the quadratic bracket is more subtle. It was shown in [30] that the bracket (4.3) satisfies (2.4) iff the following relations hold:

$$\begin{aligned} &r^{\lambda\sigma}_{\alpha\beta} r^{\mu\nu}_{\sigma\tau} + r^{\mu\sigma}_{\beta\tau} r^{\nu\lambda}_{\sigma\alpha} + r^{\nu\sigma}_{\tau\alpha} r^{\lambda\mu}_{\sigma\beta} = 0, \qquad && a^{\sigma\lambda}_{\alpha\beta} a^{\mu\nu}_{\tau\sigma} = a^{\mu\sigma}_{\tau\alpha} a^{\nu\lambda}_{\sigma\beta}, \\ &a^{\sigma\lambda}_{\alpha\beta} a^{\mu\nu}_{\sigma\tau} = a^{\mu\sigma}_{\alpha\beta} r^{\lambda\nu}_{\tau\sigma} + a^{\mu\nu}_{\alpha\sigma} r^{\sigma\lambda}_{\beta\tau}, && a^{\lambda\sigma}_{\alpha\beta} a^{\mu\nu}_{\tau\sigma} = a^{\sigma\nu}_{\alpha\beta} r^{\lambda\mu}_{\sigma\tau} + a^{\mu\nu}_{\sigma\beta} r^{\sigma\lambda}_{\tau\alpha}. \end{aligned} \tag{4.6}$$

4.1.2. Trace brackets Let us specify the form of the trace Poisson brackets.

We start with the constant trace algebra. It is somehow trivial, but fixes the notation:

$$\{x^{j}_{i,\alpha}, x^{j'}_{i',\beta}\} = c_{\alpha\beta}\, \delta^{j'}_{i}\, \delta^{j}_{i'} \qquad \Leftarrow \qquad \{\!\{x_\alpha, x_\beta\}\!\} = c_{\alpha\beta} 1 \otimes 1. \tag{4.7}$$

Here i, j, i', j' are "matrix" indices running from 1 to N whereas α, β, γ are "vector" indices running from 1 to m. One naturally understands the set of variables $x^{j}_{i,\alpha}$ as a m vector-labeled set of $N \times N$ matrices. One immediately sees that the vector index is an extra feature of this trace algebra when compared to the usual setting of classically integrable systems where the variables are encapsulated into a *single* matrix. The vector index shall later be denoted as "flavor" index using a transparent analogy with particle physics.

The linear trace algebra takes the form:

$$\{x^{j}_{i,\alpha}, x^{j'}_{i',\beta}\} = b^{\gamma}_{\alpha\beta} x^{j'}_{i,\gamma} \delta^{j}_{i'} - b^{\gamma}_{\beta\alpha} x^{j}_{i',\gamma} \delta^{j'}_{i} \quad \Leftarrow \quad \{\!\{x_\alpha, x_\beta\}\!\} = b^{\gamma}_{\alpha\beta} x_\gamma \otimes 1 - b^{\gamma}_{\beta\alpha} 1 \otimes x_\gamma. \tag{4.8}$$

The trace Poisson bracket corresponding to the general quadratic double Poisson bracket (4.3) can be defined on $\mathbb{C}[\mathrm{Rep}_N(A)]$ in the following way [29]:

$$\{x^{j}_{i,\alpha}, x^{j'}_{i',\beta}\} = r^{\gamma\epsilon}_{\alpha\beta} x^{j'}_{i,\gamma} x^{j}_{i',\epsilon} + a^{\gamma\epsilon}_{\alpha\beta} x^{k}_{i,\gamma} x^{j'}_{k,\epsilon} \delta^{j}_{i'} - a^{\gamma\epsilon}_{\beta\alpha} x^{k}_{i',\gamma} x^{j}_{k,\epsilon} \delta^{j'}_{i}, \tag{4.9}$$

where $x_{i,\alpha}^{j}$ are entries of the matrix x_α and δ_i^j is the Kronecker delta-symbol. Relations (4.4) and (4.6) hold iff (4.9) is a Poisson bracket.

It is finally interesting to note that these brackets also take a Hamiltonian form following:

Remark 4.1. Using the observation at the end of the Example 2.4 one writes (4.9) as

$$\{x_{i,\alpha}^{j}, x_{i',\beta}^{j'}\} = \{\mathrm{Tr}(e_{ij}M_\alpha), \mathrm{Tr}(e_{i'j'}M_\beta)\}. \tag{4.10}$$

The constant bracket can be rewritten as

$$\{x_\alpha, x_\beta\} = \mathrm{Tr}(e_{ij}\, c_{\alpha,\beta}\, e_{i'j'}). \tag{4.11}$$

The linear trace-Poisson brackets then read:

$$\{x_{i,\alpha}^{j}, x_{i',\beta}^{j'}\} = \mathrm{Tr}(e_{ij}\Theta_{\alpha,\beta}(e_{i'j'})). \tag{4.12}$$

Following [25], we have introduced the Hamiltonian operator $\Theta \in \mathcal{A}(A) \otimes \mathbb{M}_N(\mathbb{C})$ for the algebra $\mathcal{A}(A)$ generated by left- and right multiplications in $A = \mathbb{C} < x_1, ..., x_m >$:

$$\Theta_{\alpha,\beta} = b_{\alpha\beta}^{\sigma} L_{x_\sigma} - b_{\beta\alpha}^{\sigma} R_{x_\sigma}, \tag{4.13}$$

where $b_{\alpha\beta}^{\sigma}$ are structure constants of an associative algebra as above in (4.2), and

$$L_{x_\alpha} : \begin{cases} A \to A \\ y \to L_{x_\alpha}(y) = x_\alpha y \end{cases} \quad \text{and} \quad R_{x_\beta} : \begin{cases} A \to A \\ y \to R_{x_\beta} y = y x_\beta. \end{cases} \tag{4.14}$$

It is not difficult to write the Hamiltonian operator for the quadratic trace-Poisson brackets (4.3).

$$\Theta_{\alpha,\beta} = a_{\alpha\beta}^{\sigma\epsilon} L_{x_\sigma} L_{x_\epsilon} - a_{\beta\alpha}^{\epsilon\sigma} R_{x_\sigma} R_{x_\epsilon} + r_{\alpha\beta}^{\sigma\epsilon} L_{x_\sigma} R_{x_\epsilon}, \tag{4.15}$$

where $a_{\alpha\beta}^{\sigma\epsilon}$ and $r_{\alpha\beta}^{\sigma\epsilon}$ satisfy the relations (4.4).

Our main purpose now is to reformulate the linear and quadratic trace Poisson algebra in a fully algebraic notation involving the relevant generalization of a classical linear or quadratic r matrix structure.

4.2. Linear trace-brackets: the r-matrix formulation. The linear trace-Poisson algebra is re-expressed using a matrix-Poisson formula, with notations derived from the canonical classical r-matrix formalism [40,41] but augmented by the flavor indices. Accordingly one introduces two auxiliary vector spaces $\mathbb{C}^N$ and $\mathbb{C}^m$ and define the following objects embedded in the general tensorized structure $\mathbb{M}_m(\mathbb{C}) \otimes \mathbb{M}_N(\mathbb{C}) \otimes \mathbb{C}^m \otimes \mathbb{M}_N(\mathbb{C})$, where $\mathbb{M}_m(\mathbb{C}) = \mathrm{End}(\mathbb{C}^m)$:

$$B_{12} = \sum_{\alpha,\beta,\gamma=1}^{m} \sum_{ij=1}^{N} b_{\alpha\beta}^{\gamma}\, e_{\alpha\gamma} \otimes e_{ij} \otimes e_\beta \otimes e_{ji} \in \mathbb{M}_m(\mathbb{C}) \otimes \mathbb{M}_N(\mathbb{C}) \otimes \mathbb{C}^m \otimes \mathbb{M}_N(\mathbb{C}), \tag{4.16}$$

$$B_{21} = \sum_{\alpha,\beta,\gamma=1}^{m} \sum_{ij=1}^{N} b_{\alpha\beta}^{\gamma}\, e_\beta \otimes e_{ij} \otimes e_{\alpha\gamma} \otimes e_{ji} \in \mathbb{C}^m \otimes \mathbb{M}_N(\mathbb{C}) \otimes \mathbb{M}_m(\mathbb{C}) \otimes \mathbb{M}_N(\mathbb{C}), \tag{4.17}$$

$$X = \sum_{\alpha=1}^{m} \sum_{ij=1}^{N} x_{i\alpha}^{j} e_\alpha \otimes e_{ji} \in \mathbb{C}^m \otimes \mathbb{M}_N(\mathbb{C}), \tag{4.18}$$

$$X_1 = X \otimes \mathbb{I}_m \otimes \mathbb{I}_N \in \mathbb{C}^m \otimes \mathbb{M}_N(\mathbb{C}) \otimes \mathbb{M}_m(\mathbb{C}) \otimes \mathbb{M}_N(\mathbb{C}), \tag{4.19}$$

$$X_2 = \sum_{\alpha=1}^{m} \sum_{ij=1}^{N} x_{i\alpha}^{j}\, \mathbb{I}_m \otimes \mathbb{I}_N \otimes e_\alpha \otimes e_{ji} \in \mathbb{C}^m \otimes \mathbb{M}_N(\mathbb{C}) \otimes \mathbb{M}_m(\mathbb{C}) \otimes \mathbb{M}_N(\mathbb{C}). \tag{4.20}$$

$\mathbb{I}$ denotes the identity operator in the corresponding vector space.

We define:

$$\{X_1 \overset{\otimes}{,} X_2\} \equiv \{x_{i,\alpha}^{j}, x_{i',\beta}^{j'}\}\, e_\alpha \otimes e_{ji} \otimes e_\beta \otimes e_{j'i'} \in \mathbb{C}^m \otimes \mathbb{M}_N(\mathbb{C}) \otimes \mathbb{C}^m \otimes \mathbb{M}_N(\mathbb{C}) \tag{4.21}$$

which yields:

$$\{X_1 \overset{\otimes}{,} X_2\} = b_{\alpha\beta}^{\gamma}\, x_{i,\gamma}^{j'}\, e_\alpha \otimes e_{ji} \otimes e_\beta \otimes e_{j'j} - b_{\beta\alpha}^{\gamma}\, x_{i',\gamma}^{j}\, e_\alpha \otimes e_{ji} \otimes e_\beta \otimes e_{ii'}. \tag{4.22}$$

One easily verifies:

$$B_{12} X_1 = b_{\alpha\beta}^{\gamma} x_{i,\gamma}^{j}\, e_\alpha \otimes e_{i'i} \otimes e_\beta \otimes e_{ji'}, \tag{4.23}$$

$$B_{21} X_2 = b_{\alpha\beta}^{\gamma} x_{i,\gamma}^{j}\, e_\beta \otimes e_{ji'} \otimes e_\alpha \otimes e_{i'i}. \tag{4.24}$$

One then deduces the matrix form of the trace Poisson algebra :

Proposition 4.2. *The notation $\{X_1 \overset{\otimes}{,} X_2\} = B_{12}X_1 - B_{21}X_2$ reproduces the relations of the linear brackets for*

$$X = \sum_{\alpha=1}^{m} \sum_{ij=1}^{N} x_{i\alpha}^{j} e_\alpha \otimes e_{ji}, \quad B_{12} = \sum_{\alpha,\beta,\gamma=1}^{m} \sum_{ij=1}^{N} b_{\alpha\beta}^{\gamma}\, e_{\alpha\gamma} \otimes e_{ij} \otimes e_\beta \otimes e_{ji} \sim b_{12} \otimes P,$$

where $P \in \mathbb{M}_N(\mathbb{C}) \otimes \mathbb{M}_N(\mathbb{C})$ is the flip operator and $b_{12} = b_{\alpha\beta}^{\gamma}\, e_{\alpha\gamma} \otimes e_\beta$.

We have used the symbol $\sim$ for an equality valid up to re-ordering in the tensor product of spaces.

The proof is by direct identification.

Let us now discuss the associativity of this linear Poisson bracket in relation with the σ-associativity of its corresponding double bracket.

Proposition 4.3. *The algebra structure on the $\mathbb{C}-$vector space $V =< e_1, .., e_m >$ given by the tensor $b_{\alpha\beta}^{\gamma}$:*

$$e_\alpha e_\beta = b_{\alpha\beta}^{\gamma} e_\gamma$$

satisfies the associativity constraint iff

$$b_{\alpha\beta}^{\mu} b_{\mu\gamma}^{\sigma} = b_{\alpha\mu}^{\sigma} b_{\beta\gamma}^{\mu} \iff b_{12}b_{13} = b_{23}b_{12} \iff B_{12}B_{13} = B_{23}B_{12}. \tag{4.25}$$

This condition ensures that the trace-Poisson bracket of proposition 4.2 obeys the Jacobi identity.

The proof is immediate. Indeed,

$$b_{12}b_{13} = b^{\gamma}_{\alpha\beta}b^{\gamma'}_{\gamma\beta'}e_{\alpha\gamma'} \otimes e_{\beta} \otimes e_{\beta'} = b_{23}b_{12}.$$

For the second equivalence it is enough to observe that

$$B_{12}B_{13} \sim b_{12}b_{13} \otimes P_{12}P_{13}, \quad B_{23}B_{12} \sim b_{23}b_{12} \otimes P_{23}P_{12}$$

but $P_{23}P_{12} = P_{12}P_{13}$ which proves the first claim.

The Jacobi identity for the trace-Poisson bracket is implied by the condition

$$B_{12}B_{13} - B_{23}B_{12} + B_{32}B_{13} - B_{13}B_{12} = 0 \tag{4.26}$$

which is obviously a weaker consequence of (4.25).

4.3. Quadratic trace brackets: the r-matrix formulation. We recall the form of the quadratic Poisson brackets (4.9):

$$\{x^{j}_{i,\alpha}, x^{j'}_{i',\beta}\} = r^{\gamma\epsilon}_{\alpha\beta}x^{j'}_{i,\gamma}x^{j}_{i',\epsilon} + a^{\gamma\epsilon}_{\alpha\beta}x^{k}_{i,\gamma}x^{j'}_{k,\epsilon}\delta^{j}_{i'} - a^{\gamma\epsilon}_{\beta\alpha}x^{k}_{i',\gamma}x^{j}_{k,\epsilon}\delta^{j'}_{i}, \tag{4.27}$$

$$\{\!\{x_{\alpha}, x_{\beta}\}\!\} = r^{\gamma\epsilon}_{\alpha\beta}x_{\gamma} \otimes x_{\epsilon} + a^{\epsilon\gamma}_{\alpha\beta}x_{\gamma}x_{\epsilon} \otimes 1 - a^{\gamma\epsilon}_{\beta\alpha}1 \otimes x_{\epsilon}x_{\gamma}. \tag{4.28}$$

Using the same embedding as before one introduces the following objects:

$$\mathfrak{r}_{12} = \sum_{\alpha,\beta,\gamma,\epsilon}^{m}\sum_{ij=1}^{N} r^{\gamma\epsilon}_{\alpha\beta}e_{\alpha\gamma} \otimes e_{ij} \otimes e_{\beta\epsilon} \otimes e_{ji} \;\sim\; r_{12} \otimes P, \tag{4.29}$$

$$\mathfrak{a}_{12} = \sum_{\alpha,\beta,\gamma,\epsilon}^{m}\sum_{ij=1}^{N} a^{\gamma\epsilon}_{\alpha\beta}e_{\alpha\gamma} \otimes e_{ij} \otimes e_{\beta\epsilon} \otimes e_{ji} \sim a \otimes P. \tag{4.30}$$

One then has

$$\mathfrak{r}_{12}X_1X_2 = r^{\gamma\epsilon}_{\alpha\beta}x^{j}_{i'\gamma}x^{i}_{k\epsilon}\, e_{\alpha} \otimes e_{ii'} \otimes e_{\beta} \otimes e_{jk}, \tag{4.31}$$

$$X^{t}_{2}\mathfrak{a}_{12}X_1 = a^{\gamma\epsilon}_{\alpha\beta}x^{j}_{i'\gamma'}x^{l}_{j\epsilon}\, e_{\alpha} \otimes e_{ii'} \otimes e^{t}_{\beta} \otimes e_{li}, \tag{4.32}$$

$$X^{t}_{1}\mathfrak{a}_{21}X_2 = a^{\gamma\epsilon}_{\alpha\beta}x^{j'}_{i\epsilon}x^{i}_{k\gamma}\, e^{t}_{\beta} \otimes e_{j'j} \otimes e_{\alpha} \otimes e_{jk}. \tag{4.33}$$

It is crucial to emphasize here that the transposition x^t is a *partial* transposition taking place in the flavor space $\mathbb{C}^m$. A "transposed" vector is of course now a co-vector or a linear form, identified by canonical duality.

The Poisson brackets can be expressed as

$$\{X_1 \overset{\otimes}{,} X_2\} = r^{\gamma\epsilon}_{\alpha\beta}x^{j'}_{i\gamma}x^{j}_{i'\epsilon}e_{\alpha} \otimes e_{ji} \otimes e_{\beta} \otimes e_{j'j} + a^{\gamma\epsilon}_{\alpha\beta}x^{k}_{i\gamma}x^{j'}_{k\epsilon}e_{\alpha} \otimes e_{ji} \otimes e_{\beta} \otimes e_{j'j} - a^{\gamma\epsilon}_{\beta\alpha}x^{k}_{i'\gamma}x^{j}_{k\epsilon}e_{\alpha} \otimes e_{ji} \otimes e_{\beta} \otimes e_{ii'}. \tag{4.34}$$

The new r-matrices R and A only carry non-trivial indices of flavor type. In other words the non-trivial contributions to Poisson structure arise solely between any two X matrices with different flavors whilst the same-flavor Poisson structure is trivial. This is again an important distinction with respect to standard quadratic Poisson structure formulation.

The formulas (4.22) and (4.29)–(4.33) finally result in the following

Proposition 4.4. *The quadratic Poisson brackets* (4.27) *can be rewritten as*

$$\{X_1 \overset{\otimes}{,} X_2\} = \mathfrak{r}_{12} X_1 X_2 + (X_2^t \mathfrak{a}_{12} X_1)^{t_2} - (X_1^t \mathfrak{a}_{21} X_2)^{t_1} \tag{4.35}$$

where

$$X = \sum_{\alpha=1}^{m} \sum_{ij=1}^{N} x_{i\alpha}^{j}\, e_\alpha \otimes e_{ji}, \tag{4.36}$$

$$\mathfrak{r}_{12} = \sum_{\alpha,\beta,\gamma,\epsilon}^{m} \sum_{ij=1}^{N} r_{\alpha\beta}^{\gamma\epsilon}\, e_{\alpha\gamma} \otimes e_{ij} \otimes e_{\beta\epsilon} \otimes e_{ji} \tag{4.37}$$

$$\mathfrak{a}_{12} = \sum_{\alpha,\beta,\gamma,\epsilon}^{m} \sum_{ij=1}^{N} a_{\alpha\beta}^{\gamma\epsilon} e_{\alpha\gamma} \otimes e_{ij} \otimes e_{\beta\epsilon} \otimes e_{ji}. \tag{4.38}$$

The properties of r and a implies the following relations for $\mathfrak{r}$ and $\mathfrak{a}$:

$$r_{12} = -r_{21} \Rightarrow \mathfrak{r}_{12} = -\mathfrak{r}_{21}, \quad a_{12} = a_{21} \Rightarrow \mathfrak{a}_{12} = \mathfrak{a}_{21}.$$

4.4. Classical YBE for the full r-matrix structure $(\mathfrak{r}, \mathfrak{a})$. Consider now the Yang–Baxter equations for the full structure matrices $\mathfrak{r}$ and $\mathfrak{a}$ deduced from the compatibility equations for the components r, a previously obtained. We first have:

Proposition 4.5. *If r is a skew-symmetric solution of the AYBE* (4.6) *then $\mathfrak{r}$, as defined in Proposition* 4.4*, is a solution of the classical Yang–Baxter equation*

$$[\mathfrak{r}_{12}, \mathfrak{r}_{13} + \mathfrak{r}_{23}] + [\mathfrak{r}_{13}, \mathfrak{r}_{23}] = 0. \tag{4.39}$$

It is indeed easy to see that:

$$\begin{aligned}[\mathfrak{r}_{12}, \mathfrak{r}_{13} + \mathfrak{r}_{23}] + [\mathfrak{r}_{13}, \mathfrak{r}_{23}] &= (r_{12}r_{13} - r_{23}r_{12} + r_{13}r_{23}) \otimes P_{12}P_{13} \\ &\quad + (r_{12}r_{23} - r_{13}r_{12} - r_{23}r_{13}) \otimes P_{12}P_{23} = 0.\end{aligned}$$

When a classical r-matrix is not skew-symmetric it obeys a generalized version of this better-known CYB equation [21,22,40]. It is however more relevant, in particular with respect to quantization issues, to formulate the CYB conditions for a split pair, here $(\mathfrak{r}, \mathfrak{a})$ where one assumes in addition that $\mathfrak{r}$ is skew-symmetric and $\mathfrak{a}$ is symmetric (i.e. $\mathfrak{a}_{12} = \mathfrak{a}_{21}$). The skew-symmetric part $\mathfrak{r}$ as we have just establish, obeys the canonical skew-symmetric YB equation (4.39). The other condition now is the adjoint $(\mathfrak{r}, \mathfrak{a})$ equation [21,22] which reads:

$$\begin{aligned}[\mathfrak{r}_{12}, \mathfrak{a}_{13} + \mathfrak{a}_{23}] + [\mathfrak{a}_{13}, \mathfrak{a}_{23}] &= r_{12}a_{13} \otimes P_{12}P_{13} + r_{12}a_{23} \otimes P_{12}P_{23} + a_{13}a_{23} \otimes P_{13}P_{23} \\ &\quad - a_{13}r_{12} \otimes P_{13}P_{12} - a_{23}r_{12} \otimes P_{23}P_{12} - a_{23}a_{13} \otimes P_{23}P_{13} \\ &= 0.\end{aligned} \tag{4.40}$$

Note that the sum $\mathfrak{r} + \mathfrak{a}$ then obeys the non-skew-symmetric classical YB equation given some general properties of $\mathfrak{r}$, $\mathfrak{a}$ which we shall not detail here.

One now establishes easily that:

Proposition 4.6. *If* (r, a) *is a solution of the AYBE relations* (4.4) *such that* $a_{12} = a_{21}$ *then* $(\mathfrak{r}, \mathfrak{a})$*, as defined in Proposition (*4.4), is a solution of the adjoint classical Yang–Baxter equation (4.40).

This is a direct consequence of the formula

$$[\mathfrak{r}_{12}, \mathfrak{a}_{13} + \mathfrak{a}_{23}] + [\mathfrak{a}_{13}, \mathfrak{a}_{23}] = (r_{12}a_{13} - a_{23}r_{12} + a_{13}a_{23}) \otimes P_{12}P_{13} + (r_{12}a_{23} - a_{13}r_{12} - a_{23}a_{13}) \otimes P_{12}P_{23}$$

and the implication:

$$r_{13}a_{12} - a_{32}r_{13} = a_{32}a_{12} \Rightarrow r_{12}a_{13} - a_{23}r_{12} = a_{23}a_{13}$$

(via the permutation of spaces $2 \to 3$). We use also $a_{12}a_{31} = a_{31}a_{12}$ and $a_{12} = a_{21}$ which imply $a_{12}a_{31} = a_{13}a_{21}$. Hence $a_{23}a_{13} = a_{32}a_{13} = a_{31}a_{23}$.

4.5. Reflection Algebra Poisson brackets. The formulation of quadratic trace Poisson structure still lacks one type of term as can be seen from the famous general quadratic Poisson bracket ansatz of Freidel and Maillet [18]

$$\{l_1 \overset{\otimes}{,} l_2\} = \tilde{a}_{12}l_1l_2 + l_1\tilde{b}_{12}l_2 - l_1l_2\tilde{d}_{12} + l_2\tilde{c}_{12}l_1. \tag{4.41}$$

To get a full quadratic classical Poisson structure, hereafter denoted reflection algebra (RA), it then appears that we need to add to our formulas some analogue of the term $\tilde{d}$. This is achieved by introducing the transposed matrix of $\mathfrak{r}$ defined by:

$$\tilde{\mathfrak{r}}_{12} = \mathfrak{r}_{12}^{t_{12}} = \sum_{\alpha,\beta,\gamma,\epsilon}^{m} \sum_{ij=1}^{N} r_{\alpha\beta}^{\gamma\epsilon} e_{\gamma\alpha} \otimes e_{ij} \otimes e_{\epsilon\beta} \otimes e_{ji}.$$

Then one has the identification:

$$(X_2^t X_1^t \tilde{\mathfrak{r}}_{12})^{t_{12}} = \mathfrak{r}_{12}X_1X_2$$

so that we end up with the fully quadratic formulation a la Freidel–Maillet.

Proposition 4.7. *The reflection classical trace Poisson algebra is defined as:*

$$\{X_1 \overset{\otimes}{,} X_2\} = \frac{1}{2}\mathfrak{r}_{12}X_1X_2 - \frac{1}{2}(X_2^t X_1^t \tilde{\mathfrak{r}}_{21})^{t_{12}} + (X_2^t \mathfrak{a}_{12} X_1)^{t_2} - (X_1^t \mathfrak{a}_{21} X_2)^{t_1}. \tag{4.42}$$

We have used the notation of Proposition 4.4.

This now clearly suggests a consistent quantization of the trace Poisson algebra as a reflection algebra type structure.

5. Quantum Reflection Trace Algebra

5.1. Quantisation of the trace-Poisson algebra. We now move to the formulation of a quantization for these Poisson algebras. We first consider the quadratic trace-Poisson algebra discussed above and propose a quantized form. We then introduce a general quantum quadratic algebra and define its associated Yang–Baxter equations, using a generalization of the bivector formulation of Freidel–Maillet.

The quantization of a single-flavor general quadratic Poisson bracket has been done in [18]. Here, we need to take into account the delicate issue of transposition with respect to the flavor indices, as it occurred in the previously derived formulae. By analogy, we come up with the following proposition of a quantum quadratic trace algebra:

Proposition 5.1. *Let R and A be two matrices acting on the tensor product of two copies of an auxiliary space $V = \mathbb{C}^N \otimes \mathbb{C}^m$. The space $\mathbb{C}^N$ (resp. $\mathbb{C}^m$) will be called the color (resp. flavor) space.*

We define an associative algebra $\mathcal{A}$ through the following relation

$$(R_{12}(K_1{}^t A_{21} K_2)^{t_1})^{t_2} = ((K_2{}^t A_{12} K_1)^{t_1} R_{12}{}^{t_1 t_2})^{t_1}, \tag{5.1}$$

where the transposition acts solely on the flavor indices, and $K \in \mathcal{A} \otimes End(\mathbb{C}^N) \otimes \mathbb{C}^m$.

Then $\mathcal{A}$ is a quantization of the trace Poisson algebra defined in Proposition 4.7.

Note that the following relation, deduced from (5.1) by exchanging the auxiliary spaces 1 and 2,

$$(R_{21}(K_2^t A_{12})^{t_2} K_1)^{t_1} = (K_1^t (A_{21} K_2)^{t_2} R_{21}^{t_1 t_2})^{t_2} \tag{5.2}$$

is a priori *not*[1] equivalent to (5.1) and must also be considered simultaneously.

Consistency of proposition 5.1 is now proved by defining a quasi-classical limit, assuming the existence of the following $\hbar$ expansions:

$$R = \mathbb{I} - \hbar \mathfrak{r} + \cdots, \quad A = \mathbb{I} + \hbar \mathfrak{a} + \cdots, \quad K = X + \hbar(Y);$$
$$[X_1, X_2] = \hbar \{X_1, X_2\} + \cdots$$

There is no contribution to order $\hbar^0$. The term in $\hbar^1$ reads:

$$\{X_1 \overset{\otimes}{,} X_2^t\} = (\mathfrak{r}_{12} X_1 X_2)^{t_2} + (X_1^t \mathfrak{a}_{21} X_2)^{t_{12}} - (X_2^t \mathfrak{a}_{12} X_1) - (X_2^t X_1^t \tilde{\mathfrak{r}}_{12})^{t_1}.$$

If we apply now the transposition $()^{t_2}$ we get:

$$\{X_1 \overset{\otimes}{,} X_2\} = (\mathfrak{r}_{12} X_1 X_2) + (X_1^t \mathfrak{a}_{21} X_2)^{t_1} - (X_2^t \mathfrak{a}_{12} X_1)^{t_2} - (X_2^t X_1^t \tilde{\mathfrak{r}}_{12})^{t_{12}}$$

which is exactly the classical quadratic trace algebra. We thereby prove the consistency of the choice of (5.1) as a quantization of (4.42).

Proposition 5.1 suggests to introduce what appears as the most general quadratic quantum exchange algebra of trace-type, introducing four a priori distinct structure constant matrices A, B, C, D in a notation directly borrowed from [18].

[1] Of course, this is true only when the flavor space is not trivial: when the flavor space is one-dimensional, the two relations become equivalent.

Definition 5.2. The quantum trace reflection algebra relations read:

$$(A_{12}(K_1^t B_{12})^{t_1} K_2)^{t_2} = (K_2^t (C_{12} K_1)^{t_1} D_{12})^{t_1}. \tag{5.3}$$

Again, we recall that each of the auxiliary spaces 1 and 2 is itself a tensor space of a flavor and a color space. The transposition acts solely on the flavor indices. As before, the following relation, deduced from (5.3) by exchanging the auxiliary spaces 1 and 2,

$$(A_{21}(K_2^t B_{21})^{t_2} K_1)^{t_1} = (K_1^t (C_{21} K_2)^{t_2} D_{21})^{t_2} \tag{5.4}$$

is a priori *not*[2] equivalent to (5.3) and must be considered simultaneously.

5.2. Freidel–Maillet RA formulation and YB equation. In order to derive sufficient conditions for associativity of this RA we need to introduce an alternative representation. For single-flavor RA, it was originally proposed in [18] and possibly related (in this context) to the interpretation of RA as twists of a tensor product of several quantum algebras [14]. This representation interprets the K matrices as partially transposed bivectors. It yielded a completely bivector form for the RA of ZF algebra type: $RKK = KK$ [15,44].

In our context, we must bi-vectorialize the color space $End(\mathbb{C}^N)$ of K, which yields the following proposition:

Proposition 5.3. *The quantum reflection trace algebra can be reformulated as:*

$$\mathcal{R}^{\mathrm{I},\mathrm{II}}_{11',22'} \mathcal{K}^{\mathrm{I}}_{11'} \mathcal{K}^{\mathrm{II}}_{22'} = \mathcal{K}^{\mathrm{II}}_{22'} \mathcal{K}^{\mathrm{I}}_{11'} \quad \textit{where} \quad \mathcal{R}^{\mathrm{I},\mathrm{II}}_{11',22'} = (C_{12'}^{T_{2'}})^{-1} (D_{1'2'}^{T_{1'} T_{2'}})^{-1} A_{12} B_{1'2}^{T_{1'}}, \tag{5.5}$$

As a convention, $11'$ *and* $22'$ *denote the color spaces that are bivectorialized, while* I *and* II *label the flavor spaces. The transpositions are defined as* $T_{1'} \equiv t_{1'} t_{\mathrm{I}}$ *and* $T_{2'} \equiv t_{2'} t_{\mathrm{II}}$.

For the sake of simplicity we have omitted the flavor labels I *and* II *in the matrices* A, B, C, D.

This is easily seen by expanding the relation on the canonical basis. The matrices A, B, C, D are generically expanded as

$$M = \sum_{i,j,k,l=1}^{N} \sum_{\alpha,\beta,\gamma,\delta=1}^{m} M^{ij,kl}_{\alpha\beta,\gamma\delta}\, e_{ij} \otimes e_{\alpha\beta} \otimes e_{kl} \otimes e_{\gamma\delta}$$

and $\mathcal{K}$ is obtained from K as

$$K = \sum_{i,j=1}^{N} \sum_{\alpha=1}^{m} K^{ij}_{\alpha}\, e_{ij} \otimes e_{\alpha} \quad \Rightarrow \quad \mathcal{K} = \sum_{i,j=1}^{N} \sum_{\alpha=1}^{m} \mathcal{K}^{i,j}_{\alpha}\, e_i \otimes e_j \otimes e_{\alpha} \quad \text{with} \quad K^{ij}_{\alpha} = \mathcal{K}^{i,j}_{\alpha}.$$

RA projected on $e_i \otimes e_q \otimes e_\mu \otimes e_k \otimes e_s \otimes e_\nu$ now reads:

$$\sum_{j,l,n,p=1}^{N} \sum_{\alpha,\beta,\gamma,\delta=1}^{m} A^{ij,kl}_{\mu\beta,\nu\gamma} B^{nq,lp}_{\alpha\beta,\gamma\delta} \mathcal{K}^{j,n}_{\alpha} \mathcal{K}^{p,s}_{\delta} = \sum_{j,n,u,r=1}^{N} \sum_{\alpha,\beta,\gamma,\delta=1}^{m} D^{qn,us}_{\mu\alpha,\nu\delta} C^{ij,ru}_{\alpha\beta,\gamma\delta} \mathcal{K}^{k,r}_{\gamma} \mathcal{K}^{j,n}_{\beta}. \tag{5.6}$$

[2] Again, this is true only when the flavor space is not trivial.

A careful reinterpretation of the indices in (5.6) allows us to give it a matricial form:

$$A_{12} B_{1'2}^{T_{1'}} \mathcal{K}_{11'}^{\mathrm{I}} \mathcal{K}_{22'}^{\mathrm{II}} = D_{1'2'}^{T_{1'}T_{2'}} C_{12'}^{T_{2'}} \mathcal{K}_{22'}^{\mathrm{II}} \mathcal{K}_{11'}^{\mathrm{I}} \tag{5.7}$$

Multiplying by the inverse matrices of $D_{1'2'}^{T_{1'}T_{2'}}$ and $C_{12'}^{T_{2'}}$, one gets the ZF algebra type relation (5.5).

Note that $\mathcal{R}$ is now expanded on the canonical basis as

$$\mathcal{R}_{11',22'}^{\mathrm{I},\mathrm{II}} = \sum_{\substack{i,j',p,q'\\ r,s',k,l'}=1}^{N} \sum_{\alpha,\alpha',\gamma,\gamma'=1}^{m} \mathcal{R}_{\alpha\alpha',\gamma\gamma'}^{ij',pq';rs',kl'} \underbrace{e_{ij'}}_{1} \otimes \underbrace{e_{pq'}}_{1'} \otimes \underbrace{e_{\alpha\alpha'}}_{\mathrm{I}} \otimes \underbrace{e_{rs'}}_{2} \otimes \underbrace{e_{kl'}}_{2'} \otimes \underbrace{e_{\gamma\gamma'}}_{\mathrm{II}} \tag{5.8}$$

with

$$\mathcal{R}_{\alpha\alpha',\gamma\gamma'}^{ij',pq';rs',kl'} = \sum_{j,l,q,s=1}^{N} \sum_{\substack{\beta,\beta',\beta''\\ \delta,\delta',\delta''}=1}^{m} \widetilde{C}_{\alpha\beta,\gamma\delta}^{ij,kl} \widetilde{D}_{\beta\beta',\delta\delta'}^{pq,ll'} A_{\beta'\beta'',\delta'\delta''}^{jj',rs} B_{\alpha'\beta'',\delta''\gamma'}^{qq',ss'} \tag{5.9}$$

where $\widetilde{C}_{\alpha\beta,\gamma\delta}^{ij,kl}$ corresponds to the expansion of $(C_{12'}^{T_{2'}})^{-1}$

$$\sum_{j,l=1}^{N} \sum_{\beta,\delta=1}^{m} \widetilde{C}_{\alpha\beta,\gamma\delta}^{ij,kl} C_{\beta\beta',\delta'\delta}^{jj',l'l} = \delta_{\alpha\beta'} \delta_{\gamma\delta'} \delta^{ij'} \delta^{kl'}, \tag{5.10}$$

and $\widetilde{D}_{\alpha\beta,\gamma\delta}^{ij,kl}$ to the expansion of $(D_{12'}^{T_{1'}T_{2'}})^{-1}$

$$\sum_{j,l=1}^{N} \sum_{\beta,\delta=1}^{m} \widetilde{D}_{\alpha\beta,\gamma\delta}^{ij,kl} D_{\beta'\beta,\delta'\delta}^{j'j,l'l} = \delta_{\alpha\beta'} \delta_{\gamma\delta'} \delta^{ij'} \delta^{kl'}. \tag{5.11}$$

5.3. Consistency conditions.

5.3.1. Conditions of unitarity. So-called "Unitarity conditions" follow from requiring that the RA and its rewriting by exchange of the auxiliary space labels 1 and 2 have the same content. Indeed in the $RTT = TTR$ case this condition only yields unitarity conditions on the quantum R matrix hence the name.

The Freidel–Maillet formulation now provides a simple way to get this unitary condition. Comparing both writings leads immediately to

$$\mathcal{R}_{11',22'}^{\mathrm{I},\mathrm{II}} \mathcal{R}_{22',11'}^{\mathrm{II},\mathrm{I}} = \mathbb{I}, \tag{5.12}$$

where $\mathcal{R}_{11',22'}^{\mathrm{I},\mathrm{II}}$ is given in (5.5). Then, one gets the following sufficient conditions:

$$C_{12}^{\mathrm{I},\mathrm{II}} = B_{21}^{\mathrm{II},\mathrm{I}} \quad \text{and} \quad \left((D_{1'2'}^{\mathrm{I},\mathrm{II}})^{T_{1'}T_{2'}}\right)^{-1} A_{12}^{\mathrm{I},\mathrm{II}} = (A_{21}^{\mathrm{II},\mathrm{I}})^{-1} (D_{2'1'}^{\mathrm{II},\mathrm{I}})^{T_{1'}T_{2'}}. \tag{5.13}$$

5.3.2. Conditions of associativity (YBE). Let us now examine the associativity property of the RA

Theorem 5.4. *A sufficient condition for associativity of the RA is given by the Quantum Yang–Baxter equation for $\mathcal{R}$:*

$$\mathcal{R}^{\mathrm{I},\mathrm{II}}_{11',22'}\,\mathcal{R}^{\mathrm{I},\mathrm{III}}_{11',33'}\,\mathcal{R}^{\mathrm{II},\mathrm{III}}_{22',33'} = \mathcal{R}^{\mathrm{II},\mathrm{III}}_{22',33'}\,\mathcal{R}^{\mathrm{I},\mathrm{III}}_{11',33'}\,\mathcal{R}^{\mathrm{I},\mathrm{II}}_{11',22'}, \tag{5.14}$$

where $\mathcal{R}^{\mathrm{I},\mathrm{II}}_{11',22'}$ is the R-matrix introduced in (5.5).

Due to the mixing of flavor indices between A, B, C, D in the definition of $\mathcal{R}^{\mathrm{I},\mathrm{II}}_{11',22'}$ it is more delicate to separate (5.14) into four distinct QYB equations (schematically written as AAA, ABB, DCC and DDD) as was done in the Freidel–Maillet RA We give below some examples where it can nevertheless be done.

5.3.3. Sufficient conditions for consistency. In this section, we shall assume that the flavor and color indices are completely decoupled:

$$M^{\mathrm{I},\mathrm{II}}_{11',22'} = M_{\mathrm{I},\mathrm{II}} \otimes M_{11',22'} \quad \text{for} \quad M = A, B, C, D.$$

Unitarity condition for $\mathcal{R}^{\mathrm{I},\mathrm{II}}_{11',22'}$ is fulfilled when:

$$C_{12} = B_{21}; \quad D_{1'2'}\,D_{2'1'} = \mathbb{I}; \quad A_{12}\,A_{21} = \mathbb{I}, \tag{5.15}$$

$$C_{\mathrm{I},\mathrm{II}} = B_{\mathrm{II},\mathrm{I}}; \quad \left(D^{t_{\mathrm{I}}t_{\mathrm{II}}}_{\mathrm{I},\mathrm{II}}\right)^{-1} A_{\mathrm{I},\mathrm{II}} = (A_{\mathrm{II},\mathrm{I}})^{-1}\,D^{t_{\mathrm{I}}t_{\mathrm{II}}}_{\mathrm{II},\mathrm{I}}. \tag{5.16}$$

The Yang–Baxter equation for $\mathcal{R}^{\mathrm{I},\mathrm{II}}_{11',22'}$ then splits into color and flavor relations. The color relations read

$$A_{12}A_{13}A_{23} = A_{23}A_{13}A_{12}, \quad D_{12}D_{13}D_{23} = D_{23}D_{13}D_{12}; \tag{5.17}$$

$$A_{12}C_{13}C_{23} = C_{23}C_{13}A_{12}, \quad D_{12}B_{13}B_{23} = B_{23}B_{13}D_{12}. \tag{5.18}$$

The matrices realizes the well-known reflection Yang–Baxter equations which need not be discussed here.

For the flavor spaces, we get

$$\begin{aligned}&\left(C^{t_{\mathrm{II}}}_{\mathrm{I},\mathrm{II}}\right)^{-1} R_{\mathrm{I},\mathrm{II}}\,B^{t_{\mathrm{I}}}_{\mathrm{I},\mathrm{II}}\left(C^{t_{\mathrm{III}}}_{\mathrm{I},\mathrm{III}}\right)^{-1} R_{\mathrm{I},\mathrm{III}}\,B^{t_{\mathrm{I}}}_{\mathrm{I},\mathrm{III}}\left(C^{t_{\mathrm{III}}}_{\mathrm{II},\mathrm{III}}\right)^{-1} R_{\mathrm{II},\mathrm{III}}\,B^{t_{\mathrm{II}}}_{\mathrm{II},\mathrm{III}}\\ &\quad= \left(C^{t_{\mathrm{III}}}_{\mathrm{II},\mathrm{III}}\right)^{-1} R_{\mathrm{II},\mathrm{III}}\,B^{t_{\mathrm{II}}}_{\mathrm{II},\mathrm{III}}\left(C^{t_{\mathrm{III}}}_{\mathrm{I},\mathrm{III}}\right)^{-1} R_{\mathrm{I},\mathrm{III}}\,B^{t_{\mathrm{I}}}_{\mathrm{I},\mathrm{III}}\left(C^{t_{\mathrm{II}}}_{\mathrm{I},\mathrm{II}}\right)^{-1} R_{\mathrm{I},\mathrm{II}}\,B^{t_{\mathrm{I}}}_{\mathrm{I},\mathrm{II}}.\end{aligned} \tag{5.19}$$

We have introduced

$$R_{\mathrm{I},\mathrm{II}} = \left(D^{t_{\mathrm{I}}t_{\mathrm{II}}}_{\mathrm{I},\mathrm{II}}\right)^{-1} A_{\mathrm{I},\mathrm{II}}, \tag{5.20}$$

which is unitary, $R_{\mathrm{I},\mathrm{II}}R_{\mathrm{II},\mathrm{I}} = \mathbb{I}$ by (5.16). Remark that the matrices A and D appear only through R, showing a freedom

$$A_{\mathrm{I},\mathrm{II}} \;\to\; G_{\mathrm{I},\mathrm{II}}A_{\mathrm{I},\mathrm{II}} \quad \text{and} \quad D_{\mathrm{I},\mathrm{II}} \;\to\; D_{\mathrm{I},\mathrm{II}}G^{t_{\mathrm{I}}t_{\mathrm{II}}}_{\mathrm{I},\mathrm{II}} \tag{5.21}$$

where $G_{\mathrm{I},\mathrm{II}}$ is any invertible matrix.

The flavor part has the form of a Yang–Baxter equation with the following twisted R matrix:

$$\widetilde{R}_{\mathrm{I},\mathrm{II}} \;=\; \left(C^{t_{\mathrm{II}}}_{\mathrm{I},\mathrm{II}}\right)^{-1} R_{\mathrm{I},\mathrm{II}}C^{t_{\mathrm{I}}}_{\mathrm{II},\mathrm{I}} \quad \text{with} \quad \widetilde{R}_{\mathrm{I},\mathrm{II}}\,\widetilde{R}_{\mathrm{I},\mathrm{III}}\,\widetilde{R}_{\mathrm{II},\mathrm{III}} = \widetilde{R}_{\mathrm{II},\mathrm{III}}\,\widetilde{R}_{\mathrm{I},\mathrm{III}}\,\widetilde{R}_{\mathrm{I},\mathrm{II}}, \tag{5.22}$$

where we have used (5.16) to get the result. It is easy to see that $\widetilde{R}_{\text{I},\text{II}}$ is also unitary.

Starting now from any unitary solution to the Yang–Baxter equation $\widetilde{R}_{\text{I},\text{II}}$ and any invertible matrix $F_{\text{I},\text{II}}$, one reconstructs the matrices R, B and C through

$$B_{\text{II},\text{I}} = C_{\text{I},\text{II}} = F_{\text{I},\text{II}}^{t_{\text{II}}} \quad ; \quad R_{\text{I},\text{II}} \; = \; F_{\text{I},\text{II}}^{-1}\,\widetilde{R}_{\text{I},\text{II}}\,F_{\text{II},\text{I}}. \tag{5.23}$$

Remarkably, the splitting of color and flavor spaces induces the usual *reflection* Yang–Baxter equations for the color part, whereas we identify a *twisted* Yang–Baxter equation for the flavor part.

6. Dynamisation

The next issue is now to define consistent dynamical extensions of the trace reflection algebra. Dynamical extensions of quantum algebra have a long story going back to the Gervais–Neveu–Felder equation (also called "dynamical Yang–Baxter equation") [17,19], characterizing the Belavin–Baxter statistical mechanics R-matrix [3] for IRF models. The general idea is there to introduce a dependance of the R-matrix, and the matrix T encapsulating the algebra generators, in so-called "dynamical" non-operatorial parameters interpreted as coordinates on the dual of some Lie subalgebra of the underlying Lie algebra (or affine algebra) in the quantum structure. In the GNF case, which will be the basis of our derivation here, the subalgebra is the abelian Cartan subalgebra of this Lie/affine algebra (for non-abelian cases see e.g. [43]). The RTT relations and associativity conditions are accordingly modified to yield the GNF-type dynamical Yang–Baxter equations.

This notion naturally extends to RA. The first dynamical RA was identified as consistency conditions [9] for the boundary matrix defining open IRF models. It was later studied as "dynamical boundary algebra" in [16] and [27]. It is identified with a dynamical twist of a quadruple tensor product of quantum affine RTT algebras [20]. The second one was identified [4,5] in the quantum formulation of Ruijsenaar–Schneider models [38]. It was later studied in [27] and characterized in [6] as a deformation of a non-dynamical RTT algebra by a dynamical semi-gauge action. A third one was recently constructed in [7] and seems related to twisted Yangian structures instead of quantum affine algebras.

6.1. Freidel–Maillet formulation of the dynamical reflection algebras.

In order to define dynamical versions of the trace RA we will use the bivector formulation a la Freidel–Maillet. We must first of all construct such a formulation for the single-flavor dynamical RA described above. To the best of our knowledge this has never been done. The formulation which we propose follows on these lines:

All three dynamical RA's are represented by similar-looking quadratic exchange algebra relations:

$$\begin{aligned} &A_{12}(\lambda)K_1(\lambda - \epsilon_R h_2)B_{12}(\lambda)K_2(\lambda + \epsilon_L h_1) \\ &\quad = K_2(\lambda - \epsilon_R h_1)C_{12}(\lambda)K_1(\lambda + \epsilon_L h_2)D_{12}(\lambda) \end{aligned} \tag{6.1}$$

where ϵ_L and ϵ_R are some complex number characterizing the different reflection algebras (see below).

The dynamical variables are encapsulated in an n dimensional vector λ which will be omitted whenever no ambiguity of notation arises. It is assumed that the auxiliary vector space on which the structure matrices act is a diagonalizable, fully reducible module of the Cartan algebra, hence notations such as $K_1(\lambda - \epsilon_R h_2)$ are self explanatory. They will be used everytime the shift operates along a copy of the dual Cartan algebra *not* acted upon by the matrix inside which it appears. Shifts along copies of the dual Cartan algebra acted upon by the matrix itself must be defined in a more specific context. This brings us to define precisely the notions of "external" and "internal" shifts which will be of use throughout our derivation.

Definition 6.1. We recall the well-known notation of outside action of a shift operator [i.e. shift along a Cartan algebra copy (a) not acted upon by the matrix $M(\lambda)$]. It reads:

$$e^{(\epsilon h_a \partial)} M_{\dots}(\lambda) e^{(-\epsilon h_a \partial)} = M_{\dots}(\lambda + \epsilon h_a). \tag{6.2}$$

Definition 6.2. Consider now the problem of inside action. Denote by $M_{\dots a \dots}$ a matrix acted upon by $e^{(\epsilon h_a \partial)}$. In order to obtain a pure c-number matrix without explicit difference operators after action of the shifts we must then consider only the following objects:

$$((e^{(\epsilon h_a \partial)} M)^{t_a} e^{(-\epsilon h_a \partial)})^{t_a} := M^{sr(a)}, \tag{6.3}$$

$$((e^{(\epsilon h_a \partial)} (M e^{(-\epsilon h_a \partial)})^{t_a} := M^{sc(a)}, \tag{6.4}$$

where, for conciseness, we have omitted the λ-dependence in M.

One also defines a natural extension of the shift-row procedure to single vector indices:

$$K_{aa'}^{\mathrm{s}_{a'}(-\epsilon_L)} := \left((e^{(\epsilon h_{a'} \partial)} K_{aa'}\right)^{t'_a} e^{(-\epsilon h_{a'} \partial)})^{t_{a'}}, \quad a \in \{1, 2\}. \tag{6.5}$$

In the following, we will use the notation $\bar{K}_{aa'} = K_{aa'}^{\mathrm{s}_{a'}(-\epsilon_L)}$.

This defines the notations $sc(a)$, $sr(a)$ and s_a. We see of course that their application to a matrix depending on λ mean that the matrix elements are transformed by a shift of their dynamical variables as $\lambda_i \to \lambda_i + \delta_{i,k}$ where k is column (resp. row) index in their tensorial factor (a). In this way we have taken care of the situation where shifts occur along copies of the dual Cartan algebra acted upon by the matrix itself.

A number of identities must now be established as interplay between different types of shifts. First of all one has:

$$(M^{sr(a)})^{t_a} = (M^{t_a})^{sc(a)}. \tag{6.6}$$

This identity does not follow in a manifest way from (6.3), (6.4) but must be checked directly by computing the matrix elements.

Then one can in fact reinterpret outside shift as inside shift of a 'completed" matrix as follows:

$$M_{\dots}(\lambda + \epsilon h_a) := (M_{\dots} \otimes \mathbb{I}_a)^{sc(a)} := (M_{\dots} \otimes \mathbb{I}_a)^{sr(a)}. \tag{6.7}$$

Shift operations along a space (a) factor out on matrix products only when one of the factor matrices acts diagonally on this space. One has:

$$(M_{\dots a \dots} D_{\dots a \dots})^{sc(a)} := (M_{\dots a \dots}^{sc(a)} D_{\dots a \dots})^{sc(a)} \tag{6.8}$$

and

$$(D_{\dots a\dots}M_{\dots a\dots})^{sr(a)} := (D^{sr(a)}_{\dots a\dots}M_{\dots a\dots})^{sr(a)}, \tag{6.9}$$

where M is any matrix and D is diagonal on space (a). Of course shift-row and shift-column are identical operations on D. Combining (6.7), (6.9) and (6.8) yields the identification:

$$(N_{\dots}(\lambda + \epsilon h_a)(M_{\dots a\dots})^{sr(a)} := (N_{\dots}M_{\dots a\dots})^{sr(a)} \tag{6.10}$$

and dually

$$M^{sc(a)}_{\dots a\dots}N_{\dots}(\lambda + \epsilon h_a) := (M_{\dots a\dots}N_{\dots})^{sc(a)}. \tag{6.11}$$

Let us now consider the particular case of structure matrices. A key consistency property for dynamical reflection algebras are the zero-weight conditions of the structure matrices A, B, C, D. They must indeed obey:

$$\epsilon_R\,[h^{(1)} + h^{(2)},\ A_{12}] \;=\; \epsilon_L\,[h^{(1)} + h^{(2)},\ D_{12}] \;=\; 0, \tag{6.12}$$
$$[\epsilon_R\,h^{(1)} - \epsilon_L\,h^{(2)},\ C_{12}] \;=\; [\epsilon_L\,h^{(1)} - \epsilon_R\,h^{(2)},\ B_{12}] \;=\; 0. \tag{6.13}$$

The three dynamical RA's respectively correspond to the choice $\epsilon_R = -1, \epsilon_L = 1$ (DBA); $\epsilon_R = -1, \epsilon_L = 0$ (so-called semi-dynamical RA) ; $\epsilon_R = -1, \epsilon_L = -1$ (twisted Yangian RA).

We now introduce the notion of zero-weight shift which will be in fact particularly relevant to such structure matrices:

Definition 6.3. Consider a matrix $M_{\dots ab\dots}$ obeying a zero-weight condition

$$[\epsilon_a\,h^{(a)} + \epsilon_b\,h^{(b)},\ M_{\dots ab\dots}] = 0. \tag{6.14}$$

The exponential of the "zero-weighted" shifts $e^{(\epsilon_a h_a + \epsilon_b h_b \partial)}$ does act on the relevant matrix to yield again a pure c-number matrix (no shift term remains) with shifts inside the matrix elements given in terms of the sr, sc notions. This action yields a crossing shift formula:

$$e^{(\epsilon_a h_a \partial)}\tilde{M}_{ab}e^{(-\epsilon_b h_b \partial)} := e^{(-\epsilon_b h_b \partial)}M_{ab}e^{(\epsilon_a h_a \partial)}. \tag{6.15}$$

In particular from the zero-weight conditions on A, B, C, D one has:

$$e^{(\epsilon_R h_1 \partial)}\tilde{A}_{12}e^{(-\epsilon_R h_2 \partial)} := e^{(\epsilon_R h_2 \partial)}A_{12}e^{(-\epsilon_R h_1 \partial)}, \tag{6.16}$$
$$e^{(\epsilon_L h_1 \partial)}\tilde{D}_{12}e^{(-\epsilon_L h_2 \partial)} := e^{(\epsilon_L h_2 \partial)}D_{12}e^{(-\epsilon_L h_1 \partial)}, \tag{6.17}$$
$$e^{(\epsilon_L h_1 \partial)}\tilde{B}_{12}e^{(\epsilon_R h_2 \partial)} := e^{(\epsilon_R h_2 \partial)}B_{12}e^{(\epsilon_L h_1 \partial)}, \tag{6.18}$$
$$e^{(\epsilon_L h_2 \partial)}\tilde{C}_{12}e^{(\epsilon_R h_1 \partial)} := e^{(\epsilon_R h_1 \partial)}C_{12}e^{(\epsilon_L h_2 \partial)}. \tag{6.19}$$

An important property of the shift of a product involving a zero-weight matrix is the following:

Proposition 6.4. *Given a A_N zero-weight matrix M_{ab} such that*

$$[h^{(a)} + h^{(b)},\ M_{ab}]\ = 0$$

(the possible other space labels are omitted) and a matrix C_{ab} (without any weight conditions), then:

$$(C_{ab}M_{ab})^{sc(a)sc(b)} := C_{ab}^{sc(a)sc(b)} M_{ab}^{sc(a)sc(b)} \tag{6.20}$$

and

$$(M_{ab}C_{ab})^{sr(a)sr(b)} := M_{ab}^{sr(a)sr(b)} C_{ab}^{sr(a)sr(b)}. \tag{6.21}$$

The proof is by direct computation of the respective matrix elements, using the fact that the set of row and column indices of a A_N zero-weight matrix are identified.

We are now able to prove the following:

Theorem 6.5. *The dynamical RA (6.1) is represented in the bivector formalism by the following expression:*

$$\begin{aligned} &A_{12}(\lambda - \epsilon_L(h_{2'} + h_{1'}))\big(B_{1'2}^{t_{1'}}(\lambda - \epsilon_L h_{2'})\big)^{\mathrm{sr}_{1'}(\epsilon_L)} \bar{K}_{11'}(\lambda - \epsilon_R h_2 - \epsilon_L h_{2'})\bar{K}_{22'}(\lambda) \\ &= \big(D_{1'2'}^{t_{1'}t_{2'}}(\lambda)\big)^{\mathrm{sr}_{1'}(\epsilon_L)\mathrm{sr}_{2'}(\epsilon_L)} \big(C_{12'}^{t_{2'}}(\lambda - \epsilon_L h_{1'})\big)^{\mathrm{sr}_{2'}(\epsilon_L)} \bar{K}_{22'}(\lambda - \epsilon_R h_1 - \epsilon_L h_{1'})\bar{K}_{11'}(\lambda). \end{aligned} \tag{6.22}$$

The proof is a long and delicate (but rather straightforward) computation. It requires first of all to rewrite (6.1) using explicit shift operators of the general form $e^{(-\epsilon_{L/R} h_{2/1}\partial)}$.

$$\begin{aligned} &A_{12} e^{(-\epsilon_R h_2 \partial)} K_{11'}^{t_{1'}} e^{(\epsilon_R h_2 \partial)} B_{1'2} e^{(\epsilon_L h_{1'}\partial)} K_{22'}^{t_{2'}} \\ &= e^{(-\epsilon_R h_1 \partial)} K_{22'}^{t_{2'}} e^{(\epsilon_R h_1 \partial)} C_{12'} e^{(\epsilon_L h_{2'}\partial)} K_{11'}^{t_{1'}} e^{(-\epsilon_L h_{2'}\partial)} D_{1'2'} e^{(\epsilon_L h_{1'}\partial)}. \end{aligned}$$

Partial transposition with respect to space indices $1'$ and $2'$ redefines as before K as bivectors instead of matrices. Using the cross-shift properties (6.17), (6.18), (6.19) and (6.16) allows to rewrite the previous equality as:

$$\begin{aligned} &A_{12} e^{(-\epsilon_R h_2 \partial)} (\tilde{B}_{1'2}^{t_{1'}})^{\mathrm{sc}_{1'}(\epsilon_\mathrm{L})} e^{(\epsilon_L h_{1'}\partial)} K_{11'}^{\mathrm{s}_{1'}(-\epsilon_\mathrm{L})} e^{(\epsilon_R h_2 \partial)} e^{(\epsilon_L h_{2'}\partial)} K_{22'}^{\mathrm{s}_{2'}(-\epsilon_L)} \\ &= ((\tilde{C}_{12'}(\tilde{D}_{1'2'}(\epsilon_R h_1))^{\mathrm{sc}_{1'}(\epsilon_\mathrm{L})t_{2'}})^{\mathrm{sc}_{2'}(\epsilon_\mathrm{L})} e^{(\epsilon_L h_{2'}\partial)} K_{22'}^{\mathrm{s}_{2'}(-\epsilon_L)} e^{(\epsilon_R h_1 \partial)} e^{(\epsilon_L h_{1'}\partial)} K_{11'}^{\mathrm{s}_{1'}(-\epsilon_L)}. \end{aligned}$$

In the equation above and in the following, only shifts in the dynamical parameter are indicated, and for instance $D_{1'2'}(\epsilon_R h_1)$ stands for $D_{1'2'}(\lambda + \epsilon_R h_1)$.

Pushing the shift operators $e^{(\epsilon_R h_2 \partial)} e^{(\epsilon_L h_{2'}\partial)}$ and $e^{(\epsilon_R h_1 \partial)} e^{(\epsilon_L h_{1'}\partial)}$ to the left, using the definition of the internal shifts allows for undoing the cross-shift of A, B, C, D to yield:

$$\begin{aligned} &A_{12}(-\epsilon_L(h_{2'} + h_{1'}))(B_{1'2}^{t_{1'}}(-\epsilon_L h_{2'}))^{\mathrm{sr}_{1'}(\epsilon_L)} \bar{K}_{11'}(-\epsilon_R h_2 - \epsilon_L h_{2'})\bar{K}_{22'} \\ &= [(C_{12'}(-\epsilon_L h_{1'})(D_{1'2'}^{\mathrm{sc}_{1'}(-\epsilon_L)})^{t_{1'}})^{\mathrm{sr}_{2'}(-\epsilon_L)}]^{t_{2'}} \bar{K}_{22'}(-\epsilon_R h_1 - \epsilon_L h_{1'})\bar{K}_{11'}. \end{aligned} \tag{6.23}$$

The structure matrices on the l.h.s. are now decoupled. To achieve a similar decoupling of $[(C_{12'}(-\epsilon_L h_{1'})(D_{1'2'}^{\mathrm{sc}_{1'}(-\epsilon_L)})^{t_{1'}})^{\mathrm{sr}_{2'}(-\epsilon_L)}]^{t_{2'}}$ on the r.h.s. we essentially use Eqs. (6.6), (6.7), Proposition 6.4 and Eq. (6.11) to finally yield the decoupled terms:

$$(D_{1'2'}^{t_{1'}t_{2'}})^{\mathrm{sr}_{1'}(\epsilon_L)\mathrm{sr}_{2'}(\epsilon_L)} (C_{12'}^{t_{2'}}(-\epsilon_L h_{1'}))^{\mathrm{sr}_{2'}(\epsilon_L)}.$$

This representation of the dynamical reflection algebras allows to identify some key features of dynamical reflection algebras which shall be crucial guidelines in our conjectured formulation of a dynamical quantum reflection algebra. The bivector representation is indeed essential to identify these features and represent probably a deeper formulation of reflection algebras in general.

Criterion 1. The shifts separate into shifts labeled by the tensorial factors generated by the original vector-type indices in K (row indices, unprimed) weighted by $-\epsilon_R$; and shifts labeled by the tensorial factors corresponding to the original covector-type indices in K (column indices, primed) weighted by $-\epsilon_L$.

Criterion 2. The structure matrices A, B, C, D are zero-weighted according to the nature of their tensorial labels with the proviso that the zero-weight conditions be written for the *partially transposed* matrices such as occur in the bivector formulation.

Criterion 3. All four structure matrices are shifted along both primed-labeled directions, weighted by $-\epsilon_L$. Depending whether these labels occur or not in the matrix the shifts are inside shifts (resp. outside).

Criterion 4. The K matrices are shifted along three directions: the two respective outside shifts and the inside prime (transposed covector) shift occur with their respective consistent weights of Criterion 1.

6.2. Conjectural dynamical quantum reflection trace algebra. The quantum reflection trace algebra structure essentially differs from the QRA by the occurrence of the flavor vector index in K and the pair of corresponding extra auxiliary spaces in A, B, C, D. Our key hypothesis is to treat this extra vector index in K as a supplementary 'true" vector index (unprimed). The dynamization will now also contain a deformation parametrized by coordinates on the dual of the abelian Cartan subalgebra of the new flavor Lie algebra, a priori here A_{m-1}. Once again one assumes that the supplementary flavor vector spaces are diagonalizable fully reducible modules of this flavor Cartan algebra.

Accordingly we now introduce a third weight ϵ_f for the associated shifts (Criterion 1); complement the zero-weight conditions on A, B, C, D by this extra weight and the extra generators of the abelian Cartan subalgebra of the new flavor Lie algebra (Criterion 2); do not modify the shift structure on the matrices A, B, C, D themselves (since this extra vector index is a 'true" index, not a transposed covector) (Criterion 3); shift the K matrices additionally along their *outside* flavor space (Criterion 4).

We thus propose the following form where the flavor space labels have been omitted for the sake of simplicity, and as in (5.5) the transpositions are defined as $T_{1'} \equiv t_{1'}t_{\mathrm{I}}$ and $T_{2'} \equiv t_{2'}t_{\mathrm{II}}$.

$$\begin{aligned} &A_{12}(\lambda - \epsilon_L(h_{2'} + h_{1'}))\big(B_{1'2}^{T_{1'}}(\lambda - \epsilon_L h_{2'})\big)^{\mathrm{sr}_{1'}(\epsilon_L)} \bar{K}_{11'}(\lambda - \epsilon_R h_2 - \epsilon_L h_{2'} - \epsilon_f h_{\mathrm{II}})\bar{K}_{22'}(\lambda) \\ &\quad = \big(D_{1'2'}^{T_{1'}T_{2'}}(\lambda)\big)^{\mathrm{sr}_{1'}(\epsilon_L)\mathrm{sr}_{2'}(\epsilon_L)}\big(C_{12'}^{T_{2'}}(\lambda - \epsilon_L h_{1'})\big)^{\mathrm{sr}_{2'}(\epsilon_L)} \bar{K}_{22'}(\lambda - \epsilon_R h_1 - \epsilon_L h_{1'} - \epsilon_f h_{\mathrm{I}})\bar{K}_{11'}(\lambda). \end{aligned} \tag{6.24}$$

Although the transpositions T contain transpositions of the *flavor*-labeled components in the structure matrices A, B, C, D we have decided here to formulate and apply Criterion 3 so as to not shift the corresponding matrices along these *flavor* transposed space, since we interpret the shifts of B, C, D in (6.23) to only operate in directions corresponding to transposed covector indices in K. There is however a possible ambiguity since one could also interpret more broadly these shifts as occurring along all directions characterized by a transposed label. In this case one should reinterpret $\mathrm{sr}_{1'}, \mathrm{sr}_{2'}$ as $\mathrm{sr}_{1'} + \mathrm{sr}_{\mathrm{I}}, \mathrm{sr}_{2'} + \mathrm{sr}_{\mathrm{II}}$ in B, C, D and $\bar{K}$ wherever they occur in (6.24). Lifting the ambiguity requires an in-depth study of these structures which we shall leave for a later investigation.

7. Conclusion

We have established a consistent form for the quantum trace reflection algebra, and conjectured a form for an abelian dynamical deformation, on the lines defined by the flavorless dynamical reflection algebras. We wish to emphasize once again that the rewriting of reflection algebras (particularly dynamical) in a bivector formalism plays a crucial role in that it has allowed us to extract what appear to be the key features of this type of dynamical deformation. It suggests that the bivector Freidel–Maillet formulation is possibly the relevant frame to understand in depth the quantum reflection algebra (a point always defended by the authors of [18] and consistent with the construction of reflection algebras by twisting a quadrupled algebra in [14, 20]).

It is worth stressing here that our approach to the quantization problem of trace-Poisson brackets and the very formulation of this problem has no immediate straightforward application to the natural question of a proper deformational quantization of the *Van den Bergh double Poisson structure* itself as it was addressed by Calaque[3]: "Is there a notion of "quantization", or "double star-product" for double Poisson algebras, so that it would induce genuine star-products quantizing the above mentioned (i.e. coordinate rings of the representation moduli spaces) Poisson varieties?" Or, in other words "what kind of algebraic structure on an algebra A ensures that one will get star-products on $\mathrm{Rep}_{\mathrm{N}}(\mathrm{A})$?". Anyway, we hope that the scheme proposed here will help to clarify some aspects of the problem.

Let us also stress that we have not addressed here the case of linear double brackets and their subsequent trace-Poisson brackets. Their structure is described (4.16) by a mixed object B carrying both matrix and vector flavor indices. By contrast the quadratic double brackets yield quadratic trace-Poisson "structure constants" with pure matrix flavor indices, allowing for a quantization and dynamization on more familiar lines as Yang–Baxter R-type matrices. It is not clear at this stage how to address the linear case and we shall postpone its discussion for the time being.

Besides this issue several deep questions are now opened following our derivations. A few have already been mentioned: how about the parametric AYB algebras? This of course immediately begs the question of elliptic-type deformations (i.e., along the d generator of an underlying *affine* algebra). In addition we should ask the question of a co-algebra or (more probably) co-ideal structure in the non-dynamical case, in relation with the quasi-Hopf structure, which it seems to exhibit. Finally in the dynamical case one must now address the issues both of consistency and relevance (physical and mathematical) of this conjectured structure (whichever version of Criterion 3 turns out to be consistent). The existence of a full rewriting procedure, reversing the previous bivectorializations, of this structure in terms of K *matrices* as an explicit reflection formula mimicking (5.5) with suitable internal and external shifts, may provide a good consistency criterion.

3 See http://mathoverflow.net/questions/29543/what-is-a-double-star-product.

Acknowledgements. This work was sponsored by CNRS; Université de Cergy-Pontoise; Université de Savoie; and ANR Project DIADEMS (Programme Blanc ANR SIMI1 2010-BLAN-0120-02). JA and VR wish to thank LAPTh for their kind hosting. The work of VR was partially supported by RFBR Grant 12-01-00525. He is thankful to A. Odesskii and V. Sokolov for discussions and collaboration on related subjects.

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Communicated by N. Reshetikhin